Flavored QCD axion and Modular invariance

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Abstract

A four-dimensional effective model with $G_{\rm SM} \times SL(2,\mathbb{Z}) \times U(1)_X$ is proposed in string-derived supergravity framework, where $G_{\rm SM}$ is the Standard Model (SM) gauge group and $U(1)_X$ is gauged. We show $SL(2,\mathbb{Z})$- and $U(1)_X$-mixed anomalies should vanish. Anomalies induced by Kähler transformations match those from gaugino chiral rotations. When SM fermions transform nontrivially under $SL(2,\mathbb{Z})$, and with vanishing gaugino contributions, the anomaly-free conditions are powerful enough to determine the quark and lepton flavor structures, set scales for $U(1)_X$ breaking, and ensure the strong CP phase remains unmodified. While the Green-Schwarz coefficient $\delta_X^{\rm GS}$ is generically non-zero, vanishing $U(1)_X$ anomalies cause gauge boson decoupling and $\delta_X^{\rm GS} \to 0$, yielding a massless global $U(1)_X$ without a Nambu-Goldstone mode. We show that the modulus vacuum expectation value stabilizes near $\langle\tau\rangle \approx i$, where exact $SL(2,\mathbb{Z})$ (T-duality) is spontaneously broken, removing residual modular symmetry. The framework predicts seesaw-generated neutrino masses and flavored axion properties, with all Yukawa coefficients constrained to unit-magnitude complex numbers. Our model reproduces current quark and lepton data, predicts an axion mass $m_a \approx 0.9 \times 10^{-2}$ eV and photon coupling $|g_{a\gamma\gamma}| \approx 1.7 \times 10^{-13}\,{\rm GeV}^{-1}$, and unlike the ordinary case, suppresses flavor-violating axion couplings to s, d quarks and μ, e leptons to $\mathcal{O}(\lambda^4)$ (with λ the Cabibbo angle). It also yields normal neutrino mass hierarchy consistent with oscillation data, $0\nu\beta\beta$-decay rate, and cosmological and astrophysical measurements.

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I. INTRODUCTION

The Standard Model (SM) gauge symmetry, while successful in describing fundamental interactions, fails to constrain the flavor structure, leaving the observed fermion mass hierarchies, mixing patterns, and strong CP invariance unexplained. This strongly motivates the existence of new symmetries beyond the SM to address these unresolved flavor puzzles. For instance, an anomalous $U(1)$ could simultaneously resolve multiple issues: (i) generating a QCD axion (via the Peccei-Quinn (PQ) mechanism [1, 2]) to solve the strong CP problem while providing a dark matter candidate, (ii) constraining flavor structures to reduce Yukawa coupling arbitrariness [3, 4] . Additionally, modular $SL(2,\mathbb{Z})$ invariance offers a minimal flavor framework where Yukawa couplings are modular forms, intrinsically constraining quark/lepton structures without excessive scalars [5–7]. Recent work [7] demonstrates that CP- and modular-invariant model can yield a vanishing QCD angle and reproduce quark (though not lepton) observables. In a separate approach, a $U(1)\times A_4$ model addressing the flavor and strong CP problems [8] utilizes an A_4 subgroup of $SL(2,\mathbb{Z})$. See also Ref.[9].

Within the framework of string-derived supergravity[1], we propose a modular- and gauge-invariant model within a four-dimensional (4D) effective action. This model incorporates the symmetry group $G_{\rm SM}\times SL(2,\mathbb{Z})\times U(1)_X$, where $G_{\rm SM}=SU(3)_C\times SU(2)_L\times U(1)_Y$ is the SM gauge group. The modular symmetry $SL(2,\mathbb{Z})$, particularly its T-duality transformation, enforces strict invariance of the superpotential, Kähler potential, and gauge kinetic function under modular transformation. The gauged $U(1)_X$ acts as a flavored PQ symmetry [3], distinguished by flavor-dependent PQ charges. However, these symmetries can be violated by quantum anomalies which must cancel for the consistency of the theory, even though $SL(2,\mathbb{Z})$ is a global symmetry. Modular forms – holomorphic functions of τ acting as Yukawa couplings – are constrained to polynomials in the Eisenstein series E_4 and E_6 that transform as $SL(2,\mathbb{Z})$ singlets [7]. Their holomorphicity across the fundamental domain (including $\tau=i\infty$) yields nearly unique, weight-specific forms.

In this framework, we demonstrate that the anomalies induced by Kähler transformations align with those generated by the chiral rotation of gauginos. And, we argue that when SM

[1] In the global supersymmetry limit ($M_P\to\infty$), modular symmetry is generally not required to be preserved at the quantum level, see Eqs.(11) and (23). We argue that, although the SM fermions transform nontrivially under $SL(2,\mathbb{Z})$, this does not necessarily affect the SM strong CP phase.

fermions transform nontrivially under $SL(2,\mathbb{Z})$, the cancellation of modular anomalies – with gaugino contributions vanishing – imposes stringent constraints. These anomaly-free conditions, along with the guaranteed cancellation of mixed $SL(2,\mathbb{Z}) \times \{[SU(3)_C]^2, [U(1)_{\rm EM}]^2\}$ anomalies, can determine the flavor structure of both quarks and leptons, set scales for $U(1)_X$ symmetry breaking, and ensure the strong CP phase remains unmodified. The resulting framework can predict physical quantities, including hierarchical quark and lepton masses and mixings, seesaw-generated neutrino masses [10], and flavored axion properties, with Yukawa coefficients restricted to unit-magnitude complex numbers.

We construct a simple moduli superpotential to determine Yukawa couplings (especially, the vacuum expectation value (VEV) of the modulus τ), gauge couplings, SUSY-breaking scale, and cosmological constant. Previous studies on modulus τ stabilization have found that the VEV of τ often approaches specific fixed points, such as i, $e^{i2\pi/3}$ and $i\infty$ [7, 11–13]. We demonstrate that the modulus τ VEV stabilizes near a fixed point (particularly $\tau \approx i$). Although $SL(2,\mathbb{Z})$ is treated as an exact discrete gauge symmetry, it becomes spontaneously broken when τ develops a VEV. Notably, at $\langle\tau\rangle \approx i$, no non-trivial subgroup of the modular group survives at low energies.

The anomaly coefficients for $U(1)_X \times [G_{\rm SM}]^2$—determined by the $U(1)_X$ charges of SM fermions—can either vanish or remain finite, depending on the specific charge assignments across SM fermions (see Eq.(31)). Upon spontaneous breaking of the $U(1)_X$ gauge symmetry and subsequent decoupling of the associated gauge boson, a protected global $U(1)_X$ symmetry emerges, which remains robust against quantum gravitational effects [14]. We argue that this residual global (non)-anomalous symmetry, which exhibits its flavor dependence, can be identified with the flavored $U(1)$ PQ symmetry (Froggatt-Nielsen $U(1)$ [15]) or with baryon-lepton number $U(1)_{B-L}$.

The rest of this paper is organized as follows. Section II demonstrates the anomaly cancellation conditions for the symmetry group $G_{\rm SM} \times SL(2,\mathbb{Z}) \times U(1)_X$ and their constraints on the chiral spectrum. We also examine the flavored $U(1)_X$ symmetry and the VEV of the modulus τ. In Section III, we construct the superpotentials for the quark, lepton, and scalar fields. In Section IV, we visualize the interactions between quarks, leptons, and the flavored-QCD axion, and present a numerical analysis of their phenomenological implications. The final section provides a summary of our work.

II. MINIMAL SET-UP

We work in 4D $\mathcal{N}=1$ string-derived supergravity with chiral superfields $\Phi=(\varphi,\tau,...)$. The generalized Kähler potential is $G(\Phi,\bar{\Phi})=K(\Phi,\bar{\Phi})/M_P^2+\ln(|W(\Phi)|^2/M_P^6)$, where $M_P=2.436\times10^{18}$ GeV, K is gauge-invariant, and W is holomorphic. The framework includes a gauge kinetic function $f(\Phi)$. In type IIA intersecting D-brane models (T-dual to magnetized branes), the theory must be invariant under $SL(2,\mathbb{Z})$ modular transformations [16]

$$\tau\to\frac{a\tau+b}{c\tau+d}=\gamma\tau\,,\qquad(a,b,c,d\in Z,ad-bc=1)\,,\tag{1}$$

which is acting on the modular group $SL(2,\mathbb{Z})$ of the complex modulus τ, with $\mathrm{Im}(\tau)>0$. Modular forms are holomorphic in the fundamental domain (including $\tau=i\infty$) and constructed from Eisenstein series E_4 and E_6 that transform as $SL(2,\mathbb{Z})$ singlets [5, 7].

Our model has $G_{\rm SM}\times SL(2,\mathbb{Z})\times U(1)_X$ symmetry, where $G_{\rm SM}\times U(1)_X$ may arise from D-branes. The supersymmetry (SUSY) action is given by

$$\mathcal{S}=\int d^4xd^2\theta d^2\bar{\theta}\,K(\Phi,\bar{\Phi}e^{2V})+\Big\{\int d^4xd^2\theta\Big(W(\Phi)+\frac{f_{ab}(\Phi)}{4}\mathcal{W}^{\alpha a}\mathcal{W}^b_\alpha\Big)+\text{h.c.}\Big\}\,,\tag{2}$$

where $V\equiv V^aT^a$ is the gauge multiplet for all gauge groups containing Yang-Mills multiplet and T^a are the gauge group generators, and $\mathcal{W}_\alpha$ is a gauge-invariant chiral spinor superfield that contains the Yang-Mills field strength, as well as the $U(1)_X$ gauge field strength. Under the modular transformation, the action remains invariant with Kähler transformations

$$\begin{aligned}K(\Phi,\bar{\Phi}e^{2V})&\to K(\Phi,\bar{\Phi}e^{2V})+\big(g(\tau)+g(\bar{\tau})\big)M_P^2\,,\\ W(\Phi)&\to W(\Phi)e^{-g(\tau)}\,,\\ f(\Phi)\mathcal{W}^\alpha\mathcal{W}_\alpha&\to f(\Phi)\mathcal{W}^\alpha\mathcal{W}_\alpha\,.\end{aligned}\tag{3}$$

At the quantum level, the symmetry $G_{\rm SM}\times SL(2,\mathbb{Z})\times U(1)_X$ can be violated through both modular and gauge anomalies. However, such anomalies should be absent. Especially, since modular symmetry is exact in string theory, any low-energy supergravity derived from it must exhibit vanishing modular anomaly (see Sec.II A).

