

Poncelet property of planar elliptic integrable Kepler billiards

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Abstract

We consider the integrable dynamics of a Kepler billiard in the plane bounded by a branch of a conic section focused at the Kepler center. We show that in this case, for non-zero-energy orbits, the lines of consecutive second orbital foci along a billiard trajectory are all tangent to a fixed circle. Based on this observation we analyse in details the integrable dynamics of a planar Kepler billiard inside or outside an elliptic reflection wall, with the Kepler center occupying one of its foci. We identify the associated elliptic curve on which the dynamics is linearized, and the shift defined thereon. We also discuss explicit conditions on n -periodicity using Cayley's criteria.

Keywords: billiards, central potential, Kepler problem, Poncelet porism

MSC-Classification: 14H70, 37C79, 37J99, 37N05

1 Introduction

The classical Poncelet theorem asserts that in the plane, giving a pair of conic sections, a polygon inscribed in one conic section and circumscribing another always appears in an infinite family of such polygons [15]. A modern review due to Griffiths and Harris [9] explained that this theorem is ultimately linearized on an elliptic curve with a shift of finite order defined thereon. Starting with any point on the elliptic curve, by successive application of the shift we get an orbit of finite order in the complex domain. When the whole orbit is real, then the shift is real and this corresponds to a polygon having the prescribed property. Shifting the initial point in the real part we get an infinite family of polygons with the same property. The link of the Poncelet porism to billiard dynamics is described in [16].

Billiard systems in the plane are much studied in the research field of Dynamical Systems, both with and without a potential. Recently many studies on the integrability and dynamics of billiards in a Kepler-Coulomb potential, often referred to as Kepler billiards, have been carried out. In [4], Felder integrated out the planar Kepler billiard with a line as the wall of reflection, whose integrability is due to Gallavotti and Jauslin [7], in terms of elliptic curves with a direct computation. Some conditions on periodic points has been computed in this work as well. Conditions on n -periodicity as well as the topology of such a system have been subsequently studied in [8]. Many more integrable Kepler billiards in the plane are pointed out in [17, 18].

The aim of this article is to initiate analysis on the dynamics of integrable Kepler billiard systems with a branch of a (non-degenerate) conic section as reflection wall in the plane. In this system the particle moves on one side of the reflection wall in a Kepler potential field whose center is at a focus of the conic section. As first observed in [7] for the line case and later also in the cases with elliptic or hyperbolic reflection walls [14, 17, 18], along a billiard trajectory, the second orbital foci all lie on a circle centered at the second focus of the reflection wall.

In the case with a line reflection wall, Felder [4] has analyzed the integrable dynamics via the use of elliptic curves and elliptic function. Analogously it is natural to expect a link between the integrable dynamics of these billiard-type systems with elliptic curves. It is theoretical possible to identify this elliptic curve by direct parametrizing the elliptic boundary and the (elliptic) Keplerian orbits. We nevertheless adapt a different way based on a simple but for us unexpected geometric fact: Along a Kepler billiard orbit, the lines containing consecutive second orbital foci are all tangent to a circle, that we call the *foci-caustic circle*. Therefore by considering how the second orbital foci is iterated we reduce the analysis into the much familiar, “standard” situation of the Poncelet porism. In this situation the underlying elliptic curve, the shift, and the condition for n -periodicity are exclusively discussed in [9, 10]. An application of these classical works yields our desired results. We carry out the detailed analysis in this article for Kepler problem with negative energy with an elliptic boundary. We leave the detailed analysis of several other cases in our future upcoming works.

In a recent preprint [1] the authors showed in particular that at high energy, the only integrable Kepler billiards inside an ellipse are those with the Kepler center getting placed at a focus. Through the Hooke-Kepler correspondence, the billiard systems under our study are also directly related to centered Hooke billiard inside an ellipse, for which some analysis are available [2, 6, 11].

We organize this article as follows: In Section 2 we set up our systems under study and discuss the foci-circle, a notion first introduced in [13]. We complete the discussion for the parabolic case, in which the foci-circle should

be replaced by a *foci-line*. In Section 3 we analyse the geometric properties of the iterated second foci and show the existence of the foci-caustic circle. The billiard dynamics is now reduced to the dynamics of line segments tangent to a circle with end-points in another circle. The associated elliptic curve, the corresponding shift, and the relevant real part are computed in Section 4. Conditions on n -periodicity are computed in 5, in which we also include some explicit discussions for small n .

2 Integrable Kepler billiards and some of their geometrical properties

We consider a billiard system in the plane $\mathbb{R}^2$, for which the particle moves under the influence of a Kepler(-Coulomb) potential field and gets reflected at a branch of a conic section $\mathbf{K}$ with a focus at the Kepler center F . Let F' be the second focus of the conic section boundary: This leaves out only the case of a parabola but include two important degenerate cases of a circle for which $F = F'$ and a line for which F' is mirrored from F through the line (the case implicitly investigated by Boltzmann [3]).

We set $F' = O$ as the origin of our coordinate system. We put the Kepler center at $F = (-2c_{\mathbf{K}}, 0)$, so $c_{\mathbf{K}} \geq 0$ is the linear eccentricity. The semi major axis of $\mathbf{K}$ is denoted by $a_{\mathbf{K}}$. The eccentricity of $\mathbf{K}$ is $e_{\mathbf{K}} = c_{\mathbf{K}}/a_{\mathbf{K}}$. The degenerate case of a circle and a line correspond respectively to $c_{\mathbf{K}} = 0$ and $a_{\mathbf{K}} = \infty$, two cases of no eccentricity.

The Keplerian orbits of the underlying system are conic section branches with a focus at F . At elastic reflections the total energy is preserved, which gives us a conserved quantity of the system. We consider the cases that the total energy is non-zero. The orbital semi major axis is thus well-defined and is invariant under reflections. We denote it by a . The orbit is now completely determined by its second focus up to orientation. Thus to a billiard trajectory of Keplerian arcs reflecting at $\mathbf{K}$ there is an associated sequence of second foci $F_1 F_2 \dots$, which depends on the Kepler billiard trajectory and can be finite or infinite.

A major geometrical fact on the integrability of this mechanical billiard system is the following, which is established in [4, 7, 17, 18]:

Proposition 2.1. *The sequence of second foci associated to a Kepler billiard trajectory lies on a circle centered at F' .*

The circle is called the *foci-circle* of the system, and is denoted by $S(F', R)$ for which $R > 0$ is the radius. The quantity R is another conserved quantity of the system. For a graphical representation of the setup see Fig. 1.

The parabolic case needs to be considered separately, as it possesses only one focus. We set $R' = 2c_{\mathbf{K}} - R$, which is naturally conserved in the

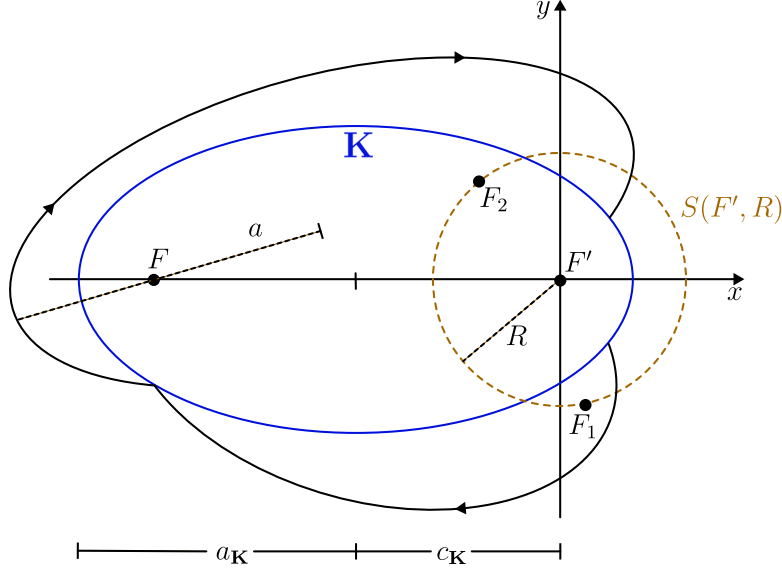


Figure 1: Setup of the system with elliptic boundary and foci-circle on which all consecutive second Kepler foci lie.

elliptic/hyperbolic boundary cases. We can rewrite the above proposition in the following form to include the parabolic boundary case:

Proposition 2.2. *The sequence of second foci associated to a Kepler billiard trajectory lies on a circle centered at F' in the elliptic/hyperbolic boundary cases. For a parabolic boundary with focus F the second Kepler billiard foci lie on a line perpendicular to its axis of symmetry.*

Proof. We complete the proof for both parabolic cases starting with the one enclosing F .

