CLOSED-STRING MIRROR SYMMETRY FOR DIMER MODELS

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Abstract. For all punctured Riemann surfaces arising as mirror curves of toric Calabi–Yau threefolds, we show that their symplectic cohomology is isomorphic to the compactly supported Hochschild cohomology of the noncommutative Landau–Ginzburg model defined on the NCCR of the associated toric Gorenstein singularities. This mirror correspondence is established by analyzing the closed–open map with boundaries on certain combinatorially defined immersed Lagrangians in the Riemann surface, yielding a ring isomorphism. We give a detailed examination of the properties of this isomorphism, emphasizing its relationship to the singularity structure.

Contents

1. Introduction

Dimer models are quivers $\mathcal{Q}$ embedded in a Riemann surface Σ such that the embedding partitions Σ into faces with opposite boundary orientations. This structure gives rise to nontrivial relations in the path algebra of $\mathcal{Q}$, resulting in an interesting noncommutative algebra called the Jacobi algebra $J = \mathrm{Jac}\,(\mathcal{Q})$. The algebra J comes equipped with a distinguished central element W, so that (J, W) defines a *noncommutative Landau–Ginzburg (LG) model*, which we take as our mirror B-model throughout.

In particular, when $\mathcal{Q}$ satisfies a suitable consistency condition and Σ is a torus, it is known that J provides a noncommutative crepant resolution (NCCR) of a 3-dimensional toric Gorenstein singularity $Y_{\mathcal{Q}}$. Hence, (J, W) is equivalent to a Landau–Ginzburg model on a toric crepant resolution $\widetilde{Y_{\mathcal{Q}}}$ of the same singularity. In fact, the potential $\tilde{W}$ on the commutative resolution $\widetilde{Y_{\mathcal{Q}}}$ is induced by the height function for which all vectors in the toric fan have height 1.

$$\begin{tikzcd}[column sep=small] (J,W) \arrow[dr, "\text{NCCR}"'] & & (\widetilde{Y_{\mathcal{Q}}},\tilde{W}) \arrow[dl, "\text{CCR}"] \\ & (Y_{\mathcal{Q}},\underline{W}) & \end{tikzcd}$$

Locally, on each toric $\mathbb{C}^3$-chart, the potential is given by the product of three affine coordinates.

The mirror of the Landau–Ginzburg (LG) model $(\widetilde{Y_{\mathcal{Q}}},\tilde{W})$ is known to be a hypersurface in $(\mathbb{C}^\times)^2$, namely a punctured Riemann surface X whose tropical degeneration can be described combinatorially by the fan data of $\widetilde{Y_{\mathcal{Q}}}$. In this paper, we study the mirror symmetry of X, incorporating the noncommutative LG model (J,W) on the mirror side. Specifically, we begin with a dimer model $\mathcal{Q}$ on a torus and investigate the Lagrangian Floer theory of the punctured Riemann surface X, which serves as the mirror A-model to the noncommutative LG model (J,W).

Mirror symmetry in this direction—taking the commutative LG model $(\widetilde{Y_{\mathcal{Q}}},\tilde{W})$ as the B-model—has been extensively studied in the literature; see, for example, [Lee16, CHL24, PS19]. The opposite direction of mirror symmetry, where $(\widetilde{Y_{\mathcal{Q}}},\tilde{W})$ plays the role of the A-model, was investigated in [AA24], where the authors construct and compute the (partially wrapped) Fukaya category associated with $(\widetilde{Y_{\mathcal{Q}}},\tilde{W})$.

On the other hand, the punctured Riemann surface X can, in fact, be obtained combinatorially via *dimer duality*. Given a dimer $\mathcal{Q}$, one performs a simple operation—consisting of cutting, flipping, and pasting—to obtain another dimer $\mathcal{Q}^\vee$, called the *dual dimer*, which is embedded in another surface $\Sigma^\vee$ (see 3.4). This process is involutive: applying the same operation to $\mathcal{Q}^\vee$ recovers the original dimer $\mathcal{Q}$. The punctured surface $X = X_{\mathcal{Q}^\vee}$ is then obtained by removing the vertices of $\mathcal{Q}^\vee$ from $\Sigma^\vee$.

Bocklandt [Boc16b] explained the homological mirror symmetry for $X_{\mathcal{Q}^\vee}$ using this dimer duality, by comparing the wrapped Floer theory of the edges of $\mathcal{Q}^\vee$—which define noncompact Lagrangians in $X_{\mathcal{Q}^\vee}$ by construction—with certain matrix factorizations of W.

Theorem 1.1. [Boc16b, Corollary 9.4.] *Let $\mathcal{Q}$ be a consistent dimer which admits a perfect matching (see 3.3), and $X_{\mathcal{Q}^\vee} := \Sigma^\vee \setminus \mathcal{Q}_0^\vee$. Then there is a fully faithful embedding*

$$\mathrm{WFuk}(X_{\mathcal{Q}^\vee}) \hookrightarrow \mathrm{mf}(J,W).$$

where the right hand side is the matrix factorization category of the noncommutative Landau-Ginzburg model $(J,W) = (\mathrm{Jac}\,(\mathcal{Q}),W)$.

Alternatively, the dimer duality can be interpreted through the mirror construction based on Maurer–Cartan deformations of immersed Lagrangians via weak bounding cochains. In this approach, we start with the dual dimer $\mathcal{Q}^\vee$. Certain cyclic paths in the path algebra of $\mathcal{Q}^\vee$ canonically determine an immersed Lagrangian

$$\mathbb{L} = \bigoplus_i L_i$$

in $X_{\mathcal{Q}^\vee}$, where each L_i is an immersed circle in $\Sigma^\vee$.

In [CHL21], the second-named author established a local mirror construction by allowing weak bounding cochains (in the sense of [FOOO09]) to vary with noncommutative coefficients. This yields a noncommutative LG model as the weak Maurer–Cartan moduli space parameterizing Floer-theoretic deformations of $\mathbb{L}$. Applying this construction to $\mathbb{L}$ recovers precisely the same noncommutative LG model (J, W); see Section 2 for further details on the corresponding formal deformation theory.

One of the advantages of this new interpretation is that it enables us to derive B-model invariants of (J, W) directly from the Floer-theoretic data of $\mathbb{L}$, thereby providing a geometric framework for comparing them with A-model invariants purely through the Floer theory of $\mathbb{L} \subset X$. For instance, the compactly supported Hochschild cohomology $HH^*_c(\mathrm{mf}(W))$—which serves as the closed-string B-model invariant of (J, W)—can be obtained from a suitable variant of the Floer complex of $\mathbb{L}$, as described below.

In the same spirit as [CT13], one may alternatively define $HH^*_c(\mathrm{mf}(W))$ as the Hochschild cohomology $HH^*(J, W)$ of the curved algebra (J, W), which is computed explicitly in [Won21] via a spectral sequence argument. Our first result shows that $HH^*(J, W)$ can be realized as a boundary deformation of the Floer complex $CF(\mathbb{L}, \mathbb{L})$, where the (weak) bounding cochain is allowed to vary over the entire (noncommutative) weak Maurer–Cartan space.

Theorem 1.2. *There is a chain-level identification which induces*

$$HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)) \;\cong\; HH^*(J, W) \;=\; HH^*_c(\mathrm{mf}(W)).$$

Here, "cyc" indicates that $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ carries a natural quiver structure arising from the composability of morphisms between L_i and L_j, and we collect cyclic elements with respect to this structure. (The precise definition of $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ will be given in Section 2.)

With this in mind, one can naturally construct a geometric map relating $HH^*(J, W)$ to the corresponding A-model invariants of X, namely, the symplectic cohomology $SH^*(X)$. Indeed, one has a chain-level map

$$SC^*(X) \;\longrightarrow\; CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)), \tag{1.1}$$

which can be viewed as an adaptation of the Kodaira–Spencer map of [FOOO16], counting pseudo-holomorphic disks with interior insertions from ambient chains in $SC^*(X)$. The map (1.1) is closer in spirit to that of [HJL25], in the sense that the mirror critical locus is non-isolated and thus involves higher-degree information as well. The construction presented here provides a noncommutative generalization of this.

On the cohomology level, the induced map

$$\mathsf{KS} : SH^*(X) \;\longrightarrow\; HH^*(J, W)$$

is automatically a ring homomorphism, whose proof essentially goes back to [FOOO16] and relies on degeneration arguments in the TQFT formalism. In this paper, we establish the closed-string mirror symmetry between X and (J, W) by showing that KS is an isomorphism. Our main result is as follows.

Theorem 1.3. *The Kodaira–Spencer map*

$$\mathsf{KS} : SH^*(X) \;\xrightarrow{\cong}\; HH^*_c(\mathrm{mf}(W))$$

is a ring isomorphism.

The Kodaira–Spencer map KS considered here differs substantially from the one introduced in [FOOO16]. Even when restricted to the degree-zero component (with respect to the usual $\mathbb{Z}/2$-grading), its image is no longer a scalar multiple of the unit class

$$\mathbf{1}_{\mathbb{L}} \in HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)) = HH^*_c(\mathrm{mf}(W)).$$

In fact, the subring generated by the unit class $\mathbf{1}_{\mathbb{L}} = \sum_i \mathbf{1}_{L_i}$ is strictly smaller than $SH^{\text{even}}(X)$ whenever the associated toric variety $Y_{\mathcal{Q}}$ has singularities along codimension-two strata. Intuitively, this phenomenon can be understood from the perspective of the commutative crepant resolution $\widetilde{Y_{\mathcal{Q}}} \to Y_{\mathcal{Q}}$, as illustrated below.

The smooth case. Consider first the nonsingular situation, where $X = X_{\mathcal{Q}^\vee}$ is a pair of pants and $Y_{\mathcal{Q}} = \mathbb{C}^3$. In this case, $W : Y \to \mathbb{C}$ is given by $W = xyz$. There is a natural correspondence between three components of conical ends at punctures of X and the three coordinate axes of the mirror $\mathbb{C}^3$, which together form the critical locus of W. The subring of $SH^*(X)$ spanned by even and odd $\mathbb{Z}_{>0}$-familes of Reeb orbits encircling each puncture of X is isomorphic to $\mathbb{C}[x, dx]$, which can be interpreted as the holomorphic de Rham complex of the corresponding coordinate axis in $\mathbb{C}^3$.

The singular case. When $Y_{\mathcal{Q}}$ carries nontrivial singularities, the situation becomes more subtle. In Figure 1, the surface $X = X_{\mathcal{Q}^\vee}$ is an elliptic curve with seven punctures, and its mirror is a Landau–Ginzburg model on the NCCR J of the toric Gorenstein singularity $Y_{\mathcal{Q}}$. The toric fan of $Y_{\mathcal{Q}}$ is the cone spanned by lattice vectors lying in a polygon $P_{\mathcal{Q}}$ (the pentagon in Figure 1) contained in the height-one hyperplane (of N).

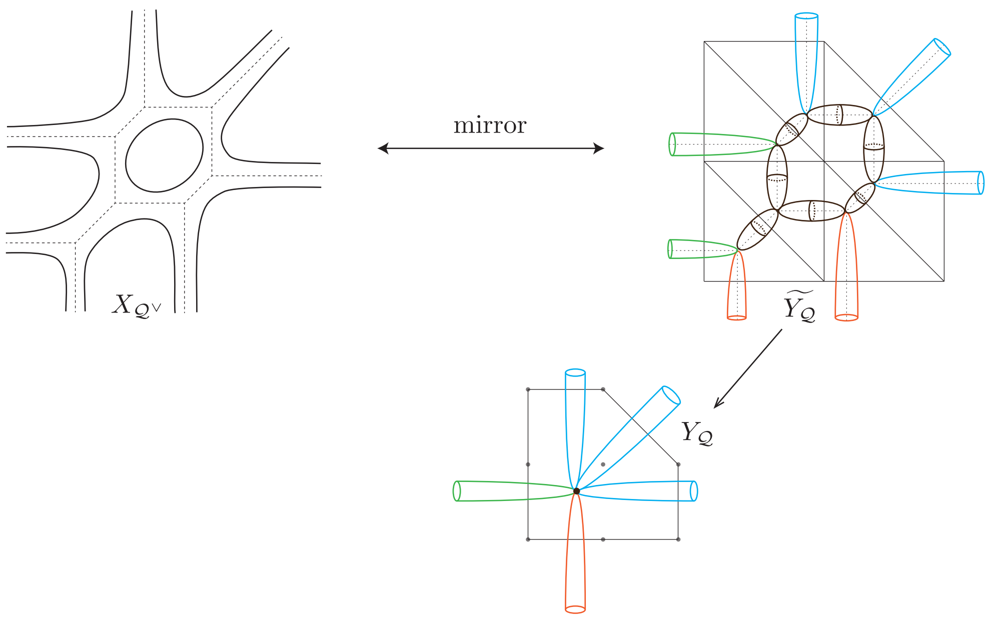

Figure 1. A punctured Riemann surface, the mirror toric Gorenstein singularity, and its toric crepant resolution

Unlike the smooth case, $Y_{\mathcal{Q}}$ possesses only five irreducible codimension-two strata, so two expected components are "missing." The subring generated by the unit class in $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ can captures these five irreducible components, only. Examining a toric crepant resolution $\widetilde{Y_{\mathcal{Q}}}$ obtained by any simplicial decomposition of $P_{\mathcal{Q}}$, we find that $\widetilde{Y_{\mathcal{Q}}}$ indeed contains the expected number of codimension-two strata (each isomorphic to $\mathbb{C}$); however, under the resolution map,

$$\widetilde{Y_{\mathcal{Q}}} \longrightarrow Y_{\mathcal{Q}},$$

certain components are identified, creating the singularities of $Y_{\mathcal{Q}}$.

Therefore, by comparing the orbits around the punctures of X with their KS-images, one can identify precisely the additional elements in $HF^{\mathrm{even}}_{\mathrm{cyc}}((\mathbb{L},b),(\mathbb{L},b))$ that correspond to the singularities along the codimension-two strata of $Y_{\mathcal{Q}}$. Roughly, these elements are the special classes $\Psi_{i,j} \in HH^{\mathrm{even}}_c(\mathrm{mf}(W))$ appearing in Wong's computation [Won21]. The odd-degree component of KS exhibits a similar feature, reflecting the singularity of $Y_{\mathcal{Q}}$ at the torus fixed point and giving rise to distinguished classes $\Theta_v \in HH^{\mathrm{odd}}_c(\mathrm{mf}(W))$. This behavior in the odd-degree part is closely related to the fact that the immersed Lagrangian $\mathbb{L}$ has multiple irreducible components, as will be clarified in 7.5.

Remark 1.4. *Our previous work* [HJL25] *deals with the special case where $Y_{\mathcal{Q}}$ is an orbifold given as an abelian quotient of $\mathbb{C}^3$. The classes $\Psi_{i,j}$ and Θ_v are analogous to the* twisted sectors *of the (closed-string) B-model invariants introduced in* [HJL25]*, which account for the orbifold singularities of the toric orbifold $Y_{\mathcal{Q}}$ in this case.*

We conclude by noting that the methods developed in this paper may provide a general framework for geometrically relating the closed-string A-model invariants of a symplectic manifold X to those of its (local) noncommutative mirror B-model $(Y_{\mathbb{L}}, W_{\mathbb{L}})$, constructed from the Maurer–Cartan deformation theory of any given Lagrangian$\mathbb{L}$. We conjecture that $HF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b))$ computes the Hochschild-type invariant of $(Y_{\mathbb{L}}, W_{\mathbb{L}})$. This can indeed be verified in the uncurved case (i.e., when $W_{\mathbb{L}} = 0$), up to certain completion issues (see [HLT]). Furthermore, we expect this invariant to coincide with the "local piece" of the symplectic cohomology $SH^*(X)$ that can be probed by $\mathbb{L}$ via the open–closed string maps.

Notations

- $\mathcal{Q}$: a zigzag consistent dimer embedded in a torus (B-model quiver)
- $\mathbb{C}\mathcal{Q}$: the path algebra of $\mathcal{Q}$ over $\mathbb{C}$
- $\mathbb{C}\mathcal{Q}_{cyc} = \mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}]$: the set of cyclic words in $\mathbb{C}\mathcal{Q}$
- $\mathcal{Q}_2 = \mathcal{Q}_2^+ \cup \mathcal{Q}_2^-$: the set of faces, positive faces, and negative faces
- $\Phi_{\mathcal{Q}} = \sum_{F \in \mathcal{Q}_2^+} \partial F - \sum_{F \in \mathcal{Q}_2^-} \partial F$: the spacetime superpotential in $\mathbb{C}\mathcal{Q}_{cyc}$
- $J = \mathbb{C}\mathcal{Q}/(\partial_e \Phi_{\mathcal{Q}} : e \in \mathcal{Q}_1)$: the Jacobi algebra for the dimer $\mathcal{Q}$
- $\mathcal{Z}J$: the center of J
- J_{cyc} : cyclic subalgebra of J
- $\mathcal{Q}^\vee$: the dual dimer obtained from $\mathcal{Q}$ (A-model quiver)
- $\Sigma^\vee$: the compact Riemann surface where the dimer $\mathcal{Q}^\vee$ is embedded
- $X = X_{Q^\vee} = \Sigma^\vee \setminus \mathcal{Q}_0^\vee$: the punctured Riemann surface for the pair $(\Sigma^\vee, \mathcal{Q}_0^\vee)$

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2. Floer theory preliminaries: noncommutative mirror construction

In this section, we briefly review the general construction of a (local) noncommutative mirror arising from the algebraic deformation of the A_∞-algebra $CF^*(\mathbb{L},\mathbb{L})$ associated to a compact exact Lagrangian $\mathbb{L}$. In our main application, we will employ the pearl trajectory model for $CF^*(\mathbb{L},\mathbb{L})$, whose technical details are provided in [Sei11]. When $\mathbb{L}$ is an immersed Lagrangian with transverse self-intersections, $CF^*(\mathbb{L},\mathbb{L})$ acquires additional generators coming from each self-intersection point, which record the branch jumps between the local sheets of $\mathbb{L}$ meeting at that point. We will give a concrete description of their roles as inputs of the A_∞-operations in Section 3.4 for the case $\dim \mathbb{L} = 1$. For more general aspects of immersed Lagrangian Floer theory, see [AJ10].

2.1. **Noncommutative deformation of $CF^*(\mathbb{L},\mathbb{L})$ by (weak) bounding cochains.** Let $(E, \omega = d\lambda)$ be a Liouville manifold, and let $\mathcal{F}(E)$ denote the compact Fukaya category. We fix some possibly immersed, mutually transversal, closed, spin, oriented, unobstructed Lagrangians $L_1, \dots, L_k \in \mathrm{Ob}(\mathcal{F}(E))$ and let

$$\mathbb{L} = \bigoplus_{i=1}^{k} L_i.$$

Following [AJ10], its (immersed) Lagrangian Floer complex takes the form

$$CF^*(\mathbb{L},\mathbb{L}) = \bigoplus_{i=1}^{k} \left[H^*(L_i;\mathbb{C}) \oplus \bigoplus_{p \in L_i} \mathrm{Span}_{\mathbb{C}}\{(p_-,p_+),(p_+,p_-)\} \right] \oplus \bigoplus_{\substack{i,j=1,\dots,k \\ i \neq j}} CF^*(L_i, L_j) \tag{2.1}$$

where $p \in L_i$ runs over all immersed points of L_i, and $p_\pm$ are points in the normalization so that $(p_-,p_+),(p_+,p_-)$ indicate corresponding branch-jumps. For $H^*(L_i;\mathbb{C})$ in (2.1), we take a Morse complex of a perfect Morse function on L_i. For later use, we denote by $\mathbf{1}_{L_i} \in C^*(L_i)$ the unit in $H^*(L_i)$. In our setting, $\mathbf{1}_{L_i}$ is simply the maximum of the chosen perfect Morse function on L_i.

We have a structure of unital A_∞-algebra

$$(CF^*(\mathbb{L},\mathbb{L}), \{m_k\}_{k\geq 1}) \tag{2.2}$$

over $\mathbb{C}$. Below, we apply the noncommutative base change as in [CHL21] to (2.2). Let us first fix a few notations and conventions.

Definition 2.1 ([CHL21, Definition 6.1])**.** *We define the quiver $\mathcal{Q}^{\mathbb{L}}$ as follows. It has a vertex v_i for each irreducible component L_i of $\mathbb{L}$, and an edge x from v_i to v_j for each odd-degree immersed Floer generator $X \in CF^*(L_i, L_j)$.*

The edge x in $\mathcal{Q}^{\mathbb{L}}$ corresponding to X will later be regarded as a formal coordinate function (dual to X) on the noncommutative mirror. For each such variable x from v_i to v_j, we define its **head** and **tail** by

$$h(x) = v_j, \qquad t(x) = v_i.$$

For each generator $X \in CF^*(L_i, L_j) \subset CF^*(\mathbb{L},\mathbb{L})$, we take the **head** and **tail** as

$$h(X) = v_i, \qquad t(X) = v_j.$$

For a Morse–Bott generator in $H^*(L_i)$, both the head and the tail are simply v_i.

To be more precise, let $\{X_e\}_{e\in I}$ be the set of all odd degree immersed generators indexed by some set I. Consider a set of noncommutative formal variables x_e. We label an edge in $\mathcal{Q}^{\mathbb{L}}$ by x_e when it corresponds to X_e, and hence we have

$$h(x_e) = t(X_e), \;\; t(x_e) = h(X_e).$$

[1] We assign degrees for them as $|x_e| = 1 - |X_e|$. Juxtaposition of these formal variables x_e defines the product structure, for which we use the *(backward) concatenation*

$$x_{e'} \cdot x_e = \begin{cases} x_{e'}x_e & \text{if } t(x_{e'}) = h(x_e), \\ 0 & \text{otherwise} \end{cases}$$

This results in the path algebra of the quiver $\mathcal{Q}^{\mathbb{L}}$

$$K := \mathbb{C}\mathcal{Q}^{\mathbb{L}} = \mathbb{C}\langle x_{e_l} \cdots x_{e_1} : e_i \in I \rangle$$

generated by words or paths forms by x_e's. The maps h and t have natural extensions to K,

$$h(x_{e_l} \cdots x_{e_1}) = h(x_{e_l}), \;\; t(x_{e_l} \cdots x_{e_1}) = t(x_{e_1}).$$

Note that K is a bimodule over the semisimple ring $\Bbbk = \mathbb{C}\pi_1 \oplus \cdots \oplus \mathbb{C}\pi_k$ spanned by idempotents π_i (length zero paths) with $h(\pi_i) = t(\pi_i) = v_i$ such that

$$\pi_j \cdot K \cdot \pi_i = R\langle x_{e_l} \cdots x_{e_1} : h(x_{e_l} \cdots x_{e_1}) = v_j, \; t(x_{e_l} \cdots x_{e_1}) = v_i \rangle.$$

Similarly, $CF^*(\mathbb{L}, \mathbb{L})$ also has a $\Bbbk$-bimodule structure, but with the opposite convention:

$$\pi_i \cdot CF^*(\mathbb{L}, \mathbb{L}) \cdot \pi_j = CF^*(L_i, L_j).$$

We then make the base change of A_∞-algebra $CF^*(\mathbb{L}, \mathbb{L})$ to obtain

$$K \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L}). \tag{2.3}$$

Therefore, for a word $f \in K$ and a generator $X \in CF^*(\mathbb{L}, \mathbb{L})$ is a nontrivial element $fX (= f \otimes X) \in K \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})$ only if $h(X) = t(f)$.

The A_∞-structure on (2.3) is a linear extension on $\{m_k\}_{k \geq 1}$ in (2.2), but it is sensitive to the order in which coefficients are pulled out, due to their noncommutativity.

Definition 2.2 ([CHL21, Definition 2.6])**.** *Let $f_1X_1, \ldots, f_kX_k \in K \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})$. The A_∞-structure $\{m_k\}_{k\geq 1}$ on* (2.2) *extends to an A_∞-structure on* (2.3) *by the formula*

$$m_k(f_1X_1, \ldots, f_kX_k) := f_k \cdots f_2 \cdot f_1 m_k(X_1, \ldots, X_k). \tag{2.4}$$

It is not difficult to see that $\{m_k\}_{k\geq 1}$ in Definition 2.2 satisfies A_∞-equations (see [CHL21, Lemma 2.9]).

Nontrivial generators $fX \in K \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})$ such that $h(f) = t(X)$ will be referred to as *cyclic paths* or simply *cycles*. The span of all cyclic paths in (2.3) is denoted by $CF^*_{cyc}(\mathbb{L}, \mathbb{L})$. Visually, for a cycle $fX \in K \otimes_{\Bbbk} CF^*(L_i, L_j)$, the assigned arrows are depicted as

$$L_i \bullet \underset{X}{\overset{f}{\rightleftarrows}} \bullet L_j.$$

Proposition 2.3. *$CF^*_{cyc}(\mathbb{L}, \mathbb{L})$ is an A_∞-subalgebra of $K \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})$.*

Proof. Suppose that there is a nontrivial m_k-operation taking generators $X_1, \ldots, X_k$ as inputs, and its output involves a generator Y. Then we have $h(X_1) = h(Y)$, $t(X_k) = t(Y)$, and $t(X_i) = h(X_{i+1})$ for $1 \leq i < k$. If $f_1, \ldots, f_k \in K$ are words for which f_iX_i are all cycles($1 \leq i \leq k$), then by Definition 2.2

$$m_k(f_1X_1, \ldots, f_kX_k) = \cdots + (f_k \cdots f_2 \cdot f_1)Y + \cdots =: \cdots + fY + \cdots .$$

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[1] This means that we assign an arrow from i (tail) to j (head) for a morphism from L_j to L_i. This seems a little unnatural, but is unavoidable since our convention of composing morphisms in the A_∞-structure follows [FOOO09] which is opposite to the usual convention of composing functions.