Under the modular group $SL(2,\mathbb{Z})$ and the gauged $U(1)_X$, for the action (2) to be invariant we consider a low-energy Kähler potential K, superpotential W, and gauge kinetic function f_{ab}:

$$\begin{aligned} K &= -M_P^2 \ln\Big\{(-i\tau + i\bar{\tau})^h \big(S + \bar{S} - \frac{h}{4\pi^2}\ln(-i\tau+i\bar{\tau})\big)\Big(U_X + \bar{U}_X - \frac{\delta_X^{\rm GS}}{16\pi^2}V_X\Big)^h\Big\} \\ &+ (-i\tau+i\bar{\tau})^{-k}|\varphi|^2 + Z_X\varphi_X^\dagger e^{-XV_X}\varphi_X + \dots\,, \\ W &= \frac{1}{3}Y(\tau)\,\varphi_i\varphi_j\varphi_k + W(S,U_X,\tau)\,, \\ f_{ab} &= \delta_{ab}(S+U_X)\,, \end{aligned} \tag{4}$$

with $h=3$ and the $U(1)_X$ charge $|X|=1$ determined by the transformation of φ_X, where $-k$ is the modular weight of φ, Z_X is the normalization factor, S denotes the axio-dilaton, τ represents the Kähler modulus, U_X corresponds to the complex structure modulus, and the modular- and gauge-invariant superpotential $W(S,U_X,\tau)$ is given by Eq.(33). The dots in Eq.(4) denote the contributions of non-renomalizable terms scaled by an UV cutoff M_P. The $U(1)_X$ charged matter fields φ_X and complex structure modulus U_X and the vector superfield V_X of the gauged $U(1)_X$ containing the gauge field A^μ_X participate in the 4D Green-Schwarz (GS) mechanism [17]. The axio-dilaton S, the $U(1)_X$ charged modulus U_X, and the $U(1)_X$ charged scalar field φ_X can be decomposed as [2]

$$S = \frac{1}{g_s^2} + i\theta_s\,,\qquad U_X = \sigma + i\theta_X\,,\qquad \varphi_X\big|_{\theta=\bar{\theta}=0} = \frac{1}{\sqrt{2}}e^{i\frac{A_X}{v_X}}(v_X+h_X)\,, \tag{5}$$

where [3] $\sigma = 1/g_X^2$ with g_X being the 4D gauge coupling of $U(1)_X$, and A_X, v_X, and h_X are the Nambu-Goldstone (NG) mode, VEV, and Higgs boson of scalar components, respectively. The GS parameter $\delta_X^{\rm GS}$ characterizes the coupling of the anomalous gauge boson to the closed string axion θ_X.

A. $SL(2,\mathbb{Z})$ modular anomaly cancellation

Under the modular group $SL(2,\mathbb{Z})$, invariance of the 4D action Eq.(2) requires that the matter fields φ_i and the modulus S transform as

$$\varphi_i \to (c\tau+d)^{-k_i}\varphi_i\,,\qquad S \to S - \frac{1}{4\pi^2}\ln(c\tau+d)^h\,, \tag{6}$$

[2] The superpotential $W(S,U_X,\tau)$ of Eq.(33) explicitly breaks the S-shift symmetry.

[3] Assuming $\mathrm{Re}(U_X) \gg \mathrm{Re}(S)$, the dilaton S couples to all SM gauge groups with equal strength at tree level, as set by the string scale. However, this universality is broken by quantum effects, including threshold corrections and renormalization group running.

where $-k_i$ is the modular weight of the matter field φ_i. Under the modular transformation Eq.(1) with Eq.(6), the Kähler potential K transforms as in Eq.(3), yielding $g(\tau) = \ln(c\tau + d)^h$. This redundancy in the Kähler transformation induces a modular anomaly [8, 18, 22, 23], see below Eq.(13). For the superpotential $W(\Phi)$ to remain modular-invariant under the Kähler transformation in Eq.(3), the modular form $Y(\tau)$ must transform as a modular form of weight $-k_Y$:

$$Y(\tau) \to Y(\gamma\tau) = (c\tau + d)^{-k_Y} Y(\tau) \tag{7}$$

where $k_Y = h - (k_i + k_j + k_k)$. Canonically normalized fields $\hat{\varphi}$, defined by $\varphi_i = (K^{-1/2})_{ij}\hat{\varphi}_j$, ensures that the modular forms are normalized. The field transformation is given by

$$\varphi_i \to (-i\tau + i\bar{\tau})^{\frac{k_i}{2}} \hat{\varphi}_i \,. \tag{8}$$

This normalization leads to $W = \frac{1}{3}\hat{Y}(\tau)\hat{\varphi}_i\hat{\varphi}_j\hat{\varphi}_k$ with the normalized modular form given [4] by $\hat{Y}(\tau) = e^{K/2M_P^2} Y(\tau)(-i\tau + i\bar{\tau})^{(k_i+k_j+k_k)/2}$ within the framework of supergravity. Under the modular transformations of Eqs.(1) and (6), the normalized fields and modular forms transform as

$$\hat{\varphi}_i \to \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{-\frac{k_i}{2}} \hat{\varphi}_i \,, \qquad \hat{Y}(\tau) \to \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{\frac{1}{2}(k_i+k_j+k_k-h)} \hat{Y}(\tau) \,. \tag{9}$$

In the framework where the Einstein-Hilbert term is canonically normalized, the kinetic and mass terms of normalized SM fermions $\hat{\psi}$ and gauginos $\hat{\lambda}$ arise as functions determined by the Kähler potential K and superpotential W. The relevant Lagrangian can be written in two-component spinor notation as (see also Refs.[7, 19, 20])

$$\begin{aligned} &-\frac{1}{2} e^{\frac{K}{2M_P^2}} (K^{-1/2})_i{}^k (K^{-1/2})_j{}^l (\mathcal{D}_k D_l W) \hat{\psi}^i \hat{\psi}^j + \frac{1}{4} (\mathrm{Re}\, f)^{-1}_{ac}\, F^i \partial_i f_{cb}\, \hat{\lambda}^a \hat{\lambda}^b + \text{h.c.} \\ &+ \Big(-\frac{i}{2} \bar{\hat{\psi}}^{\bar{i}} \bar{\sigma}^\mu \Gamma^i_{jk} \partial_\mu \phi^j \hat{\psi}^k + \text{h.c.} \Big) - i \bar{\hat{\psi}}^{\bar{i}} \bar{\sigma}^\mu D_\mu \hat{\psi}^j - i \bar{\hat{\lambda}} \bar{\sigma}^\mu D_\mu \hat{\lambda} \,, \end{aligned} \tag{10}$$

where $F^i = e^{K/2M_P^2} K^{i\bar{j}} \bar{D}_{\bar{j}} \bar{W}$, $\sigma^\mu = (1, \sigma^k)$ and $\bar{\sigma}^\mu = (1, -\sigma^k)$ with σ^k the Pauli matrices, $\Gamma^i_{jk} = K^{i\bar{\ell}} \partial_j K_{k\bar{\ell}}$ is the modular connection (Kähler Christoffel symbol), $K^{i\bar{j}} = (\partial_i \partial_{\bar{j}} K)^{-1}$ is the inverse Kähler metric, the Kähler-covariant derivative of the superpotential is $D_i W =$

[4] The physical modular forms can be expressed in a good approximation for $M_P \gg \langle \varphi_X \rangle, \langle \varphi \rangle$ and $V_X = 0$ as $\hat{Y}(\tau) \simeq Y(\tau)(-i\tau + i\bar{\tau})^{(k_i+k_j+k_k-h)/2}(U_X + \bar{U}_X)^{-\frac{h}{2}}(S + \bar{S} - \frac{h}{4\pi^2}\ln(-i\tau + i\bar{\tau}))^{-\frac{1}{2}}$. Without loss of generality, the term $(U_X + \bar{U}_X)^{-\frac{h}{2}}(S + \bar{S} - \frac{h}{4\pi^2}\ln(-i\tau + i\bar{\tau}))^{-\frac{1}{2}}$ can be absorbed to Yukawa coefficients and normalized to one, leading to $\hat{Y}(\tau) \simeq Y(\tau)(-i\tau + i\bar{\tau})^{(k_i+k_j+k_k-h)/2}$.

$W_i + \frac{K_i}{M_P^2} W$, and the fully covariant second derivative[5] is $\mathcal{D}_i D_j W = W_{ij} + \frac{K_{ij}}{M_P^2} W + \frac{K_i}{M_P^2} D_j W + \frac{K_j}{M_P^2} D_i W - \frac{K_i K_j}{M_P^4} W - \Gamma^k_{ij} D_k W$. The spacetime covariant derivatives acting on SM fermions and gauginos are given by

$$\begin{aligned} D_\mu \hat{\psi}^j &= \partial_\mu \hat{\psi}^j + i q_K K_\mu \hat{\psi}^j \,, \\ D_\mu \hat{\lambda}^a &= \partial_\mu \hat{\lambda}^a + f^{abc} A^b_\mu \hat{\lambda}^c + i q_K K_\mu \hat{\lambda}^a \,, \end{aligned} \tag{11}$$

for flat spacetime, where $K_\mu = -\frac{i}{2M_P^2}(K_i \partial_\mu \phi^i - K_{\bar{i}} \partial_\mu \bar{\phi}^{\bar{i}})$ is the Kähler connection[6]. For the gauginos $\hat{\lambda}^a$, which transform in the adjoint representation of the gauge group, the f^{abc} are the totally antisymmetric structure constants. The Kähler charge q_K takes the values $-1/2$ for SM fermions, and $+1/2$ for gauginos in Eq.(11). Under the Kähler transformation Eq.(3) the Kähler connection transforms as

$$K_\mu \to K_\mu - i\frac{h}{2}\Big(\frac{c\partial\tau}{c\tau + d} - \frac{c\partial\bar{\tau}}{c\bar{\tau} + d}\Big)\,, \tag{12}$$

where $\partial_\mu g(\tau) = \frac{\partial g}{\partial \tau}\partial_\mu \tau$ is used. To cancel this variation, the SM fermions and gauginos should transform with their respective Kähler charge q_k as

$$\Psi \to e^{-q_K \frac{g - \bar{g}}{2}} \Psi\,, \qquad \bar{\Psi} \to e^{-q_K \frac{g - \bar{g}}{2}} \bar{\Psi} \qquad \text{with } \Psi = \hat{\psi}, \hat{\lambda}\,. \tag{13}$$

In addition, under the modular transformation Eq.(1) the term including the modular connection in Eq.(10), that is, $\frac{1}{2}(\Gamma^\varphi_{\tau\varphi}\partial_\mu \tau - \Gamma^\varphi_{\bar{\tau}\varphi}\partial_\mu \bar{\tau})$ transforms as

$$\frac{k_i}{2}\Big(\frac{c\partial\tau}{c\tau + d} - \frac{c\partial\bar{\tau}}{c\bar{\tau} + d}\Big) - \frac{k_i}{2}\Big(\frac{\partial_\mu \tau + \partial_\mu \bar{\tau}}{\tau - \bar{\tau}}\Big)\,, \tag{14}$$

where we have used that for a diagonal Kähler metric $\Gamma^i_{ji} = \partial_j \ln K_{i\bar{i}}$. The second term in Eq.(14) vanishes in the cusp limit $\tau \to i\infty$, while the first term must be cancelled by the

[5] The covariant derivative $\mathcal{D}_i$ acting on an object V_j with Kähler weight (p, q), meaning it transforms as $V_j \to e^{-(pg + q\bar{g})} V_j$ under the Kähler transformation Eq.(3), is given by $\mathcal{D}_i V_j = \partial_i V_j - \Gamma^k_{ij} V_k + \frac{p}{M_P^2} K_i V_j$. Here, under the Kähler transformation Eq.(3), the object $D_j W$ transforms as $D_j W \to e^{-g(\Phi)} D_j W$.