Call F the focus of the parabolic boundary equal to the Kepler force center, F_1 the second orbital focus, P_1 the point of reflection and ℓ_{P_1} the tangent line at P_1 to the parabolic boundary. Then the mirror point F_{P_1} of F with respect to ℓ_{P_1} is such that $F_{P_1}P_1$ is parallel to the symmetry axis of the parabolic boundary. The next second orbital focus F_2 lies on two circles, one centered at P_1 with radius $|F_1P_1|$ and the other one centered at F_{P_1} with radius $|F_{P_1}F_1|$. The first follows from optical definition of conic section and the second follows from a theorem of [7] when the reflection is considered to happen along ℓ_{P_1} . As both circles are symmetric with respect to the line $F_{P_1}P_1$, the point F_2 is just the mirror point of F_1 and thus F_1F_2 is perpendicular to the major axis of the parabolic boundary. Thus under iterations all F'_i s lie in the same line perpendicular to the symmetry axis of the parabolic boundary. See Fig. 2 for a graphical representation of the proof. This can be interpreted as the limiting case of a hyperbolic boundary with $F' \rightarrow \infty$ and $c_K - a_K = \text{const}$. In this case the curvature of the foci-

circle tends to zero with center tending toward ∞ , and in the limit we get the foci-line. $\square$

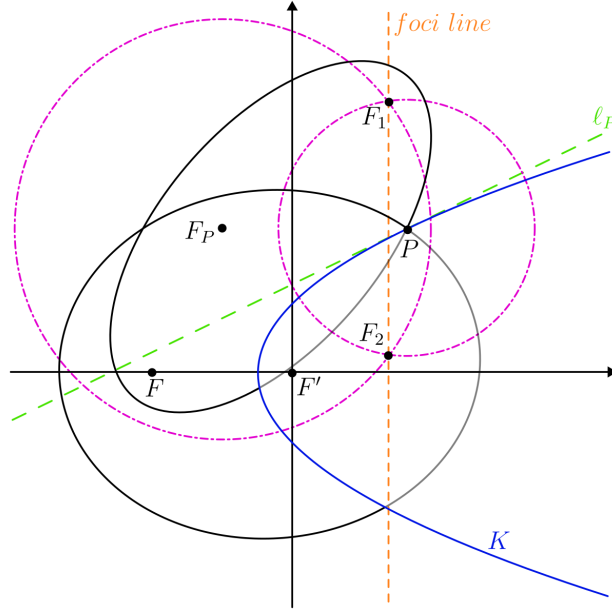


Figure 2: Foci line construction for the parabolic boundary.

The existence of R (respectively R') provides a conserved quantity independent of the total energy, thus makes these billiard systems integrable. This allows us to make detailed analysis of their (regular) dynamics.

3 The foci-chords and the foci-caustic circle

3.1 The Foci-Caustic Circle

Associated to a foci sequence $F_1 F_2 \dots$ there is a sequence of line segments $F_1 F_2, F_2 F_3, \dots$ that we call the *foci-chords*. The lines containing them are denoted by $\ell(F_i F_{i+1}), i = 1, 2, 3 \dots$ respectively. We have

Theorem 3.1. *If the reflection wall is a focused ellipse or a branch of a focused hyperbola, then along a Kepler billiard trajectory, the iterated line of foci-chords $\ell(F_1 F_2), \ell(F_2 F_3), \dots$ are tangent to the circle $S(x_0, |r_0|)$, called the foci-caustic circle, in which $x_0 = x_0(a, R)$ and $r_0 = r_0(a, R)$ are given by quadratic functions in terms of the semi major axis a of the Kepler orbits and the radius R of the foci-circle.*

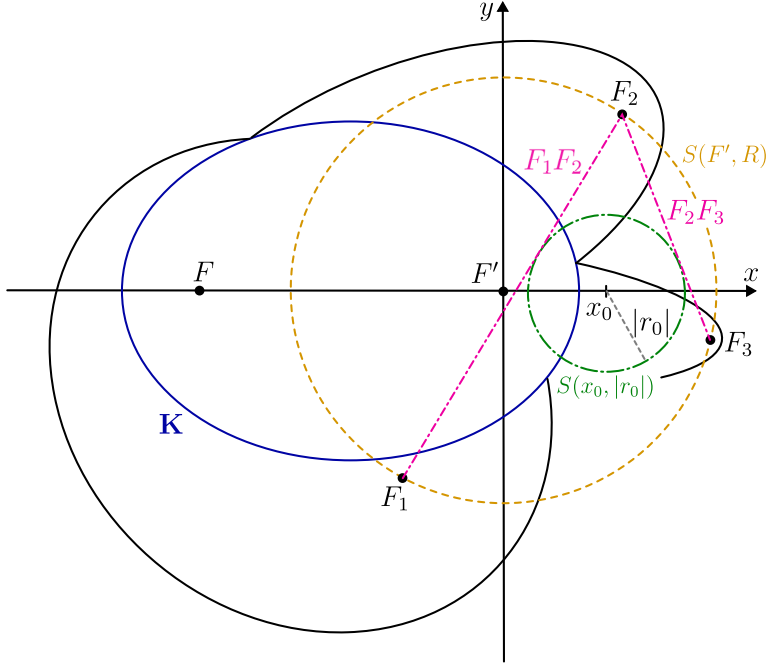


Figure 3: Foci-chords tangent to the foci-caustic circle.

For illustration see Fig. 3.

Thus the iterated foci-chords satisfy the usual setting of the Poncelet porism: The line containing the foci-chords are all tangent to the foci-caustic circle and have their ends in the foci-circle.

To proceed we distinguish cases: for the reflection boundary, we write $(\mathbf{E})$, $(\mathbf{P})$, $(\mathbf{H}_{F'})$, $(\mathbf{H}_F)$ respectively for the cases of elliptic boundary, parabolic boundary, hyperbolic branch enclosing F' , and hyperbolic branch enclosing F . We write respectively $(\mathcal{E})$, $(\mathcal{H}_{F'})$, $(\mathcal{H}_F)$ for orbit types: elliptic orbits, hyperbolic orbits enclosing F' , and hyperbolic orbits enclosing F . Note that hyperbolic orbits enclosing the empty focus F' appear when the Kepler-Coulomb center is repulsive.

The case (P) is special as we know that there is a line containing all the F'_i s along a billiard trajectory. Consequently the foci-chords all lie in the same line, and the notion of foci-caustic is not well-defined in this case.

Proof. We consider the case that an orbit with second focus F_1 is reflected to an orbit with second focus F_2 at the point $P_1 \in \mathbf{K}$. We assume $\mathbf{K}$ is not parabolic.

We set

$$|FP_1| = A_1, \quad |F'P_1| = B_1, \quad |F_1P_1| = C_1^+, \quad |F_2P_1| = C_1^-.$$

Depending on the type of $\mathbf{K}$ we have by its two-foci definition

$$(\mathbf{E}) : A_1 + B_1 = 2a_{\mathbf{K}};$$

$$(\mathbf{H}_{F'}) : A_1 - B_1 = 2a_{\mathbf{K}};$$

$$(\mathbf{H}_F) : B_1 - A_1 = 2a_{\mathbf{K}}.$$

Then for the orbits we have as well

$$(\mathcal{E}) : A_1 + C_1^+ = A_1 + C_1^- = 2a;$$

$$(\mathcal{H}_{F'}) : A_1 - C_1^- = A_1 - C_2 = 2a;$$

$$(\mathcal{H}_F) : A_1 - C_1^+ = A_1 - C_1^- = -2a.$$

So we have in all cases

$$C_1^+ = C_1^- := C_1,$$

and we obtain the following relationships in different cases:

$$(\mathbf{E}, \mathcal{E}) : C_1 - B_1 = 2(a - a_{\mathbf{K}});$$

$$(\mathbf{E}, \mathcal{H}_{F'}) : C_1 + B_1 = 2(a_{\mathbf{K}} - a);$$

$$(\mathbf{E}, \mathcal{F}) : C_1 + B_1 = 2(a + a_{\mathbf{K}});$$

$$(\mathbf{H}_{F'}, \mathcal{E}) : C_1 + B_1 = 2(a - a_{\mathbf{K}});$$

$$(\mathbf{H}_{F'}, \mathcal{H}_{F'}) : C_1 - B_1 = 2(a_{\mathbf{K}} - a);$$

$$(\mathbf{H}_{F'}, \mathcal{H}_F) : C_1 - B_1 = 2(a_{\mathbf{K}} + a);$$

$$(\mathbf{H}_F, \mathcal{E}) : C_1 + B_1 = 2(a + a_{\mathbf{K}});$$

$$(\mathbf{H}_F, \mathcal{H}_{F'}) : B_1 - C_1 = 2(a_{\mathbf{K}} + a);$$

$$(\mathbf{H}_F, \mathcal{H}_F) : B_1 - C_1 = 2(a_{\mathbf{K}} - a).$$

To continue, we denote the polar angle of P_1 by φ . The direction $F'P_1$ is thus given by the unit vector $(\cos \varphi, \sin \varphi)$. As mentioned in the proof of [14, Thm 2], F_1, F_2 are symmetric with respect to $F'P_1$. Thus the line $\ell(F_1F_2)$ is perpendicular to $F'P_1$ and is given by an equation of the form

$$x \cos \varphi + y \sin \varphi + k = 0,$$

which has signed distance k to F .

