

An Algebraic Abstraction of the Localic Sheafification via the Tripos-to-Topos Construction

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Abstract

Localic and realizability toposes are two central classes of toposes in categorical logic, both arising through the Hyland–Johnstone–Pitts tripos-to-topos construction.

We investigate their shared geometric features by providing an algebraic abstraction of the notions of localic presheaves, sheafification and their connection to supercompactification of a locale via an instance of the Comparison Lemma. This can be applied to a broad class of toposes obtained to the tripos-to-topos constructions, including all those generated from a tripos based on the classical category of ZFC-sets.

These results provide a unified geometric framework for understanding localic and realizability toposes.

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1. Introduction

Localic and realizability toposes represent two of the most fundamental classes of toposes in categorical logic, whose key distinction lies in the fact that localic toposes are Grothendieck toposes of sheaves, whereas realizability toposes are not.

In the 1980s, Hyland, Johnstone, and Pitts introduced the notion of *tripos* [19] to explain, from an abstract perspective, in what sense Higgs' description of localic sheaf toposes as $\mathbf{H}$ -valued sets [16] and Hyland's Effective topos $\mathbf{Eff}$ [18] are both instances of the same general construction.

Inspired by the works of Higgs [16] and of Fourman and Scott [11], they defined a particular family of Lawvere's hyperdoctrines [25, 24, 23], called triposes, together with the *tripos-to-topos* construction, producing a topos $\mathbf{T}_P$ from a given tripos $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Hey}$. Both localic and realizability toposes can then be shown to arise as instances of this general construction for suitable triposes.

The main goal of this work is to further investigate the common geometric structures underlying these two classes of toposes from a more geometric and broader categorical perspective.

To this end, it is useful to recall some well-known fundamental features of locales and localic toposes: given a locale $\mathbf{L}$, applying the construction that freely adds suprema to a meet-semilattice yields a supercoherent locale $D(\mathbf{L})$ [1], i.e. the *supercompactification* of $\mathbf{L}$. Moreover, by the classical Comparison

Lemma, we obtain an equivalence

$$\mathbf{PSh}(\mathbf{L}) \equiv \mathbf{Sh}(D(\mathbf{L})).$$

Summarizing, we have the following situation:

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\mathbf{Sh}(-)} & \mathbf{Sh}(\mathbf{L}) \\ \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) \\ D(\mathbf{L}) & \xrightarrow{\mathbf{Sh}(-)} & \mathbf{Sh}(D(\mathbf{L})) \equiv \mathbf{PSh}(\mathbf{L}) \end{array}$$

Our goal is to generalize the previous picture to triposes. In particular, we aim to abstract the notions of localic presheaf category, sheafification, supercompactification of a locale and the equivalence $\mathbf{PSh}(\mathbf{L}) \simeq \mathbf{Sh}(D(\mathbf{L}))$, in a broader context of triposes including also realizability ones.

We start by identifying the categorical counterpart of the localic presheaf category. To this purpose, we recall that any localic presheaf topos $\mathbf{PSh}(\mathbf{L})$ on a locale $\mathbf{L}$ can be described as the **ex/lex**-completion $(\mathcal{G}_{\mathbf{L}(-)})_{\mathbf{ex/lex}}$, in the sense of Carboni [7, 6], of the category of points $\mathcal{G}_{\mathbf{L}(-)}$ associated to the localic tripos $\mathbf{L}^{(-)}: \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Hey}$. Inspired by this fact, we generalize the notion of localic presheaf category $\mathbf{E}_P$ for a tripos P by identifying it with the exact completion $\mathbf{E}_P := (\mathcal{G}_P)_{\mathbf{ex/lex}}$ of the category of points $\mathcal{G}_P$.

Then, to provide an algebraic rendering of the localic instance of the Comparison Lemma, we identify the localic tripos of the supercompactification of a locale with the construction known as the *full existential completion* [33], originally introduced in [45], which freely adds existential quantifiers to a given doctrine.

This observation is motivated by the fact that, for the localic tripos on a locale $\mathbf{L}$, its full existential completion coincides with the localic tripos associated to the supercompactification $D(\mathbf{L})$, as shown in [33, Thm. 7.32]. Intuitively, the generalization of the notion of a supercompact element of a locale to the setting of a tripos corresponds to replacing the role of arbitrary joins (not necessarily available in a tripos) with existential quantification (see also [34, Sec. 3.2]). And, hence this kind of generalization differs from the one developed in the context of Grothendieck toposes [43, 5, 39], whose development crucially relies on arbitrary disjoint coproducts—structures that are not generally available in a tripos.

Furthermore, there is a crucial difference between our generalized notion of supercompactification for a tripos and its localic instance: while the supercompactification of a locale is again a locale, the full existential completion of an arbitrary tripos is not, in general, a tripos. Whenever this property holds, we say that the tripos is $\exists$ -supercompactifiable.

Finally, we employ all the above notions to show an analogue of the localic instance of the Comparison Lemma, stating that

$$\mathbf{E}_P \equiv \mathbf{T}_{P^\exists}$$

where $P^\exists$ is the full existential completion of P , and $\mathbf{T}_{P^\exists}$ its tripos to-topos. This lets us conclude that the category $\mathbf{E}_P$ is a topos if and only if P is $\exists$ -supercompactifiable.

Whenever P is $\exists$ -supercompactifiable, we also show that the topos $\mathbf{T}_P$ generated from P is a category of j_P -sheaves for a Lawvere–Tierney topology j_P on $\mathbf{E}_P$; hence, $\mathbf{T}_P$ arises as the result of an *abstract localic sheafification*.

The following diagram can summarize the situation:

$$\begin{array}{ccc} P & \xrightarrow{\quad \tau \quad} & \mathbf{T}_P \\ \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \uparrow \\ \dashv \\ \downarrow \end{array} \right) \\ P^\exists & \xrightarrow{\quad \tau \quad} & \mathbf{T}_{P^\exists} \equiv \mathbf{E}_P \end{array}$$

where $\mathbf{E}_P := (\mathcal{G}_P)_{\text{ex/lex}}$, and P is a $\exists$ -supercompactifiable tripos.

Then we *characterize the class of $\exists$ -supercompactifiable triposes* by identifying it with the class of *triposes whose base category has weak dependent products and a generic proof* in the sense of [36, 38], employing the analysis of the tripos-to-topos construction and exact completions developed in [41, 31, 28].

This characterization plays a crucial role in showing that the class of $\exists$ -supercompactifiable triposes includes all triposes over the usual category of sets within Zermelo–Fraenkel set theory with Choice, and hence all realizability triposes, besides localic ones. Among the main examples belonging to this class we recall: the Modified Realizability tripos [20, 47], the Extensional Realizability tripos [46], the Dialectica tripos [3], the Krivine tripos [44]. As a further significant example of an $\exists$ -supercompactifiable tripos that is not **Set**-based, we mention the tripos of extended Weihrauch degrees recently introduced in [35].

Finally, to demonstrate the breadth of the class of $\exists$ -supercompactifiable triposes, we prove an analogue of the fundamental theorem of toposes, showing that these triposes are closed under localization. This constitutes a necessary step toward extending our framework to fibrations of toposes, including those formalized within a predicative metalanguage, as in [27], where the reference to Lawvere–Tierney sheaves is essential.

2. Preliminaries on doctrines and triposes

In this section, we briefly provide some categorical background for analyzing the tripos-to-topos construction via completions of doctrines and their relationship to geometric morphisms between toposes. We begin by recalling the completions of doctrines that we use to prove the main theorems of the paper.

2.1. Lex primary and full existential doctrines

The notion of a hyperdoctrine was introduced by F. W. Lawvere in a series of seminal papers [25, 24]. We recall here some definitions that will be useful in what follows. Further details on the theory of elementary and existential doctrines can be found in [30, 29, 31, 28, 10].

We indicate with **Set** the category of sets formalizable within the classical axiomatic set theory ZFC.

Definition 2.1. A **lex primary doctrine** is a functor $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ from the opposite of a category $\mathcal{C}$ with finite limits to the category of inf-semilattices.

Definition 2.2. Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ and $R: \mathcal{D}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be two lex primary doctrines. A **lex primary morphism** of doctrines is given by a pair $(F, \flat)$

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \downarrow F^{\text{op}} & \searrow P & \\
 & \flat & \mathbf{InfSl} \\
 \mathcal{D}^{\text{op}} & \nearrow R &
 \end{array}$$

where

- $F: \mathcal{C} \rightarrow \mathcal{D}$ is a finite limits preserving functor;
- $\mathfrak{b}: P \rightarrow R \circ F$ is a natural transformation.

Definition 2.3. Let us consider two lex primary doctrines $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ and $R: \mathcal{D}^{\text{op}} \rightarrow \mathbf{InfSl}$, and two lex primary morphisms $(F, \mathfrak{b}), (G, \mathfrak{c}): P \rightarrow R$. A **doctrine transformation** is a natural transformation $\theta: F \rightarrow G$ such that

$$\mathfrak{b}_A(\alpha) \leq R_{\theta_A}(\mathfrak{c}_A(\alpha))$$

for every α of $P(A)$.

We denote by **LexDoc** the 2-category of lex primary doctrines, lex primary morphisms and doctrine transformations.

The following example of primary doctrine is introduced in [17].

Example 2.4. Let $\mathbb{A}$ be a partial combinatory algebra (pca). We can define a lex primary doctrine $\mathbb{A}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSl}$ assigning to a set X the set of $\mathbb{A}^X$ of functions from X to $\mathbb{A}$. Given two elements $f, g \in \mathbb{A}^X$, we have that $\alpha \leq \beta$ if there exists an element $a \in \mathbb{A}$ such that for every $x \in X$ we have that $a \cdot \alpha(x)$ is defined and $a \cdot \alpha(x) = \beta(x)$.

Example 2.5. Let $\mathbf{S}$ be a inf-semilattice. We can define a lex primary doctrine $\mathbf{S}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSl}$ assigning to a set X the set of $\mathbf{S}^X$ of functions from X to $\mathbf{S}$. Given two elements $f, g \in \mathbf{S}^X$, we have that $\alpha \leq \beta$ if $f(x) \leq g(x)$ for every $x \in X$.

We recall here the notion of *full existential doctrine* from [33], that is a specific instance of the notion of elementary and existential doctrine used in [30, 29].

Definition 2.6. A lex primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is **full existential** if, for every object A and B in $\mathcal{C}$ for any product projection $f: A \rightarrow B$, the functor $P_f: P(B) \rightarrow P(A)$ has a left adjoint $\exists_f$, and these satisfy *Beck-Chevalley condition* and *Frobenius reciprocity*.

Remark 2.7. Notice that in a full existential doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ we can define an equality predicate $\delta_X := \exists_{\langle \text{id}_X, \text{id}_X \rangle}(\top_X)$ in $P(X \times X)$ for every object X of $\mathcal{C}$.

Definition 2.8. A lex primary morphism is said **full existential** whenever the natural transformation $\mathfrak{b}: P \rightarrow F \circ R$ commutes with left adjoints along every arrow.