[6] In the global SUSY limit $M_P \to \infty$, the Kähler connection vanishes. Consequently, the associated modular anomalies disappear (see below Eq.(16)), and the chiral transformations of Eq.(15) effectively reduce to the case $h = 0$. Meanwhile, the term containing the modular connection in Eq.(10) is required to maintain covariance under SM fermion reparameterizations on the Kähler manifold. The quantum anomalies associated with the modular connection are benign in this limit, as it pertains to a spacetime-like symmetry rather than an internal gauge symmetry, and thus do not jeopardize the consistency of the quantum theory.

transformation of the first term in Eq.(9). To achieve the cancellation of Eqs.(12) and (14), the canonically normalized SM fermions $\hat{\psi}$ and gauginos $\hat{\lambda}$ should transform as

$$\hat{\psi}_i \to \left(\frac{c\tau+d}{c\bar{\tau}+d}\right)^{k_{\hat{\psi}_i}} \hat{\psi}_i\,, \qquad \hat{\lambda} \to \left(\frac{c\tau+d}{c\bar{\tau}+d}\right)^{-\frac{h}{4}} \hat{\lambda} \qquad \text{with } k_{\hat{\psi}_i} = \frac{h}{4} - \frac{1}{2}k_i\,, \tag{15}$$

which[7] ensures that the kinetic and mass terms of Eq.(10) are invariant under the Kähler transformation in Eq.(4) and modular transformation Eq.(1) (up to total derivative). However, these transformations Eq.(15) induce chiral rotations in the fermionic path-integral measure, generating modular anomalies – triangle anomalies analogous to the Adler-Bell-Jackiw anomaly [21]. At the quantum level, this anomaly appears in the effective action under the Kähler transformation in Eq.(4) and modular transformation Eq.(1) as

$$SL(2,\mathbb{Z}) \times \left\{[U(1)_X]^2,\ [U(1)_Y]^2,\ [SU(2)_L]^2,\ [SU(3)_C]^2\right\}. \tag{16}$$

First, the anomaly generated by the chiral rotation of gauginos, which exactly matches that from the Kähler transformation[8], takes the form

$$-\frac{C}{32\pi^2}\Big\{-\big(g(\tau)+g(\bar{\tau})\big)Q^{\mu\nu}Q_{\mu\nu} + i\big(g(\tau)-g(\bar{\tau})\big)Q^{\mu\nu}\tilde{Q}_{\mu\nu}\Big\}\,, \tag{17}$$

where the first term in the brackets corresponds to the gauge boson kinetic term, while the second, CP-odd term involves the dual field strength $\tilde{Q}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}Q^{\rho\sigma}$. The gauge field strengths Q are given by $\{G, W, Y, F_X\}$ for $SU(3)_C$, $SU(2)_L$, $U(1)_Y$, and $U(1)_X$, respectively. In Eq.(17), the coefficients take values $C = 3, 2, 1, 1$ for gluino, wino, bino, and $U(1)_X$ gaugino, respectively. The anomalies are cancelled via the transformation of the modulus S in Eq.(6), which enters the one-loop corrected gauge kinetic function,

$$f_{ab}^{1-\text{loop}}(\Phi) \supset \delta_{ab}\Big\{S - \frac{1}{4\pi^2}\ln(c\tau+d)^h\Big\}\,, \tag{18}$$

as discussed in Refs.[8, 18, 22, 23]. The gaugino masses $M_{\hat{\lambda}}$ transform according to $M_{\hat{\lambda}} \to \left(\frac{c\tau+d}{c\bar{\tau}+d}\right)^{\frac{h}{2}} M_{\hat{\lambda}}$ due to Eq.(15). Since gauginos do not mix with SM fermions, their contribution to the modular anomalies cancels through the one-loop corrected gauge kinetic function [8], yielding

$$\arg(M_{\hat{\lambda}}) = 0\,. \tag{19}$$

[7] See also Ref.[7].

[8] Under the Kähler transformation, the modular anomaly manifests via the variation of the action $\delta S = -\tilde{c}\frac{1}{4}\int d^4x d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha g(\tau) + \text{h.c.}$ [23].

For example, the gluino contribution to the strong CP phase vanishes: $\arg(M_3) = 0$. Consequently, the effective strong CP phase then reduces to

$$\vartheta_{\rm eff} = \vartheta_{\rm QCD} + A_C \arg\left(\frac{c\tau + d}{c\bar{\tau} + d}\right) + \arg\left[\det(M_u M_d)\right], \tag{20}$$

where $\vartheta_{\rm QCD}$ comes from θ_s and θ_X of Eq.(5), and A_C denotes the sum of the modular weights of the quark fields, under the assumption[9] $k_{H_{u(d)}} = 0$ for the Higgs doublets $H_{u(d)}$.

Second, since the modular group is treated as a discrete-gauge symmetry in our framework, see below Eq.(11), any anomalies that would break this invariance must be canceled. The anomaly generated by the rotation of chiral fermions in Eq.(15) also takes the form presented in Eq.(16). The QCD anomaly coefficient for $SL(2,\mathbb{Z}) \times [SU(3)_C]^2$ is given by $A_C = 2{\rm Tr}[k_i\, T^2_{SU(3)_C}]$:

$$A_C = \sum_{i=1}^{3}\left(2k_{\hat{Q}_i} + k_{\hat{U}^c_i} + k_{\hat{D}^c_i}\right), \tag{21}$$

where $k_{\hat{Q}_i}$ and $k_{\hat{U}^c_i}(k_{\hat{D}^c_i})$ denote the weights for the normalized left-handed quarks and right-handed up (down)-type quarks, respectively. The $U(n)$ generators ($n \geq 2$) are normalized to ${\rm Tr}[T^a T^b] = \delta^{ab}/2$. Similarly, the electromagnetic anomaly coefficient for $SL(2,\mathbb{Z}) \times [U(1)_{\rm EM}]^2$ is given by $A_E = 2{\rm Tr}[k_i\,(Q^{\rm em}_i)^2]$:

$$A_E = \frac{2}{3}\sum_{i=1}^{3}\left\{3(k_{\hat{L}_i} + k_{\hat{\ell}^c_i}) + 5k_{\hat{Q}_i} + 4k_{\hat{U}^c_i} + k_{\hat{D}^c_i}\right\}, \tag{22}$$

where $k_{\hat{L}_i}$ and $k_{\hat{\ell}^c_i}$ denote the weights for the normalized left-handed leptons and right-handed leptons, respectively. Due to the modular- and SM gauge-invariant structure of the superpotential with non-negative weight modular forms, from Eqs.(21) and (22) we obtain

$$\sum_{i=1}^{3}(k_{\hat{Q}_i} + k_{\hat{U}^c_i}) = 0\,, \qquad \sum_{i=1}^{3}(k_{\hat{Q}_i} + k_{\hat{D}^c_i}) = 0\,, \qquad \sum_{i=1}^{3}(k_{\hat{L}_i} + k_{\hat{\ell}^c_i}) = 0\,, \tag{23}$$

for $k_{H_{u(d)}} = 0$, which means $A_C = A_E = 0$. And the anomaly coefficients for $SL(2,\mathbb{Z}) \times [SU(2)_L]^2$ and $SL(2,\mathbb{Z}) \times [U(1)_Y]^2$ are given, respectively, by $A_L = 2{\rm Tr}[k_i\, T^2_{SU(2)}]$ and $A_Y = 2{\rm Tr}[k_i\, Y^2]$, which expand explicitly as

$$A_L = \sum_{i=1}^{3}\left(k_{\hat{L}_i} + 3k_{\hat{Q}_i}\right), \quad A_Y = \frac{1}{3}\sum_{i=1}^{3}\left(k_{\hat{Q}_i} + 8k_{\hat{U}^c_i} + 2k_{\hat{D}^c_i} + 3k_{\hat{L}_i} + 6k_{\hat{\ell}^c_i}\right), \tag{24}$$

[9] We assume that all symmetry-breaking scalars have modular weight zero; otherwise their VEVs would have to be zero in order to remain invariant under the modular transformation, see Eqs.(6) or (9) and Eq.(47).

which satisfy $A_L = -A_Y$ using Eq.(23). We note that the conditions $A_C = 0 = A_E$ and $A_L = -A_Y$ persist in the global SUSY limit. If the GS counterterms are introduced to cancel Eq.(24), certain anomalies of the form shown in Eq.(16), that were already cancelled, are reintroduced, and additional anomalous contributions arise. So, $A_L = -A_Y$ should be zero:

$$\sum_{i=1}^{3} \left(k_{\hat{L}_i} + 3k_{\hat{Q}_i}\right) = 0\,. \tag{25}$$

Using Eq.(23) the anomaly coefficient for $SL(2,\mathbb{Z}) \times [U(1)_X]^2$ is given by

$$A_{SX} = \sum_{i=1}^{3} \left\{ 3k_{\hat{Q}_i}\left(2X^2_{Q_i} - X^2_{U^c_i} - X^2_{D^c_i}\right) + k_{\hat{L}_i}\left(2X^2_{L_i} - X^2_{\ell^c_i}\right) + k_{\hat{N}^c_i} X^2_{N^c_i} \right\}, \tag{26}$$

which must also vanish, especially, constraining the flavor-dependent $U(1)_X$ charges and the modular weights of right-handed neutrinos N^c_i. Here, X_{Q_i} (X_{L_i}) represent the $U(1)_X$ charges of the left-handed quark (lepton) doublets, $X_{D^c_i}$ $(X_{U^c_i})$ represent the charges of the gauge singlet right-handed down (up)-type quarks, and $X_{\ell^c_i}$ $(X_{N^c_i})$ represent the charges of the gauge singlet right-handed charged-leptons (neutrinos).