Since $h(f_i) = t(X_i) = h(X_{i+1}) = t(f_{i+1})$, $f = f_k \cdots f_2 \cdot f_1$ is not zero, and we see that

$$h(f) = h(f_k) = t(X_k) = t(Y) \text{ and } t(f) = t(f_1) = h(X_1) = h(Y).$$

Thus the output fY is also a cycle, and the assertion follows by applying the same argument to each generator Y appearing in the output. □

In particular, the following element of $CF^*_{cyc}(\mathbb{L}, \mathbb{L})$ will play an central role:

$$b = \sum_{e \in I} x_e X_e \in CF^*_{cyc}(\mathbb{L}, \mathbb{L}).$$

We solve the weak Maurer–Cartan equation for b in order to determine the relations among $\{x_e\}$ that make b a weak bounding cochain. Specifically, we compute

$$m(e^b) = m_1(b) + m_2(b, b) + \cdots, \tag{2.5}$$

and seek for conditions under which the above expression becomes a K-linear combination of units. In practice, $m_k(b, \cdots, b)$ in (2.5) is by definition given as the sum of terms of the form $m_k(x_{e_1} X_{e_1}, \cdots, x_{e_k} X_{e_k})$ which can be computed using the rule (2.4). Since b is in odd degree, we have

$$m_1(b) + m_2(b, b) + \cdots = \sum_{i=1}^{k} W_{L_i} \mathbf{1}_{L_i} + \sum_f P_f X_f \in K \otimes_{\Bbbk} CF^{\text{even}}(\mathbb{L}, \mathbb{L})$$

for some series P_f and W_{L_i} in x_e's, where X_f runs over all even degree generators.

Remark 2.4. *In general, the element P_f appearing in the equation is an infinite series, so one typically needs to complete K in order to include P_f. The standard approach is to use Novikov coefficients and consider T-adic convergence of the series. However, in our setting, this procedure is unnecessary, as all series that arise in our main applications are actually finite. See Lemma 3.15.*

We denote the sum of componentwise units $\mathbf{1}_{L_i}$ by

$$\mathbf{1}_{\mathbb{L}} := \sum_{i=1}^{k} \mathbf{1}_{L_i} \in CF^*(\mathbb{L}, \mathbb{L}),$$

and it serves as the unit of a A_∞-algebra $CF^*(\mathbb{L}, \mathbb{L})$. We then define

$$W_{\mathbb{L}} := \sum_{i=1}^{k} W_{L_i} \in K,$$

and call it the *worldsheet superpotential* or simply the *potential* of $\mathbb{L}$. Since $W_{L_i} \cdot \mathbf{1}_{L_j} = \delta_{ij}$, we have

$$m_1(b) + m_2(b, b) + \cdots + m_k(b, \ldots, b) + \cdots = W_{\mathbb{L}} \mathbf{1}_{\mathbb{L}} \tag{2.6}$$

modulo the relations $P_f \equiv 0$ for each f. More precisely, if we denote by $A_{\mathbb{L}}$ the quotient of K by the two-sided ideal $(P_f : f \in F)$, and make the further base change

$$A_{\mathbb{L}} \otimes_K (K \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})) \cong A_{\mathbb{L}} \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L}), \tag{2.7}$$

then $b = \sum x_e X_e$ viewed as an element of (2.7) now solves the weak Maurer-Cartan equation (2.6). The noncommutative algebra $A_{\mathbb{L}}$ will be called the (weak) Maurer-Cartan algebra of $\mathbb{L}$.

Remark 2.5. *When the A_∞-structure is cyclic, one can identify $A_{\mathbb{L}}$ with the* Jacobi algebra *$J(\Phi_{\mathbb{L}})$ of the quiver $\mathcal{Q}^{\mathbb{L}}$ with the cyclic (spacetime) potential*

$$\Phi_{\mathbb{L}} = \sum_k \frac{1}{k+1} \langle m_k(b, \ldots, b), b \rangle,$$

defined by

$$J(\Phi_{\mathbb{L}}) := K/(\partial_{x_e} \Phi_{\mathbb{L}} : e \in I).$$

We abuse the notation W to also denote its induced element in $A_{\mathbb{L}}$, which consists of cyclic paths in $\mathcal{Q}^{\mathbb{L}}$.[2] Then we have

Proposition 2.6. [CHL21, Theorem 3.10]*W lies in the center of $A_{\mathbb{L}}$. (In this case, $(A_{\mathbb{L}}, W)$ is often called a noncommutative Landau-Ginzburg model.)*

We next deform the A_∞-structure on $A_{\mathbb{L}} \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})$ by weak bounding cochain b. It gives rise to a new A_∞ structure $\{m_k^{b,\cdots,b}\}$ defined by

$$\begin{aligned} m_k^{b,\ldots,b}(f_1X_1, \ldots, f_kX_k) &= m(e^b, f_1X_1, e^b, \ldots, e^b, f_kX_k, e^b) \\ &= \sum_{l_0,\cdots,l_k \geq 0} m_{k+l_0+\cdots+l_k}(b^{\otimes l_0}, f_1X_1, b^{\otimes l_1}, \ldots, f_kX_k, b^{\otimes l_k}). \end{aligned} \tag{2.8}$$

A similar convergence issue potentially arises in the expansion of $m_k^{b,\ldots,b}$, as discussed in Remark 2.4. However, this can again be resolved in our main application due to the finiteness property of m_k operations (see Lemma 3.15).

Remark 2.7. *Readers are warned that the A_∞-structure $m_k^{b,\ldots,b}$ is not linear over $A_{\mathbb{L}}$ since variables are no longer commutative. However, it is still linear over the center of $A_{\mathbb{L}}$, which is obvious from* (2.4) *and* (2.8).

Note that we now have a nontrivial curvature $m_0^b(1) = m(e^b) = W_{\mathbb{L}} \mathbf{1}_{\mathbb{L}}$. Following the standard argument as in [FOOO09, §3.6], this gives us a curved A_∞-algebra

$$CF^*((\mathbb{L}, b), (\mathbb{L}, b)) := \Big(A_{\mathbb{L}} \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L}), \{m_k^{b,\ldots,b}\}_{k \geq 0} \Big) \tag{2.9}$$

over $A_{\mathbb{L}}$. Obviously, $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ is unital, and $(m_1^{b,b})^2 = 0$ since b is a weak bounding cochain.

Analogously to $CF^*_{cyc}(\mathbb{L}, \mathbb{L})$, one can collect cyclic elements in $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$.

Definition 2.8. *We denote this by $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ a subspace of of $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ generated by elements $fX (= f \otimes X) \in A_{\mathbb{L}} \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L})$ such that $h(f) = t(X)$.*

Since b itself is cyclic, the same proof as in Proposition 2.3 tells us that $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ is an A_∞ subalgebra of $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$. Moreover, it contains the potential $W_{\mathbb{L}}$ and the unit $\mathbf{1}_{\mathbb{L}}$ (hence it is unital). We will relate this with some closed-string invariant of the noncommutative LG model $(A_{\mathbb{L}}, W)$.

---

[2] The quiver structure on K descends to $A_{\mathbb{L}}$ since each P_f is homogenous, i.e., all of its summands share the same head and tail.

3. Dimer models and nc mirror symmetry for punctured Riemanns surfaces

We briefly recall the theory of dimer models, which are quivers $\mathcal{Q}$ embedded in a Riemann surface Σ in a particularly structured way. In particular, Bocklandt [Boc16b] provides a combinatorial construction of a noncommutative mirror to the punctured Riemann surface $\Sigma \setminus \mathcal{Q}_0$ using dimer duality. This construction agrees with the noncommutative Landau–Ginzburg model given in Proposition 2.6, as will be reviewed in Section 3.4 below. We will also specify the closed-string B-model invariant of this noncommutative mirror that we will work with in our main application.

Our exposition will largely follow [Won21], including its notation.

3.1. **Dimer models and noncommutative LG mirror.**

Dimer models are a special class of quivers that naturally fit into the geometry of punctured Riemann surfaces, and are defined as follows.

Definition 3.1. *A* **dimer model** $\mathcal{Q}$ *on a compact Riemann surface* Σ *is an embedded quiver*

$$\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1) \subset \Sigma$$

such that every boundary of each component F of $\Sigma \setminus \mathcal{Q}$ consists of at least 3 quiver edges that are altogether arranged either in the direction of the orientation (positive) of ∂F or in the opposite direction (negative).

We call a component of $\Sigma \setminus \mathcal{Q}$ as a *face*. We denote the collection of faces by $\mathcal{Q}_2$. Faces of a dimer on a surface Σ can be divided into two classes, *positive faces* $\mathcal{Q}_2^+$ and *negative faces* $\mathcal{Q}_2^-$, depending on whether the alignment of their boundaries and quiver arrows is positive or negative. Note that, given a dimer model $\mathcal{Q}$ on a surface Σ, every edge $e \in \mathcal{Q}_1$ is adjacent to exactly one face in each of $\mathcal{Q}_2^\pm$.

We define the cyclic (spacetime) potential of $\mathcal{Q}$ by

$$\Phi_{\mathcal{Q}} = \sum_{F \in \mathcal{Q}_2^+} \partial F - \sum_{F \in \mathcal{Q}_2^-} \partial F$$

regarded as an element in $\mathbb{C}\mathcal{Q}_{cyc} = \mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}]$, where

$$\mathbb{C}\mathcal{Q} := \mathbb{C}\langle x_{e_k} \cdots x_{e_1} : h(x_{e_i}) = t(x_{e_{i+1}}), e_i \in \mathcal{Q}_1 \rangle$$

is the path algebra of $\mathcal{Q}$. Then the Jacobi algebra $J = \mathrm{Jac}\,(\mathcal{Q})$ for $(\mathcal{Q}, \Phi_{\mathcal{Q}})$ is defined by

$$J = \mathrm{Jac}\,(\mathcal{Q}, \Phi_{\mathcal{Q}}) = \mathbb{C}\mathcal{Q}/\left(\partial_{x_e}\Phi : e \in \mathcal{Q}_1\right)$$

where ∂_{x_e} is the cyclic derivative at $e \in \mathcal{Q}_1$

$$\partial_{x_e} = \frac{\partial}{\partial x_e} : \mathbb{C}\mathcal{Q}_{cyc} \to \mathbb{C}\mathcal{Q}, \qquad x_{a_k} \cdots x_{a_1} \mapsto \sum_{a_i = e} x_{a_{i-1}} \cdots x_{a_1} x_{a_k} \cdots x_{a_{i+1}}.$$

The relation $\partial_{x_e}\Phi = 0$ in J is ofter referred to as the Jacobi relation.

There is a special central element $W \in \mathcal{Z}J$ which is defined as follows. To each $v \in \mathcal{Q}_0$, we fix an arbitrary face $F_v \in \mathcal{Q}_2$ that has v in its corner. Let $W_v \in J$ be the image of a boundary word of F_v that is starting and ending at v. It is easy to see that W_v is independent from the choice F_v due to Jacobi relations. We set

$$W = \sum_{v \in \mathcal{Q}_0} W_v$$

and call it the potential of J. One can show that W lies in the center of J.

Cyclic paths in J form a subalgebra, defined by

$$J_{cyc} = \bigoplus_{v \in \mathcal{Q}_0} v \cdot J \cdot v \subset J,$$

where v denote the idempotent (length 0 path) at v. It is known that J_{cyc} is a semisimple algebra over the center of J, since every cyclic path in J can be uniquely extended to a central element whose component at each vertex v is the original cyclic path. (In particular, its rank as a semisimple algebra equals the number of vertices in $\mathcal{Q}$.)

3.2. **Zigzag paths and consistency.** We now recall the zigzag cycles in a dimer $\mathcal{Q}$. If two consecutive edges $a'a$(i.e., $t(a') = h(a)$) lies on the boundary of a positive face, they are called a *zigzag*. (a is a *zig* and a' is a *zag*.) Likewise, if $a'a$ lies on the boundary of a negative face, they are called a *zagzig*, where in this case a is a *zag* and a' is a *zig*. Fix a universal cover $\widetilde{\Sigma} \to \Sigma$ and let $\widetilde{\mathcal{Q}}$ be a lifting of the quiver $\mathcal{Q}$ in $\widetilde{\Sigma}$, which is obviously a dimer model $\widetilde{\Sigma}$. We write $\tilde{e} \in \widetilde{\mathcal{Q}}_1$ for any lift of an edge $e \in \mathcal{Q}_1$.

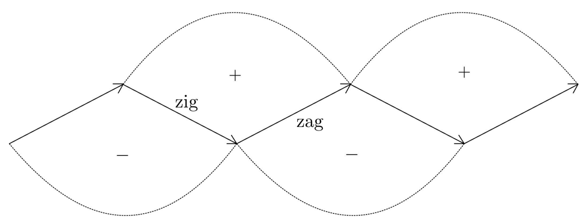

FIGURE 2. Zigs and zags

Definition 3.2. *A* **zig ray** *(resp. a* **zag ray***) at $\tilde{e}$ is a path*

$$\cdots \tilde{b}_l \tilde{a}_l \cdots \tilde{b}_1 \tilde{a}_1 \tag{3.1}$$

in $\widetilde{\mathcal{Q}}$ such that $\tilde{a}_1 = \tilde{e}$ and $\tilde{b}_l \tilde{a}_l$ are zigzags (resp. zagzigs), $\tilde{a}_{l+1}\tilde{b}_l$ are zagzigs (resp. zigzags) for all $l \geq 1$.

We will focus on dimers whose zigzag rays satisfy the following condition, which guarantees several desirable algebraic properties of J.

Definition 3.3. *A dimer $\mathcal{Q}$ is called to be* **zigzag consistent** *if for every edge $e \in \mathcal{Q}_1$ the zig and zag rays in $\widetilde{\mathcal{Q}}_1$ that are emanating from $\tilde{e}$ does not intersect in edges other than the initial edge $\tilde{e}$.*

As $\mathcal{Q}$ is finite, any zig or zag ray (3.1) projects down to a periodic path

$$\cdots b_l a_l \cdots b_1 a_1 \tag{3.2}$$

in $\mathcal{Q}$. The shortest repeat in (3.2) gives rise to a closed path of even length, say $2L$,

$$Z = b_L a_L \ldots b_1 a_1. \tag{3.3}$$

It will be referred to as a *zigzag cycle* in $\mathcal{Q}$. Indeed, zigzag cycles give rise to cycles in Σ from the embedding $\mathcal{Q} \subset \Sigma$. It is often convenient to regard zigzag cycles as cyclic words in $\mathbb{C}\mathcal{Q}_{cyc}$ so that

$$Z = a_1 b_L a_L \cdots a_2 b_1 = b_1 a_1 b_L \cdots b_2 a_2 = \cdots = a_L b_{L-1} \cdots a_1 b_L \in \mathbb{C}\mathcal{Q}_{cyc}$$

We will always use $\{a_l\}$ to denote the set of zigs whenever a zigzag cycle is represented in the form (3.3).

A zigzag cycle Z can be assigned with a homology class $[Z] \in H_1(\Sigma; \mathbb{Z})$. If $\mathcal{Q}$ is zigzag consistent, $[Z]$ is always a nonzero generator in $H_1(\Sigma; \mathbb{Z})$ known to be primitive, meaning that it is a part of some $\mathbb{Z}$-basis. We will focus particularly on consistent dimer models embedded in a torus. In this setting, the following proposition characterizes the intersection behavior of zigzag cycles in terms of their homology classes.

Proposition 3.4 ([Bro12], [Boc12])**.** *Let $\mathcal{Q}$ be a zigzag consistent dimer on a torus T^2 and Z_1 and Z_2 be two distinct zigzag cycles in $\mathcal{Q}$.*

(i) *If $[Z_1]$ and $[Z_2]$ are linearly independent homology classes in $H_1(T^2; \mathbb{Z})$, then there exists at least one edge $e \in \mathcal{Q}$ that appear in both Z_1 and Z_2.*
(ii) *If $[Z_1] = [Z_2]$, then they do not intersect.*

Nonintersecting zigzag cycles will play an important role later.

Definition 3.5. *For a zigzag consistent dimer $\mathcal{Q}$ on a torus, we say that two zigzag cycles are* **parallel** *if they represent the same homology class.*

This terminology is justified by property (ii) of Proposition 3.4.

Next, we introduce the notion of anti-zigzags. Let Z be a zigzag cycle in a dimer $\mathcal{Q}$, and consider all the faces in $\mathcal{Q}_2^+$ adjacent to Z. Each zigzag in Z forms part of the boundary of a face in $\mathcal{Q}_2^+$. Since these faces share the same sign, they cannot share common edges. The boundary paths of these faces complementary to Z form a cycle. We denote this by $\mathcal{O}^+(Z)$ and call it a *positive anti-zigzag cycle* of Z. The cycle analogously defined from negative adjacent disks of Z is denoted by $\mathcal{O}^-(Z)$ and is called *negative anti-zigzag cycle*.

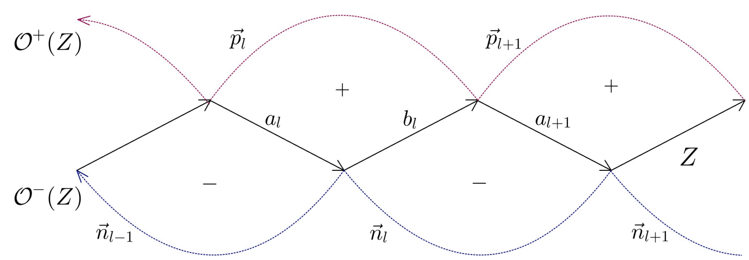

FIGURE 3. A zigzag cycle Z and its two anti-zigzags $\mathcal{O}^{\pm}(Z)$

When zigzag cycle Z is given as in (3.3), we write

$$\mathcal{O}^+(Z) = \vec{p}_1 \cdots \vec{p}_L \text{ and } \mathcal{O}^-(Z) = \vec{n}_1 \cdots \vec{n}_L \tag{3.4}$$

for anti-zigzags $\mathcal{O}^+(Z)$ and $\mathcal{O}^-(Z)$ (as elements in $\mathbb{C}\mathcal{Q}_{cyc}$). The indices are arranged in such a way that $\vec{p}_l b_l a_l$ are boundary words from positive disks adjacent to Z and $\vec{n}_l a_{l+1} b_l$ are boundary words from negative disks adjacent to Z. Since concatenating $\mathcal{O}^{\pm}(Z)$ with Z results in contractible loops in Σ, we see that $[\mathcal{O}^{\pm}(Z)] = -[Z]$ in $H_1(\Sigma; \mathbb{Z})$.

3.3. **Perfect matchings.** Perfect matchings on a dimer is not only a tool to grade the Jacobi algebra J, but also provides a strong link between theory of dimers and NCCR (noncommutative crepant resolution) of a 3 dimensional toric Gorenstein singularities.

Definition 3.6. *A set $\mathcal{P}$ of edges in $\mathcal{Q}$ is called a* **perfect matching** *if for any face $F \in \mathcal{Q}_2$ there is exactly one edge $e \in \mathcal{P}$ such that $e \in \partial F$.*

Any perfect matching $\mathcal{P}$ defines an obvious map

$$\deg_{\mathcal{P}} : \mathcal{Q}_1 \to \mathbb{Z}, \qquad e \mapsto \begin{cases} 1 & e \in \mathcal{P} \\ 0 & e \notin \mathcal{P}, \end{cases}$$

which can be easily shown to induce

$$\deg_{\mathcal{P}} : J \to \mathbb{C}. \tag{3.5}$$

Observe that $\mathcal{Q}$ fixes a cellular decomposition of Σ, which is just $(\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2)$. Let us consider the associated cellular cochain complex

$$0 \longrightarrow \hom(Q_0, \mathbb{Z}) \xrightarrow{\partial} \hom(Q_1, \mathbb{Z}) \xrightarrow{\partial} \hom(Q_2, \mathbb{Z}) \longrightarrow 0.$$

The coboundary map can be explicitly written as defined by $v^* \mapsto \sum_{h(e)=v} e^* - \sum_{t(e)=v} e^*$ and $e^* \mapsto \sum_{e \in \partial F} F^*$ in terms of dual basis elements. If we identify $\mathcal{P} \subset \mathcal{Q}_1$ with a cellular 1-cochain $\sum_{e \in \mathcal{P}} e^*$, then $\mathcal{P}$ is a perfect matching if and only if we have

$$\partial \mathcal{P} = \sum_{F \in \mathcal{Q}_2} F^*.$$

Therefore the difference of two perfect matching defines an element in $H^1(\Sigma, \mathbb{Z}))$, and the set

$$\{[\mathcal{P} - \mathcal{P}_o] : \mathcal{P} \in PM(\mathcal{Q})\} \subset H^1(\Sigma; \mathbb{Z}) \tag{3.6}$$

for some fixed perfect matching $\mathcal{P}_o$ represents a finite subset in $H^1(\Sigma; \mathbb{Z})$.

When $\Sigma \simeq T^2$, we have $H^1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$, and (3.6) defines a finite subset of this rank-2 lattice. The convex hull of (3.6) in the plane $H^1(\Sigma; \mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}^2$ is called the *matching polytope* of $\mathcal{Q}$, denoted by $MP(\mathcal{Q})$. We classify perfect matchings as *corner*, *boundary*, or *internal*, depending on the location of $[\mathcal{P} - \mathcal{P}_o]$ within $MP(\mathcal{Q})$. These types reflect the behavior of perfect matchings with respect to zigzag cycles. More specifically, let $\mathcal{Q}$ be a zigzag consistent dimer on a torus, and let

$$-\eta_1, \ldots, -\eta_N \in H_1(\Sigma; \mathbb{Z}) \tag{3.7}$$

denote the homology classes associated to the zigzag cycles. We assume these are ordered according to their counterclockwise cyclic order in the lattice $H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^2$.[3]

Theorem 3.7 ([Gul08, §3.3, §3.4],[Boc16a, Theorem 1.47]). *Suppose $\mathcal{Q}$ is a zigzag consistent dimer on a torus and $-\eta$ be one among (3.7).*

(i) *$MP(\mathcal{Q})$ is a lattice polytope with N edges, and the vectors $-\eta_1, \ldots, -\eta_N$ are the outward-pointing normals to these edges.*
(ii) *Each corner of $MP(\mathcal{Q})$ is represented by a unique perfect matching.*
(iii) *For $-\eta_i$, let $\mathcal{P}_i, \mathcal{P}_{i+1} \in MP(\mathcal{Q})$ be two consecutive corner matchings joined by the edge normal to $-\eta_i$. Then we have*

$$\mathcal{P}_i = \mathcal{J}_{\eta_i} \cup \bigcup_{j=1}^{m_i} \mathrm{zig}(Z_{i,j}), \quad \mathcal{P}_{i+1} = \mathcal{J}_{\eta_i} \cup \bigcup_{j=1}^{m_i} \mathrm{zag}(Z_{i,j}).$$

where $Z_{i,1}, \ldots, Z_{i,m_i}$ are the parallel zigzag cycles in the class $-\eta_i$, and $\mathcal{J}_{\eta_i} := \mathcal{P}_i \cap \mathcal{P}_{i+1}$ does not intersect $Z_{i,j}$'s.
(iv) *The set*

$$\left\{ \mathcal{P}_I = \mathcal{J}_{\eta_i} \cup \bigcup_{j \in I} \mathrm{zig}(Z_{i,j}) \cup \bigcup_{j \notin I} \mathrm{zag}(Z_{i,j}) \in PM(\mathcal{Q}) : I \subseteq \{1, \ldots, m_i\} \right\} \tag{3.8}$$

exhausts the boundary matchings lying between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$.

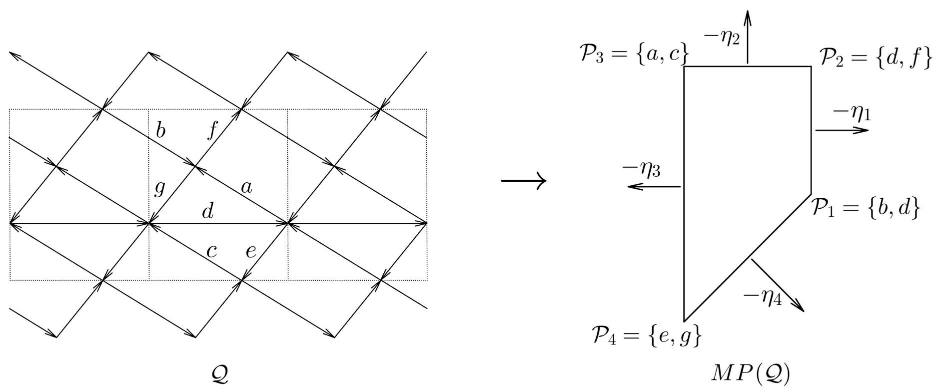

FIGURE 4. A suspended pinchpoint $\mathcal{Q}$ and its matching polytope $MP(\mathcal{Q})$

On the other hand, one can associate a 3-dimensional affine toric Gorenstein singularity $Y_{\mathcal{Q}}$ to the lattice polygon $MP(\mathcal{Q})$. The fan of $Y_{\mathcal{Q}}$ consists of a single cone, constructed by placing $MP(\mathcal{Q})$ in the hyperplane $\{z = 1\} \subset \mathbb{R}^3_{(x,y,z)}$ and taking the cone over $MP(\mathcal{Q})$ with vertex at the origin $(0, 0, 0)$. The singular locus of $Y_{\mathcal{Q}}$ lies along codimension-2 strata corresponding to edges of $MP(\mathcal{Q})$ that contain interior lattice points.

Theorem 3.8 ([Boc16a, Theorem 3.20])**.** *Let $\mathcal{Q}$ be a zigzag consistent dimer on a torus. In the sense of* [vdB04], *J is the NCCR of its center $\mathcal{Z}J$ which is the coordinate ring of the affine toric Gorenstein threefold $Y_{\mathcal{Q}}$ defined by the matching polytope $MP(\mathcal{Q})$.*

Moreover Broomhead proved that every affine toric Gorenstein singularity admits an NCCR from a consistent dimer model on a torus [Bro12, Theorem 8.6].