We denote by **ExDoc** the 2-category of full existential doctrines, full existential morphisms and doctrine transformation.

The following examples are discussed in [25, 31].

Example 2.9 (subobjects doctrine). Let $\mathcal{C}$ be a category with finite limits. The functor $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ assigns to an object A in $\mathcal{C}$ the poset $\text{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$ and, for an arrow $f: B \rightarrow A$ the morphism $\text{Sub}_{\mathcal{C}}(f): \text{Sub}_{\mathcal{C}}(A) \rightarrow \text{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f . This is a full existential elementary doctrine if and only if the category $\mathcal{C}$ is regular.

Example 2.10 (weak-subobjects doctrine). Let $\mathcal{C}$ be a category with finite limits. The functor $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ assigns to an object A in $\mathcal{C}$ the poset reflection of the slice category $\mathcal{C}/A$, and for an arrow $f: B \rightarrow A$, the homomorphism $\Psi_{\mathcal{C}}(f): \Psi_{\mathcal{C}}(A) \rightarrow \Psi_{\mathcal{C}}(B)$ is defined via the pullback. This doctrine is full existential, and the existential left adjoint are given by the post-composition.

The following examples are discussed in [19, 41].

Example 2.11 (localic doctrine). Let $\mathbb{L}$ be a locale. The lex primary doctrine $\mathbb{L}^{(-)}: \text{Set}^{\text{op}} \rightarrow \text{InfSI}$ defined in Theorem 2.5 is a full existential doctrine.

Example 2.12 (realizability doctrine). Let $\mathbb{A}$ be a pca, we can consider the realizability doctrine $\mathcal{P}: \text{Set}^{\text{op}} \rightarrow \text{InfSI}$: for each set X , $(\mathcal{P}(\mathbb{A})^X, \leq)$ is defined as the set of functions from X to the powerset $\mathcal{P}(\mathbb{A})$ of $\mathbb{A}$ and, given two elements α and β of $\mathcal{P}(\mathbb{A})^X$, we say that $\alpha \leq \beta$ if there exists an element $\bar{a} \in \mathbb{A}$ such that for all $x \in X$ and all $a \in \alpha(x)$, $\bar{a} \cdot a$ is defined and it is an element of $\beta(x)$. As in Theorem 2.4, this defines a preorder, so we have consider its poset reflection.

2.2. Hyperdoctrines and triposes

The notion of *tripos* and the *tripos-to-topos construction* were originally introduced in [41] and revisited in [19] in order to generalize the construction of the category of sheaves of a locale. Over the past few years, there has been an increasing focus on its universal properties in [31, 28, 12], which we will exploit in our work. In particular, we will make use of the fact, shown in [31], that the tripos-to-topos construction coincides with the *exact completion* of a tripos viewed as *an elementary existential doctrine*.

Furthermore, in [19] it was also introduced a notion of geometric morphism between arbitrary triposes, extending a more restrictive one in [41], which induces a geometric morphism between their generated toposes. This also specializes to the notion of geometric embedding from a tripos $\mathbb{T}_P$ to a tripos $\mathbb{T}_R$, which extends to a geometric embedding between the corresponding toposes, to view the topos generated by $\mathbb{T}_P$ as a topos of internal sheaves on the topos generated by $\mathbb{T}_R$ for the Lawvere-Tierney topology induced by the geometric embedding, see e.g. [26]. We start recalling the main concepts regarding first-order hyperdoctrines and triposes, mainly following the notation used in [41], and the definition of pre-equipment $\mathbf{Trip}$ of triposes from [12].

Definition 2.13. A **full first order hyperdoctrine** is a full existential doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ such that

- for every object A of $\mathcal{C}$ the fibre $P(A)$ is a Heyting algebra, and for every arrow $f: A \rightarrow B$ of $\mathcal{C}$, $P_f: P(B) \rightarrow P(A)$ is a morphism of Heyting algebras;
- for any arrow $f: A \rightarrow B$, the functor

$$P_f: P(B) \rightarrow P(A)$$

has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$. Moreover, these adjoints have to satisfy *Beck-Chevalley condition*.

Definition 2.14. A lex primary doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ has a **weak predicate classifier** if there exists an object Ω of $\mathcal{C}$ together with an element $\in$ of $P(\Omega)$ such that for every object A of $\mathcal{C}$ and every α of $P(A)$ there exists a morphism $\{\alpha\}: A \rightarrow \Omega$ such that $P_{\{\alpha\}}(\in) = \alpha$.

Definition 2.15. A lex primary doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ has **weak power objects** if for every object X of $\mathcal{C}$ there exists an object PX and an element $\in_X$ of $P(X \times PX)$ such that for every β of $P(X \times Y)$ there exists an arrow $\{\alpha\}_X: Y \rightarrow PX$ such that $\beta = P_{\text{id}_X \times \{\alpha\}_X}(\in_X)$.

Notice that if $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ has weak power objects, then it has a weak predicate classifier given by $P1$ and $\in_1$. The vice versa holds when the base category is cartesian closed [41], and this extends also to the weakly case:

Lemma 2.16. *If $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ has a weak predicate classifier and the base category is weakly cartesian closed then, for every object A of $\mathcal{C}$, we can define an object $PA := \Omega^A$ and an element $\in_A := P_{\text{ev}}(\in)$ of $P(A \times PA)$ where $\text{ev}: A \times \Omega^A \rightarrow \Omega$ is the evaluation arrow, and these assignments give to P the structure of weak power objects.*

Definition 2.17. A full first-order hyperdoctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is called **full tripos** if it has weak power objects.

Example 2.18. The localic doctrine $\mathbf{L}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSl}$ defined in Theorem 2.11 is a full tripos, usually called localic tripos. The weak predicate classifier given by $\Omega := \mathbf{L}$ and $\in := \text{id}_{\mathbf{L}}$.

Example 2.19. The realizability doctrine $\mathcal{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSl}$ defined in Theorem 2.12 is a full tripos. The weak predicate classifier given by $\Omega := \mathbb{A}$ and $\in := \text{id}_{\mathbb{A}}$.

Example 2.20. If $\mathcal{E}$ is a topos, then the functor $\mathbf{Sub}: \mathcal{E}^{\text{op}} \rightarrow \mathbf{InfSl}$, as defined in Theorem 2.9, is a tripos.

Example 2.21. If $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is a lex primary doctrine with a weak predicate classifier given by an object Ω and $\in$ element of $P(\Omega)$ then the slice doctrine $P_{/X}: \mathcal{C}/X^{\text{op}} \rightarrow \mathbf{InfSl}$ as defined in Theorem 7.13 has a weak predicate classifier, given by the object $\pi_X: \Omega \times X \rightarrow X$ and the element $P_{\pi_X}(\in)$. Similarly, if P is a full tripos then $P_{/X}$ is a full tripos.

To guarantee that a doctrine of weak subobjects of a finite limit category is a tripos it is enough to check that it has weak dependent products in the sense of [4] and generic proof in the sense of [37]. We recall the main definitions and briefly outline the key structures of this functor.

Definition 2.22. A **weak dependent product** of an arrow $f: X \rightarrow J$ of $\mathcal{C}$ along a map $g: J \rightarrow I$ consists of a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{e} & E & \longrightarrow & Z \\ & \searrow f & \downarrow \lrcorner & & \downarrow h \\ & & J & \xrightarrow{g} & I \end{array}$$

such that it has the universal property of being weakly terminal in the category of diagrams with such a shape, namely for every other diagram

$$\begin{array}{ccccc} X & \xleftarrow{e'} & E' & \longrightarrow & Z' \\ & \searrow f & \downarrow \lrcorner & & \downarrow h' \\ & & J & \xrightarrow{g} & I \end{array}$$

there are arrows $w : Z' \rightarrow Z$ $k : E' \rightarrow E$ such that $e' = ek$ and $h' = hw$.

Remark 2.23. We recall from [9] that if a category has finite limits and weak dependent then every slice category has weak dependent products.

It is a known fact that if a category $\mathcal{C}$ has weak dependent products, then the doctrine $\Psi_{\mathcal{C}} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ is universal and implicational. The proof is straightforward, and the main idea is that we can define the universal quantifier $\forall_g(f) : Z \rightarrow I$ as the weak dependent product of f along g and then define the implication as $f \rightarrow g := \forall_f \Psi_f(g)$.

Now we recall from [38, 36] the notion of *generic proof*.

Definition 2.24. A **generic proof** is a morphism $\theta : \Theta \rightarrow \Lambda$ such that for every arrow $f : Y \rightarrow X$ there exists a map $v_f : X \rightarrow \Lambda$ such that f factors through $v_f^* \theta$ and $v_f^* \theta$ factors through f .

$$\begin{array}{ccccc} Y & \xrightleftharpoons[e_1]{e_2} & E & \longrightarrow & \Theta \\ & \searrow f & \downarrow \lrcorner & & \downarrow \theta \\ & & X & \xrightarrow{v_f} & \Lambda \end{array}$$

It is immediate to check that the doctrine $\Psi_{\mathcal{C}} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ has a weak predicate classifier if and only if $\mathcal{C}$ has a generic proof.

Remark 2.25. Notice that if $\mathcal{C}$ has a generic proof $\theta : \Theta \rightarrow \Lambda$, it is straightforward to check that every slice category $\mathcal{C}/X$ has a generic proof given by

$$\begin{array}{ccc} X \times \Theta & \xrightarrow{\text{id}_X \times \theta} & X \times \Lambda \\ \pi_X \searrow & & \swarrow \pi_X \\ & X & \end{array}$$

2.3. Tripos-to-topos construction

Now we recall the tripos-to-topos construction [41] in the general context of full existential doctrines [31, 28].

Tripos-to-topos. Given a full existential doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$, the category $\mathbb{T}_P$ consists of:

objects: pairs (A, ρ) such that $\rho \in P(A \times A)$ satisfies

- *symmetry:* $\rho \leq P_{\langle \pi_2, \pi_1 \rangle}(\rho)$;
- *transitivity:* $P_{\langle \pi_1, \pi_2 \rangle}(\rho) \wedge P_{\langle \pi_2, \pi_3 \rangle}(\rho) \leq P_{\langle \pi_1, \pi_3 \rangle}(\rho)$, where π_i are the projections from $A \times A \times A$;

arrows: $\phi: (A, \rho) \rightarrow (B, \sigma)$ are objects $\phi \in P(A \times B)$ such that

- (i) $\phi \leq P_{\langle \pi_1, \pi_1 \rangle}(\rho) \wedge P_{\langle \pi_2, \pi_2 \rangle}(\sigma)$;
- (ii) $P_{\langle \pi_1, \pi_2 \rangle}(\rho) \wedge P_{\langle \pi_1, \pi_3 \rangle}(\phi) \leq P_{\langle \pi_2, \pi_3 \rangle}(\phi)$ where π_i are projections from the object $A \times A \times B$;
- (iii) $P_{\langle \pi_2, \pi_3 \rangle}(\sigma) \wedge P_{\langle \pi_1, \pi_2 \rangle}(\phi) \leq P_{\langle \pi_1, \pi_3 \rangle}(\phi)$ where π_i are projections from the object $A \times B \times B$;
- (iv) $P_{\langle \pi_1, \pi_2 \rangle}(\phi) \wedge P_{\langle \pi_1, \pi_3 \rangle}(\phi) \leq P_{\langle \pi_2, \pi_3 \rangle}(\sigma)$ where π_i are projections from the object $A \times B \times B$;
- (v) $P_{\Delta_A}(\rho) \leq \exists_{\pi_1}(\phi)$ where π_1 is the first projection from $A \times B$.

