

Linear independence of values of hypergeometric functions and arithmetic Gevrey series

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Abstract

We prove new linear independence results for the values of generalized hypergeometric functions ${}_pF_q$ at several distinct algebraic points, over arbitrary algebraic number fields. Our approach combines constructions of type II Padé approximants with a novel non-vanishing argument for generalized Wronskians of Hermite type. The method applies uniformly across all parameter regimes. Even for $p = q + 1$, we extend known results from single-point to multi-point settings over general number fields, in the both complex and p -adic settings. When $p < q + 1$, we establish linear independence results over arbitrary number fields; and for $p > q + 1$, we confirm that the values do not satisfy global linear relations in the p -adic setting. Our results generalize and strengthen earlier work by Chudnovsky's, Nesterenko, Sorokin, Delaygue and others, and demonstrate the flexibility of our Padé construction for families of contiguous hypergeometric values.

Key words: Arithmetic Gevrey series, Generalized hypergeometric function, Euler series, linear independence, transcendence, Padé approximation.

1 Introduction

The generalized hypergeometric function is a mathematical object that ubiquitously appears in various fields such as number theory, differential equations and mathematical physics. Among the families of series, it is the Gevrey series that emerges as formal solutions to differential equations, singular perturbations and difference equations, emphasized by M. Gevrey [41] in 1918. In [2], Y. André defined the notion of *arithmetic Gevrey series* to build on the theory of E -functions and G -functions introduced by C. L. Siegel [71] in his study of transcendental number theory.

In this article, we investigate the arithmetic properties of values of the generalized hypergeometric functions, indeed a most significant class of arithmetic Gevrey series, relying on explicit Padé approximations of type II. Our construction of Padé approximations for the generalized hypergeometric function ${}_pF_q$ does not depend on the choice of p and q ; covering uniformly all the cases $p = q + 1$, $p < q + 1$ and $p > q + 1$. Despite the arithmetic and analytic behavior fundamentally differ according to the cases, our approach allows a universal and systematic construction of Padé approximants without distinction. We present a relevant proof of the non-vanishing

property for the generalized Wronskian of Hermite type to achieve the linear independence, that is crucial for the universality.

Algebraic relations among solutions of general hypergeometric differential equations are studied via their differential Galois groups. The differential Galois groups of *irreducible* hypergeometric differential equations are determined in the case $p = q + 1$ in [12], while for $p < q + 1$ they are described in [47].

When $p = q + 1$, let us consider the solutions of hypergeometric differential equations supposed to be G -functions (*e.g.* the polylogarithm function $\text{Li}_s(z) = \sum_k \frac{z^k}{k^s}$). It is widely expected that values of G -functions at algebraic points are linearly independent over $\mathbb{Q}$ modulo obvious obstructions*. In the both cases, archimedean and non-archimedean, arithmetic properties of the generalized hypergeometric functions have been studied by many authors, including E. Bombieri [13], A. I. Galochkin [39], D. V. Chudnovsky and G. V. Chudnovsky [25], W. Zudilin [77], P. Débes [32] and Y. André [1].

Moreover, for specific hypergeometric G -functions such as the polylogarithms [51], refined results have been obtained through explicit constructions of Padé approximations, notably by E. M. Nikishin [60], D. V. Chudnovsky [18], V. N. Sorokin [73], M. Hata [45, 46], T. Rivoal [63], R. Marcovecchio [52], by V. Merilä [54, 55] and the authors [28, 29, 30].

When $p < q + 1$, in particular $p = q$, the solutions belong to the class of E -functions, the most significant example being the classical exponential series $e^z = \sum_k z^k/k!$. For E -functions, the Siegel–Shidlovsky theorem (*cf.* [72, 70]) determines the algebraic relations among the functions and their values. The whole theory has been rewritten by the seminal work of André and by Beukers [5, 9]. Based on their works, Delaygue recently obtained a linear independence result over $\overline{\mathbb{Q}}$ [33].

General independence criteria for the values of general hypergeometric functions in the case $p < q + 1$, have been also obtained by notably Salikhov [67, 68], Galochkin [40], and more recently by Gorelov [42, 43]. All of these results are over the field of complex numbers.

In the case $p > q + 1$, the functions have been referred to as Z -series since D. Bertrand, in reference to Euler and his series $Z(z) = \sum_k k!z^k$; in particular, when $p = q + 2$, they are called *Euler-type series*. Over Archimedean fields, these series have radius of convergence zero, and their values cannot be studied. Over non-Archimedean fields, however, the radius of convergence is positive, making it possible to study their values. In this case, it has been shown that there are no global relations among the values (see [13]). The general theory in the case $p = q + 2$ has been studied by T. Yebbou (unpublished), D. Bertrand, V. Chirskii, and J. Yebbou [7], and V. G. Chirskii has investigated the case $p > q + 1$ [14, 15, 16].

Improved results over those in [7] have been obtained for explicit general hypergeometric Euler-type series via the construction of Padé approximations by T. Matala-aho and W. Zudilin [53], K. Väänänen [75], and L. Seppälä [69].

*Remark that $\text{Li}_1(3/4) - 2\text{Li}_1(1/2) = 0$.

In this paper, we provide a new linear independence criterion for the values of *several* contiguous ${}_pF_q$ for any p, q at several distinct points, over a given algebraic number field of any finite degree.

Our statement extends previous ones due to D. V. Chudnovsky or D. V. Chudnovsky-G. V. Chudnovsky in [18, Theorem 3.1] [22, Theorem I], [24, Theorem 0.3] [25, Theorem I] and Yu. Nesterenko [57, Theorem 1] [58, Theorem 1], which all dealt with values at one point and over the rational number field.

Our approach is inspired by our previous series that included a formal explicit construction of Padé approximants. However, standard derivation or primitivation (as in [28, 29, 30] (*refer* [48, 49, 50])) can no longer be used as in the case of polylogarithms, and we introduce an appropriate operator that mimics their property for the given set of hypergeometric functions.

Thus this work follows the classical philosophy (Prove Padé approximation is good enough and achieve a zero estimate) due to A. I. Galochkin in [38, 39], M. Hata in [45], V. N. Sorokin in [73], K. Väänänen in [74] and W. Zudilin in [77], giving linear independence criteria, either over the field of rational numbers or quadratic imaginary fields. However, in our situation (several functions and several special values simultaneously) an actual zero estimate has to be proved as there seems to be no trivial way of proving the linear independence of the set of approximation constructed. In [7] (Euler function case), the authors actually prove a zero estimate (after a construction of auxiliary functions via a Siegel lemma). Our proof involves the non vanishing of a determinant thus achieving optimal non vanishing conditions. In the case $p \leq q + 1$, V. Merilä sketched an approach involving several points [54] and Padé type II approximations.

As related works, we refer to the algebraic independence sketched in [26, Theorem 3.4] of the two special values of Gauss' hypergeometric functions ${}_2F_1\left(\begin{smallmatrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{smallmatrix} \middle| \alpha\right)$ and ${}_2F_1\left(\begin{smallmatrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{smallmatrix} \middle| \alpha\right)$ when α is a non-zero algebraic number supposed to be of small module, that was later proved by Y. André in [1] along with the p -adic analogue. We also mention that the work by F. Beukers involves several algebraicity of values of the function [8, 10]. A historical survey for further reference is given in [28, 29], with comparison concerning with earlier works.

Our criterion indeed shows the linear independence of values of full generalized hypergeometric functions including the contiguous ones, whose functional linear independence has been discussed in [57, 58]. Our contribution if any, is an uncharted non-vanishing property for the generalized Wronskian of Hermite type, along with a formal construction that allows a systematic treatment of Euler type, G and E -functions simultaneously. It should also be noted that our construction allows us to replace highly technical analytic estimates by simple norm operator evaluation for often better or at least competitive quantitative estimates[†].

2 Notations and main results

Let us first define Gevrey series.

[†]Quantitative estimates are discussed in Section 6.

DEFINITION 2.1. Let j be a rational number. A formal power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$$

is a *Gevrey of order j* if and only if the associated series

$$f^{[j]}(z) := \sum_{k=0}^{\infty} \frac{a_k}{k!^j} z^k$$

has a positive radius of convergence. It is called *Gevrey of precise order j* if and only if $f^{[j]}$ has positive and finite radius of convergence.

Now we introduce arithmetic Gevrey series, defined by André (*confer* [4, 2.1.1. Definition]).

DEFINITION 2.2. Let j be a rational number, and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ an embedding. A formal power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in \overline{\mathbb{Q}}[[z]]$$

is an *arithmetic Gevrey series of order j* if and only if

- (i) There exists $C_1 > 0$ such that for all $n \geq 0$ and $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\left| \sigma \left(\frac{a_n}{n!^j} \right) \right| \leq C_1^{n+1}$,
where $|\cdot|$ is the usual complex absolute value induced by the chosen embedding;
- (ii) There exists $C_2 > 0$ such that for all $n \geq 0$, $\text{den} \left(\frac{a_0}{0!^j}, \dots, \frac{a_n}{n!^j} \right) \leq C_2^{n+1}$.

Here $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$. Siegel's G -functions (respectively E -functions) [71] are nothing but holonomic[‡] arithmetic Gevrey series of order 0 (respectively order -1) and holonomic arithmetic Gevrey series of order 1 are called Euler-type series.

Let p, q be non-negative integers and $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{Q} \setminus \{0\}$, where none of them are negative integers. We define the generalized hypergeometric function with parameters a_i, b_j by

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where $(a)_k$ is the Pochhammer symbol: $(a)_0 = 1$, $(a)_k = a(a+1) \cdots (a+k-1)$. Whenever $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{Q} \setminus \{0\}$, the function ${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right)$ is indeed a holonomic arithmetic Gevrey series of order $p - q - 1$.

We collect some notations which we use throughout the article. Let K be an algebraic number field of arbitrary degree $[K : \mathbb{Q}] < \infty$. Let us denote by $\mathbb{N}$ the set of strictly positive integers[§]. The set of places of K is denoted by $\mathfrak{M}_K$ (with $\mathfrak{M}_K^\infty$ and $\mathfrak{M}_K^f$ representing the set of

[‡]A power series in $K[[z]]$ over a field K is said to be *holonomic* if it satisfies a linear differential equation over $K[z]$.

[§]Note that this convention is not the one commonly used in Europe where $\mathbb{N}$ would include 0

infinite places and finite places, respectively). For $v \in \mathfrak{M}_K$, we denote the completion of K with respect to v by K_v . Let us denote the normalized absolute value $|\cdot|_v$ for $v \in \mathfrak{M}_K$:

$$|p|_v = p^{-\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} \text{ if } v \in \mathfrak{M}_K^f \text{ and } v \mid p, \quad |x|_v = |\sigma_v(x)|^{\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} \text{ if } v \in \mathfrak{M}_K^\infty,$$

where p is a prime number and σ_v the embedding $K \hookrightarrow \mathbb{C}$ corresponding to v . On K_v^n , the norm $|\cdot|_v$ denotes the norm of the supremum.

Let m be a positive integer and $\beta = (\beta_0, \dots, \beta_m) \in K^{m+1}$. The absolute affine height of β is defined by

$$H(\beta) = \prod_{v \in \mathfrak{M}_K} \max\{1, |\beta_0|_v, \dots, |\beta_m|_v\},$$

and the logarithmic absolute height by $h(\beta) = \log H(\beta)$. We denote the local logarithmic absolute $\log \max\{1, |\beta_i|_v\}$ by $h_v(\beta)$ for each $v \in \mathfrak{M}_K$. Then, $h(\beta) = \sum_{v \in \mathfrak{M}_K} h_v(\beta)$.

Define the denominator of S by

$$\text{den}(S) = \min\{n \in \mathbb{Z} \mid n > 0 \text{ such that } n\alpha \text{ is an algebraic integer for all } \alpha \in S\}$$

for a finite set S of algebraic numbers. Let y be a real number. Write the least (respectively the greatest) integer greater (respectively less) than or equal to y by $\lceil y \rceil$ (resp. $\lfloor y \rfloor$). Denote by

$$\mu_n(x) = \text{den}(x)^n \prod_{\substack{q:\text{prime} \\ q|\text{den}(x)}} q^{\lfloor \frac{n}{q-1} \rfloor}, \quad \mu(x) = \text{den}(x) \prod_{\substack{q:\text{prime} \\ q|\text{den}(x)}} q^{\frac{1}{q-1}}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{Q}$. We also denote by[¶]

$$\mu_v(x) = \begin{cases} 1 & \text{if } v \in \mathfrak{M}_K^\infty \text{ or } v \in \mathfrak{M}_K^f \text{ \& } |x|_v \leq 1, \\ |\text{den}(x)|_v |p|_v^{\frac{1}{p-1}} & \text{if } v \in \mathfrak{M}_K^f \text{ \& } |x|_v > 1 \text{ where } p \text{ is the prime below } v. \end{cases}$$

Now we are ready to state our main theorems. Let p, q be positive integers and $a_1, \dots, a_p, b_1, \dots, b_q$ be non-zero rational numbers such that none of them is a negative integer. We now fix an algebraic number field K and a place $v \in \mathfrak{M}_K$. We denote the radius of convergence of ${}_pF_q\left(\begin{smallmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{smallmatrix} \middle| z\right)$ in K_v by r_v . The following provides a table of r_v .

	$v \in \mathfrak{M}_\mathbb{Q}^\infty$	$v \in \mathfrak{M}_\mathbb{Q}^f$
$p < q + 1$	$r_v = \infty$	$r_v < \infty$
$p = q + 1$	$r_v < \infty$	$r_v < \infty$
$p > q + 1$	$r_v = 0$	$r_v < \infty$

Let us fix $\alpha = (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$. Additionally, we now assume

- (1) neither a_k nor $a_k + 1 - b_j$ ($1 \leq k \leq p$ and $1 \leq j \leq q$) is a strictly positive integer.

[¶]Note that $\mu_v(x)$ mimics $|\mu(x)|_v$ for $v \in \mathfrak{M}_K^f$, however since $\mu(x)$ needs not be in the field K it has to be defined accordingly.

Under the assumption (1), our main results describe the arithmetic properties of the values of the generalized hypergeometric functions

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{\alpha_i}{z}\right)$$

within their respective radii of convergence. The following table indicates which theorem corresponds to each arithmetic Gevrey series and to which type of place:

	$v \in \mathfrak{M}_{\overline{\mathbb{Q}}}^{\infty}$	$v \in \mathfrak{M}_{\overline{\mathbb{Q}}}^f$
$p < q + 1$	Theorem 2.4	—
$p = q + 1$	Theorem 2.3	Theorem 2.3
$p > q + 1$	—	Theorem 2.5

First let us consider the case where $p = q + 1$ (denoted by d). Let $\varepsilon_v = 1$ if $v | \infty$ and 0 otherwise. For $\beta \in K \setminus \{0\}$ and $v \in \mathfrak{M}_K$, define a real number:

$$\begin{aligned} V_v(\alpha, \beta) &= \log |\beta|_v + dm(h_v(\alpha) - h(\alpha, \beta)) - (dm + 1)h_v(\alpha) \\ &\quad - \left(dm \log(2) + d \left(\log(dm + 1) + dm \log \left(\frac{dm + 1}{dm} \right) \right) \right) \\ &\quad - dm \sum_{j=1}^d \text{den}(a_j) - (dm + 1) \sum_{j=1}^d \log \mu(b_j) + \sum_{j=1}^d \log \mu_v(a_j) . \end{aligned}$$

THEOREM 2.3. *Assume that each coordinate of α is pairwise distinct and Equation (1) holds. Suppose $V_v(\alpha, \beta) > 0$. Then the $dm + 1$ elements in K_v :*

$${}_dF_{d-1}\left(\begin{matrix} a_1, \dots, a_d \\ b_1, \dots, b_{d-1} \end{matrix} \middle| \frac{\alpha_i}{\beta}\right) , \quad {}_dF_{d-1}\left(\begin{matrix} a_1 + 1, \dots, a_s + 1, a_{s+1}, \dots, a_d \\ b_1, \dots, b_{d-1} \end{matrix} \middle| \frac{\alpha_i}{\beta}\right)$$

$(1 \leq i \leq m, 1 \leq s \leq d - 1)$ and 1 are linearly independent over K .

Next we consider the case $p < q + 1$. Here, we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

THEOREM 2.4. *Assume $p < q + 1$ and each coordinate of α is pairwise distinct and Equation (1) holds. Then, the $(q + 1)m + 1$ complex numbers:*

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \alpha_i\right) , \quad {}_pF_q\left(\begin{matrix} a_1 + 1, \dots, a_p + 1 \\ b_1 + 1, \dots, b_s + 1, b_{s+1}, \dots, b_q \end{matrix} \middle| \alpha_i\right)$$

$(1 \leq i \leq m, 1 \leq s \leq q)$ and 1 are linearly independent over $\overline{\mathbb{Q}}$.

Finally, we address the case $p > q + 1$. Assume all α_i are algebraic integers with

$$|\alpha_i|_v < \prod_{j=1}^d \mu_v(a_j) |p_v|_v^{\frac{d-d'}{p_v-1}}$$

for any $v \in \mathfrak{M}_K^f$ above a rational prime p_v that divides $\prod_{j=1}^p \text{den}(a_j)$.

THEOREM 2.5. Assume that each coordinate of α is pairwise distinct and Equation (1) holds. Let $\lambda = (\lambda_0, \lambda_{s,i})_{\substack{1 \leq s \leq p \\ 1 \leq i \leq m}} \in K^{dm+1} \setminus \{0\}$. Then there exists an effectively computable positive real number H_0 such that, whenever $H(\lambda) \geq H_0$, for any $H \geq H(\lambda)$, there exists a prime

$$p' \in \left[\left(\frac{3pm \log H}{(p-q-1) \log \log H} \right)^{\frac{1}{8dm}}, \frac{12pm \max_{1 \leq j \leq q} \{\text{den}(b_j)\} \log H}{(p-q-1) \log \log H} \right]$$

and a place $v \in \mathfrak{M}_K^f$ above p' for which the linear forms in hypergeometric values in K_v satisfies

$$\lambda_0 + \sum_{i=1}^m \lambda_{p,i} \cdot {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \alpha_i \right) + \sum_{s=1}^{p-1} \sum_{i=1}^m \lambda_{s,i} \cdot {}_pF_q \left(\begin{matrix} a_1 + 1, \dots, a_s + 1, a_{s+1}, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \alpha_i \right) \neq 0 .$$

It can be remarked that the range of primes needed to ensure non vanishing is a *short* interval in the sense that both left and right hand side are proportional to $\log(H)/\log \log(H)$. Whereas in Matala-aho-Zudilin [53], Väänänen [75] and L. Seppälä [69] (special case of the Euler series $\sum_k k! z^k$), and in Bertrand-Chirskii-Yebbou [7] (special case $p = q+2$ but Gevrey not necessarily contiguous) or the later works of Chirskii ([14, 15, 16]), the required size of the prime interval is a *large* one: the left hand side is of the order of the logarithm of the right hand side (which is similar to ours, of the order of $\log(H)/\log \log(H)$). This is due to our optimal construction and a factorial is therefore not lost in the estimates.

This article is organized as follows. In Section 3.1, we describe our setup for generalized hypergeometric functions. In Section 3.2, we proceed with our construction of Padé approximants, generalizing the method used in [28, 29, 30]. Section 4 is devoted to showing the non-vanishing property of the crucial determinant by the study of kernels of linear maps associated with contiguous hypergeometric functions. In Section 6, we give the proof of Theorem 2.3, 2.4 and 2.5. A more general statements, together with totally effective linear independence measures in case of $p = q + 1$, are also provided in this section, as given by Theorem 6.3, 6.5 and 6.10.

3 Padé approximation of generalized hypergeometric functions

Throughout this section, denote by K a field of characteristic 0. Denote the ring of K -linear endomorphisms (respectively automorphism) of $K[t]$ by $\text{End}_K(K[t])$ (respectively $\text{Aut}_K(K[t])$). We embed the Weyl algebra $K[t, \frac{d}{dt}]$ into $\text{End}_K(K[t])$ in a natural way.

3.1 Preliminaries

3.1.1 Linear properties of differential operators

NOTATION 3.1. (i) For $\alpha \in K$, denote by Eval_α the linear evaluation map $K[t] \longrightarrow K$, $P \longmapsto P(\alpha)$. Whenever there is an ambiguity in a setting of variables, we will denote the map by $\text{Eval}_{t \rightarrow \alpha}$.

