MODULAR ELEMENTS IN THE LATTICE OF MONOID VARIETIES

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ABSTRACT. An element x of a lattice L is modular if L has no five-element sublattice isomorphic to the pentagon in which x would correspond to the lonely midpoint. In the present work, we classify all modular elements of the lattice of all monoid varieties.

1. INTRODUCTION AND SUMMARY

An element x of a lattice L is *modular* if it makes the formula

$$\forall y,z \in L:\ y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y$$

hold true. Modular elements enjoy a distinguished position in lattices. The interest in them is explained by the fact that modular elements of L are exactly ones that are not the central elements of any pentagon (that is, a five-element non-modular sublattice shown in Fig. 1) of L (see [7, Proposition 2.1]).

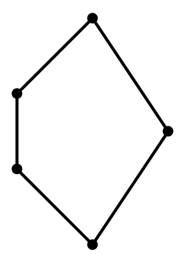

FIGURE 1

Modular elements played a crucial role in the study of first-order definability in the lattice $\mathbb{SEM}$ of all semigroup varieties [8]. Although a complete description of modular elements in $\mathbb{SEM}$ is still unknown, a number of profound results have been obtained in this direction. In particular, the set of all modular elements in $\mathbb{SEM}$ is uncountably infinite. More information can be found in the comprehensive survey article [9] in the context of studying special elements of various types in the lattice $\mathbb{SEM}$.

The present article is concerned with modular elements of the lattice $\mathbb{MON}$ of all varieties of monoids, i.e., semigroups with an identity element. Even though monoids are very similar to semigroups, the situation turns out to be very different. Compared with lattice $\mathbb{SEM}$, the systematic study of lattice $\mathbb{MON}$ has began relatively recently, although the first results in this direction were obtained back in the late 1960s. In particular, the problem of describing modular elements of the lattice $\mathbb{MON}$ remained open (see Section 9.1 in the recent survey [4]). The goal of the present note is a complete solution to this problem. We

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present an exhaustive countably infinite list of monoid varieties that are modular elements of the lattice $\mathbb{MON}$.

Let us briefly recall a few notions that we need to formulate our main result. Let $\mathscr{X}$ be a countably infinite set called an *alphabet*. As usual, let $\mathscr{X}^*$ denote the free monoid over the alphabet $\mathscr{X}$. Elements of $\mathscr{X}$ are called *letters* and elements of $\mathscr{X}^*$ are called *words*. We treat the identity element of $\mathscr{X}^*$ as *the empty word*, which is denoted by 1. Words and letters are denoted by small Latin letters. However, words unlike letters are written in bold. An identity is written as $\mathbf{u} \approx \mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathscr{X}^*$; it is *non-trivial* if $\mathbf{u} \neq \mathbf{v}$. A variety $\mathbf{V}$ *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$, if for any monoid $M \in \mathbf{V}$ and any substitution $\varphi\colon \mathscr{X} \to M$, the equality $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$ holds in M.

For any set $\mathscr{W}$ of words, let $M(\mathscr{W})$ denote the Rees quotient monoid of $\mathscr{X}^*$ over the ideal of all words that are not subwords of any word in $\mathscr{W}$. Given a set $\mathscr{W}$ of words, let $\mathbf{M}(\mathscr{W})$ denote the monoid variety generated by $M(\mathscr{W})$. For brevity, if $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathscr{X}^*$, then we write $M(\mathbf{w}_1, \dots, \mathbf{w}_k)$ [respectively, $\mathbf{M}(\mathbf{w}_1, \dots, \mathbf{w}_k)$] rather than $M(\{\mathbf{w}_1, \dots, \mathbf{w}_k\})$ [respectively, $\mathbf{M}(\{\mathbf{w}_1, \dots, \mathbf{w}_k\})$].

As usual, let $\mathbb{N}$ denote the set of all natural numbers. For any $n \in \mathbb{N}$, we denote by S_n the full symmetric group on the set $\{1, \dots, n\}$. For any $n, m \in \mathbb{N}$ and $\rho \in S_{n+m}$, we define the words:

$$\mathbf{a}_{n,m}[\rho] := \left(\prod_{i=1}^{n} z_i t_i\right) x \left(\prod_{i=1}^{n+m} z_{i\rho}\right) x \left(\prod_{i=n+1}^{n+m} t_i z_i\right),$$
$$\mathbf{a}'_{n,m}[\rho] := \left(\prod_{i=1}^{n} z_i t_i\right) \left(\prod_{i=1}^{n+m} z_{i\rho}\right) x^2 \left(\prod_{i=n+1}^{n+m} t_i z_i\right).$$

We denote by $\mathbf{MON}$ the variety of all monoids.

Our main result is the following

Theorem 1. *For a monoid variety* $\mathbf{V}$ *the following are equivalent:*

(i) $\mathbf{V}$ *is a modular element of the lattice* $\mathbb{MON}$*;*
(ii) $\mathbf{V}$ *is either the variety* $\mathbf{MON}$ *or satisfies the identities*

$$x^2 \approx x^3,\ x^2y \approx yx^2,\ xyzxty \approx yxzxty,\ xzytxy \approx xzytyx,$$
$$xzxtxysy \approx xzxtyxsy,\ xzxytysy \approx xzyxtysy,\ \mathbf{a}_{n,m}[\rho] \approx \mathbf{a}'_{n,m}[\rho]$$

for all $n, m \in \mathbb{N}$ *and* $\rho \in S_{n+m}$.
(iii) $\mathbf{V}$ *coincides with one of the varieties*

$$\mathbf{M}(\emptyset), \mathbf{M}(1), \mathbf{M}(x), \mathbf{M}(xy), \mathbf{M}(xt_1x), \dots, \mathbf{M}(xt_1x \cdots t_nx), \dots, \mathbf{M}(\{xt_1x \cdots t_nx \mid n \in \mathbb{N}\}),$$
$$\mathbf{M}(xzxyty, xt_1x), \dots, \mathbf{M}(xzxyty, xt_1x \cdots t_nx), \dots, \mathbf{M}(\{xzxyty, xt_1x \cdots t_nx \mid n \in \mathbb{N}\}), \mathbf{MON}.$$

In general, the set of modular elements in a lattice need not form a sublattice. For example, the elements x and y of the lattice in Fig. 2 are modular but their join $x \vee y$ is not. However, Theorem 1 shows that the set of all modular elements of the lattice $\mathbb{MON}$ forms a sublattice and, moreover, all proper monoid varieties that are modular elements in $\mathbb{MON}$ constitute an order ideal in $\mathbb{MON}$.