The following result is presented in [31]:

Theorem 2.26. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be a full existential doctrine. Then $\mathbb{T}_P$ is an exact category.*

The following result is presented in [41, Thm. 3.4]:

Theorem 2.27. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be a full first order hyperdoctrine. Then $\mathbb{T}_P$ is a logos.*

A necessary and sufficient condition for making $\mathbb{T}_P$ a topos is provided by a careful analysis in [41]. In particular, given a first order hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$, we have $\mathbb{T}_P$ is a topos precisely when P satisfies in its internal language the form of *Comprehension Axiom*:

Axiom 2.28 (CA). A first order hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ satisfies the **Comprehension Axiom** if for any object X of $\mathcal{C}$ there exists an object $P(X)$ and an element $\in_X$ of $P(X \times P(X))$ such that for every object Y of $\mathcal{C}$ and every $\alpha \in P(X \times Y)$, P satisfies the following sentence

$$\forall i : Y. \exists s : P(X). \forall x : X. \in_X (x, s) \longleftrightarrow \alpha(x, i)$$

in its internal language.

Hence, we have the following result, see [41, Thm. 4.2].

Theorem 2.29. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be a first-order hyperdoctrine. Then T_P is a topos if and only if P satisfies (CA).*

Since every tripos satisfies (CA) we have the following corollary:

Corollary 2.30. *If $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ is a tripos, then T_P is a topos.*

The following examples are discussed in [19], and show that realizability and localic toposes can be presented as instances of the same construction, i.e. the tripos-to-topos one.

Example 2.31. The topos $\mathsf{T}_{\mathsf{L}^{(-)}}$ associated to the tripos $\mathsf{L}^{(-)}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSl}$ is equivalent to the category of sheaves $\mathbf{Sh}(\mathsf{L})$ of the locale.

Example 2.32. Given a pca $\mathbb{A}$, the tripos-to-topos $\mathsf{T}_{\mathcal{P}}$ of the realizability tripos $\mathcal{P}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSl}$ is equivalent to the realizability topos $\mathbf{RT}(\mathbb{A})$.

Example 2.33. By Theorem 5.5, we have that the exact completion $(\mathcal{C})_{\text{ex/lex}}$ of a category $\mathcal{C}$ with finite limits happens to be equivalent to the tripos-to-topos $\mathsf{T}_{\Psi_{\mathcal{C}}}$ of the doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ of weak subobjects of $\mathcal{C}$.

The notion of *geometric morphism* of triposes was originally introduced in [19, Def. 3.4] for $\mathbf{Set}$ -based triposes. Here we consider a slightly generalization for arbitrary based triposes that can be found in [12].

Definition 2.34. Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ and $R: \mathcal{D}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be two full triposes. A **geometric morphism** of triposes $(F, \mathfrak{b}): P \rightarrow R$ is a lex primary morphism of doctrines that has a left adjoint. A geometric morphism is said to be a **geometric embedding** if the counit of the adjunction is an iso.

Example 2.35. Assuming the axiom of choice, we have a geometric inclusion of the tripos $\text{Sub}_{\text{Set}}: \text{Set}^{\text{op}} \longrightarrow \text{InfSI}$ into the realizability tripos $\mathcal{P}: \text{Set}^{\text{op}} \longrightarrow \text{InfSI}$ sending a mono $m: X \rightarrow Y$ into $\exists_f(\top_X)$.

Now we recall a useful result, showing that a geometric morphism (and also a geometric inclusion) of triposes induces a geometric morphism (and a geometric inclusion) between their tripos-to-topos. This fact was already proved in the original work [19, Prop. 3.5] for **Set**-based triposes, and it has been analyzed in the more general setting of arbitrary based triposes by Frey in [12]

Theorem 2.36. *Every $(F, \flat) \dashv (G, \flat)$ is a geometric morphism of triposes induces a geometric morphism $\mathbb{T}(F, \flat) \dashv \mathbb{T}(G, \flat)$ of toposes between their tripos-to-topos construction. The same result holds for geometric inclusions.*

Remark 2.37. Notice that the left adjoint of a geometric morphism of triposes is a morphism of full existential doctrines.

3. Abstracting localic presheaves for lex primary doctrines

The main purpose of this section is to generalize the notion of *localic presheaves* to doctrines. To this purpose recall from [21, 29, 31] that:

Definition 3.1. The **Grothendieck construction** or **category of points** $\mathcal{G}_P$ of a lex primary doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSI}$ has defined as follows:

- objects of $\mathcal{G}_P$ are pairs (A, α) , where A is an object of $\mathcal{C}$ and $\alpha \in P(A)$;
- a morphism $f: (A, \alpha) \rightarrow (B, \beta)$ in $\mathcal{G}_P$ is an arrow $f: A \rightarrow B$ of $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$.

It is direct to check that the category of points of a lex primary doctrine inherits finite limits from the base category of the doctrine:

Proposition 3.2. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSI}$ be a lex primary doctrine. Then $\mathcal{G}_P$ has finite limits.*

Example 3.3. Let $\mathbb{L}$ be a locale. The category of points of $\mathcal{G}_{\mathbb{L}^{(-)}}$ of the localic tripos $\mathbb{L}^{(-)}: \text{Set}^{\text{op}} \longrightarrow \text{InfSI}$ is the category of *fuzzy sets* [2]. Moreover this is equivalent to the coproduct completion $\mathbb{L}_+$ of the locale $\mathbb{L}$.

Example 3.4. The category $\mathcal{P}asm(\mathbb{A})$ of partitioned assemblies (see e.g. [40, 42]) of a partial combinatory algebra $\mathbb{A}$ is equivalent to the category of points $\mathcal{G}_{\mathbb{A}(-)}$ of the doctrine $\mathbb{A}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ defined in Theorem 2.4.

Now the main idea for generalizing the construction of the category of localic presheaves is to look at its categorical properties in terms of the generating triposes. In particular, recall from [6, 36, 38] that the category of presheaves $\mathbf{PSh}(\mathbf{L})$ of a locale $\mathbf{L}$ can be equivalently described as the *exact on lex completion* of the coproduct completion $\mathbf{L}_+$ of the locale $\mathbf{L}$

$$\mathbf{PSh}(\mathbf{L}) \equiv (\mathbf{L}_+)_{\text{ex/lex}}$$

or, even better, as

$$\mathbf{PSh}(\mathbf{L}) \equiv (\mathcal{G}_{\mathbf{L}^{(-)}})_{\text{ex/lex}}.$$

since the coproduct completion $\mathbf{L}_+$ is given by the category of points $\mathcal{G}_{\mathbf{L}^{(-)}}$ of the localic tripos $\mathbf{L}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ as observed in Theorem 3.3.

This crucial observation suggests us how to abstract the notion of category of localic presheaves to arbitrary doctrines:

Definition 3.5. Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a lex primary doctrine. We define the **exact category of points of P** as the category $\mathbf{E}_P := (\mathcal{G}_P)_{\text{ex/lex}}$.

Example 3.6. Let $\mathbf{L}$ be a locale and the localic tripos $\mathbf{L}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$. As anticipated at the beginning of this section, we have the equivalence $\mathbf{E}_{\mathbf{L}^{(-)}} = (\mathbf{L}_+)_{\text{ex/lex}} \equiv \mathbf{PSh}(\mathbf{L})$.

Example 3.7. The category $\mathbf{E}_{\top}$ associated with the trivial lex primary doctrine $\top: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ defined in Theorem 4.4 is precisely the exact completion $(\mathcal{C})_{\text{ex/lex}}$.

Example 3.8. By Theorem 3.4, we have that the category $\mathcal{P}asm(\mathbb{A})$ of partitioned assemblies of a given $\mathbb{A}$ is equivalent to the category of points $\mathcal{G}_{\mathbb{A}^{(-)}}$ of the doctrine $\mathbb{A}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ defined in Theorem 2.4. Hence The category $\mathbf{E}_{\mathbb{A}^{(-)}}$ associated with the doctrine $\mathbb{A}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ is precisely $\mathbf{E}_{\mathbb{A}^{(-)}} = (\mathcal{P}asm(\mathbb{A}))_{\text{ex/lex}} \equiv \mathbf{RT}(\mathbb{A})$.

Example 3.9. Let $\mathbb{A}$ be a pca, and let us consider the realizability tripos $\mathcal{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$. Objects and morphisms of the category of points $\mathcal{G}_{\mathcal{P}}$ can be described as follows: they are pairs (X, α) , where X is a set and $\alpha \subseteq X \times \mathbb{A}$ is a relation. A morphism $f: (X, \alpha) \rightarrow (Y, \beta)$ is given by a function $f: X \rightarrow Y$ such that there exists an element $a \in \mathbb{A}$ that *tracks* f , namely for every x in X and for every b in $\mathbb{A}$, we have that if $x\alpha b$ then $f(x)\beta(a \cdot b)$.

4. Abstracting localic supercompactification for lex primary doctrines

The full existential completion introduced in [45] and studied in [33, 13] is a construction that freely adds existential quantifiers to a given lex primary doctrine, and it captures, as a particular instance, the construction of a supercoherent locale from an inf-semilattice in the sense of [1].

This result, proved in [33, Sec. 7.4], motivates identifying this construction as the doctrinal abstraction of the notion of *supercompactification*.

We briefly recall the definition of the full existential completion from [45] and its application to the construction of a supercoherent locale.

Full existential completion. Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ be a lex primary doctrine. For every object A of $\mathcal{C}$ consider the following preorder:

- **objects:** pairs $(B \xrightarrow{f} A, \alpha)$, where A and B are objects of $\mathcal{C}$ and $\alpha \in P(B)$.
- **order:** $(B \xrightarrow{f} A, \alpha) \leq (C \xrightarrow{g} A, \beta)$ if there exists an arrow $h: B \rightarrow C$ of $\mathcal{C}$ such that $f = gh$ and $\alpha \leq P_h(\beta)$.

It is easy to see that the previous data give a preorder, and we denote by $P^\exists(A)$ the partial order obtained by its poset reflection.

Given a morphism $f: A \rightarrow B$ in $\mathcal{C}$, let $P_f^\exists(C \xrightarrow{g} B, \beta)$ be the object $(D \xrightarrow{f^*g} A, P_{g^*f}(\beta))$ where f^*g and g^*f are defined by the pullback of f and g as usual.

The assignment $P^\exists: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is called the **full existential completion** of P .