- (ii) For $P \in K[t]$, we denote by $[P]$ the multiplication by P (the map $Q \mapsto PQ$). If there is no ambiguity, we will sometimes omit the brackets.
- (iii) For a K -automorphism φ of a K -module M and an integer k , put

$$\varphi^k = \begin{cases} \overbrace{\varphi \circ \dots \circ \varphi}^{k\text{-times}} & \text{if } k > 0 \\ \text{id}_M & \text{if } k = 0 \\ \overbrace{\varphi^{-1} \circ \dots \circ \varphi^{-1}}^{-k\text{-times}} & \text{if } k < 0 \end{cases} .$$

The following are elementary remarks on the action of the differential on polynomials, formal series that we use several times. We regroup them for the convenience of the reader

FACTS 3.2. (i) The linear operator on $K[t]$ defined by $A(t) \mapsto t \frac{d}{dt}(A(t))$ has eigenvalue k on the element t^k of the canonical basis of $K[t]$.

(ii) Let $\alpha \in K$ and $A(t) \in K[t]$, then $A(t \frac{d}{dt} + \alpha)$ has eigenvalue $A(k + \alpha)$ on the element t^k of the canonical basis of $K[t]$. In particular, if we assume moreover that $\alpha + k$ is not a root of A for any $k \geq 0$, then $A(t \frac{d}{dt} + \alpha) \in \text{Aut}_K(K[t])$. Moreover, the operator $A(t \frac{d}{dt} + \alpha) \in \text{End}_K(K[t])$ leaves stable all the ideals (t^n) , $n \geq 0$ viewed as K -vector spaces.

(iii) Let $H(t) \in K[t]$. For any $k \geq 0$, we have $[t^k] \circ H(t \frac{d}{dt}) = H(t \frac{d}{dt} - k) \circ [t^k]$.

PROOF. (i) and (ii) do not require proof. For (iii), apply (ii) to $A = H$, $\alpha = 0$, for any non-negative integer m , the left hand side is

$$[t^k] \circ H\left(t \frac{d}{dt}\right)(t^m) = H(m)t^{m+k} .$$

Whereas, the right hand side, again by (ii), with $A = H$ and $\alpha = -k$.

$$H\left(t \frac{d}{dt} - k\right) \circ [t^k](t^m) = H(k + m - k)t^{m+k} = H(m)t^{m+k} .$$

□

3.1.2 Generalized contiguous hypergeometric functions

In this subsection, we introduce the generalized hypergeometric function. First, let us introduce polynomials $A(X), B(X) \in K[X]$ satisfying $\max\{\deg A, \deg B\} > 0$. Assume

$$(2) \quad A(k)B(k) \neq 0 \quad (k \geq 0) .$$

Consider the differential equation

$$(\partial E_{A,B}) \quad \left(B\left(-z \frac{d}{dz}\right) z - A\left(-z \frac{d}{dz}\right) \right) f(z) = B(0) .$$

FACTS 3.3. The equation $(\partial E_{A,B})$ has a unique solution with residue 1 in $(1/z) \cdot K[[1/z]]$ given by

$$F_{A,B} \left(\frac{1}{z} \right) = F \left(\frac{1}{z} \right) = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}}$$

where the sequence $\mathbf{c} = (c_k)_{k \geq 0}$ is inductively defined by:

$$(3) \quad c_0 = 1, \quad c_{k+1} = c_k \cdot \frac{A(k)}{B(k+1)} \quad (k \geq 0) .$$

PROOF. Hypothesis (2) ensures that the sequence c_k is well defined and $c_k \neq 0$, $k \geq 0$. Moreover, using Facts 3.2, (ii), one readily checks that any solution in $(1/z) \cdot K[[1/z]]$ necessarily satisfies (3) which uniquely defines $F(1/z)$. $\square$

We now construct series called contiguous to F in the sense that they are linked to F by a order 1 differential operator. Let $\gamma \in K$, and introduce for a given choice of A, B as above, the series $F_{A(X+\gamma), B(X+\gamma)}$ (well defined provided γ is not a rational integer ≤ 0) which satisfies

$$(4) \quad F_{A(X+\gamma), B(X+\gamma)}(1/z) = \left(-z \frac{d}{dz} + \gamma - 1 \right) (F_{A,B}(1/z)) .$$

In other words, for each sequence of elements in K , it is possible to construct a chain of functions each linked to the next one by an order one differential operator. For our purpose, it is enough to restrict ourselves to finite chains.

Put $d = \max\{\deg A, \deg B\}$ and take $\gamma = (\gamma_1, \dots, \gamma_{d-1}) \in K^{d-1}$. Let s be an integer with $1 \leq s \leq d$. We define the power series $F_s(\gamma, z)$ by

$$(5) \quad F_d(\gamma, z) = F(z), \quad F_s(\gamma, z) = \sum_{k=0}^{\infty} (k + \gamma_1) \cdots (k + \gamma_{d-s}) c_k z^{k+1} \text{ for } 1 \leq s \leq d-1 .$$

We denote $F_s(\gamma, z)$ by $F_s(z)$ when no confusion may arise. Notice that $F_s(1/z)$ satisfies

$$F_s(1/z) = \left(-z \frac{d}{dz} + (\gamma_1 - 1) \right) \circ \cdots \circ \left(-z \frac{d}{dz} + (\gamma_{d-s} - 1) \right) (F_d(1/z)) .$$

REMARK 3.4. Let p, q be positive integers and $a_1, \dots, a_p, b_1, \dots, b_q \in K \setminus \{0\}$ such that none of them is a negative integer. Put

$$A(X) = (X + a_1 + 1) \cdots (X + a_p + 1), \quad B(X) = (X + b_1) \cdots (X + b_q)(X + 1)$$

$$c_k = \frac{(a_1)_{k+1} \cdots (a_p)_{k+1}}{(b_1)_{k+1} \cdots (b_q)_{k+1} (k+1)!} \quad (k \geq 0) .$$

Then, $(c_k)_{k \geq 0}$ satisfies

$$c_{k+1} = c_k \cdot \frac{A(k)}{B(k+1)} .$$

For this sequence,

$$F \left(\frac{1}{z} \right) = {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{1}{z} \right) - 1 .$$

In the case of $p \geq q + 1$ and $\gamma_1 = a_1 + 1, \dots, \gamma_{p-1} = a_{p-1} + 1$, the series $F_s(1/z)$ has the expression:

(6)

$$F_p(1/z) = {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{1}{z} \right) - 1, \quad F_s(1/z) = a_1 \cdots a_{p-s} \left({}_pF_q \left(\begin{matrix} a_1 + 1, \dots, a_{p-s} + 1, a_{p-s+1}, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{1}{z} \right) - 1 \right),$$

for $1 \leq s \leq p - 1$.

In the case of $p < q + 1$ and $\gamma_1 = 1, \gamma_2 = b_q, \dots, \gamma_q = b_2$, the series $F_s(1/z)$ has the expression:

(7)

$$F_{q+1}(1/z) = {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{1}{z} \right) - 1, \quad F_s(1/z) = \frac{a_1 \cdots a_p}{b_1 \cdots b_s z} \cdot {}_pF_q \left(\begin{matrix} a_1 + 1, \dots, a_p + 1 \\ b_1 + 1, \dots, b_s + 1, b_{s+1}, \dots, b_q \end{matrix} \middle| \frac{1}{z} \right),$$

for $1 \leq s \leq q$.

Throughout this section, we fix $A(X)$ and $B(X)$ such that (2) holds, and ensure that $\max\{\deg A, \deg B\} > 0$ and recall $d = \max\{\deg A, \deg B\}$, we moreover set $\deg B = d'$. We denote by $\mathbf{c} = (c_k)_{k \geq 0}$ the sequence satisfying (3), where $c_k \in K \setminus \{0\}$ defining F .

Additionally, we fix $\gamma_1, \dots, \gamma_{d-1} \in K$ (at this stage, it is not necessary to assume that γ_i is not an integer ≤ 0). The chain of power series $F_s(z)$, defined in (5) for the given sequence $\mathbf{c}$ and $\gamma_1, \dots, \gamma_{d-1}$ in K now fixed, is denoted by $F_s(z)$. Take m as a strictly positive integer and $\alpha_1, \dots, \alpha_m$ in $K \setminus \{0\}$ (at this stage it is not necessary to assume them pairwise distinct).

We are now in a position to define the operators that will play a role analogous to derivation and primitivation (which were enough to deal with simpler classes of functions like the polylogarithms, *confer* [28, 29, 30]).

DEFINITION 3.5. (i) Let $\mathbf{c} = (c_k)_{k \geq 0}$ be a sequence of elements of $K \setminus \{0\}$.

Define $\mathcal{T}_{\mathbf{c}} \in \text{Aut}_K(K[t])$ by

$$(8) \quad \mathcal{T}_{\mathbf{c}} : K[t] \longrightarrow K[t]; \quad t^k \mapsto \frac{t^k}{c_k}.$$

(ii) Let $\gamma_1, \dots, \gamma_{d-1} \in K$ and $\alpha_1, \dots, \alpha_m \in K \setminus \{0\}$. We define

$$(9) \quad \varphi_{i,d} = [\alpha_i] \circ \text{Eval}_{\alpha_i} \circ \mathcal{T}_{\mathbf{c}}^{-1} \text{ for } 1 \leq i \leq m.$$

$$(10) \quad \varphi_{i,s} = \varphi_{i,d} \circ \left(t \frac{d}{dt} + \gamma_1 \right) \circ \cdots \circ \left(t \frac{d}{dt} + \gamma_{d-s} \right) \text{ for } 1 \leq i \leq m, 1 \leq s \leq d - 1,$$

The following statement is one of our new ingredients.

LEMMA 3.6. (i) The operators $\mathcal{T}_{\mathbf{c}}$ and $t \frac{d}{dt}$ commute.

(ii) Let k be a positive integer. We have:

$$[t^k] \circ \mathcal{T}_{\mathbf{c}} = \mathcal{T}_{\mathbf{c}} \circ A \left(t \frac{d}{dt} - 1 \right) \circ \cdots \circ A \left(t \frac{d}{dt} - k \right) \circ B \left(t \frac{d}{dt} \right)^{-1} \circ \cdots \circ B \left(t \frac{d}{dt} - (k - 1) \right)^{-1} \circ [t^k].$$

Note that $B \left(t \frac{d}{dt} - j \right)$ is not necessarily invertible on the whole of $K[t]$. However, by Hypothesis (2), and Facts (3.2) its restriction to the ideal (t^k) (stable subvector space) is and the right hand side is thus well defined since the morphism $[t^k]$ maps onto (t^k) .

PROOF. (i) is clear, we move to (ii). Let m be a non-negative integer. Since t^m is an eigenvector for all the operators involved (except multiplication by t^k), one gets its image by multiplication of eigenvalues:

$$[t^k] \circ \mathcal{T}_{\mathbf{c}}(t^m) = \frac{1}{c_m} t^{k+m} ,$$

similarly for the right hand side,

$$\begin{aligned} & \mathcal{T}_{\mathbf{c}} \circ A\left(t \frac{d}{dt} - 1\right) \circ \cdots \circ A\left(t \frac{d}{dt} - k\right) \circ B\left(t \frac{d}{dt}\right)^{-1} \circ \cdots \circ B\left(t \frac{d}{dt} - (k-1)\right)^{-1} \circ [t^k](t^m) \\ &= \frac{1}{c_{m+k}} \frac{A(m+k-1) \cdots A(m)}{B(m+k) \cdots B(m+1)} t^{k+m} \end{aligned}$$

Equality then follows from the recurrence relation (3) which yields

$$\frac{1}{c_{m+k}} = \frac{B(m+k) \cdots B(m+1)}{A(m+k-1) \cdots A(m)} \cdot \frac{1}{c_m} ,$$

which achieves the proof of (ii). $\square$

3.2 Construction of Padé approximants

We are now ready for our construction of Padé approximants, of the hypergeometric functions at distinct points. We define the order function ord_{∞} at $z = \infty$ by

$$\text{ord}_{\infty} : K((1/z)) \rightarrow \mathbb{Z} \cup \{\infty\}; \quad \sum_k \frac{c_k}{z^k} \mapsto \min\{k \in \mathbb{Z} \mid c_k \neq 0\} .$$

We first recall the following fact (see [35]):

LEMMA 3.7. *Let r be a positive integer, $f_1(z), \dots, f_r(z) \in (1/z) \cdot K[[1/z]]$ and $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{N}^r$. Put $N := \sum_{i=1}^r n_i$. Let M be a positive integer with $M \geq N$. Then, there exists a family of polynomials $(P_0(z), P_1(z), \dots, P_r(z)) \in K[z]^{r+1} \setminus \{\mathbf{0}\}$ satisfying the following conditions:*

- (i) $\deg P_0(z) \leq M$,
- (ii) $\text{ord}_{\infty}(P_0(z)f_j(z) - P_j(z)) \geq n_j + 1$ for $1 \leq j \leq r$.

DEFINITION 3.8. We say that a vector of polynomials $(P_0(z), P_1(z), \dots, P_r(z)) \in K[z]^{r+1}$ satisfying the properties (i) and (ii) a weight $\mathbf{n}$ and degree M Padé-type approximant of $(f_1, \dots, f_r)$. For such approximants $(P_0(z), P_1(z), \dots, P_r(z))$ of $(f_1, \dots, f_r)$, we call the formal Laurent series $(P_0(z)f_j(z) - P_j(z))_{1 \leq j \leq r}$ weight $\mathbf{n}$ degree M Padé-type approximations of $(f_1, \dots, f_r)$.

The following statements provides for Padé approximation in our situation.

PROPOSITION 3.9. (confer [27, Theorem 5.5]) *We use the same notation as above. For a non-negative integer ℓ , we define polynomials:*

$$(11) \quad H_{\ell}(t) = t^{\ell} \prod_{i=1}^m (t - \alpha_i)^{dn} ,$$

$$(12) \quad P_{\ell}(z) = \left[\frac{1}{(n-1)!d'} \right] \circ \text{Eval}_z \circ \mathcal{T}_{\mathbf{c}} \circ \bigcirc_{j=1}^{n-1} B\left(t \frac{d}{dt} + j\right) (H_{\ell}(t)) ,$$

$$(13) \quad P_{\ell, i, s}(z) = \varphi_{i, s} \left(\frac{P_{\ell}(z) - P_{\ell}(t)}{z - t} \right) \text{ for } 1 \leq i \leq m, 1 \leq s \leq d ,$$

where $\mathcal{T}_{\mathbf{c}}$ and $\varphi_{i,s}$ are defined in Definition 3.5. Then, $(P_{\ell}(z), P_{\ell,i,s}(z))_{1 \leq i \leq m, 1 \leq s \leq d}$ forms a weight $(n, \dots, n) \in \mathbb{N}^{dm}$ and degree $dmn + \ell$ Padé-type approximant of $(F_s(\alpha_i/z))_{1 \leq i \leq m, 1 \leq s \leq d}$.

PROOF. By the definition of $P_{\ell}(z)$,

$$\deg P_{\ell}(z) = dmn + \ell .$$

Hence the required condition on the degree is verified.

Let k be an integer with $0 \leq k \leq n-1$. Using [48, Lemma 2.3], it is sufficient to prove

$$(*) \quad \varphi_{i,s} \left(t^k P_{\ell}(t) \right) = 0 \quad \text{for } 1 \leq i \leq m, 1 \leq s \leq d .$$

To ease notation, set:

$$\begin{aligned} \mathcal{A} &= \bigcirc_{j=1}^k A \left(t \frac{d}{dt} - j \right), \quad \mathcal{B} = \bigcirc_{j=1}^{n-1-k} B \left(t \frac{d}{dt} + j \right), \\ \mathcal{C} &= \bigcirc_{j=1}^{n-1} B \left(t \frac{d}{dt} + j \right), \quad \mathcal{D} = \bigcirc_{j=0}^{k-1} B \left(t \frac{d}{dt} - j \right), \end{aligned}$$

and note that

$$\bigcirc_{j=1}^{n-1} B \left(t \frac{d}{dt} + (j-k) \right) \circ \mathcal{D}^{-1} = \mathcal{B} .$$

By Lemma 3.6 (ii),

$$\begin{aligned} (n-1)!^{d'} t^k P_{\ell}(t) &= [t^k] \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{C} \left(t^{\ell} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) \\ &= \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{D}^{-1} \circ [t^k] \circ \mathcal{C} \left(t^{\ell} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) . \end{aligned}$$

Now, taking into account Facts 3.2 (iii) applied to $[t^k] \circ \mathcal{C}$,

$$\begin{aligned} (n-1)!^{d'} t^k P_{\ell}(t) &= \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{D}^{-1} \circ \bigcirc_{j=1}^{n-1} B \left(t \frac{d}{dt} + (j-k) \right) \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) \\ &= \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{B} \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) , \end{aligned}$$

Therefore, taking into account Definition 3.5, and Lemma 3.6 (i) for the last equality:

$$\begin{aligned} \varphi_{i,s}((n-1)!^{d'} t^k P_{\ell}(t)) &= \varphi_{i,s} \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{B} \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) \\ &= \varphi_{i,d} \circ \bigcirc_{j=1}^{d-s} \left(t \frac{d}{dt} + \gamma_j \right) \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{B} \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) \\ &= [\alpha_i] \circ \text{Eval}_{\alpha_i} \circ \mathcal{T}_{\mathbf{c}}^{-1} \circ \bigcirc_{j=1}^{d-s} \left(t \frac{d}{dt} + \gamma_j \right) \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{B} \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) \\ &= [\alpha_i] \circ \text{Eval}_{\alpha_i} \circ \bigcirc_{j=1}^{d-s} \left(t \frac{d}{dt} + \gamma_j \right) \circ \mathcal{A} \circ \mathcal{B} \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right) . \end{aligned}$$

Since

$$\deg \left(\prod_{j''=1}^{d-s} (X + \gamma_{j''}) \prod_{j'=1}^k A(X - j') \prod_{j=1}^{n-1-k} B(X + j) \right) \leq d - s + dk + d(n - 1 - k) \leq dn - 1 ,$$

thanks to the Leibniz rule, the differential operator $\bigcirc_{j=1}^{d-s} (t \frac{d}{dt} + \gamma_j) \circ \mathcal{A} \circ \mathcal{B}$ is of order at most $dn - 1$, hence, the polynomial

$$\bigcirc_{j''=1}^{d-s} (t \frac{d}{dt} + \gamma_{j''}) \circ \mathcal{A} \circ \mathcal{B} \left(t^{\ell+k} \prod_{i=1}^m (t - \alpha_i)^{dn} \right)$$

belongs to the ideal $(t - \alpha_i) = \ker \text{Eval}_{\alpha_i}$. Consequently we have (*), hence completing the proof of the proposition^{||}. $\square$

REMARK 3.10. The polynomial $P_\ell(z)$ does not depend on the choice of $\gamma_1, \dots, \gamma_{d-1} \in K$. By contrast, the polynomials $P_{\ell,i,s}(z)$ depend on them.

REMARK 3.11. Let d, m be strictly positive integers. Let $x \in K$, supposed to be non-negative integer and $\alpha_1, \dots, \alpha_m \in K \setminus \{0\}$ be pairwise distinct. Put $A(X) = B(X) = (X + x + 1)^d$ and $c_k = 1/(k + x + 1)^d$. Then,

$$c_{k+1} = c_k \cdot \frac{A(k)}{B(k+1)} .$$

Put $\gamma_1 = \dots = \gamma_{d-1} = x + 1$. This gives us

$$(14) \quad F_s(\alpha_i/z) = \sum_{k=0}^{\infty} \frac{1}{(k + x + 1)^s} \cdot \frac{\alpha_i^{k+1}}{z^{k+1}} = \Phi_s(x, \alpha_i/z) \quad (1 \leq i \leq m, 1 \leq s \leq d) ,$$

where $\Phi_s(x, 1/z)$ is the s -th Lerch function (generalized polylogarithmic function, *confer* [29]). In this case, we have $\mathcal{T}_{\mathbf{c}} = (t \frac{d}{dt} + x + 1)^d / (x + 1)^d$ and

$$P_\ell(z) = \left[\frac{1}{(x + 1)^d \cdot (n - 1)!^d} \right] \circ \text{Eval}_z \bigcirc_{j=1}^n (t \frac{d}{dt} + x + j)^d \left(t^\ell \prod_{i=1}^m (t - \alpha_i)^{dn} \right) .$$

The polynomial $(x + 1)^d / n^d P_\ell(z)$ gives Padé-type approximant of Lerch functions in [28, Theorem 3.8].