B. Gauged $U(1)_X$ anomaly-free and global $U(1)_X$

The 4D action of Eq.(2), combined with the Kähler potential Eq.(4), must remain $U(1)_X$ gauge invariant. Under the $U(1)_X$ gauge transformation $V_X \to V_X + i(\Lambda_X - \bar{\Lambda}_X)$, the fields transform as $\varphi_X \to e^{iX\Lambda_X}\varphi_X$ and $U_X \to U_X + i\frac{\delta^{\rm GS}_X}{16\pi^2}\Lambda_X$, respectively, where $\Lambda_X(\bar{\Lambda}_X)$ are (anti)chiral supefields parametrizing $U(1)_X$ transformation in superspace. So, the axionic modulus θ_X (from U_X) and axion A_X (from the matter sector) have shift symmetries

$$\theta_X \to \theta_X - \frac{\delta^{\rm GS}_X}{16\pi^2}\xi_X\,, \qquad A_X \to A_X + \alpha^Q_X\, f_X \xi_X\,, \tag{27}$$

where $\xi_X = -{\rm Re}\Lambda_X|_{\theta=\bar{\theta}=0}$, $f_X = Xv_X$ is the $U(1)_X$ breaking scale, and α^Q_X are transformation constants. Then, the $U(1)_X$ gauge field A^μ_X transforms as

$$A^\mu_X \to A^\mu_X - \partial^\mu \xi_X\,. \tag{28}$$

Then the 4D gauge-invariant effective action for θ_X, A_X, and A^μ_X reads [4, 24]

$$\begin{aligned} &K_{U_X\bar{U}_X}\left(\partial^\mu\theta_X - \frac{\delta^{\rm GS}_X}{16\pi^2}A^\mu_X\right)^2 - \frac{1}{4g^2_X}F^{\mu\nu}_X F_{X\mu\nu} + g_X\xi^{\rm FI}_X D_X - D_X g_X X|\varphi_X|^2 \\ &+\frac{1}{2}D^2_X + |D_\mu\varphi_X|^2 + \theta_X{\rm Tr}(Q^{\mu\nu}\tilde{Q}_{\mu\nu}) + \frac{A_X}{f_X}\frac{\delta^Q_X}{16\pi^2}{\rm Tr}(Q^{\mu\nu}\tilde{Q}_{\mu\nu})\,, \end{aligned} \tag{29}$$

where $Q = G, W, Y, F_X$ denote the gauge field strengths for $SU(3)_C$, $SU(2)_L$, $U(1)_Y$, and $U(1)_X$, respectively, with gauge couplings absorbed into their definitions. The first, third, fourth, and sixth terms result from expanding the Kähler potential Eq.(4), and the second, fifth, seventh, and eighth terms result from the gauge interaction term in Eq.(2). $F_X^{\mu\nu} = \partial^\mu A_X^\nu - \partial^\nu A_X^\mu$ is the $U(1)_X$ gauge field strength. The $U(1)_X$ gauge covariant derivative $D^\mu\varphi_X = \partial^\mu\varphi_X - iXA_X^\mu\varphi_X$ governs the coupling of the scalar component of φ_X to the $U(1)_X$ gauge boson, where the gauge coupling g_X is absorbed into A_X^μ. The first and seventh terms together, and the sixth and eighth terms in Eq.(29), are gauge invariant under the anomalous $U(1)_X$ transformations of Eqs.(27) and (28). Under the gauge transformations of Eqs.(27) and (28), it requires

$$\delta_X^{\rm GS} = \alpha_X^Q\,\delta_X^Q\,,\qquad \partial_\mu J_X^\mu = \frac{\delta_X^{\rm GS}}{16\pi^2}{\rm Tr}(Q^{\mu\nu}\tilde{Q}_{\mu\nu}) = -\partial_\mu J_\theta^\mu\,, \tag{30}$$

where the anomalous current J_X^μ and J_θ^μ couplings to A_X^μ are represented by $J_\mu^\theta = K_{U_X\bar{U}_X}\frac{\delta_X^{\rm GS}}{8\pi^2}\partial_\mu\theta_X$ and $J_\mu^X = -iX\varphi_X^\dagger\overleftrightarrow{\partial_\mu}\varphi_X$. The coefficients δ_X^Q of the mixed $U(1)_X\times[SU(3)_C]^2$, $U(1)_X\times[SU(2)_L]^2$, $U(1)_X\times[U(1)_Y]^2$, and $[U(1)_X]^3$ anomalies are given, respectively, by

$$\begin{aligned}
\delta_X^G &= 2{\rm Tr}[X_\psi T^2_{SU(3)}] = \sum_{i=1}^{3}\left(2X_{Q_i} + X_{D_i^c} + X_{U_i^c}\right),\\
\delta_X^W &= 2{\rm Tr}[X_\psi T^2_{SU(2)}] = \sum_{i=1}^{3}\left(X_{L_i} + 3X_{Q_i}\right),\\
\delta_X^Y &= 2{\rm Tr}[X_\psi Y^2] = \sum_{i=1}^{3}\left(\frac{1}{3}X_{Q_i} + \frac{8}{3}X_{U_i^c} + \frac{2}{3}X_{D_i^c} + X_{L_i} + 2X_{\ell_i^c}\right),\\
\delta_X^{F_X} &= 2{\rm Tr}[X_\psi^3] = 2\sum_{i=1}^{3}\left(3(2X^3_{Q_i} + X^3_{D_i^c} + X^3_{U_i^c}) + 2X^3_{L_i} + X^3_{\ell_i^c} + X^3_{N_i^c}\right).
\end{aligned} \tag{31}$$

where the trace is over all fermions ψ carrying $U(1)_X$ charges, $T_{SU(3)}$, $T_{SU(2)}$ are gauge group generators, Y is the hypercharge operator, and X_ψ denotes the $U(1)_X$ charge of ψ. Here, for convenience, δ_X^Y and $\delta_X^{F_X}$ are defined as above for hypercharge and $U(1)_X$, respectively. The Fayet-Iliopoulos (FI) term $\mathcal{L}_X^{\rm FI} = \xi_X^{\rm FI}\int d^2\theta d^2\bar{\theta}V_X = \xi_X^{\rm FI}g_X D_X$ with $D_X = g_X(-\xi_X^{\rm FI} + X|\varphi_X|^2)$ leads to D-term potential $V_D = \frac{g_X^2}{2}(-\xi_X^{\rm FI} + X|\varphi_X|^2)^2$ for the anomalous $U(1)_X$. Here $\xi_X^{\rm FI}$ is the FI factor $\xi_X^{\rm FI} = \frac{\partial K}{\partial V_X}|_{V_X=0,\sigma=\sigma_0}\Delta\sigma$ produced by expanding the Kähler potential Eq.(4) in components linear in V_X:

$$\xi_X^{\rm FI} = hM_P^2\frac{\delta_X^{\rm GS}}{16\pi^2}\frac{\Delta\sigma}{\sigma_0}\,, \tag{32}$$

where $\Delta\sigma = \sigma - \sigma_0$. Since the FI term is controlled by the string coupling (σ in Eq.(5)), in general, it can not be zero for $\delta_X^{\rm GS} \neq 0$, see Appendix B.

The effective action (29), after canonical normalization with $\theta_X = a_\theta/8\pi^2 f_\theta$ where $f_\theta = \sqrt{2K_{U_X\bar{U}_X}}/8\pi^2$, and incorporating the gauge kinetic function, yields kinetic terms for both axions along with their couplings to the topological term ${\rm Tr}(Q^{\mu\nu}\tilde{Q}_{\mu\nu})$. The $U(1)_X$ gauge boson acquires mass $m_X = \sqrt{2K_{U_X\bar{U}_X}(\delta_X^{\rm GS}/16\pi^2)^2 + 2f_X^2}$ through the super-Higgs mechanism, while the D-term potential V_D remains. The open string axion A_X (with its decay constant f_X) is mixed linearly with the closed string axion a_θ (with its decay constant f_θ) such that the orthogonal combinations $G \approx a_\theta$ (NG mode, absorbed by the gauge boson) and $\tilde{A} \approx A_X$ (pseudo-NG mode, remaining as the physical axion) emerge when $f_\theta \gg f_X$. Below the scale m_X, the gauge boson decouples, leaving an anomalous [10] global $U(1)_X$. See the details in Ref.[4, 8].

Interestingly, the vanishing $U(1)_X$ anomaly coefficient ($\delta_X^{\rm GS} = 0$) ensures the gauge boson decouples at low energies, leaving an anomaly-free global symmetry without a massless NG mode. Flavor-dependent charges satisfying $\delta_X^Q = 0$ cancel the gauge anomaly, making both $\partial_\mu J_X^\mu$ and $\partial_\mu J_\theta^\mu$ exactly conserved. This simultaneously nullifies the FI term ($\xi_X^{\rm FI} = 0$) and enforces D-flatness, stabilizing the scalar potential. The superpotential $W(U_X, \tau)$ (for $\alpha = 0$ in Eq.(33)) lifts a_θ's mass while A_X becomes the longitudinal mode of the massive $U(1)_X$ gauge boson, leaving only a non-anomalous global $U(1)_X$ below m_X. For instance, this fundamental symmetry could correspond to baryon-lepton number. This symmetry exactly reproduces $U(1)_{B-L}$ when charges are assigned as $X_{Q_i} = 1/3$, $X_{D_i^c} = X_{U_i^c} = -1/3$, $X_{L_i} = -1$, $X_{\ell_i^c} = 1$, and $X_{N_i^c} = 1$, yielding $\delta_X^G = \delta_X^W = \delta_X^Y = \delta_X^{F_X} = 0$ and ensuring complete anomaly cancellation [11].

[10] In certain gravitational backgrounds, non-perturbative quantum gravitational anomaly effects (specifically, instantons) can lead to an anomalous non-conservation of an axial vector current: $\partial_\mu J^\mu \propto R\tilde{R}$ where R is the Riemann curvature tensor and $\tilde{R}$ is its dual [25]. In this work, for simplicity, we assume that the gravitational mixed anomalies are canceled by the GS mechanism.

[11] Generically, for example, by considering three-stacks of D-branes with gauge symmetry $U(3)\times U(2)\times U(1)$, one may realize the three $U(1)_X \times U(1)_Y \times U(1)_{B-L}$.