Theorem 4.1. *The doctrine $P^\exists: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is a full existential doctrine. Moreover the assignment $P \mapsto P^\exists$ extends to a 2-functor*

$$\begin{array}{ccc} & \xrightarrow{(-)^\exists} & \\ \text{LexDoc} & \xrightarrow{\quad \perp \quad} & \text{ExDoc} \\ & \xleftarrow{\quad} & \end{array}$$

from the 2-category **LexDoc** of lex primary doctrines to the 2-category **ExDoc** of full existential doctrine.

We define canonical inclusion

$$(\text{id}_C, \mathbf{i}): P \rightarrow P^\exists$$

where $\mathbf{i}: P(A) \rightarrow P^\exists(A)$ sends $\alpha \mapsto (A \xrightarrow{\text{id}_A} A, \alpha)$. Notice that $(\text{id}_C, \mathbf{i})$ is a lex primary morphism of doctrines, but it does not preserve existential quantifiers, i.e. it is not full existential, in general.

As direct application of the universal property of the full existential completion we have the following corollary.

Corollary 4.2. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ be a full existential doctrine. Then there exists a full existential morphism $(\text{id}_C, \bar{\mathbf{i}}): P^\exists \rightarrow P$ such that $(\text{id}_C, \bar{\mathbf{i}}) \dashv (\text{id}_C, \mathbf{i})$ and $(\text{id}_C, \bar{\mathbf{i}})(\text{id}_C, \mathbf{i}) \cong \text{id}_P$.*

Corollary 4.3. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ be a lex primary doctrine. Then there is an adjunction of full existential doctrine whose counit is an iso*

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \downarrow \text{id}_C^{\text{op}} & \searrow \Psi_C & \\
 \mathcal{C}^{\text{op}} & \xrightarrow{P^\exists} & \text{InfSI}
 \end{array}
 \quad
 \begin{array}{c}
 \bar{\mathbf{b}} \quad \dashv \quad \mathbf{b} \\
 \curvearrowright \quad \quad \quad \curvearrowleft
 \end{array}$$

Proof. It is enough to observe that for every lex primary doctrine P we have and adjunction of lex primary doctrine

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \downarrow \text{id}_C^{\text{op}} & \searrow \top & \\
 \mathcal{C}^{\text{op}} & \xrightarrow{P} & \text{InfSI}
 \end{array}
 \quad
 \begin{array}{c}
 \bar{\mathbf{t}} \quad \dashv \quad \mathbf{t} \\
 \curvearrowright \quad \quad \quad \curvearrowleft
 \end{array}$$

where $\mathbf{t}$ sends the top element into the top, and $\bar{\mathbf{t}}$ sends every element of the fibre of P into the unique element of the fibre of $\top$. Hence, when we apply the full existential completion to each component of diagram and we obtain the desired adjunction, since by Theorem 4.4, the full existential completion of the trivial doctrine is the weak subobject doctrine, i.e. $\top^\exists = \Psi_C$. $\square$

Our main examples of full existential completions are the weak subobject doctrine, realizability triposes and localic triposes associated with a supercoherent locale. We refer to [33] and [13] for more details:

Example 4.4. Let $\mathcal{C}$ be a category with finite limits. Then the weak subobject doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ is the full existential completion of the trivial doctrine on $\top: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ which sends every object A into the poset $\top(A) := \{\top\}$ with only an element.

Example 4.5. Given a pca $\mathbb{A}$, the realizability doctrine $\mathcal{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ defined in Theorem 2.12 is the full existential completion of the lex primary doctrine $\mathbb{A}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ defined in Theorem 2.4.

While all realizability triposes are full existential completions, not all localic triposes are so. We refer to [33] for more details.

Theorem 4.6. *The localic doctrine $\mathbf{L}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ is a full existential completion if and only if $\mathbf{L}$ is supercoherent in the sense of [1]. In particular, when $\mathbf{L}$ is supercoherent, the localic tripos $\mathbf{L}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ is the full existential completion of the lex primary doctrine $\mathbf{S}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ associated to the inf-semilattice $\mathbf{S}$ of the supercompact elements of $\mathbf{L}$. In particular, we have the following isomorphisms of doctrines*

$$(\mathbf{S}^{(-)})^{\exists} \cong D(\mathbf{S})^{(-)} \cong \mathbf{L}^{(-)}$$

where $D(\mathbf{S})$ is the supercoherent local generated from $\mathbf{S}$.

Now we provide a result that will be useful in the following sections. First we recall the following *rule of choice* from [32, 28].

Rule 4.7 (RC). A first order hyperdoctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ satisfies the **Rule of Choice** when, for every element α of $P(A \times B)$, if $\top_A \leq \exists_{\pi_A}(\alpha)$ then there exists an arrow $f: A \rightarrow B$ of $\mathcal{C}$ such that $\top_A \leq P_{\langle \text{id}_A, f \rangle}(\alpha)$;

We recall the following result from [33, Thm. 4.16]:

Proposition 4.8. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a lex primary doctrine. Then $P^{\exists}$ satisfies the Rule of Choice.*

Example 4.9. The weak subobject doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ satisfies the rule of choice, see [28].

Example 4.10. The realizability tripos $\mathcal{P}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSI}$ satisfies the rule of choice, see [33].

Now we present a useful characterization of first-order hyperdoctrines satisfying the rule of choice:

Proposition 4.11. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a first order hyperdoctrine such that:*

1. *P satisfies the Comprehension Axiom (CA);*
2. *P satisfies the Rule of Choice (RC).*

Then $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is a tripos.

Proof. Let α be an element of the fibre $P(X \times Y)$. Since P satisfies (CA), we have that the inequality

$$\top_1 \leq \forall_{!_Y} \exists_{\pi_Y} \forall_{\langle \pi_{P(X)}, \pi_Y \rangle} (P_{\langle \pi_X, \pi_{P(X)} \rangle}(\in_X) \longleftrightarrow P_{\langle \pi_X, \pi_Y \rangle}(\alpha))$$

holds in $P(1)$, where the domain of the projections is $X \times P(X) \times Y$. Hence the inequality

$$\top_Y \leq \exists_{\pi_Y} \forall_{\langle \pi_{P(X)}, \pi_Y \rangle} (P_{\langle \pi_X, \pi_{P(X)} \rangle}(\in_X) \longleftrightarrow P_{\langle \pi_X, \pi_Y \rangle}(\alpha))$$

holds in $P(Y)$. Now, since P satisfies the Rule of Choice we have that there exists an arrow $f: Y \rightarrow P(X)$ such that

$$\top_Y \leq P_{\langle f, \text{id}_Y \rangle} \forall_{\langle \pi_{P(X)}, \pi_Y \rangle} (P_{\langle \pi_X, \pi_{P(X)} \rangle}(\in_X) \longleftrightarrow P_{\langle \pi_X, \pi_Y \rangle}(\alpha)).$$

By (BCC), we have that

$$\top_Y \leq \forall_{\pi_Y} P_{\langle \pi_X, f\pi_Y, \pi_Y \rangle} (P_{\langle \pi_X, \pi_{P(X)} \rangle}(\in_X) \longleftrightarrow P_{\langle \pi_X, \pi_Y \rangle}(\alpha)).$$

Therefore

$$\top_{X \times Y} \leq P_{\langle \pi_X, f\pi_Y, \pi_Y \rangle} (P_{\langle \pi_X, \pi_{P(X)} \rangle}(\in_X) \longleftrightarrow P_{\langle \pi_X, \pi_Y \rangle}(\alpha))$$

in $P(X \times Y)$. Hence

$$\top_{X \times Y} \leq P_{\text{id}_X \times f}(\in_X) \longleftrightarrow \alpha.$$

Thus, we can conclude that for every element α of $P(X \times Y)$ there exists an arrow $\{\alpha\} := f: Y \rightarrow P(X)$ such that $P_{\text{id}_X \times \{\alpha\}}(\in_X) = \alpha$, i.e. that P is a tripos. $\square$

Combining this result with Theorem 2.29 we obtain the following useful corollary:

Corollary 4.12. *Let us suppose that $P^\exists: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSI}$ is a first-order hyperdoctrine. Then category $\mathbb{T}_{P^\exists}$ is a topos if and only if $P^\exists$ is a tripos.*

5. Abstracting Localic Comparison Lemma for lex primary doctrines

The main purpose of this section is to show that the equivalence

$$\text{Sh}(D(\mathbf{L})) \equiv \text{PSh}(\mathbf{L})$$

obtained by applying the comparison lemma to the categories $D(\mathbf{L})$ and $\mathbf{L}$, equipped with a suitable coverage, see e.g. [22, Ex. 2.2.4 (d)], can be generalized to an equivalence

$$\mathbb{T}_{P^\exists} \equiv \mathbb{E}_P.$$

To reach this goal, we need to recall and characterize the regular and exact completions of doctrines.

5.1. Preliminaries on the regular and exact completion of a doctrine

We recall from [31, 28] the construction of a *free regular category* from a full existential doctrine. This construction can be applied in the more general context of elementary and existential doctrines, but for the purpose of this work, we just need to recall the construction for full existential doctrines:

Definition 5.1. Let $P: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSI}$ be a full existential doctrine. We define the category $\text{Reg}(P)$ as follows:

- the **objects** of $\text{Reg}(P)$ are pairs (A, α) , where A is an object of $\mathcal{C}$ and α is an element of $P(A)$;
- an **arrow** of $\text{Reg}(P)$ from (A, α) to (B, β) is given by an element ϕ of $P(A \times B)$ such that:
 1. $\phi \leq P_{\pi_A}(\alpha) \wedge P_{\pi_B}(\beta)$;
 2. $\alpha \leq \exists_{\pi_A}(\phi)$;
 3. $P_{\langle \pi_1, \pi_2 \rangle}(\phi) \wedge P_{\langle \pi_1, \pi_3 \rangle}(\phi) \leq P_{\langle \pi_2, \pi_3 \rangle}(\delta_B)$.

The compositions of morphisms of $\mathbf{Reg}(P)$ is given by the usual *relational composition*: the composition of $\phi: (A, \alpha) \rightarrow (B, \beta)$ and $\psi: (B, \beta) \rightarrow (C, \gamma)$ is given by

$$\exists_{\langle \pi_1, \pi_3 \rangle} (P_{\langle \pi_1, \pi_2 \rangle}(\gamma) \wedge P_{\langle \pi_2, \pi_3 \rangle}(\sigma))$$

where π_i are projections from $A \times B \times C$.

Definition 5.2. Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a full existential doctrine. We call the category $\mathbf{Reg}(P)$ the **regular completion** of P .

The choice of the name in Theorem 5.2 is justified by the following result, which is a specialization of [28, Thm. 3.3] for the case of full existential doctrines:

Theorem 5.3. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be an elementary, existential doctrine. Then the category $\mathbf{Reg}(P)$ is regular and the assignment $P \mapsto \mathbf{Reg}(P)$ extends to a 2-functor*

$$\begin{array}{ccc} & \xrightarrow{\mathbf{Reg}(-)} & \\ \mathbf{ExDoc} & \begin{array}{c} \perp \\ \xleftarrow{\mathbf{Sub}(-)} \end{array} & \mathbf{RegCat} \end{array}$$

which is left biadjoint to the inclusion of the 2-category $\mathbf{RegCat}$ of regular categories in the 2-category $\mathbf{ExDoc}$ of full existential doctrines acting as $\mathcal{C} \mapsto \mathbf{Sub}_{\mathcal{C}}$.