4 Non-vanishing of the generalized Wronskian of Hermite type

Throughout this section, we consider the following setting: K is a field of characteristic 0 and $A, B \in K[X]$ are monic polynomials satisfying (2) with $\min\{\deg A, \deg B\} > 0$. Put

$$\max\{\deg A, \deg B\} = d, \quad \deg A = d'', \quad \deg B = d' .$$

^{||}Note that a similar construction was also considered by D. V. Chudnovsky and G. V. Chudnovsky in [27, Theorem 5.5], but without arithmetic application. See also a related work by Matala-aho [56].

By replacing K with an appropriate finite extension, we may assume that $A(X)$ and $B(X)$ are decomposable in K and put

$$A(X) = (X + \eta_1) \cdots (X + \eta_{d''}), \quad B(X) = (X + \zeta_1) \cdots (X + \zeta_{d'}) ,$$

where $\eta_i, \zeta_j \in K \setminus \mathbb{Z}_{\leq 0}$.

We fix the sequence $\mathbf{c} = (c_k)_{k \geq 0}$ satisfying (3) for the given polynomials $A(X)$ and $B(X)$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{d-1}) \in K^{d-1}$. Let us fix a positive integer n . For a non-negative integer ℓ with $0 \leq \ell \leq dm$, recall the polynomials $P_\ell(z), P_{\ell,i,s}(z)$ defined in Proposition 3.9 for these choices of $A, B, \boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. We define column vectors $\vec{p}_\ell(z) \in K[z]^{dm+1}$ by

$$\vec{p}_\ell(z) = {}^t \left(P_\ell(z), P_{\ell,1,1}(z), \dots, P_{\ell,1,d}(z), \dots, P_{\ell,m,1}(z), \dots, P_{\ell,m,d}(z) \right) ,$$

and put

$$\Delta_n(z) = \Delta(z) = \det \left(\vec{p}_0(z) \cdots \vec{p}_{dm}(z) \right) .$$

The aim of this section is to prove the following proposition.

PROPOSITION 4.1. *Assume $\alpha_1, \dots, \alpha_m$ are pairwise distinct and*

$$\eta_i - \zeta_j \quad (1 \leq i \leq d'', 1 \leq j \leq d'),$$

is not a positive integer. Then $\Delta(z) \in K \setminus \{0\}$.

REMARK 4.2. In this remark, we emphasize that though the choices of A, B are crucial, since each set defines fundamentally different special functions, as is the choice of $\boldsymbol{\alpha}$ that fixes the special values studied, in contrast the choice of $\boldsymbol{\gamma}$ is not that significative, indeed the differential operators that $\boldsymbol{\gamma}$ define are linked by simple linear transformations. In particular, the non-vanishing of $\Delta(z)$ *does not* depend on the choice of $\boldsymbol{\gamma}$.

Take $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_{d-1}) \in K^{d-1}$. Denote the K -morphism $\varphi_{F_s(\tilde{\boldsymbol{\gamma}}, \boldsymbol{\alpha}_i/z)}$ by $\tilde{\varphi}_{i,s}$, the polynomial $\tilde{\varphi}_{i,s}((P_\ell(z) - P_\ell(t))/(z - t))$ by $\tilde{P}_{\ell,i,s}(z)$ for $1 \leq i \leq m$ and $1 \leq s \leq d$,

$$\tilde{p}_\ell(z) = {}^t \left(P_\ell(z), \tilde{P}_{\ell,1,1}(z), \dots, \tilde{P}_{\ell,1,d}(z), \dots, \tilde{P}_{\ell,m,1}(z), \dots, \tilde{P}_{\ell,m,d}(z) \right) ,$$

$$\tilde{\Delta}(z) = \det \left(\tilde{p}_0(z) \cdots \tilde{p}_{dm}(z) \right) .$$

Set** for $1 \leq s \leq d$, $D_{s,\boldsymbol{\gamma}} = (X + \gamma_1) \cdots (X + \gamma_{d-s})$ with empty product (for $s = d$) equal to 1, There exists a $(d \times d)$ upper triangular matrix $A(\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}})$ with all the diagonal entries are 1 such that

$$(15) \quad {}^t(D_{1,\tilde{\boldsymbol{\gamma}}}, \dots, D_{d,\tilde{\boldsymbol{\gamma}}}) = A(\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}) \cdot {}^t(D_{1,\boldsymbol{\gamma}}, \dots, D_{d,\boldsymbol{\gamma}}) .$$

**When no confusion can occur, we omit the subscript $\boldsymbol{\gamma}$.

Put the $(dm + 1) \times (dm + 1)$ upper triangular matrix with all diagonal entries are 1 by

$$B(\gamma, \tilde{\gamma}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & A(\gamma, \tilde{\gamma}) & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & A(\gamma, \tilde{\gamma}) \end{pmatrix} .$$

Equation (15) implies

$${}^t(\tilde{\varphi}_{i,1}, \dots, \tilde{\varphi}_{i,d}) = B(\gamma, \tilde{\gamma}) \cdot {}^t(\varphi_{i,1}, \dots, \varphi_{i,d}) \quad 1 \leq i \leq m ,$$

therefore $\tilde{p}_\ell(z) = B(\gamma, \tilde{\gamma})\vec{p}_\ell(z)$ for any $0 \leq \ell \leq dm$. This yields the equality:

$$\tilde{\Delta}(z) = \det B(\gamma, \tilde{\gamma}) \cdot \Delta(z) = \Delta(z) .$$

4.1 First Step

In this subsection, we establish the determinant satisfies $\Delta(z) \in K$. Define column vectors $\vec{q}_\ell \in K^{dm}$ by

$$\vec{q}_\ell = {}^t \left(\varphi_{1,1}(t^n P_\ell(t)), \dots, \varphi_{1,d}(t^n P_\ell(t)), \dots, \varphi_{m,1}(t^n P_\ell(t)), \dots, \varphi_{m,d}(t^n P_\ell(t)) \right)$$

for $0 \leq \ell \leq dm - 1$ and a determinant

$$(16) \quad \Theta = \det \begin{pmatrix} \vec{q}_0 & \cdots & \vec{q}_{dm-1} \end{pmatrix} .$$

LEMMA 4.3. (*confer* [30, Lemma 4.2]). *There exists a non-zero element $c \in K$ with $\Delta(z) = c \cdot \Theta$.*

PROOF. Put $R_{\ell,i,s}(z) = P_\ell(z)F_s(\alpha_i/z) - P_{\ell,i,s}(z)$ and

$$c = \frac{1}{(dm(n+1))!} \left(\frac{d}{dz} \right)^{dm(n+1)} P_{dm}(z) ,$$

be the coefficient of highest degree ($= dm(n+1)$) of the polynomial $P_{dm}(z)$. Consider

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ F_1(\alpha_1/z) & -1 & 0 & \vdots \\ 0 & \ddots & -1 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & F_d(\alpha_m/z) & -1 \end{pmatrix} \begin{pmatrix} \vec{p}_0(z) & \cdots & \vec{p}_{dm}(z) \end{pmatrix} = \begin{pmatrix} P_0(z) & \cdots & P_{dm}(z) \\ R_{0,1,1}(z) & \cdots & R_{dm,1,d}(z) \\ \vdots & \ddots & \vdots \\ R_{0,m,1}(z) & \cdots & R_{dm,m,d}(z) \end{pmatrix} .$$

Since the entries of the first line are (by definition on $P_\ell(z)$) polynomials of degree $dm(n+1)$ and entries of the other lines (by Theorem 3.9) are of valuation at least $n+1$, we can apply [30,

Lemma 3.11 (ii)]. We need only to check the coefficients of highest degree (for the first line) and of minimal valuation (for all the other lines). By construction, the vector of highest degree ($= dm n + dm$) for the first line is $(0, \dots, 0, c)z^{dm n + dm}$ and

$$R_{\ell, i, s}(z) = \sum_{k=n}^{\infty} \frac{\varphi_{i, s}(t^k P_{\ell}(t))}{z^{k+1}} ,$$

for $0 \leq \ell \leq dm$, $1 \leq i \leq m$ and $1 \leq s \leq d$. So, by [30, Lemma 3.11 (ii)]

$$\Delta(z) = \pm c \cdot \det(\varphi_{i, s}(t^n P_{\ell}(t)))_{\substack{0 \leq \ell \leq dm-1 \\ 1 \leq i \leq m, 1 \leq s \leq d}} ,$$

as claimed. $\square$

4.2 Second step

Relying on Lemma 4.3, we study here the value Θ defined in (16). From this subsection, we specify the choice of $\gamma_1, \dots, \gamma_{d-1} \in K$ and take $\gamma_d \in K$ as follows.

CHOICE 4.4. We fix $\gamma_i = \zeta_i$ for $1 \leq i \leq d'$, and choose $\gamma_{d'+1}, \dots, \gamma_d$ arbitrarily (if $d' < d$).

We recall that Proposition 4.1 does not depend on this choice (see Remark 4.2).

Let the dm by dm matrix

$$(17) \quad \mathcal{M}_n := \begin{pmatrix} \text{Eval}_{\alpha_i} \circ \bigcirc_{w=d-s+1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right)^{-1} (t^n H_{\ell}(t)) \\ \text{Eval}_{\alpha_i} \circ \left(t \frac{d}{dt} \right)^{dn+s'-1} (t^n H_{\ell}(t)) \end{pmatrix}_{\substack{0 \leq \ell \leq dm-1 \\ 1 \leq i \leq m \\ d-d'+1 \leq s \leq d \\ 1 \leq s' \leq d-d'}} ,$$

where $H_{\ell}(t)$ is defined in Proposition 3.9. We then simplify the determinant Θ to prove its non-vanishing property.

LEMMA 4.5. *There exist elements $a_{s,0} \in K$ for $d - d' + 1 \leq s \leq d$ such that*

$$\Theta = \frac{\prod_{i=1}^m \alpha_i^d \prod_{s=d-d'+1}^d a_{s,0}^m}{(n-1)!d^{2m}} \cdot \det \mathcal{M}_n .$$

PROOF. Put K -endomorphisms

$$\mathcal{D}_s = \bigcirc_{j=1}^{d-s} \left(t \frac{d}{dt} + \gamma_j \right) , \quad \mathcal{A} = \bigcirc_{j=1}^n A \left(t \frac{d}{dt} - j \right) ,$$

$$\mathcal{B} = \bigcirc_{j=0}^{n-1} B \left(t \frac{d}{dt} - j \right)^{-1} , \quad \mathcal{B}' = \bigcirc_{j=1}^{n-1} B \left(t \frac{d}{dt} + j \right) ,$$

with the convention $\mathcal{D}_d = \text{Id}$.

$$\begin{aligned} (n-1)!^{d'} \varphi_{i, s}(t^n P_{\ell}) &= \varphi_{i, d} \circ \mathcal{D}_s \circ [t^n] \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{B}'(H_{\ell}) \\ &= \varphi_{i, d} \circ \mathcal{D}_s \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{B} \circ [t^n] \circ \mathcal{B}'(H_{\ell}) \\ &= [\alpha_i] \circ \text{Eval}_{\alpha_i} \circ \mathcal{T}_{\mathbf{c}}^{-1} \circ \mathcal{D}_s \circ \mathcal{T}_{\mathbf{c}} \circ \mathcal{A} \circ \mathcal{B} \circ [t^n] \circ \mathcal{B}'(H_{\ell}) \\ &= [\alpha_i] \circ \text{Eval}_{\alpha_i} \circ \mathcal{D}_s \circ \mathcal{A} \circ \mathcal{B} \circ \bigcirc_{j=1}^{n-1} B \left(t \frac{d}{dt} + (j - n) \right) (t^n H_{\ell}) \\ &= [\alpha_i] \circ \text{Eval}_{\alpha_i} \circ \mathcal{D}_s \circ \mathcal{A} \circ B \left(t \frac{d}{dt} \right)^{-1} (t^n H_{\ell}) . \end{aligned}$$

We now consider the euclidean division of $\mathcal{A}$ by $B(t\frac{d}{dt})/\mathcal{D}_s$ in $K[t, \frac{d}{dt}]$ with the convention $B(t\frac{d}{dt})/\mathcal{D}_s = B(t\frac{d}{dt})$ if $d' < d - s$:

$$\mathcal{A} = \mathcal{Q}_s \circ B(t\frac{d}{dt})/\mathcal{D}_s + \mathcal{R}_s ,$$

so that

$$\varphi_{i,s}(t^n P_\ell) = \left[\frac{\alpha_i}{(n-1)!d'} \right] \circ \text{Eval}_{\alpha_i} \circ \left[\mathcal{Q}_s + \mathcal{D}_s \circ \mathcal{R}_s \circ B\left(t\frac{d}{dt}\right)^{-1} \right] (t^n H_\ell) .$$

Note that

$$\text{ord}(\mathcal{Q}_s) = n \deg(A) + \text{ord}(\mathcal{D}_s) - \deg(B) = d''n - d' + d - s ,$$

and distinguish two cases:

Case I: $d - s < d'$. In this case, $\text{ord}(\mathcal{Q}_s) < d''n \leq dn$, so $\mathcal{Q}_s$ is a differential operator of order $\leq dn - 1$, so, by Leibniz rule,

$$\mathcal{Q}_s(t^n H_\ell) \in (t - \alpha_i)$$

since $t^n H_\ell$ belongs to the ideal $(t - \alpha_i)^{dn}$ and since $\ker \text{Eval}_{\alpha_i} = (t - \alpha_i)$,

$$\text{Eval}_{\alpha_i} \circ \mathcal{Q}_s(t^n H_\ell) = 0$$

and we can simplify

$$\varphi_{i,s}(t^n P_\ell) = \left[\frac{\alpha_i}{(n-1)!d'} \right] \circ \text{Eval}_{\alpha_i} \circ \left[\mathcal{D}_s \circ \mathcal{R}_s \circ B\left(t\frac{d}{dt}\right)^{-1} \right] (t^n H_\ell) .$$

Put the polynomials $D_s(X) = \prod_{j=1}^{d-s}(X + \gamma_j)$ and $R_s(X) \in K[X]$ such that $R_s(t\frac{d}{dt}) = \mathcal{R}_s$. We now choose the following natural K -basis for the quotient $K[X]/(B/D_s)$:

$$e_0 = 1, e_1 = (X + \gamma_{d-s+1}), \dots, e_j = \prod_{\ell=1}^j (X + \gamma_{d-s+\ell}), \dots, e_{d'+s-d-1} = (X + \gamma_{d-s+1}) \dots (X + \gamma_{d'-1})$$

and write R_s in this basis:

$$R_s = \sum_{j=0}^{d'+s-d-1} a_{s,j} e_j$$

so that

$$D_s R_s B^{-1} = \sum_{j=0}^{d'+s-d-1} a_{s,j} \left[\prod_{\ell=d-s+j+1}^{d'} (X + \gamma_\ell) \right]^{-1} .$$

Define a column vector:

$$C = \begin{pmatrix} (X + \gamma_{d'})^{-1} \\ [(X + \gamma_{d'-1})(X + \gamma_{d'})]^{-1} \\ \vdots \\ \left[\prod_{\ell=j+1}^{d'} (X + \gamma_\ell) \right]^{-1} \\ \vdots \\ [(X + \gamma_1) \dots (X + \gamma_{d'})]^{-1} \end{pmatrix} .$$

Then, the column vector

$$(D_s R_s B^{-1})_{d-d'+1 \leq s \leq d} = \begin{pmatrix} a_{d-d'+1,0} & 0 & 0 & \cdots & 0 \\ a_{d-d'+2,1} & a_{d-d'+2,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ a_{d,d'-1} & \cdots & \cdots & \cdots & a_{d,0} \end{pmatrix} \cdot C .$$

Denote by T the above upper triangular matrix and note that the diagonal coefficients satisfy:

$$(18) \quad a_{s,0} = R_s(-\gamma_{d-s+1}) = \prod_{j=0}^{n-1} A(-\gamma_{d-s+1} - j) .$$

We can now compute taking into account the fact that T is a scalar matrix, hence commutes with $[\alpha_i] \circ \text{Eval}_{\alpha_i}$,

$$\Theta = \det(\varphi_{i,s}(t^n P_\ell))_{i,s} = \frac{\prod_{i=1}^m \alpha_i^d \prod_{s=d-d'+1}^d a_{s,0}^m}{(n-1)! d d' m} \left(\text{Eval}_{\alpha_i} \circ_{w=d-s+1}^d \left(t \frac{d}{dt} + \gamma_w \right)^{-1} (t^n H_\ell(t)) \right)_{\substack{0 \leq \ell \leq dm-1 \\ 1 \leq i \leq m \\ 1 \leq s \leq d}} .$$

Case II: $d - s \geq d'$. Note that this case can occur only if $d = d'' > d'$. In this case, the polynomial B divides D_s , hence, the operator $\mathcal{A} \circ \mathcal{D}_s \circ B(t \frac{d}{dt})^{-1} \in K[t \frac{d}{dt}]$ that we can readily write in the basis $((t \frac{d}{dt})^n)_{n \geq 0}$,

$$\varphi_{i,s}(t^n P_\ell) = \mathcal{A} \mathcal{D}_s B^{-1}(t^n H_\ell(t)) = \sum_{k=0}^{dn+d-d'-s} b_{s,k} \left(t \frac{d}{dt} \right)^k (t^n H_\ell(t)) .$$

Again, the terms of degree $\leq dn - 1$ lie in the kernel of Eval_{α_i} and this simplifies in

$$\varphi_{i,s}(t^n P_\ell) = \sum_{k=dn}^{dn+d-d'-s} b_{s,k} \left(t \frac{d}{dt} \right)^k (t^n H_\ell(t)) .$$

Since all the polynomials $\mathcal{A}, \mathcal{D}_s, B(t \frac{d}{dt})$ are monic, we again can transform the expression via a triangular matrix (but this time with diagonal entries 1 since the terms of highest degree $b_{s,dn+d-d'+s} = 1$). This yields the lemma in case II. $\square$

We now make the following.

CHOICE 4.6. Assume that for any $1 \leq i \leq d''$ and any $1 \leq j \leq d'$,

$$\eta_i - \zeta_j$$

is not a positive integer.

COROLLARY 4.7. Under Choice 4.4 and 4.6 the factor $\prod_{s=d-d'+1}^d a_{s,0}^m$ appearing in Lemma 4.5 is non-zero.

PROOF. By Equation (18), $a_{s,0} = \prod_{j=1}^{n-1} A(-\gamma_{d-s+1} - j)$ for $d - d' + 1 \leq s \leq d$. Therefore the choice of γ_i implies the assertion. $\square$

4.3 Final step

We keep the notation of Subsection 4.2 and assume $\alpha_1, \dots, \alpha_m$ are pairwise distinct. We now prove the non-vanishing of $\det \mathcal{M}_n$ without explicitly computing its exact value. The strategy we employ is a differential analogue of the approach used in [50, Section 8]. We continue Choice 4.4 for γ_w , in particular we have $\gamma_w \notin \mathbb{Z}_{\leq 0}$ for $1 \leq w \leq d'$.

Denote the K -morphisms in the definition of $\mathcal{M}_n$ (see Equation (17)) by

$$\psi_{i,s} : K[t] \longrightarrow K; \quad t^k \mapsto \begin{cases} \prod_{w=d-s+1}^{d'} \frac{\alpha_i^k}{k + \gamma_w} & d - d' + 1 \leq s \leq d \\ k^{dn+s-1} \alpha_i^k & 1 \leq s \leq d - d'. \end{cases}$$

Notice

$$(19) \quad \psi_{i,s} = \begin{cases} \text{Eval}_\alpha \circ \bigcirc_{w=d-s+1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right)^{-1} & d - d' + 1 \leq s \leq d \\ \text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^{dn+s-1} & 1 \leq s \leq d - d'. \end{cases}$$

Let $\mathbf{q} = {}^t(q_\ell)_{0 \leq \ell \leq dm-1} \in K^{dm}$ be a vector satisfying

$$(20) \quad \mathcal{M}_n \cdot \mathbf{q} = 0.$$

Put

$$(21) \quad Q(t) := t^n \sum_{\ell=0}^{dm-1} q_\ell H_\ell(t).$$

Using linearity of the morphisms $\psi_{i,s}$, Equation (20) implies

$$Q(t) \in \bigcap_{i=1}^m \bigcap_{s=1}^d \ker \psi_{i,s}.$$

The regularity of the matrix $\mathcal{M}_n$ is equivalent to the following statement:

PROPOSITION 4.8. *We have $Q(t) = 0$.*

To prove Proposition 4.8, we study the kernel of $\psi_{i,s}$. First we consider the case $d - d' + 1 \leq s \leq d$.