C. The VEV of the modulus τ

To determine Yukawa couplings, gauge couplings, SUSY-breaking scale, and cosmological constant, we consider a simple modular- and gauge-invariant superpotential $W(S,U_X,\tau)$ in terms of the Dedekind η-function, which[12] is a modular form of weight 1/2, $\eta(\tau)\rightarrow(c\tau+d)^{1/2}\eta(\tau)$, and non-perturbative effect for gaugino condensation:

$$W(S,U_X,\tau)=C_0\frac{e^{-\alpha S}M_P^3}{[\eta(\tau)]^{2h(1+\alpha/4\pi^2)}}+\frac{M_P^3}{[\eta(\tau)]^{2h}}\big(Ae^{-aU_X}-Be^{-bU_X}\big)\,, \tag{33}$$

where C_0, α,a,b are constants. In the limit $\alpha\rightarrow0$, this superpotential takes a racetrack type for U_X [28]. Under the modular transformation Eq.(1), the modular invariance of the generalized Kähler potential implies that the superpotential transforms as $W\rightarrow We^{-g(\tau)}$ with Eq.(6). Under the $U(1)_X$ transformation of U_X, see above Eq.(27), the $A(\varphi_X/M_P)$ and $B(\varphi_X/M_P)$, which are analytic function of φ_X, transform as

$$A\Big(\frac{\varphi_X}{M_P}\Big)\rightarrow A\Big(\frac{\varphi_X}{M_P}\Big)e^{i\frac{a}{16\pi^2}\delta_X^{\rm GS}\Lambda_X}\,,\qquad B\Big(\frac{\varphi_X}{M_P}\Big)\rightarrow B\Big(\frac{\varphi_X}{M_P}\Big)e^{i\frac{b}{16\pi^2}\delta_X^{\rm GS}\Lambda_X}\,. \tag{34}$$

The F-term potential has the form $V_F=e^{K/M_P^2}\big\{K^{I\bar{J}}D_IW\bar{D}_{\bar{J}}\bar{W}-\frac{3}{M_P^2}|W|^2\big\}$, where I,J stand for U_X,τ,S, and matter fields are set to zero. From Eq.(33) the covariant derivatives $D_{U_X}W$, $D_\tau W$, and D_SW are expressed as

$$\begin{aligned}
D_{U_X}W&=\frac{M_P^3}{[\eta(\tau)]^6}\big(-aAe^{-aU_X}+bBe^{-bU_X}\big)-\frac{3W}{U_X+\bar{U}_X}\,,\\
D_\tau W&=-3W\Big[\frac{1-1/(4\pi^2y)}{\tau-\bar{\tau}}+2\frac{\eta'(\tau)}{\eta(\tau)}\Big]-\frac{3}{2\pi^2}\frac{\eta'(\tau)}{\eta(\tau)}\alpha C_0\frac{e^{-\alpha S}M_P^3}{[\eta(\tau)]^{6(1+\alpha/4\pi^2)}}\,,\\
D_SW&=-\frac{W}{y}-\alpha C_0\frac{e^{-\alpha S}M_P^3}{[\eta(\tau)]^{6(1+\alpha/4\pi^2)}}\,,
\end{aligned} \tag{35}$$

where $y=S+\bar{S}-\frac{3}{4\pi^2}\ln(-i\tau+i\bar{\tau})$. For small α, the superpotential Eq.(33) becomes

$$\begin{aligned}
W\;&=\;W(U_X,\tau)-\alpha\,C_0\frac{M_P^3}{[\eta(\tau)]^6}\big(S+\frac{3}{2\pi^2}\ln\eta(\tau)\big)\\
&+\;\frac{1}{2}\alpha^2C_0\frac{M_P^3}{[\eta(\tau)]^6}\big(S+\frac{3}{2\pi^2}\ln\eta(\tau)\big)^2-\frac{1}{6}\alpha^3C_0\frac{M_P^3}{[\eta(\tau)]^6}\big(S+\frac{3}{2\pi^2}\ln\eta(\tau)\big)^3+\ldots.
\end{aligned} \tag{36}$$

[12] Here, for simplicity, the Dedekind multiplier $e^{i\epsilon(a,b,c,d)}$ is omitted, where $\epsilon(a,b,c,d)$ is a moduli-independent phase, which can depend on the $SL(2,\mathbb{Z})$ transformation. Under the modular transformation given in Eq.(1), in general, the Dedekind eta function $\eta(\tau)$ transforms as $\eta(\tau)\rightarrow e^{i\epsilon}(c\tau+d)^{1/2}\eta(\tau)$ and the superpotential W transforms as $W\rightarrow e^{-g(\tau)-i2h\epsilon}W$ where the matter fields φ_i transform as: $\varphi_i\rightarrow e^{-i\epsilon_i}(c\tau+d)^{-k_i}\varphi_i$ with the condition $\sum_i\epsilon_i=\epsilon$, and the dilaton transforms as $S\rightarrow S-\frac{1}{4\pi^2}\{g(\tau)+i2h\epsilon\}$.

Then the potential V_F is expanded as $V_F = V_F^{(0)} + \alpha V_F^{(1)} + \alpha^2 V_F^{(2)} + ...$, where $V_F^{(0)}$ is the potential at $\alpha = 0$ and $V_F^{(n)}$ $(n \neq 0)$ is the n-th order correction due to small $\alpha \neq 0$. In the limit $\alpha \to 0$ (where $V_F \to V_F^{(0)}$, see Eq.(A1)), the scalar potential for the fields σ, τ has a local minimum at σ_0, τ_0. It is supersymmetric and Minkowski, *i.e.*,

$$W(\sigma_0, \tau_0) = 0\,, \qquad D_I W(\sigma_0, \tau_0) = 0\,, \qquad V(\sigma_0, \tau_0) = 0\,. \tag{37}$$

And C_0 and σ_0 are determined by the conditions $V_F^{(0)}(\sigma_0) = 0$ and $\partial V_F^{(0)}/\partial\sigma|_{\sigma=\sigma_0} = 0$:

$$C_0 = -A_0\Big(\frac{aA_0}{bB_0}\Big)^{-\frac{a}{a-b}} + B_0\Big(\frac{aA_0}{bB_0}\Big)^{-\frac{b}{a-b}}\,, \qquad \sigma_0 = \frac{1}{a-b}\ln\Big(\frac{aA_0}{bB_0}\Big)\,, \tag{38}$$

where A_0 and B_0 are the values of $A(\varphi_X/M_P)$ and $B(\varphi_X/M_P)$ at $\langle\varphi_X\rangle$, respectively.

The spontaneous breaking of modular symmetry is governed by the VEV of the modulus ($\mathrm{Im}\tau > 0$), which can always be constrained to lie within the fundamental domain $\mathcal{D}$ of the modular group. This domain is defined as

$$\mathcal{D} \equiv \Big\{\tau \in \mathcal{H} : -\frac{1}{2} \leq \mathrm{Re}\,\tau < \frac{1}{2}, |\tau| > 1\Big\} \cup \Big\{\tau \in \mathcal{H} : -\frac{1}{2} < \mathrm{Re}\,\tau \leq 0, |\tau| = 1\Big\}\,, \tag{39}$$

where $\mathcal{H}$ denotes the upper half-plane of complex numbers $\mathcal{H} \equiv \{\tau \in \mathbb{C}|\mathrm{Im}(\tau) > 0\}$. While no specific value of τ preserves the full modular symmetry, partial modular symmetries are retained at special symmetric points such as $\tau = i, i\infty, e^{i2\pi/3}$ [11, 13]. As shown in Ref.[26], all extrema of the potential $V(\tau, \bar{\tau})$ must lie either on the boundary of the fundamental domain $\mathcal{D}$ or on the imaginary axis.

When $\alpha \neq 0$, the VEV of τ is determined by solving $\partial V_F/\partial\tau = 0$. For small α, the VEV shifts to $\tau_0(\alpha) \simeq \tau_0 + \alpha\delta\tau$ where $\delta\tau$ represents the first-order correction. At $\tau \equiv \tau_0(\alpha)$, we expand $\partial V_F/\partial\tau$ to n-th oder in α:

$$\frac{\partial V_F}{\partial\tau}\Big|_{\tau=\tau_0(\alpha)} = \alpha^2\frac{\partial V_F^{(2)}}{\partial\tau}\Big|_{\tau_0} + \alpha^3\Big(\delta\tau\frac{\partial^2 V_F^{(2)}}{\partial\tau^2}\Big|_{\tau_0} + \frac{\partial V_F^{(3)}}{\partial\tau}\Big|_{\tau_0}\Big) + \mathcal{O}(\alpha^4)\,. \tag{40}$$

where the vanishing conditions $\frac{\partial V_F^{(0)}}{\partial\tau}\big|_{\tau_0,s_0,\sigma_0} = 0$, $\frac{\partial^2 V_F^{(0)}}{\partial\tau^2}\big|_{\tau_0,s_0,\sigma_0} = 0$, $\frac{\partial^3 V_F^{(0)}}{\partial\tau^3}\big|_{\tau_0,s_0,\sigma_0} = 0$, $\frac{\partial V_F^{(1)}}{\partial\tau}\big|_{\tau_0,s_0,\sigma_0} = 0$, $\frac{\partial^2 V_F^{(1)}}{\partial\tau^2}\big|_{\tau_0,s_0,\sigma_0} = 0$ are used. Note that both $V_F^{(0)}$ and $V_F^{(1)}$ expressed in Eq.(A1) vanish at σ_0 as a consequence of Eq.(38). Consequently, the minimization condition reduces to $\partial V_F^{(2)}/\partial\tau\big|_{\tau_0,s_0,\sigma_0} = 0$, which yields

$$\tau_0(\alpha) \approx i\,, \tag{41}$$

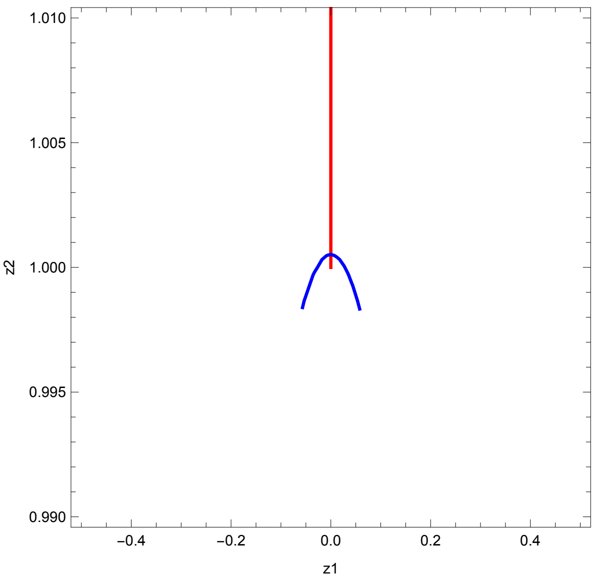

FIG. 1: Contour plot of $\partial V_F^{(2)}/\partial\tau\big|_{\tau_0,s_0,\sigma_0}=0$, where $\tau=z_1+iz_2$. Blue curve is for $\partial V_F^{(2)}/\partial z_2=0$ and red line for $\partial V_F^{(2)}/\partial z_1=0$, where $s_0=1$ is used for simplicity.

as shown by the cross point in Fig.1. Here, the second-order potential $V_F^{(2)}$ is given by

$$\begin{aligned}
V_F^{(2)} &= e^{K/M_P^2}\frac{M_P^6}{|\eta(\tau)|^{12}}|C_0|^2\Big\{|S+\frac{3}{2\pi^2}\ln\eta(\tau)|^2\big(K^{\tau\bar\tau}|H|^2+\frac{1}{M_P^2}\big)\\
&- \frac{y}{M_P^2}\big(S+\bar S+\frac{3}{2\pi^2}\ln\eta(\tau)\eta(\bar\tau)\big)+\frac{y^2}{M_P^2}\\
&- \frac{3}{2\pi^2}K^{\tau\bar\tau}\big((S+\frac{3}{2\pi^2}\ln\eta(\tau))H\frac{\eta'(\bar\tau)}{\eta(\bar\tau)}+h.c.\big)+\frac{2}{\pi^4}K^{\tau\bar\tau}\big|\frac{\eta'(\tau)}{\eta(\tau)}\big|^2\Big\}, \qquad (42)
\end{aligned}$$

where $H=\frac{3}{\tau-\bar\tau}(1-\frac{1}{4\pi^2y})+6\frac{\eta'(\tau)}{\eta(\tau)}$ and $K^{\tau\bar\tau}=-(\tau-\bar\tau)^2/\{M_P^2(3(1-\frac{1}{4\pi^2y})+\frac{9}{16\pi^4y^2})\}$, and Eq.(38) has been used to simplify the expression. The shift $\delta\tau$ is given by[13] $\delta\tau=-\frac{\partial V_F^{(3)}}{\partial\tau}/\frac{\partial^2V_F^{(2)}}{\partial\tau^2}\big|_{\tau_0,s_0,\sigma_0}\sim\mathcal{O}(1)$.