The exact completion of a full existential doctrine is obtained combining the regular completion of a doctrine with the $(-)\text{ex/reg}$ exact completion of a regular category introduced by Freyd in [15]. We recall from [31] the following result:

Corollary 5.4. *The assignment $P \mapsto (\mathbf{Reg}(P))_{\text{ex/reg}}$ extends to a 2-functor from the 2-category $\mathbf{ExDoc}$ of full existential doctrines to the 2-category $\mathbf{ExCat}$ of exact categories, and it is left adjoint to the functor sending an exact category $\mathcal{C}$ to doctrine $\mathbf{Sub}_{\mathcal{C}}$ of its subobjects.*

The following example appears in [28].

Example 5.5. The regular completion $(\mathcal{C})_{\text{reg/lex}}$ of category with finite limits $\mathcal{C}$ is equivalent to the regular completion $\mathbf{Reg}(\Psi_{\mathcal{C}})$ of the lex primary doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ of weak subobjects of $\mathcal{C}$. Hence, the exact completion $(\mathcal{C})_{\text{ex/lex}}$ of a category $\mathcal{C}$ is equivalent to the exact completion of the doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ of weak subobjects of $\mathcal{C}$.

We recall from [31, Sec. 2]:

Theorem 5.6. *For any full existential doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$, the category T_P is exact and the following equivalence holds $\mathsf{T}_P \equiv (\mathbf{Reg}(P))_{\text{ex/}\mathbf{reg}}$.*

These results can be combined with Menni's characterization of lex categories whose exact completions are toposes [37] to show that having weak dependent products and a generic proof are not only sufficient, but also necessary for making $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ a tripos:

Theorem 5.7 (Menni [37]). *Let $\mathcal{C}$ be a category with finite limits. Then $(\mathcal{C})_{\text{ex/}\mathbf{lex}}$ is a topos if and only if $\mathcal{C}$ has weak dependent products and a generic proof.*

Combining this result with Theorem 2.33 and Theorem 2.16, we obtain the following corollary:

Corollary 5.8. *Let $\mathcal{C}$ be category with finite limits. Then the weak subobject doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is a full tripos if and only if $\mathcal{C}$ has weak dependent products and a generic proof.*

5.2. Abstracting the localic Comparison Lemma

The first result we are going to show is that the regular completion of the full existential completion $P^{\exists}$ of a lex primary doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is equivalent to the regular category associated to the category of points $\mathcal{G}_P$ of P

$$\mathbf{Reg}(P^{\exists}) \equiv (\mathcal{G}_P)_{\text{reg/}\mathbf{lex}}.$$

Observe now that there is a strong link between the fibres of a full existential completion of a doctrine P and the weak subobject doctrine of the Grothendieck construction of P . To this purpose, let us denote by $I_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{G}_P$ be the canonical functor sending $A \mapsto I_{\mathcal{C}}(A) := (A, \top)$.

Lemma 5.9. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a lex primary doctrine. Then $P^{\exists} = \Psi_{\mathcal{G}_P} \circ I_{\mathcal{C}}$.*

Lemma 5.10. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a lex primary doctrine. Every object of $(\mathcal{G}_P)_{\text{reg/}\mathbf{lex}}$ is isomorphic to one of the form $f: (A, \alpha) \rightarrow (B, \top_B)$.*

Proof. It follows by the definition of arrows in the reg/lex -completion [7, 6]. Indeed, if we consider an object $(A, \alpha) \xrightarrow{f} (B, \beta)$ of the category $(\mathcal{G}_P)_{\text{reg/lex}}$, then it is direct to check that

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{f} & (B, \beta) \\ & \downarrow [\text{id}] & \\ (A, \alpha) & \xrightarrow{f} & (B, \top) \end{array}$$

is an arrow of $(\mathcal{G}_P)_{\text{reg/lex}}$ and that is an isomorphism. □

Combining Theorem 5.10 with Theorem 5.5 we obtain the following corollary:

Corollary 5.11. *Let us consider the lex doctrine $\Psi_{\mathcal{G}_P}: \mathcal{G}_P^{\text{op}} \rightarrow \text{InfSI}$, where $P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ is a lex primary doctrine. Then for every object $((B, \beta), f)$ of $\text{Reg}(\Psi_{\mathcal{G}_P})$ is isomorphic to one of the form $((B, \top), f)$.*

Now we observe that the regular completion of a full existential doctrine depends only on the objects of the base category and on the fibres:

Lemma 5.12. *If $(F, f): P \rightarrow R$ is a morphism of full existential doctrines such that $f_A: P(A) \rightarrow R(FA)$ is an isomorphism for every object A of the base, then $\text{Reg}(F, f): \text{Reg}(P) \rightarrow \text{Reg}(R)$ is full and faithful functor.*

Proof. Since $(F, f): P \rightarrow R$ is a morphism of full existential doctrines and f is a natural transformation whose components $f_A: P(A) \rightarrow R(FA)$ are isomorphisms we have that an element ϕ of the fibre $P(A \times B)$ provides a morphism $\phi: (A, \alpha) \rightarrow (B, \beta)$ of $\text{Reg}(P)$ if and only if the element $f_{A \times B}(\phi)$ of the fibre $RF(A \times B)$ (that is isomorphic to the fibre $R(FA \times FB)$ since F is a finite limits preserving functor) provides a morphism $f_{A \times B}(\phi): (FA, f_A(\alpha)) \rightarrow (FB, f_B(\beta))$. Therefore, we have that the functor $\text{Reg}(F, f): \text{Reg}(P) \rightarrow \text{Reg}(R)$ is a full and faithful. □

Theorem 5.13. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ be a lex primary doctrine. Then*

$$\text{Reg}(P^\exists) \equiv (\mathcal{G}_P)_{\text{reg/lex}}$$

Proof. By Theorem 5.9 we have that $\mathbf{Reg}(P^\exists) \cong \mathbf{Reg}(\Psi_{\mathcal{G}_P} \circ I_C)$. To conclude the proof it is enough to observe that by Theorem 5.12 the category $\mathbf{Reg}(\Psi_{\mathcal{G}_P} \circ I_C)$ is a full sub-category of $\mathbf{Reg}(\Psi_{\mathcal{G}_P})$. In particular, by definition of $\mathbf{Reg}(-)$ (see Theorem 5.1) it is the full subcategory whose objects are of the form $(B, \top, f)$. Hence we can apply Theorem 5.11 to conclude that

$$\mathbf{Reg}(\Psi_{\mathcal{G}_P} \circ I_C) \equiv \mathbf{Reg}(\Psi_{\mathcal{G}_P}).$$

Therefore, combining these equivalences with Theorem 5.5 we can conclude that

$$\mathbf{Reg}(P^\exists) \equiv \mathbf{Reg}(\Psi_{\mathcal{G}_P} \circ I_C) \equiv \mathbf{Reg}(\Psi_{\mathcal{G}_P}) \equiv (\mathcal{G}_P)_{\text{reg/lex}}.$$

□

Example 5.14. By combining Theorem 5.13 with Theorem 3.4 and Theorem 4.5 we obtain that the regular completion of the realizability tripos is equivalent to the category of assemblies:

$$\mathbf{Reg}(\mathcal{P}) \equiv (\mathcal{G}_{\mathbb{A}(-)})_{\text{reg/lex}} \equiv (\mathcal{P}\mathcal{A}sm(\mathbb{A}))_{\text{reg/lex}}$$

i.e. the category $\mathbf{Reg}(\mathcal{P})$ is equivalent to the category of assemblies $\mathcal{A}sm(\mathbb{A})$ [40, 42].

Combining Theorem 5.6 with Theorem 5.13 we obtain our first main result, that will provide a generalization in the context of doctrines of the (canonical) localic instance of the Comparison Lemma:

Theorem 5.15. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be a lex primary doctrine. Then we have the isomorphism*

$$\top_{P^\exists} \equiv (\mathcal{G}_P)_{\text{ex/lex}} = \mathbf{E}_P.$$

Theorem 5.15 and Theorem 3.6 show the tripos-to-topos construction is able to capture as particular cases both localic sheaf and presheaf toposes.

Combining the universal properties of the tripos-to-topos and the full existential completion we can conclude that the construction $\mathbf{E}_-$ has the universal property of being the *exact completion of lex primary doctrines*:

Corollary 5.16 (exact completion of lex primary doctrines). *The category $\mathbf{E}_P$ is an exact category, and the assignment $P \mapsto \mathbf{E}_P$ extends to a 2-functor*

$$\begin{array}{ccc} & \xrightarrow{\mathbf{E}_-} & \\ \text{LexDoc} & \xrightarrow{\quad \perp \quad} & \text{ExCat} \\ & \xleftarrow{\text{Sub}_{(-)}} & \end{array}$$

from the 2-category $\mathbf{LexDoc}$ of lex primary doctrines to the 2-category $\mathbf{ExCat}$ of exact categories, and it is left adjoint to the functor sending an exact category $\mathcal{C}$ to doctrine $\mathbf{Sub}_{\mathcal{C}}$ of its subobjects.

Corollary 5.17. *Let $\mathbf{L}$ be a locale. Then we have the following equivalence*

$$\mathbf{Sh}(D(\mathbf{L})) \equiv \mathbf{PShL}$$

where $D(\mathbf{L})$ is the supercoherent locale constructed from $\mathbf{L}$.