Remark 3.9. *By Theorem 3.8, each corner matching $\mathcal{P}_i$ corresponds to a torus-invariant divisor in $Y_{\mathcal{Q}}$. For a central element f, its degree $\deg_{\mathcal{P}_i}(f)$ (as defined in* (3.5)*) with respect to $\mathcal{P}_i$ measures the vanishing order of f, viewed as a regular function on $Y_{\mathcal{Q}}$, along the toric divisor associated to $\mathcal{P}_i$. This is essentially a reinterpretation of the proof of* [Boc16a, Theorem 3.20] *in the language of toric geometry.*

Example 3.10. *Figure 4 depicts a dimer model in a torus (that resolves the toric Gorenstein singularity called a suspended pinchpoint), which has three vertices and six edges. The fundamental domain of the torus is given by dotted lines. There are five zigzag cycles, namely,*

$$fb,\ adcf,\ ec,\ ga,\ dgbe$$

in four different homology classes:

$$-\eta_1 = [fb],\ -\eta_2 = [adcf],\ -\eta_3 = [ec] = [ga](= -[fb]),\ -\eta_4 = [dgbe].$$

(Hence $N = 4$.) From the picture and Theorem 3.7, one can find the matching polytope and corner matchings $\mathcal{P}_i$. There are two boundary matchings $\{a, e\}$ and $\{c, g\}$, both sitting at the unique lattice point between $\mathcal{P}_3$ and $\mathcal{P}_4$.

3.4. **Zigzag Lagrangians in the dual dimer.** We now turn to the symplectic geometry associated with the dual dimer $\mathcal{Q}^\vee$ of $\mathcal{Q}$. Following [CHL21, §10.1], we recall how to construct immersed circles, known as *zigzag Lagrangians*, from zigzag cycles in $\mathcal{Q}^\vee$. We begin with the definition of dual dimers.

[3]The negative signs are chosen so that $\eta_1, \ldots, \eta_N$ correspond to the homology classes of the associated anti-zigzags.

Definition 3.11. *Suppose $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1) \subset \Sigma$ is a dimer model. We define a quiver*

$$\mathcal{Q}^\vee = (\mathcal{Q}_0^\vee, \mathcal{Q}_1^\vee)$$

as follows. First, we set $\mathcal{Q}_0^\vee$ to be the set of all zigzag cycles in $\mathcal{Q}$ and $\mathcal{Q}_1^\vee = \mathcal{Q}_1$. Each $e \in \mathcal{Q}_1^\vee$ has the head and tail $t(e) = v_{Z_1}$ and $h(e) = v_{Z_2}$ where Z_1 and Z_2 are zigzag cycles obtained from the projections of zag and zig rays at some lift $\tilde{e} \in \widetilde{\mathcal{Q}}$, respectively. It is called the **dual dimer** *of $\mathcal{Q}$.*

We orient the arrows $\mathcal{Q}_1^\vee$ in the dual dimer $\mathcal{Q}^\vee$ according to the orientations of the arrows in $\mathcal{Q}_1$. It is not difficult to see that the boundary cycle of a positive face in $\mathcal{Q}$ corresponds to a cycle in $\mathcal{Q}^\vee$, along which we attach a positive face. The boundary cycle of a negative face in $\mathcal{Q}$ appears in $\mathcal{Q}^\vee$ with reversed orientation, and we attach a negative face along this reversed cycle. Let $\Sigma^\vee$ denote the resulting Riemann surface, into which $\mathcal{Q}^\vee$ embedds. Thus, there is a one-to-one correspondence between the faces of the two dimers that preserves the set of boundary labels (though not the orientation in the case of negative faces). Obviously,

Lemma 3.12. *Given a dimer model $\mathcal{Q}$, a set of edges $\mathcal{P} \subset \mathcal{Q}_1 = \mathcal{Q}_1^\vee$ is a perfect matching of $\mathcal{Q}$ iff it is a perfect matching of $\mathcal{Q}^\vee$.*

Figure 5 shows the arrangement of faces of $\mathcal{Q}^\vee$ near a vertex $v_Z \in \mathcal{Q}_0^\vee$ which is dual to zigzag cycle Z in $\mathcal{Q}$. It is worthwhile to compare this with their corresponding faces in $\mathcal{Q}$ drawn in Figure 3.

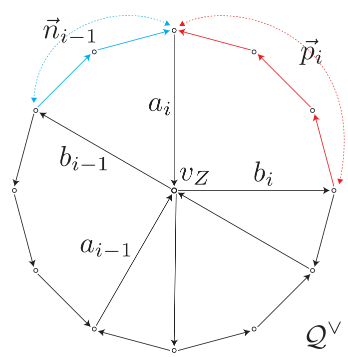

FIGURE 5. Neighboring faces near the vertex $v_Z \in \mathcal{Q}_0^\vee$ dual to the zigzag path Z in $\mathcal{Q}$

Removing $\mathcal{Q}_0^\vee$ from $\Sigma^\vee$ gives a punctured Riemann surface, which we denote by $X_{\mathcal{Q}^\vee} := \Sigma^\vee \setminus \mathcal{Q}_0^\vee$. We equip $X_{\mathcal{Q}^\vee}$ with a Liouville structure such that near each puncture, the Liouville vector field points outward toward the puncture. As a result, $X_{\mathcal{Q}^\vee}$ has conical ends modeled on $S^1 \times \mathbb{R}$ at each puncture.

We now introduce the *zigzag Lagrangians* $\mathbb{L} = \bigoplus_{i=1}^k L_i$. Let $Z_i^\vee$ be a zigzag cycle in the dual dimer $\mathcal{Q}^\vee$. For each such cycle, we select the midpoints of all edges contained in $Z_i^\vee$ and connect consecutive midpoints by segments lying inside the face enclosed by the corresponding pair of edges (see Figure 6). The union of these segments, after a slight smoothing at the corners, forms a closed curve. Since this curve avoids the punctures, it defines a Lagrangian circle L_i in $X_{\mathcal{Q}^\vee}$.

Repeating this construction for each zigzag cycle yields an immersed Lagrangian $\mathbb{L} = \bigoplus_{i=1}^k L_i$ in $X_{\mathcal{Q}^\vee}$. By construction, the components of $\mathbb{L}$ correspond bijectively to the zigzag cycles in the dimer $\mathcal{Q}^\vee$. Moreover, by applying a suitable smooth isotopy, these circles can be arranged to be exact Lagrangians [CHL21, Lemma 10.6].

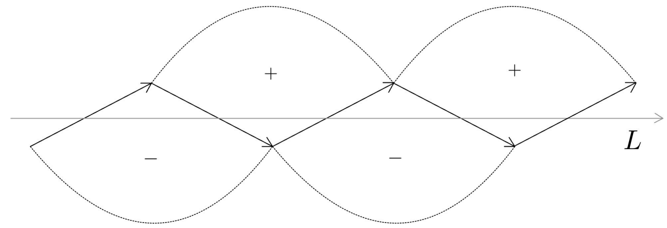

FIGURE 6. A zigzag Lagrangian component L

We now describe the quiver $\mathcal{Q}^{\mathbb{L}}$ introduced in Definition 2.1, in the context of the zigzag Lagrangians $\mathbb{L}$. There is a one-to-one correspondence between the vertices of $\mathcal{Q}^{\mathbb{L}}$ (that is, the components L_i) and the zigzag cycles in $\mathcal{Q}^{\vee}$, which in turn correspond to the vertices of the original quiver $Q = (\mathcal{Q}^{\vee})^{\vee}$. It is easy to see that there is an arrow x from vertex L_i to L_j in $\mathcal{Q}^{\mathbb{L}}$ whenever the zigzag cycles Z_i and Z_j intersect (it is possible that $i = j$).

Let X_e denotes the intersection point of L_i and L_j corresponding to an edge $e \in \mathcal{Q}_1^{\vee}$. Then we obtain two immersed generators associated with this intersection: X_e, of odd degree, and $\bar{X}_e$, of even degree. These represent two distinct types of branch jumps between L_i and L_j, as illustrated in Figure 8. When they appear as inputs of the m_k-operations, the boundary of a contributing holomorphic polygon turns at the corners formed by the pair (L_i, L_j) (at X_e).

Moreover, the correspondence between $\mathcal{Q}^{\mathbb{L}}$ and $\mathcal{Q}$ identifies the two algebras $A_{\mathbb{L}}$ (constructed on $\mathcal{Q}^{\mathbb{L}}$) and J.

Proposition 3.13 ([CHL21, Lemma 10.13 and 10.21, Corollary 10.16 and 10.20])**.** *The quiver $\mathcal{Q}^{\mathbb{L}}$ is canonically identified with $\mathcal{Q}$. Furthermore, with an appropriate choice of spin structures on $\mathbb{L}$, this identification induces an isomorphism between the (weak) Maurer–Cartan algebra $A_{\mathbb{L}}$ and the Jacobi algebra of $\mathcal{Q}$:*

$$A_{\mathbb{L}} \cong J,$$

where each variable dual to a generator X_e corresponds naturally to the element $x_e \in J$. Under this correspondence, the superpotential $W_{\mathbb{L}} \in A_{\mathbb{L}}$ matches the potential $W \in J$.

Therefore, the noncommutative mirror $(A_{\mathbb{L}}, W_{\mathbb{L}})$ associated to the zigzag Lagrangians $\mathbb{L} \subset X_{\mathcal{Q}^{\vee}}$ coincides with the noncommutative Landau–Ginzburg model (J, W). From this point on, we will not strictly distinguish between the two, and use the notation (J, W) to refer to either model.

Example 3.14. *The dual dimer $\mathcal{Q}^{\vee}$ (dotted arows inin Figure 7) of $\mathcal{Q}$ in the Example 3.10 is embedded in a sphere. The zigzag Lagrangian consists of three irreducible components, which are dual to three vertices in $\mathcal{Q}$ of the Example 3.10.*

Finally, we remark that for the zigzag Lagrangian $\mathbb{L}$, the boundary-deformed complex

$$CF^*((\mathbb{L}, b), (\mathbb{L}, b)) = \left(A_{\mathbb{L}} \otimes_{\Bbbk} CF^*(\mathbb{L}, \mathbb{L}), \{m_k^{b,\ldots,b}\}_{k\geq 0}\right)$$

is well-defined without any convergence issues in the terms $m_k^{b,\ldots,b}$. In fact, the output of $m_k^{b,\ldots,b}$ is a finite sum, meaning that each coefficient therein is computed from finitely many contributions. While this may be intuitively clear from the geometric construction, it can be rigorously justified using the grading structure.

Lemma 3.15. *For zigzag Lagrangians $\mathbb{L}$, the operations $m_k^{b,\ldots,b}$ ($k \geq 0$) are well-defined on $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ without the need for completion.*

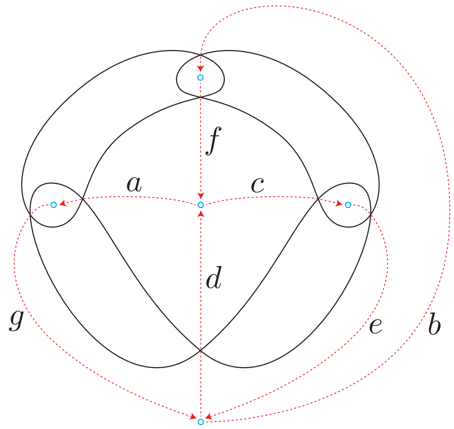

FIGURE 7. The dual dimer $\mathcal{Q}^\vee$ for the dimer $\mathcal{Q}$ in Figure 4

Proof. Recall from [CHL21, 10.5] that each perfect matching $\mathcal{P}$ determines a quadratic volume η_P such that all odd-degree immersed generators have degree 1, except those lying on the perfect matching, which have degree -1. Choose enough many perfect matchings, say $\mathcal{P}_1, \cdots, \mathcal{P}_l$, so that each odd-degree immersed generator X has degree 1 for at least one $\eta_{\mathcal{P}_i}$. Now consider the fractional grading induced by the tensor product

$$\eta := \eta_{\mathcal{P}_1} \otimes \cdots \otimes \eta_{\mathcal{P}_l} \left(\in \Gamma(T^* X_{\mathcal{Q}^\vee}^{\otimes l}) \right).$$

It is not difficult to see that the degree of X with respect to η averages out those with respect to η_i's, i.e,.

$$\deg_\eta(X) = \frac{1}{l}(\deg_{\eta_1}(X) + \cdots + \deg_{\eta_l}(X)),$$

and by construction, we have $\deg_\eta(X) < 1$. Therefore its shifted degree is negative, and hence $m_k^{b,\cdots,b}$ should be a finite sum (since the shifted degree of the operation is always 1). ∎

4. Closed-string B-model invariants for (J, W)

The discussion naturally leads to the mirror pair arising from the dual pair of dimers $\mathcal{Q}^\vee \subset \Sigma^\vee$ and $\mathcal{Q} \subset \Sigma$:

$$\begin{array}{c} \boxed{A} \quad \text{punctured Riemann surface } X_{\mathcal{Q}^\vee} := \Sigma^\vee \setminus \mathcal{Q}_0^\vee \\ \updownarrow \\ \boxed{B} \quad \text{(nc) Landau–Ginzburg model } (J, W) = (A_{\mathbb{L}}, W_{\mathbb{L}}) \end{array}$$

Here, $\mathbb{L}$ denotes the zigzag Lagrangian in $X_{\mathcal{Q}^\vee}$. The homological mirror symmetry for this pair has been studied in [Boc16b, CHL21], where it is shown that the wrapped Fukaya category of $X_{\mathcal{Q}^\vee}$ is equivalent to a full subcategory of the matrix factorization category associated to (J, W).

Our goal is to compare the closed-string A-model invariants of $X_{\mathcal{Q}^\vee}$ with the B-model invariants of (J, W). On the A-model side, we consider the symplectic cohomology of $X_{\mathcal{Q}^\vee}$; on the B-model side, it is natural to consider the (compactly supported) Hochschild cohomology of the matrix factorization category of (J, W). The latter has been explicitly computed in [Won21], which we now review.

4.1. **The compactly supported Hochschild cohomology $HH_c^*(\mathrm{mf}(W))$.** Let (J, W) be a noncommutative Landau–Ginzburg (LG) model, where J is a $\Bbbk$-algebra and W is a central element. A general idea, going back to [CT13], is that one can instead study the Hochschild invariants of (J, W) viewed as a *curved algebra*.

In this setting, the Hochschild cochain complex of (J, W) is obtained by modifying the usual Hochschild differential d_J of the algebra J to include an additional component d_W, which encodes the twist by W. Although we will primarily work with a Koszul-type bimodule resolution of J later, we first present the explicit formula for the differential using the bar resolution.

For each $k \geq 0$, the compactly supported Hochschild cochain complex $CC_c^*(J, W)$, defined via the bar resolution, is given by:

$$CC_c^k(J, W) = \bigoplus_{m+n=k} \mathrm{Hom}_{\Bbbk}^m(J^{\otimes n}, J).$$

Note that we take the *direct sum* rather than the *direct product*, which motivates the term "compactly supported." As shown in [CT13], taking the product instead would lead to trivial cohomology. It has the differential $d = d_J + d_W$ given by

$$\begin{aligned} d_J f(a_1 \otimes \cdots \otimes a_{n+1}) =& (-1)^{|f||a_1|} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} \end{aligned}$$

and

$$d_W f(a_1 \otimes \cdots \otimes a_{n-1}) = \sum_{i=1}^{n-1} (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i \otimes W \otimes a_{i+1} \otimes \cdots \otimes a_{n-1}). \tag{4.1}$$

Definition 4.1 ([CT13, Definition 3.10], [PP12, Section 2.4])**.** *We call*

$$HH_c^*(J, W) = H^*(CC_c^*(J, W); d)$$

the **compactly supported Hochschild cohomology** *of a curved algebra* (J, W)*. We identify this with* $HH_c^*(\mathrm{mf}(W))$ *throughout.*

Note that $(CC_c^*(J), d)$ forms a double complex, whose first page is precisely the Hochschild cohomology $HH^*(J, J)$ of the algebra J. The differential d_W then descends to the bracket $\{W, -\}$ on $HH^*(J, J)$, where $\{-, -\}$ denotes the Gerstenhaber bracket on Hochschild cohomology. We have

Proposition 4.2. [Won21, Proposition 5.2] *The spectral sequence of the double complex* $(CC_c^*(J), d = d_J + d_W)$ *induces an additive isomorphism*

$$HH_c^*(J, W) (= HH_c^*(\mathrm{mf}(W))) \cong H^*(HH^*(J, J); d_W = \{W, -\}).$$

In our main application, we will work with an alternative projective resolution $J^\bullet$ of $J = J$ (see 4.3), which is better suited for this spectral sequence calculation due to its finite length.[4] This resolution was originally introduced by Ginzburg [Gin06], and, interestingly, has a direct connection to the Floer theory of the zigzag Lagrangian $\mathbb{L}$. In fact, Wong [Won21] computed $HH^*(J)$ primarily using this same resolution $J^\bullet$, largely relying on Calabi–Yau duality.

[4]While it is possible to transfer the differential d to $J^\bullet$ via the standard comparison theorem (Theorem A.1), we will not need an explicit formula for the transferred differential.

4.2. **Calabi-Yau algebras and BV-structure on $HH_c^*(J,W)$.** Recall from [Gin06] that a homologically smooth (DG-)algebra J is called a *Calabi-Yau algebra of dimension* $d(\geq 1)$, or simply a CYd-algebra if

$$J \cong J^\vee[d] := \mathrm{RHom}_{J\text{-}\mathsf{Bimod}}(J, J \otimes_{\mathbb{C}} J)[d]$$

as J-bimodules. In general, we define $M^\vee := \hom_{J\text{-}\mathsf{Bimod}}(M, J \otimes_{\mathbb{C}} J)$ to be the bimodule dual to a J-bimodule M. The target $J \otimes_{\mathbb{C}} J$ is viewed as the outer J-Bimod bimodule when taking the bimodule hom, and $M^\vee$ is again a J-Bimod bimodule via the inner structure on $J \otimes_{\mathbb{C}} J$.

Remark 4.3. *For any algebra A, the tensor product $A \otimes_{\mathbb{C}} A$ can be equipped with two A-bimodule structures, called the outer and the inner bimodule structures, defined as*

$$b(a \otimes a')c = ba \otimes a'c, \qquad b(a \otimes a')c = ac \otimes ba',$$

respectively.

If J is Calabi-Yau of dimentions d, one can define a Van den Bergh duality map [vdB98]

$$\mathbb{D} : HH_*(J) \xrightarrow{\sim} HH^{d-*}(J)$$

which, coupled with the Connes B-operator $B : HH_*(J) \to HH_{*+1}(J)$ (see (A.2) for the precise formula), induces a *Batalin-Vilkovisky structure*

$$\Delta : HH^*(A) \to HH^{*-1}(A) \tag{4.2}$$

on the Gerstenhaber algebra $(HH^*(A), \cup, \{-,-\})$, making it into a BV-algebra. See [Gin06, 3.4] for more details.

One of great advantages from consistency condition on $\mathcal{Q}$ is the following.

Theorem 4.4 ([Boc16b, Theorem 4.4])**.** *The Jacobi algebra J of a zigzag consistent dimer $\mathcal{Q}$ is a Calabi-Yau algebra of dimension 3.*

In particular, we can define a BV-structure on $HH^*(J,J)$. In what follows, we introduce a new projective resolution of J that manifests this Calabi-Yau structure.

4.3. **The Koszul resolution of J.** It is known that J is a Calabi-Yau algebra (of dimension 3) if and only if the following complex of J-bimodules is exact (hence it is a free resolution of J as the diagonal bimodule).

$$J^\bullet : 0 \longrightarrow (J \otimes_{\mathbb{C}} J)^{\Bbbk} \xrightarrow{j} J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J \xrightarrow{c} J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \xrightarrow{j^\vee} J \otimes_{\Bbbk} J \longrightarrow J \longrightarrow 0 \tag{4.3}$$

where E is the $\mathbb{C}$-vector space spanned by $\mathcal{Q}_1$, and $E^* = \hom_{\mathbb{C}}(E, \mathbb{C})$. Notice that the resolution has a finite length. Each map in the sequence is given as follows.

(i) By definition, $(J \otimes_{\mathbb{C}} J)^{\Bbbk}$ consists of $a \otimes_{\mathbb{C}} b$ such that $h(a) = t(b)$. This is opposite to $J \otimes_{\Bbbk} J$ whose element is $a \otimes_{\Bbbk} b$ satisfying $t(a) = h(b)$. Indeed, we have $(J \otimes_{\mathbb{C}} J)^{\Bbbk} = (J \otimes_{\Bbbk} J)^\vee$. As before, the inner bimodule structure on $J \otimes_{\mathbb{C}} J$ makes $(J \otimes_{\mathbb{C}} J)^{\Bbbk}$ a J-bimodule, and it is generated $\sum_i \pi_i \otimes \pi_i$ over J. The first map j is defined as

$$j : (J \otimes_{\mathbb{C}} J)^{\Bbbk} \to J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J, \qquad a \otimes_{\mathbb{C}} b \mapsto \sum_{x \in \mathcal{Q}_1} (bx \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} a - b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} xa)\,.$$

where $\{\bar{x} : x \in \mathcal{Q}_1\}$ is the basis for E^* dual to $\mathcal{Q}_1$.

(ii) The second map $c : J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J \to J \otimes_{\Bbbk} E \otimes_{\Bbbk} J$ in the sequence is given by

$$c(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} b) := \sum_{y \in \mathcal{Q}_1} a \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)' \otimes_{\Bbbk} y \otimes_{\Bbbk} \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)'' b.$$

for the cyclic potential $\Phi = \Phi_{\mathbb{L}}$. Here, the second derivative (Hessian) is the composion of the usual cyclic derivatives

$$\frac{\partial}{\partial y} : \mathbb{C}\mathcal{Q}_{cyc} := \mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}] \to \mathbb{C}\mathcal{Q}$$

with

$$\frac{\partial}{\partial x} : \mathbb{C}\mathcal{Q} \to \mathbb{C}\mathcal{Q} \otimes \mathbb{C}\mathcal{Q}, \qquad \frac{\partial f}{\partial x} := \sum \left(\frac{\partial f}{\partial x}\right)' \otimes \left(\frac{\partial f}{\partial x}\right)'',$$

where we write

$$df = \sum_x \left(\frac{\partial f}{\partial x}\right)' \otimes dx_j \otimes \left(\frac{\partial f}{\partial x}\right)''.$$

(Here, $df := 1 \otimes f - f \otimes 1$ for $f \in J$.) In practice, it can be easily calculated using the formula

$$\frac{\partial}{\partial x}(x_{\vec{v}}\, x\, x_{\vec{w}}) = x_{\vec{v}} \otimes x_{\vec{w}}.$$

when $x_{\vec{v}}$ and $x_{\vec{w}}$ are words not containing x.

(iii) Finally, $j^\vee : J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \to J \otimes_{\Bbbk} J$ takes the form of

$$j^\vee(a \otimes_{\Bbbk} x \otimes_{\Bbbk} b) := ax \otimes_{\Bbbk} b - a \otimes_{\Bbbk} xb.$$

This agrees with the differential in the bar resolution in the sense that $j^\vee$ factors through the obvious inclusion

$$J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \hookrightarrow J \otimes_{\Bbbk} J \otimes_{\Bbbk} J \to J \otimes_{\Bbbk} J$$

with the second map being the differential in the bar resolution.

The self-duality can be clearly seen from (4.3). Thus it is straightforward that J is Calabi-Yau if this gives a resolution.

4.4. **Hochschild invariants from the Koszul resolution.** Let us now compute the Hochschild invariants of the algebra J using the resolution $J^\bullet$. Applying $\hom_{J\text{-}\mathsf{Bimod}}(J^\bullet, -)$ to a J-bimodule M, we get the Hochschild cochain complex in coefficient M

$$\begin{aligned}&\hom_{J\text{-}\mathsf{Bimod}}(J \otimes_{\Bbbk} J, M) \longrightarrow \hom_{J\text{-}\mathsf{Bimod}}(J \otimes_{\Bbbk} E \otimes_{\Bbbk} J, M)\\ &\longrightarrow \hom_{J\text{-}\mathsf{Bimod}}(J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J, M) \longrightarrow \hom_{J\text{-}\mathsf{Bimod}}((J \otimes_{\mathbb{C}} J)^{\Bbbk}, M)\end{aligned} \tag{4.4}$$

which reduces to

$$\hom_{\Bbbk}(\Bbbk, M) \longrightarrow \hom_{\Bbbk}(E, M) \longrightarrow \hom_{\Bbbk}(E^*, M) \longrightarrow \hom_{\Bbbk}(\Bbbk, M).$$

The above complex can be further simplified to

$$CC^*(J, M): \ M_{cyc} \xrightarrow{d_0} (M \otimes_{\Bbbk} E^*)_{cyc} \xrightarrow{d_1} (M \otimes_{\Bbbk} E)_{cyc} \xrightarrow{d_2} M_{cyc}. \tag{4.5}$$

In particular, the Hochschild cochain complex in coefficient J is given by

$$CC^*(J, J): \ J_{cyc} \longrightarrow (J \otimes_{\Bbbk} E^*)_{cyc} \longrightarrow (J \otimes_{\Bbbk} E)_{cyc} \longrightarrow J_{cyc}. \tag{4.6}$$

To clarify the connection with the Floer complex later, we will write $X := \bar{x}$ and $\bar{X} := x$ for the generators of E^* and E appearing in (4.5). Then the identification between (4.4) and (4.5) is given as

$$\hom_{J\text{-}\mathsf{Bimod}}(J \otimes_{\Bbbk} J, M) \to M_{cyc} \qquad f \mapsto f(1 \otimes_{\Bbbk} 1),$$

where 1 denotes the unit $\sum_i \pi_i$ of J, and

$$\begin{array}{rcllcl} \hom_{J\text{-}\mathsf{Bimod}}(J \otimes_{\Bbbk} E \otimes_{\Bbbk} J, M) & \cong & (M \otimes_{\Bbbk} E^*)_{cyc} & f & \mapsto & \sum_{x \in Q_1} f(1 \otimes_{\Bbbk} x \otimes_{\Bbbk} 1) \otimes_{\Bbbk} X \\ \hom_{J\text{-}\mathsf{Bimod}}(J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J, M) & \cong & (M \otimes_{\Bbbk} E)_{cyc} & f & \mapsto & \sum_{x \in Q_1} f(1 \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} \bar{X} \\ \hom_{J\text{-}\mathsf{Bimod}}((J \otimes_{\mathbb{C}} J)^{\Bbbk}, M) & \cong & M_{cyc} & f & \mapsto & f\left(\sum_i e_i \otimes_{\mathbb{C}} e_i\right), \end{array}$$

and the differential on (4.5) can be written as

$$\begin{array}{rcl} d_0(m) & = & \sum_{x \in Q_1} (xm - mx) \otimes_{\Bbbk} X, \\ d_1(m \otimes_{\Bbbk} Y) & = & \sum_{x \in Q_1} \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)' m \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)'' \otimes_{\Bbbk} \bar{X}, \\ d_2(m \otimes_{\Bbbk} \bar{Y}) & = & ym - my = [y, m]. \end{array} \tag{4.7}$$

It is easy to check that the above coincides with the complex $J^\bullet \otimes_{J\text{-}\mathsf{Bimod}} M$, or more precisely,

$$\hom_{J\text{-}\mathsf{Bimod}}(J^\bullet, M) \cong J^{3-\bullet} \otimes_{J\text{-}\mathsf{Bimod}} M$$

which establishes the duality

$$HH^*(J, M) \cong HH_{3-*}(J, M). \tag{4.8}$$

(This duality holds for any Calabi-Yau algebra.)