The article consists of three sections. Some definitions, notation and auxiliary results are given in Section 2, while Section 3 is devoted to the proof of Theorem 1.

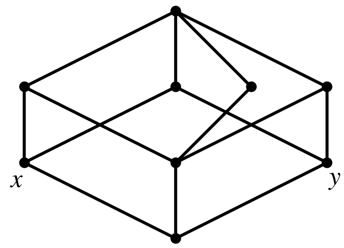

FIGURE 2

2. PRELIMINARIES

An identity $\mathbf{u} \approx \mathbf{v}$ is *directly deducible* from an identity $\mathbf{s} \approx \mathbf{t}$ if there exist some words $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$ such that $\{\mathbf{u}, \mathbf{v}\} = \{\mathbf{a}\varphi(\mathbf{s})\mathbf{b}, \mathbf{a}\varphi(\mathbf{t})\mathbf{b}\}$. A non-trivial identity $\mathbf{u} \approx \mathbf{v}$ is *deducible* from a set Σ of identities if there exists some finite sequence $\mathbf{u} = \mathbf{w}_0, \ldots, \mathbf{w}_m = \mathbf{v}$ of words such that each identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is directly deducible from some identity in Σ.

Proposition 2 (Birkhoff's Completeness Theorem for Equational Logic; see [1, Theorem II.14.19])**.** *A monoid variety defined by a set Σ of identities satisfies an identity $\mathbf{u} \approx \mathbf{v}$ if and only if $\mathbf{u} \approx \mathbf{v}$ is deducible from Σ.* $\square$

Given a variety $\mathbf{V}$, a word $\mathbf{u}$ is called an *isoterm for* $\mathbf{V}$ if the only word $\mathbf{v}$ such that $\mathbf{V}$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$ is the word $\mathbf{u}$ itself.

Lemma 3 ([6, Lemma 3.3])**.** *Let $\mathbf{V}$ be a monoid variety and $\mathscr{W}$ a set of words. Then $M(\mathscr{W}) \in \mathbf{V}$ if and only if each word in $\mathscr{W}$ is an isoterm for $\mathbf{V}$.* $\square$

The *alphabet* of a word $\mathbf{w}$, i.e., the set of all letters occurring in $\mathbf{w}$, is denoted by $\mathrm{alph}(\mathbf{w})$. For a word $\mathbf{w}$ and a letter x, let $\mathrm{occ}_x(\mathbf{w})$ denote the number of occurrences of x in $\mathbf{w}$. A letter x is called *simple* [*multiple*] *in a word* $\mathbf{w}$ if $\mathrm{occ}_x(\mathbf{w}) = 1$ [respectively, $\mathrm{occ}_x(\mathbf{w}) > 1$]. The set of all simple [multiple] letters in a word $\mathbf{w}$ is denoted by $\mathrm{sim}(\mathbf{w})$ [respectively, $\mathrm{mul}(\mathbf{w})$].

Lemma 4 ([5, Lemma 2.17])**.** *Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $M(xy)$. If $\mathbf{u} = \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_m \mathbf{u}_m$, where $\mathrm{sim}(\mathbf{u}) = \{t_1, \ldots, t_m\}$, then $\mathbf{v} = \mathbf{v}_0 t_1 \mathbf{v}_1 \cdots t_m \mathbf{v}_m$, $\mathrm{alph}(\mathbf{u}_0 \cdots \mathbf{u}_m) = \mathrm{alph}(\mathbf{v}_0 \cdots \mathbf{v}_m)$ and $\mathrm{sim}(\mathbf{v}) = \{t_1, \ldots, t_m\}$.* $\square$

The following statement was established in the proof of Lemma 3.5 in [5].

Lemma 5. *Let $\mathbf{V}$ be a monoid variety such that $M(xt_1x \cdots t_nx) \in \mathbf{V}$. If $M(\mathbf{p}xy\mathbf{q}) \notin \mathbf{V}$, where $\mathbf{p} := a_1t_1 \cdots a_kt_k$ and $\mathbf{q} := t_{k+1}a_{k+1} \cdots t_{k+\ell}a_{k+\ell}$ for some $k, \ell \geq 0$ and $a_1, \ldots, a_{k+\ell}$ are letters such that $\{a_1, \ldots, a_{k+\ell}\} = \{x, y\}$ and $\mathrm{occ}_x(\mathbf{pq}), \mathrm{occ}_y(\mathbf{pq}) \leq n$, then $\mathbf{V}$ satisfies the identity $\mathbf{p}xy\mathbf{q} \approx \mathbf{p}yx\mathbf{q}$.* $\square$

If $\mathbf{w}$ is a word and $\mathscr{X} \subseteq \mathrm{alph}(\mathbf{w})$, then we denote by $\mathbf{w}_{\mathscr{X}}$ [respectively, $\mathbf{w}(\mathscr{X})$] the word obtained from $\mathbf{w}$ by removing all occurrences of letters from $\mathscr{X}$ [respectively, $\mathrm{alph}(\mathbf{w}) \setminus \mathscr{X}$]. If $\mathscr{X} = \{z\}$, then we write $\mathbf{w}_z$ rather than $\mathbf{w}_{\{z\}}$.

The expression ${}_{i\mathbf{w}}x$ means the ith occurrence of a letter x in a word $\mathbf{w}$. If the ith occurrence of x precedes the jth occurrence of y in a word $\mathbf{w}$, then we write $({}_{i\mathbf{w}}x) < ({}_{j\mathbf{w}}y)$.

3. PROOF OF THEOREM 1

We will prove Theorem 1 following the scheme (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let $\mathbf{V}$ be a proper monoid variety which is a modular element of the lattice $\mathbb{MON}$. Then the variety $\mathbf{V}$ satisfies the identities

$$x^2 \approx x^3,\ x^2y \approx yx^2 \tag{1}$$

by Proposition 4.3 in [3].