While the α-dependent term in the superpotential modifies the potential and induces a shift in τ, this term acts as a small perturbation when α is sufficiently small. Consequently, the shape of the potential for τ is not significantly affected, and the location of the minimum (the VEV of τ) remains close to its value at $\alpha=0$. Supersymmetry is broken in the direction of τ and S for $\alpha\neq0$. Meanwhile, in the U_X direction, a deeper supersymmetric Anti-de Sitter (AdS) minimum emerges. As in Ref.[28, 30], an uplift term (e.g., from a D-term) ΔV

[13] $V_F^{(3)}=e^{K/M_P^2}\frac{M_P^6}{|\eta(\tau)|^{12}}|C_0|^2\{-\frac12|S+\frac{3}{2\pi^2}\ln\eta(\tau)|^2(S+\bar S+\frac{3}{2\pi^2}\ln\eta(\tau)\eta(\bar\tau))(K^{\tau\bar\tau}|H|^2+\frac{1}{M_P^2})+\frac12[\frac{3}{2\pi^2}K^{\tau\bar\tau}((S+\frac{3}{2\pi^2}\ln\eta(\tau))^2H\frac{\eta'(\bar\tau)}{\eta(\bar\tau)}+h.c.)+\frac{y}{M_P^2}((S+\frac{3}{2\pi^2}\ln\eta(\tau))^2+h.c.)]-(S+\bar S+\frac{3}{2\pi^2}\ln\eta(\tau)\eta(\bar\tau))(\frac{2}{\pi^4}K^{\tau\bar\tau}|\frac{\eta'(\tau)}{\eta(\tau)}|^2+\frac{y^2}{M_P^2})\}$ where Eq.(38) has been used for simplification.

slightly shifts U_X, see Appendix B, making $D_{U_X}W \neq 0$ of Eq.(B3), but this contribution is strongly suppressed by m_σ of Eq.(B3). At the shifted minimum $\tilde{\sigma}_0, \tilde{\tau}_0, s_0$, Eqs.(35) and (B3) lead to

$$\langle D_\tau W\rangle \neq 0\,, \qquad \langle D_S W\rangle \neq 0\,, \qquad \langle D_{U_X} W\rangle \neq 0\,. \tag{43}$$

Comparing the F-term magnitudes:

$$\left|\frac{F^\tau}{F^S}\right| \approx \frac{3}{2\pi^2}\frac{\langle K^{\tau\bar{\tau}}\rangle}{\langle K^{S\bar{S}}\rangle}\left|\frac{\eta'(\tau_0)}{\eta(\tau_0)}\right|\left|\frac{2s_0 - \frac{3}{4\pi^2}\ln(2\mathrm{Im}\,\tau_0)}{-s_0 + \frac{3}{4\pi^2}\big(2\ln\eta(\tau_0) + \ln(2\mathrm{Im}\,\tau_0)\big)}\right| \ll 1\,, \tag{44}$$

where $K^{S\bar{S}} = y^2/M_P^2$, $K^{\tau\bar{\tau}} = -(\tau-\bar{\tau})^2/\{6M_P^2\big(1 - 1/(4\pi^2 y) + 3/(8\pi^4 y^2)\big)\}$, and $|\eta'(i)| \approx 0.192$. This implies that supersymmetry is broken predominantly by the dilaton S and slightly by the modulus τ, induced by $e^{-\alpha S}$ term in Eq.(33). This is decoupled from the AdS minimum of the U_X direction. When including the uplifting term and $\alpha \neq 0$, the gravitino mass $m_{3/2}^2 = e^{K/M_P^2}|W|^2/M_P^4$ becomes

$$m_{3/2}^2 \simeq \frac{|V_F|}{3M_P^2} \approx \frac{|V_{\rm AdS}| + \alpha^2|V_F^{(2)}|}{3M_P^2}\,, \tag{45}$$

evaluated at the shifted minimum $\tilde{\sigma}_0, \tilde{\tau}_0, s_0$, where $V_{\rm AdS}$ and $V_F^{(2)}$ are given by Eq.(B2) and Eq.(42). The gravitino mass emerges through the uplifting potential and when $\alpha \neq 0$, both of which break supersymmetry and lift the minimum to a de Sitter (dS) vacuum. For $\alpha \to 0$, the gravitino mass becomes, as expected in Ref.[28], $m_{3/2} \approx \frac{1}{\sqrt{8\langle y\rangle(2\mathrm{Im}\tau_0)^3}}\big(\frac{a-b}{\ln\frac{aA_0}{bB_0}}\big)^{\frac{3}{2}}\frac{|\Delta W|}{M_P^2}$. For $\alpha \neq 0$, the gravitino mass gains an explicit α-dependence via the second term of Eq.(45). When the α-term dominates,

$$m_{3/2} \approx \alpha|C_0|M_P\frac{|s_0 + \frac{3}{2\pi^2}\ln\eta(\tau_0)|}{2|\eta(\tau_0)|^6\sqrt{6\langle y\rangle(2\mathrm{Im}\tau_0)^3}}\Big(\frac{a-b}{\ln\frac{aA_0}{bB_0}}\Big)^{\frac{3}{2}}\,. \tag{46}$$

For particularly simple parameter choices $A_0 = B_0 = 1$, $a = 2\pi/100$, $b = 2\pi/101$, $s_0 = 1$, $|\eta(i)| \approx 0.768$, the resulting values are $C_0 = 3.66\times10^{-3}$ and $\sigma_0 = 15.99$. This leads to a gravitino mass of $m_{3/2} \sim \alpha \times 7.13\times10^{-9}M_P$. The corresponding gravitino mass scales are 1.74×10^5 TeV ($\alpha = 10^{-2}$), 1.74×10^3 TeV ($\alpha = 10^{-4}$), and 1 TeV ($\alpha = 6\times10^{-8}$).

III. SUPERPOTENTIAL FOR QUARK AND LEPTON

Since $SL(2,\mathbb{Z})$ acts as a flavor symmetry (e.g. modular transformations on fields) the associated anomalies must cancel over the fermion spectrum. To build quark and lepton

Yukawa superpotentials consistent with the anomaly cancellations (including $A_C = A_E = 0$ (Eq.(23)), $A_L = -A_Y = 0$ (Eq.(25)), and $A_{SX} = 0$ (Eq.(26)), we assign the $U(1)_X$ quantum numbers and modular weights k_I to quark and lepton fields under $SL(2,\mathbb{Z})\times U(1)_X$. A viable charge assignment is presented in Table-I and II. Such viable assignments subsequently can naturally constrain the $U(1)_X$-breaking scale, as all Yukawa coefficients are restricted to unit-magnitude complex numbers. This scale connects to physical scales like the neutrino seesaw scale or the axion mass scale via the PQ mechanism.

To construct a unique supersymmetric, modular-, and gauge-invariant scalar potential, we consider minimal supermultiplets. Under $SL(2,\mathbb{Z}) \times U(1)_X$, we assign Higgs doublets $H_{u,d}$ as $(0,0)$ and SM singlets χ, $\tilde{\chi}$, χ_0 as $(0,+1)$, $(0,-1)$, $(3,0)$, respectively. The modular weight-3 field χ_0 ensures that modular forms $Y(\tau)$ are τ-independent constants [8]. The leading-order superpotential invariant under $SL(2,\mathbb{Z}) \times U(1)_X$ is

$$W_v \;=\; g_{\chi_0}\chi_0\, H_u H_d + \chi_0(g_\chi \chi\tilde{\chi} - \mu_\chi^2)\;, \tag{47}$$

where g_{χ_0} and g_χ are initially set to unity but receive corrections from higher-order terms (see Eq.(51)). The scale μ_χ sets the spontaneous $U(1)_X$ breaking scale. Minimizing the F-term potential (Appendix C) yields [14]

$$\langle\chi\rangle = \langle\tilde{\chi}\rangle = \frac{v_\chi}{\sqrt{2}}\,, \qquad \mu_\chi = v_\chi\sqrt{\frac{g_\chi}{2}} \tag{48}$$

assuming $\langle\chi\rangle, \langle\tilde{\chi}\rangle \gg \langle H_{u,d}\rangle$. This supersymmetric solution satisfies the D-flatness condition for $\xi_X^{\rm FI} = 0$. For $\xi_X^{\rm FI} \neq 0$, SUSY must be broken by the F-term to allow the D-term to uplift the potential from the AdS minimum to dS [29, 30] (see Sec.II C).

After spontaneous $U(1)_X$ breaking ($\langle\chi\rangle \neq 0$), the NG mode A_X emerges. Decomposing the complex scalar fields [3, 31, 32]

$$\chi = \frac{v_\chi}{\sqrt{2}} e^{i\frac{A_X}{f_A}}\left(1 + \frac{h_\chi}{f_A}\right), \quad \tilde{\chi} = \frac{v_{\tilde{\chi}}}{\sqrt{2}} e^{-i\frac{A_X}{f_A}}\left(1 + \frac{h_{\tilde{\chi}}}{f_A}\right) \qquad \text{with } f_A = \sqrt{v_\chi^2 + v_{\tilde{\chi}}^2}\,, \tag{49}$$

with $v_\chi = v_{\tilde{\chi}}$ and $h_\chi = h_{\tilde{\chi}}$ in the SUSY limit.

---

[14] From the vanishing of the F-terms associated to $\chi(\tilde{\chi})$ and $H_{u,d}$, the VEV of χ_0 is determined as $\langle\chi_0\rangle = 0$.

TABLE I: Representations of the SM quark fields under $SL(2,\mathbb{Z}) \times U(1)_X$ and modular weight k_I with $h=3$, where Q_i ($i=1,2,3$) represent the left-handed quark doublets, (d^c, s^c, b^c) are the right-handed down-type quarks, and (u^c, c^c, t^c) are the right-handed up-type quarks.