4.5. **BV operators in the Hochschild cohomology in view of different resolutions.** Our upcoming argument requires an explicit identification between the generators of $HH^*(J,J)$ used in [Won21] and those of a certain Floer cohomology. Although the identification of their underlying complexes will turn out to be straightforward, some of the generators in [Won21] involve the BV operator Δ (see (4.2)) in their definitions. To address this, we need to compute the BV operator on a specific generator in the Hochschild cohomology of J (obtained from the Koszul resolution).

Lemma 4.5. *For an element $a \in HH^3(J,J)$, choose its representative $x_1 \cdots x_k \in \mathbb{C}\mathcal{Q}$ (i.e., a cyclic path which lifts a). Then we have*

$$\Delta(a) = \sum_i (x_{i+1} \cdots x_k)(x_1 \cdots x_{i-1}) \otimes_{\Bbbk} \bar{X}_i,$$

which defines a class in $HH^2(J,J)$.

The proof of Lemma 4.5, together with further details on the BV operators, will be provided in Appendix A.

Recall that there is an additional component d_W of the Hochschild differential (for $HH^*(J,W)$), which produces a differential acting on $HH^*(J,J)$. We remark that d_W (see (4.1)) can be expressed entirely in terms of the BV operator Δ as

$$d_W(f) = \{W, f\} = \Delta(Wf) - W\,\Delta(f), \tag{4.9}$$

using the algebraic relations among the operators Δ, $\cup$, and $\{-,-\}$. This formula will be useful later for explicitly computing d_W on cocycles in $HH^3(J,J)$.

4.6. **Explicit calculation for $HH^*(J,J)$ and d_W.** We present several explicit computational results on $HH^*(J,J)$ and $HH^*_c(mf(W)) = HH^*(J,W)$ that will be used in the later part of the paper. For details about the calculations, we refer readers to [Won21, Section]. We first list generators of $HH^*(J,J)$ according to their homological degree (i.e., the resolution degree from $J^\bullet$). Recall that the dimer $\mathcal{Q}$ is embedded in the torus T^2.

(deg 0) Each nontrivial $\alpha \in H_1(T^2;\mathbb{Z})$ defines a Hochschild cycle $x_\alpha \in HH^0(J,J) = \mathcal{Z}(J)$ whose underlying path in $\mathbb{C}\mathcal{Q}$ belongs to the class α. By requiring it to be a non-multiple of W, it is uniquely determined (up to Jacobi relations). In particular, each antizigzag class η_i defines a Hochschild cocycle x_{η_i}.

(deg 1) Each corner matching $\mathcal{P}_i$ defines a Hochschild 1-cocycle $\partial_{\mathcal{P}_i}$ defined (on the chain level) by

$$\partial_{\mathcal{P}_i} \in (J \otimes E^*)_{cyc} : e \mapsto \begin{cases} e & \text{if } e \in \mathcal{P}_i \\ 0 & \text{otherwise} \end{cases}$$

where we regard this as a map $E \to J$ (hence an element of $(J \otimes_{\Bbbk} E^*)_{cyc} \subset CC^*(J,J)$ (4.6)), which can be extended to a derivation $J \to J$. Let σ_i be the cone in $H_1(T^2;\mathbb{Z})$ spanned by η_i and η_{i+1}. For $\alpha \in \operatorname{int}\sigma_i$, we define

$$\partial_\alpha := x_\alpha W^{-1} \partial_{\mathcal{P}_i} \in (J \otimes E^*)_{cyc}.$$

The expression involves W^{-1}, and hence it a priori only gives a map $J \to J[W^{-1}]$, but one can prove that it factors through J by some grading argument using perfect matchings.

(deg 3) Any vertex $v \in \mathcal{Q}_0$ (or the corresponding idempotent) defines an element of $[v] \in HH_0(J,J)$. Denote by $\theta_v \in HH^3(J,J)$ the element dual to $[v]$ under the duality $HH^3(J,J) \cong HH_0(J,J)$ (4.8). On the chain level, it is simply the idempotent at v in J_{cyc} .

(deg 2) Suppose $Z_{i,1} \cdots Z_{i,m_i}$ are parallel zigzag cycles in the class η_i arranged in such a way that they divide T into closed strips $\mathcal{E}_{i,j}$ bounded by $\mathcal{O}^+(Z_{i,j}) \cup \mathcal{O}^-(Z_{i,j+1})$. For $v \in \mathcal{E}_{i,j}$, we set $\psi_{i,j} := \Delta(x_{\eta_i}\theta_v) \in HH^2(J,J)$ where Δ is the BV operator on $HH^*(J,J)$.

One can specify a representative of $\psi_{i,j}$ as follows. Note that any vertex v contained in the antizigzag $\mathcal{O}^+(Z_{i,j})$ lies in $\mathcal{E}_{i,j}$ by definition. Moreover, $x_{\eta_i}\theta_v$ regarded as as element of J_{cyc} in this case is nothing but the labelling of the antizigzag $\mathcal{O}^+(Z_{i,j})$ itself that ends at v. Therefore, by Lemma 4.5, we have

$$\psi_{i,j} = \Delta(x_{\eta_i}\theta_v) = \Delta(x_1 \cdots x_k) = \sum_i (x_{i+1} \cdots x_k)(x_1 \cdots x_{i-1}) \otimes_{\Bbbk} \bar{X}_i, \tag{4.10}$$

where $x_1 \cdots x_k$ represents the antizigzag $\mathcal{O}^+(Z_{i,j})$ such that $t(x_k) = v$.

In terms of the above cocycles, $HH^*(J,J)$ is given as follows.

Proposition 4.6. [Won21, Theorem 1.2] *For a zigzag consistent dimer model $\mathcal{Q}$, the Hochschild cohomology of J is given by*

$$\begin{aligned}
HH^0(J,J) &\cong \mathcal{Z} \\
HH^1(J,J) &\cong \bigoplus_{1\le i\le N} \mathcal{Z} \cup \partial_{\mathcal{P}_i} \oplus \bigoplus_{\alpha\in \operatorname{int}\sigma_i} \mathbb{C}\partial_\alpha \\
HH^2(J,J) &\cong \bigoplus_{1\le i,j\le N} \mathcal{Z} \cup \partial_{\mathcal{P}_i} \cup \partial_{\mathcal{P}_j} \oplus \bigoplus_{\substack{1\le j\le N \\ \alpha\in \operatorname{int}\sigma_i}} \mathbb{C}\partial_\alpha \cup \partial_{\mathcal{P}_j} \oplus \bigoplus_{\substack{1\le i\le N \\ 1\le j\le m_i, n\in\mathbb{Z}_{>0}}} \mathbb{C}x_{\eta_i}^n \psi_{i,j} \\
HH^3(J,J) &\cong \bigoplus_{1\le i,j,k\le N} \mathcal{Z} \cup \partial_{\mathcal{P}_i} \cup \partial_{\mathcal{P}_j} \cup \partial_{\mathcal{P}_k} \oplus \bigoplus_{\substack{1\le j,k\le N \\ \alpha\in \operatorname{int}\sigma_i}} \mathbb{C}\partial_\alpha \cup \partial_{\mathcal{P}_j} \cup \partial_{\mathcal{P}_k} \oplus \bigoplus_{\substack{1\le i\le N \\ v\in\mathcal{E}_{i,j}, n\in\mathbb{Z}_{>0}}} \mathbb{C}x_{\eta_i}^n \theta_v
\end{aligned}$$

where $\mathcal{Z} = \mathcal{Z}(J)$ denotes the center of the Jacobi algebra.

The following formulas from [Won21] will be used later.

Lemma 4.7. [Won21, Proposition 4.21, 4.23] *The following holds for the Gerstenharber bracket $\{-,-\}$ and the cup product $\cup$ on $HH^*(J,J)$.*

(i) *The Gerstenharber bracket between $f \in HH^0(J,J)$ and $\partial_{\mathcal{P}_i} \in HH^1(J,J)$ is given by*
$$\{\partial_{\mathcal{P}_i}, f\} = \deg_{P_i}(f)f$$
(ii) *The cup product between $\partial_{\mathcal{P}_k} \in HH^1(J,J)$ and $\psi_{i,j} \in HH^2(J,J)$ is given by*
$$\partial_{\mathcal{P}_k} \cup \psi_{i,j} = \deg_{P_k}(x_{\eta_i})\, x_{\eta_i}\, \theta_v$$
for $v \in \mathcal{E}_{i,j}$.

5. Hochschild invariants of (J,W) from Floer theory of zigzag Lagrangians

In this section, we show that the Hochschild invariant $HH^*_c(\mathrm{mf}(W))$ of the noncommutative LG-mirror in Section 4 can be obtained from the deformed Floer complex $CF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b))$ defined in 2.8 for the zigzag Lagrangian $\mathbb{L}$. Namely, we prove

Proposition 5.1. *For a zigzag consistent dimer $\mathcal{Q}$, there is an (additive) isomorphism*
$$HF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b)) \cong HH^*_c(\mathrm{mf}(W))$$
where $\mathbb{L}$ is the zigzag Lagrangian $\mathbb{L}$ obtained from the dual dimer $\mathcal{Q}^\vee$.

We will use a certain nongeometric $\mathbb{Z}$-grading on this Floer complex to make it a double complex, and use associated spectral sequence to show that the cohomology of $CF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b))$ computes $HH^*_c(\mathrm{mf}(W))$. This indeed coincides with the spectral sequence for $HH^*_c(\mathrm{mf}(W))$ itself, appearing in Proposition 4.2.

5.1. $CF^*((\mathbb{L},b),(\mathbb{L},b))$ for the zigzag Lagrangians $\mathbb{L} \subset X_{\mathcal{Q}^\vee}$.

Let $\mathcal{Q}$ be a dimer, and $\mathbb{L}$ the zigzag Lagrangians in $X_{\mathcal{Q}^\vee}$. Recall from 3.4 that the self-intersections in $\mathbb{L}$ occur along an edge of $\mathcal{Q}^\vee$, which, by dimer duality, leads to an one-to-one correspondence between the edges $e \in \mathcal{Q}_1$ and the self-intersections in $\mathbb{L}$. Such a self-intersection point gives rise to two generators of $CF(\mathbb{L},\mathbb{L})$ indicating branch jumps between local Lagrangian components at the intersection point. For $e \in \mathcal{Q}_1$, we denote the corresponding generators in the odd and even degrees by X_e and $\bar{X}_e$, respectively. The immersed Floer complex (2.1) can be written as
$$CF^*(\mathbb{L},\mathbb{L}) = \bigoplus_{i=1}^{k} C^*(L_i) \oplus \bigoplus_{e\in\mathcal{Q}_1} \mathrm{Span}\{X_e, \bar{X}_e\},$$
where $C^*(L_i)$ is a Morse complex of a Lagrangian circle L_i (more precisely, the domain circle of a possibly immersed circle L_i) with respect to a chosen perfect Morse function. Pictorially, X_e corresponds to the corner formed by two branches of $\mathbb{L}$ whose orientations are aligned (either positively or negatively), and $\bar{X}_e$ is complementary to X_e. See Figure 8.

Solving (weak) Maurer-Cartan equation for $b = \sum_{e\in\mathcal{Q}_1} x_e X_e$ produces a noncommutative LG model $(A_{\mathbb{L}}, W)$. We know that it agrees with the Jacobi algebra J together with the central element W (Proposition 3.13).

We define a grading on $CF^*((\mathbb{L},b),(\mathbb{L},b)) = A_{\mathbb{L}} \otimes_{\Bbbk} CF^*(\mathbb{L},\mathbb{L})$ so that generators listed below have degree $0, 1, 2$ and 3:
$$\oplus_i A_{\mathbb{L}} \cdot \mathbf{1}_i, \qquad \oplus_{e\in\mathcal{Q}_1} A_{\mathbb{L}} \cdot X_e, \qquad \oplus_{e\in\mathcal{Q}_1} A_{\mathbb{L}} \cdot \bar{X}_e, \qquad \oplus_i A_{\mathbb{L}} \cdot \mathbf{pt}_i, \tag{5.1}$$
where $\mathbf{1}_i$ is the maximum of the perfect Morse function on L_i (hence it gives the unit class), and $\mathbf{pt}_i$ is the minimum. We will often call this the "exterior grading" due to its resemblance

with the grading on the exterior algebra (especially in the case of pair-of=pants), and write $|\mathbf{1}_i| = 0$, $|X_e| = 1$, $|\bar{X}_e| = 2$ and $|\mathbf{pt}_i| = 3$ to indicate their exterior degrees.

This agrees with the Floer grading only in modulo 2, and the Floer differential $m_1^{b,b}$ on $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ decomposes as $m_1^{b,b} = m_{1,+1}^{b,b} + m_{1,-1}^{b,b}$ according to the exterior grading. Writing $\delta = m_1^{b,b}$, $\delta_+ = m_{1,+1}^{b,b}$ and $\delta_- = m_{1,-1}^{b,b}$ for simplicity, we have

$$CF^*((\mathbb{L}, b), (\mathbb{L}, b)) : \ \oplus_i A_{\mathbb{L}} \cdot \mathbf{1}_i \underset{\xleftarrow{\delta_-}}{\xrightarrow{\delta_+}} \oplus_{e \in \mathcal{Q}_1} A_{\mathbb{L}} \cdot X_e \underset{\xleftarrow{\delta_-}}{\xrightarrow{\delta_+}} \oplus_{e \in \mathcal{Q}_1} A_{\mathbb{L}} \cdot \bar{X}_e \underset{\xleftarrow{\delta_-}}{\xrightarrow{\delta_+}} \oplus_i A_{\mathbb{L}} \cdot \mathbf{pt}_i,$$

i.e., $|\delta_+| = 1$ and $|\delta_-| = -1$. Here, the component $\bigoplus A_{\mathbb{L}} \cdot \mathbf{1}_i \to \bigoplus A_{\mathbb{L}} \cdot \mathbf{pt}_i$ of δ vanish due to unitality. By choosing Morse functions suitably, one can make $\bigoplus A_{\mathbb{L}} \cdot \mathbf{pt}_i \to \bigoplus A_{\mathbb{L}} \cdot \mathbf{1}_i$ also vanish. For instance, we may locate $\mathbf{1}_i$ on the boundary of a positive disk for all i, and $\mathbf{pt}_i$ on the boundary of a negative disk so that they never lie on the boundary of the same disk.

Finally, we restrict ourselves to the subcomplex $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ of $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ generated by cyclic element, and the decomposition $d = \delta_+ + \delta_-$ descends to $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$:

$$\oplus_i \pi_i A_{\mathbb{L}} \pi_i \cdot \mathbf{1}_i \underset{\xleftarrow{\delta_-}}{\xrightarrow{\delta_+}} \oplus_{e \in \mathcal{Q}_1} (A_{\mathbb{L}} \cdot X_e)_{cyc} \underset{\xleftarrow{\delta_-}}{\xrightarrow{\delta_+}} \oplus_{e \in \mathcal{Q}_1} (A_{\mathbb{L}} \cdot \bar{X}_e)_{cyc} \underset{\xleftarrow{\delta_-}}{\xrightarrow{\delta_+}} \oplus_i \pi_i A_{\mathbb{L}} \pi_i \cdot \mathbf{pt}_i. \tag{5.2}$$

where $(A_{\mathbb{L}} \cdot X_e)_{cyc} = \pi_{t(X_e)} A_{\mathbb{L}} \cdot X_e$ and $(A_{\mathbb{L}} \cdot \bar{X}_e)_{cyc} = \pi_{h(X_e)} A_{\mathbb{L}} \cdot \bar{X}_e$. This makes $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ into a double complex. In the main application, we will compare this double complex with the one in [Won21] computing $HH^*_c(\mathrm{mf}(W))$.

5.2. **δ_+-cohomology and $HH^*(J, J)$.** Recall that $HH^*_c(\mathrm{mf}(W))$, identified as the Hochschild cohomology of the curved algebra (J, W), can be (additively) computed as the cohomology of $HH^*(J, J)$ with respect to the differential $d_W = \{W, -\} : HH^*(J, J) \to HH^*(J, J)$. We first claim that the δ_+-cohomology of $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ (which is the first page of the associated spectral sequence) agrees with $HH^*(J, J)$. In fact, one can easily identify their underlying complexes.

Lemma 5.2. *There is a chain-level isomorphism*

$$(CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)), \delta_+) \cong (\hom_{J\text{-}Bimod}(J^\bullet, J), d_A), \tag{5.3}$$

where d_A denotes the Hochschild differential with respect to the Koszul resolution $J^\bullet$ of J (see 4.3).

Proof. We write $J = J$ for simplicity. Recall from 4.4 that the Hochschild cochain complex $\hom_{J\text{-}\mathsf{Bimod}}(J^\bullet, M)$ simplifies to

$$J_{cyc} \xrightarrow{d_0} (J \otimes_{\Bbbk} E^*)_{cyc} \xrightarrow{d_1} (J \otimes_{\Bbbk} E)_{cyc} \xrightarrow{d_2} J_{cyc}. \tag{5.4}$$

where E is the vector space formally generated by $\mathcal{Q}_1$, and E^* is its dual. Given that $A_{\mathbb{L}} = J$ by Proposition 3.13, there is an obvious graded vector space isomorphism between (5.4) and (5.2). The identification for degree 0 and 3 components are simply the trivial identify $J_{cyc} = \oplus_i \pi_i J \pi_i$.

Now the image generators in (5.1) under δ_+ can be explicitly calculated as follows. Firstly, the even degree generators are mapped to

$$\delta_+(f \cdot \mathbf{1}_i) = \sum_{\substack{x \in Q_1 \\ h(X) = i}} x f X - \sum_{\substack{x \in Q_1 \\ t(X) = i}} f x X \tag{5.5}$$

$$\delta_+(f \cdot \bar{X}_e) = x_e f \mathbf{pt}_{t(e)} - f x_e \mathbf{pt}_{h(e)}. \tag{5.6}$$

The former (5.5) is determined by the unital property $m_2(\mathbf{1}_i, X) = (-1)^{|X|} m_2(X, \mathbf{1}_j) = X$ for $X \in CF(L_i, L_j)$, while the latter (5.6) is due to $m_2(X, \bar{X}) = -\mathbf{pt}_i, m_2(\bar{X}, X) = \mathbf{pt}_j$ when $X \in CF(L_i, L_j)$ and $\bar{X} \in CF(L_j, L_i)$. This is essentially the same calculation as [Sei11, (10.2)].

For a degree 1 generator $f \cdot X_e$, $\delta_+(f \cdot X_e)$ counts all polygons P having X_e as a corner. Let $w_p = x_k \cdots x_1 x_e$ be the (clockwise) labeling of corners appearing in ∂P up to cyclic permutations. See Figure 8. Note that these are precisely two summands of the cyclic potential Φ involving x_e.

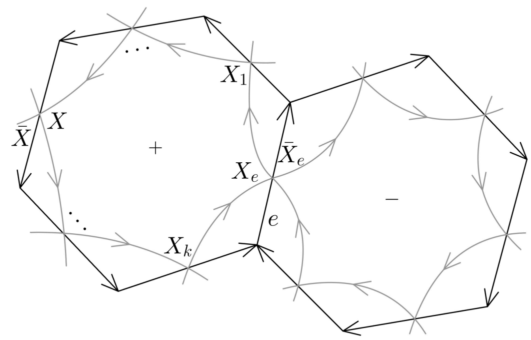

FIGURE 8. Holomorphic disks bounded by $\mathbb{L}$ (grey) with X_e on their corner

Since δ_+ produces an output of exterior degree 2, each corner X_i of P other than X_e contributes a term of the form $x_{i-1} \cdots x_1 f x_k \cdots x_{i+1} \bar{X}_i$. Therefore

$$\begin{aligned} \delta_+(f \cdot X_e) &= \sum_{\substack{P \text{ with} \\ w_P = x_k \cdots x_1 x_e}} x_{i-1} \cdots x_1 f x_k \cdots x_{i+1} \bar{X}_i \\ &= \sum_{e \in \mathcal{Q}_1} \left(\frac{\partial^2 \Phi}{\partial x \partial x_e} \right)' f \left(\frac{\partial^2 \Phi}{\partial x \partial x_e} \right)'' \otimes_{\mathbb{k}} \bar{X} \end{aligned} \tag{5.7}$$

(5.5), (5.6) and (5.7) agree with the calculation of d_A given in (4.7), and the lemma follows. □

It is straightforward to match generators of $HH^*(J, J)$ appearing in Proposition 4.6 and Floer generators of $H^*(CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)), \delta_+)$. Let us write

$$F : HH^*(J, J) \xrightarrow{\cong} H^*(CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)), \delta_+) \tag{5.8}$$

for the isomorphism in the proof of Lemma 5.2 (or its chain level map from the identification of the underlying complexes). Then we have

$$F(x_\alpha) = x_\alpha \mathbf{1}_{\mathbb{L}}, \quad F(\partial_{\mathcal{P}_i}) = \sum_{e \in \mathcal{P}_i} x_e X_e, \tag{5.9}$$

and $F(\partial_\alpha) = \sum_{e \in \mathcal{P}_i} x_\alpha W^{-1} x_e X_e$ for $\alpha \in \operatorname{int} \sigma_i$. Here, $x_\alpha W^{-1} x_e$ belongs to J since it has nonnegative degrees with respect to all perfect matchings (see [Won21, Lemma 4.5]). Also,

$$F(\theta_v) = \mathbf{pt}_i \tag{5.10}$$

where $L_i = L_{i_v}$ is the component of $\mathbb{L}$ corresponding to the vertex $v \in \mathcal{Q}_0$. Finally, from (4.10), we have

$$F(\psi_{i,j}) = \sum_i (x_{i+1} \cdots x_k)(x_1 \cdots x_{i-1}) \otimes_{\mathbb{k}} \bar{X}_i, \tag{5.11}$$

where $x_1 \cdots x_k$ represents the antizigzag $\mathcal{O}^+(Z_{i,j})$.

Remark 5.3. *The resolution $J^\bullet$ itself can also be expressed in terms of the Floer complex of $\mathbb{L}$. Let us write $V := CF^*(\mathbb{L}, \mathbb{L})$. As modules, we have*

$$J^\bullet \cong J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J,$$

which identifies E^* with the dual of $\mathbb{C}\langle \bar{X}_e : e \in \mathcal{Q}_1\rangle$, and E with the dual of $\mathbb{C}\langle X_e : e \in \mathcal{Q}_1\rangle$. Under this identification, the differential on the right hand side is a J-bimodule map obtained by taking dual of the A_∞-operations (the "deformed m_1") on V. Namely, it is given by

$$a \in V^* \mapsto \sum \lambda\, x_1 \cdots x_i \otimes a' \otimes y_j \cdots y_1$$

where the sum in the right hand side runs over all A_∞-operations satsifying

$$m_k(Y_1, \cdots, Y_j, A', X_i, \cdots, X_1) = \cdots + \lambda^{-1} A + \cdots$$

for degree 1 immersed generators X's and Y's (i.e., A appears as a summand of the output with a nontrivial coefficient). Here A, A' are Floer generators in the basis chosen in 5.1 with $|A| = |A'| + 1$, and a, a' are their corresponding elements in the dual basis. With this perspective, the identification between d_A and δ_+ becomes manifest.

In order to finish the proof of Proposition 5.1, it remains to compare $d_W = \{W, -\}$ and δ_- on the cohomologies of (5.3). This is automatic for (exterior) degree 0 (d_A- or δ_+-)cocycles. To simplify the comparison for positive degree cocycles, we need some preparations.

5.3. **The cup product on $HH^*(J,J)$.** Recall that $HH^*(J,J)$ carries the cup product $\cup$. We show that this can be obtained from the "leading order" Floer product on $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$. The Floer product $m_2^{b,b,b}$ on $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ also decomposes according to the exterior grading as

$$m_2^{b,b,b} = m_{2,0}^{b,b,b} + m_{2,<0}^{b,b,b}$$

($m_{2,\geq 2}^{b,b,b}$ vanishes due to obvious degree reason and unitality.)