If $M(xyx) \notin \mathbf{V}$, then it follows from Lemma 3.3(i) in [5] that $\mathbf{V}$ satisfies the identities $x^2y \approx xyx \approx yx^2$ which, evidently, imply all the identities listed in Item (ii) of Theorem 1. So, we may further assume that $M(xyx) \in \mathbf{V}$.

The rest of the proof proceeds in three steps.

Step 1: $\mathbf{V}$ satisfies the identities $xyzxty \approx yxzxty$ and $xzytxy \approx xzytyx$.

Arguing by contradiction, suppose that $\mathbf{V}$ violates the identity $xyzxty \approx yxzxty$. Then $M(xyzxty) \in \mathbf{V}$ by Lemma 5. Let $\mathbf{X}'$ denote the monoid variety defined by the identity

$$\mathbf{u} := z_1t_1z_2t_2\,c^2z_1bz_2\,xcy\,b\,s_1xs_2y \approx z_1t_1z_2t_2\,c^2z_1bz_2\,ycx\,b\,s_1xs_2y =: \mathbf{v}.$$

We need the following auxiliary result.

Claim 1. *The words* $\mathbf{u}$ *and* $\mathbf{v}$ *can only form an identity of* $\mathbf{X}'$ *with each other.*

Proof. Take $\mathbf{w} \in \{\mathbf{u},\mathbf{v}\}$ and consider an arbitrary identity of the form $\mathbf{w} \approx \mathbf{w}'$ that holds in $\mathbf{X}'$. We are going to verify that $\mathbf{w}' \in \{\mathbf{u},\mathbf{v}\}$. By Proposition 2 and evident induction, we may assume without any loss that the identity $\mathbf{w} \approx \mathbf{w}'$ is directly deducible from the identity $\mathbf{u} \approx \mathbf{v}$, i.e., there exist words $\mathbf{a},\mathbf{b} \in \mathscr{X}^*$ and a substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$ such that $(\mathbf{w},\mathbf{w}') = (\mathbf{a}\varphi(\mathbf{u})\mathbf{b}, \mathbf{a}\varphi(\mathbf{v})\mathbf{b})$. Notice that every subword of $\mathbf{w}$ of length > 1 occurs in $\mathbf{w}$ exactly once and each letter occurs in $\mathbf{w}$ at most thrice. It follows that

($*$) $\varphi(v)$ is either the empty word or a letter for any $v \in \mathrm{mul}(\mathbf{u}) = \mathrm{mul}(\mathbf{v})$.

Suppose first that $\varphi(c) = 1$. In this case, if $\varphi(x) = 1$ or $\varphi(y) = 1$, then $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$ and so $\mathbf{w}' \in \{\mathbf{u},\mathbf{v}\}$, as required. So, we may assume that $\varphi(x) \neq 1$ and $\varphi(y) \neq 1$. Then $\varphi(x)$ and $\varphi(y)$ are letters by ($*$). These letters must be distinct since no letter occurs in $\mathbf{w}$ more than thrice. However, the word $\mathbf{w}$ does not contain any subword consisting exclusively of non-last occurrences of two distinct letters in $\mathbf{w}$, while the first occurrences of the letters $\varphi(x)$ and $\varphi(y)$ are adjacent in $\mathbf{w}$, a contradiction.

Suppose now that $\varphi(c) \neq 1$. Then $\varphi(c) = c$ because c is the only letter which occurs thrice in both $\mathbf{u}$ and $\mathbf{w}$. Hence

$$\varphi(z_1bz_2x) = \begin{cases} z_1bz_2x & \text{if } \mathbf{w} = \mathbf{u},\\ z_1bz_2y & \text{if } \mathbf{w} = \mathbf{v}.\end{cases}$$

Further, it follows from ($*$) that $\varphi(b) = b$, whence

$$(\varphi(x),\varphi(y)) = \begin{cases} (x,y) & \text{if } \mathbf{w} = \mathbf{u},\\ (y,x) & \text{if } \mathbf{w} = \mathbf{v}.\end{cases}$$

Since $({}_{1\mathbf{u}}x) < ({}_{1\mathbf{u}}y) < ({}_{2\mathbf{u}}x) < ({}_{2\mathbf{u}}y)$ and $({}_{1\mathbf{v}}y) < ({}_{1\mathbf{v}}x) < ({}_{2\mathbf{v}}x) < ({}_{2\mathbf{v}}y)$, it follows that the case when $\mathbf{w} = \mathbf{v}$ is impossible. Therefore, $(\varphi(x),\varphi(y)) = (x,y)$ in either case. This implies that $\mathbf{w}' = \mathbf{v}$, and we are done. $\square$

Put $\mathbf{X} := \mathbf{M}(\mathbf{u},\mathbf{v}) \wedge \mathbf{X}'$. Consider an arbitrary identity of the form $\mathbf{u} \approx \mathbf{w}$ that is satisfied by the variety $\mathbf{X} \vee \mathbf{V}$. In view of Claim 1, $\mathbf{w} \in \{\mathbf{u},\mathbf{v}\}$. Since $M(xyzxty) \in \mathbf{V}$, Lemma 3 implies that $\mathbf{w}(x,y,s_1,s_2) = \mathbf{u}(x,y,s_1,s_2) = xys_1xs_2y$. Hence $\mathbf{w} = \mathbf{u}$. We see that $\mathbf{u}$ is an isoterm for the variety $\mathbf{X} \vee \mathbf{V}$. By a similar argument, we can show that $\mathbf{v}$ is an isoterm for $\mathbf{X} \vee \mathbf{V}$ as well. Then

$$(\mathbf{X} \vee \mathbf{V}) \wedge \mathbf{M}(\mathbf{u},\mathbf{v}) = \mathbf{M}(\mathbf{u},\mathbf{v})$$