Write $h = |F_1F_2|/2$. By construction of the foci-circle, we have

$$k^2 + h^2 = R^2. \tag{1}$$

Applying Pythagorean theorem to P_1 , F_1 and the middle point

$S = (-k \cos(\varphi), -k \sin(\varphi))$ of $F_1 F_2$, we have the formula

$$(B_1 + k)^2 + h^2 = C_1^2. \quad (2)$$

We thus get from (1) and (2) that

$$k = \frac{C_1^2 - B_1^2 - R^2}{2B_1}.$$

In any of the nine cases listed above we have

$$\begin{aligned} C_1^2 - B_1^2 - R^2 &= (C_1 - B_1)(C_1 + B_1) - R^2 \\ &= 2(\pm a_{\mathbf{K}} \pm a) \cdot (2(\pm a_{\mathbf{K}} \pm a) \pm 2B_1) - R^2 = 4(a_{\mathbf{K}} \pm a)^2 - R^2 \pm 4(a_{\mathbf{K}} \pm a)B_1, \end{aligned}$$

with a corresponding choice of signs, with the note that the signs are chosen in the same way in the expressions $(\pm a_{\mathbf{K}} \pm a)$.

So we have

$$k = \frac{4(a_{\mathbf{K}} \pm a)^2 - R^2}{2B_1} \pm 2(a_{\mathbf{K}} \pm a).$$

As P_1 lies on $\mathbf{K}$ we have

$$B_1 = \frac{a_{\mathbf{K}}^2 \pm c_{\mathbf{K}}^2}{a_{\mathbf{K}} \pm c_{\mathbf{K}} \cdot \cos \varphi},$$

in which the sign in the numerator is $+$ when $\mathbf{K}$ is a hyperbolic branch, and is $-$ when $\mathbf{K}$ is an ellipse; the sign in the denominator is $-$ for $\mathbf{H}_F$ and is $+$ for the other two cases.

So we can now write

$$k = \frac{(4(a_{\mathbf{K}} \pm a)^2 - R^2)(a_{\mathbf{K}} \pm c_{\mathbf{K}} \cdot \cos \varphi)}{2(a_{\mathbf{K}}^2 \pm c_{\mathbf{K}}^2)} \pm 2(a_{\mathbf{K}} \pm a)$$

and thus the equation of the line $\ell(F_1 F_2)$ can be written as

$$\left(x + \frac{(4(a_{\mathbf{K}} \pm a)^2 - R^2)c_{\mathbf{K}}}{2(a_{\mathbf{K}}^2 \pm c_{\mathbf{K}}^2)}\right) \cos \varphi + y \sin \varphi + \frac{a_{\mathbf{K}}(4(a_{\mathbf{K}} \pm a)^2 - R^2)}{2(a_{\mathbf{K}}^2 \pm c_{\mathbf{K}}^2)} \pm 2(a_{\mathbf{K}} \pm a) = 0,$$

in which the choice of the sign is the same in the expressions $(a_{\mathbf{K}} \pm a)$.

Now if we write the above equation as

$$(x - x_0) \cos \varphi + y \sin \varphi + r_0 = 0$$

then this line, and as well as any of the other lines $\ell(F_i F_{i+1}) \dots$, is tangent to the circle $S(x_0, r_0)$ centered at x_0 with radius $|r_0|$, for which both x_0 and r_0 are uniquely determined by the reflection boundary $\mathbf{K}$ and the fixed conserved quantities (a, R) .

□

3.2 The Foci and Foci-caustic Circles: The $(\mathbf{E}, \mathcal{E})$ Case

We consider the $(\mathbf{E}, \mathcal{E})$ case in more details. Remind that as focused ellipses intersect at most at two points, the particle may move either inside or outside $\mathbf{E}$ and leads to the same dynamics up to a time reversal [14].

3.2.1 The Expressions of x_0, r_0 and the Parameter Range

We have

$$x_0 = -\frac{(4(a - a_{\mathbf{K}})^2 - R^2)c_{\mathbf{K}}}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)}, \quad r_0 = \frac{(4a_{\mathbf{K}}(a - a_{\mathbf{K}})^2 - a_{\mathbf{K}}R^2)}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)} + 2(a - a_{\mathbf{K}}). \quad (3)$$

Once the elliptic boundary $\mathbf{K}$ of type $(\mathbf{E})$ is given, we have

Proposition 3.2. *To effectively define a Kepler billiard system, the conserved quantities $(a, R) \in \mathbb{R}_+^2$ has to satisfy*

$$2a > a_{\mathbf{K}} - c_{\mathbf{K}} > 0, \quad |2a_{\mathbf{K}} - 2a| \leq R \leq 2a + 2c_{\mathbf{K}}.$$

For fixed $a_{\mathbf{K}} > c_{\mathbf{K}} > 0$, we call such a pair of (a, R) *admissible*.

Proof. The Hill's region of the Kepler problem with semi major axis a is the disk $D(F, 2a)$ centered at F , which has to contain some point $\mathbf{K}$ and in particular its pericenter, which has distant $a_{\mathbf{K}} - c_{\mathbf{K}}$ from F . We thus have the first condition.

Also the Hill's region $D(F, 2a)$ has to contain the second orbital focus, so it has to intersect the foci circle $S(F', R)$. This is equivalent to say that the closest point from $S(F', R)$ to F is in $D(F, 2a)$. We thus get

$$-2a \leq 2c_{\mathbf{K}} - R \leq 2a$$

and thus

$$2(c_{\mathbf{K}} - a) < R < 2(a + c_{\mathbf{K}}).$$

On a point of reflection P , if a corresponding second orbital focus is F_1 , then we have by the two-foci definition of ellipses

$$|FP| + |F'P| = 2a_{\mathbf{K}}, \quad |FP| + |F_1P| = 2a,$$

so we have

$$R = |F'F_1| \geq 2|a - a_{\mathbf{K}}|.$$

As

$$c_{\mathbf{K}} - a < a_{\mathbf{K}} - a \leq |a - a_{\mathbf{K}}|$$

we get the second condition. The combined conditions are represented in Fig. 4. $\square$

This condition is also sufficient, in the sense that for any value of (a, R) satisfying Prop. 3.2, there exist billiard trajectories of elliptic Keplerian orbits with these parameter values. This follows from the following analysis on orbits of the second foci, which asserts that for all these parameter values there are orbits of the second foci $\{F_i\}_{i \in \mathbb{Z}}$ and the following simple observation: If F_i and F_{i+1} are both real, then by the previous construction, F_i is reflected to F_{i+1} by the line perpendicular to $F_i F_{i+1}$ and pass their middle point. Moreover by construction, this line is tangent to $\mathbf{K}$ and thus in particular the corresponding elliptic Keplerian orbits are reflected to each other at the point of tangency. So the analysis in the sequel faithfully reflects the dynamics of the Kepler billiard system.