LEMMA 4.9. *Let $r \in \mathbb{N}$, $\alpha \in K \setminus \{0\}$ and $\gamma_1, \dots, \gamma_r \in K \setminus \mathbb{Z}_{\leq 0}$. Then the following identity holds.*

$$\bigcap_{s=1}^r \bigcirc_{w=1}^s \left(t \frac{d}{dt} + \gamma_w \right) \circ [t - \alpha](K[t]) = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right) \circ [(t - \alpha)^r](K[t]) .$$

PROOF. The Leibniz formula yields

$$\bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right) \circ [(t - \alpha)^r](K[t]) \subseteq \bigcap_{s=1}^r \bigcirc_{w=1}^s \left(t \frac{d}{dt} + \gamma_w \right) \circ [t - \alpha](K[t]) .$$

Let us show the opposite inclusion. Let $P(t) \in K[t]$. Assume there exist polynomials $P_s(t)$ for $1 \leq s \leq r$ such that

$$(22) \quad P(t) = \bigcirc_{w=1}^s \left(t \frac{d}{dt} + \gamma_w \right) \circ [t - \alpha](P_s(t)) \quad (1 \leq s \leq r) .$$

It is sufficient to prove

$$(23) \quad P_s(t) \in (t - \alpha)^{s-1} K[t] \quad (1 \leq s \leq r) ,$$

since above relation implies

$$P(t) = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right) \circ [t - \alpha](P_r(t)) \in \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right) \circ [(t - \alpha)^r](K[t]) .$$

Let us show Equation (23) by induction on r . There is nothing to prove for $r = 1$. Assume Equation (23) holds for $r \geq 1$. Let us take $r + 1$. Then the induction hypothesis for

$$\begin{aligned} P(t) &= \bigcirc_{w=1}^s \left(t \frac{d}{dt} + \gamma_w \right) \circ [t - \alpha](P_s(t)) \quad (1 \leq s \leq r) , \\ \left(t \frac{d}{dt} + \gamma_1 \right)^{-1} (P(t)) &= \bigcirc_{w=2}^s \left(t \frac{d}{dt} + \gamma_w \right) \circ [t - \alpha](P_s(t)) \quad (2 \leq s \leq r + 1) , \end{aligned}$$

assert that $P_r(t), P_{r+1}(t) \in (t - \alpha)^{r-1} K[t]$. Put

$$P_r(t) = (t - \alpha)^{r-1} \tilde{P}_r(t) \quad \text{and} \quad P_{r+1}(t) = (t - \alpha)^{r-1} \tilde{P}_{r+1}(t) .$$

Equation (22) for $r + 1$ implies that

$$(t - \alpha)^r \tilde{P}_r(t) = \left(t \frac{d}{dt} + \gamma_{r+1} \right) \circ [(t - \alpha)^r](\tilde{P}_{r+1}(t)) = rt(t - \alpha)^{r-1} \tilde{P}_{r+1}(t) + (t - \alpha)^r (t \tilde{P}'_{r+1}(t) + \gamma_r) .$$

Since $\alpha \neq 0$, this allows us to get $\tilde{P}_{r+1}(t)$ is divisible by $(t - \alpha)$ and thus we get $P_{r+1}(t) \in (t - \alpha)^r K[t]$. This completes the proof of Equation (23). We complete the proof of the lemma. $\square$

LEMMA 4.10. *Let $d - d' + 1 \leq s \leq d$ be an integer. Then we have*

$$\ker \psi_{i,s} = \bigcirc_{w=d-s+1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) (t - \alpha_i) K[t] .$$

PROOF. It is easy to see that

$$(24) \quad \bigcirc_{w=d-s+1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) (t - \alpha_i) K[t] \subset \ker \psi_{i,s} .$$

Let us take $P(t) \in \ker \psi_{i,s}$. Since $\gamma_w \notin \mathbb{Z}_{\leq 0}$, we notice $(t \frac{d}{dt} + \gamma_w) \in \text{Aut}_K(K[t])$. This shows there exists a polynomial $\tilde{P}(t)$ with

$$P(t) = \bigcirc_{w=d-s+1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) (\tilde{P}(t)) .$$

By definition of $\psi_{i,s}$, we have

$$0 = \psi_{i,s}(P(t)) = \text{Eval}_{\alpha_i}(\tilde{P}(t)) .$$

This implies $\tilde{P}(t) \in (t - \alpha_i) K[t]$ and thus, we obtain the desire equality. $\square$

COROLLARY 4.11. *The following equalities hold.*

$$\bigcap_{i=1}^m \bigcap_{s=d-d'+1}^d \ker \psi_{i,s} = \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ \left[\prod_{i=1}^m (t - \alpha_i)^{d'} \right] (K[t]) .$$

PROOF. By combining Lemma 4.9 (i) and Lemma 4.10 (i), it is sufficient to prove

$$\bigcap_{i=1}^m \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ [(t - \alpha_i)^{d'}](K[t]) = \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ \left[\prod_{i=1}^m (t - \alpha_i)^{d'} \right] (K[t]) .$$

By definition we see that the right hand side is contained in the left hand side. Let $P \in \bigcap_{i=1}^m \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ [(t - \alpha_i)^{d'}](K[t])$. Then there exist polynomials $P_i \in K[t]$ for $1 \leq i \leq m$ such that

$$P = \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) ((t - \alpha_i)^{d'} P_i(t)) .$$

Since $\bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \in \text{Aut}_K(K[t])$, the above equality implies

$$(t - \alpha_1)^{d'} P_1(t) = \cdots = (t - \alpha_m)^{d'} P_m(t) ,$$

and thus $P_i(t) \in \prod_{j \neq i} (t - \alpha_j)^{d'} K[t]$ for $1 \leq i \leq m$. This leads us to get

$$P \in \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ \left[\prod_{i=1}^m (t - \alpha_i)^{d'} \right] (K[t]) .$$

□

LEMMA 4.12. *Let $P(t), Q(t) \in K[t]$ and $r, n \in \mathbb{N}$. Let $\alpha, \gamma_1, \dots, \gamma_r \in K \setminus \{0\}$ and $\beta \in \{0, \alpha\}$. Assume*

$$Q(t)(t - \beta)^n = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right)^r \circ [(t - \alpha)^r](P(t)) .$$

Then $P(t) \in (t - \beta)^n K[t]$.

PROOF. Let us prove the lemma by induction on n . Firstly we consider the case $\beta = 0$. Let $n = 1$. Then, since $t \frac{d}{dt}(P(t)) \in tK[t]$, we have

$$Q(t)t = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right)^r \circ [(t - \alpha)^r](P(t)) \in \prod_{w=1}^r \gamma_w (t - \alpha)^r P(t) + tK[t].$$

Since $\alpha \in K \setminus \{0\}$, the above equality leads us to get $P(t) \in tK[t]$. Assume the statement holds for $n \geq 1$. Let us take $n+1$. The induction hypothesis implies $P(t) \in t^n K[t]$. Put $P(t) = t^n \tilde{P}(t)$. This allows to show

$$Q(t)t^{n+1} = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right)^r \circ [(t - \alpha)^r](P(t)) \in \prod_{w=1}^r \gamma_w (t - \alpha)^n t^n \tilde{P}(t) + t^{n+1} K[t].$$

Since $\alpha \in K \setminus \{0\}$, the above equality leads us to get $\tilde{P}(t) \in tK[t]$ and therefore $P(t) \in t^{n+1} K[t]$.

Next we consider the case $\beta = \alpha$. Let $n = 1$. A straightforward computation yields

$$Q(t)(t - \alpha) = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right)^r \circ [(t - \alpha)^r](P(t)) \in r!t^r P(t) + (t - \alpha)K[t].$$

This allows us to get $P(t) \in (t - \alpha)K[t]$. Assume the statement holds for $n \geq 1$. Let us take $n + 1$. The induction hypothesis implies $P(t) \in (t - \alpha)^n K[t]$. Put $P(t) = (t - \alpha)^n \tilde{P}(t)$. We thus obtain

$$Q(t)(t - \alpha)^{n+1} = \bigcirc_{w=1}^r \left(t \frac{d}{dt} + \gamma_w \right)^r \circ [(t - \alpha)^{r+n}](\tilde{P}(t)) \in (n+1)_r t^r (t - \alpha)^n \tilde{P}(t) + (t - \alpha)^{n+1} K[t].$$

The above equality implies $\tilde{P}(t) \in (t - \alpha)K[t]$ and thus $P(t) \in (t - \alpha)^{n+1} K[t]$. This completes the proof of the lemma. $\square$

Next, we consider the kernel of $\varphi_{i,s}$ for $1 \leq s \leq d - d'$.

LEMMA 4.13. *Let N, r be positive integers and $\alpha \in K \setminus \{0\}$. Then we have*

$$\bigcap_{s=1}^r \ker \text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^{N+s} \cap (t - \alpha)^N K[t] \subseteq (t - \alpha)^{N+r} K[t] .$$

PROOF. It suffices to show that

$$(25) \quad P(t) = \sum_{j=0}^{r-1} p_j (t - \alpha)^{N+j} \in \bigcap_{s=1}^r \ker \text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^{N+s-1} \Rightarrow P(t) = 0 .$$

We prove $p_j = 0$ by induction on j . Applying $\text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^N$ to $P(t)$ and using the Leibniz rule, we obtain

$$\text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^N (P) = p_0 \alpha^N N! = 0 ,$$

so $p_0 = 0$. Now fix $1 \leq j < r - 1$ and assume $p_0 = p_1 = \dots = p_j = 0$. Then applying $\text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^{N+j+1}$ to $P(t)$ yields

$$\text{Eval}_\alpha \circ \left(t \frac{d}{dt} \right)^{N+j+1} (P) = p_{j+1} \alpha^{N+j+1} (N + j + 1)! = 0 ,$$

so $p_{j+1} = 0$. This completes the induction and proves (25). $\square$

COROLLARY 4.14. *Let $P(t) \in \bigcap_{i=1}^m \bigcap_{s=1}^{d-d'} \ker \psi_{i,s}$. Assume $P(t)$ is divisible by $\prod_{i=1}^m (t - \alpha_i)^{dn}$. Then we have*

$$P(t) \in \prod_{i=1}^m (t - \alpha_i)^{dn+d-d'} K[t] .$$

PROOF. From (19), we have

$$P \in \bigcap_{s=1}^{d-d'} \ker \text{Eval}_{\alpha_i} \circ \left(t \frac{d}{dt} \right)^{dn+s-1}$$

for each $1 \leq i \leq m$. By Lemma 4.13, it follows that

$$P \in (t - \alpha_i)^{dn+d-d'} K[t] \quad \text{for all } i = 1, \dots, m .$$

Since the α_i are pairwise distinct, we conclude the assertion. $\square$

Proof of Proposition 4.8. We recall the polynomial $Q(t) \in \bigcap_{i=1}^m \bigcap_{s=1}^d \ker \psi_{i,s}$ (see Equation (21)). By definition, $Q(t)$ is divisible by t^n . Applying Corollary 4.14 to $Q(t) \in \bigcap_{i=1}^m \bigcap_{s=1}^{d-d'} \ker \psi_{i,s}$, we deduce that $Q(t)$ is divisible by

$$t^n \prod_{i=1}^m (t - \alpha_i)^{dn+d-d'} .$$

Corollary 4.11, combined with the above divisibility, implies that

$$(26) \quad Q(t) \in t^n \prod_{i=1}^m (t - \alpha_i)^{dn+d-d'} K[t] \cap \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ \left[\prod_{i=1}^m (t - \alpha_i)^{d'} \right] (K[t]) .$$

From (26), there exist polynomials $P(t), \tilde{Q}(t) \in K[t]$ such that

$$Q(t) = t^n \prod_{i=1}^m (t - \alpha_i)^{dn+d-d'} \tilde{Q}(t) = \bigcirc_{w=1}^{d'} \left(t \frac{d}{dt} + \gamma_w \right) \circ \left[\prod_{i=1}^m (t - \alpha_i)^{d'} \right] (P(t)) .$$

Applying Lemma 4.12 repeatedly with $r = d'$ and using the fact that the $(t - \alpha_i)$ are pairwise coprime (since the α_i are distinct), we deduce that

$$P(t) \in t^n \prod_{i=1}^m (t - \alpha_i)^{dn+d-d'} K[t] .$$

Hence, if $Q \neq 0$, the degree of Q satisfies

$$\deg Q \geq n + (dn + d - d')m + d'm = dm(n + 1) + n .$$

On the other hand, by the definition of Q (see Equation (21)), we have

$$\deg Q \leq n + (dm - 1) + dm n = dm(n + 1) + n - 1 .$$

This contradiction implies $Q(t) = 0$. □

We now finish the proof of Proposition 4.1.

Proof of Proposition 4.1. Combining Lemmas 4.3 and 4.5 yields that there exist $c \in K \setminus \{0\}$ and $a_{0,s} \in K$ for $d - d' + 1 \leq s \leq d$ such that

$$\Delta(z) = c \cdot \frac{\prod_{i=1}^m \alpha_i^d \prod_{s=d-d'+1}^d a_{s,0}^m}{(n-1)!^{d^2 m}} \cdot \det \mathcal{M}_n .$$

Corollary 4.7 ensures that the non-vanishing of the term except for $\det \mathcal{M}_n$. Finally, since the α_i are pairwise distinct, Proposition 4.8 ensures $\det \mathcal{M}_n \neq 0$. □

5 Estimates

We keep the notations of Section 4. We further assume that K is a number field and $\eta_1, \dots, \eta_{d''}, \zeta_1, \dots, \zeta_{d'}$ be rational numbers which are not negative integers with $\eta_i - \zeta_j \notin \mathbb{N}$ for $1 \leq i \leq d'', 1 \leq j \leq d'$.

Assume

$$A(X) = (X + \eta_1) \cdots (X + \eta_{d''}), \quad B(X) = (X + \zeta_1) \cdots (X + \zeta_{d'})$$

when $d'd'' > 0$. Let $\mathbf{c} = (c_k)_{k \geq 0}$ be the sequence satisfying $c_0 = 1$ and (3) for the given polynomials $A(X)$ and $B(X)$. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{d-1}) \in K^{d-1}$. Unless otherwise stated, let us consider $\alpha_1, \dots, \alpha_m$ as variables. For a non-negative integer ℓ with $0 \leq \ell \leq dm$, recall the polynomials $P_\ell(z), P_{\ell,i,s}(z)$ defined in Proposition 3.9 for the given data.

We start this section with elementary considerations, useful during intermediary estimates for the norm of our auxiliary polynomials.

LEMMA 5.1. *Let k be a positive integer and η, ζ be a strictly positive rational numbers.*

(i) *One has*

$$\frac{1}{(\eta)_k} \leq \frac{(\lfloor \eta \rfloor + k) \lfloor \eta \rfloor!}{\eta \cdot (\lfloor \eta \rfloor + k)!}.$$

(ii) *One has*

$$k! \leq (1 + \zeta)_k \leq \frac{(\lceil \zeta \rceil + k)!}{\lceil \zeta \rceil!}.$$

(iii) *For any positive integer a , one has*

$$\frac{1}{(a + k)!} \leq \frac{1}{k! k^a}.$$

LEMMA 5.2. *Let n, k be positive integers and η, ζ be non-zero rational numbers. Recall $\mu_n(\zeta) = \text{den}(\zeta)^n \cdot \prod_{\substack{q:\text{prime} \\ q|\text{den}(\zeta_j)}} q^{\lfloor \frac{n}{q-1} \rfloor}$, $\mu(\zeta) = \text{den}(\zeta) \prod_{q:\text{prime}} q^{\frac{1}{q-1}}$.*

(i) *One has*

$$\mu_n(\zeta) \cdot \frac{(\zeta)_k}{k!} \in \mathbb{Z} \quad \text{for } 0 \leq k \leq n.$$

(ii) *One has*

$$\mu_n(\zeta) = \mu_n(\zeta + k) \quad \text{and} \quad \mu_{n+k}(\zeta) \text{ is divisible by } \mu_n(\zeta) \mu_k(\zeta).$$

(iii) *For a non-negative integer n , put*

$$D_n = \text{den} \left(\frac{(\eta)_0}{(\zeta)_0}, \dots, \frac{(\eta)_n}{(\zeta)_n} \right).$$

One has

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log D_n \leq \log \mu(\eta) + \text{den}(\zeta)^{\dagger\dagger}.$$

^{††}Using Dirichlet's prime number theorem on arithmetic progression (see [6]), we may improve the upper bound as

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log D_n \leq \log \mu(\eta) + \frac{\text{den}(\zeta)}{\varphi(\text{den}(\zeta))} \sum_{\substack{j=1 \\ (j, \text{den}(\zeta))=1}}^{\text{den}(\zeta)} \frac{1}{j},$$

where φ denotes Euler's totient function.

(iv) Put $\zeta = c/d$ where c, d are coprime integers with $d > 0$. Put $N_n = c(c+d) \cdots (c+d(n-1))$. Let p be a prime number with $p \mid N_n$. One has

$$1 \leq \left| \frac{n!}{(\zeta)_n} \right|_p \leq |c| + d(n-1) .$$

PROOF. (i) This property is proven in [10, Lemma 2.2].

(ii) We directly obtain the assertion by the definition of $\mu_n(\zeta)$.

(iii) Put

$$D_{1,n} = \text{den} \left(\frac{(\eta)_0}{0!}, \dots, \frac{(\eta)_n}{n!} \right), \quad D_{2,n} = \text{den} \left(\frac{0!}{(\zeta)_0}, \dots, \frac{n!}{(\zeta)_n} \right) .$$

Since inequality $D_n \leq D_{1,n} D_{2,n}$ holds, the assertion is deduced from

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log D_{1,n} \leq \log \mu(\eta), \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log D_{2,n} \leq \text{den}(\zeta) .$$

The first inequality is a consequence of (i). Second inequality is shown in [49, Lemma 4.1], however, we explain here this proof in an abbreviated form, to let our article be self-contained. This proof is originally indicated by Siegel [72, p.57,58]. Put $d = \text{den}(\zeta)$, $c = d \cdot \zeta$. We set $N_k = c(c+d) \cdots (c+(k-1)d)$. Let p be a prime number with $p \mid N_k$. The following three properties hold.

(a) We have $\text{GCD}(p, d) = 1$. For any integers i, ℓ with $\ell > 0$, there exists exactly one integer ν with $0 \leq \nu \leq p^\ell - 1$ and such that $p^\ell \mid c + (i + \nu)d$.

(b) Let ℓ be a strictly positive integer with $|c| + (k-1)d < p^\ell$. Then, $c + id$ for $0 \leq i \leq k-1$ is not divisible by p^ℓ .

(c) Set $C_{p,k} = \lfloor \log(|c| + (k-1)d) / \log(p) \rfloor$. Then,

$$v_p(k!) = \sum_{\ell=1}^{C_{p,k}} \left\lfloor \frac{k}{p^\ell} \right\rfloor \leq v_p(N_k) \leq \sum_{\ell=1}^{C_{p,k}} \left(1 + \left\lfloor \frac{k}{p^\ell} \right\rfloor \right) = v_p(k!) + C_{p,k} ,$$

where v_p denotes the p -adic valuation. This allows us

$$(27) \quad \log D_{2,n} = \sum_{p \mid N_n} \max_{0 \leq k \leq n} \log \left| \frac{k!}{(\zeta)_k} \right|_p \leq \log(|c| + d(n-1)) \sum_{p \leq c+d(n-1)} 1 .$$

Denote $\pi(x) = \#\{p : \text{prime} \mid p < x\}$ for $x > 0$. Then by prime number theorem

$$\limsup_{n \rightarrow \infty} \frac{\log(|c| + d(n-1)) \pi(c + d(n-1))}{n} = d ,$$

and we deduce desire inequality (*confer* [66]).

(iv) We keep the notation in the proof of (iii). The assertion (c) in the proof of (iii) yields

$$1 \leq \left| \frac{n!}{N_n} \right|_p \leq p^{C_{p,n}} ,$$

as claimed. □

Throughout the section, the small o -symbol $o(1)$ and $o(n)$ and the large O -symbol $O(n)$ refer when n tends to infinity. Put $\varepsilon_v = 1$ if $v \mid \infty$ and 0 otherwise.

Let I be a non-empty finite set of indices, $R = K[X_i]_{i \in I}$ be the polynomial ring over K in indeterminate X_i . We set $\|P\|_v = \max\{|a|_v\}$ where a runs in the coefficients of P for any place $v \in \mathfrak{M}_K$. The degree of an element of R is as usual the total degree.