by Lemma 3. It is easy to see that the identity $xyzxty \approx yxzxty$ holds in $M(\mathbf{u},\mathbf{v})$. Then $\mathbf{V}\wedge\mathbf{M}(\mathbf{u},\mathbf{v})$ satisfies $\mathbf{u} \overset{\mathbf{V}}{\approx} \mathbf{u}_c c^2 \overset{\mathbf{M}(\mathbf{u},\mathbf{v})}{\approx} \mathbf{v}_c c^2 \overset{\mathbf{V}}{\approx} \mathbf{v}$. By the very definition, the identity $\mathbf{u}\approx\mathbf{v}$ is satisfied by $\mathbf{X}$ as well. Since $\mathbf{X}\subset\mathbf{M}(\mathbf{u},\mathbf{v})$, we have

$$\mathbf{X}\vee(\mathbf{V}\wedge\mathbf{M}(\mathbf{u},\mathbf{v}))\subset(\mathbf{X}\vee\mathbf{V})\wedge\mathbf{M}(\mathbf{u},\mathbf{v})=\mathbf{M}(\mathbf{u},\mathbf{v}),$$

contradicting the fact that $\mathbf{V}$ is a modular element of $\mathbb{MON}$. Therefore, $\mathbf{V}$ satisfies $xyzxty \approx yxzxty$. By the dual argument, we can show that $xzytxy \approx xzytyx$ holds in $\mathbf{V}$.

Step 2: $\mathbf{V}$ satisfies the identity $\mathbf{a}_{n,m}[\rho]\approx\mathbf{a}'_{n,m}[\rho]$ for all $n,m\in\mathbb{N}$ and $\rho\in S_{n+m}$.

Arguing by contradiction, suppose that $\mathbf{V}$ violates $\mathbf{a}_{p,q}[\pi]\approx\mathbf{a}'_{p,q}[\pi]$ for some $p,q\in\mathbb{N}$ and $\pi\in S_{p+q}$. Then $M(\mathbf{a}_{n,m}[\rho])\in\mathbf{V}$ for some $n,m\in\mathbb{N}$ and $\rho\in S_{n+m}$ by Lemma 4.8 in [2]. For brevity, put

$$\mathbf{p}:=\begin{cases} z_1t_1\cdots z_nt_n, & \text{if } 1\le 1\rho\le n,\\ y_1s_1y_2s_2\,z_1t_1\cdots z_nt_n, & \text{if } n<1\rho\le n+m,\end{cases}$$

$$\mathbf{r}:=\begin{cases} z_{n+1}t_{n+1}\cdots z_{n+m}t_{n+m}\,s_1y_1s_2y_2, & \text{if } 1\le 1\rho\le n,\\ z_{n+1}t_{n+1}\cdots z_{n+m}t_{n+m}, & \text{if } n<1\rho\le n+m.\end{cases}$$

Let

$$\mathbf{X}:=\begin{cases}\mathbf{M}(y^2xtx,xy^2tx) & \text{if } 1\le 1\rho\le n,\\ \mathbf{M}(xtxy^2,xty^2x) & \text{if } n<1\rho\le n+m.\end{cases}$$

Consider an arbitrary identity of the form

$$\mathbf{u}:=\mathbf{p}x\left(\prod_{i=1}^{n+m-1}z_{i\rho}\right)y_1z^2y_2z_{(n+m)\rho}x\mathbf{r}\approx\mathbf{v}$$

that is satisfied by the variety $\mathbf{X}\vee\mathbf{V}$. In view of Lemma 3,

$$\mathbf{v}_{\{y_1,y_2,s_1,s_2,z\}}=\mathbf{u}_{\{y_1,y_2,s_1,s_2,z\}}=\mathbf{a}_{n,m}[\rho],$$

$$\mathbf{v}_{\{x,z\}}=\mathbf{u}_{\{x,z\}}=\mathbf{p}\left(\prod_{i=1}^{n+m-1}z_{i\rho}\right)y_1y_2z_{(n+m)\rho}\mathbf{r},$$

$$\mathbf{v}(y_1,s_1,z)=\mathbf{u}(y_1,s_1,z)=\begin{cases}y_1z^2s_1y_1 & \text{if } 1\le 1\rho\le n,\\ y_1s_1y_1z^2 & \text{if } n<1\rho\le n+m,\end{cases}$$

$$\mathbf{v}(y_2,s_2,z)=\mathbf{u}(y_2,s_2,z)=\begin{cases}z^2y_2s_2y_2 & \text{if } 1\le 1\rho\le n,\\ y_2s_2z^2y_2 & \text{if } n<1\rho\le n+m.\end{cases}$$

It follows that $\mathbf{v}=\mathbf{u}$. We see that the word $\mathbf{u}$ is an isoterm for the variety $\mathbf{X}\vee\mathbf{V}$. Then

$$(\mathbf{X}\vee\mathbf{V})\wedge\mathbf{M}(\mathbf{u})=\mathbf{M}(\mathbf{u})$$

by Lemma 3.

Now we are going to verify that $M(\mathbf{u})$ satisfies the identity

$$\mathbf{a}:=\mathbf{u}_z=\mathbf{p}x\left(\prod_{i=1}^{n+m-1}z_{i\rho}\right)y_1y_2z_{(n+m)\rho}x\mathbf{r}\approx\mathbf{p}z_{1\rho}x\left(\prod_{i=2}^{n+m-1}z_{i\rho}\right)y_1y_2z_{(n+m)\rho}x\mathbf{r}=:\mathbf{a}'.$$

Consider an arbitrary substitution $\psi\colon\mathscr{X}\to M(\mathbf{u})$ and show that $\psi(\mathbf{a})=\psi(\mathbf{a}')$. We may suppose that at least one of the elements $\psi(\mathbf{a})$ or $\psi(\mathbf{a}')$ is non-zero and forms a subword of $\mathbf{u}$. If $\psi(x)=1$, then $\psi(\mathbf{a})=\psi(\mathbf{a}')$, and we are done. So, we may further assume that $\psi(x)\ne 1$. Clearly, none of the letters $s_1,s_2,t_1,\dots,t_{n+m}$ belongs to $\mathrm{alph}(\psi(x))$ because all these letters are simple in $\mathbf{u}$, while $x\in\mathrm{mul}(\mathbf{a})=\mathrm{mul}(\mathbf{a}')$. Further, none of the letters