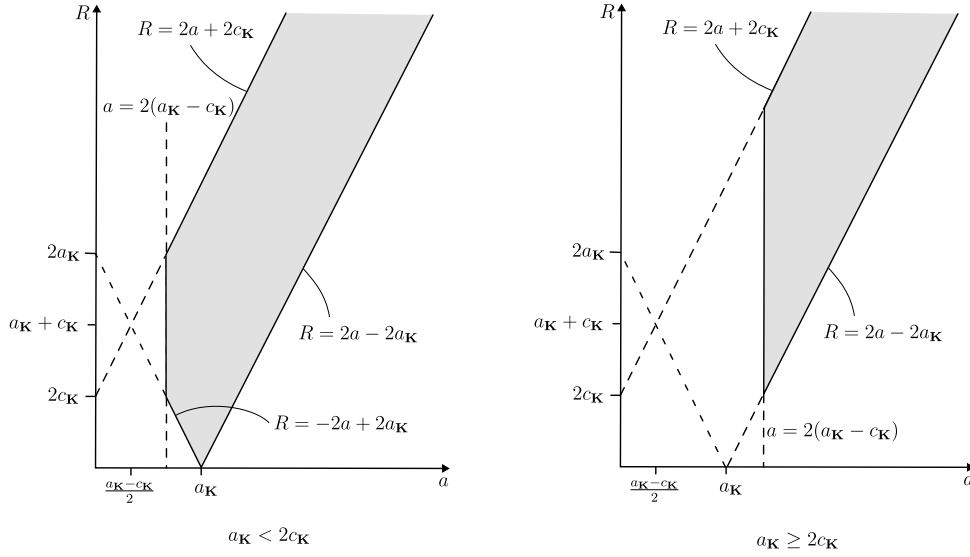


Figure 4: Gray regions displaying the admissible values (a, R) for physical reflections with the elliptic boundary.

3.2.2 Scenarios of the Two Circles

We fix an admissible $a > (a_K - c_K)/2$ and increase R from 0. We came across different scenarios.

When $R = 0$, we have

$$x_0 = -\frac{2(a - a_K)^2 c_K}{(a_K^2 - c_K^2)}, r_0 = \frac{2a_K(a - a_K)^2}{(a_K^2 - c_K^2)} + 2(a - a_K).$$

As $a_K > c_K$, we have $r_0 > |x_0|$ and thus the degenerate $S(F', 0) = F'$ lies inside $S(x_0, r_0)$, unless $a = a_K$.

When $a \neq a_K$, we increase R from 0: When R is sufficiently small, the foci-caustic circle lies inside the foci-circle. This is the prohibited case, as no lines intersecting $S(F', R)$ can be tangent to $S(x_0, r_0)$.

This situation does not change till $R = 2|a - a_{\mathbf{K}}|$, for which there hold $x_0 = 0, r_0 = 2(a - a_{\mathbf{K}}) > 0$. In this case the two circles $S(F', R)$ and $S(x_0, r_0)$ coincide. So we have $R > 2|a - a_{\mathbf{K}}|$ for admissible R as in Prop. 3.2.

Moreover by (3), for $R > 2|a - a_{\mathbf{K}}|$, always we have $x_0 > 0$. For $R > 2|a - a_{\mathbf{K}}|$, we consider the quantity

$$x_0 + |r_0| - R.$$

If $r_0 > 0$, then this quantity is

$$\begin{aligned} x_0 + r_0 - R &= \frac{4(a - a_{\mathbf{K}})^2 - R^2}{2(a_{\mathbf{K}} + c_{\mathbf{K}})} + 2(a - a_{\mathbf{K}}) - R \\ &= \frac{(2(a - a_{\mathbf{K}}) - R)(2(a + c_{\mathbf{K}}) - R)}{2(a_{\mathbf{K}} + c_{\mathbf{K}})}. \end{aligned}$$

If $r_0 < 0$, then it is

$$\begin{aligned} x_0 - r_0 - R &= -\frac{4(a - a_{\mathbf{K}})^2 - R^2}{2(a_{\mathbf{K}} - c_{\mathbf{K}})} - 2(a - a_{\mathbf{K}}) - R \\ &= -\frac{(2(a - a_{\mathbf{K}}) + R)(2(a - c_{\mathbf{K}}) - R)}{2(a_{\mathbf{K}} - c_{\mathbf{K}})}. \end{aligned}$$

We distinguish two cases:

Case 1: $a - c_{\mathbf{K}} \geq |a - a_{\mathbf{K}}|$, or equivalently $2a \geq a_{\mathbf{K}} + c_{\mathbf{K}}$, since we have $a_{\mathbf{K}} > c_{\mathbf{K}} > 0$.

In this case, we have $x_0 + |r_0| < R$ for $2|a - a_{\mathbf{K}}| < R < 2(a - c_{\mathbf{K}})$. As $x_0 > 0$, geometrically this implies that $S(x_0, r_0)$ lies inside $S(F', R)$.

In the particular case $r_0 = 0$, the foci-caustic circle degenerates into a point, and the corresponding foci-orbit are 2-periodic. This is given by

$$r_0 := \frac{4a_{\mathbf{K}}(a - a_{\mathbf{K}})^2 - a_{\mathbf{K}}R^2}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)} + 2(a - a_{\mathbf{K}}) = 0,$$

so

$$R = R_2 := 2\sqrt{(a - a_{\mathbf{K}})(a - c_{\mathbf{K}}^2/a_{\mathbf{K}})}.$$

As $c_{\mathbf{K}} < a_{\mathbf{K}}$, we have $R_2 > 2|a - a_{\mathbf{K}}|$. We also have $R_2 < 2|a - c_{\mathbf{K}}|$, which follows from the inequality

$$(a - a_{\mathbf{K}})(a - c_{\mathbf{K}}^2/a_{\mathbf{K}}) < (a - c_{\mathbf{K}})^2,$$

which after expansion is equivalent to

$$-a_{\mathbf{K}} - c_{\mathbf{K}}^2/a_{\mathbf{K}} < -2c_{\mathbf{K}},$$

which holds true multiplying both sides by $a_{\mathbf{K}}$ and then completing the square.

We thus have $2|a - a_{\mathbf{K}}| < R_2 < 2(a - c_{\mathbf{K}})$.

When $R = 2(a - c_{\mathbf{K}})$, we have

$$r_0 = \frac{(4a_{\mathbf{K}}(a - a_{\mathbf{K}})^2 - 4a_{\mathbf{K}}(a - c_{\mathbf{K}})^2)}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)} + 2(a - a_{\mathbf{K}}) = \frac{2a(c_{\mathbf{K}} - a_{\mathbf{K}})}{a_{\mathbf{K}} + c_{\mathbf{K}}} < 0,$$

$$x_0 = \frac{2c_{\mathbf{K}}(2a - a_{\mathbf{K}} - c_{\mathbf{K}})}{a_{\mathbf{K}} + c_{\mathbf{K}}}.$$

So in this case we have $x_0 - r_0 = x_0 + |r_0| = R$, meaning that the two circles are tangent to each other, with $S(x_0, r_0)$ lies inside $S(F', R)$.

Note that for this parameter value there exists a particular focal reflection property [14]: The ellipses tends monotonically to the degenerate limit of a collinear bouncing orbit at the pericenter of the boundary ellipse along its major axis. Now this property has an almost trivial geometrical proof: The iterated (oriented) foci-chords are well-defined outside the point of tangency of these two circles, and tend to the point of tangency.

For $2(a - c_{\mathbf{K}}) < R < 2(a + c_{\mathbf{K}})$, the two circles $S(x_0, r_0)$ and $S(F', R)$ intersect transversely. Only on the part of $S(F', R)$ lying outside $S(x_0, r_0)$ there defines the dynamics of our iterated foci-chords.

When $R = 2(a + c_{\mathbf{K}})$, we have $x_0 + r_0 = x_0 - |r_0| = R$. This means an outer tangency of the two circles $S(F', R)$ and $S(x_0, r_0)$.

The case that $R > 2(a + c_{\mathbf{K}})$, so that the two circles $S(F', R)$ and $S(x_0, r_0)$ are separated, is disallowed by Prop. 3.2.

Case 2: $a - c_{\mathbf{K}} < |a - a_{\mathbf{K}}|$, or equivalently $a_{\mathbf{K}} - c_{\mathbf{K}} < 2a < a_{\mathbf{K}} + c_{\mathbf{K}}$.

We have for $2|a - a_{\mathbf{K}}| < R < 2(a + c_{\mathbf{K}})$, there holds

$$x_0 + r_0 < R, \quad x_0 - r_0 > R,$$

so the two circles $S(x_0, r_0)$ and $S(F', R)$ intersect transversely.

For $R = 2(a + c_{\mathbf{K}})$ the two circles are tangent to each other. The case $R > 2(a + c_{\mathbf{K}})$ is disallowed by Prop. 3.2.

4 The Associated Elliptic Curve of the $(\mathbf{E}, \mathcal{E})$ Case

4.1 The Griffiths-Harris Construction [9]

Write $\mathcal{C}_1 = S(O, R)$, $\mathcal{C}_2 = S(x_0, r_0)$ for the two circles, where $O = F'$. When considering periodic trajectories of an integrable Kepler billiard being one of the cases listed before, considering foci-chords leads us to a particular case of the Poncelet porism. An intrinsic link of the Poncelet theorem and the

theory of elliptic curves is pointed out by Griffiths-Harris in their classical work [9].