Field	Q_1	Q_2	Q_3	d^c	s^c	b^c	u^c	c^c	t^c
k_I	$\frac{h}{2}+2$	$\frac{h}{2}+6$	$\frac{h}{2}+10$	$\frac{h}{2}-2$	$\frac{h}{2}-6$	$\frac{h}{2}-10$	$\frac{h}{2}-2$	$\frac{h}{2}-6$	$\frac{h}{2}-10$
$U(1)_X$	-3	-3	9	23	13	-5	26	13	-9

A. Modular-invariant Yukawa superpotential for quark

According to Table-I, the quark Yukawa superpotential reads

$$\begin{aligned} W_q &= \Big[\alpha_t\, t^cQ_3 + \alpha_c\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{10} c^cQ_2 + \alpha_u\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{23} u^cQ_1 \\ &+ \alpha_{t2}\Big(\frac{\chi}{\Lambda}\Big)^{12} Y^{(4)}_{\mathbf{1}} t^cQ_2 + \alpha_{t1}\Big(\frac{\chi}{\Lambda}\Big)^{12} Y^{(8)}_{\mathbf{1}} t^cQ_1 + \alpha_{c1}\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{10} Y^{(4)}_{\mathbf{1}} c^cQ_1\Big] H_u \\ &+ \Big[\alpha_b\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{4} b^cQ_3 + \alpha_{b2}\Big(\frac{\chi}{\Lambda}\Big)^{8} Y^{(4)}_{\mathbf{1}} b^cQ_2 + \alpha_{b1}\Big(\frac{\chi}{\Lambda}\Big)^{8} Y^{(8)}_{\mathbf{1}} b^cQ_1 \\ &+ \alpha_s\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{10} s^cQ_2 + \alpha_{s1}\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{10} Y^{(4)}_{\mathbf{1}} s^cQ_1 + \alpha_d\Big(\frac{\tilde{\chi}}{\Lambda}\Big)^{20} d^cQ_1\Big] H_d + \dots\,, \end{aligned} \tag{50}$$

where all Yukawa coefficients α_i are complex numbers with unit magnitude, and dots represent higher-order contributions compactly expressed as $\sum_{n=1}^{\infty}(\frac{\chi\tilde{\chi}}{\Lambda^2})^n \times$ *leading terms*. Here the flavor dynamics scale Λ can be identified with the stabilization scale of the lightest modulus $m_{\rm modulus} \sim \Lambda$, ensuring that fluctuations of heavy moduli are suppressed below Λ. These corrections modify the effective Yukawa coefficients α_i, constrained by

$$1 - \frac{\Delta_\chi^2}{1-\Delta_\chi^2} \leq |\alpha_i| \leq 1 + \frac{\Delta_\chi^2}{1-\Delta_\chi^2} \qquad \text{with } \Delta_\chi \equiv \frac{v_\chi}{\sqrt{2}\,\Lambda}\,. \tag{51}$$

According to the canonically normalized fields in Eq.(8), the Yukawa coefficients transform as

$$\begin{aligned} &\alpha_{c1} \to (2{\rm Im}\,\tau)^{-2}\,\alpha_{c1}\,, \quad \alpha_{t1} \to (2{\rm Im}\,\tau)^{-4}\,\alpha_{t1}\,, \quad \alpha_{t2} \to (2{\rm Im}\,\tau)^{-2}\,\alpha_{t2}\,, \\ &\alpha_{s1} \to (2{\rm Im}\,\tau)^{-2}\,\alpha_{s1}\,, \quad \alpha_{b1} \to (2{\rm Im}\,\tau)^{-4}\,\alpha_{b1}\,, \quad \alpha_{b2} \to (2{\rm Im}\,\tau)^{-2}\,\alpha_{b2}\,, \end{aligned} \tag{52}$$

while $\alpha_{u,c,t}$ and $\alpha_{d,s,b}$ remain unchanged. The modular forms of weights 4 and 8, under $SL(2,\mathbb{Z})$, read [6, 33] (see Appendix D).

Under chiral rotation of the quark fields, the QCD anomaly term reduces to

$$\mathcal{L}_\vartheta = \Big(\vartheta_{\rm eff} + \frac{A_X}{F_a}\Big)\frac{\alpha_s'}{8\pi} G^{a\mu\nu}\tilde{G}^a_{\mu\nu} \qquad \text{with } F_a = \frac{f_A}{\delta^G_X}\,, \tag{53}$$

where $\alpha'_s = g_s^2/4\pi$, F_a is the axion decay constant with f_A Eq.(49), and $\vartheta_{\rm eff}$ is the effective strong CP phase of Eq.(20) with the vanishing modular anomaly conditions, $\arg(M_3) = 0$ of Eq.(19) and $A_C = 0$ of Eq.(21). At low energies A_X will get a VEV, $\langle A_X\rangle = -F_a\vartheta_{\rm eff}$, eliminating the constant $\vartheta_{\rm eff}$ term. The QCD axion then is the excitation of the A_X field, $a = A_X - \langle A_X\rangle$. The quark quantum numbers in Table-I yield the color anomaly coefficient for $U(1)_X \times [SU(3)_C]^2$ (defined in Eq.(31)) as

$$\delta^G_X = 67\,, \tag{54}$$

determining the domain-wall number $N_{\rm DW} = |\delta^G_X|$. To avoid cosmological domain walls, either $N_{\rm DW} = 1$ or the PQ transition must occur during/before inflation for $N_{\rm DW} > 1$.

B. Modular-invariant Yukawa superpotential for lepton

TABLE II: Representations of the lepton fields under $SL(2,\mathbb{Z}) \times U(1)_X$ and modular weight k_I with $h = 3$, where L_i $(i = e, \mu, \tau)$ represent the left-handed lepton doublets, (e^c, μ^c, τ^c) are the right-handed charged-leptons, and N^c_j $(j = 1, 2, 3)$ are the right-handed neutrinos.

Field	L_e	L_μ	L_τ	e^c	μ^c	τ^c	N^c_1	N^c_2	N^c_3
k_I	$\frac{h}{2}-18$	$\frac{h}{2}-22$	$\frac{h}{2}-14$	$\frac{h}{2}+18$	$\frac{h}{2}+22$	$\frac{h}{2}+14$	$\frac{h}{2}-6$	$\frac{h}{2}-6$	$\frac{h}{2}-6$
$U(1)_X$	$\frac{1}{2}+4$	$\frac{1}{2}+4$	$\frac{1}{2}+4$	$-\frac{1}{2}+20$	$-\frac{1}{2}-15$	$-\frac{1}{2}-8$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

According to Table-II, the charged-lepton Yukawa superpotential reads

$$\begin{aligned} W_\ell \;=\; & \Big[\alpha_\tau\Big(\frac{\chi}{\Lambda}\Big)^4\tau^c L_\tau + \alpha_\mu\Big(\frac{\chi}{\Lambda}\Big)^{11}\mu^c L_\mu + \alpha_e\Big(\frac{\tilde\chi}{\Lambda}\Big)^{24}e^c L_e \\ & +\alpha_{\tau 2}\Big(\frac{\chi}{\Lambda}\Big)^4 Y^{(8)}_{\mathbf{1}}\tau^c L_\mu + \alpha_{\tau 1}\Big(\frac{\chi}{\Lambda}\Big)^4 Y^{(4)}_{\mathbf{1}}\tau^c L_e + \alpha_{e2}\Big(\frac{\tilde\chi}{\Lambda}\Big)^{24} Y^{(4)}_{\mathbf{1}} e^c L_\mu\Big] H_d + \dots\,. \end{aligned} \tag{55}$$

And the neutrino Yukawa superpotential reads

$$\begin{aligned} W_\nu \;=\; & \Big(\frac{\tilde\chi}{\Lambda}\Big)^4\Big[\beta_{1e}Y^{(24)}_{\mathbf{1}}N^c_1 L_e + \beta_{1\mu}Y^{(28)}_{\mathbf{1}}N^c_1 L_\mu + \beta_{1\tau}Y^{(20)}_{\mathbf{1}}N^c_1 L_\tau \\ & +\beta_{2e}Y^{(24)}_{\mathbf{1}}N^c_2 L_e + \beta_{2\mu}Y^{(28)}_{\mathbf{1}}N^c_2 L_\mu + \beta_{2\tau}Y^{(20)}_{\mathbf{1}}N^c_2 L_\tau \\ & +\beta_{3e}Y^{(24)}_{\mathbf{1}}N^c_3 L_e + \beta_{3\mu}Y^{(28)}_{\mathbf{1}}N^c_3 L_\mu + \beta_{3\tau}Y^{(20)}_{\mathbf{1}}N^c_3 L_\tau\Big] H_u \\ & + \frac{1}{2}\Big[\gamma_{11}Y^{(12)}_{\mathbf{1}}N^c_1 N^c_1 + \gamma_{12}Y^{(12)}_{\mathbf{1}}N^c_1 N^c_2 + \gamma_{13}Y^{(12)}_{\mathbf{1}}N^c_1 N^c_3 + \gamma_{22}Y^{(12)}_{\mathbf{1}}N^c_2 N^c_2 \\ & +\gamma_{33}Y^{(12)}_{\mathbf{1}}N^c_3 N^c_3 + \gamma_{23}Y^{(12)}_{\mathbf{1}}N^c_2 N^c_3\Big]\chi + \dots\,. \end{aligned} \tag{56}$$

Recall that, analogous to the quark sector, all Yukawa coefficients are effectively determined by Eq.(51), after accounting for the contributions of all higher-dimensional operators induced by $\chi\tilde{\chi}$. According to the canonically normalized fields in Eq.(8), the Yukawa coefficients transform as

$$\begin{aligned}
&\alpha_{\tau 2} \to (2\mathrm{Im}\,\tau)^{-4}\,\alpha_{\tau 2}\,, \qquad \alpha_{\tau 1} \to (2\mathrm{Im}\,\tau)^{-2}\,\alpha_{\tau 1}\,, \qquad \alpha_{e2} \to (2\mathrm{Im}\,\tau)^{-2}\,\alpha_{e2}\,,\\
&\beta_{ie} \to (2\mathrm{Im}\,\tau)^{-12}\,\beta_{ie}\,, \qquad \beta_{i\mu} \to (2\mathrm{Im}\,\tau)^{-14}\,\beta_{i\mu}\,, \qquad \beta_{i\tau} \to (2\mathrm{Im}\,\tau)^{-10}\,\beta_{i\tau}\,,\\
&\gamma_{ij} \to (2\mathrm{Im}\,\tau)^{-6}\,\gamma_{ij}\,, \qquad \text{others} = \text{invariant}\,, \qquad (i,j=1,2,3)\,.
\end{aligned} \tag{57}$$

The modular forms of weights 12, 20, 24, and 28 under $SL(2,\mathbb{Z})$ read [6, 33] (see Appendix D).