$y_1, y_2, z_1, \dots, z_{n+m}$ belongs to $\mathrm{alph}(\psi(x))$ because there is a simple letter between the first and the second occurrences of any of these letters in $\mathbf{u}$, while there are no simple letters between the first and the second occurrences of x in both $\mathbf{a}$ and $\mathbf{a}'$. Finally, $x \notin \mathrm{alph}(\psi(x))$ because

$$\psi\left(\left(\prod_{i=1}^{n+m-1} z_{i\rho}\right) y_1 y_2 z_{(n+m)\rho}\right)$$

cannot contain z^2 as a subword. Therefore, $\psi(x) = z$. Then $\psi(y_1) = \psi(y_2) = \psi(z_{2\rho}) = \cdots = \psi(z_{(n+m)\rho}) = 1$. Assume that $\psi(z_{1\rho}) \ne 1$. Then $\psi(\mathbf{a}')$ is a subword of $\mathbf{u}$, and if $1 \le 1\rho \le n$ [respectively, $n < 1\rho \le n+m$] the image of the second [respectively, first] occurrence of $z_{1\rho}$ in $\mathbf{a}'$ under ψ must contain the first [respectively, second] occurrence of y_1 in $\mathbf{u}$, a contradiction. Therefore, $\psi(z_{1\rho}) = 1$, whence $\psi(\mathbf{a}) = \psi(\mathbf{a}')$. Thus, we have proved that the identity $\mathbf{a} \approx \mathbf{a}'$ holds in $M(\mathbf{u})$.

Then $\mathbf{V} \wedge \mathbf{M}(\mathbf{u})$ satisfies

$$\mathbf{u} \stackrel{\mathbf{V}}{\approx} \mathbf{u}_z z^2 = \mathbf{a} z^2 \stackrel{\mathbf{M}(\mathbf{u})}{\approx} \mathbf{a}' z^2 \stackrel{\mathbf{V}}{\approx} \mathbf{p} z_{1\rho} x \left(\prod_{i=2}^{n+m-1} z_{i\rho}\right) y_1 z^2 y_2 z_{(n+m)\rho} x \mathbf{r} =: \mathbf{u}'.$$

It follows from the very definition of $\mathbf{X}$ that $\mathbf{X} \subset \mathbf{M}(\mathbf{u})$ and the identity $\mathbf{u} \approx \mathbf{u}'$ is satisfied by $\mathbf{X}$. Then

$$\mathbf{X} \vee (\mathbf{V} \wedge \mathbf{M}(\mathbf{u})) \subset (\mathbf{X} \vee \mathbf{V}) \wedge \mathbf{M}(\mathbf{u}) = \mathbf{M}(\mathbf{u}),$$

contradicting the fact that $\mathbf{V}$ is a modular element of $\mathbb{MON}$. Therefore, $\mathbf{V}$ satisfies $\mathbf{a}_{n,m}[\rho] \approx \mathbf{a}'_{n,m}[\rho]$ for all $n, m \in \mathbb{N}$ and $\rho \in S_{n+m}$.

Step 3: $\mathbf{V}$ satisfies the identities $xzxtxysy \approx xzxtyxsy$ and $xzxytysy \approx xzyxtysy$.

If $M(xyxzx) \notin \mathbf{V}$, then, by Lemma 3, the variety $\mathbf{V}$ satisfies an identity $xyxzx \approx x^p y x^q z x^r$ with either $p > 1$ or $q > 1$ or $r > 1$. In any case, this identity together with the identities (1) imply the identity $xyxzx \approx x^2 yz$ and so the identities $xzxtxysy \approx xzxtyxsy$ and $xzxytysy \approx xzyxtysy$. So, we may further assume that $M(xyxzx) \in \mathbf{V}$.

Arguing by contradiction, suppose that $\mathbf{V}$ violates the identity $xzxtxysy \approx xzxtyxsy$. Then $M(xzxtxysy) \in \mathbf{V}$ by Lemma 5. Let $\mathbf{X}'$ denote the monoid variety defined by the identity

$$\mathbf{u} := xs_1xs_2z_1t_1z_2t_2\,cz_1bz_2\,xcy\,bs_3y \approx xs_1xs_2z_1t_1z_2t_2\,cz_1bz_2\,ycx\,bs_3y =: \mathbf{v}.$$

We need the following auxiliary result.

Claim 2. *The words* $\mathbf{u}$ *and* $\mathbf{v}$ *can only form an identity of* $\mathbf{X}'$ *with each other.*

Proof. Take $\mathbf{w} \in \{\mathbf{u}, \mathbf{v}\}$ and consider an arbitrary identity of the form $\mathbf{w} \approx \mathbf{w}'$ that holds in $\mathbf{X}'$. We are going to verify that $\mathbf{w}' \in \{\mathbf{u}, \mathbf{v}\}$. By Proposition 2 and evident induction, we may assume without any loss that the identity $\mathbf{w} \approx \mathbf{w}'$ is directly deducible from the identity $\mathbf{u} \approx \mathbf{v}$, i.e., there exist some words $\mathbf{a}, \mathbf{b} \in \mathscr{X}^*$ and substitution $\varphi\colon \mathscr{X} \to \mathscr{X}^*$ such that $(\mathbf{w}, \mathbf{w}') = (\mathbf{a}\varphi(\mathbf{u})\mathbf{b}, \mathbf{a}\varphi(\mathbf{v})\mathbf{b})$. Notice that every subword of $\mathbf{w}$ of length > 1 occurs in $\mathbf{w}$ exactly once. It follows that

($*$) $\varphi(v)$ is either the empty word or a letter for any $v \in \mathrm{mul}(\mathbf{u})$.

Suppose first that $\varphi(c) = 1$. In this case, if $\varphi(x) = 1$ or $\varphi(y) = 1$, then $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$ and so $\mathbf{w}' \in \{\mathbf{u}, \mathbf{v}\}$, as required. So, we may assume that $\varphi(x) \ne 1$ and $\varphi(y) \ne 1$. Then $\varphi(x)$ and $\varphi(y)$ are letters by ($*$). Since x is the only letter that occurs thrice in both $\mathbf{u}$ and $\mathbf{v}$, we have $\varphi(x) = x$. It follows that the image of ${}_{1\mathbf{w}}y$ under φ must be either ${}_{2\mathbf{w}}c$ (if $\mathbf{w} = \mathbf{u}$) or ${}_{2\mathbf{w}}b$ (if $\mathbf{w} = \mathbf{v}$), a contradiction.