According to [9], we should consider the lifted quadrics in $\mathbf{CP}^2$. We write $[z_1 : z_2 : z_3]$ for the homogeneous coordinates of points in $\mathbf{CP}^2$. The complexifications of the curves $\mathcal{C}_1, \mathcal{C}_2$ are given respectively by the homogeneous equations

$$\mathcal{C}_1 : z_1^2 + z_2^2 - R^2 z_3^2 = 0, \quad (4)$$

$$\mathcal{C}_2 : z_1^2 - 2x_0 \cdot z_1 \cdot z_3 + z_2^2 + (x_0^2 - r_0^2)z_3^2 = 0. \quad (5)$$

These are associated respectively to the symmetric matrices

$$Q_1[z_1, z_2, z_3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -R^2 \end{pmatrix}, \quad (6)$$

$$Q_2[z_1, z_2, z_3] = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & 0 \\ -x_0 & 0 & x_0^2 - r_0^2 \end{pmatrix}. \quad (7)$$

These curves are singular if and only if there associated symmetric matrices are singular, corresponding respectively to the cases $R = 0$ and $r_0 = 0$. When $R = 0$, the foci-circle degenerates into a point and thus all points are fixed (1-periodic). This is possible only if $a = a_{\mathbf{K}}$. The case $r_0 = 0$ corresponds to the case when the foci-caustic circle degenerates into a point, and the corresponding orbits are all 2-periodic: This is the case $R = R_2$.

Otherwise, both curves are non-singular. So we assume $R \notin \{0, R_2\}$ from now on.

When $\mathcal{C}_1, \mathcal{C}_2$ are non-singular, their dual conics $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$, consisting respectively of their tangent lines, are also non-singular in the dual projective space $\mathbf{CP}^{2*}$, associated with symmetric matrices Q_1^{-1} and Q_2^{-1} respectively.

Define the *incidence variety* as

$$E = \{(p, \xi) \in \mathcal{C}_2 \times \mathcal{C}_1^*, p \in \xi\}.$$

When $\mathcal{C}_1$ and $\mathcal{C}_2$ are transverse, then E is a smooth algebraic curve of genus 1 in $\mathbf{CP}^2 \times \mathbf{CP}^{2*}$. A proper choice of a base point as identity for the group law makes it into an elliptic curve.

A handy way to describe this elliptic curve is introduced in [10], which also leads to Cayley's criterion for n -periodic trajectories. This is to consider the following elliptic curve which is birationally equivalent to E^1 , defined through the pencil of quadrics through $\mathcal{C}_1$ and $\mathcal{C}_2$, with ∞ as the identity:

$$\mathcal{D} : y^2 = -\det(t \cdot Q_1 + Q_2).$$

¹Moreover, this birational equivalence preserves realness.

Both Q_i and $-Q_i$ define the same quadric. The negative sign is nevertheless preferred to help us tracking the real part of the associated elliptic curves.

We have

$$t \cdot Q_1 + Q_2 := \begin{pmatrix} t+1 & 0 & -x_0 \\ 0 & t+1 & 0 \\ -x_0 & 0 & -R^2t + x_0^2 - r_0^2 \end{pmatrix}, \quad (8)$$

whose determinant is computed as

$$\begin{aligned} \det(t \cdot Q_1 + Q_2) &= (t+1)^2(-R^2t + x_0^2 - r_0^2) - x_0^2(t+1), \\ &= -R^2(t+1)(t+t_1)(t+t_2). \end{aligned} \quad (9)$$

So we obtain the following expression for $\mathcal{D}$:

$$\mathcal{D} : y^2 = R^2(t+1)(t+t_1)(t+t_2),$$

in which

$$\begin{aligned} t_1 &= -\frac{1}{2}\left(\frac{x_0^2 - r_0^2}{R^2} - 1\right) - \frac{1}{2}\sqrt{\left(\frac{x_0^2 - r_0^2}{R^2} - 1\right)^2 - 4\frac{r_0^2}{R^2}}, \\ t_2 &= -\frac{1}{2}\left(\frac{x_0^2 - r_0^2}{R^2} - 1\right) + \frac{1}{2}\sqrt{\left(\frac{x_0^2 - r_0^2}{R^2} - 1\right)^2 - 4\frac{r_0^2}{R^2}}. \end{aligned}$$

4.2 Determinant and different cases of real forms

The sign of the normalized determinant of the quadratic equation

$$(t+t_1)(t+t_2) = 0$$

is

$$\Delta := (x_0^2 - r_0^2 - R^2)^2 - 4r_0^2R^2,$$

which determines the number of ramification points on the real part of $\mathcal{D}$. A short computation leads to the factorization

$$\Delta = (x_0 + r_0 - R)(x_0 - r_0 + R)(x_0 - r_0 - R)(x_0 + r_0 + R).$$

For $2|a - a_{\mathbf{K}}| < R < 2(a - c_{\mathbf{K}})$, the circle $S(x_0, r_0)$ lies inside $S(F', R)$. We thus have

$$-R < x_0 + |r_0| < R$$

and thus $\Delta > 0$ in this case. As $t_1 < t_2 < 0 < 1$ in this case, the curve is regular. All the three finite ramification points of $\mathcal{D}$ are real and thus its real part is the union of two circles.

For $R = 2(a - c_{\mathbf{K}})$, the circle $S(x_0, r_0)$ is tangent from inside to $S(F', R)$. In this case $\Delta = 0$ and the elliptic curve $\mathcal{D}$ is singular.

For $2(a - c_{\mathbf{K}}) < R < 2(a + c_{\mathbf{K}})$, the two circles intersect transversally. In this case we have $x_0 > 0, r_0 < 0$ and

$$x_0 - r_0 > R, x_0 + r_0 > -R,$$

with moreover

$$x_0 - r_0 > 0 > -R, x_0 + r_0 < R.$$

So in this case we have $\Delta < 0$, and two of the three finite ramification points are not real. The real part is thus a circle.

We summarize these discussions in the following theorem:

Theorem 4.1. *The incidence variety E is birationally equivalent to the elliptic curve given by*

$$\mathcal{D}' : y^2 = (t + 1)(t + t_1)(t + t_2)$$

in which

$$t_1 = \frac{1}{2}\left(1 - \frac{x_0^2 - r_0^2}{R^2}\right) - \frac{1}{2R^2}\sqrt{\Delta},$$

$$t_2 = \frac{1}{2}\left(1 - \frac{x_0^2 - r_0^2}{R^2}\right) + \frac{1}{2R^2}\sqrt{\Delta},$$

with

$$\Delta := (x_0^2 - r_0^2 - R^2)^2 - 4r_0^2 R^2 = (x_0 + r_0 - R)(x_0 - r_0 + R)(x_0 - r_0 - R)(x_0 + r_0 + R).$$

Moreover, the real part $E_{\mathbb{R}}$ of E is mapped to the real part $\mathcal{D}'_{\mathbb{R}}$ of $\mathcal{D}'$, which we characterize as follows:

- $2|a - a_{\mathbf{K}}| < R < 2(a - c_{\mathbf{K}})$: the union of two circles;
- $R = 2(a - c_{\mathbf{K}})$: two circles pinched at one point;
- $2(a - c_{\mathbf{K}}) < R < 2(a + c_{\mathbf{K}})$: a circle.

Next we identify the corresponding shift which linearizes the dynamics on the elliptic curve, which is denoted by σ . According to [9] this is given by one of the two points over $t = 0$ in $\mathcal{D}$: We easily find that in $\mathcal{D}'$:

$$y^2 = t_1 t_2 = r_0^2 / R^2, \quad y = \pm r_0 / R.$$

When $2|a - a_{\mathbf{K}}| < R < 2(a - c_{\mathbf{K}})$, the real part $\mathcal{D}'_{\mathbb{R}}$ is defined over $(-1, -t_2)$ and $(-t_1, \infty)$ and the shift σ sends a point $t = \infty$ to a point over $t = 0$ lying in different circle components. By the theory of elliptic curves, σ is independent of the initial points, and thus always permutes the two circle components, so $\sigma \circ \sigma$ is a rotation when restricted to one of the two circles. When $2(a - c_{\mathbf{K}}) < R < 2(a + c_{\mathbf{K}})$, the real part is a circle, with a nonlinear linear shift σ defined there on.