We start estimating Padé approximants, but the method differs from the previous ones in [29, Lemma 5.4] and [30, Lemma 5.2]. The previous method involved estimating the norm of the operator using submultiplicativity, while this time we are estimating the norm taking advantage of the fact that most of the operators involved in the construction of P_ℓ , $P_{\ell,i,s}$, $R_{\ell,i,s}$ and related polynomials are defined via diagonally acting linear operators (see remark 4 of [29]). Though not necessary for P_ℓ this becomes necessary for the others.

Recall our kernel polynomial (defined in Proposition 3.9)

$$H_\ell(t) = t^\ell \prod_{i=1}^m (t - \alpha_i)^{d_n} ,$$

and

$$P_\ell(t) = \mathcal{T}_c \circ_{j=1}^{d'} S_{n-1, \zeta_j} (H_\ell(t)) ,$$

where $S_{n-1, \zeta_j} = \frac{1}{(n-1)!} \prod_{l=1}^{n-1} (t \frac{d}{dt} + \zeta_j + l)$.

Also recall that by definition, $\mathcal{T}_c$ and S_{n-1, ζ_j} both act diagonally on the standard basis of $K[t]$ with eigenvalues respectively

$$\lambda_{\mathcal{T}_c}(k) = \frac{\prod_{j=1}^{d'} (\zeta_j + 1)_k}{\prod_{j=1}^{d''} (\eta_j)_k} , \quad \lambda_{S_{n-1, \zeta_j}}(k) = \frac{(k + \zeta_j + 1)_{n-1}}{(n-1)!} .$$

DEFINITION 5.3. An admissible error term is a sequence $(c_{n,v})_{v \in \mathfrak{M}_K, n \in \mathbb{N}}$ of positive real numbers such that:

$$\frac{1}{n} \sum_{v \in \mathfrak{M}_K} \log(c_{n,v}) \xrightarrow{n \rightarrow \infty} 0 .$$

The product of two admissible error terms $(c_{n,v}) \cdot (c'_{n,v})$ is done term by term and is equal to $(c_{n,v} \cdot c'_{n,v})_{n,v}$. The product of finitely many admissible error terms is still admissible, and if $c_{n,v} = 1$ for all but finitely many $v \in \mathfrak{M}_K$ and is a $o(n)$ for those places where it is not = 1, it is obviously admissible.

We now put the constant

$$(28) \quad B(d, d', m) = B = dm \log(2) + d' \left(\log(dm + 1) + dm \log \left(\frac{dm + 1}{dm} \right) \right) ,$$

which depends on d, d' and m .

LEMMA 5.4. *There is an admissible effectively computable error term $(c_{n,v} = c_{n,v}(\zeta, \eta))_{v \in \mathfrak{M}_K, n \in \mathbb{N}}$ depending only on the given data ζ, η (also depends implicitly on the parameter m) such that*

(i) Assume v is Archimedean, then

$$\max\{\|P_\ell\|_v, \|P_{\ell,i,s}\|_v\} \leq c_{n,v} \exp(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n) |(dmn)!|_v^{\max\{0, d'-d''\}} .$$

(ii) Assume v is ultrametric, then, setting

$$D_{k,l,n}(\zeta, \eta) = \frac{\prod_{j=1}^{d'} (\zeta_j + l + 1)_{k-l+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j + l)_{k-l}} ,$$

and

$$A_{n,v}(\zeta, \eta) = \max\{|D_{k,l,n}(\zeta, \eta)|_v ; 0 \leq l \leq k-1, 0 \leq k \leq dmn + dm\} .$$

One has:

$$\max\{\|P_\ell\|_v, \|P_{\ell,i,s}\|_v\} \leq c_{n,v} A_{n,v}(\zeta, \eta) .$$

PROOF. We start with the case of P_ℓ . Now, by the observation above, the eigenvalue of $\mathcal{T}_c \circ \bigcirc_{j=1}^{d'} S_{n-1, \zeta_j}$ corresponding to the monomial t^k is:

$$\frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} .$$

Hence,

$$\|P_\ell\|_v \leq \max_{0 \leq k \leq dmn+l} \left| \frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} \right|_v \|H_\ell\|_v .$$

Since for finite places, $\|H_\ell\|_v \leq 1$, (ii) is proven for P_ℓ , with $c_{n,v} = 1$. We now turn to the numerators of the Padé approximation system *i. e.* the polynomials $P_{\ell,i,s}$.

By the definition, we have $P_{\ell,i,s}(z) = \varphi_{i,s} \left(\frac{P_\ell(z) - P_\ell(t)}{z-t} \right)$ and

$$\varphi_{i,s} = [\alpha_i] \circ \text{Eval}_{t=\alpha_i} \circ \mathcal{T}_c^{-1} \bigcirc_{u=1}^{d-s} (t \frac{d}{dt} + \gamma_u) .$$

Setting $\Gamma(Q) = \frac{Q(z) - Q(t)}{z-t}$, we have

$$P_{\ell,i,s}(z) = [\alpha_i] \circ \text{Eval}_{t=\alpha_i} \circ \mathcal{T}_c^{-1} \bigcirc_{u=1}^{d-s} (t \frac{d}{dt} + \gamma_u) \circ \Gamma(P_\ell) .$$

Note that the operators $[\alpha_i]$, $\text{Eval}_{t=\alpha_i}$ are isometries (right shift, substitution of variables). As for the operator Γ , it takes the monomial t^δ to $\sum_{j=0}^{\delta-1} t^{\delta-j-1} z^j$ and is of norm 1 ([29], Lemma 5.2 (iii)). The action of the operators on the monomial t^k

$$\mathcal{T}_c^{-1} \bigcirc_{u=1}^{d-s} (t \frac{d}{dt} + \gamma_u) \Gamma \circ \mathcal{T}_c \circ \bigcirc_{j=1}^{d'} S_{n-1, \zeta_j}$$

is thus given by their eigenvalue

$$\prod_{j=1}^{d'} \frac{(k + \zeta_j + 1)_{n-1}}{(n-1)!} \lambda_{\mathcal{T}_c}(k) \prod_{u=1}^{d-s} (l + \gamma_u) \lambda_{\mathcal{T}_c}^{-1}(l)$$

where l varies between 0 and $k - 1$. Without prejudice, one can multiply $c_{n,v}$ by

$$\max \left\{ \left| \prod_{u=1}^{d-s} (l + \gamma_u) \right|_v ; 0 \leq l \leq dm n + dm - 1 \right\}$$

which is admissible since $\gamma_u \in K$ are fixed and the height of l grows linearly with $\log n$.

We are thus left with estimating:

$$\frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} \frac{\prod_{j=1}^{d''} (\eta_j)_l}{\prod_{j=1}^{d'} (\zeta_j + 1)_l} = \frac{\prod_{j=1}^{d'} (\zeta_j + l + 1)_{k-l+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j + l)_{k-l}} .$$

By definition of $D_{k,l,n}(\zeta, \eta)$ and taking $c_{n,v} = \prod_{w=1}^d \max\{1, |\gamma_w|_v\}$ to account for the neglected factors, (ii) is now completely proven.

For now on, we assume v is archimedean and temporarily assume that $\eta_j + l, \zeta_j + l$ are all > 0 .

By Lemma 5.1 (i),

$$\frac{1}{(\eta + l)_{k-l}} \leq \frac{(\lfloor \eta \rfloor + k)(\lfloor \eta \rfloor + l)!}{(\eta + l)(\lfloor \eta \rfloor + k)!}$$

and up to an admissible error term (polynomial in n), the right hand side is equal to $\frac{l!}{k!}$. Similarly, using this time Lemma 5.1 (ii),

$$\frac{(\zeta + l + 1)_{k-l+n-1}}{(n-1)!} \leq \frac{(\lceil \zeta \rceil + k + n - 1)!(\lceil \zeta \rceil + k)!}{(n-1)!(\lceil \zeta \rceil + k)!(\lceil \zeta \rceil + l)!}$$

and the right hand side is up to an admissible error term $\binom{k+n}{n} \frac{k!}{l!}$.

Putting together and taking the product over ζ_j, η_j one deduces that

$$\Theta(k, l) = \frac{\prod_{j=1}^{d'} (\zeta_j + l + 1)_{k-l+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j + l)_{k-l}} \leq \exp(o(n)) \binom{k+n}{n}^{d'} \frac{k!^{d'-d''}}{l!^{d'-d''}}$$

When $d' \geq d''$, the right hand side is maximal for $l = 0, k = dm n + dm$, whence if $d'' \geq d'$ it is maximal for $l = k - 1$ and $k = dm n + dm$.

Taking into account (since $k \leq dm n + dm$)

$$\log \binom{k+n}{n} \leq n \left(\log(dm + 1) + dm \log \left(\frac{dm + 1}{dm} \right) \right) + o(n) ,$$

one gets

$$\log \Theta(k, l) \leq \max\{0, d' - d''\} \log(dm n)! + nd' \left(\log(dm + 1) + dm \log \left(\frac{dm + 1}{dm} \right) \right) + o(n) .$$

To finish the proof, we turn to the case where all the ζ_j, η_j might not be strictly positive. Let $k^* = \max\{-\lfloor \zeta_j \rfloor, -\lfloor \eta_j \rfloor\}$. If $k > k^*$, note that

$$(\zeta + l + 1)_{k-l+n-1} = ((\zeta + l) \dots (\zeta + l + k^* - 1)) (\zeta + l + k^* + 1)_{k-l-k^*+n-1}$$

and

$$(\eta + l)_{k-l} = ((\eta + l) \dots (\eta + l + k^* - 1)) (\eta + l + k^*)_{k-l-k^*} .$$

So, upto finitely many factors (depending only on ζ, η), the above bound is valid, the maximum of these finitely many factors being admissible.

Finally, using Lemma 5.2 (iv) of [29], one gets that

$$\|H_\ell\|_v \leq \left(2^{dmn+dm}(dmn+dm)^m\right)^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]} .$$

Therefore, one gets

$$\max\{\|P_\ell\|_v, \|P_{\ell,i,s}\|_v\} \leq c_{n,v} \exp(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n) |(dmn)!|_v^{\max\{0, d'-d''\}} ,$$

and part (i) of the lemma is also proven.

We start with (i) for the case P_ℓ . Using Lemma 5.2 (iv) of [29], one gets that

$$\|H_\ell\|_v \leq \left(2^{dmn+dm}(dmn+dm)^m\right)^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]} .$$

Now, the k -th eigenvalue of $\mathcal{T}_c \circ \bigcirc_{j=1}^{d'} S_{n-1, \zeta_j}$ is by the observation above:

$$\frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} = \prod_{j=1}^{d'} \frac{(\lceil |\zeta_j| \rceil + k + n - 1)! (\zeta_j + 1)_{k+n-1} (\lceil |\zeta_j| \rceil + k)!}{(\lceil |\zeta_j| \rceil + k + n - 1)! (n-1)! (\lceil |\zeta_j| \rceil + k)!} \prod_{j=1}^{d''} \frac{(\lceil |\eta_j| \rceil + k)!}{(\lceil |\eta_j| \rceil + k)! (\eta_j)_k} .$$

We now make use of the fact that v is Archimedean and get using the previous Lemma 5.1, (i) and then (iii):

$$\frac{(\lceil |\eta| \rceil + k)!}{(\lceil |\eta| \rceil + k)! (\eta)_k} \leq \frac{k}{|\eta|} \binom{k + \lceil |\eta| \rceil}{\lceil |\eta| \rceil} \cdot \frac{1}{(\lceil |\eta| \rceil + k)!} \leq \frac{k}{|\eta|} \binom{k + \lceil |\eta| \rceil}{k} \cdot \frac{1}{k! \max\{1, k\}^{\lceil |\eta| \rceil}}$$

and by Lemma 5.1 (ii)

$$\begin{aligned} \frac{(\lceil |\zeta| \rceil + k + n - 1)! (\zeta + 1)_{k+n-1} (\lceil |\zeta| \rceil + k)!}{(\lceil |\zeta| \rceil + k + n - 1)! (n-1)! (\lceil |\zeta| \rceil + k)!} &\leq \binom{\lceil |\zeta| \rceil + k + n - 1}{n-1} \frac{(\lceil |\zeta| \rceil + k)!}{\lceil |\zeta| \rceil!} \\ &= \binom{\lceil |\zeta| \rceil + k + n - 1}{n-1} k! \binom{\lceil |\zeta| \rceil + k}{\lceil |\zeta| \rceil} . \end{aligned}$$

Putting together and recalling that $0 \leq k \leq dmn + dm$, one gets and taking into account the fact that the absolute value $|\cdot|_v$ is normalized so that it coincides with the power $|\cdot|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ of the usual absolute value in $\mathbb{Q}$

$$\|\mathcal{T}_c \circ \bigcirc_{j=1}^{d'} S_{n-1, \zeta_j}\|_v = \max_{0 \leq k \leq dmn+dm} \left\{ |\lambda_{\mathcal{T}_c}(k)|_v \prod_{j=1}^{d'} |\lambda_{S_{n-1, \zeta_j}}(k)|_v \right\} ,$$

and so

$$\|P_\ell\|_v \leq \max_{0 \leq k \leq dmn+dm} \left\{ \left| \frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} \right| \right\}^{\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} \|H_\ell\|_v .$$

We estimate the max for k for the product over all the η_j, ζ_j of the upper bounds above:

$$A = \max_{0 \leq k \leq dm n + dm} \left\{ \prod_{j=1}^{d'} \left[\binom{\lceil |\zeta_j| \rceil + k + n - 1}{n-1} k! \binom{\lceil |\zeta_j| \rceil + k}{\lceil |\zeta_j| \rceil} \right] \prod_{j=1}^{d''} \left[\frac{k}{|\eta_j|} \binom{k + \lceil |\eta_j| \rceil}{k} \cdot \frac{1}{k! \max\{1, k\}^{\lceil |\eta_j| \rceil}} \right] \right\}$$

and observe the maximum for each term is either obtained for $k = 0$ or $k = dm n + dm$. Thus,

$$\begin{aligned} A &\leq (dm n + dm)!^{\max\{0, d' - d''\}} \prod_{j=1}^{d'} \binom{\lceil |\zeta_j| \rceil + dm n + dm + n - 1}{n-1} \\ &\quad \times \prod_{j=1}^{d'} \left[\binom{\lceil |\zeta_j| \rceil + dm n + dm}{\lceil |\zeta_j| \rceil} \right] \prod_{j=1}^{d''} \left[\frac{dm n + dm}{|\eta_j|} \binom{dm n + dm + \lceil |\eta_j| \rceil}{dm n + dm} \right]. \end{aligned}$$

Notice the standard Stirling formula implies

$$(29) \quad \log \left(\binom{(dm+1)n + dm + \lceil |\zeta| \rceil}{n-1} \right) = n \left(\log(dm+1) + dm \log \left(\frac{dm+1}{dm} \right) \right) + o(n).$$

We now define

$$c'_{n,v} = \prod_{j=1}^{d''} \frac{dm n + dm}{|\eta_j|} \prod_{j=1}^{d''} \binom{dm n + dm + \lceil |\eta_j| \rceil}{\lceil |\eta_j| \rceil} \prod_{j=1}^{d'} \binom{dm n + dm + \lceil |\zeta_j| \rceil}{\lceil |\zeta_j| \rceil} \cdot \left(\frac{(dm n + dm)!}{(dm n)!} \right)^{\max\{0, d' - d''\}} \cdot e^{o(1)}.$$

We deduce,

$$\begin{aligned} \|P_\ell\|_v &\leq A^{\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} \|H_\ell\|_v \\ &\leq c'_{n,v} \exp \left(n \left(\log(dm+1) + dm \log \left(\frac{dm+1}{dm} \right) \right) \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} \right) |(dm n)!|_v^{\max\{0, d' - d''\}} \|H_\ell\|_v. \end{aligned}$$

Inputting the upper bound for the norm of H_ℓ , one first simplifies the constants, defining for Archimedean v :

$$c_{n,v} = \left(c'_{n,v} 2^{dm} (dm n + dm)^m \right)^{\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} ,$$

then, consequently, we get

$$\|P_\ell\|_v \leq c_{n,v} \exp(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n) |(dm n)!|_v^{\max\{0, d' - d''\}}.$$

The same estimates provide for ultrametric bounds; the fact that the eigen values are

$$\frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k},$$

and taking into account the fact that $\|H_\ell\|_v = 1$, one gets the ultrametric part of the lemma^{††}.

One notes that for finite places, it is enough to choose $c_{n,v} = 1$ at this stage.

^{††}In view of Lemma 5.1, this is a bound in n very similar to the Archimedean case, however, it would be premature to simplify the quantity since then one would lose convergence ensured by the product formula when summing over all places.

Then, since $c_{n,v}$ is by definition polynomial in n ,

$$\frac{1}{n} \sum_{v \in \mathfrak{M}_K} \log(c_{n,v}) = \frac{1}{n} \sum_{v \in \mathfrak{M}_K^\infty} \log(c_{n,v}) \xrightarrow{n \rightarrow \infty} 0 .$$

Hence $c_{n,v}$ is an admissible error term and the lemma is completely proven for P_ℓ . ^{§§} $\square$

We now turn to the remainder term.

LEMMA 5.5. *Let $u \geq 0$ and integer. Let $(c_{n,v})_{v \in \mathfrak{M}_K, n \in \mathbb{N}}$ be the admissible error terms defined in Lemma 5.4. Then there exists a rational function $A(n, u)$ in n and u depending on ζ and η such that:*

(i) *If v is Archimedean and $d'' \leq d'$,*

$$\|\varphi_{i,s}(t^{n+u} P_\ell(t))\|_v \leq c_{n,v} |A(n, u)|_v \exp(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n) \left| \frac{1}{n!u!} \right|_v^{d'-d''} .$$

(ii) *If v is ultrametric,*

$$\|\varphi_{i,s}(t^{n+u} P_\ell(t))\|_v \leq c_{n,v} |n!|_v^{-d'} \max \left\{ \left| \frac{\prod_{j=1}^{d''} (\eta_j + k)_{n+u}}{\prod_{j=1}^{d'} (\zeta_j + k + n)_{u+1}} \right|_v ; 0 \leq k \leq dm n + dm \right\} .$$

PROOF. We can follow the same approach, as

$$\varphi_{i,s}(t^{n+u} P_\ell(t)) = [\alpha_i] \circ \text{Eval}_{t=\alpha_i} \circ \mathcal{T}_{\mathbf{c}}^{-1} \circ \bigcirc_{u=1}^{d-s} (t \frac{d}{dt} + \gamma_u) \circ [t^{n+u}] \circ \mathcal{T}_{\mathbf{c}} \circ \bigcirc_{j=1}^{d'} S_{n-1, \zeta_j} (H_\ell(t)) .$$

This time, we are only left with diagonally acting operators, substitutions of variables (morphisms Eval) and shifts (multiplication by $[\alpha_i]$ and $[t^{n+u}]$) and we have a direct estimation of the norm via the eigenvalues.

We need to estimate

$$(30) \quad \max_{0 \leq k \leq dm n + dm} \left\{ \left| \frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} \frac{\prod_{j=1}^{d''} (\eta_j)_{k+n+u}}{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n+u}} \prod_{w=1}^{d-s} (k+n+u+\gamma_w) \right|_v \right\} .$$

Notice

$$(31) \quad \frac{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j)_k} \frac{\prod_{j=1}^{d''} (\eta_j)_{k+n+u}}{\prod_{j=1}^{d'} (\zeta_j + 1)_{k+n+u}} = \frac{1}{(n-1)!^{d'}} \frac{\prod_{j=1}^{d''} (\eta_j + k)_{n+u}}{\prod_{j=1}^{d'} (\zeta_j + k + n)_{u+1}} .$$

^{§§}to be recomputed with the corrections Taking into account the above consideration, we conclude

$$c_{n,v} = \begin{cases} e^{o(1)} \left(2^{dm} \cdot (dmn)^{\max_{j',j''} \{ \lceil \lceil \eta_{j'} \rceil, \lceil \lceil \zeta_{j''} \rceil \} (d' + d'') + m + d + 1 + dm \max\{0, d' - d''\} } \right)^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]} & \text{if } v \mid \infty , \\ \left| \prod_{w=1}^d \gamma_w \right|_v & \text{if } v \nmid \infty . \end{cases}$$

Now, we start the proof of (i). In Equation (30), the factor $\prod_{w=1}^{d-s} (k+n+u+\gamma_w)$ is a polynomial in n and u and can conveniently be absorbed into the definition of the error term $A(n, u)$. Using Lemma 5.1 (ii) again,

$$|(\eta+k)_{n+u}| \leq \frac{(\lceil |\eta| \rceil + k + n + u - 1)!}{(\lceil |\eta| \rceil + k - 1)!}.$$

By definition of $\lfloor \cdot \rfloor$

$$\left| \frac{1}{(\zeta + k + n)_{u+1}} \right| \leq \frac{(\lfloor |\zeta| \rfloor + k + n - 1)!}{(\lfloor |\zeta| \rfloor + k + n + u)!}.$$

Now the terms $(\lfloor |\zeta| \rfloor + k + n - 1)!$, $(\lfloor |\zeta| \rfloor + k + n + u)!$ are $(k+n)!$ and $(k+n+u)!$ respectively upto an error term and similarly $(\lceil |\eta| \rceil + k + n + u - 1)!$ and $(\lceil |\eta| \rceil + k - 1)!$ are $(k+n+u)!$ and $k!$ respectively which can be put in the definition of $A(n, u)$, and moreover $(n-1)!$ is $n!$ upto an admissible error.