By breaking down the equation $\left[m_1^{b,b}, m_2^{b,b,b}\right] = 0$ degreewise, we see that $\delta_+ = m_{1,+1}^{b,b}$ is a derivation with respect to $m_{2,0}^{b,b,b}$, and hence $m_{2,0}^{b,b,b}$ induces a product on

$$H^*(CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)), \delta_+) = HH^*(J,J)$$

We claim that this product coincides with the cup product on $HH^*(J,J)$.

Lemma 5.4. *The product on $HH^*(J,J)$ induced from $m_{2,0}^{b,b,b}$ agrees with the cup (Yoneda) product on the Hochschild cohomology.*

Proof. Let us first recall the following well-known fact about the cup product on the Hochschild cohomology. Consider a resolution $J^\bullet$ of J, and suppose $\tilde{\Delta} : J^\bullet \to J^\bullet \otimes_J J^\bullet$ lifts the diagonal map, i.e,

$$\begin{tikzcd} J^\bullet \arrow[r] \arrow[d, "\tilde{\Delta}"'] & J \arrow[d, "\Delta"] \\ J^\bullet \otimes_J J^\bullet \arrow[r] & J \otimes_J J \cong J, \end{tikzcd} \tag{5.12}$$

Given such $\tilde{\Delta}$, the cup product on $HH^*(J) = H^* \hom_{J\text{-}\mathsf{Bimod}}(J^\bullet, J)$ can be computed by

$$\hom_{J\text{-}\mathsf{Bimod}}(J^\bullet, J)^{\otimes 2} \to \hom_{J\text{-}\mathsf{Bimod}}(J^\bullet \otimes J^\bullet, J) \to \hom_{J\text{-}\mathsf{Bimod}}(J^\bullet, J), \tag{5.13}$$

where the second map is the dual of $\tilde{\Delta}$.

In our situation, we have $J^\bullet = J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J$ for $V = CF^*(\mathbb{L},\mathbb{L})$ as explained in Remark 5.3, and hence, we need to find a J-bimodule map

$$\tilde{\Delta} : J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J \to (J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J) \otimes_J (J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J) = J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J \otimes_{\Bbbk} V^* \otimes_{\Bbbk} J.$$

Analogously to Remark 5.3, it can be obtained by daulising A_∞-operations (the "deformed m_2") in the sense that

$$\tilde{\Delta} : a \in V^* \mapsto \sum \lambda\, x_1 \cdots x_i \otimes a_1 \otimes y_1 \cdots y_j \otimes a_2 \otimes z_1 \cdots z_k$$

where the sum is taken for all A_∞-operations satisfying

$$m_k(Z_k, \cdots, Z_1, A_2, Y_j, \cdots, Y_1, A_1, X_i, \cdots, X_1) = \cdots + \lambda^{-1} A + \cdots \tag{5.14}$$

for any degree 1 immersed generators X, Y and Z. It is easy to check that $\tilde{\Delta}$ satisfies (5.12) (the only A_∞-operation involved here is $m_2(\mathbf{1}_{\mathbb{L}}, \mathbf{1}_{\mathbb{L}}) = \mathbf{1}_{\mathbb{L}}$). Thus the product in (5.13) is essentially the double dual of (5.14) and can be easily identified with $m_{2,0}^{b,b,b}$. □

The following remark explains the origin of the discrepancy between the product structure on $SH^*(X)$ and that on $HH^*(\mathrm{mf}(W))$, which is used in the comparison carried out in [Won21].

Remark 5.5. *Observe that*

$$\left[m_{1,-1}^{b,b}, m_{2,0}^{b,b,b}\right] = -\left[m_{1,+1}^{b,b}, m_{2,<0}^{b,b,b}\right]$$

where the right hand side vanishes on the $\delta_+ (= m_{1,+1}^{b,b})$-cohomology. Therefore $m_{2,0}^{b,b,b}$ also induces a product on the E_2-page of the spectral sequence of the double complex $d_A + d_W$. This is precisely the product appearing in [Won21]. *Although the E_2-page agrees with $HH_c^*(\mathrm{mf}(W))$ as vector spaces, $m_{2,0}^{b,b,b}$ does not seem to be an intrinsic product on $HH_c^*(\mathrm{mf}(W))$, nor is it compatible with the product on $SH^*(X)$.*

5.4. **Proof of $HF_{cyc}^*((\mathbb{L},b),(\mathbb{L},b)) \cong HH_c^*(\mathrm{mf}(W))$.** We now prove Propostion 5.1. By Lemma 5.2, it suffices to show that δ_- equals $d_W = \{W, -\}$, or more precisely F in (5.8) intertwines $d_W = \{W, -\}$

$$\begin{tikzcd}
HH^*(J,J) \arrow[r, "d_W"] \arrow[d, "F"', "\cong"] & HH^{*-1}(J,J) \arrow[d, "\cong"', "F"] \\
H^*(CF_{cyc}^*((\mathbb{L},b),(\mathbb{L},b)),\delta_+) \arrow[r, "\delta_-"] & H^{*-1}(CF_{cyc}^*((\mathbb{L},b),(\mathbb{L},b)),\delta_+).
\end{tikzcd}$$

This is trivial for exterior degree 0 generators in $HH^*(J)$. We begin with the following set of generators of the degree 1 component

$$\partial_{\mathcal{P}_i},\ \partial_\alpha \quad \text{for } 1 \le i \le N \text{ and } \alpha \in H_1(T^2, \mathbb{Z})$$

(as a module over the ring spanned by degree 0 elements, see 4.6). The image of d_W evaluated at these degree 1 generators are explicitly computed in [Won21]. Lemma 4.7 applying to $f = W$ tells us that

$$d_W(\partial_{\mathcal{P}_i}) = -\{\partial_{\mathcal{P}_i}, W\} = -W, \tag{5.15}$$

and using the definition $\partial_\alpha = x_\alpha W^{-1} \partial_{\mathcal{P}_i}$, we have

$$d_W(\partial_\alpha) = -\{\partial_\alpha, W\} = -x_\alpha. \tag{5.16}$$

for $\alpha \in \operatorname{int} \sigma_i$.

Lemma 5.6. *d_W and δ_- agree on the degree 1 component $HH^1(J,J) \cong H^1(CF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b)),\delta_+)$.*

Proof. Since both δ_- and d_W are derivations with respect to $\cup = m_{2,0}^{b,b,b}$, it suffices to check if (5.15) and (5.16) hold for their corresponding Floer generators, i.e., if the following hold true:

$$\delta_-\left(\sum_{e\in\mathcal{P}_i} x_e X_e\right) = F(-W) = -W_{\mathbb{L}}\mathbf{1}_{\mathbb{L}}, \qquad \delta_-\left(\sum_{e\in\mathcal{P}_i} x_\alpha W^{-1} x_e X_e\right) = F(-x_\alpha) = -x_\alpha \mathbf{1}_{\mathbb{L}}.$$

The former counts a polygon with odd-degree corners containing X_e, assigning its output to one of the $\mathbf{1}_{L_i}$'s. By the definition of a perfect matching, exactly one such polygon exists for each e, and the conclusion follows. For the latter, we first need to invert W (by taking localization), and use the linearly of operators (since they are trivial on degree 0 components). This is possible since $HH^1(J,J) \to HH^1(J[W^{-1}],J[W^{-1}])$ is known to be injective, or equivalently $HH^1(J,J)$ is torsion free. (See the proof of [Won21, Theorem 4.6].) $\square$

It follows that δ_- and d_W agree on the subspace multiplicatively generated by degree 0 and degree 1 elements. By Proposition 4.6, the complement of this subspace is generated over the degree 0 component by θ_v and $\theta_{i,j}$. Therefore, the following lemma completes the proof of Proposition 5.1.

Lemma 5.7. *d_W and δ_- agree on the generators $\psi_{i,j}$ and θ_v in $HH^*(J,J)$ and their corresponding generators in $H^*(CF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b)),\delta_+)$.*

Proof. Making use of the algebraic compatibility between the BV-operator Δ and the bracket (4.9), one has

$$d_W(\theta_v) = \{W,\theta_v\} = \Delta(W\theta_v) - W\Delta(\theta_v) = \Delta(W\theta_v).$$

By Lemma 4.5, we have $\Delta(\theta_v) = 0$ since through Calabi-Yau duality it can be represented by a constant path in HH_*.

Note that $W\theta_v$ can be represented by any boundary cycle in $\mathcal{Q}$ (a summand of W) starting at the vertex v. In particular, one may choose $W\theta_v = x_{i_1}\cdots x_{i_k}$, which is one such cycle obtained from the labeling of the holomorphic polygon P (with boundary on $\mathbb{L}$) passing through $\mathbf{pt}_{i_v}$. Here, i_v is chosen such that the component L_{i_v} of $\mathbb{L}$ is dual to the vertex $v \in \mathcal{Q}_0$. Then, by Lemma 4.5,

$$d_W(\theta_v) = \Delta(W\theta_v) = \sum_i (x_{i_a+1}\cdots x_{i_k})(x_{i_1}\cdots x_{i_a-1})\bar{X}_{i_a}.$$

This is precisely $\delta_-(F(\theta_v)) = \delta_-(\mathbf{pt}_{i_v})$, contributed solely by the same polygon P.

For $\psi_{i,j}$, we use

$$\partial_{\mathcal{P}_k} \cup \psi_{i,j} = \deg_{P_k}(x_{\eta_i})\, x_{\eta_i}\, \theta_v$$

for all $1 \le k \le N$ ((ii) of Lemma 4.7), where $v \in \mathcal{E}_{i,j}$. Apart from $\psi_{i,j}$, we know that d_W and δ_- agree on all the other factors. Thus we see that the difference of $d_W(\psi_{i,j}$ and $\delta_-(F(\psi_{i,j}))$ is anahilated by by $\partial_{\mathcal{P}_k}\cup$. More precisely, we have

$$\partial_{\mathcal{P}_k} \cup (d_W(\psi_{i,j}) - F^{-1}\delta_-(F(\psi_{i,j}))) = 0$$

Hence, if we write $d_W(\psi_{i,j}) - F^{-1}\delta_-(F(\psi_{i,j}))) := \sum w_i \partial_{\mathcal{P}_i} + \sum_\alpha c_\alpha \partial_\alpha$, then the above implise

$$\sum_{i\neq k} w_i \partial_{\mathcal{P}_i} \cup \partial_{\mathcal{P}_k} + \sum_\alpha c_\alpha \partial_\alpha \cup \partial_{\mathcal{P}_k} = 0$$

for all k. On the other hand, one knows explicitly an additive basis of $HH^2(J)$ from Proposition 4.6, and this forces $w_i = c_\alpha = 0$ for all i and α. Therefore, we have $F(d_W(\psi_{i,j})) = \delta_-(F(\psi_{i,j}))$. $\square$

6. The Kodaira-Spencer map $\mathsf{KS} : SH^*(X_{\mathcal{Q}^\vee}) \to HH^*_c(\mathrm{mf}(W))$

The closed string mirror symmetry in our geometric setting is expressed as an equivalence between the symplectic cohomology $SH^*(X_{\mathcal{Q}^\vee})$ and the invariant $HH^*_c(\mathrm{mf}(W))$, introduced in Section 4.1, which is isomorphic to $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ by Proposition 5.1. To establish this equivalence, we use a geometrically constructed map

$$\mathsf{KS} : SH^*(X_{\mathcal{Q}^\vee}) \to HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)),$$

called the *Kodaira–Spencer map.* This map is analogous to the one introduced in [FOOO16], but adapted to the noncompact setting. Its construction is essentially provided in [HJL25, Section 4.1], though now incorporating noncommutative mirror variables. While this does not affect the definition of the relevant moduli spaces, special care must be taken with the ordering of b-inputs, as the associated variables are noncommutative.

We begin with a brief review of the symplectic cohomology of $X_{\mathcal{Q}^\vee}$, the domain of KS.

6.1. Symplectic cohomology of a punctured Riemann surface.

Let (E, ω) be an exact convex symplectic manifold. There exists a compact domain $E^{\mathrm{in}} \subset E$ such that ∂E^{in} is a contact boundary with contact 1-form α (defined by $\alpha = \lambda|_{\partial E^{\mathrm{in}}}$, where $\omega = d\lambda$). The complement of E^{in} is the conical end of E, given by the symplectomorphism

$$(E \setminus E^{\mathrm{in}}, \omega) \cong \big(\partial E^{\mathrm{in}} \times (1, \infty), \, d(r\alpha)\big).$$

Let $H : E \to \mathbb{R}$ be a generic autonomous Hamiltonian, which is C^2-small and Morse on E^{in}, and becomes linear with some slope $\epsilon > 0$ for sufficiently large r on the conical end:

$$H(y, r) = \epsilon r \quad \text{for } r \gg 1, \tag{6.1}$$

where $(y, r) \in \partial E^{\mathrm{in}} \times (1, \infty)$ are the cylindrical coordinates. The slope ϵ is chosen generically so that (i) there are no Reeb orbits on ∂E^{in} with period less than or equal to ϵ, and (ii) no positive multiple of ϵ is the period of a Reeb orbit.

For each positive integer w, consider the Hamiltonian Floer complex $CF^*(wH)$, generated over a coefficient ring $\mathbb{C}$ by the 1-periodic orbits of the Hamiltonian vector field X_{wH} on E. To achieve transversality and break the S^1-symmetry of the periodic orbits, one replaces H with a small time-dependent perturbation, still denoted by H. We use a $\mathbb{Z}/2$-grading on $CF^*(wH)$, induced by the Conley–Zehnder index. Introducing a formal variable $\mathbf{q}$ of degree -1, with $\mathbf{q}^2 = 0$, the symplectic cochain complex of E is defined as

$$SC^*(E) = \bigoplus_{w=1}^{\infty} CF^*(wH)[\mathbf{q}].$$

The differential d on $SC^*(E)$ is given by

$$d(x + \mathbf{q}y) = (-1)^{|x|}\mathfrak{d}x + (-1)^{|y|}(\mathbf{q}\mathfrak{d}y + \mathfrak{K}y - y), \tag{6.2}$$

where $\mathfrak{d} : CF^*(wH) \to CF^{*+1}(wH)$ is the Floer differential, and $\mathfrak{K} : CF^*(wH) \to CF^*((w+1)H)$ is the continuation map.

Finally, the symplectic cohomology is then defined as the direct limit

$$SH^*(E) = \varinjlim HF^*(wH), \tag{6.3}$$

where the limit is taken over the continuation maps, which are canonical at the level of cohomology. Symplectic cohomology is equipped with a graded-commutative product, known as the *pair-of-pants product*. Moreover, one can define an A_∞-structure $\{\mu^k\}$ on $SC^*(E)$, with $\mu^1 = d$. We refer the reader to [RS17, §4.6] for further details.

Returning to our geometric setup, let $E = X_{\mathcal{Q}^\vee}$ denote the punctured Riemann surface obtained by removing the vertices of the dual dimer $\mathcal{Q}^\vee$ from the surface $\Sigma^\vee$ in which it is embedded. We equip E with a canonical conical end structure near each puncture. Then $\mathbb{Z}_{>0}$ noncontractible Hamiltonian orbits appear near each puncture, where $\mathbb{Z}_{>0}$ encodes their winding numbers.

On the other hand, a puncture in $\Sigma^\vee$ corresponds to a vertex v_Z of the dimer $\mathcal{Q}^\vee$, and hence to a zigzag cycle Z in $\mathcal{Q}$ by dimer duality. Therefore noncontractible generators of $SH^*(E)$ can be parametrized in terms of zigzag cycles in $\mathcal{Q}$ together with winding numbers. In fact, it can be explicitly computed as the *logarithmic cohomology* of the pair $(\Sigma^\vee, \mathcal{Q}_0^\vee)$, as shown in [GP20]:

$$SH^*(E) \cong H^*_{\log}(\Sigma^\vee, \mathcal{Q}_0^\vee) = H^*(\Sigma^\vee \setminus \mathcal{Q}_0^\vee; \mathbb{C}) \oplus \bigoplus_Z t_Z H^*(\mathring{S}_Z; \mathbb{C})[t_Z], \tag{6.4}$$

where the sum is taken over all zigzag cycles Z in $\mathcal{Q}$ (or equivalently for all $v_Z \in \mathcal{Q}_0^\vee$). t_Z are distinct formal variables, each recording the winding number around the puncture $v_Z \in \mathcal{Q}_0^\vee$ (dual to Z), and each $\mathring{S}_Z$ is diffeomorphic to S^1, arising as the boundary of the real-oriented blow-up at the point $Z \subset \Sigma^\vee$. Consequently,

$$H^0(\mathring{S}_Z; \mathbb{Z}) = \langle e_Z \rangle, \quad H^1(\mathring{S}_Z; \mathbb{Z}) = \langle f_Z \rangle.$$

The summand $H^*(\Sigma^\vee \setminus \mathcal{Q}_0^\vee; \mathbb{C}) = H^*(X_{\mathcal{Q}^\vee}; \mathbb{C})$ is computed using the Morse cohomology of the Hamiltonian H. A specific choice of Morse function will be made in the proof of Lemma 7.8.

The ring structure on $SH^*(E)$ is readily described. The nontrivial products among the noncontractible generators are given by:

$$e_Z t_Z^n \cdot e_{Z'} t_{Z'}^m = \delta_{ZZ'}\, e_Z t_Z^{n+m}, \quad e_Z t_Z^n \cdot f_{Z'} t_{Z'}^m = \delta_{ZZ'}\, f_Z t_Z^{n+m}, \quad f_Z t_Z^n \cdot f_{Z'} t_{Z'}^m = 0.$$

In addition, for an odd-degree critical point p of H, we have the product:

$$p \cdot e_Z t_Z^n = f_Z t_Z^n,$$

where Z is the puncture toward which the gradient trajectory of p asymptotically flows. This, together with the unital property of the unit $\mathbf{1}_E \in H^*(X_{\mathcal{Q}^\vee}; \mathbb{C})$ (corresponding to the maximum of H), completely determines the ring structure on $SH^*(E)$.

6.2. **Construction of KS.** The *Kodaira–Spencer map*

$$\mathsf{KS} : SH^*(E) \to HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)), \tag{6.5}$$

is a variant of the closed-open map constructed in [RS17][5]:

$$\mathcal{CO} : SH^*(E) \to HH^*(\mathcal{W}(E)), \tag{6.6}$$

but incorporates an arbitrary number of boundary insertions taken from a *noncommutative* weak bounding cochain b. The map (6.5) is essentially the same as the Kodaira–Spencer map constructed in [HJL25, Sec 4.1], except that here we record the order of boundary inputs more precisely due to their noncommutativity. In (6.6), $\mathcal{W}(E)$ denotes the wrapped Fukaya category of the exact convex symplectic manifold E, whose objects are exact Lagrangians

[5]Their construction also applies in the monotone setting.

(equipped with suitable brane structures), and whose morphism spaces are given by the wrapped Floer complex:

$$\hom_{\mathcal{W}(E)}(L, L') = CW^*(L, L') := \bigoplus_{w \geq 1} CF^*(L, L'; wH)[\mathbf{q}],$$

where $\mathbf{q}$ is a formal variable of degree -1 satisfying $\mathbf{q}^2 = 0$. The differential involves continuation maps between Hamiltonians $wH \to (w+1)H$, analogously to (6.2). We refer to [AS10] for further details.

In our case, the target of KS is the Floer complex of a compact Lagrangian $\mathbb{L}$, which uses a *slope zero* Hamiltonian. (This is modeled using a Morse–Bott setup; see §5.1.) To accommodate this, we introduce a slight modification of the wrapped Fukaya category. Define the extended wrapped Fukaya category $\mathcal{W}_\Diamond(E)$ to have the same objects as $\mathcal{W}(E)$, but with morphism spaces between compact Lagrangians L, L' given by:

$$\hom_{\mathcal{W}_\Diamond(E)}(L, L') := CF^*(L, L')[\mathbf{q}] \oplus CW^*(L, L').$$

Here, $CF^*(L, L')$ is computed using a C^2-small Hamiltonian (of slope zero). Alternatively, a Morse–Bott model may be used if L and L' intersect cleanly. Since continuation maps are quasi-isomorphisms for compact Lagrangians, it does not alter the equivalence class of the category: $\mathcal{W}_\Diamond(E) \simeq \mathcal{W}(E)$. Also, this construction defines a natural functor from the compact Fukaya category $\mathcal{F}(E)$ to $\mathcal{W}(E)$.

The object $\mathbb{L} \in \mathcal{W}_\Diamond(E)$ thus has endomorphism algebra

$$\hom_{\mathcal{W}_\Diamond(E)}(\mathbb{L}, \mathbb{L}) = CF^*(\mathbb{L}, \mathbb{L})[\mathbf{q}] \oplus CW^*(\mathbb{L}, \mathbb{L}).$$

The natural inclusion

$$f : CF^*(\mathbb{L}, \mathbb{L}) \hookrightarrow \hom_{\mathcal{W}_\Diamond(E)}(\mathbb{L}, \mathbb{L})$$

into the first summand is an A_∞ quasi-isomorphism (with trivial higher-order maps), and hence allows one to transfer the deformation theory by weak bounding cochains, as in (2.9). The resulting deformation of $\hom_{\mathcal{W}_\Diamond(E)}(\mathbb{L}, \mathbb{L})$ by the noncommutative weak bounding cochain $b = \sum x_e X_e$ is denoted by

$$\hom_{\mathcal{W}_\Diamond(E)}((\mathbb{L}, b), (\mathbb{L}, b)). \tag{6.7}$$

The quasi-isomorphism from $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ to (6.7) follows directly from f.

Accordingly, we extend the symplectic cochain complex to include a slope-zero component

$$SC^*_\Diamond(E) := H^*(E)[\mathbf{q}] \oplus \bigoplus_{w=1}^{\infty} CF^*(wH)[\mathbf{q}],$$

where the second summand is the usual symplectic cochain complex $SC^*(E)$, and the first term accounts for the slope-zero (compact) sector. The closed-open map can be adapted to this modification, and we have

$$\mathcal{CO}_\Diamond : SC^*_\Diamond(E) \to CC^*(\mathcal{W}_\Diamond(E)).$$

Within this technical framework, we first define

$$\tilde{\mathsf{ks}} : SC^*_\Diamond(E) \to \mathrm{Hom}_{\mathcal{W}_\Diamond(E)}((\mathbb{L}, b), (\mathbb{L}, b))$$

as the count of punctured pseudo-holomorphic disks with boundary and interior data specified as in Figure 9. More precisely, consider such a punctured disk with boundary inputs at odd-degree corners labeled by $X_1, \ldots, X_k$, an output labeled by some Y, and a puncture asymptotic to a Hamiltonian orbit α (see Figure 10). This disk contributes to $\tilde{\mathsf{ks}}(\alpha)$ a term of the form:

$$x_k \cdots x_1 \cdot Y \in \mathrm{Hom}_{\mathcal{W}_\Diamond(E)}((\mathbb{L}, b), (\mathbb{L}, b)).$$

Here, each input $x_i X_i$ represents a deformation input corresponding to the chosen bounding cochain, and the scalar coefficients x_i are factored out according to the sign and ordering conventions analogous to (2.4). Thus, the coefficient for Y is $x_k \cdots x_1$.

Note that $\mathcal{CO}_\Diamond$ (the closed-open map in this setting) uses the same moduli spaces of punctured disks, although the way contributions are extracted differs. The detailed description of the data carried by pseudo-holomorphic disks in the moduli space can be found in [HJL25, Section 4.1], and will not be repeated here for the sake of conciseness.

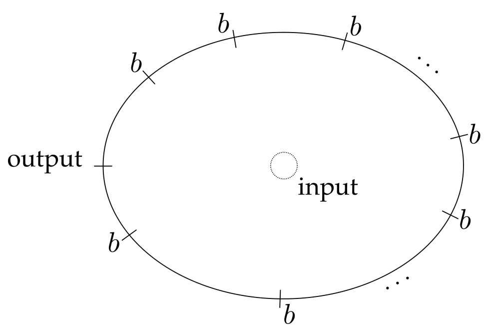

Figure 9. The general configuration of a disk in the Kodaira-Spencer moduli

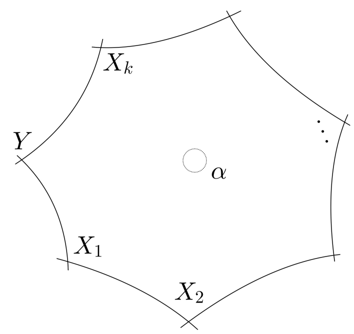

Figure 10. A contributing disk for $\mathsf{ks}(\alpha)$ with an output Y

Finally we define $\mathsf{KS} : SC^*(E) \to CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ by the following commutative diagram

$$\begin{tikzcd}
SC^*(E) \arrow[r, hook, "\sim", "qis"'] \arrow[d, "\mathsf{ks}"] & SC^*_\Diamond(E) \arrow[d, "\tilde{\mathsf{ks}}"] \\
CF^*((\mathbb{L}, b), (\mathbb{L}, b)) \arrow[r, hook, "\sim", "f"'] & \hom_{\mathcal{W}_\Diamond(E)}((\mathbb{L}, b), (\mathbb{L}, b)).
\end{tikzcd}$$

The target of KS is a priori $CF^*((\mathbb{L}, b), (\mathbb{L}, b))$, but it lands on $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$.

Proposition 6.1. *The map* $\mathsf{ks} : SC^*(E) \to CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ *is a ring homomorphism, and it factors through* $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$*:*

$$\mathsf{ks} : SC^*(E) \to CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)) (\subset CF^*((\mathbb{L}, b), (\mathbb{L}, b))).$$

Proof. The fact that $\mathsf{ks} : SC^*(E) \to CF^*((\mathbb{L}, b), (\mathbb{L}, b))$ is a ring homomorphism follows from a standard cobordism argument. The proof is essentially the same as that of [HJL25, Lemma 4.4], as the relevant disk degenerations preserve the cyclic order of inputs, and hence the ordering of noncommutative variables.