4.3 The Weierstrass Form

We start with the curve

$$\mathcal{D}' : y^2 = (t+1)(t+t_1)(t+t_2)$$

and mark the shift σ represented by $\sigma(\infty, \infty) = (0, \pm r_0/R)$ of our relevance.

Expanding the defining equation for $\mathcal{D}'$ we have

$$y^2 = t^3 + (t_1 + t_2 + 1)t^2 + (t_1 + t_2 + t_1 t_2)t + t_1 t_2.$$

We set $t = \tilde{t} - \frac{t_1 + t_2 + 1}{3}$ to eliminate the quadratic term. We get

$$y^2 = \left(\tilde{t} - \frac{t_1 + t_2 + 1}{3}\right)^3 + (t_1 + t_2 + 1)\left(\tilde{t} - \frac{t_1 + t_2 + 1}{3}\right)^2 + (t_1 + t_2 + t_1 t_2)\left(\tilde{t} - \frac{t_1 + t_2 + 1}{3}\right) + t_1 t_2.$$

Expanding this equation and writing $y = \tilde{y}/2$, we obtain

$$\begin{aligned} \tilde{y}^2 &= 4\tilde{t}^3 + 4\left((t_1 + t_2 + t_1 t_2) - \frac{(t_1 + t_2 + 1)^2}{3}\right)\tilde{t} \\ &+ 4\left(\frac{2(t_1 + t_2 + 1)^3}{27} - \frac{t_1 + t_2 + 1}{3}(t_1 + t_2 + t_1 t_2) + t_1 t_2\right), \end{aligned}$$

which represents $\tilde{\mathcal{D}}$ in the Weierstrass form

$$\tilde{\mathcal{D}} : \tilde{y}^2 = 4\tilde{t}^3 - g_2\tilde{t} - g_3$$

with

$$\begin{aligned} g_2 &= -4\left((t_1 + t_2 + t_1 t_2) - \frac{(t_1 + t_2 + 1)^2}{3}\right), \\ g_3 &= -4\left(\frac{2(t_1 + t_2 + 1)^3}{27} - \frac{t_1 + t_2 + 1}{3}(t_1 + t_2 + t_1 t_2) + t_1 t_2\right). \end{aligned}$$

This curve is parametrized by a Weierstrass $\wp$ -function $\wp$ such that $(\tilde{t}, \tilde{y}) = (\wp, \wp')$ is associated to a lattice $\Lambda \in \mathbb{C}$ uniquely determined by the equation of $\tilde{\mathcal{D}}$.

On the other hand from the equation of $\mathcal{D}'$ we have

$$\tilde{\mathcal{D}} : \tilde{y}^2 = 4(\tilde{t} - e_1)(\tilde{t} - e_2)(\tilde{t} - e_3),$$

in which

$$e_1 = \frac{t_1 + t_2 - 2}{3}, e_2 = \frac{t_1 + 1 - 2t_2}{3}, e_3 = \frac{t_2 + 1 - 2t_1}{3}.$$

We have in any case $e_1 \in \mathbb{R}$. When $\Delta > 0$, we have $t_1 < t_2 < 0 < 1$ and thus $e_1 < e_2 < e_3$ in this case. When $\Delta < 0$ the pair e_2, e_3 are complex conjugate.

A pair of fundamental periods $\{\omega_1, \omega_2\}$ of the lattice Λ associated to the elliptic curve can be computed by integrating the invariant differential $\frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}$ over the fundamental cycles. Explicitly, when $\Delta > 0$, we may use the formulas

$$\omega_1/2 = \int_{e_3}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}},$$

$$\omega_2/2 = \int_{-\infty}^{e_1} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}$$

to compute the periods, in which ω_1 is real and ω_2 is purely imaginary. The fundamental domain is a rectangle. The real part of $\tilde{D}$ is the union of two circles, each having period ω_1 .

The shift on $\tilde{D}$ is given by integration from the point $(\tilde{t}, \tilde{y}) = (\infty, \infty)$ to $(\tilde{t}, \tilde{y}) = (\frac{t_1+t_2+1}{3}, \pm 2r_0/R)$ of the invariant differential $\frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}$. Explicitly this shift can be computed as one of the two choices of $\pm\xi$, with:

$$\xi := \int_{\frac{t_1+t_2+1}{3}}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}.$$

As $\frac{t_1+t_2+1}{3} < e_3 = \frac{t_2+1-2t_1}{3}$, we have $\xi > \omega_1/2$, so shifting by ξ exchange the two circles.

When $\Delta < 0$, we know that e_1 remains real, e_2, e_3 are non-real and they are complex conjugate. In this case the fundamental domain of the periodic lattice is a rhombus, generated by $\omega_1 \in \mathbb{R}$ and ω_2 such that $\Re\{\omega_2/\omega_1\} = 1/2$. The real part of $\tilde{D}$ is a circle, whose period is computed from

$$\omega_1/2 = \int_{e_1}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}.$$

A complex half-period can be computed by

$$\int_{e_2}^{e_3} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}$$

along a (vertical) path in $\mathbb{C}$ connecting e_2 to e_3 .

The associated shift is $\pm\xi$ with

$$\xi := \int_{\frac{t_1+t_2+1}{3}}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}}.$$

As now $e_1 < \frac{t_1+t_2+1}{3}$, the shift is within a half real period.

4.4 Torsion in the sense of KAM theory

The frequency of the mapping is given by ξ/ω_1 . For given a , when this quantity is given by a non-constant function on R , then this effective change of frequency allows us to apply Moser's version of the KAM theorem on this energy hypersurface to the billiard mapping to conclude that sufficiently small change, say in the C^∞ topology does not completely destroy the invariant curves of the system. Instead, a set of positive measure of these invariant curves remain.

When $\Delta > 0$, we explicitly obtain that ξ/ω_1 is given by

$$\int_{\frac{t_1+t_2+1}{3}}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}} / \left(2 \int_{\frac{t_1+t_2+1}{3}-t_1}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}} \right),$$

When $\Delta < 0$, the frequency is given by

$$\int_{\frac{t_1+t_2+1}{3}}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}} / \left(2 \int_{\frac{t_1+t_2+1}{3}-1}^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}} \right),$$

in both of which the integrands are positive.

Proposition 4.2. *For almost all admissible choices of the orbital semi major axis a , the frequency ξ/ω_1 is a non-constant function of R .*

With the explicit formula it is possible to do concrete computations. For our purpose we are nevertheless content to have the following non-explicit, “soft” argument.

Proof. By analyticity we only need to show that for an admissible fixed a , the frequency is different for two different values of R . We choose $a > a_{\mathbf{K}} + c_{\mathbf{K}}$ to avoid the Case 2 situation in Section 3.2.2. We know that in this situation, we can have $R = 2(a - c_{\mathbf{K}})$, for which there exists a singular orbit which bounces in the primary axis with the pericenter, which is a 1-periodic orbit. This corresponds to the case $\Delta = 0$ and is the limiting case of both $\Delta > 0$ and $\Delta < 0$. In the case $\Delta > 0$, when $R = R_2$ the corresponding orbits are two-periodic. When $\Delta < 0$ the foci-circle and the foci-caustic circles are intersected and in particular there are no 1-periodic orbits.

So for this value of a , the frequency mapping changes with R in a non-trivial manner. Our assertion follows. $\square$

5 Condition on n -Periodicity

5.1 Existence of n -periodic orbits in the elliptic case

We follow the classical argument presented in [16] for the existence of n -periodic orbits in the $(\mathbf{E}, \mathcal{E})$ case. The following proposition is rather weak

and can certainly be generalized to many other cases, while here we merely want to establish the basic fact of their existence.

As in [16], we consider the cyclic configurations space of n points on $\mathbf{E}$:

$$G(\mathbf{E}, n) = \{(q_1, q_2, \dots, q_n) : q_i \in \mathbf{E}, q_i \neq q_j \ \forall i \neq j\}.$$

This space is not connected. Its connected components are further labelled by their rotation number, which is an integer counting how many times the closed collection of chords $\{q_1q_2, q_2q_3, \dots, q_nq_1\}$ winds around $\mathbf{E}$.