Taking into account the fact that the norm of H_ℓ is bounded by 2^{dmn} up to an admissible error, we deduce that, for Archimedean places, we have:

$$\|\varphi_{i,s}(t^{n+u}P_\ell(t))\|_v \leq c_{n,v}|A(n, u)|_v|2|_v^{dmn} \max \left\{ \left| \frac{(k+n)!^{d'}}{(k+n+u)!^{d'-d''}n!^{d'}k!^{d''}} \right|_v ; 0 \leq k \leq dmn + dm \right\}.$$

Notice that

$$\frac{(k+n)!^{d'}}{(k+n+u)!^{d'-d''}n!^{d'}k!^{d''}} = \binom{k+n}{n}^{d'} \binom{k+n+u}{k}^{d''-d'} (n+u)!^{d''-d'}.$$

From this expression, it is evident that $\binom{k+n}{n}^{d'}$ is maximal for $k = dmn + dm$ and, since $d' \geq d''$, $\binom{k+n+u}{k}^{d''-d'}$ is maximal for $k = 0$, and replacing this with $k = dmn$ introduces only an admissible error; moreover $(n+u)! \geq n!u!$. Thus, the maximum becomes upto admissible error:

$$\left| \binom{(dm+1)n}{n}^{d'} \left(\frac{1}{n!u!} \right)^{d'-d''} \right|_v.$$

Taking into account Equation (28), part (i) of the lemma follows.

We prove (ii). Assume v is ultrametric. Relying on (31), Equation (30) is bounded by

$$\prod_{w=1}^d |\gamma_w|_v |n!|_v^{-d'} \max \left\{ \left| \frac{\prod_{j=1}^{d''} (\eta_j + k)_{n+u}}{\prod_{j=1}^{d'} (\zeta_j + k + n)_{u+1}} \right|_v ; 0 \leq k \leq dmn + dm \right\}.$$

Since the norm of $H_\ell(t)$ is equal to 1, the above estimate yields the desired conclusion for part (ii). $\square$

Recall that if P is a homogeneous polynomial in some variables $y_i, i \in I$, for any point $\alpha = (\alpha_i)_{i \in I} \in K^{\text{Card}(I)}$ where I is any finite set, and $\|\cdot\|_v$ stands for the sup norm in $K_v^{\text{Card}(I)}$, with

$$C_v(P) = (\deg(P) + 1)^{\frac{\varepsilon_v[K_v:\mathbb{R}](\text{Card}(I))}{[K:\mathbb{Q}]}} ,$$

one has

$$(32) \quad |P(\alpha)|_v \leq C_v(P) \|P\|_v \cdot H_v(\alpha)^{\deg(P)} .$$

So, the preceding lemma yields trivially estimates for the v -adic norm of the above given polynomials. Moreover, since all the polynomials involved here $(P_\ell, P_{\ell,i,s}, \varphi_{i,s}(t^{n+u}P_\ell))$ are of degree polynomial in n , the Archimedean error term $C_v(P)$ above is an admissible error in n independent of u ^{¶¶}.

Recall that by Proposition 3.9 and Definition 3.5 the polynomials $P_\ell, P_{\ell,i,s}$ are of degree at most $dmn + \ell$ and $dmn + \ell - 1$ respectively and $\varphi_{i,s}(t^{n+u}P_\ell)$ is of degree at most $dmn + \ell + n + u + 1$.

We now turn to the issue of convergence. For $v \in \mathfrak{M}_K$, we denote the embedding K into K_v , and the extension to the Laurent series ring by

$$\sigma_v : K[[1/z]] \longrightarrow K_v[[1/z]]; \quad f(z) \mapsto f_v(z) := \sigma_v(f(z)) .$$

LEMMA 5.6. *Let $\alpha = (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$, $\beta \in K \setminus \{0\}$ and $v \in \mathfrak{M}_K$. Let $(c_{n,v}(\zeta, \eta))_{v \in \mathfrak{M}_K, n \in \mathbb{N}}$ be the admissible error terms defined in Lemma 5.4. Let i, ℓ, s be integers such that $1 \leq i \leq m, 0 \leq \ell \leq dm, 1 \leq s \leq d$. Recall $\deg A = d'', \deg B = d'$ and $\max\{d', d''\} = d$. Then^{***}*

(i) *Assume $d' = d''$, $v \in \mathfrak{M}_K^\infty$ and $|\alpha_i|_v < |\beta|_v$. Then the series $R_{\ell,i,s,v}(z)$ converges to an element of K_v at $z = \beta$ and there exists an admissible constant $c_{n,v}$ such that*

$$|R_{\ell,i,s,v}(\beta)|_v \leq c_{n,v} H_v(\alpha)^{(dm+1)n+\ell+1} \exp\left(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n\right) |\beta|_v^{-n-1} .$$

(ii) *Assume $d'' < d'$ and $v \in \mathfrak{M}_K^\infty$. Then the series $R_{\ell,i,s,v}(z)$ converges to an element of K_v at $z = 1$ and*

$$|R_{\ell,i,s,v}(1)|_v \leq \exp(C_v n) \left| \frac{1}{n!} \right|_v^{d'-d''} ,$$

where C_v is a constant depending on α, η, ζ and v .

(iii) *Assume $d'' \geq d'$, $v \in \mathfrak{M}_K^f$ and*

$$(33) \quad \left| \frac{\alpha_i}{\beta} \right|_v < \prod_{j=1}^{d''} \mu_v(\eta_j) |p|_v^{\frac{d''-d'}{p-1}} ,$$

where p is the prime below v . Then the series $R_{\ell,i,s,v}(z)$ converges to an element of K_v at $z = \beta$.

^{¶¶}The number of non zero terms is at most the number of coefficients of H_ℓ for $\varphi_{i,s}(t^{n+u}P_\ell)$ and is thus independent of u since $\varphi_{i,s}(t^{n+u}P_\ell)$ factors by α_i^{n+u} .

^{***}One may note that the series does not converge at Archimedean places if $d' < d''$ and that if $d' > d''$, the ultrametric series do not provide valuable enough information to offset the norm of the Padé approximants. Hence, there is no loss of generality to restrict ourselves to these cases from now on. Also, the behavior of the functions differ fundamentally depending on $d' > d''$, $d' = d''$ and $d' < d''$ and it makes sense to distinguish cases from now on.

(iv) Assume $d'' = d'$, $v \in \mathfrak{M}_K^f$ and Equation (33). Then there exists an admissible constant $c_{n,v}$ such that

$$|R_{\ell,i,s,v}(\beta)|_v \leq e^{o(n)} c_{n,v} H_v(\alpha)^{(dm+1)n} |\beta|_v^{-n} \prod_{j=1}^{d''} |\mu_n(\eta_j)|_v^{-1} .$$

(v) Assume $d'' > d'$ and each α_i is an algebraic integer. Let $v \in \mathfrak{M}_K^f$ with $|\eta_j|_v \leq 1$ for any $1 \leq j \leq d'$. Denote p the prime number below v . Assume $p \geq e^{\frac{d''}{d''-d'}}$ and

$$p \leq \Delta_n := \max_{1 \leq j \leq d'} \left\{ |\text{den}(\zeta_j) \zeta_j| + \text{den}(\zeta_j)((dm+1)n + dm) \right\} .$$

Then there exists an admissible constant $c_{n,v}$ such that

$$|R_{\ell,i,s,v}(1)|_v \leq c_{n,v} \delta_v(n) p^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} \prod_{j=1}^{d''} |\mu_n(\eta_j)|_v^{-1} \cdot |n!|_v^{d''-d'} ,$$

where

$$\delta_v(n) = \prod_{j=1}^{d'} (|\text{den}(\zeta_j) \zeta_j| + \text{den}(\zeta_j)((dm+1)n + dm)) \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]} .$$

PROOF. Recall we have

$$(34) \quad R_{\ell,i,s}(z) = \sum_{u=0}^{\infty} \frac{\varphi_{i,s}(t^{u+n} P_{\ell}(t))}{z^{u+n+1}} .$$

We now start the proof of (i). Let v be an Archimedean place. Combining Lemma 5.5 (i) with $d' = d''$ and Equation (32) yields

$$(35) \quad \begin{aligned} |\varphi_{i,s}(t^{n+u} P_{\ell}(t))|_v &\leq \|\varphi_{i,s}(t^{n+u} P_{\ell}(t))\|_v |\alpha_i|_v^{dmn+\ell+n+u+1} \\ &\leq c_{n,v}(\zeta, \eta) \exp\left(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n\right) H_v(\alpha)^{dmn+\ell} |\alpha_i|_v^{u+n+1} |A(n, u)|_v \end{aligned}$$

Since $A(n, u)$ is a rational function with respect n and u ,

$$(36) \quad |A(n, u)|_v \leq C \cdot |n|_v^{\deg_n A} \cdot |u|_v^{\deg_u A}$$

where C is a constant. Since $|\alpha_i|_v < |\beta|_v$, Equation (35), together with above inequality, yields:

$$\begin{aligned} |R_{\ell,i,s,v}(\beta)|_v &\leq c_{n,v}(\zeta, \eta) \exp\left(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n\right) H_v(\alpha)^{dmn+\ell} \sum_{u=0}^{\infty} |A(n, u)|_v \left| \frac{\alpha_i}{\beta} \right|^{n+u+1} \\ &\leq c_{n,v} H_v(\alpha)^{(dm+1)n+\ell+1} \exp\left(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n\right) |\beta|_v^{-n-1} . \end{aligned}$$

This proves (i).

(ii) Let v be an Archimedean place. Since $F_{s,v}(z)$ are entire function on K_v , the series $R_{\ell,i,s,v}(z)$ converges to an element of K_v at $z = 1$. Combining Lemma 5.5 (i) and Equation (32) together with (36) yields

$$\begin{aligned} |R_{\ell,i,s,v}(1)|_v &\leq \sum_{u=0}^{\infty} c_{n,v}(\zeta, \eta) H_v(\alpha)^{dmn+\ell} |A(n, u)|_v \exp\left(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n\right) \left| \frac{1}{n!u!} \right|_v^{d'-d''} |\alpha_i|_v^{n+u+1} \\ &\leq c_{n,v}(\zeta, \eta) H_v(\alpha)^{(dm+1)n+\ell+1} |n|_v^{\deg_n A} \exp\left(B \frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]} n\right) \left| \frac{1}{n!} \right|_v^{d'-d''} \sum_{u=0}^{\infty} |u|_v^{\deg_u(A)} \left| \frac{\alpha_i^u}{u!^{d'-d''}} \right|_v. \end{aligned}$$

Since the series $\sum_{u=0}^{\infty} |u|_v^{\deg_u(A)} \left| \alpha_i^u / u!^{d'-d''} \right|_v$ converges in $\mathbb{R}$, we see that there exists a constant C_v such that

$$|R_{\ell,i,s,v}(1)|_v \leq \exp(C_v n) \left| \frac{1}{n!} \right|_v^{d'-d''}.$$

This completes the proof of (ii).

(iii) Let v be a non-Archimedean place. Denote by p the rational prime lying below v . Let us show $\lim_{u \rightarrow \infty} |\varphi_{i,s}(t^{n+u} P_{\ell}(t)) / \beta^{n+u+1}|_v = 0$ under the assumption (33). Combining Lemma 5.5 (ii) and Equation (32) yields that

$$\begin{aligned} \left| \frac{\varphi_{i,s}(t^{n+u} P_{\ell}(t))}{\beta^{n+u+1}} \right|_v &\leq \\ c_{n,v}(\zeta, \eta) H_v(\alpha)^{dmn+\ell} &\max_{0 \leq k \leq dm+n+dm} \left\{ \left| \frac{1}{(n)!^{d'}} \frac{\prod_{j=1}^{d''} (\eta_j + k)_{n+u}}{\prod_{j=1}^{d'} (\zeta_j + k + n)_{u+1}} \left(\frac{\alpha_i}{\beta} \right)^{n+u+1} \right|_v \right\}. \end{aligned}$$

By Lemma 5.2 (i) and (ii),

$$|(\eta + k)_{n+u}|_v \leq |\mu_{n+u}(\eta)|_v^{-1} |(n+u)!|_v,$$

and, by Lemma 5.2 (iv), if $p \mid \prod_{l=0}^{k+n+u} (\text{den}(\zeta)\zeta + \text{den}(\zeta)l)$

$$\left| \frac{(u+1)!}{(\zeta + k + n)_{u+1}} \right|_v \leq (|\text{den}(\zeta)\zeta| + \text{den}(\zeta)(k+n+u))^{\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} ,$$

and of course if v does not divide the above quantity, $\left| \frac{(u+1)!}{(\zeta + k + n)_{u+1}} \right|_v \leq 1$.

We now set

$$\delta_v(n, u) = \prod_{j=1}^{d'} (|\text{den}(\zeta_j)\zeta_j| + \text{den}(\zeta_j)((dm+1)n + dm + u))^{\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}}.$$

Taking the product over all η_j, ζ_j , one deduces (using $\mu_n(\zeta)$ is increasing in n)

$$\max_{0 \leq k \leq dm+n+dm} \left\{ \left| \frac{1}{n!^{d'}} \frac{\prod_{j=1}^{d''} (\eta_j + k)_{n+u}}{\prod_{j=1}^{d'} (\zeta_j + k + n)_{u+1}} \right|_v \right\} \leq \frac{\delta_v(n, u) \prod_{j=1}^{d''} |\mu_{n+u}(\eta_j)|_v^{-1} |(n+u)!|_v^{d''}}{|n!|_v^{d'} |(u+1)!|_v^{d'}}.$$

We now simplify the combinatorial factors

$$\frac{(n+u)!^{d''}}{n!^{d'} (u+1)!^{d'}} = \frac{1}{n^{d'}} \binom{n+u}{u+1}^{d'} \binom{n+u}{u}^{d''-d'} n!^{d''-d'} u!^{d''-d'}.$$

We conclude for all place v above a prime number p satisfying

$$p \mid \prod_{j=1}^{d'} \prod_{k=0}^{dmn+dm} [\text{den}(\zeta_j)\zeta_j + \text{den}(\zeta_j)(k+n+u)] ,$$

then we have

$$(37) \quad \left| \frac{\varphi_{i,s}(t^{n+u}P_\ell(t))}{\beta^{n+u+1}} \right|_v \leq c_{n,v}(\zeta, \eta) |n!|_v^{d''-d'} \prod_{j=1}^{d'} |\mu_n(\eta_j)|_v^{-1} \mathbf{H}_v(\alpha)^{(dm+1)n+\ell+1} |\beta|_v^{-n-1} \cdot \left(|u!|_v^{d''-d'} \prod_{j=1}^{d''} |\mu_u(\eta_j)|_v^{-1} \delta_v(n, u) \left| \frac{\alpha_i}{\beta} \right|_v^u \right) .$$

Now, since $\delta_v(n, u)$ is a polynomial in n, u , assuming $|\alpha_i/\beta|_v < |p|_v^{\frac{d''-d'}{p-1}} \prod_{j=1}^{d''} \mu_v(\eta_j)$, one deduces

$$\lim_{u \rightarrow \infty} \left(|u!|_v^{d''-d'} \prod_{j=1}^{d''} |\mu_u(\eta_j)|_v^{-1} \delta_v(n, u) \left| \frac{\alpha_i}{\beta} \right|_v^u \right) = 0 .$$

This shows that $R_{\ell,i,s,v}(z)$ converges to an element of K_v at $z = \beta$.

The same argument works if v is not above a prime dividing

$$p \mid \prod_{j=1}^{d'} \prod_{k=0}^{dmn+dm} [\text{den}(\zeta_j)\zeta_j + \text{den}(\zeta_j)(k+n+u)] .$$

We now turn to bounding the series.

(iv) Under the assumption (33), since $\delta_v(n, u)$ is a polynomial in n, u , we have

$$\max_{0 \leq u} \left\{ |u!|_v^{d''-d'} \prod_{j=1}^{d''} |\mu_u(\eta_j)|_v^{-1} \delta_v(n, u) \left| \frac{\alpha_i}{\beta} \right|_v^u \right\} = e^{o(n)} .$$

Combining (37) with $d' = d''$ and above equality yields

$$|R_{\ell,i,s,v}(\beta)|_v \leq \max_{0 \leq u} \left\{ \left| \frac{\varphi_{i,s}(t^{n+u}P_\ell(t))}{\beta^{n+u+1}} \right|_v \right\} \leq c_{n,v} \mathbf{H}_v(\alpha)^{(dm+1)n} |\beta|_v^{-n} \prod_{j=1}^{d''} |\mu_n(\eta_j)|_v^{-1} .$$

(v) The definition of $\delta_v(n, u)$ yields

$$\delta_v(n, u) \leq 2\delta_v(n) \max\{1, u\}^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} .$$

Hence, at places where $|\alpha_i|_v \leq 1$ and $|\text{den}(\eta_j)|_v = 1$,

$$\max_{0 \leq u} \left\{ |u!|_v^{d''-d'} \prod_{j=1}^{d''} |\mu_u(\eta_j)|_v^{-1} \delta_v(n, u) \left| \frac{\alpha_i}{\beta} \right|_v^u \right\} \leq 2\delta_v(n) \max_{1 \leq u} \left\{ u^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} |p|_v^{(d''-d')v_p(u!)} \right\} .$$

Under the assumption $p \geq e^{\frac{d''}{d''-d'}}$, let us show

$$(38) \quad \max_{1 \leq u} \left\{ u^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} |p|_v^{(d''-d')v_p(u!)} \right\} \leq p^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}}.$$

Notice, by taking $u = p - 1$,

$$(p-1)^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} \leq \max_{1 \leq u} \left\{ u^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} |p|_v^{(d''-d')v_p(u!)} \right\}.$$

We now consider $u \geq p - 1$. Denote the p -adic expansion of u by $\sum_{i=0}^{\ell} a_i p^i$ with $0 \leq a_i < p$ and $S_u = \sum_{i=0}^{\ell} a_i$. Using the identity $v_p(u!) = \frac{u-S_u}{p-1}$ with $S_u \leq (p-1) \log_p(u) + \frac{(p-1)}{u \log p}$, we obtain the estimate

$$u^{d'} |p|_p^{(d''-d')v_p(u!)} \leq u^{d''} p^{-\frac{(d''-d')u}{p-1}} \exp\left(\frac{d''-d'}{u}\right) = \exp\left(d'' \log u - \log p \frac{(d''-d')u}{p-1} + \frac{d''-d'}{u}\right).$$

The function $u \mapsto 1/u$ is maximal at $u = 1$ and $u \mapsto d'' \log u - \log p \frac{(d''-d')u}{p-1}$ achieves its maximum on $u > 0$ at

$$u = \frac{(p-1)d''}{(d''-d') \log p},$$

and both functions are decreasing right of their maximal value. The assumption $p \geq e^{\frac{d''}{d''-d'}}$ yields

$$\frac{(p-1)d''}{(d''-d') \log p} \leq p-1.$$

In the range $u \leq p-1$, we have already seen the desired bound (38) holds. Thus, we can assume $u = p$ and readily check the bound also holds. Combining above considerations concludes

$$|R_{\ell,i,s,v}(1)|_v \leq c_{n,v} \delta_v(n) p^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} \prod_{j=1}^{d''} |\mu_n(\eta_j)|_v^{-1} \cdot |n!|_v^{d''-d'}.$$

□

6 Proof of Main theorems

In this section, we keep notations of Section 5. Recall K is a number field and $\eta_1, \dots, \eta_{d''}, \zeta_1, \dots, \zeta_{d'}$ rational numbers which are not negative integers with $\eta_i - \zeta_j \notin \mathbb{N}$ for $1 \leq i \leq d'', 1 \leq j \leq d'$. Put $d = \max\{d', d''\}$. Assume

$$A(X) = (X + \eta_1) \cdots (X + \eta_{d''}), \quad B(X) = (X + \zeta_1) \cdots (X + \zeta_{d'})$$

where $d'd'' > 0$. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$ whose coordinates are pairwise distinct. We also recall the Padé approximants $P_\ell(z), P_{\ell,i,s}(z)$ defined in Proposition 3.9 for the above data. We now prove our main theorems stated in Section 2 by considering the cases for the relationship between d'' and d' , namely $d'' = d'$, $d'' < d'$ or $d'' > d'$.