Combining Theorem 4.3 with Theorem 5.15 we have the following corollary, relating the exact completion of the base category of a lex primary doctrine with the tripos-to-topos of its full existential completion:

Corollary 5.18. *Let $P: \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{InfSI}$ be a lex primary doctrine. Then we have an adjunction of exact categories whose co-unit is an iso:*

$$\mathbf{E}_P \equiv \mathbf{T}_{P\exists} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} (\mathcal{C})_{\mathrm{ex}/\mathrm{lex}}$$

Example 5.19. A relevant topos having the presentation of Theorem 5.15 is provided by the realizability topos $\mathbf{RT}(\mathbb{A})$ associated to a pca $\mathbb{A}$ [19, 42, 18]. In particular, we recall from [42] that we have an equivalence $\mathbf{RT}(\mathbb{A}) \equiv (\mathcal{P}\mathcal{A}sm(\mathbb{A}))_{\mathrm{ex}/\mathrm{lex}}$ between a realizability topos and the exact completion of the lex category of partition assemblies. Given a realizability doctrine $\mathcal{P}: \mathbf{Set}^{\mathrm{op}} \rightarrow \mathbf{InfSI}$ associated to a pca $\mathbb{A}$, we have that combining Theorem 5.15 with Theorem 3.4 and Theorem 4.5 we can conclude that

$$\mathbf{T}_{\mathcal{P}} \equiv (\mathcal{G}_{\mathbb{A}(-)})_{\mathrm{ex}/\mathrm{lex}} \equiv (\mathcal{P}\mathcal{A}sm(\mathbb{A}))_{\mathrm{ex}/\mathrm{lex}} \equiv \mathbf{RT}(\mathbb{A}).$$

Example 5.20. As observed in [19, Ex. 2.15], the topos $\mathbf{T}_{\mathbf{L}(-)}$ obtained by the localic doctrine associated to a locale $\mathbf{L}$ has to be equivalent to the category $\mathbf{Sh}(\mathbf{L})$ of canonical sheaves over $\mathbf{L}$. When $\mathbf{L}$ is supercoherent then, by Theorem 4.6, we have that the localic doctrine $\mathbf{L}^{(-)}: \mathbf{Set}^{\mathrm{op}} \rightarrow \mathbf{InfSI}$ is the full existential completion of the doctrine $\mathbf{S}^{(-)}: \mathbf{Set}^{\mathrm{op}} \rightarrow \mathbf{InfSI}$ associated to the inf-semilattice $\mathbf{S}$ of the supercompact elements of $\mathbf{L}$. Therefore, by Theorem 5.15, we have

$$\mathbf{T}_{\mathbf{L}(-)} \equiv \mathbf{Sh}(\mathbf{L}) \equiv (\mathcal{G}_{\mathbf{S}(-)})_{\mathrm{ex}/\mathrm{lex}}$$

where $\mathbf{S}$ is the inf-semilattice of the supercompact elements of $\mathbf{L}$.

6. A characterization of abstract localic presheaves toposes

Observe that while the notion of a tripos is sufficient to ensure that its tripos-to-topos construction yields a topos, it does not, in general, guarantee that the full existential completion of a tripos is itself a tripos, and hence that the corresponding category $\mathbf{E}_P$ of a tripos P is again a topos.

The underlying intuition is that the latter construction depends not only on the elements of the fibres of a tripos, but also on the arrows of its base category.

Definition 6.1 ($\exists$ -supercompactifiable doctrine). A lex primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is said $\exists$ -**supercompactifiable** if the category of points $\mathcal{G}_P$ has weak dependent products and a generic proof.

Employing our previous analysis, we can provide the following characterization of $\exists$ -supercompactifiable doctrines.

Theorem 6.2. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ be a lex primary doctrine. Then the following are equivalent:*

1. $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is a $\exists$ -supercompactifiable doctrine;
2. $\Psi_{\mathcal{G}_P}: \mathcal{G}_P^{\text{op}} \rightarrow \mathbf{InfSl}$ is a full tripos;
3. $P^\exists: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSl}$ is a full tripos;
4. $\mathbf{E}_P$ is a topos.

Proof. ($1 \Rightarrow 2$) It follows by Theorem 5.8 and by Theorem 6.1.

($2 \Rightarrow 3$) If $\Psi_{\mathcal{G}_P}$ is a full tripos, then by Theorem 5.9 we have that by $P^\exists$ is a first-order hyperdoctrine. Combining Theorem 5.15 with Theorem 5.7 we have that the category $\mathbf{T}_{P^\exists} \equiv (\mathcal{G}_P)_{\text{ex/lex}}$ is a topos. Now we can conclude that $P^\exists$ is a full tripos applying Theorem 4.12.

($3 \Rightarrow 4$) By Theorem 5.15 we have that $\mathbf{T}_{P^\exists} \equiv (\mathcal{G}_P)_{\text{ex/lex}} \equiv \mathbf{E}_P$, and hence, by Theorem 2.29, we can conclude that $\mathbf{E}_P$ is a topos, since $P^\exists$ is a full tripos.

($4 \Rightarrow 1$) It follows by Theorem 3.5 and by Theorem 5.7. $\square$

Corollary 6.3. *The $\mathbf{E}_{(-)}$ -construction extends to a 2-functor*

$$\begin{array}{ccc}
 & \mathbf{E}_{(-)} & \\
 \exists\text{-LexDoc} & \xrightarrow{\quad} & \mathbf{Top}_{\text{reg}} \\
 & \text{Sub}_{(-)} & \\
 & \perp &
 \end{array}$$

from the 2-category $\exists$ -supercompactifiable of lex primary doctrines to the 2-category of toposes categories and regular functor, and it is left adjoint to the functor sending an exact category $\mathcal{C}$ to doctrine $\mathbf{Sub}_{\mathcal{C}}$ of its subobjects.

Corollary 6.4. *If a lex primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ happens to be $\exists$ -supercompactifiable, then $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ is a tripos, and hence $(\mathcal{C})_{\text{ex/lex}}$ is a topos.*

Proof. This happens because the weak dependent products and the generic proof of $\mathcal{G}_P$ induce weak dependent products and a generic proof for $\mathcal{C}$ $\square$

Combining the previous remark with Theorem 6.2, Theorem 4.3 and Theorem 5.15 we obtain the following:

Corollary 6.5. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a $\exists$ -supercompactifiable doctrine. Then we have a geometric inclusion of triposes $\Psi_{\mathcal{C}} \rightarrow P^{\exists}$. Moreover, the right adjoint is a full existential morphism.*

Corollary 6.6. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a $\exists$ -supercompactifiable doctrine. Then we have a geometric embedding of toposes $(\mathcal{C})_{\text{ex/lex}} \rightarrow \mathbf{E}_P$ whose right adjoint is a morphism of exact categories.*

Example 6.7. By Theorem 4.5, we have that the lex primary doctrine $\mathbb{A}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ defined in Theorem 2.4 is $\exists$ -supercompactifiable, since its full existential completion is the realizability tripos $\mathcal{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$. In this case, the geometric embedding of Theorem 6.6 is exactly well-known geometric inclusion $\mathbf{Set} \rightarrow \mathbf{RT}(\mathbb{A})$.

Example 6.8. Let $\mathbf{S}$ be an inf-semilattice. By Theorem 4.6, we have that the lex primary doctrine $\mathbf{S}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$ is $\exists$ -supercompactifiable, because its full existential completion is the localic tripos $D(\mathbf{S}): \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$. In this case, the geometric inclusion of Theorem 6.6 is exactly geometric inclusion $\mathbf{Set} \rightarrow \mathbf{Sh}(D(\mathbf{S}))$.

Example 6.9. By Theorem 5.7 and Theorem 5.8 we have that every trivial doctrine $\top: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ is $\exists$ -supercompactifiable tripos if and only if $\mathcal{C}$ has weak dependent products and a generic proof.

Example 6.10. A non-trivial example of a lex primary doctrine that is non $\mathbf{Set}$ -based is the elementary instance reducibility doctrine presented in [35, Def 4.1], since its full existential completion is a tripos [35, Thm. 4.9].

This is a doctrine whose base category is that of partitioned assemblies, and whose full existential completion abstract the notion of extended Weihrauch degrees. In this case, the geometric embedding of Theorem 6.6 provides exactly the geometric inclusion of the effective topos into the topos of extended Weihrauch degrees proved in [35, Thm. 6.8].

It is important to mention that within the literature of categorical realizability, there are already two results which can be seen as instances of the general theorem Theorem 6.2, one provided by P. Hofstra in [17] and the other shown by J. Frey in [14]. These are two combinatorial characterizations of **Set**-indexed preorders associated with suitable pre-realizability notions whose full existential completion is a tripos. While in both these works the main focus is on the features of pre-realizability notion which is taken as starting point for defining doctrines and triposes (basic combinatory objects and filtered ordered combinatory algebras in [17], uniform preorder and discrete combinatory object in [14]), our characterization is purely categorical and designed for arbitrary lex primary doctrines.

7. A characterization of $\exists$ -supercompactifiable triposes

In many concrete examples, showing that a given tripos $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is a $\exists$ -supercompactifiable tripos may be tricky.

The main purpose of this section is to provide a simple characterization of $\exists$ -supercompactifiable tripos, namely Theorem 7.6 where we prove that a tripos is $\exists$ -supercompactifiable if and only if its base category has weak dependent products and a generic proof. This result is useful also to identify examples of triposes which are not $\exists$ -supercompactifiable.

We will then deduce that *every Set-base full tripos is a $\exists$ -supercompactifiable tripos* by a crucial use of the axiom of choice in the meta-theory.

We first study when a category of points has weak dependent products. For sake of generality, we will present all the results with the minimal assumption on the doctrines.

Lemma 7.1. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a implicational and universal doctrine whose base category has finite limits. If the diagram*

$$\begin{array}{ccccc}
(X, \alpha) & \xleftarrow{e'} & (E, P_{g^*(h')}(\beta) \wedge P_{(h')^*g}(\sigma')) & \longrightarrow & (Z', \sigma') \\
& \searrow f & \downarrow \lrcorner & & \downarrow h' \\
& & (J, \beta) & \xrightarrow{g} & (I, \gamma)
\end{array}$$

commutes in $\mathcal{G}_P$, then we have that

$$\sigma' \leq \forall_{(h')^*g}(P_{g^*(h')}(\beta) \rightarrow P_{e'}(\alpha)) \wedge P_{h'}(\gamma).$$

Proof. By definition, we have that $h': (Z', \sigma') \rightarrow (I, \gamma)$ is an arrow of $\mathcal{G}_P$ implies $\sigma' \leq P_{h'}(\gamma)$. Similarly, since $e': (E, P_{g^*(h')}(\beta) \wedge P_{(h')^*g}(\sigma')) \rightarrow (X, \alpha)$ is an arrow of $\mathcal{G}_P$ we have that $P_{g^*(h')}(\beta) \wedge P_{(h')^*g}(\sigma') \leq P_{e'}(\alpha)$, and hence $\sigma' \leq \forall_{(h')^*g}(P_{g^*(h')}(\beta) \rightarrow P_{e'}(\alpha))$, because P is universal and implicational. Therefore we can conclude that $\sigma' \leq \forall_{(h')^*g}(P_{g^*(h')}(\beta) \rightarrow P_{e'}(\alpha)) \wedge P_{h'}(\gamma)$. $\square$

Theorem 7.2. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSI}$ be a lex primary doctrine whose base category has weak dependent products. If P is implicational and universal, then $\mathcal{G}_P$ has weak dependent products. In particular, if*

$$\begin{array}{ccccc}
X & \xleftarrow{e} & E & \longrightarrow & Z \\
& \searrow f & \downarrow \lrcorner & & \downarrow h \\
& & J & \xrightarrow{g} & I
\end{array}$$

is a weak dependent product of f along g in $\mathcal{C}$, then the diagram

$$\begin{array}{ccccc}
(X, \alpha) & \xleftarrow{e} & (E, P_{g^*h}(\beta) \wedge P_{h^*g}(\sigma)) & \longrightarrow & (Z, \sigma) \\
& \searrow f & \downarrow \lrcorner & & \downarrow h \\
& & (J, \beta) & \xrightarrow{g} & (I, \gamma)
\end{array}$$

with

$$\sigma := \forall_{h^*g}(P_{g^*h}(\beta) \rightarrow P_e(\alpha)) \wedge P_h(\gamma)$$

is a weak dependent product of $f: (X, \alpha) \rightarrow (J, \beta)$ along $g: (J, \beta) \rightarrow (I, \gamma)$ in $\mathcal{G}_P$.