Proposition 5.1. *In the $(\mathbf{E}, \mathcal{E})$ case, for a given orbital semi major axis a such that $a > a_{\mathbf{K}} + c_{\mathbf{K}}$, for any integer $n \geq 2$, there exists R such that all the orbits of (a, R) are n -periodic.*

Proof. We transform the problem into the equivalent mechanical billiard problem of an attractive Hooke center lying at the center of an elliptic reflection wall, still denoted by $\mathbf{E}$, based on the complex square mapping

$$\mathbb{C} \rightarrow \mathbb{C}, z \mapsto z^2.$$

See [14, 17] for details. In particular, the negative of the energy of the Kepler problem appears as the string constant in the corresponding Hooke problem having normalized energy 1. The relevant corresponding cyclic configurations space is the symmetric cyclic configuration space

$$\tilde{G}(\mathbf{E}, n) = \{z_1, -z_1, z_2, -z_2, \dots, z_n, -z_n, z_i^2 \neq z_j^2 \ \forall i \neq j\},$$

in which $z_i^2 = q_i, i = 1, 2, \dots, n$.

The energy being given, by assumption $\mathbf{E}$ lies entirely in the corresponding Hill's region of the planar Hooke problem. Thus the corresponding Jacobi metric is a non-degenerate Riemannian metric in the region bounded by $\mathbf{E}$, and the billiard system is equivalent up to a time change to the billiard system defined by geodesics of the Jacobi metric. We denote by $d_J(x, y)$ the Jacobi-distance between two points x, y . For the latter, we consider the maximum of the total length function $\sum_{i=1}^n d_J(z_i, z_{i+1})$ with the natural convention $z_{n+1} := z_1$. By triangle inequality the maximum must be achieved in the interior of a connected component of $\tilde{G}(\mathbf{E}, n)$ and thus gives an n -periodic orbit, which in turn gives an n -periodic orbit of the Kepler billiard we have started with. Finally, due to its link with the Poncelet porism, for the value of (a, R) corresponds to such an n -periodic orbit, all orbits are n -periodic.

□

In the above proposition, the assertion $a > a_{\mathbf{K}} + c_{\mathbf{K}}$ is rather strong. It is only made to have $\mathbf{E}$ lying entirely in the interior of the Hill's region, for which we do not have to analyse the intersection of the zero-velocity curve

(the boundary of the Hill's region) with $\mathbf{E}$ and thus the argument takes its most simple form. On the other hand, already the analysis of the 2-periodic case in Sec 3.2.2, which only exist in Case 1, shows that when $\mathbf{E}$ intersects the zero-velocity curve, the situation is more involved.

5.2 Cayley's Criteria for n -Periodicity

Explicit conditions on Poncelet n -gon have been found by Cayley and reviewed in modern terms by Griffiths-Harris [9]. This can be applied directly in our setting.

To state the criteria, we expand the function $\sqrt{\det(t \cdot C_1 + C_2)}$ into Taylor series around $t = 0$:

$$\sqrt{\det(t \cdot C_1 + C_2)} = A_0 + A_1 t + A_2 t^2 + \dots$$

Then the condition for having a Poncelet n -gon for $n \geq 3$ circumscribed by $\mathcal{C}_2$ and inscribed in $\mathcal{C}_1$ is

$$\det \begin{pmatrix} A_2 & A_3 \dots & A_{m+1} \\ A_3 & A_4 \dots & A_{m+2} \\ \dots & \dots & \dots \\ A_{m+1} & \dots & A_{2m} \end{pmatrix} = 0 \quad \text{for odd } n = 2m + 1, \quad (10)$$

$$\det \begin{pmatrix} A_3 & A_4 & \dots & A_{m+1} \\ A_4 & A_5 & \dots & A_{m+2} \\ \dots & \dots & \dots & \dots \\ A_{m+1} & A_{m+2} & \dots & A_{2m-1} \end{pmatrix} = 0 \quad \text{for even } n = 2m. \quad (11)$$

$$(12)$$

5.2.1 Computations of A_i 's

From previous discussions, we have

$$t \cdot Q_1 + Q_2 := \begin{pmatrix} t+1 & 0 & -x_0 \\ 0 & t+1 & 0 \\ -x_0 & 0 & -R^2 t + x_0^2 - r_0^2 \end{pmatrix}, \quad (13)$$

whose determinant is

$$\det(t \cdot Q_1 + Q_2) = -R^2(t+1) \left(t^2 - \left(\frac{x_0^2 - r_0^2}{R^2} - 1 \right) t + \frac{r_0^2}{R^2} \right).$$

We do not need to bother with the negative factor -1 in the discussions of periodicity. We set

$$F = R^2(t+1) \left(t^2 - \left(\frac{x_0^2 - r_0^2}{R^2} - 1 \right) t + \frac{r_0^2}{R^2} \right).$$

We have

$$F = \sum_{i=0}^3 C_i t^i,$$

with

$$C_0 = r_0^2, C_1 = (2r_0^2 + R^2 - x_0^2), C_2 = 2R^2 - x_0^2 + r_0^2, C_3 = R^2.$$

So if we set

$$\sqrt{F} = \sum_{i=0}^{\infty} A_i t^i,$$

then by comparing coefficients we obtain

$$\begin{aligned} A_0^2 &= C_0, \\ A_0 A_1 &= C_1, \\ 2A_0 A_2 + A_1^2 &= C_2, \\ 2A_0 A_3 + 2A_1 A_2 &= C_3, \\ 2A_0 A_4 + 2A_1 A_3 + A_2^2 &= 0, \\ 2A_0 A_5 + 2A_1 A_4 + 2A_2 A_3 &= 0, \\ 2A_0 A_6 + 2A_1 A_5 + 2A_2 A_4 + A_3^2 &= 0, \\ &\dots\dots\dots \end{aligned}$$

After choosing one of the two roots from the first relation, the other coefficients are determined consecutively in an explicit manner.

The two choices of A_0 results in just a sign change in all that follows, so we may just choose $A_0 = r_0$. We thus find:

$$\begin{aligned} A_1 &= C_1/(2A_0) = \frac{2r_0^2 + R^2 - x_0^2}{2r_0}, \\ A_2 &= \frac{C_2 - A_1^2}{2A_0} \\ A_3 &= \frac{C_3 - 2A_1 A_2}{2A_0} = \frac{R^2 - 2A_1 A_2}{2r_0}. \end{aligned}$$

Substituting A_1 and A_2 in we get

$$A_3 = \frac{8R^2 r_0^4 - (2r_0^2 + R^2 - x_0^2)(4R^2 r_0^2 - (x_0^2 - R^2)^2)}{16r_0^5}.$$

We may then find the other A'_i s iteratively:

$$\begin{aligned}
A_4 &= -\frac{2A_1A_3 + A_2^2}{2A_0}, \\
A_5 &= -\frac{2A_1A_4 + 2A_2A_3}{2A_0}, \\
A_6 &= -\frac{2A_1A_5 + 2A_2A_4 + A_3^2}{2A_0}.
\end{aligned}$$

In general for $m \geq 2$, we have

$$A_{2m} = -\frac{2A_1A_{2m-1} + 2A_2A_{2m-2} + \cdots + A_m^2}{2A_0},$$

and

$$A_{2m+1} = -\frac{2A_1A_{2m} + 2A_2A_{2m-1} + \cdots + A_mA_{m+1}}{2A_0}.$$

Below we analyse the case corresponds to small n :

5.2.2 $n = 2$

The case $n = 2$ corresponding to the degenerate case when the foci-caustic circle becomes a point, and has been considered before in Sec 3.2.2. Recall that the condition $R = R_2$ is obtained from

$$r_0 := \frac{(4a_{\mathbf{K}}(a - a_{\mathbf{K}})^2 - a_{\mathbf{K}}R^2)}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)} + 2(a - a_{\mathbf{K}}) = 0.$$

This results into a real $R_2 = 2\sqrt{(a - a_{\mathbf{K}})(a - c_{\mathbf{K}}^2/a_{\mathbf{K}})}$ only in Case 1 in Sec 3.2.2. Fig. 5 displays two Poncelet configurations for a 2-periodic szenario.

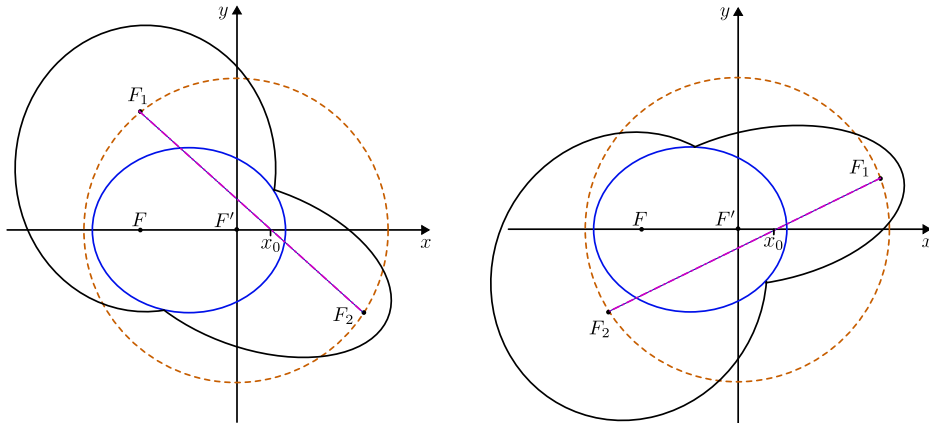


Figure 5: 2-periodic orbits and the Poncelet property.