6.1 Proof of Theorem 2.3

We consider the G -function case that is $d' = d'' = d$ which is the situation of Theorem 2.3. We want to apply the qualitative linear independence criterion [29, Proposition 5.6].

Let us define notation. For $\beta \in K \setminus \{0\}$, we put the $dm + 1$ by $dm + 1$ matrices M_n by

$$M_n = \begin{pmatrix} P_\ell(\beta) \\ P_{\ell,i,s}(\beta) \end{pmatrix}.$$

Remark that Proposition 4.1 ensures $M_n \in \text{GL}_{dm+1}(K)$. Recall

$$B = dm \log(2) + d \left(\log(dm + 1) + dm \log \left(\frac{dm + 1}{dm} \right) \right),$$

and, for $v \in \mathfrak{M}_K$ and $p \in \mathfrak{M}_\mathbb{Q}$ the prime below v , define the functions $F_v : \mathbb{N} \rightarrow \mathbb{R}$ by

$$F_v(n) = n \left(\varepsilon_v B \frac{[K_v : \mathbb{Q}_p]}{[K : \mathbb{Q}]} + \left(dm + \frac{dm+1}{n} \right) h_v(\alpha, \beta) + (1 - \varepsilon_v) \log A_{n,v}(\zeta, \eta) \right) + \log c_{n,v},$$

where the real numbers $A_{n,v}(\zeta, \eta)$, $c_{n,v}$ are defined in Lemma 5.4. So combining Lemma 5.4 and Equation (32) yields,

$$\|M_n\|_v \leq e^{F_v(n)} \quad \text{for } v \in \mathfrak{M}_K.$$

We now choose a place v_0 of K and define a real number

$$\mathbb{A}_{v_0}(\beta) = \log |\beta|_{v_0} - (dm + 1) h_{v_0}(\alpha) - \varepsilon_{v_0} B \frac{[K_{v_0} : \mathbb{Q}_p]}{[K : \mathbb{Q}]} + \sum_{j=1}^d \log \mu_{v_0}(\eta_j).$$

Then Lemma 5.6 (i) and (iv) derive

$$\log |R_{\ell,i,s}(\beta)|_{v_0} \leq -\mathbb{A}_{v_0}(\beta)n + o(n).$$

We check the condition of the qualitative linear independence criterion [29, Proposition 5.6].

LEMMA 6.1. *One has*

$$\sum_{v \in \mathfrak{M}_K^f \setminus \{v_0\}} \log A_{n,v}(\zeta, \eta) \leq dm \sum_{j=1}^d \text{den}(\eta_j) + (dm + 1) \sum_{j=1}^d \log \mu(\zeta_j).$$

PROOF. By Lemma 5.2 (ii) joined with (i) $\mu_{n-1}(\zeta) \frac{(\zeta+l+1)_{n-1}}{(n-1)!}$ is an integer and by Lemma 5.2 (ii) joined with (iii)

$$D_{k,l} = \text{den} \left(\frac{(\zeta + l + n + 1)_{k-l}}{(\eta + l)_{k-l}} \right)_{0 \leq l \leq k-1, 0 \leq k \leq dm n + dm}$$

satisfies

$$\limsup_n \frac{1}{n} \log(D_{k,l}) \leq dm \log \mu(\zeta) + dm \text{den}(\eta).$$

Putting together, we deduce

$$\limsup_n \frac{1}{n} \sum_{v \in \mathfrak{M}_K^f \setminus \{v_0\}} \log A_{n,v}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \leq dm \sum_{j=1}^d \text{den}(\eta_j) + (dm+1) \sum_{j=1}^d \log \mu(\zeta_j) .$$

□

LEMMA 6.2. ^{†††} We have

$$\begin{aligned} \limsup_n \frac{1}{n} \sum_{v \neq v_0} F_v(n) &\leq \mathbb{B}_{v_0}(\beta) := dm(\text{h}(\boldsymbol{\alpha}, \beta) - \text{h}_{v_0}(\boldsymbol{\alpha}, \beta)) + B \left(1 - \varepsilon_{v_0} \frac{[K_{v_0} : \mathbb{Q}_p]}{[K : \mathbb{Q}]} \right) \\ &\quad + dm \sum_{j=1}^d \text{den}(\eta_j) + (dm+1) \sum_{j=1}^d \log \mu(\zeta_j) . \end{aligned}$$

PROOF. By the definition of $F_v(n)$, and Lemma 6.1 and summation over all places, one gets the statement. □

Define

$$\begin{aligned} V_{v_0}(\beta) &:= \mathbb{A}_{v_0}(\beta) - \mathbb{B}_{v_0}(\beta) = \log |\beta|_{v_0} + dm(\text{h}_{v_0}(\boldsymbol{\alpha}) - \text{h}(\boldsymbol{\alpha}, \beta)) - (dm+1)\text{h}_{v_0}(\boldsymbol{\alpha}) - B \\ &\quad - \left[dm \sum_{j=1}^d \text{den}(\eta_j) + (dm+1) \sum_{j=1}^d \log \mu(\zeta_j) + \sum_{j=1}^d (\log \mu_{v_0}(\eta_j)) \right] . \end{aligned}$$

By direct application of [29, Proposition 5.6] implies the following result.

THEOREM 6.3. *Let $v_0 \in \mathfrak{M}_K$ such that $V_{v_0}(\beta) > 0$. For any $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{d-1}) \in K^{d-1}$, the functions $F_s(\boldsymbol{\gamma}, z)$, $1 \leq s \leq d$ converge around α_i/β in K_{v_0} , $1 \leq i \leq m$ and for any positive number ε with $\varepsilon < V_{v_0}(\beta)$, there exists an effectively computable positive number H_0 depending on ε and the given data such that the following property holds. For any $\boldsymbol{\lambda} = (\lambda_0, \lambda_{i,s})_{\substack{1 \leq i \leq m \\ 1 \leq s \leq d}} \in K^{dm+1} \setminus \{\mathbf{0}\}$ satisfying $H_0 \leq H(\boldsymbol{\lambda})$, then*

$$\left| \lambda_0 + \sum_{i=1}^m \sum_{s=1}^d \lambda_{i,s} F_s(\boldsymbol{\gamma}, \alpha_i/\beta) \right|_{v_0} > C(\beta, \varepsilon) H_{v_0}(\boldsymbol{\lambda}) H(\boldsymbol{\lambda})^{-\mu(\beta, \varepsilon)} ,$$

where

$$\begin{aligned} \mu(\beta, \varepsilon) &= \frac{\mathbb{A}_{v_0}(\beta) + U_{v_0}(\beta)}{V_{v_0}(\beta) - \varepsilon} , \\ C(\beta, \varepsilon) &= \exp \left(- \left(\frac{\log(2)}{V_{v_0}(\beta) - \varepsilon} + 1 \right) (\mathbb{A}_{v_0}(\beta) + U_{v_0}(\beta)) \right) , \\ U_{v_0}(\beta) &= \limsup_n \frac{1}{n} F_{v_0}(n) . \end{aligned}$$

^{†††}We easily see that the criterion [29, Proposition 5.6] is also verified replacing $\lim_n \frac{1}{n} \sum_v F_v(n) < \infty$ by $\limsup_n \frac{1}{n} \sum_v F_v(n) < \infty$.

Proof of Theorem 2.3. In Theorem 6.3, we take $\boldsymbol{\eta} = (a_1, \dots, a_d), \boldsymbol{\zeta} = (b_1 - 1, \dots, b_{d-1} - 1, 0)$ and $\boldsymbol{\gamma} = (a_{d-1}, \dots, a_1)$. Then $V_{v_0}(\boldsymbol{\alpha}, \beta) \leq V_{v_0}(\beta)$. Combining Equation (6) and Theorem 6.3 yields the assertion of Theorem 2.3. $\square$

EXAMPLE 6.4. Applying Theorem 2.3 for $d = 2$, $(a_1, a_2) = (-1/2, 1/2)$ and $b_1 = 1$ yields a linear independence criterion concerning the following solutions of the Gauss-Manin connection for the Legendre family of elliptic curves (*confer* [1, 7.1]):

$${}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| z\right), {}_2F_1\left(\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| z\right).$$

Here we consider $K = \mathbb{Q}$ and take v_0 a place of $\mathbb{Q}$, $m = 10$ and $\boldsymbol{\alpha} = (1, 2, \dots, 10)$. Now, let us establish a sufficient condition for $\beta \in \mathbb{Q}$, considering the 21-elements

$$1, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{j}{\beta}\right), {}_2F_1\left(\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| \frac{j}{\beta}\right) \in \mathbb{Q}_{v_0} \quad \text{for } 1 \leq j \leq 10$$

are linearly independent over $\mathbb{Q}$. In case of $v_0 = \infty$ and $\beta \in \mathbb{Z} \setminus \{0\}$, we have $V_\infty(\boldsymbol{\alpha}, \beta) > \log |\beta| - 150.2579$. Theorem 2.3 implies that if β satisfies $|\beta| \geq e^{150.2579}$ then

$$1, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{j}{\beta}\right), {}_2F_1\left(\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| \frac{j}{\beta}\right) \in \mathbb{R} \quad \text{for } 1 \leq j \leq 10$$

are linearly independent over $\mathbb{Q}$.

In case of $v_0 = p$ where p is a rational prime and $\beta = p^{-k}$ for a positive integer k , we have $V_p(\boldsymbol{\alpha}, p^{-k}) > k \log p - 150.2579 + 4 \log |2|_p$. Theorem 2.3 asserts that if p and k satisfy either $p \geq e^{150.2579 - 4 \log |2|_p}$ and $k = 1$ or

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
$k \geq$	221	137	94	78	63	59	54	52	48	45	44	42	41	40	40	38	37	37	36	36

then

$$1, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| jp^k\right), {}_2F_1\left(\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \middle| jp^k\right) \in \mathbb{Q}_p \quad \text{for } 1 \leq j \leq 10$$

are linearly independent over $\mathbb{Q}$.

6.2 Proof of Theorem 2.4

We assume $\deg A = d'' < \deg B = d$. We now fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For $v \in \mathfrak{M}_K^\infty$, denote the embedding corresponds to v by $\sigma_v : K \hookrightarrow \mathbb{C}$.

THEOREM 6.5. *Let K be an algebraic number field and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$ whose coordinates are pairwise distinct. Then, the $dm + 1$ complex numbers*

$$1, F_s(\boldsymbol{\gamma}, \alpha_i) \quad \text{for } 1 \leq i \leq m, 1 \leq s \leq d$$

are linearly independent over $\overline{\mathbb{Q}}$.

To prove Theorem 6.5, we rely on the following remarkable results of F. Beukers [9, Proposition 4.1] and Fischler and Rivoal [36, Proposition 1], which are grounded in the theory of E -operators developed by André [2].

PROPOSITION 6.6. (*confer [9, Proposition 4.1]*) *Let j be a negative integer. Let $f(z) \in K[[z]]$ be an arithmetic Gevrey series of order j and $\xi \in \overline{\mathbb{Q}} \setminus \{0\}$ such that $f(\xi) = 0$. Then $f(z)/(z - \xi)$ is again an arithmetic Gevrey series of order j .*

PROOF. Assuming f has rational coefficients, by applying [4, Theorem 3.4.1] and using the same arguments as in the proof of [9, Corollary 2.2], we can ensure Proposition 6.6. For the general case, Proposition 6.6 is proved by the same arguments as in [9, Proposition—4.1]. $\square$

For a power series $f(z) \in \overline{\mathbb{Q}}[[z]]$ and an embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we denote the image of f of the natural extension of σ to $\overline{\mathbb{Q}}[[z]]$ by f^σ .

PROPOSITION 6.7. (*confer [36, Proposition 1]*) *Let j be a negative integer. Let $f \in K[[z]]$ be an arithmetic Gevrey series of order j and $\xi \in \overline{\mathbb{Q}} \setminus \{0\}$. Then the following assertions are equivalent:*

- (i) *f vanishes at ξ .*
- (ii) *There exists $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that f^σ vanishes at $\sigma(\xi)$.*
- (iii) *For any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that f^σ vanishes at $\sigma(\xi)$.*
- (iv) *There exists an arithmetic Gevrey series g of order j with coefficients in K such that*

$$f(z) = D(z)g(z) \quad \text{where } D \text{ is the minimal polynomial of } \xi \text{ over } K \text{ .}$$

PROOF. This proposition is proved using the same arguments as in [36, Proposition 1], with Proposition 6.6 used in place of [9, Proposition 4.1]. $\square$

COROLLARY 6.8. *Let $(b_0, b_{i,s})_{1 \leq i \leq m, 1 \leq s \leq q} \in K^{dm+1} \setminus \{\mathbf{0}\}$. Assume $b_0 + \sum_{i,s} b_{i,s} F_s(\gamma, \alpha_i) = 0$. Then for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have*

$$\sigma(b_0) + \sum_{i,s} \sigma(b_{i,s}) F_s^\sigma(\gamma, \sigma(\alpha_i)) = 0 \text{ .}$$

PROOF. Notice $F_s(\gamma, \alpha_i z) \in K[[z]]$ are arithmetic Gevrey series of exact order $d'' - d' < 0$ for any $1 \leq i \leq m, 1 \leq s \leq q$. This yields the power series

$$f(z) := b_0 + \sum_{i,s} b_{i,s} F_s(\gamma, \alpha_i z)$$

is also an arithmetic Gevrey series of order $d'' - d' < 0$. Using Proposition 6.7 for the above f and $\xi = 1$, we obtain the assertion. $\square$

Proof of Theorem 6.5. Let $P_\ell(z), P_{\ell,i,s}(z)$ be polynomials defined in (12), (13) respectively for $F_s(\gamma, \alpha_i/z)$. Set

$$a_\ell^{(n)} = P_\ell(1), \quad a_{\ell,i,s}^{(n)} = P_{\ell,i,s}(1) \quad \text{for } 0 \leq \ell \leq dm, \quad 1 \leq i \leq m, \quad 1 \leq s \leq d .$$

By Proposition 4.1 and Remark 4.2, the matrix $M_n = \begin{pmatrix} a_\ell^{(n)} \\ a_{\ell,i,s}^{(n)} \end{pmatrix}$ is invertible. Assume 1, $F_s(\gamma, \alpha_i)$ are linearly dependent over K . Then there exists a non-zero vector $\mathbf{b} = (b_0, b_{i,s})_{i,s} \in K^{dm+1}$ such that

$$b_0 + \sum_{i,s} b_{i,s} F_s(\gamma, \alpha_i) = 0 .$$

Corollary 6.8 ensures, for any $v \in \mathfrak{M}_K^\infty$,

$$(39) \quad \sigma_v(b_0) + \sum_{i,s} \sigma_v(b_{i,s}) F_s^{\sigma_v}(\gamma, \sigma_v(\alpha_i)) = 0 .$$

Since M_n is non-singular, there exists $0 \leq \ell_n \leq dm$ such that

$$B_{\ell_n} := b_0 a_{\ell_n}^{(n)} + \sum_{i,s} b_{i,s} a_{\ell_n,i,s}^{(n)} \in K \setminus \{0\} .$$

Notice (39) implies

$$(40) \quad \sigma_v(B_{\ell_n}) = - \sum_{i,s} \sigma_v(b_{i,s}) (\sigma_v(a_{\ell_n}^{(n)}) F_s^{\sigma_v}(\gamma, \sigma_v(\alpha_i)) - \sigma_v(a_{\ell_n,i,s}^{(n)})) \quad \text{for } v \in \mathfrak{M}_K^\infty .$$

Lemma 5.6 (ii) ensures

$$\max_{\substack{0 \leq \ell \leq dm \\ 1 \leq i \leq m, 1 \leq s \leq d}} \{ |\sigma_v(a_\ell^{(n)}) F_s^{\sigma_v}(\gamma, \sigma_v(\alpha_i)) - \sigma_v(a_{\ell,i,s}^{(n)})|_v \} \leq e^{C_v n} (n!)^{\frac{(d''-d)[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} \quad \text{for } v \in \mathfrak{M}_K^\infty .$$

thus, combining above inequality and Equation (40) leads us

$$(41) \quad |B_{\ell_n}|_v \leq e^{C_v n} (n!)^{\frac{(d''-d)[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} \quad \text{for } v \in \mathfrak{M}_K^\infty .$$

LEMMA 6.9. *There is a constant $C > 0$ depending only on ζ, η such that*

$$\frac{1}{n} \sum_{v \in \mathfrak{M}_K^f} \log A_{n,v}(\zeta, \eta) \leq C ,$$

where $A_{n,v}(\zeta, \eta)$ is defined in Lemma 5.4 .

PROOF. Let $0 \leq k \leq dm n + dm$ and $0 \leq l \leq k - 1$ then recall (Lemma 5.4)

$$D_{k,l,n}(\zeta, \eta) = \frac{\prod_{j=1}^{d'} (\zeta_j + l + 1)_{k-l+n-1}}{(n-1)!^{d'} \prod_{j=1}^{d''} (\eta_j + l)_{k-l}} = \prod_{j=1}^{d'} \frac{(\zeta_j + l + 1)_{n-1}}{(n-1)!} \prod_{j=1}^{d''} \frac{(\zeta_j + l + n)_{k-l}}{(\eta_j + l)_{k-l}} \prod_{j=d''+1}^{d'} (\zeta_j + l + n)_{k-l} .$$

Then, Lemma 5.2 (i) ensures

$$\sum_{v \in \mathfrak{M}_K^f} \log \left| \prod_{j=1}^{d'} \frac{(\zeta_j + l + 1)_{n-1}}{(n-1)!} \right|_v \leq n \sum_{j=1}^{d'} \log \mu(\zeta_j) ;$$

similarly, Lemma 5.2 (iii) ensures

$$\sum_{v \in \mathfrak{M}_K^f} \log \left| \prod_{j=1}^{d''} \frac{(\zeta_j + l + n)_{k-l}}{(\eta_j + l)_{k-l}} \right|_v \leq dmn \left(\sum_{j=1}^{d''} (\log \mu(\zeta_j) + \text{den}(\eta_j)) \right) + o(n)$$

and trivially,

$$\sum_{v \in \mathfrak{M}_K^f} \log \left| \prod_{j=d''+1}^{d'} (\zeta_j + l + n)_{k-l} \right|_v \leq dmn \sum_{j=d''+1}^{d'} \text{den}(\zeta_j) .$$

This completes the proof of Lemma 6.9. $\square$

From this lemma, one deduces

$$(42) \quad \sum_{v \in \mathfrak{M}_K^f} \log |B_{\ell_n}|_v \leq Cn .$$

The product formula for $B_{\ell_n} \in K \setminus \{0\}$ together with Equations (41), (42) implies

$$1 = \prod_{v \in \mathfrak{M}_K} |B_{\ell_n}|_v \leq e^{O(n)} (n!)^{d''-d} \rightarrow 0 \quad (n \rightarrow \infty) .$$

This gives a contradiction. Since we may take any algebraic number field K containing α_i , we get the assertion. $\square$

Proof of Theorem 2.4. We use the same notation as in Theorem 2.4. In Theorem 6.5, we take $d'' = p$, $d = q + 1$, $\boldsymbol{\eta} = (a_1, \dots, a_p)$, $\boldsymbol{\zeta} = (b_1 - 1, \dots, b_q - 1, 0)$ and $\boldsymbol{\gamma} = (1, b_q, \dots, b_2)$. Then, Theorem 6.5 and Equation (7) allow us to obtain the assertion of Theorem 2.4. $\square$

6.3 Proof of Theorem 2.5

We now consider the case $\deg B = d' < \deg A = d$. Let $\mathcal{O}_K$ denote the ring of integers of K . For each $v \in \mathfrak{M}_K^f$, by Lemma 5.6 (iii), for any algebraic integer $\alpha \in K$ satisfying

$$(43) \quad |\alpha|_v < \prod_{j=1}^d \mu_v(\eta_j) |p|_v^{\frac{d-d'}{p-1}} ,$$

where p is the rational prime below v , the series $F_{s,v}(\boldsymbol{\gamma}, \alpha)$ converges to an element of K_v .