Proof. First, notice that the diagram is well-defined in $\mathcal{G}_P$ because, by definition, we have that $\sigma \leq P_h(\gamma)$ and $P_{g^*h}(\beta) \wedge P_{h^*g}(\sigma) \leq P_e(\alpha)$. Now we show

that $h: (Z, \omega) \rightarrow (I, \gamma)$ satisfies the universal property of weak dependent products. Let us consider another diagram

$$\begin{array}{ccccc}
 (X, \alpha) & \xleftarrow{e'} & (E, P_{g^*(h')}(\beta) \wedge P_{(h')^*g}(\sigma')) & \longrightarrow & (Z', \sigma') \\
 & \searrow f & \downarrow \lrcorner & & \downarrow h' \\
 & & (J, \beta) & \xrightarrow{g} & (I, \gamma).
 \end{array}$$

Since $h: Z \rightarrow I$ is a weak dependent product of f along g in $\mathcal{C}$, there exists an arrow $w: Z' \rightarrow Z$ (and an arrow $k: E' \rightarrow E$) such that the diagram

$$\begin{array}{ccccccc}
 & & & E' & \longrightarrow & Z' & \\
 & \swarrow e' & & \downarrow k & & \swarrow w & \\
 X & \xleftarrow{e} & E & \longrightarrow & Z & & \\
 & \searrow f & \downarrow \lrcorner & & \downarrow h & \swarrow h' & \\
 & & J & \xrightarrow{g} & I & &
 \end{array}$$

commutes. We have to show that the arrow $w: Z' \rightarrow Z$ induces an arrow $w: (Z', \sigma') \rightarrow (Z, \sigma)$ in $\mathcal{G}_P$, i.e. that $\sigma' \leq P_w(\sigma)$. By Theorem 7.1 we have that

$$\sigma' \leq \forall_{(h')^*g} (P_{g^*(h')}(\beta) \rightarrow P_{e'}(\alpha)) \wedge P_{h'}(\gamma).$$

Hence, since $h' = hw$ and $e' = ek$, we have that

$$\sigma' \leq \forall_{(hw)^*g} (P_{g^*(hw)}(\beta) \rightarrow P_{ek}(\alpha)) \wedge P_{hw}(\gamma).$$

Therefore, since we have that $g^*(hw) = (g^*(h))k$, we have that

$$\sigma' \leq \forall_{(hw)^*g} P_k(P_{g^*h}(\beta) \rightarrow P_e(\alpha)) \wedge P_{hw}(\gamma).$$

Now, by Beck-Chevalley condition (and the standard property of pullbacks), we have that $\forall_{(hw)^*(g)} P_k = P_w \forall_{h^*(g)}$, and hence we can conclude

$$\sigma' \leq P_w(\forall_{h^*(g)} (P_{g^*h}(\beta) \rightarrow P_e(\alpha)) \wedge P_h(\gamma)) = P_w(\sigma).$$

Finally, it is straightforward that the arrow k is well-defined in $\mathcal{G}_P$. $\square$

Example 7.3. The category of points of any **Set**-base tripos has weak dependent products.

Example 7.4. The category of points of the doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ has weak dependent products whenever $\mathcal{C}$ has them.

Now we are going to study the generic proof in the category of points:

Theorem 7.5. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a lex primary doctrine. If P has a weak predicate classifier and $\mathcal{C}$ has weak dependent products and a generic proof then P is a $\mathcal{G}_P$ has a generic proof.*

Proof. Let us consider an arrow $f: (Y, \alpha) \rightarrow (X, \beta)$ of $\mathcal{G}_P$. By definition of weak predicate classifier of P , we have an arrow $\{\alpha\}: Y \rightarrow \Omega$ such that $\alpha = P_{\{\alpha\}}(\epsilon)$. Now, by Theorem 5.8 we have that $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ is a full tripes, and hence we have weak power objects. Hence we can consider the arrow $\langle \{\alpha\}, f \rangle: Y \rightarrow \Omega \times X$ and, by definition of power objects for $\Psi_{\mathcal{C}}$, we have a commutative diagram

$$\begin{array}{ccccc}
 Y & \xrightleftharpoons[e_2]{e_1} & E & \xrightarrow{a_2} & \Theta_{\Omega} \\
 & \searrow \langle \{\alpha\}, f \rangle & \downarrow a_1 \lrcorner & & \downarrow \epsilon_{\Omega} \\
 & & \Omega \times X & \xrightarrow{\text{id}_{\Omega} \times \{\langle \{\alpha\}, f \rangle\}_{\Omega}} & \Omega \times P(\Omega) \\
 & \searrow f & \downarrow \pi_X \lrcorner & & \downarrow \pi_{P(\Omega)} \\
 & & X & \xrightarrow{\{\langle \{\alpha\}, f \rangle\}_{\Omega}} & P(\Omega)
 \end{array} \tag{1}$$

in $\mathcal{C}$ where the arrows $\epsilon_{\Omega}: \Theta_{\Omega} \rightarrow \Omega \times P(\Omega)$ and $\{\langle \{\alpha\}, f \rangle\}_{\Omega}: X \rightarrow P(\Omega)$ are defined by the power objects in $\Psi_{\mathcal{C}}$. Now we show that the diagram:

$$\begin{array}{ccccc}
 (Y, \alpha) & \xrightleftharpoons[e_2]{e_1} & (E, P_{\pi_X a_1}(\beta) \wedge P_{\pi_{\Omega} \epsilon_{\Omega} a_2}(\epsilon)) & \xrightarrow{a_2} & (\Theta_{\Omega}, P_{\pi_{\Omega} \epsilon_{\Omega}}(\epsilon)) \\
 & \searrow \langle \{\alpha\}, f \rangle & \downarrow a_1 \lrcorner & & \downarrow \epsilon_{\Omega} \\
 & & (\Omega \times X, P_{\pi_X}(\beta)) & \xrightarrow{\text{id}_{\Omega} \times \{\langle \{\alpha\}, f \rangle\}_{\Omega}} & (\Omega \times P(\Omega), \top_{\Omega \times P(\Omega)}) \\
 & \searrow f & \downarrow \pi_X \lrcorner & & \downarrow \pi_{P(\Omega)} \\
 & & (X, \beta) & \xrightarrow{\{\langle \{\alpha\}, f \rangle\}_{\Omega}} & (P(\Omega), \top_{P(\Omega)})
 \end{array}$$

is a well-defined diagram in $\mathcal{G}_P$, namely that the arrow

$$\pi_{P\Omega} \in_{\Omega} : (\Theta_{\Omega}, P_{\pi_{\Omega} \in_{\Omega}}(\in)) \rightarrow (P(\Omega), \top_{P(\Omega)})$$

is a generic proof for $\mathcal{G}_P$. Notice that we need only to check that e_1 and e_2 are well-defined arrows of $\mathcal{G}_P$ because both the pullbacks are well-defined in $\mathcal{G}_P$ just by definition.

Now we start by showing that

$$P_{\pi_X a_1}(\beta) \wedge P_{\pi_{\Omega} \in_{\Omega} a_2}(\in) = P_{e_1}(\alpha).$$

This follows since (1) is a commutative diagram in $\mathcal{C}$, and in particular we have that $a_1 = \langle \{\alpha\}, f \rangle_{e_1}$ and $\pi_{\Omega} \in_{\Omega} a_2 = \pi_{\Omega}(\text{id}_{\Omega} \times \{\langle \{\alpha\}, f \rangle\}_{\Omega})a_1$, hence

$$\pi_{\Omega} \in_{\Omega} a_2 = \pi_{\Omega}(\text{id}_{\Omega} \times \{\langle \{\alpha\}, f \rangle\}_{\Omega})\langle \{\alpha\}, f \rangle_{e_1} = \{\alpha\}_{e_1}.$$

In particular, using also the fact that $\alpha = P_{\{\alpha\}}(\in)$, we have that

$$P_{\pi_X a_1}(\beta) \wedge P_{\pi_{\Omega} \in_{\Omega} a_2}(\in) = P_{\pi_X \langle \{\alpha\}, f \rangle_{e_1}}(\beta) \wedge P_{\{\alpha\}_{e_1}}(\in) = P_{e_1}(P_f(\beta) \wedge \alpha) = P_{e_1}(\alpha)$$

where the last equality follows from the assumption that $f: (Y, \alpha) \rightarrow (X, \beta)$ of $\mathcal{G}_P$, i.e. $\alpha \leq P_f(\beta)$. Hence, we have that $e_1: (E, P_{e_1}(\alpha)) \rightarrow (Y, \alpha)$ is a well-defined arrow of $\mathcal{G}_P$.

Now we pass to show that

$$\alpha = P_{e_1 e_2}(\alpha).$$

By definition of weak predicate classifier of P , we have that $\alpha = P_{\{\alpha\}}(\in)$. Now, since (1) is a commutative diagram, we have that $\alpha = P_{\pi_{\Omega} a_1 e_2}(\in)$ and $\langle \{\alpha\}, f \rangle = a_1 e_2$ and $a_1 = \langle \{\alpha\}, f \rangle_{e_1}$. Therefore we can conclude that

$$\alpha = P_{\pi_{\Omega} a_1 e_2}(\in) = P_{\pi_{\Omega} \langle \{\alpha\}, f \rangle_{e_1} e_2}(\in) = P_{e_1 e_2}(P_{\{\alpha\}}(\in) = P_{e_1 e_2}(\alpha) = P_{e_2}(P_{e_1}(\alpha))$$

and hence that $e_2: (Y, \alpha) \rightarrow E, P_{e_1}(\alpha))$ is a well-defined arrow of $\mathcal{G}_P$ $\square$

Now, we are ready to obtain our main result:

Corollary 7.6. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSI}$ be a full tripos. Then the following conditions are equivalent:*

1. *P is a $\exists$ -supercompactifiable;*

2. $\mathcal{C}$ has weak dependent products and a generic proof;

3. $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is a full tripos.

Proof. 1) $\Rightarrow$ 2) Follows by Theorem 6.4 and Theorem 5.8. 2) $\Rightarrow$ 1) follows by Theorem 7.5 and Theorem 7.2. 2) $\Longleftrightarrow$ 3) follows by Theorem 5.8 $\square$

Corollary 7.7. *Every full tripos whose base category is a topos with splitting epis is $\exists$ -supercompactifiable.*

In particular, we can conclude that every tripos as originally introduced in [19] is $\exists$ -supercompactifiable:

Corollary 7.8. *Every full tripos whose base category is $\mathbf{Set}$ (with the axiom of choice) is $\exists$ -supercompactifiable.*

Example 7.9. Relevant examples of $\mathbf{Set}$ -based triposes, which are in particular $\exists$ -supercompactifiable by Theorem 7.8, are localic and realizability triposes [19, 41], the modified realizability tripos [20, 47], the extensional realizability tripos [46], dialectica tripos [3], Krivine tripos [44].