Now, let α be any generator in the $\mathbf{q}^0$-component of $SC^*(E)$. Suppose u is a pseudo-holomorphic punctured disk contributing to $\mathsf{ks}(\alpha)$, with input α, output Y, and boundary insertions $x_1 X_1, \ldots, x_k X_k$ from the weak bounding cochain b. By examining the boundary conditions of the punctured disk, it is straightforward to verify that

$$h(X_1) = h(Y), \quad t(X_i) = h(X_{i+1}) \text{ for } 1 \le i \le k-1, \quad t(X_k) = t(Y).$$

Otherwise, the corresponding moduli space is empty. The resulting summand $x_k \cdots x_1 Y$ in $\mathsf{ks}_u(\alpha)$ thus forms a cyclic path in $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$.

$\square$

7. KS is an isomorphism

We now establish the closed-string mirror symmetry for punctured Riemann surfaces $X_{\mathcal{Q}^\vee} := \Sigma^\vee \setminus \mathcal{Q}_0^\vee$, namely, we prove that the map

$$\mathsf{KS} : SH^*(X_{\mathcal{Q}^\vee}) \to HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))(\cong HH^*_c(\mathrm{mf}(W))), \tag{7.1}$$

constructed in the previous section, is a ring isomorphism. The proof naturally divides into two parts. First, we restrict KS to the even-degree component and compute the image of *primitive orbits* (that is, noncontractible Hamiltonian orbits of winding number 1). An interesting feature is that these orbits do not map to multiples of the unit $\mathbf{1}_{\mathbb{L}}$ when there exist parallel zigzag cycles (Definition 3.5) within the same homology class. This phenomenon explains the appearance of the elements $\psi_{i,j}$ on the mirror side. The odd-degree component forms a finite-rank module over the even-degree component. After a careful analysis of its module structure, the argument reduces to comparing the corresponding ranks.

In 7.5, we also give a detailed account of the behavior of the Kodaira–Spencer map in relation to the Landau–Ginzburg models

$$(Y, \underline{W}) \quad \text{and} \quad (\tilde{Y}_{\text{crep}}, \tilde{W}),$$

where Y denotes the toric Gorenstein singularity resolved by J, and $\tilde{Y}_{\text{crep}}$ its toric crepant resolution. Here $\tilde{W} : \tilde{Y}_{\text{crep}} \to \mathbb{C}$ is simply given as the product of the three toric affine coordinates on each local chart $\mathbb{C}^3$.

$$\begin{tikzcd}
(\text{“Spec” } J, W) \arrow[dr, "\text{NCCR}"'] & & (\tilde{Y}_{\text{crep}}, \tilde{W}) \arrow[dl, "\text{CCR}"] \\
& (Y, \underline{W}) &
\end{tikzcd}$$

We begin with a more detailed calculation on $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ using the spectral sequence associated with the double complex $\delta(= m_1^{b,b}) = \delta_+ + \delta_-$.

7.1. The E_2 page of the spectral sequence and $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$.

Recall that we have an additive isomorphism

$$HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)) \cong E_2 \tag{7.2}$$

where E_2 denotes the E_2-page of the spectral sequence associated with the double complex $\delta_+ + \delta_- = d_A + d_W$.[6] We will frequently identify generators of $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ and $CC^*(J, J)$ (4.6) via the chain-level identification F (5.9), (5.10) and (5.11).

Following [Won21], E_2 (originally for $(CC^*(J, J), d_A + d_W))$ can be additively described as follows: for a fixed $v_0 \in \mathcal{Q}_0$, and $1 \le r < s < t \le N$,

$$\begin{aligned}
E_2^{\text{even}} &= \mathbb{C}\langle x^n_{\eta_i},\ x^n_{\eta_i}\Psi_{i,j} : 1 \le i \le N, 1 < j \le m_i, n \ge 0\rangle, \\
E_2^{\text{odd}} &= \mathbb{C}\langle x^n_{\eta_i}U,\ x^n_{\eta_i}V,\ x^n_{\eta_i}\Theta_v : 1 \le i \le N, v(\neq v_0) \in \mathcal{E}_{i,j>1})\rangle / V(\eta_i)x_{\eta_i}U = U(\eta_i)x_{\eta_i}V
\end{aligned} \tag{7.3}$$

where

$$U = \partial_{\mathcal{P}_r} - \partial_{\mathcal{P}_t}, \quad V = \partial_{\mathcal{P}_s} - \partial_{\mathcal{P}_t}, \quad \Psi_{i,j} = \psi_{i,j} - \psi_{i,1}, \quad \Theta_v = \theta_v - \theta_{v_0}.$$

[6]The actual E_2-page is a periodic repetition of what we refer to as the E_2-page here, where ours corresponds precisely to a single period (one component) of this repeated structure. See [Won21, 5.1].

By cyclically permuting the index j in $\mathcal{E}_{i,j}$, we may assume that $v_0 \in \mathcal{E}_{i,1}$ for each i.

A few remarks are in order.

Remark 7.1. *Continuing from Remark 5.5, we emphasize that the product in the above expression is induced by the cup product on $HH^*(J,J)$, which corresponds to the operation $m_{2,0}^{b,b,b}$ on the chain level, that is, on the E_0-page. By contrast, the map KS in (7.1) is a ring homomorphism with respect to the full product $m_2^{b,b,b}$ on the target.*

Remark 7.2. *U and V, originally in $(J \otimes_{\Bbbk} E^*)_{cyc} \subset CC^*(J,J)$, can be seen as elements of $H^1(T^2;\mathbb{Z})$ where T^2 is the torus that $\mathcal{Q}$ embeds in. $U(\eta_i)$ and $V(\eta_i)$ are simply their evaluations at the 1-cycle η_i. Choose $W := aU + bV$ for $a,b \in \mathbb{Z}$ such that $\ker W$ does not contain any η_i's. When $x_{\eta_i}U \neq 0$, then $U(\eta_i) \neq 0$ from the relation in E_2^{odd} (7.3), and*

$$x_{\eta_i}U = cx_{\eta_i}W$$

with $c = \frac{U(\eta_i)}{aU(\eta_i)+bV(\eta_i)} (\neq 0)$. Similarly any nonzero $x_{\eta_i}V$ can be expressed as a scalar multiple of $x_{\eta_i}W$. Therefore

$$\{U, V, x_{\eta_i}^n W, x_{\eta_i}^m \Theta_v : v(\neq v_0) \in \mathcal{E}_{i,j}, 1 \leq i \leq N, 1 < j \leq m_i, n \geq 1, m \geq 0\} \tag{7.4}$$

forms a basis of E_2^{odd}. Readers are cautioned that the expression in (7.4) contains a redundancy, in the sense that

$$x_{\eta_i}^m \Theta_v = x_{\eta_i}^m \Theta_{v'}$$

for distinct vertices $v \neq v'$ whenever (and only when) $m \geq 1$ and $v, v' \in \mathcal{E}_{i,j}$. To avoid a potential confusion, we write $y_{i,j}^m = x_{\eta_i}^m \Theta_v$ for any $v(\neq v_0) \in \mathcal{E}_{i,j}$.

Note that E_2-page is $\mathbb{Z}$-graded, and each can be identified with the component of the associated graded of the homology. The precise identification goes as follows:

$$(E_2)_{\deg 0} = \{[a_0] \in HF_{cyc}^{\mathrm{even}}((\mathbb{L},b),(\mathbb{L},b)) : (\text{exterior}) \deg a_0 = 0\},$$

$$(E_2)_{\deg 1} = \{[a_1] \in HF_{cyc}^{\mathrm{odd}}((\mathbb{L},b),(\mathbb{L},b)) : (\text{exterior}) \deg a_1 = 1\},$$

$$(E_2)_{\deg 2} = \frac{HF_{cyc}^{\mathrm{even}}((\mathbb{L},b),(\mathbb{L},b))}{\{[a_0] \in HF_{cyc}^{\mathrm{even}}((\mathbb{L},b),(\mathbb{L},b)) : (\text{exterior}) \deg a_0 = 0\}},$$

$$(E_2)_{\deg 3} = \frac{HF_{cyc}^{\mathrm{odd}}((\mathbb{L},b),(\mathbb{L},b))}{\{[a_1] \in HF_{cyc}^{\mathrm{odd}}((\mathbb{L},b),(\mathbb{L},b)) : (\text{exterior}) \deg a_1 = 1\}},$$

and hence, we have short exact sequences

$$0 \to (E_2)_{\deg 0} \to HF_{cyc}^{\mathrm{even}} \to (E_2)_{\deg 2} \to 0, \tag{7.5}$$

$$0 \to (E_2)_{\deg 1} \to HF_{cyc}^{\mathrm{odd}} \to (E_2)_{\deg 3} \to 0. \tag{7.6}$$

The isomorphism (7.2) can be obtained by choosing (noncanonical) splitting of these short exact sequences.

7.2. KS **image of the sum of "parallel" orbits.** One distinguishing feature of our KS-map, compared to [FOOO16], is that its image (in even degrees) is not necessarily a scalar multiple of the unit class $\mathbf{1}_{\mathbb{L}}$. While this introduces some additional complexity in later analysis, there is nevertheless a particular combination of Hamiltonian orbits that always maps to the unit component in $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$.

Let $\eta \in H_1(T^2; \mathbb{Z})$ be a homology class such that the zigzag cycles in $\mathcal{Q}$ are supported on $-\eta$. In general, there may be several zigzag cycles in this class, say $Z_1, \ldots, Z_m$ (see Definition 3.5). Consider the degree 0 Hamiltonian orbits $e_{Z_i} t_{Z_i}$ which wind once around the punctures $v_{Z_i} \in \mathcal{Q}_0^\vee \subset \Sigma^\vee$, the vertex dual to the cycle Z_i. Via this duality, we may refer to these as *parallel orbits*. We will show that the sum of these parallel orbits is always mapped to a multiple of $\mathbf{1}_{\mathbb{L}}$ under the Kodaira–Spencer map.

Proposition 7.3. *Let $\{Z_1, Z_2, \ldots, Z_m\}$ be the set of all zigzag cycles in the homology class $-\eta$ for some $\eta \in H_1(T^2; \mathbb{Z})$. Then we have, modulo $m_1^{b,b}$-coboundary,*

$$\sum_{i=1}^m \mathsf{ks}(e_{Z_i} t_{Z_i}) = x_\eta \mathbf{1}_{\mathbb{L}} \mod \mathrm{Im}(m_1^{b,b}). \tag{7.7}$$

We first need some preparation.

Lemma 7.4. *Let Z be a zigzag cycle in $\mathcal{Q}$. Let $e_Z t_Z^n$ denote the symplectic cohomology generator around $v_Z \in \mathcal{Q}_0^\vee$ in the presentation* (6.4). *Let $\eta = -[Z] \in H_1(T^2; \mathbb{Z})$, the homology class of an anti-zigzag. We have*

$$W_{\mathbb{L}} \mathsf{ks}(e_Z t_Z) = \sum_{e \in \mathrm{zig}(Z)} x_\eta m_1^{b,b}(x_e X_e) = \sum_{e \in \mathrm{zag}(Z)} x_\eta m_1^{b,b}(x_e X_e). \tag{7.8}$$

Proof. We will only prove the first equality, and the second follows by a symmetric argument. Choose a representation $Z = b_L a_L \cdots b_1 a_1$ (unique up to cyclic permutations) such that a_i's are zigs, and and $\mathcal{O}^{\pm}(Z)$ be anti-zigzags of Z as presented in

$$\mathcal{O}^+(Z) = \vec{p}_1 \cdots \vec{p}_L \text{ and } \mathcal{O}^-(Z) = \vec{n}_1 \cdots \vec{n}_L$$

arranged as in (3.4) (see Figure 3). We want to show

$$W_{\mathbb{L}} \mathsf{ks}(e_Z t_Z) = \sum_{i=1}^L x_\eta m_1^{b,b}(x_{a_i} X_{a_i}). \tag{7.9}$$

Let us first compare the degree 2 components (with respect to the exterior grading) of both sides. The possible degree 2 outputs of the terms $m_1^{b,b}(x_{a_i} X_{a_i})$ are of the following types:

$$① \frac{\partial \Phi}{\partial x_{e_p}} \cdot \bar{X}_{e_p}, \quad ② \frac{\partial \Phi}{\partial x_{e_n}} \cdot \bar{X}_{e_n}, \quad \text{or} \quad ③ \frac{\partial \Phi}{\partial x_{b_{i-1}}} \cdot \bar{X}_{b_{i-1}}$$

for an edge e_p lying in $\vec{p}_i$ and e_n in $\vec{n}_{i-1}$. See Figure 11, which is related via dimer duality to the face arrangement near the zigzag cycle Z drawn in Figure 3. For example, the coefficient $\frac{\partial \Phi}{\partial x_{e_n}}$ reads the edge labelling of the polygon u_0 (contributing to $W_{\mathbb{L}}$ also) that omits x_{e_n}.

Observe that the type ③ terms cancel among themselves in the sum $\sum_{i=1}^L m_1^{b,b}(x_{a_i} X_{a_i})$. In Figure 11, the polygons u_0 and u_0' form a cancelling pair for the output $\bar{X}_{b_{i-1}}$. As a result, we are left with the following degree 2 terms in the output of the sum $\sum_{i=1}^L x_\eta m_1^{b,b}(x_{a_i} X_{a_i})$:

$$x_\eta \frac{\partial \Phi}{\partial x_{e_p}} \cdot \bar{X}_{e_p}, \quad x_\eta \frac{\partial \Phi}{\partial x_{e_n}} \cdot \bar{X}_{e_n}.$$

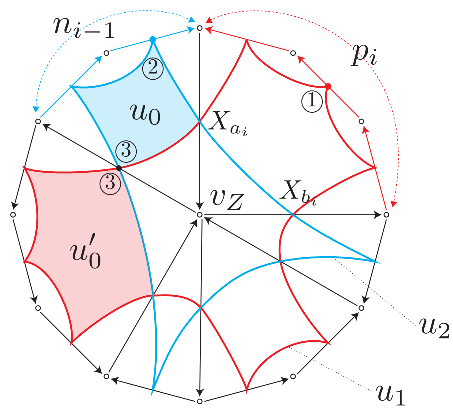

FIGURE 11. Holomorphic polygon counting for $\sum_{i=1}^{L} x_\eta m_1^{b,b}(x_{a_i} X_{a_i})$

Note that $x_\eta \frac{\partial \Phi}{\partial x_{e_p}}$ corresponds to the path obtained by removing x_{e_p} from the cyclic path

$$h(\bar{X}_{e_p}) \cdot W_{\mathbb{L}} x_\eta \cdot h(\bar{X}_{e_p}),$$

that is, the component of $W_{\mathbb{L}} x_\eta$ based at the vertex $h(\bar{X}_{e_p})$. This precisely matches the summand of $W_{\mathbb{L}} \mathsf{ks}(e_Z t_Z)$ with output $\bar{X}_{e_p}$, which is contributed by the polygon u_1 depicted in Figure 11.[7] Similarly, the term $x_\eta \frac{\partial \Phi}{\partial x_{e_n}} \cdot \bar{X}_{e_n}$ appears in $W_{\mathbb{L}} \mathsf{ks}(e_Z t_Z)$ as the contribution from the polygon u_2 in Figure 11.

This establishes the equality (7.9) when restricted to the degree 2 components. To compare the degree 0 components, a parallel argument applies, with the only difference being that the outputs (maximum points of Morse functions on L_i) now lie along the edges of the polygons. We omit the details. □

Now we are ready to prove Proposition 7.3.

Proof of Proposition 7.3. Observe that, for every perfect matching $\mathcal{P}$, we have

$$\sum_{e \in \mathcal{P}} m_1^{b,b}(x_e X_e) = W_{\mathbb{L}} \mathbf{1}_{\mathbb{L}}, \tag{7.10}$$

since each holomorphic polygon contributing to $W_{\mathbb{L}}$ contains exactly one X_e in $\{X_e : e \in \mathcal{P}\}$, and every $\bar{X}$-term appears twice in the computation of $m_1^{b,b}$, with canceling coefficients due to the Jacobi relations. By Theorem 3.7 (iv), there exists a subset $\mathcal{J}_\eta \subset \mathcal{Q}_1$ intersecting none of Z_i's such that for any subset $I \subseteq \{1, \ldots, m\}$,

$$\mathcal{P}_I := \mathcal{J}_\eta \cup \bigcup_{i \in I} \mathrm{zig}(Z_i) \cup \bigcup_{i \notin I} \mathrm{zag}(Z_i) \subset \mathcal{Q}_1 \tag{7.11}$$

defines a boundary matching that sits on the edge of $PM(\mathcal{Q})$ perpendicular to η.

Taking $\mathcal{P}$ in (7.10) to be any such $\mathcal{P}_I$ (7.11), we have

$$W_{\mathbb{L}} \mathbf{1}_{\mathbb{L}} = \sum_{e \in \mathcal{P}_I} m_1^{b,b}(x_e X_e) = \sum_{\substack{i \in I \\ e \in \mathrm{zig}(Z_i)}} m_1^{b,b}(x_e X_e) + \sum_{\substack{i \notin I \\ e \in \mathrm{zag}(Z_i)}} m_1^{b,b}(x_e X_e) + \sum_{e \in \mathcal{J}_\eta} m_1^{b,b}(x_e X_e)$$

[7] Here, one requires a more careful analysis of the contributing *pseudo-holomorphic polygon* to confirm that the count matches the topological picture. This can be justified by applying the same domain-stretching argument used in the proof of Lemma 4.6, which addresses an essentially identical situation.

holds. Multiplying the central elelement x_η in $A_{\mathbb{L}}$ yields

$$W_{\mathbb{L}} x_\eta \mathbf{1}_{\mathbb{L}} = \sum_{i\in I}\sum_{e\in \mathrm{zig}(Z_i)} x_\eta m_1^{b,b}(x_e X_e) + \sum_{i\notin I}\sum_{e\in \mathrm{zag}(Z_i)} x_\eta m_1^{b,b}(x_e X_e) + \sum_{e\in \mathcal{J}_\eta} x_\eta m_1^{b,b}(x_e X_e). \tag{7.12}$$

On the other hand, Proposition 7.4 tells us that for all $1 \le i \le m$,

$$\sum_{e\in \mathrm{zig}(Z_i)} x_\eta m_1^{b,b}(x_e X_e) = W_{\mathbb{L}} \mathsf{ks}(e_{Z_i} t_{Z_i}), \quad \sum_{e\in \mathrm{zag}(Z_i)} x_\eta m_1^{b,b}(x_e X_e) = W_{\mathbb{L}} \mathsf{ks}(e_{Z_i} t_{Z_i}),$$

and, combining with (7.12), we have

$$W_{\mathbb{L}} x_\eta \mathbf{1}_{\mathbb{L}} = W_{\mathbb{L}} \sum_{i=1}^m \mathsf{ks}(e_{Z_i} t_{Z_i}) + \sum_{e\in \mathcal{J}_\eta} m_1^{b,b}(x_\eta x_e X_e). \tag{7.13}$$

Here, we used the fact that $m_1^{b,b}$ is linear over the center of $A_{\mathbb{L}}$ (Remark 2.7). We claim that the second term in (7.13) is divisible by $W_{\mathbb{L}}$. By [Boc16a, Lemma 3.19], it is enough to show that $x_\eta x_e$ has a strictly positive degree with respect to any corner matching in $PM(\mathcal{Q})$. First of all, this is true for the two corner matchings whose join contains boundary matchings $\mathcal{P}_I$ (7.11).[8] This is because these two should also contain $\mathcal{J}_\eta$ by Theorem 3.7 (iii), but $e \in \mathcal{J}_\eta$ in the second summand of (7.13). For any corner matching $\mathcal{P}$ other than these two, we have $\deg_{\mathcal{P}}(x_\eta) \ge 1$ due to Remark 3.9, and hence the claim follows.

Therefore there exists an element $x_b \in J$ such that $W_{\mathbb{L}} \cdot x_b = x_\eta x_e$, and in particular $x_b X_e$ is a well defined element in $CF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$. We have

$$W_{\mathbb{L}} x_\eta \mathbf{1}_{\mathbb{L}} = W_{\mathbb{L}} \sum_{i=1}^m \mathsf{ks}(e_{Z_i} t_{Z_i}) + W_{\mathbb{L}} \sum_{e\in \mathcal{J}_\eta} m_1^{b,b}(x_b X_e) \tag{7.14}$$

Because $\mathcal{Q}$ is zigzag consistent, both J is cancellative, and hence, multiplying $W_{\mathbb{L}}$ is injective. Therefore we obtain

$$x_\eta \mathbf{1}_{\mathbb{L}} = \sum_{i=1}^m \mathsf{ks}(e_{Z_i} t_{Z_i}) + \sum_{e\in \mathcal{J}} m_1^{b,b}(x_b X_e).$$

□

Passing to cohomology and using the fact that KS is a ring homomorphism, we have:

Corollary 7.5. *In the setting of Lemma 7.3,*

$$\mathsf{KS}\left(\sum_{i=1}^m e_{Z_i} t_{Z_i}^n\right) = x_\eta^n \mathbf{1}_{\mathbb{L}}.$$

7.3. **KS on the even degree cohomology.** From the above discussion, we see that $x_\eta \mathbf{1}_{\mathbb{L}}$ solely cannot distinguish parallel orbits, and it will turn out that the additional even cycles $\Psi_{i,j}$ remedy the situation.

Let $Z_{i,j}$ be parallel zigzag cycles in $\mathcal{Q}$ in class $-\eta_i \in H_1(T^2, \mathbb{Z})$ for $1 \le j \le m_i$. For simplicity, we denote their corresponding degree 0 generators in $SH^*(X)$ by

$$\alpha_{i,j} := e_{Z_{i,j}} t_{Z_{i,j}} \qquad \text{for } 1 \le j \le m_i.$$

We set

$$\alpha_i = \alpha_{\eta_i} := \sum_{l=1}^{m_i} \alpha_{i,l}, \qquad \tau_{i,j} := \sum_{l=1}^{j-1} \alpha_{i,l} \quad (\text{for } 2 \le j \le m_i), \tag{7.15}$$

[8] These are the cases when $I = \phi$ or $I = \{1, \cdots, m\}$ in (7.11).

and write $(SH^{\text{even}}(X))_0$ for the subspace of $SH^{\text{even}}(X)$ (multiplicatively) generated by α_i's. Thus

$$0 \to (SH^{\text{even}}(X))_0 \to SH^{\text{even}}(X) \to \overline{SH^{\text{even}}(X)} \to 0.$$

We have shown in Proposition 7.3 that

$$\mathsf{KS} : \alpha_i^n \mapsto x_{\eta_i}^n \mathbf{1}_{\mathbb{L}}$$

which gives the isomorphism $(SH^{\text{even}}(X))_0 \xrightarrow{\cong} (E_2)_{\deg 0}$. Comparing with (7.6), we have

$$\begin{tikzcd} 0 \arrow[r] & (SH^{\text{even}}(X))_0 \arrow[r] \arrow[d, "\cong"] & SH^{\text{even}}(X) \arrow[r] \arrow[d, "\mathsf{KS}|_{\text{even}}"] & \overline{SH^{\text{even}}(X)} \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & (E_2)_{\deg 0} \arrow[r] & HF_{cyc}^{\text{even}}((\mathbb{L}, b), (\mathbb{L}, b)) \arrow[r] & (E_2)_{\deg 2} \arrow[r] & 0 \end{tikzcd} \quad (7.16)$$

On the other hand, using the explicit description of the ring structure on $SH^{\text{even}}(X)$ given in 6.1, we see that $(SH^{\text{even}}(X))_0$ is a subring of $SH^{\text{even}}(X)$, and the set

$$\left\{ \alpha_i^n \cdot \tau_{i,j} = \sum_{l=1}^{j-1} \alpha_{i,l}^{n+1} = \tau_{i,j}^{n+1} : 1 \leq i \leq N, 1 < j \leq m_i, n \in \mathbb{Z}_{\geq 0} \right\} (\subset SH^{\text{even}}(X))$$

projects to a basis of the quotient $\overline{SH^{\text{even}}(X)}$. It follows from convergence of the spectral sequence that the map h on the last column of (7.16) is simply extracting the degree 2 component from the actual image of KS. Namely, if $\mathsf{KS}(\alpha) = [a_0 + a_2] \in HF_{cyc}^{\text{even}}$ then, it maps $\overline{\alpha} \in \overline{SH^{\text{even}}(X)}$ to $[a_2] \in (E_2)_{\deg 2}$. We claim that this also gives an isomorphism.

Lemma 7.6. *$[\tau_{i,j}^{n+1}] \in \overline{SH^{\text{even}}(X)}$ maps to $x_{\eta_i}^n \Psi_{i,j} \in (E_2)_{\deg 2}$ under the induced map h in* (7.16).