5.2.3 $n = 3$

The condition on 3-periodic points is

$$A_2 = \frac{4R^2r_0^2 - (x_0^2 - R^2)^2}{8r_0^3} = 0,$$

which is equivalent to

$$(2Rr_0 + x_0^2 - R^2)(2Rr_0 - x_0^2 + R^2) = 0, \quad (14)$$

so in general we have two sets of conditions for 3-periodicity, obtained by equalizing each factor to 0.

From (3) we have

$$x_0 = -\frac{(4(a - a_{\mathbf{K}})^2 - R^2)c_{\mathbf{K}}}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)},$$

and

$$r_0 = \frac{(4a_{\mathbf{K}}(a - a_{\mathbf{K}})^2 - a_{\mathbf{K}}R^2)}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)} + 2(a - a_{\mathbf{K}}) = \frac{4(a - a_{\mathbf{K}})(a_{\mathbf{K}}a - c_{\mathbf{K}}^2) - a_{\mathbf{K}}R^2}{2(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)}.$$

Substituting these in each factor of (14) we obtain the condition as one of the following two equations should be satisfied:

$$\frac{(4(a - a_{\mathbf{K}})^2 - R^2)^2 c_{\mathbf{K}}^2}{4(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)^2} - R^2 \pm R \cdot \frac{4(a - a_{\mathbf{K}})(a_{\mathbf{K}}a - c_{\mathbf{K}}^2) - a_{\mathbf{K}}R^2}{a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2} = 0.$$

We know from Prop 5.1 that 3-periodic orbits exist for $a > a_{\mathbf{K}} + c_{\mathbf{K}}$. This should not be optional for the parameter range for the existence of 3-periodic orbits, but it does not seem either easy or illuminating to identify the precise threshold. In Fig. 6 one can see a 3-periodic orbit including the foci-circle, the foci-caustic circle and the tangent lines formed by the consecutive second Kepler foci.

5.2.4 $n = 4$

For 4-periodic orbits the condition is $A_3 = 0$, which is equivalent to the following condition :

$$(R^2 - x_0^2)(2r_0^2(R^2 + x_0^2) - (R^2 - x_0^2)^2) = 0.$$

So we have 4-periodic orbits when one of the following two conditions hold

$$R = |x_0|,$$

$$2r_0^2(R^2 + x_0^2) = (R^2 - x_0^2)^2.$$

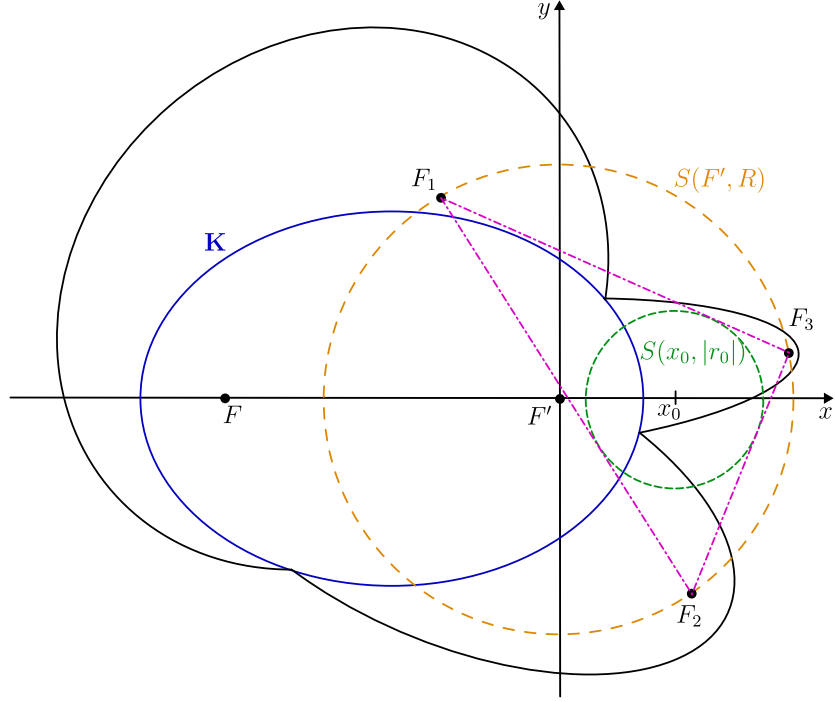


Figure 6: Example for a 3-periodic configuration.

The second equation is cubic in R^2 and it does not seem easy to make a detailed analysis.

The equation

$$R = |x_0|,$$

can be solved explicitly as

$$R = \frac{a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2}{c_{\mathbf{K}}} \cdot \left(1 + \sqrt{1 + \frac{4c_{\mathbf{K}}^2(a - a_{\mathbf{K}})^2}{(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)^2}} \right).$$

As the parameter range is given by

$$2a > a_{\mathbf{K}} - c_{\mathbf{K}} > 0, \quad |2a_{\mathbf{K}} - 2a| \leq R \leq 2a + 2c_{\mathbf{K}},$$

We have

$$|2a_{\mathbf{K}} - 2a| \leq \frac{a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2}{c_{\mathbf{K}}} \cdot \left(1 + \sqrt{1 + \frac{4c_{\mathbf{K}}^2(a - a_{\mathbf{K}})^2}{(a_{\mathbf{K}}^2 - c_{\mathbf{K}}^2)^2}} \right) \leq 2a + 2c_{\mathbf{K}}.$$

For the inequality on the left, after squaring both sides we easily see that it always holds. For the inequality on the right, after squaring both sides and simplifying the expressions, we obtain the much simplified equivalent form giving restriction for the shape of the reflection wall:

$$3c_{\mathbf{K}} \geq a_{\mathbf{K}}.$$

A Computation of integrals

In the computation of periods there appear the integrals of the type

$$\int_a^{+\infty} \frac{d\tilde{t}}{\sqrt{4\tilde{t}^3 - g_2\tilde{t} - g_3}} = \int_a^{+\infty} \frac{d\tilde{t}}{\sqrt{4(\tilde{t} - e_1)(\tilde{t} - e_2)(\tilde{t} - e_3)}}.$$

Here we indicate a standard procedure to reduce this computation to Jacobi's elliptic integrals, see e.g. [12]. We shall not need to express the periods as well as other related integrals in elliptic integrals but this might be useful for analysis of different questions.

Indeed we can set $\hat{t} = \tilde{t} - e_1$ to transform the above integral into the form

$$\int_{a-e_1}^{+\infty} \frac{d\hat{t}}{\sqrt{4\hat{t}(\hat{t} + e_1 - e_2)(\hat{t} + e_1 - e_3)}}.$$

Further we set $\hat{t} = \frac{e_2 - e_1}{\check{t}}$ so $d\hat{t} = -\frac{e_2 - e_1}{\check{t}^2} d\check{t}$ to transform the integral into the form

$$\int_0^{\frac{e_2 - e_1}{a - e_1}} \frac{(e_2 - e_1)d\check{t}}{\check{t}^{1/2} \sqrt{4(e_2 - e_1)((e_2 - e_1)(1 - \check{t}))((e_2 - e_1) + (e_1 - e_3)\check{t})}}.$$

Choosing the branch of the square root for $(e_2 - e_1)^{1/2}$ as the analytic continuation of the positive square root on the positive real axis and further with the coordinate change $\check{t} = s^2$ we get

$$\frac{1}{(e_2 - e_1)^{1/2}} \int_0^{\sqrt{\frac{e_2 - e_1}{a - e_1}}} \frac{ds}{\sqrt{(1 - s^2) \left(1 - \frac{e_1 - e_3}{e_1 - e_2} s^2\right)}} = \frac{1}{(e_2 - e_1)^{1/2}} F\left(\sqrt{\frac{e_2 - e_1}{a - e_1}}, k\right),$$

in which $F(x, k)$ denotes the incomplete elliptic integral of the first kind in Jacobi form, and

$$k^2 = \frac{e_1 - e_3}{e_1 - e_2},$$

which allows to evaluate the period, the shift and the frequency in terms of Jacobi's elliptic integrals.

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