Suppose now that $\varphi(c) \ne 1$. Then $\varphi(c) \in \{b,c\}$ because b and c are the only multiple letters of $\mathbf{w}$ between the first and second occurrences of which there are no simple letters. If $\varphi(c) = b$, then

$$\varphi(z_1bz_2x) = \begin{cases} z_2xcy & \text{if } \mathbf{w} = \mathbf{u}, \\ z_2ycx & \text{if } \mathbf{w} = \mathbf{v}. \end{cases}$$

and, by $(*)$, $\varphi(b) \in \{x,y\}$. However, this contradicts the fact that there are simple letters between ${}_{2\mathbf{w}}x$ and ${}_{3\mathbf{w}}x$ as well as between ${}_{1\mathbf{w}}y$ and ${}_{2\mathbf{w}}y$ in $\mathbf{w}$. Hence $\varphi(c) = c$ and so

$$\varphi(z_1bz_2x) = \begin{cases} z_1bz_2x & \text{if } \mathbf{w} = \mathbf{u}, \\ z_1bz_2y & \text{if } \mathbf{w} = \mathbf{v}. \end{cases}$$

Further, it follows from $(*)$ that $\varphi(b) = b$, whence

$$(\varphi(x),\varphi(y)) = \begin{cases} (x,y) & \text{if } \mathbf{w} = \mathbf{u}, \\ (y,x) & \text{if } \mathbf{w} = \mathbf{v}. \end{cases}$$

Since $({}_{1\mathbf{u}}x) < ({}_{2\mathbf{u}}x) < ({}_{3\mathbf{u}}x) < ({}_{1\mathbf{u}}y) < ({}_{2\mathbf{u}}y)$ and $({}_{1\mathbf{v}}x) < ({}_{2\mathbf{v}}x) < ({}_{1\mathbf{v}}y) < ({}_{3\mathbf{v}}x) < ({}_{2\mathbf{v}}y)$, it follows that the case when $\mathbf{w} = \mathbf{v}$ is impossible. Therefore, $(\varphi(x),\varphi(y)) = (x,y)$ in either case. This implies that $\mathbf{w}' = \mathbf{v}$, and we are done. $\square$

Put $\mathbf{X} := \mathbf{M}(\mathbf{u},\mathbf{v}) \wedge \mathbf{X}'$. Consider an arbitrary identity of the form $\mathbf{u} \approx \mathbf{w}$ that is satisfied by the variety $\mathbf{X} \vee \mathbf{V}$. In view of Claim 2, $\mathbf{w} \in \{\mathbf{u},\mathbf{v}\}$. Since $M(xzxtxysy) \in \mathbf{V}$, Lemma 3 implies that $\mathbf{w}(x,y,s_1,s_2,s_3) = \mathbf{u}(x,y,s_1,s_2,s_3) = xs_1xs_2xys_3y$. Hence $\mathbf{w} = \mathbf{u}$. We see that $\mathbf{u}$ is an isoterm for the variety $\mathbf{X} \vee \mathbf{V}$. By a similar argument, we can show that $\mathbf{v}$ is an isoterm for $\mathbf{X} \vee \mathbf{V}$ as well. Then

$$(\mathbf{X} \vee \mathbf{V}) \wedge \mathbf{M}(\mathbf{u},\mathbf{v}) = \mathbf{M}(\mathbf{u},\mathbf{v})$$

by Lemma 3. It is easy to see that $xzxtxysy \approx xzxtyxsy$ holds in $M(\mathbf{u},\mathbf{v})$. Recall that $\mathbf{V}$ satisfies the identities (1) and $\mathbf{a}_{n,m}[\rho] \approx \mathbf{a}'_{n,m}[\rho]$ for all $n,m \in \mathbb{N}$ and $\rho \in S_{n+m}$. Then $\mathbf{V} \wedge \mathbf{M}(\mathbf{u},\mathbf{v})$ satisfies $\mathbf{u} \stackrel{\mathbf{V}}{\approx} \mathbf{u}_c c^2 \stackrel{\mathbf{M}(\mathbf{u},\mathbf{v})}{\approx} \mathbf{v}_c c^2 \stackrel{\mathbf{V}}{\approx} \mathbf{v}$. By the very definition, the identity $\mathbf{u} \approx \mathbf{v}$ is satisfied by $\mathbf{X}$ as well. Since $\mathbf{X} \subset \mathbf{M}(\mathbf{u},\mathbf{v})$, we have

$$\mathbf{X} \vee (\mathbf{V} \wedge \mathbf{M}(\mathbf{u},\mathbf{v})) \subset (\mathbf{X} \vee \mathbf{V}) \wedge \mathbf{M}(\mathbf{u},\mathbf{v}) = \mathbf{M}(\mathbf{u},\mathbf{v}),$$

contradicting the fact that $\mathbf{V}$ is a modular element of $\mathbb{MON}$. Therefore, $\mathbf{V}$ satisfies $xzxtxysy \approx xzxtyxsy$. By the dual argument, we can show that $xzxytysy \approx xzyxtysy$ holds in $\mathbf{V}$.

Implication (i) $\Rightarrow$ (ii) is thus proved.

(ii) $\Rightarrow$ (iii). If $M(xyx) \notin \mathbf{V}$, then it follows from Lemma 3.3(i) in [5] that $\mathbf{V}$ coincides with one of the varieties $\mathbf{M}(\emptyset)$, $\mathbf{M}(1)$, $\mathbf{M}(x)$ or $\mathbf{M}(xy)$, and we are done. Assume now that $M(xyx) \in \mathbf{V}$. Denote by $\mathscr{W}$ the set of all words in $\{xzxyty, xt_1x\cdots t_nx \mid n \in \mathbb{N}\}$ that are isoterms for $\mathbf{V}$. Notice that, in this case, the variety $\mathbf{M}(\mathscr{W})$ coincides with one of the varieties

$$\mathbf{M}(xt_1x), \ldots, \mathbf{M}(xt_1x\cdots t_nx), \ldots, \mathbf{M}(\{xt_1x\cdots t_nx \mid n \in \mathbb{N}\}),$$
$$\mathbf{M}(xzxyty, xt_1x), \ldots, \mathbf{M}(xzxyty, xt_1x\cdots t_nx), \ldots, \mathbf{M}(\{xzxyty, xt_1x\cdots t_nx \mid n \in \mathbb{N}\}).$$