Proof. We first prove $h([\tau_{i,j}]) = \Psi_{i,j}$. Let us find the image of $\alpha_{i,j}$ in $(E_2)_{\deg 2}$ under the induced map h from KS. If we write the antizigzags associated with $\alpha_{i,j}$ appearing in the boundary of $\mathcal{E}_{i,j}$ and $\mathcal{E}_{i.j-1}$, respectively as $x_1 \cdots x_k$ and $y_1 \cdots y_l$ (up to cyclic permutation), then,

$$\overline{\alpha_{i,j}} \overset{h}{\mapsto} \sum x_{a+1} \cdots x_k x_1 \cdots x_{a-1} \bar{X}_a - \sum y_{b+1} \cdots y_l x_1 \cdots x_{b-1} \bar{X}_b.$$

(By definition of h, degree 0 part of the actual image of KS is ignored.) As shown in Figure 12, take a vertex v and v' on these antizigzags. In particular, we have $v \in \mathcal{E}_{i,j}$ and $v' \in \mathcal{E}_{i,j-1}$. Therefore,

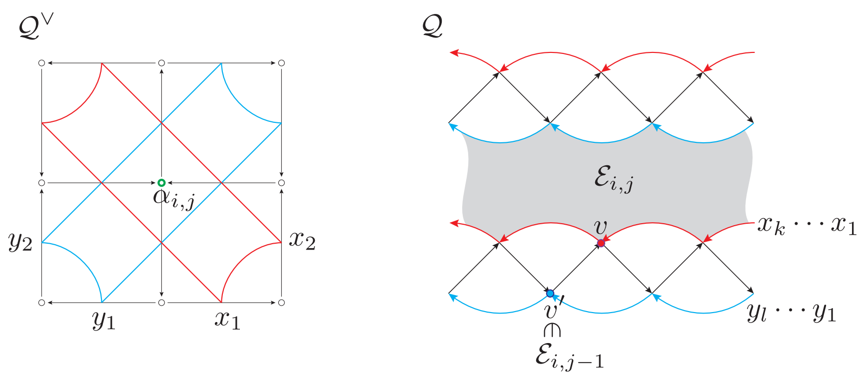

FIGURE 12

$$\psi_{i,j} = \Delta(x_{\eta_i}\theta_v) = \sum x_{a+1}\cdots x_k x_1 \cdots x_{a-1}\bar{X}_a,$$

$$\psi_{i,j-1} = \Delta(x_{\eta_i}\theta_{v'}) = \sum y_{b+1}\cdots y_l x_1 \cdots x_{b-1}\bar{X}_b,$$

and hence $\overline{\alpha_{i,j}}$ maps to $\psi_{i,j} - \psi_{i,j-1}$ under h, and

$$h : \overline{\tau_{i,j}} \mapsto \psi_{i,2} - \psi_{i,1} + \psi_{i,3} - \psi_{i,2} + \cdots + \psi_{i,j} - \psi_{i,j-1} = \psi_{i,j} - \psi_{i,1} = \Psi_{i,j}.$$

The actual image of $\tau_{i,j}$ in HF^{even}_{cyc} is some lifting, say $\tilde{\Psi}_{i,j}$, of $\Psi_{i,j}$ obtained by adding a suitable degree 0 element to it. Since KS is a ring homomorphism and the image of α_i is a multiple of unit,

$$\alpha_i^n \cdot \tau_{i,j} \left(= \sum_{l=1}^{j-1} \alpha_{i,l}^{n+1} = \tau_{i,j}^{n+1} \right) \mapsto x_{\eta_i}^n \tilde{\Psi}_{i,j}.$$

We see that the map $\overline{SH^{\mathrm{even}}(X)} \to (E_2)_{\deg 2}$ sends $[\tau_{i,j}^{n+1}]$ to $x_{\eta_i}^n \Psi_{i,j}$. (Here, $\alpha_i^n \cdot \tau_{i,j} = \tau_{i,j}^{n+1}$ uses the knowledge of the ring structure on $SH^{\mathrm{even}}(X)$.) $\square$

Therefore, by 5-lemma, we have

Proposition 7.7. $\mathsf{KS}|_{\mathrm{even}} : SH^{\mathrm{even}}(X) \to HF^{\mathrm{even}}_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ *is a ring isomorphism.*

7.4. **KS on the odd degree cohomology.** Observe that the short exact sequence (7.6) is indeed a short exact sequence of $R := (SH^{\mathrm{even}}(X))_0 = HF^0_{cyc}$-modules:

$$0 \to (E_2)_{\deg 1} \to HF^{\mathrm{odd}}_{cyc}((\mathbb{L}, b), (\mathbb{L}, b)) \to (E_2)_{\deg 3} \to 0.$$

Recall from 7.1 that $(E_2)_{\deg 1}$ is generated over R by the two elements U and V, where we choose them to be

$$U := \partial_{\mathcal{P}_i} - \partial_{\mathcal{P}_{i-1}}, \qquad V := \partial_{\mathcal{P}_{i+1}} - \partial_{\mathcal{P}_{i-1}}$$

for some fixed corner matching $\mathcal{P}_i$. We denote by $\underline{(E_2)_{\deg 1}}$ the $\mathbb{C}$-vector space generated by U and V so that

$$(E_2)_{\deg 1} = R \cdot \underline{(E_2)_{\deg 1}}. \tag{7.17}$$

Likewise, we may write

$$(E_2)_{\deg 3} = R \cdot \underline{(E_2)_{\deg 3}} \tag{7.18}$$

where $\underline{(E_2)_{\deg 3}}$ is the $\mathbb{C}$-vector space generated by Θ_v's.

Analogously, $SH^{\mathrm{odd}}(X)$ admits a decomposition

$$SH^{\mathrm{odd}}(X) = H^1(X; \mathbb{C}) \oplus SH^{\mathrm{odd}}_+(X), \tag{7.19}$$

and $SH^{\mathrm{odd}}(X)$ is generated by $H^1(X; \mathbb{C})$ over R.[9] Recall that in our setup, $H^1(X; \mathbb{C})$ is modeled on the Morse homology of a proper Morse function on X, with a choice of Morse function to be specified shortly.

Lemma 7.8. *There exists two independent elements p and q in $H^1(X; \mathbb{C})$ that are mapped to U and V under* KS.

Proof. We choose a Morse function $h = h_{X_{\mathcal{Q}^\vee}}$ on $X_{\mathcal{Q}^\vee}$ that fits into the dimer structure of $\mathcal{Q}^\vee$, as depicted on the left of Figure 13. It has an odd degree critical point p_e for each edge $e \in \mathcal{Q}_1^\vee$, and an even degree one p_f for each face $f \in \mathcal{Q}_2^\vee$. Note that this is not a perfect Morse function, but we are interested only in odd degree critical points for which the differential vanishes. To avoid (self-)intersection points $\mathbb{L}$, we slightly perturb each p_e away from the edge e so that it lies in the positive face adjacent to e.

[9]This is from the explicit computation in 6.1 for the punctured Riemann surface $X_{\mathcal{Q}^\vee}$. In general, the three terms in (7.19) form a short exact sequence only, which may not be canonically split.

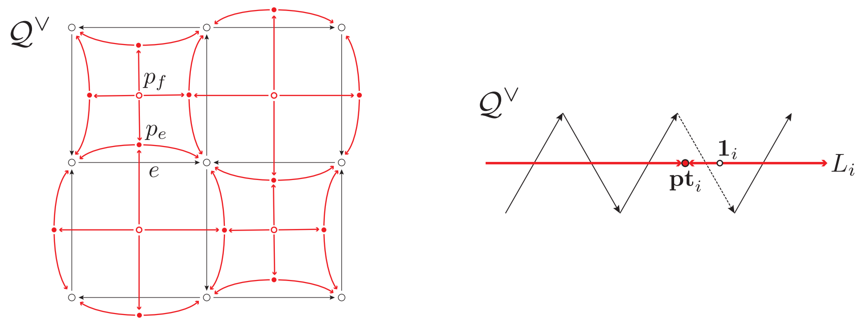

FIGURE 13. Critical points and gradient flows of the Morse functions h and f_{L_i}

We next specify a perfect Morse function $f_{L_{i_v}}$ on each Lagrangian (irreducible) component L_{i_v} for each $v \in \mathcal{Q}_0$. Recall that L_{i_v} is obtained by smoothing the zigzag path in $\mathcal{Q}^\vee$ dual to v. As shown in the right of Figure 13, we fix a zig edge in this zigzag path, and locate the maximum and the minimum, respectively, on the positive and the negative face adjacent to this zig. With this choice, the gradient trajectory flows along the orientation L_v except on a short segment that intersects this fixed zig edge.

For a fixed i_0, we consider all zigzag paths $Z_{i_0,j}$ $(1 \leq j \leq m_{i_0})$ in $\mathcal{Q}$ in class $-\eta_{i_0,j} \in H^1(T^2, \mathbb{Z})$. By dimer duality, they correspond to vertices $v_{Z_{i_0,j}} \in \mathcal{Q}_0^\vee$. We denote by $a_1, b_1, \cdots, a_L, b_L$ edges of $\mathcal{Q}^\vee$ incident to $v_{Z_{i_0,j}}$ arranged in counterclockwise order where a_i and b_i indicate incoming and outgoing, respectively (this is consistent with Figure 5). Let

$$p_{i_0,j} := \sum_{l=1}^{k} p_{a_l} + \sum_{l=1}^{k} p_{b_l}.$$

Let L_1 and L_2 be the two Lagrangian branches in $\mathbb{L}$ that intersect at X_{a_l} (lying on the edge a_l).[10] Then $\mathsf{KS}(p_{a_l})$ is computed by counting pearl trajectories, each consisting of an ambient Morse trajectory of h from p_{a_l}, followed by a Morse trajectory of f_{L_i}, which then flows to one of the following:

$$\text{①}\ \text{a self-intersection point } X, \quad \text{or} \quad \text{②}\ \text{a minimum point } \mathbf{pt}_i \text{ on } L_i,$$

See Figure 14 for type ①. Type ② is obtained by letting the gradient trajectory of f_{L_i} flow to the minimum. Case ① contributes an output of the form xX to $\mathsf{KS}(p_{a_l})$, while case ② contributes simply $\mathbf{pt}_i$. The image $\mathsf{KS}(p_{b_l})$ can be computed similarly.

Let Q be the punctured polygon formed by connecting the midpoints of the edges $a_1, b_1, \ldots, a_L, b_L$ in order, as illustrated in Figure 15 (see the shaded rectangle in the center). Since the critical points

$$p_{a_1}, p_{b_1}, \ldots, p_{a_L}, p_{b_L} \tag{7.20}$$

arise as small perturbations of the vertices of Q, the terms in $\mathsf{KS}(p_{i_0,j})$ of type $\mathbf{pt}_i$ cancel cyclically. As a result, $\mathsf{KS}(p_{i_0,j})$ lies in $(E_2)_{\deg 1} \subset SH^{\mathrm{odd}}(X)$ (i.e., It receives contributions only from case ① above.). In what follows, we compute $\mathsf{KS}(p_{i_0,j})$ by direct counting.

If the boundary edges of Q do not carry any insertions of $\mathbf{1}_{L_v}$ or $\mathbf{pt}_{L_v}$, then Figure 15 (a) shows that the image is given by

$$\mathsf{KS}(p_{i_0,j}) = \sum x_{a_l} X_{a_l} - \sum x_{b_l} X_{b_l}, \tag{7.21}$$

[10] It is possible that $L_1 = L_2$, in which case X_{a_l} is a self-intersection point of a single Lagrangian circle component.

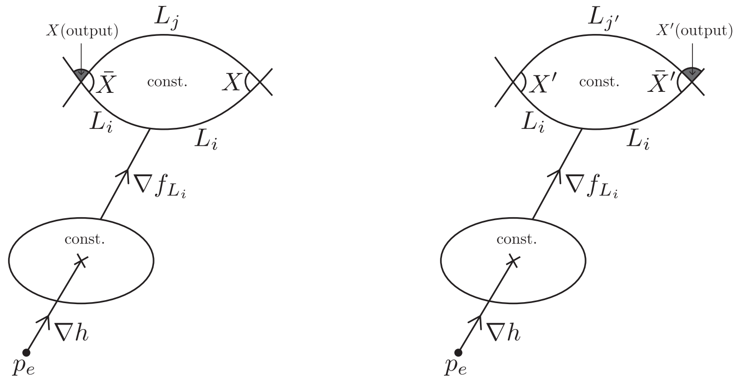

FIGURE 14. Two pearl trajectories consisting of constant disks and resulting in xX (or $x'X'$) as outputs

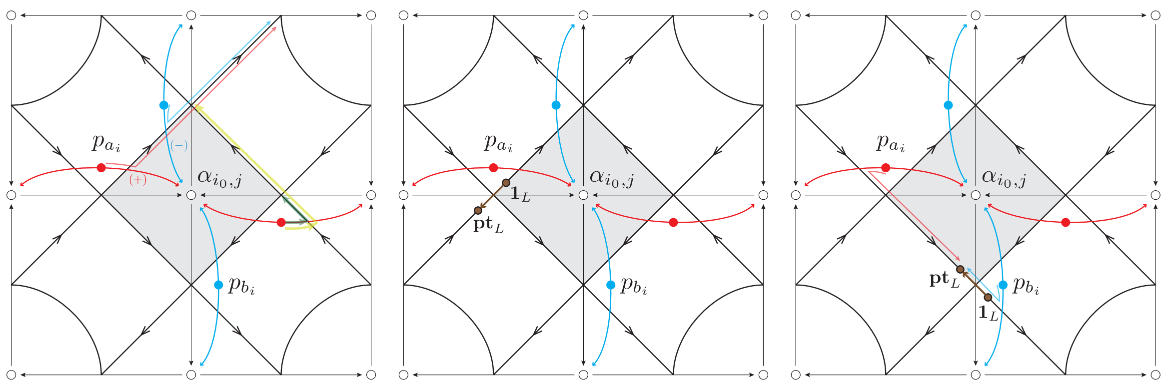

FIGURE 15. Neighboring faces of $\alpha_{i_0,j}$ in $\mathcal{Q}^\vee$ and ambient Morse flows. The red and blue trajectories in (a) are cancel pairs whereas the green and yellow ones contribute $x_{a_j}X_{a_j}$ and $x_{b_j}X_{b_j}$ (with opposite signs). (b) and (c) show the effect of presence of $\mathbf{1}_{L_v}$ and $\mathbf{pt}_{L_v}$ on ∂Q, respectively.

since all other xX-terms cancel out due to consecutive pearl trajectories of type 1, when arranged according to (7.20). The sign difference for this cancellation arises from the distinct orientations of the gradient flows of h in their contributing pearl trajectories. On the other hand, the terms $x_{a_l}X_{a_l}$ and $\sum x_{b_l}X_{b_l}$ in (7.21) appear with opposite signs due to the differing configurations of their contributing pearl trajectories, as illustrated in the two diagrams in Figure 14. (This is essentially the same reason why $m_2(X, \bar{X})$ and $m_2(\bar{X}, X)$ have opposite signs, [Sei11, (10.2)].)[11]

Even if there is $\mathbf{1}_{L_v}$ or $\mathbf{pt}_{L_v}$ on the boundary of Q, Figure 15 (b) and (c) show that it does not affect $\mathsf{KS}(p_{i_0,j})$ making use of our choice of (relative) positions of critical points of h and f_{L_i}.

[11] Adopting a different sign convention results only in an overall change of signs in (7.21), which does not affect our argument (since we may replace $p_{i_0,j}$ by $-p_{i_0,j}$ if necessary). What truly matters is the relative sign difference on the right-hand side of (7.21).

Now, let $p := \sum_j p_{i_0,j}$. Observe that $\sum x_{a_l} X_{a_l}$ in $\mathsf{KS}(p_{i_0,j})$ is the sum of all zigs in the zigzag cycle $Z_{i_0,j}$ whereas $\sum x_{b_l} X_{b_l}$ is the sum of all zags in $Z_{i_0,j}$. Therefore

$$\mathsf{KS}(p) = \partial_{\mathcal{P}_{i_0-1}} - \partial_{\mathcal{P}_{i_0}}$$

by properties of the corner matchings $\mathcal{P}_{i_0}$ and $\mathcal{P}_{i_0+1}$ in Theorem 3.7. Lastly, we take $q := \sum_j p_{i_0-1,j}$, run the same argument to have

$$\mathsf{KS}(q) = \partial_{\mathcal{P}_{i_0-2}} - \partial_{\mathcal{P}_{i_0-1}}.$$

Therefore, q and p map to U and V, respectively, for appropriate choices of r, s, t. $\square$

Let K be the subspace of $H^1(X;\mathbb{C})$ generated by p and q. Then, by Lemma 7.8, K maps isomorphically onto $\underline{(E_2)_{\deg 1}} \subset (E_2)_{\deg 1}$. Thus the restriction

$$\mathsf{KS}|_{H^1(X;\mathbb{C})} : H^1(X;\mathbb{C}) \to SH^{\mathrm{odd}}(X)$$

descends to $\tilde{r} : H^1(X;\mathbb{C})/K \to (E_2)_{\deg 3}$, i.e.,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, hook] & H^1(X;\mathbb{C}) \arrow[r] \arrow[d] & H^1(X;\mathbb{C})/K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (E_2)_{\deg 1} \arrow[r] & HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b)) \arrow[r] & (E_2)_{\deg 3} \arrow[r] & 0
\end{tikzcd}$$

We claim that the image actually lies in $\underline{(E_2)_{\deg 3}} = \langle \Theta_v : v(\neq v_0) \in \mathcal{Q}_0 \rangle$. Moreover,

Lemma 7.9. *$\tilde{r} : H^1(X;\mathbb{C})/K \to (E_2)_{\deg 3}$ factors through an isomorphism*

$$r : H^1(X;\mathbb{C})/K \xrightarrow{\cong} \underline{(E_2)_{\deg 3}}. \tag{7.22}$$

Proof. Consider an odd degree critical point p_e of the ambient Morse function h constructed in the proof of Lemma 7.8 (for $e \in \mathcal{Q}_1^\vee$). If we denote by L_1 and L_2 two Lagrangian branches in $\mathbb{L}$ meeting at X_e, then we see that the image $\bar{p_e} \in H^1(X;\mathbb{C})/K$ maps to

$$\bar{p_e} \mapsto \theta_{t(e)} - \theta_{h(e)} (= \Theta_{t(e)} - \Theta_{h(e)}) \in \underline{(E_2)_{\deg 3}}, \tag{7.23}$$

where we now view e as an edge in $\mathcal{Q}$ joining v_{L_1} and v_{L_2}. Notice that we do not have U, V terms in the image, due to the projection $HF^*_{cyc}((\mathbb{L},b),(\mathbb{L},b))^{\mathrm{odd}} \to (E_2)_{\deg 3}$. Thus r in (7.22) is well-defined.

Since $\mathcal{Q}$ is a connected graph, the map r must be surjective. Indeed, if $e_l e_{l-1} \cdots e_1$ is a path in $\mathcal{Q}$ from v_0 to v, then by (7.23) we have

$$\sum_{i=1}^{l} p_{e_i} = \theta_v - \theta_{v_0} = \Theta_v. \tag{7.24}$$

Furthermore, the combinatorial correspondence from dimer duality implies that the number $|\mathcal{Q}_0|$ of vertices of $\mathcal{Q}$ coincides with the elementary (normalized) area of its matching polygon $MP(\mathcal{Q})$. This area can be expressed as

$$|\mathcal{Q}_0| = 2I + B - 2 = 2g + N - 2,$$

where I and B denote the numbers of interior and boundary lattice points of $MP(\mathcal{Q})$, respectively, and g and N are the genus and the number of punctures of $X = \Sigma^\vee \setminus \mathcal{Q}_0^\vee$.

The first equality follows from Pick's formula, and the second from [Boc16a, Theorem 1.50]. Therefore

$$\begin{aligned}\dim H^1(X;\mathbb{C})/K &= \dim H^1(X;\mathbb{C}) - 2 = (2g+N-1)-2 \\ &= 2I + B - 3 = |\mathcal{Q}_0| - 1 \\ &= \dim \underline{(E_2)_{\deg 3}}.\end{aligned}$$

We conclude that r is an isomorphism of $\mathbb{C}$-vector spaces. $\square$

In summary, we have

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\cong"] & H^1(X;\mathbb{C}) \arrow[r] \arrow[dd, hook] & H^1(X;\mathbb{C})/K \arrow[r] \arrow[d, "\cong"] & 0 . \\ & \underline{(E_2)_{\deg 1}} \arrow[d, hook] & & \underline{(E_2)_{\deg 3}} \arrow[d, hook] & \\ 0 \arrow[r] & (E_2)_{\deg 1} \arrow[r] & HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b)) \arrow[r] & (E_2)_{\deg 3} \arrow[r] & 0 \end{tikzcd}$$

where the injectivity of the middle map follows from the 5-lemma. Therefore, $H^1(X;\mathbb{C})$ isomorphically maps onto some subspace $\underline{HF}$ of $HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b))$. From (7.17) and (7.17), we see that $HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b))$ is generated by $\underline{HF}$ over R. In particular, we have:

Lemma 7.10. $\mathsf{KS}_{\mathrm{odd}} : SH^{\mathrm{odd}}(X) \to HF^{\mathrm{odd}}_{cyc}$ *is surjective.*

In order to prove $\mathsf{KS}_{\mathrm{odd}}$ is an isomorphism, it suffices to show that the induced map

$$SH^{\mathrm{odd}}_{+}(X)(\cong SH^{\mathrm{odd}}(X)/H^1(X;\mathbb{C})) \to \overline{HF} := HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b))/\underline{HF} \tag{7.25}$$

is an isomorphism. (In fact, it suffices to show that (7.25) is injective due to Lemma 7.10.) For this, we put $\overline{HF^{\mathrm{odd}}_{cyc}}$ in the analogous short exact sequence from the spectral sequence, and look at

$$\begin{tikzcd} & & SH^{\mathrm{odd}}_{+}(X) \arrow[d] & & . \\ 0 \arrow[r] & \overline{(E_2)_{\deg 1}} \arrow[r] & \overline{HF} \arrow[r] & \overline{(E_2)_{\deg 3}} \arrow[r] & 0 \end{tikzcd}$$

The above short exact sequence is obtained by taking quotient of the first row by the second in the diagram below:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \underline{(E_2)_{\deg 1}} \arrow[r] \arrow[d] & \underline{HF} \arrow[r] \arrow[d] & \underline{(E_2)_{\deg 3}} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (E_2)_{\deg 1} \arrow[r] \arrow[d] & HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b)) \arrow[r] \arrow[d] & (E_2)_{\deg 3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{(E_2)_{\deg 1}} \arrow[r] \arrow[d] & \overline{HF} \arrow[r] \arrow[d] & \overline{(E_2)_{\deg 3}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

From Remark 7.2, we know that $\overline{(E_2)_{\deg 1}}$ and $\overline{(E_2)_{\deg 3}}$ admit free (additive) generators

$$\{x_{\eta_i}^n W : n \geq 1\} \quad \text{and} \quad \{x_{\eta_i}^m \Theta_v : v(\neq v_0) \in \mathcal{E}_{i,j>1}, m > 1\},$$

respectively. Now we are ready to prove

Proposition 7.11. $\mathsf{KS}_{\mathrm{odd}} : SH^{\mathrm{odd}}(X) \to HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b))$ *is an (R-module) isomorphism.*

Proof. From our discussion above, it only remains to show that (7.25) is injective. Recall from Remark 7.2 that

$$\{x_{\eta_i}^n W, x_{\eta_i}^m \Theta_v : v(\neq v_0) \in \mathcal{E}_{i,j>1}, n \geq 1, m \geq 0\}$$

forms a basis of $\overline{(E_2)_{\deg 1}} \oplus \overline{(E_2)_{\deg 3}}$, where $y_{i,j}^m = x_{\eta_i}^m \Theta_v$ for any $v \in \mathcal{E}_{i,j}$. We choose a particular $v = v_{i,j} \in \mathcal{E}_{i,j}$ as follows, which will help us fixing an explicit representative for $y_{i,j}^m$. As shown in Figure 16, take any zigzag path Z in $\mathcal{Q}$ starting from v_0 that is transversal to $\pm\eta_i$. This path will pass through regions $\mathcal{E}_{i,2}, \mathcal{E}_{i,3}, \cdots, \mathcal{E}_{i,j}$. For each j, we first choose $v_{i,j}$ to be the point at which the zigzag path Z entering the region $\mathcal{E}_{i,j}$. Thus we have $y_{i,j}^m = x_{\eta_i}^m \Theta_{v_{i,j}}$ in this case.