To establish Theorem 2.5, we first prove a slightly more general result:

THEOREM 6.10. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathcal{O}_K \setminus \{0\})^m$ have distinct coordinates, each α_i satisfying (43). Let $\gamma = (\gamma_1, \dots, \gamma_{d-1}) \in K^{d-1}$ and $\lambda = (\lambda_0, \lambda_{i,s})_{1 \leq i \leq m, 1 \leq s \leq d} \in \mathcal{O}_K^{dm+1} \setminus \{0\}$. Then there exists an effectively computable positive real number H_0 such that, whenever $H(\lambda) \geq H_0$, for any $H \geq H(\lambda)$, there exists a prime

$$p \in \left[\left(\frac{3dm \log H}{(d-d') \log \log H} \right)^{\frac{1}{8dm}}, \frac{12dm \max_{1 \leq j \leq d'} \{\text{den}(\zeta_j)\} \log H}{(d-d') \log \log H} \right]$$

and a place $v \in \mathfrak{M}_K^f$ lying above p such that the following linear form in hypergeometric values is nonzero in K_v :

$$\lambda_0 + \sum_{i=1}^m \sum_{s=1}^d \lambda_{i,s} F_{s,v}(\gamma, \alpha_i) \neq 0.$$

Before proving Theorem 6.10, we show how it implies Theorem 2.5.

Proof of Theorem 2.5. We retain the notation from Theorem 2.5. In Theorem 6.10, take $d' = q + 1$, $d = p$, $\eta = (a_1, \dots, a_p)$, $\zeta = (b_1 - 1, \dots, b_q - 1, 0)$, and $\gamma = (a_{p-1}, \dots, a_1)$. Then, Theorem 6.10 and Equation (6) allow us to obtain the assertion of Theorem 2.5. $\square$

We now proceed to prove Theorem 6.10. We begin with some preparatory lemmas.

LEMMA 6.11. [66] Let $x > 1$ be a real number. Then

$$\sum_{\substack{p:\text{prime} \\ p \leq x}} \frac{\log p}{p} < \log x.$$

LEMMA 6.12. [66, Corollary 1, Theorem 9] Let $x > 1$ be a real number. Define

$$\pi(x) = \sum_{\substack{p:\text{prime} \\ p \leq x}} 1, \quad \vartheta(x) = \sum_{\substack{p:\text{prime} \\ p \leq x}} \log p.$$

Then:

- (i) $\pi(x) < \frac{1.25506 x}{\log x}$.
- (ii) $\vartheta(x) < 1.01624 x$.

6.3.1 Proof of Theorem 6.10

Let $P_\ell(z), P_{\ell,i,s}(z)$ be the polynomials defined in (12), (13), respectively, for $F_s(\gamma, \alpha_i/z)$.

Define

$$a_\ell^{(n)} = P_\ell(1), \quad a_{\ell,i,s}^{(n)} = P_{\ell,i,s}(1).$$

Moreover, Proposition 4.1 and Remark 4.2 assert that the matrix

$$M_n = \begin{pmatrix} a_\ell^{(n)} \\ a_{\ell,i,s}^{(n)} \end{pmatrix}$$

is invertible. Let $\lambda = (\lambda_0, \lambda_{i,s})_{1 \leq i \leq m, 1 \leq s \leq d} \in \mathcal{O}_K^{dm+1} \setminus \{0\}$. Due to the invertibility of M_n , there exists $0 \leq \nu_n \leq dm$ such that

$$B_{\nu_n} := a_{\nu_n}^{(n)} \lambda_0 + \sum_{i=1}^m \sum_{s=1}^d a_{\nu_n, i, s}^{(n)} \lambda_{i, s} \neq 0 .$$

We denote by $\mathbf{a}_{\nu_n}$ the vector $(a_{\nu_n}^{(n)}, a_{\nu_n, i, s}^{(n)})$. Let us now estimate $|B_{\nu_n}|_v$ for $v \in \mathfrak{M}_K$.

We separate cases with $\mathfrak{M}_K = \mathfrak{M}_K^\infty \sqcup S_1 \sqcup S_2 \sqcup S_3$, where

$$\begin{aligned} S_1 &= \{v \in \mathfrak{M}_K^f; v \mid p, p \leq (dmn)^{1/8dm}\}, \\ S_2 &= \{v \in \mathfrak{M}_K^f; v \mid p, (dmn)^{1/8dm} < p < 4 \max_{1 \leq j \leq d'} \{\text{den}(\zeta_j)\} dmn\} , \\ S_3 &= \{v \in \mathfrak{M}_K^f; v \mid p, p \geq 4 \max_{1 \leq j \leq d'} \{\text{den}(\zeta_j)\} dmn\} . \end{aligned}$$

We start with a lemma to apply Lemma 5.4.

LEMMA 6.13. *One has, provided $n \geq 2$,*

$$\sum_{v \in S_1 \sqcup S_3} \log A_{n,v}(\zeta, \eta) \leq 3dmn \left(\sum_{j=1}^{d'} \log \mu(\zeta_j) + \sum_{j=1}^d \text{den}(\eta_j) \right) + (d-d') \sum_{p \leq (dmn)^{\frac{dm}{8}}} \log |(dm(n+1))!|_v^{-1} ,$$

where $A_{n,v}(\zeta, \eta)$ is defined in Lemma 5.4.

PROOF. Let $v \in \mathfrak{M}_K^f$. Recall

$$A_{n,v}(\zeta, \eta) = \max\{|D_{k,l,n}(\zeta, \eta)|_v; 0 \leq l \leq k-1, 0 \leq k \leq dmn + dm\}$$

with

$$\begin{aligned} D_{k,l,n}(\zeta, \eta) &= \frac{\prod_{j=1}^{d'} (\zeta_j + l + 1)_{k-l+n-1}}{(n-1)!^{d'} \prod_{j=1}^d (\eta_j + l)_{k-l}} \\ &= \prod_{j=1}^{d'} \frac{(\zeta_j + l + 1)_{n-1}}{(n-1)!} \prod_{j=1}^{d'} \frac{(\zeta_j + l + n)_{k-l}}{(\eta_j + l)_{k-l}} \prod_{j=d'+1}^d \frac{(k-l)!}{(\eta_j + l)_{k-l}} \cdot \frac{1}{(k-l)!^{d-d'}} . \end{aligned}$$

By Lemma 5.2 (i),

$$\sum_{v \in \mathfrak{M}_K^f} \log^+ \left| \prod_{j=1}^{d'} \frac{(\zeta_j + l + 1)_{n-1}}{(n-1)!} \right|_v \leq \sum_{j=1}^{d'} \log \mu_n(\zeta_j) \leq n \sum_{j=1}^{d'} \log \mu(\zeta_j) .$$

By Lemma 5.2, (iii) combined with Lemma 6.11 (i) we also have, provided $n \geq 2$

$$\sum_{v \in \mathfrak{M}_K^f} \log^+ \left| \prod_{j=1}^{d'} \frac{(\zeta_j + l + n)_{k-l}}{(\eta_j + l)_{k-l}} \right|_v \leq 2dmn \left(\sum_{j=1}^{d'} \log \mu(\zeta_j) + \text{den}(\eta_j) \right)$$

and using the same argument, for $n \geq 2$

$$\sum_{v \in \mathfrak{M}_K^f} \log^+ \left| \prod_{j=d'+1}^d \frac{(k-l)!}{(\eta_j + l)_{k-l}} \right|_v \leq 2dmn \left(\sum_{j=d'+1}^d \text{den}(\eta_j) \right) .$$

Summing up the three terms, one gets (since $|(k-l)!|_v = 1$ for $v \in S_3$)

$$\sum_{v \in S_1 \sqcup S_3} \log A_{n,v}(\zeta, \eta) \leq 3dmn \left(\sum_{j=1}^{d'} \log \mu(\zeta_j) + \sum_{j=1}^d \text{den}(\eta_j) \right) + (d-d') \sum_{p \leq (dmn)^{\frac{dm}{8}}} \log |(dm(n+1))!|_p^{-1} .$$

This completes the proof of Lemma 6.13. $\square$

Proof of Theorem 6.10. We now assume by contradiction

$$(44) \quad \lambda_0 + \sum_{i=1}^m \sum_{s=1}^d \lambda_{i,s} F_{s,v}(\gamma, \alpha_i) = 0 \text{ for all } v \in S_2 .$$

We are going to bound trivially $|B_{\nu_n}|_v$ by $(dm+1)^{\varepsilon_v[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]} \|\mathbf{a}_{\nu_n}\|_v \|\boldsymbol{\lambda}\|_v$ for $v \in \mathfrak{M}_K^\infty \sqcup S_1 \sqcup S_3$ and use Lemma 5.6 (iv) for primes lying in S_2 .

By Lemma 5.4 (i) together with Equation (32), one has

$$\sum_{v \in \mathfrak{M}_K^\infty} \log^+ \|\mathbf{a}_{\nu_n}\|_v \leq \sum_{v \in \mathfrak{M}_K^\infty} \log c_{n,v} + Bn + dmnh_\infty(\boldsymbol{\alpha}) ,$$

and thus

$$(45) \quad \sum_{v \in \mathfrak{M}_K^\infty} \log |B_{\nu_n}|_v \leq \sum_{v \in \mathfrak{M}_K^\infty} \log c_{n,v} + Bn + dmnh_\infty(\boldsymbol{\alpha}) + h_\infty(\boldsymbol{\lambda}) .$$

By Lemma 5.4 (ii) combined with Lemma 6.13, taking into account that $\alpha_i \in \mathcal{O}_K$, one has

$$(46) \quad \begin{aligned} \sum_{v \in S_1 \sqcup S_3} \log^+ \|\mathbf{a}_{\nu_n}\|_v &\leq \sum_{v \in S_1 \sqcup S_3} \log c_{n,v} + 3dmn \left(\sum_{j=1}^{d'} \log \mu(\zeta_j) + \sum_{j=1}^d \text{den}(\eta_j) \right) \\ &\quad + (d-d') \sum_{p \leq (dmn)^{1/8dm}} \log |(dm(n+1))!|_p^{-1} . \end{aligned}$$

Using $v_p(k!) \leq k/(p-1)$ and Lemma 6.11, one gets assuming $n \geq dm$

$$\begin{aligned} \sum_{p \leq (dmn)^{1/8dm}} \log |(dm(n+1))!|_p^{-1} &\leq 2dm(n+1) \sum_{p \leq (dmn)^{1/8dm}} \frac{\log p}{p} \\ &\leq \frac{2dm(n+1)}{8dm} \log(dmn) \leq \frac{n \log n}{8} . \end{aligned}$$

Combining Equation (46) and above inequality gives

$$(47) \quad \sum_{v \in S_1 \sqcup S_3} \log |B_{\nu_n}|_v \leq \sum_{v \in S_1 \sqcup S_3} (\log c_{n,v} + h_v(\boldsymbol{\lambda})) + 3dmn \left(\sum_{j=1}^{d'} \log \mu(\zeta_j) + \sum_{j=1}^d \text{den}(\eta_j) \right) + (d-d')n \log n .$$

We now turn to primes in S_2 . Notice, by (44), a straightforward computation gives

$$(48) \quad B_{\nu_n} = - \sum_{i=1}^m \sum_{s=1}^d \lambda_{i,s} r_{i,s}^{(n)} \in K_v ,$$

where

$$r_{i,s}^{(n)} = a_{\nu_n}^{(n)} F_{s,v}(\gamma, \alpha_i) - a_{\nu_n, i, s}^{(n)} .$$

We assume further $n \geq \exp(d)$ to ensure $p \geq e^{\frac{d}{d-d'}}$ if p is a prime below $v \in S_2$ and deduce from Equation (48) and Lemma 5.6 (iv) that

$$|B_{\nu_n}|_v \leq c_{n,v} \delta_v(n) p^{d' \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} \prod_{j=1}^d |\mu_n(\eta_j)|_v^{-1} \cdot |n!|_v^{d-d'} \cdot H_v(\lambda) ,$$

where

$$\delta_v(n) = \prod_{j=1}^{d'} (|\text{den}(\zeta_j)\zeta_j| + \text{den}(\zeta_j)((dm+1)n + dm)) \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]} .$$

One deduces

$$\begin{aligned} (49) \quad \sum_{v \in S_2} \log |B_{\nu_n}|_v &\leq \sum_{v \in S_2} \log c_{n,v} + d' \sum_{v \in S_2, p|v} \log p + n \sum_{j=1}^d \log \mu(\eta_j) \\ &\quad + \log \prod_{j=1}^{d'} (|\text{den}(\zeta_j)\zeta_j| + \text{den}(\zeta_j)((dm+1)n + dm)) \sum_{v \in S_2, p|v} 1 \\ &\quad + (d - d') \sum_{v \in S_2, p|v} \log |n!|_p . \end{aligned}$$

By Lemma 6.12

$$(50) \quad \sum_{v \in S_2, p|v} \log p \leq 8 \max_{1 \leq j \leq d} \{\text{den}(\zeta_j)\} dm n ,$$

$$\begin{aligned} (51) \quad \sum_{v \in S_2, p|v} 1 &\leq 8 \frac{\max_{1 \leq j \leq d} \{\text{den}(\zeta_j)\} dm n}{\log n} , \\ \sum_{p \in S_1} \log |n!|_p &\geq -n \sum_{p \leq (dmn)^{1/(8dm)}} \log(p)/p , \end{aligned}$$

and by formula 3.24 of [66]

$$-n \sum_{p \leq (dmn)^{1/(8dm)}} \log p/p \geq -\frac{n}{8dm} \log(dmn) = -\frac{n \log n}{8dm} + O(n) .$$

It follows since $n \leq 4 \max_{1 \leq j \leq d'} \{\text{den}(\zeta_j)\} dm n$

$$\sum_{p \in S_2} \log |n!|_p = \sum_{p \in S_1 \cup S_2} \log |n!|_p - \sum_{p \in S_1} \log |n!|_p = \log n! - \sum_{p \in S_1} \log |n!|_p \geq -n \log n + \frac{n \log n}{8dm} + O(n)$$

and summing up

$$\sum_{v \in S_2, p|v} \log |n!|_p \leq -\frac{7}{8} n \log n + O(n) .$$

Putting inequalities (50), (51) and the above one in (49), we conclude

$$(52) \quad \sum_{v \in S_2} \log |B_{\nu_n}|_v \leq -\frac{7}{8}(d - d')n \log n + C_1 n ,$$

where C_1 is some positive constant. Using Equations (45), (47) and (52), one gets

$$\sum_{v \in \mathfrak{M}_K} \log |B_{\nu_n}|_v \leq -\frac{3}{8}(d - d')n \log n + C_2 n + \log H .$$

We now choose

$$H \geq H_0 \text{ large enough, and } n = \frac{3 \log H}{(d - d') \log \log H} .$$

The above inequality cannot hold and thus the hypothesis that all the linear form vanishes for all primes in S_2 is false. $\square$

7 Corrigendum to Linear Forms in Polylogarithms

The proof of Lemma 4.8, as written in [29] was incorrect, we provide for a rectified version of Lemma 4.8, with conclusion unchanged. Since Lemma 4.8 of *loc. cit.* was used as Lemma 4.10 in our subsequent paper [30], one should apply the rectified version stated below instead and no other change in this paper is needed.

We keep all notations and conventions of [29] and state the new version of Lemma 4.8 [29]. The last line of the proof of Lemma 4.8 is incorrect since inequality $l - |\mathbf{I}| \geq 0$ is not enough to conclude that

$$|\mathbf{I}| + r(l - |\mathbf{I}|) \geq 2r^2 n + r^2 \implies |\mathbf{I}| \geq (2n + 1)r^2 .$$

LEMMA 7.1. *Let $0 \leq l$ be an integer and $\mathbf{I} = (a_1, \dots, a_r) \in \mathbb{N}^r$ such that $|\mathbf{I}| \leq l$. Assume further:*

- (i) *The $2r$ dimensional vector $(\mathbf{k}, \mathbf{k} - \mathbf{I})$ has two coordinates in common.*

Then,

$$\Delta_\alpha \circ \psi_{\beta, \mathbf{k}} \circ \psi_{\alpha, \mathbf{k} - \mathbf{I}} \left(g \frac{\partial^j f}{\partial \alpha^j} \right) = 0 \quad \text{for all } 0 \leq j \leq l - |\mathbf{I}| .$$

Moreover, assume

- (ii) *$l < (2n + 1)r^2$. Then, for every $|\mathbf{I}| \leq l$, and every j , $0 \leq j \leq l - |\mathbf{I}|$, one has*

$$\Delta_\alpha \circ \psi_{\beta, \mathbf{k}} \circ \psi_{\alpha, \mathbf{k} - \mathbf{I}} \left(g \frac{\partial^j f}{\partial \alpha^j} \right) = 0 .$$

PROOF. The first part of the statement is a subset of the first part of the original version of Lemma 7.1 unaffected by the error and needs not to be proved.

In view of condition (i), we can assume $a_s - s \geq 0$ for all s . Let $\mathbf{I}$ such that $|\mathbf{I}| \leq l$ and let $0 \leq j \leq l - |\mathbf{I}|$. For two r tuples of integers $\mathbf{a}, \mathbf{b}$ we say that $\mathbf{a} \leq \mathbf{b}$ if for every

$1 \leq s \leq r$, $a_s \leq b_s$ (partial order). By Leibnitz formula, setting $\text{Deri}^{\mathbf{m}} = \bigcirc_{s=1}^r \text{Deri}_{X_s, x}^{m_s}$ (recall $\text{Deri}_{X_s, x} = \left(\frac{\partial}{\partial X_s} + x/X_s \right) \circ [X_s]$)

$$\psi_{\alpha, \mathbf{k}-\mathbf{I}} \left(g \frac{\partial^j}{\partial \alpha^j} (f) \right) = \alpha^r \bigcirc_{s=1}^r \text{Eval}_{X_s \rightarrow \alpha} \left(\sum_{\mathbf{m} \leq \mathbf{I}-\mathbf{k}} \sum_{\mathbf{n} \leq \mathbf{I}-\mathbf{k}-\mathbf{m}} c(\mathbf{m}, \mathbf{n}) \text{Deri}^{\mathbf{m}}(g) \text{Deri}^{\mathbf{n}} \left(\frac{\partial^j}{\partial \alpha^j} (f) \right) \right)$$

where $c(\mathbf{m}, \mathbf{n})$ is some combinatorial factor. By definition of g , if $|\mathbf{m}| < \frac{r(r-1)}{2}$, one has

$$\bigcirc_{s=1}^r \text{Eval}_{X_s \rightarrow \alpha} (\text{Deri}^{\mathbf{m}}(g)) = 0 \ ,$$

since $\prod_{1 \leq i < j \leq r} (X_i - X_j)$ divides g .

There is thus no restriction to assume $|\mathbf{m}| \geq \frac{r(r-1)}{2}$. Now, since $\text{Eval}_{\beta \rightarrow \alpha}$ commutes with $\psi_{\beta, \mathbf{k}}, \psi_{\alpha, \mathbf{k}-\mathbf{I}}$, we have

$$\text{Eval}_{\beta \rightarrow \alpha} \circ \psi_{\beta, \mathbf{k}} \circ \psi_{\alpha, \mathbf{k}-\mathbf{I}} \left(g \frac{\partial^j}{\partial \alpha^j} (f) \right) = \psi_{\beta, \mathbf{k}} \circ \psi_{\alpha, \mathbf{k}-\mathbf{I}} \left(g \text{Eval}_{\beta \rightarrow \alpha} \left(\frac{\partial^j}{\partial \alpha^j} (f) \right) \right) \ .$$

We now consider

$$\bigcirc_{s=1}^r \text{Eval}_{X_s \rightarrow \alpha} \circ \text{Deri}^{\mathbf{n}} \left(\text{Eval}_{\beta \rightarrow \alpha} \left(\frac{\partial^j}{\partial \alpha^j} (f) \right) \right) \ .$$

Since $[(X_s - \alpha)(X_s - \beta)]^{r_n} \mid f$ for all $1 \leq s \leq r$, this quantity vanishes as soon as $j + |\mathbf{n}| < 2nr^2$. Hence, there is no restriction to assume

$$j + |\mathbf{n}| \geq 2nr^2 \ .$$

We can now conclude

$$2nr^2 \leq j + |\mathbf{n}| \leq j + |\mathbf{I}| - |\mathbf{k}| - |\mathbf{m}| \leq l - |\mathbf{I}| + |\mathbf{I}| - \frac{r(r+1)}{2} - \frac{r(r-1)}{2} = l - r^2 \ .$$

The lemma follows. □

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