Example 7.10. For every topos $\mathcal{B}$ with splitting epis, we have that the full tripos $\mathbf{Sub}_{\mathcal{B}}: \mathcal{B}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is $\exists$ -supercompactifiable. A relevant example of this class of toposes are presheaf toposes over a discrete category.

Example 7.11. The extended Weihrauch tripos $\mathfrak{eW}: \mathcal{P}\mathcal{A}sm(\mathbb{A})^{\text{op}} \longrightarrow \mathbf{InfSI}$ introduced in [35] is $\exists$ -supercompactifiable, since the category of partitioned assemblies $\mathcal{P}\mathcal{A}sm(\mathbb{A})$ has weak dependent products and a generic proof.

Example 7.12. Every subobject doctrine $\mathbf{Sub}_{\mathcal{B}}: \mathcal{B}^{\text{op}} \longrightarrow \mathbf{InfSI}$ on an elementary topos $\mathcal{B}$ with no generic proof provides an example of a tripos that is not $\exists$ -supercompactifiable.

7.1. Closure of $\exists$ -supercompactifiable triposes under slicing

The main purpose of this section is to prove a fundamental-like theorem for $\exists$ -supercompactifiable triposes. To achieve this goal, we start by recalling from [8, Ex. 3.15] the notion of *slice doctrine*:

Definition 7.13 (slice doctrine). Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a lex primary doctrine and X be an object of $\mathcal{C}$. The **slice doctrine** $P_{/X}: \mathcal{C}/X^{\text{op}} \longrightarrow \mathbf{InfSI}$ is the functor defined by the assignments $P_{/X}(Y \xrightarrow{f} X) := P(Y)$ and, for an arrow $h: g \rightarrow f$ in $\mathcal{C}/X$, $P_{/X}(h) := P_h$.

Proposition 7.14. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ be a lex primary doctrine and X be an object of $\mathcal{C}$. Then:*

1. *if $\mathcal{C}$ has weak dependent products and a generic proof, $\mathcal{C}/X$ has weak dependent products and a generic proof;*
2. *if P is a full first order hyperdoctrine, $P_{/X}$ is a full first-order hyperdoctrine;*
3. *if P has a weak predicate classifier, $P_{/X}$ has a weak predicate classifier.*

Proof. (1) Recall from [9] that if a category has finite limits and weak dependent then every slice category has weak dependent products. Moreover, if $\mathcal{C}$ has a generic proof $\theta: \Theta \rightarrow \Lambda$, it is straightforward to check that every slice category $\mathcal{C}/X$ has a generic proof given by

$$\begin{array}{ccc} X \times \Theta & \xrightarrow{\text{id}_X \times \theta} & X \times \Lambda \\ & \searrow \pi_X \quad \swarrow \pi_X & \\ & X & \end{array}$$

(2) It follows by definition of slice doctrine.

(3) If $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ is a lex primary doctrine with a weak predicate classifier given by an object Ω and $\in$ element of $P(\Omega)$ then the slice doctrine $P_{/X}: \mathcal{C}/X^{\text{op}} \longrightarrow \mathbf{InfSl}$ as defined in Theorem 7.13 has a weak predicate classifier, given by the object $\pi_X: \Omega \times X \rightarrow X$ and the element $P_{\pi_X}(\in)$. $\square$

Combining Theorem 7.14 with Theorem 7.6 we obtain the following result:

Corollary 7.15. *If a full tripos $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSl}$ is a $\exists$ -supercompactifiable, then $P_{/X}: \mathcal{C}/X^{\text{op}} \longrightarrow \mathbf{InfSl}$ is a $\exists$ -supercompactifiable full tripos for every object X of $\mathcal{C}$.*

8. Abstracting localic sheafification

Our previous analysis enables us to show that for a $\exists$ -supercompactifiable tripos, its tripos-to-topos construction is equivalent to a category of j -sheaves for a Lawvere-Tierney topology that abstracts the sheafification process. In particular, we prove that this topology arises from a geometric embedding of triposes:

Theorem 8.1. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a $\exists$ -supercompactifiable tripos. Then*

1. *there is a geometric embedding of triposes $P \hookrightarrow P^{\exists}$;*
2. *there is a geometric embedding of toposes $\mathbb{T}_P \hookrightarrow \mathbb{T}_{P^{\exists}}$.*

Proof. The first point follows by Theorem 4.2 and the second by Theorem 2.36. $\square$

Since geometric embeddings of toposes corresponds to Lawvere-Tierney topologies, see e.g. [26, Cor. 7, Sec. VII], we have the following corollary:

Corollary 8.2. *Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be a $\exists$ -supercompactifiable tripos. Then there exists a Lawvere-Tierney topology j on $\mathbb{T}_{P^{\exists}}$ such that $\mathbb{T}_P \equiv \mathbf{Sh}_j(\mathbb{T}_{P^{\exists}})$.*

Combining Theorem 2.33 and Theorem 5.8 with Theorem 8.2 we have that every topos $\mathcal{B}$ arising as the ex/lex -completion of a lex category $\mathcal{C}$ fits within our framework:

Corollary 8.3. *Let $\mathcal{B} \equiv (\mathcal{C})_{\text{ex/lex}}$ be a topos. Then $\mathcal{B} \equiv \mathbf{Sh}_j(\mathbb{T}_{\Psi_{\mathcal{C}}^{\exists}})$.*

Combining Theorem 8.2 with Theorem 7.7 we obtain:

Corollary 8.4. *Let $P: \mathcal{B}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be full tripos, where $\mathcal{B}$ is a topos whose epis split. Then there exists a Lawvere-Tierney topology j on $\mathbb{T}_{P^{\exists}}$ such that $\mathbb{T}_P \equiv \mathbf{Sh}_j(\mathbb{T}_{P^{\exists}})$.*

In particular, by Theorem 7.8, we have:

Corollary 8.5. *Let $P: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be full tripos. Then there exists a Lawvere-Tierney topology j on $\mathbb{T}_{P^{\exists}}$ such that $\mathbb{T}_P \equiv \mathbf{Sh}_j(\mathbb{T}_{P^{\exists}})$.*

Example 8.6 (Localic toposes). Let $L^{(-)}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSI}$ be localic tripos, as defined in Theorem 2.11. By Theorem 7.6, the doctrine $L^{\exists}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSI}$ is a full tripos and, by Theorem 8.1, we have an adjunction between the toposes

$$\begin{array}{ccc}
 & \xrightarrow{\mathbb{T}(\text{id}_{\mathbf{Set}}, \bar{i})} & \\
 \mathbf{PSh}(L) & \perp & \mathbf{Sh}(L) \\
 & \xleftarrow{\mathbb{T}(\text{id}_{\mathbf{Set}}, i)} &
 \end{array}$$

that is the localic sheafification.

Example 8.7 (Realizability toposes). Let $\mathbb{A}$ be a pca, and let us consider the realizability tripos $\mathcal{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSI}$. Then the category $\mathbf{E}_{\mathcal{P}}$ is a topos, and the inclusion $\mathbf{RT}(\mathbb{A}) \hookrightarrow \mathbf{E}_{\mathcal{P}}$ has a left adjoint. In particular, by Theorem 8.1, we have an adjunction

$$\begin{array}{ccc} & \xrightarrow{\mathsf{T}(\text{id}_{\mathbf{Set}}, \bar{\mathbf{i}})} & \\ \mathbf{E}_{\mathcal{P}} & \begin{array}{c} \perp \\ \hline \end{array} & \mathbf{RT}(\mathbb{A}). \\ & \xleftarrow{\mathsf{T}(\text{id}_{\mathbf{Set}}, \mathbf{i})} & \end{array}$$

Hence, we have that every realizability topos can be presented as a topos of j -sheaves $\mathbf{RT}(\mathbb{A}) \equiv \mathbf{Sh}_j(\mathbf{E}_{\mathcal{P}})$ for with respect to a topos and a topology abstracting the topos of localic presheaves and the localic sheafification.

Example 8.8. Since all these toposes are generated via the tripos-to-topos of a $\mathbf{Set}$ -based tripos, we have that the modified realizability topos [20, 47], the extensional realizability topos [46], dialectica topos [3], Krivine topos [44] can be all presented as toposes of j -sheaves with respect to the topos of abstract presheaves generated by their generating triposes.

Example 8.9. By Theorem 7.11, we have that the extended Weihrauch topos fits within our framework, hence it can be presented as a topos of j -sheaves with respect to the topos of abstract presheaves generated by its generating tripos.

9. Conclusion

We have shown that localic and realizability toposes share deeper structural features beyond being instances of the tripos-to-topos construction [19, 41]: they are both instances of a tripos-to-topos T_P generated from a $\exists$ -supercompactifiable tripos P . This means that they both can be presented as toposes of j -sheaves on an abstract localic presheaf category $\mathbf{E}_P = (\mathcal{G}_P)_{\text{ex/lex}}$ which enjoys a generalization of the localic instance of the Comparison Lemma, since their $\mathbf{E}_P$ coincides with the topos $\mathsf{T}_{P^\exists}$ generated by their full existential completion $P^\exists$. Diagrammatically, we have the geometric embeddings of toposes

$$\begin{array}{ccc} \mathbf{PSh}(\mathbf{L}) & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathbf{Sh}(\mathbf{L}) \end{array} \qquad \begin{array}{ccc} \mathbf{E}_{\mathcal{P}} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathbf{RT}(\mathbb{A}) \end{array}$$

where $\mathbf{PSh}(\mathbf{L}) = (\mathcal{G}_{\mathbf{L}(-)})_{\text{ex/lex}} \equiv \mathbf{T}_{\mathbf{L}^\exists}$ and $\mathbf{E}_{\mathcal{P}} = (\mathcal{G}_{\mathcal{P}})_{\text{ex/lex}} \equiv \mathbf{T}_{\mathcal{P}^\exists}$, and more generally

$$\begin{array}{ccc} & \xrightarrow{\quad} & \\ \mathbf{T}_{\mathcal{P}^\exists} & \begin{array}{c} \perp \\ \xleftarrow{\quad} \end{array} & \mathbf{T}_{\mathcal{P}} \end{array}$$

where $\mathbf{E}_{\mathcal{P}} = (\mathcal{G}_{\mathcal{P}})_{\text{ex/lex}} \equiv \mathbf{T}_{\mathcal{P}^\exists}$.

Regarding related works, we remark that in the context of Grothendieck toposes, a notion of *supercompact objects* and *supercompactly generated topos* was introduced in [43, 5], inspired by [39] but with a crucial use of arbitrary coproducts to investigate their properties and applications. In particular, for these notions we are not aware of a corresponding notion of supercompactification, and, hence, of a generalization of the localic instance of the Comparison Lemma.

In the future, we aim to extend our results to the context of fibrations of toposes, encompassing predicative localic and realizability toposes like that in [27], where the reference to theory of Lawvere doctrines and Lawvere-Tierney sheaves are compulsory.

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