According to Lemma 3, $\mathbf{M}(\mathscr{W}) \subseteq \mathbf{V}$. It remains to verify that $\mathbf{V} \subseteq \mathbf{M}(\mathscr{W})$. Arguing by contradiction, suppose that $\mathbf{M}(\mathscr{W}) \ne \mathbf{V}$. Then there is an identity σ which holds in $\mathbf{M}(\mathscr{W})$

but does not hold in $\mathbf{V}$. Proposition 3.15 in [5] and its proof allow us to assume that σ coincides with either the identity

$$x\left(\prod_{i=1}^{n} t_i x\right) \approx x^2\left(\prod_{i=1}^{n} t_i\right), \tag{2}$$

where $n \in \mathbb{N}$, or the identity

$$\left(\prod_{i=1}^{k} a_i t_i\right) xy \left(\prod_{i=k+1}^{k+\ell} t_i a_i\right) \approx \left(\prod_{i=1}^{k} a_i t_i\right) yx \left(\prod_{i=k+1}^{k+\ell} t_i a_i\right), \tag{3}$$

where $k, \ell \geq 0$ and $\{a_1, \ldots, a_{k+\ell}\} = \{x, y\}$. Notice that an identity of the form (3) does not hold in $\mathbf{V}$ if and only if it coincides (up to renaming of letters) with the identity $xzxyty \approx xzyxty$. If σ equals to (2), the word $xt_1x\cdots t_nx$ is an isoterm for $\mathbf{V}$ by Lemma 3.3(ii) in [5]. Then this word must belong to the set $\mathscr{W}$, contradicting our assumption that σ is satisfied by $\mathbf{M}(\mathscr{W})$. If σ coincides with the identity $xzxyty \approx xzyxty$, then the word $xzxyty$ is an isoterm for $\mathbf{V}$ by Lemma 5. But this contradicts our assumption that σ holds in $\mathbf{M}(\mathscr{W})$. Therefore, $\mathbf{V} = \mathbf{M}(\mathscr{W})$, and we are done.

(iii) $\Rightarrow$ (i). It follows from [3, Theorem 1.1] that the varieties $\mathbf{M}(\emptyset)$, $\mathbf{M}(1)$, $\mathbf{M}(x)$ and $\mathbf{M}(xy)$ are modular elements of $\mathbb{MON}$. So, we may further assume that $\mathbf{V} = \mathbf{M}(\mathscr{W})$ for some $\mathscr{W} \subseteq \{xzxyty, xt_1x\cdots t_nx \mid n \in \mathbb{N}\}$. In particular, $M(xyx) \in \mathbf{V}$.

Arguing by contradiction, suppose that $\mathbf{V}$ is not a modular element of the lattice $\mathbb{MON}$. This means that there are monoid varieties $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{X} \vee \mathbf{V} = \mathbf{Y} \vee \mathbf{V}$, $\mathbf{X} \wedge \mathbf{V} = \mathbf{Y} \wedge \mathbf{V}$ but $\mathbf{X} \subset \mathbf{Y}$.

We need the following auxiliary result.

Claim 3. *Let* $\mathbf{u} \approx \mathbf{v}$ *be an identity of* $\mathbf{X}$. *If* $\mathbf{u} \approx \mathbf{v}$ *holds in* $\mathbf{M}(\{xt_1x\cdots t_nx \mid n \in \mathbb{N}\})$*, then* $\mathbf{u} \approx \mathbf{v}$ *is satisfied by* $\mathbf{Y}$ *as well.*

Proof. If the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{V}$, then it is also holds in $\mathbf{X} \vee \mathbf{V}$ and so in $\mathbf{Y}$ because $\mathbf{X} \vee \mathbf{V} = \mathbf{Y} \vee \mathbf{V}$. So, we may further assume that $\mathbf{V}$ violates $\mathbf{u} \approx \mathbf{v}$. This is only possible when $M(xzxyty)$ violates $\mathbf{u} \approx \mathbf{v}$ and $M(xzxyty) \in \mathbf{V}$ because $\mathbf{V} \subseteq \mathbf{M}(\{xzxyty, xt_1x\cdots t_nx \mid n \in \mathbb{N}\})$ and $\mathbf{u} \approx \mathbf{v}$ is satisfied by $M(\{xt_1x\cdots t_nx \mid n \in \mathbb{N}\})$. Then $M(xzxyty) \notin \mathbf{Y}$ because $M(xzxyty) \in \mathbf{X}$ otherwise, contradicting the fact that $M(xzxyty)$ violates $\mathbf{u} \approx \mathbf{v}$.

If $M(xyx) \in \mathbf{Y}$, then $\mathbf{Y}$ satisfies $xzxyty \approx xzyxty$ by Lemma 5. Now let $M(xyx) \notin \mathbf{Y}$. Then Lemma 3 implies that the variety $\mathbf{Y}$ satisfies an identity $xyx \approx x^p yx^q$ with either $p > 1$ or $q > 1$. Further, it follows from Lemma 5 that $\mathbf{X} \vee \mathbf{M}(xyx)$ satisfies $xzxyty \approx xzyxty$. Then $\mathbf{X}$ satisfies also the identity $x^p zx^q yty \approx x^p zyx^q ty$. Evidently, the last identity holds in the variety $\mathbf{V}$ as well. Therefore, it is satisfied by $\mathbf{Y}$ because $\mathbf{Y} \subset \mathbf{X} \vee \mathbf{V}$. Thus, we see that the identity $xzxyty \approx xzyxty$ holds in the variety $\mathbf{Y}$ in any case.

Let $\mathbf{u} = \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_m \mathbf{u}_m$, where $t_1, \ldots, t_m$ are all the simple letters of the word $\mathbf{u}$ and $\mathbf{u}_0, \ldots, \mathbf{u}_m \in \mathscr{X}^*$. Then, by Lemma 4, $\mathrm{sim}(\mathbf{v}) = \{t_1, \ldots, t_m\}$ and $\mathbf{v} = \mathbf{v}_0 t_1 \mathbf{v}_1 \cdots t_m \mathbf{v}_m$ for some $\mathbf{v}_0, \ldots, \mathbf{v}_m \in \mathscr{X}^*$.