Suppose the zigzag path Z from v_0 to $v_{i,j}$ is written as $x_{l_j} x_{l_j - 1} \cdots x_2 x_1$. Each x_i has the corresponding edge denoted by the same letter x_i in the dual $\mathcal{Q}^\vee$. We then take

$$\xi_{v_{i,j}} := \sum_{a=1}^{l_j} p_{x_a}.$$

Observe that since $x_1, \cdots, x_{l_j}$ are a composable sequence of arrows, the argument as in the proof of Lemma 7.9 (see (7.24)) tells us that the degree 3 part of $\mathsf{KS}(\xi_{v_{i,j}})$ is given as

$$\theta_{v_{i,j}} - \theta_{v_0} = \Theta_{v_{i,j}}.$$

Therefore $\mathsf{KS}(\xi_{v_{i,j}})$ is a lift of $\Theta_{v_{i,j}} \in (E_2)_{\deg 3}$ to $HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b))$,

$$HF^{\mathrm{odd}}_{cyc}((\mathbb{L},b),(\mathbb{L},b)) \to (E_2)_{\deg 3}, \qquad \mathsf{KS}(\xi_{v_{i,j}}) \mapsto \Theta_{v_{i,j}}.$$

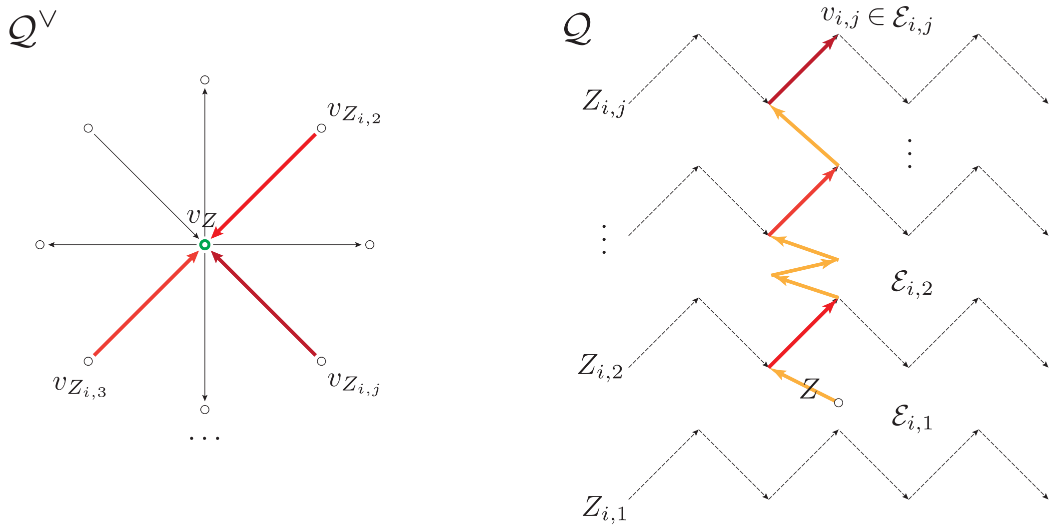

FIGURE 16

Claim. *We claim that*

$$\{\alpha_i^n(aq+bp), \alpha_i^n\xi_{v_{i,j}} : 1 \leq i \leq N, 1 < j \leq m_i, n \geq 1\} \tag{7.26}$$

is a basis of $SH^{\mathrm{odd}}_+(X)$ where a, b are coefficients appearing in Remark 7.2, and α_i is the sum of degree 0 primitive orbits winding once around $v_{Z_{i,j}}$ for all j as in (7.15)*.*

Proof of Claim. Let $\tilde{\beta}_{i,j}$ be the odd degree Hamiltonian orbit that is in the same homology class with $\alpha_{i,j}$ for $j > 2$. Then, using the explicit product structure described in 6.1, we have

$$\alpha_{i,j'} \cdot \xi_{v_{i,j}} = \begin{cases} \tilde{\beta}_{i,j'} & j' \leq j \\ 0 & \text{otherwise.} \end{cases} \tag{7.27}$$

We denote $\beta_{i,j} := \tilde{\beta}_{1,1} + \cdots + \tilde{\beta}_{i,j}$ for $1 \leq j \leq m_i - 1$, and $\beta_i := \beta_{1,1} + \cdots + \beta_{i,m_i}$. It is easy to see from the explicit ring structure of $SH^*(X)$ that

$$\{\alpha_i^n\beta_i, \alpha_i^n\beta_{i,j} : 1 \leq j \leq m_i - 1, n \geq 0\}$$

forms a basis of $SH^{\mathrm{odd}}_+(X)$. It follows from (7.27) that

$$\alpha_i^n \cdot \xi_{v_{i,j}} = \alpha_i^{n-1} \cdot \beta_{i,j}$$

for $n \geq 1$.

We now show that $\alpha_i^n\beta_i$ is a nonzero scalar multiple of $\alpha_i^n(ap+bq)$, which completes the proof of the lemma. It suffices to consider the case $n = 0$. Let us first compute $\alpha_i \cdot p$. This is a linear combination of $\tilde{\beta}_{i,j}$'s whose coefficient is the signed count of Morse trajectories from one of critical points in p asymptotic to the puncture $\alpha_{i,j}$. We denote this number by $r_{i,j;p}$ so that $\alpha_i \cdot p = \sum_j r_{i,j;p}\tilde{\beta}_{i,j}$.

Recall $p = \sum_j p_{i_0,j}$, and each $p_{i_0,j}$ is the sum of p_e for e incident to $v_{Z_{i_0,j}}$. Hence, a Morse trajectory from p to $\alpha_{i,j}$ exists whenever there is an edge e joining $v_{Z_{i,j}}$ and $v_{Z_{i_0,j'}}$ for some $1 \leq j' \leq m_{i_0}$. On the dual dimer $\mathcal{Q}$, the corresponding edge e is nothing but an intersection of zigzag cycles $Z_{i,j}$ and $Z_{i_0,j'}$. By the construction of h (see the proof of Lemma 7.8, especially Figure 15), the sign of the Morse trajectory is positive (resp. negative) if e is a zig (resp. a zag) of $Z_{i_0,j'}$. [12]

By Theorem 3.7, $V = \partial_{\mathcal{P}_{i_0-1}} - \partial_{\mathcal{P}_{i_0}}$ (after cancellation of the common edges in $\partial_{\mathcal{P}_{i_0}}$ and $\partial_{\mathcal{P}_{i_0-1}}$) consist of zigs of $Z_{i_0,j'}$ with positive signs and zags of $Z_{i_0,j'}$ with negative signs (for all j'). Thus we have

$$r_{i,j;p} = m_{i_0} V(\eta_i).$$

where the factor m_{i_0} appears since the above occurs for each $Z_{i_0,j'}$ with $1 \leq j' \leq m_{i_0}$. Therefore

$$\alpha_{i,j} \cdot p = m_{i_0} V(\eta_i) \tilde{\beta}_{i,j}$$

Proceeding with similar argument, we have

$$\alpha_{i,j} \cdot (aq + bp) = m_{i_0}(aU(\eta_i) + bV(\eta_i))\tilde{\beta}_{i,j}$$

which implies

$$\alpha_i \cdot (ap + bq) = \left(\sum_{j=1}^{m_i} \alpha_{i,j}\right) \cdot (aq + bp) = m_{i_0}(aU(\eta_i) + bV(\eta_i)) \cdot \sum_{j=1}^{m_i} \tilde{\beta}_{i,j} = c\beta_i,$$

with $c = m_{i_0}(aU(\eta_i) + bV(\eta_i))$ which is nonzero by our choice of a and b in Remark 7.2. Therefore (7.26) is a basis of $SH_+^{\mathrm{odd}}(X)$. $\square$

Now, suppose

$$\beta := \sum_i c_i \alpha_i^{n_i}(aq + bp) + \sum_{i,j} c_{i,j} \alpha_i^{n_{i,j}} \xi_{v_{i,j}}$$

for some $c_i, c_{i,j} \in \mathbb{C}$ and $n_i, n_{i,j} \in \mathbb{Z}_{\geq 1}$ belongs to the kernel of $\overline{KS} : SH_+^{\mathrm{odd}}(X) \to \overline{HF}$ (7.25). Then $\varphi(\beta) = \sum_{i,j} c_{i,j} x_{\eta_i}^{n_{i,j}} \Theta_{v_{i,j}} = 0$ must vanish, where φ is defined as the composition

$$\begin{tikzcd}
 & SH_+^{\mathrm{odd}}(X) \arrow[d, "\overline{KS}"'] \arrow[dr, "\varphi"] & & \\
0 \arrow[r] & \overline{(E_2)_{\deg 1}} \arrow[r] & \overline{HF} \arrow[r] & \overline{(E_2)_{\deg 3}} \arrow[r] & 0
\end{tikzcd}.$$

By their linear independence, we have $c_{i,j} = 0$ for all i, j. We see that the image of β lies in $\overline{(E_2)_{\deg 1}}$, and is indeed given by $\sum_i c_i x_{\eta_i} W$, which should vanish as well. Again, by linear independence, we have $c_i = 0$, and hence $\beta = 0$. $\square$

Continuing from Remark 5.5, several multiplicative relations on

$$HH_c^*(\mathrm{mf}(W)) = HF_{cyc}^*((\mathbb{L}, b), (\mathbb{L}, b))$$

stated in [Won21, Theorem 5.3] no longer hold when the more natural product $m_2^{b,b,b}$ is used—that is, the product for which the Kodaira–Spencer map KS in (7.1) becomes an isomorphism. For instance, $\Psi_{i,j} \cdot \Psi_{i,j'}$ is no longer zero with respect to $m_2^{b,b,b}$. (The precise value of this product can be determined once one chooses liftings of these elements (in E_2) to $HF_{cyc}^*((\mathbb{L}, b), (\mathbb{L}, b))$, or equivalently, a splitting of (7.5).)

[12] In fact, the zigzag consistency condition implies that if $Z_{i,j}$ intersects $Z_{i_0,j'}$ at a zig (resp. a zag), then all other intersections between them must also occur at a zig (resp. a zag). Moreover, whether the intersection is a zig or a zag is determined solely by the sign of the intersection of their homology classes.

7.5. **Singularities of W occurring along discriminant loci of $Y_{\mathcal{Q}}$.** For a given *consistent dimer model* $\mathcal{Q}$ embedded in the two-torus T^2, let $Y_{\mathcal{Q}}$ denote the associated toric Gorenstein singularity, so that the *Jacobian algebra* $J(\mathcal{Q})$ provides a noncommutative crepant resolution (NCCR) of $Y_{\mathcal{Q}}$. The *singular locus* of $Y_{\mathcal{Q}}$ lies along its toric strata of codimension ≥ 2, which can be read directly from its *toric fan*. Note that the fan of $Y_{\mathcal{Q}}$ is simply the cone over the matching polytope $MP(\mathcal{Q})$.

We are interested in the Landau–Ginzburg model on $Y_{\mathcal{Q}}$ with the superpotential

$$W \in \mathcal{Z}(J) \cong \mathbb{C}[Y_{\mathcal{Q}}],$$

viewed as a regular function on $Y_{\mathcal{Q}}$. Locally on a toric crepant resolution $\widetilde{Y_{\mathcal{Q}}} \longrightarrow Y_{\mathcal{Q}}$, the function W lifts to the product of three affine coordinates. Hence its critical locus coincides precisely with the toric strata of codimension ≥ 2 in $\widetilde{Y_{\mathcal{Q}}}$. The map $\widetilde{Y_{\mathcal{Q}}} \to Y_{\mathcal{Q}}$ identifies certain *noncompact codimension-2 strata* (each isomorphic to $\mathbb{C}$), and *contracts* all the *compact codimension-2 strata* (each isomorphic to $\mathbb{P}^1$).

If an edge of $MP(\mathcal{Q})$ contains interior lattice points, then the corresponding codimension-2 stratum acquires an *orbifold singularity*, whose order equals the number of such lattice points. In our setup, the edge of $MP(\mathcal{Q})$ normal to η_i contains $(m_i - 1)$ interior lattice points. This implies that m_i irreducible codimension-2 components (each $\cong \mathbb{C}$) are identified under the map $\widetilde{Y_{\mathcal{Q}}} \to Y_{\mathcal{Q}}$. This identification is precisely captured by the element $\Psi_{i,j}$ (for $1 < j \leq m_i$). For example, if there were no singularities along the codimension-2 strata, then all even cocycles in $HF^*_{cyc}((\mathbb{L}, b), (\mathbb{L}, b))$ would be scalar multiples of the unit.

Similarly, if the elementary (normalized) area of $MP(\mathcal{Q})$ exceeds 1, then the vertex (the unique torus fixed point) becomes singular. This can be detected by the presence of nontrivial elements Θ_v. The elementary area of $MP(\mathcal{Q})$ equals the absolute value of the Euler characteristic of the mirror curve $X_{\mathcal{Q}^\vee}$. Hence, it is natural that the number of irreducible components in $\mathbb{L}$ increases as the singularity at the torus fixed point of $Y_{\mathcal{Q}}$ becomes deeper.

Appendix A. Comparison between two resolutions of J

Our goal is to find an explicit formula of the BV operator Δ on the Hochschild cohomology of $J = J$

$$HH^*(J, J) = \hom_{J\text{-bimod}}(J^\bullet, J)$$

when applying to the degree 3 elements, where $J^\bullet$ is the length 3 resolution introduced in 4.3. The original definition of the BV operator uses the Connes operator on $HH_*(J, J)$ which is described in terms of the bar resolution. We use the following comparison theorem from elementary homological algebra to compare the standard bar resolution of $J = J$ and $J^\bullet$.

Theorem A.1 (Comparison Theorem)**.** *For $R - S$ bimodules M and N, let $\cdots P_2 \to P_1 \to P_0 \to M$ be a complex with all P_i projective, and $\cdots A_2 \to A_1 \to A_0 \to N$ an acyclic complex. Then for any $f \in \hom_{R-S}(M, N)$, there exists a chain map $f_\bullet : P_\bullet \to A_\bullet$ that extends f, i.e., $f_{-1} = f$. Moreover, any two such liftings are homotopic.*

Thus we need to find a chain map $f_\bullet$ that lifts the identify map $\underline{J} \to \underline{J}$. We present one concrete lifting here.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, "f_{\geq 4}\equiv 0"] & (J \otimes_{\mathbb{C}} J)^{\Bbbk} \arrow[r, "j"] \arrow[d, "f_3"] & J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J \arrow[r, "c"] \arrow[d, "f_2"] & J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \arrow[r, "j^\vee"] \arrow[d, "f_1"] & J \otimes_{\Bbbk} J \arrow[r] \arrow[d, "f_0"] & \underline{J} \arrow[d, "f_{-1}=id"] \\
\cdots \arrow[r] & J \otimes_{\Bbbk} \underline{J}^{\otimes 3} \otimes_{\Bbbk} J \arrow[r, "b_3"'] & J \otimes_{\Bbbk} \underline{J}^{\otimes 2} \otimes_{\Bbbk} J \arrow[r, "b_2"'] & J \otimes_{\Bbbk} \underline{J} \otimes_{\Bbbk} J \arrow[r, "b_1"'] & J \otimes_{\Bbbk} J \arrow[r] & \underline{J}
\end{tikzcd}
$$

First few maps are obvious:

$$f_0 = id : J \otimes_{\Bbbk} J \to J \otimes_{\Bbbk} J,$$

$$f_1 = incl. : J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \to J \otimes_{\Bbbk} \underline{J} \otimes_{\Bbbk} J,$$

and we set $f_{\geq 4} \equiv 0$.

We next define f_2 as follows:

$$
\begin{array}{rcl}
f_2 : J \otimes_{\Bbbk} E^* \otimes_{\Bbbk} J & \to & J \otimes_{\Bbbk} \underline{J}^{\otimes 2} \otimes_{\Bbbk} J \\
a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} b & \mapsto & \sum_y a \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)' \otimes_{\Bbbk} y \otimes_{\Bbbk} \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)'' \otimes_{\Bbbk} b.
\end{array}
$$

In other words, $f_2(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} b) = c(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} b$. To see this is a chain map,

$$
\begin{array}{rcl}
b_2(f_2(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} b)) & = & b_2\left(c(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1)) \otimes_{\Bbbk} b\right) \\
& = & \overbrace{b_1(c(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1))}^{=j^\vee(c(a\otimes_{\Bbbk}\bar{x}\otimes_{\Bbbk}1))=0} \otimes_{\Bbbk} b + c(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) b \\
& = & c(a \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} b).
\end{array}
$$

Finally, we take $f_3 : (J \otimes_{\mathbb{C}} J)^{\Bbbk} \to J \otimes_{\Bbbk} \underline{J}^{\otimes 3} \otimes_{\Bbbk} J$ to be

$$
\begin{array}{rcl}
f_3(a \otimes_{\mathbb{C}} b) & := & \displaystyle\sum_x \sum_y b \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)' \otimes_{\Bbbk} y \otimes_{\Bbbk} \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)'' \otimes_{\Bbbk} x \otimes_{\Bbbk} a \\
& = & \displaystyle\sum_x c(b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} x \otimes_{\Bbbk} a.
\end{array}
$$

Note that using $j^\vee \circ c (= b_1 \circ c) = 0$, we have

$$
\begin{array}{rcl}
b_3(f_3(a \otimes_{\mathbb{C}} b)) & = & \displaystyle\sum_x (b \cdot c(1 \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \cdot x \otimes_{\Bbbk} a - c(b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} xa) \\
& = & \displaystyle\sum_x (b \cdot c(1 \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} x) \otimes_{\Bbbk} a - c(b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} xa) \\
& = & \displaystyle\sum_x (b \cdot c(x \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} a - c(b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} 1) \otimes_{\Bbbk} xa) \\
& = & \displaystyle\sum_x (b \cdot f_2(x \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} a) - f_2(b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} xa)) \\
& = & f_2\left(\sum_x bx \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} a - b \otimes_{\Bbbk} \bar{x} \otimes_{\Bbbk} xa\right) = f_2(j(a \otimes_{\mathbb{C}} b)).
\end{array}
$$

where we used in the third line $x \cdot c(1_{\Bbbk} \otimes \bar{x} \otimes_{\Bbbk} 1) = c(1_{\Bbbk} \otimes \bar{x} \otimes_{\Bbbk} 1) \cdot x$ which is equivalent to $c \circ j = 0$.

A.1. **Homotopy inverse of f.** By comparison theorem, there also exists a homotopy inverse g of f that fits into

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "f_{\geq 4}\equiv 0"] & (J\otimes_{\mathbb{C}} J)^{\Bbbk} \arrow[r, "j"] \arrow[d, "f_3"] & J\otimes_{\Bbbk} E^*\otimes_{\Bbbk} J \arrow[r, "c"] \arrow[d, "f_2"] & J\otimes_{\Bbbk} E\otimes_{\Bbbk} J \arrow[r, "j^\vee"] \arrow[d, "f_1"] & J\otimes_{\Bbbk} J \arrow[r] \arrow[d, equal, "f_0=id"] & \underline{J} \arrow[d, equal, "f_{-1}=id"] \\
\cdots \arrow[r] \arrow[d, "0"] & J\otimes_{\Bbbk} \underline{J}^{\otimes 3}\otimes_{\Bbbk} J \arrow[r, "b_3"'] \arrow[d, "g_3"] & J\otimes_{\Bbbk} \underline{J}^{\otimes 2}\otimes_{\Bbbk} J \arrow[r, "b_2"'] \arrow[d, "g_2"] & J\otimes_{\Bbbk} \underline{J}\otimes_{\Bbbk} J \arrow[r, "b_1"'] \arrow[d, "g_1"] & J\otimes_{\Bbbk} J \arrow[r] \arrow[d, equal] & \underline{J} \arrow[d, equal] \\
0 \arrow[r] & (J\otimes_{\mathbb{C}} J)^{\Bbbk} \arrow[r, "j"] & J\otimes_{\Bbbk} E^*\otimes_{\Bbbk} J \arrow[r, "c"] & J\otimes_{\Bbbk} E\otimes_{\Bbbk} J \arrow[r, "j^\vee"] & J\otimes_{\Bbbk} J \arrow[r] & \underline{J}
\end{tikzcd}.$$

For our purpose, it is enough to choose an explicit g_1. It depends on a choice of a lifting $\sigma : J \to \mathbb{C}Q$, and we fix one here. Then we define g_1 to be a composition of σ and the following map

$$\mathfrak{c} : \mathbb{C}Q \to J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \qquad x_1x_2\cdots x_k \mapsto \sum_i x_1\cdots x_{i-1}\otimes_{\Bbbk} x_i \otimes_{\Bbbk} x_{i+1}\cdots x_k.$$

More precisely, g_1 is given as the composition of sequence of maps

$$g_1 : J\otimes_{\Bbbk} \underline{J}\otimes_{\Bbbk} J \to J\otimes_{\Bbbk} \mathbb{C}Q\otimes_{\Bbbk} J \to (J\otimes_{\Bbbk}\mathbb{C}Q)\otimes_{\Bbbk} E\otimes_{\Bbbk}(\mathbb{C}Q\otimes_{\Bbbk} J) \to J\otimes_{\Bbbk} E\otimes_{\Bbbk} J \tag{A.1}$$

where the first map and second map are $1\otimes_{\Bbbk}\sigma\otimes_{\Bbbk} 1$ and $1\otimes_{\Bbbk}\mathfrak{c}\otimes_{\Bbbk} 1$, and the last map is the projection $\mathbb{C}Q \to J$ (composed with the multiplication $J\otimes_{\Bbbk} J \to J$).

A.2. **Proof of Lemma 4.5.** We start by the Connes differential B on the Hochschild homology $HH_*(J,J)$ which dualizes to the BV operator Δ via (4.8),

$$\begin{tikzcd}
HH_{3-*}(J,J) \arrow[r, "B"] \arrow[d, "\cong"'] & HH_{4-*}(J,J) \arrow[d, "\cong"] \\
HH^*(J,J) \arrow[r, "\Delta"] & HH^{*-1}(J,J)
\end{tikzcd}$$

B is originally described in terms of the bar resolution $P^\bullet := J\otimes_{\Bbbk}\underline{J}^{\otimes\bullet}\otimes_{\Bbbk} J$, and hence, is an operator on the standard bar complex (another model for the Hochschild chain complex)

$$J\otimes_{J\text{-}\mathsf{Bimod}} P^\bullet = \oplus_{n\geq 0}\left(J\otimes_{\Bbbk}\underline{J}^{\otimes n}\right)_{cyc}.$$

Specifically, the Connes differential B is defined by

$$\begin{array}{rcl} B : \left(J\otimes_{\Bbbk}\underline{J}^{\otimes n}\right)_{cyc} & \longrightarrow & \left(J\otimes_{\Bbbk}\underline{J}^{\otimes n+1}\right)_{cyc} \\ a_0\otimes_{\Bbbk} a_1\otimes_{\Bbbk}\cdots\otimes_{\Bbbk} a_n & \mapsto & \sum_{i=0}^n (-1)^{in}\, 1\otimes a_i\otimes\cdots\otimes a_n\otimes a_0\otimes\cdots\otimes a_{i-1}. \end{array} \tag{A.2}$$

Proof of Lemma 4.5. The case of our interest is the degree 3 Hochschild cocycle in $HH^*(J,J)$, and hence, dually, we need to look at the Connes operator applying to the Hochschild cycle of degree 0 (i.e., $n=0$ in (A.2)):

$$B : \underline{J}_{cyc} \to (J\otimes_{\Bbbk}\underline{J})_{cyc} \qquad a\mapsto 1\otimes_{\Bbbk} a$$

for a cyclic path J_{cyc}. We want to read off in terms of the Koszul resolution (or the associated Hochschild chain complex with coefficient J). That is, we find $\tilde{B}$ that commutes the diagram

$$\begin{tikzcd} \underline{J} \otimes_{J\text{-}\mathsf{Bimod}} (J \otimes_{\Bbbk} J) \arrow[r, "\tilde{B}"] \arrow[d, equal] & \underline{J} \otimes_{J\text{-}\mathsf{Bimod}} (J \otimes_{\Bbbk} E \otimes_{\Bbbk} J) \arrow[d, shift right, "id\otimes f_1"'] & \cong & \underline{J}_{cyc} \arrow[r, "\tilde{B}"] \arrow[d, equal] & (\underline{J} \otimes_{\Bbbk} E)_{cyc} \arrow[d, shift right, "id\otimes f_1"'] \\ \underline{J} \otimes_{J\text{-}\mathsf{Bimod}} (J \otimes_{\Bbbk} J) \arrow[r, "B"'] & \underline{J} \otimes_{J\text{-}\mathsf{Bimod}} (J \otimes_{\Bbbk} \underline{J} \otimes_{\Bbbk} J) \arrow[u, shift right, "id\otimes g_1"'] & & \underline{J}_{cyc} \arrow[r, "B"'] & (\underline{J} \otimes_{\Bbbk} \underline{J})_{cyc} \arrow[u, shift right, "id\otimes g_1"'] \end{tikzcd}$$

where vertical maps are obtained by the comparison map between $J^\bullet$ and $P^\bullet$

$$\begin{tikzcd} J^\bullet : & J \otimes_{\Bbbk} E \otimes_{\Bbbk} J \arrow[r] \arrow[d, shift right, "f_1"'] & J \otimes_{\Bbbk} J \arrow[d, equal] \\ P^\bullet : & J \otimes_{\Bbbk} \underline{J} \otimes_{\Bbbk} J \arrow[r] \arrow[u, shift right, "g_1"'] & J \otimes_{\Bbbk} J \end{tikzcd}$$

given in (A.1). For this purpose, we only need to compute the image of $B(a) = 1 \otimes_{\Bbbk} a$ under the comparison map $id \otimes g_1$, i.e.,

$$\tilde{B}(a) = id \otimes g_1(1 \otimes_{\Bbbk} a).$$

By definition, this can be computed from (any of) its lifting $\sigma(a) = x_1 x_2 \cdots x_k \in \mathbb{C}\mathcal{Q}$ as in A.1. Since a is a cyclic path, we have $h(x_k) = t(x_1)$.

Identifying $1 \otimes_{\Bbbk} a = 1 \otimes_{J\text{-}\mathsf{Bimod}} (1 \otimes_{\Bbbk} a \otimes_{\Bbbk} 1)$ under $(\underline{J} \otimes_{\Bbbk} \underline{J})_{cyc} \cong \underline{J} \otimes_{J\text{-}\mathsf{Bimod}} (J \otimes_{\Bbbk} \underline{J} \otimes_{\Bbbk} J)$, the image $id \otimes g_1(1 \otimes_{\Bbbk} a)$ is given by

$$\begin{tikzcd} \sum_i (x_{i+1} \cdots x_k)(x_1 \cdots x_{i-1}) \otimes_{\Bbbk} x_i \arrow[r, equal] & 1 \otimes_{J\text{-}\mathsf{Bimod}} \left(\sum_i x_1 \cdots x_{i-1} \otimes_{\Bbbk} x_i \otimes_{\Bbbk} x_{i+1} \cdots x_k \right) \\ B(a) = 1 \otimes_{\Bbbk} a \arrow[r, equal] \arrow[u] & 1 \otimes_{J\text{-}\mathsf{Bimod}} (1 \otimes_{\Bbbk} a \otimes_{\Bbbk} 1) \arrow[u, "id\otimes g_1"] \end{tikzcd}$$

where x_i denotes its projection in J by abuse of notation, and the equality at the bottom is from the identification $(\underline{J} \otimes_{\Bbbk} E)_{cyc} \cong \underline{J} \otimes_{J\text{-}\mathsf{Bimod}} (J \otimes_{\Bbbk} E \otimes_{\Bbbk} J)$.

By duality 4.8, this immediately implies that

$$\Delta : HH^3(J,J) \to HH^2(J,J)$$

is given by

$$\Delta(a) = \sum_i (x_{i+1} \cdots x_k)(x_1 \cdots x_{i-1}) \otimes_{\Bbbk} \bar{X}_i$$

where we view $a \in \underline{J}$ as a degree 3 Hochschild cochain (cocycle) this time.

$\square$

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