It is easy to see that the identity $xzxyty \approx xzyxty$ implies the identities

$$\mathbf{u} \approx \mathbf{p}_0\mathbf{q}_0 t_1 \mathbf{p}_1\mathbf{q}_1 \cdots t_m \mathbf{p}_m\mathbf{q}_m =: \mathbf{u}', \quad \mathbf{v} \approx \mathbf{p}_0'\mathbf{q}_0' t_1 \mathbf{p}_1'\mathbf{q}_1' \cdots t_m \mathbf{p}_m'\mathbf{q}_m' =: \mathbf{v}'$$

where the word $\mathbf{p}_i$ [respectively, $\mathbf{q}_i$] is obtained from the word $\mathbf{u}_i$ by retaining only the first [respectively, non-first] occurrences of letters in $\mathbf{u}$, while the word $\mathbf{p}_i'$ [respectively, $\mathbf{q}_i'$] is obtained from the word $\mathbf{v}_i$ by retaining only the first [respectively, non-first] occurrences of letters in $\mathbf{v}$. By the very construction of the word $\mathbf{u}'$ and $\mathbf{v}'$, the identity $\mathbf{u}' \approx \mathbf{v}'$ holds in the monoid $M(xzxyty)$ and so in the variety $\mathbf{V}$. Since $\mathbf{X}$ satisfies $xzxyty \approx xzyxty$ and $\mathbf{u} \approx \mathbf{v}$, the identity $\mathbf{u}' \approx \mathbf{v}'$ also holds in $\mathbf{X}$. Then $\mathbf{Y}$ satisfies $\mathbf{u}' \approx \mathbf{v}'$ as $\mathbf{Y} \subseteq \mathbf{X} \vee \mathbf{V}$. It follows that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{Y}$ because the identities $\mathbf{u} \approx \mathbf{u}'$ and $\mathbf{v} \approx \mathbf{v}'$ are consequences of $xzxyty \approx xzyxty$. □

It follows from Lemma 3 and Claim 3 that there is the least $k \in \mathbb{N}$ such that the word $xt_1x \cdots t_kx$ is not an isoterm for $\mathbf{X}$. Then $\mathbf{X}$ satisfies an non-trivial identity $xt_1x \cdots t_kx \approx \mathbf{w}$ for some $\mathbf{w} \in \mathscr{X}^*$. It follows from Lemma 4 that $\mathbf{w} = x^{e_0}t_1x^{e_1} \cdots t_kx^{e_k}$ for some $e_0, \dots, e_k \geq 0$. Since k is the least natural number for which the word $xt_1x \cdots t_kx$ is not an isoterm for $\mathbf{X}$, at least one of the numbers $e_0, \dots, e_k$ must exceed 1.

Now consider an arbitrary identity $\mathbf{a} \approx \mathbf{b}$ of $\mathbf{X}$. Applying the identity $xt_1x \cdots t_kx \approx \mathbf{w}$ to the words $\mathbf{a}$ and $\mathbf{b}$, we can replace their subwords of the form $x\mathbf{f}_1x \cdots \mathbf{f}_kx$ to $x^{e_0}\mathbf{f}_1x^{e_1} \cdots \mathbf{f}_kx^{e_k}$. In other words, the identity $xt_1x \cdots t_kx \approx \mathbf{w}$ can be used to convert the words $\mathbf{a}$ and $\mathbf{b}$ into some words $\mathbf{a}'$ and $\mathbf{b}'$, respectively, so that the identity $\mathbf{a}' \approx \mathbf{b}'$ holds in $M(\{xt_1x \cdots t_nx \mid n \in \mathbb{N}\})$ (because the word $xt_1x \cdots t_{k-1}x$ is an isoterm for $\mathbf{X}$ and at least one of the numbers $e_0, \dots, e_k$ exceeds 1). Therefore, the variety $\mathbf{X}$ can be defined by $\{xt_1x \cdots t_kx \approx \mathbf{w}\} \cup \Sigma$ for some set Σ of identities holding in $M(\{xt_1x \cdots t_nx \mid n \in \mathbb{N}\})$.

Further, if $M(xt_1x \cdots t_kx) \in \mathbf{V}$, then $M(xt_1x \cdots t_kx) \notin \mathbf{Y}$ because $\mathbf{X} \wedge \mathbf{V} = \mathbf{Y} \wedge \mathbf{V}$. In this case, it is easy to deduce from Lemmas 3 and 4 that $\mathbf{Y}$ satisfies a non-trivial identity $xt_1x \cdots t_kx \approx x^{f_0}t_1x^{f_1} \cdots t_kx^{f_k} =: \mathbf{w}'$, where at least one of the numbers $f_0, \dots, f_k$ exceeds 1. Since $\mathbf{X} \subset \mathbf{Y}$, the identity $\mathbf{w} \approx \mathbf{w}'$ holds in $\mathbf{X}$. Clearly, this identity is satisfied by $\mathbf{V}$ as well. Hence $\mathbf{X} \vee \mathbf{V} = \mathbf{Y} \vee \mathbf{V}$ satisfies $\mathbf{w} \approx \mathbf{w}'$ and, therefore, $\mathbf{Y}$ satisfies $xt_1x \cdots t_kx \approx \mathbf{w}$.

Suppose now that $M(xt_1x \cdots t_kx) \notin \mathbf{V}$. In this case, it is easy to deduce from Lemmas 3 and 4 that $\mathbf{V}$ satisfies the identity $xt_1x \cdots t_kx \approx \mathbf{w}$. Since $\mathbf{X} \vee \mathbf{V} = \mathbf{Y} \vee \mathbf{V}$, we see that this identity holds in $\mathbf{Y}$.

According to Claim 3, all the identities in Σ are satisfied by $\mathbf{Y}$, contradicting $\mathbf{X} \subset \mathbf{Y}$. Therefore, $\mathbf{V}$ is a modular element of of the lattice $\mathbb{MON}$. Theorem 1 is thus proved. □

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