Below the $U(1)_X$ symmetry breaking scale (which coincides with the seesaw scale, cf. Eq.(82)), the effective interactions of QCD axion with the weak and hypercharge gauge bosons and with the photon are expressed through the chiral rotation of Eq.(60). The electromagnetic anomaly coefficient E of $U(1)_X\times[U(1)_{EM}]^2$ is defined by $E = 2\sum_{\psi_f} X_{\psi_f}(Q^{\rm em}_{\psi_f})^2$ where $Q^{\rm em}_{\psi_f}$ is the electric charge of the field ψ_f. For the $U(1)_X$ charges (see Table-I and -II), this evaluates to

$$E \;=\; \delta^W_X + \delta^Y_X = \frac{386}{3}\,, \tag{58}$$

where the anomaly coefficients δ^W_X, δ^Y_X are defined in Eq.(31). The physical quantities of QCD axion, such as axion mass m_a and axion-photon coupling $g_{a\gamma\gamma}$, depend on the ratio of electromagnetic anomaly coefficient E to the color anomaly coefficient δ^G_X (see Fig.2). This example model has the specific ratio $E/\delta^G_X = 386/201$ with the $U(1)_X$ breaking scale fixed at $f_A = 4\times 10^{10}$ GeV (as derived in Eq.(82)). This distinctive prediction makes the model testable in future axion search experiments.

IV. QUARK, LEPTON, AND FLAVORED-QCD AXION

At energies below the electroweak scale when $H_{u(d)}$ acquire non-zero VEVs all quarks and leptons obtain masses. The relevant quark and lepton interactions are given from

Eqs.(50,55,56) by

$$
\begin{aligned}
-\mathcal{L} \supset\ & \overline{q^u_R}\,\mathcal{M}_u\, q^u_L + \overline{q^d_R}\,\mathcal{M}_d\, q^d_L + \frac{g}{\sqrt{2}} W^+_\mu \overline{q^u_L}\gamma^\mu\, q^d_L \\
+\ & \overline{\ell_R}\,\mathcal{M}_\ell\,\ell_L + \frac{1}{2}\begin{pmatrix}\overline{\nu^c_L} & \overline{N_R}\end{pmatrix}\begin{pmatrix}0 & m_D^T \\ m_D & M_R\end{pmatrix}\begin{pmatrix}\nu_L \\ N^c_R\end{pmatrix} + \frac{g}{\sqrt{2}} W^-_\mu \overline{\ell_L}\gamma^\mu\,\nu_L + \text{h.c.}\,, \qquad (59)
\end{aligned}
$$

where g is the $SU(2)_L$ coupling constant, $q^u = (u, c, t)$, $q^d = (d, s, b)$, $\ell = (e, \mu, \tau)$, $\nu = (\nu_e, \nu_\mu, \nu_\tau)$, and $N = (N_1, N_2, N_3)$. M_R contains a VEV of χ in Eq.(49). The explicit forms of $\mathcal{M}_{u,d,l}$ will be given later. The above Lagrangian of the fermions, including their kinetic terms, should be invariant under $U(1)_X$:

$$
\psi_f \to e^{iX_{\psi_f}\frac{\gamma_5}{2}\beta}\psi_f\,, \quad t = \text{invariant}\,, \quad N \to e^{i\frac{\gamma_5}{2}\beta}N \qquad (60)
$$

where $\psi_f = \{u, c, d, s, b, e, \mu, \tau, \nu\}$ and β is a transformation constant parameter.

With the VEV of Eq.(48) the mass matrices $\mathcal{M}_u$ and $\mathcal{M}_d$ for up- and down-type quarks are described in terms of Δ_χ and modular forms $Y^{(4)}_{\mathbf{1}}$ and $Y^{(8)}_{\mathbf{1}}$:

$$
\mathcal{M}_u = C^u_R \begin{pmatrix} \alpha_u \Delta_\chi^{23} & 0 & 0 \\ \alpha_{c1}(2\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{10}\,Y^{(4)}_{\mathbf{1}} & \alpha_c \Delta_\chi^{10} & 0 \\ \alpha_{t1}(2\mathrm{Im}\,\tau)^{-4}\Delta_\chi^{12}\,Y^{(8)}_{\mathbf{1}} & \alpha_{t2}(2\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{12}\,Y^{(4)}_{\mathbf{1}} & \alpha_t \end{pmatrix} C^u_L\, v_u\,, \qquad (61)
$$

$$
\mathcal{M}_d = C^d_R \begin{pmatrix} \alpha_d\, \Delta_\chi^{20} & 0 & 0 \\ \alpha_{s1}(2\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{10}\,Y^{(4)}_{\mathbf{1}} & \alpha_s\, \Delta_\chi^{10} & 0 \\ \alpha_{b1}(2\mathrm{Im}\,\tau)^{-4}\Delta_\chi^{8}\,Y^{(8)}_{\mathbf{1}} & \alpha_{b2}(2\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{8}\,Y^{(4)}_{\mathbf{1}} & \alpha_b\,\Delta_\chi^4 \end{pmatrix} C^d_L\, v_d\,, \qquad (62)
$$

where $v_d \equiv \langle H_d \rangle = v\cos\beta/\sqrt{2}$, $v_u \equiv \langle H_u \rangle = v\sin\beta/\sqrt{2}$ with $v \simeq 246$ GeV, and

$$
\begin{aligned}
C^u_R &= \mathrm{diag}(e^{-i35\frac{A_X}{f_A}}, e^{-i22\frac{A_X}{f_A}}, 1)\,, & C^u_L &= \mathrm{diag}(e^{i12\frac{A_X}{f_A}}, e^{i12\frac{A_X}{f_A}}, 1)\,, \\
C^d_R &= \mathrm{diag}(e^{-i28\frac{A_X}{f_A}}, e^{-i18\frac{A_X}{f_A}}, 1)\,, & C^d_L &= \mathrm{diag}(e^{i8\frac{A_X}{f_A}}, e^{i8\frac{A_X}{f_A}}, e^{-i4\frac{A_X}{f_A}})\,. \qquad (63)
\end{aligned}
$$

Similarly, the mass matrices $\mathcal{M}_\ell$, M_R, and m_D for charged-lepton, heavy neutrino, and Dirac neutrino given in the Lagrangian (59) are derived in terms of Δ_χ and modular forms $Y^{(4)}_{\mathbf{1}}$, $Y^{(8)}_{\mathbf{1}}$, $Y^{(12)}_{\mathbf{1}}$, $Y^{(20)}_{\mathbf{1}}$, $Y^{(24)}_{\mathbf{1}}$, and $Y^{(28)}_{\mathbf{1}}$ as,

$$
\mathcal{M}_\ell = C^\ell_R \begin{pmatrix} \alpha_e\,\Delta_\chi^{24} & \alpha_{e2} Y^{(4)}_{\mathbf{1}}\Delta_\chi^{24}(2\mathrm{Im}\,\tau)^{-2} & 0 \\ 0 & \alpha_\mu\,\Delta_\chi^{11} & 0 \\ \alpha_{\tau1} Y^{(4)}_{\mathbf{1}}\Delta_\chi^{4}(2\mathrm{Im}\,\tau)^{-2} & \alpha_{\tau2} Y^{(8)}_{\mathbf{1}}\Delta_\chi^{4}(2\mathrm{Im}\,\tau)^{-4} & \alpha_\tau\,\Delta_\chi^4 \end{pmatrix} v_d\,, \qquad (64)
$$

$$M_R \;=\; e^{i\frac{A_X}{f_A}}\,(2\mathrm{Im}\,\tau)^{-6}\begin{pmatrix} \gamma_{11}\,Y^{(12)}_{\mathbf{1}} & \gamma_{12}\,Y^{(12)}_{\mathbf{1}} & \gamma_{13}\,Y^{(12)}_{\mathbf{1}} \\ \gamma_{12}\,Y^{(12)}_{\mathbf{1}} & \gamma_{22}\,Y^{(12)}_{\mathbf{1}} & \gamma_{23}\,Y^{(12)}_{\mathbf{1}} \\ \gamma_{13}\,Y^{(12)}_{\mathbf{1}} & \gamma_{23}\,Y^{(12)}_{\mathbf{1}} & \gamma_{33}\,Y^{(12)}_{\mathbf{1}} \end{pmatrix}\langle\chi\rangle\,, \tag{65}$$

$$m_D = e^{-4i\frac{A_X}{f_A}}\,\Delta^4_\chi\begin{pmatrix} \beta_{1e}\,Y^{(24)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-12} & \beta_{1\mu}\,Y^{(28)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-14} & \beta_{1\tau}\,Y^{(20)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-10} \\ \beta_{2e}\,Y^{(24)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-12} & \beta_{2\mu}\,Y^{(28)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-14} & \beta_{2\tau}\,Y^{(20)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-10} \\ \beta_{3e}\,Y^{(24)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-12} & \beta_{3\mu}\,Y^{(28)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-14} & \beta_{3\tau}Y^{(20)}_{\mathbf{1}}(2\mathrm{Im}\,\tau)^{-10} \end{pmatrix} v_u\,, \tag{66}$$

where $\gamma_{ij}\,Y^{(12)}_{\mathbf{1}}$, $\beta_{ie}\,Y^{(24)}_{\mathbf{1}}$, $\beta_{i\mu}\,Y^{(28)}_{\mathbf{1}}$, and $\beta_{i\tau}\,Y^{(20)}_{\mathbf{1}}$ are expanded to $\gamma_{ij}\,Y^{(12)}_{\mathbf{1}A}+\tilde{\gamma}_{ij}\,Y^{(12)}_{\mathbf{1}B}$, $\beta_{ie}\,Y^{(24)}_{\mathbf{1}A}+\tilde{\beta}_{ie}\,Y^{(24)}_{\mathbf{1}B}$, $\beta_{i\mu}\,Y^{(28)}_{\mathbf{1}A}+\tilde{\beta}_{i\mu}\,Y^{(28)}_{\mathbf{1}B}$, and $\beta_{i\tau}\,Y^{(20)}_{\mathbf{1}A}+\tilde{\beta}_{i\tau}\,Y^{(20)}_{\mathbf{1}B}$, respectively, and

$$C^\ell_R = \mathrm{diag}(e^{-i24\frac{A_X}{f_A}}, e^{i11\frac{A_X}{f_A}}, e^{i4\frac{A_X}{f_A}})\,. \tag{67}$$

Recall that the coefficients α_i, γ_{ij}, β_{ij} are complex numbers satisfying Eq.(51). The different corrections in Eq.(66), originating from the canonical normalization of the matter fields in Eq.(8), ensure the observed atmospheric and solar neutrino mass-squared differences, $\Delta m^2_{\rm Atm}$ and $\Delta m^2_{\rm Sol}$, and consequently determine whether the neutrino mass ordering is normal or inverted, see Sec.IV B.

A. Quark and charged-lepton masses, mixing, and QCD axion interactions

The quark mass matrices $\mathcal{M}_u$ of Eq.(61) and $\mathcal{M}_d$ of Eq.(62) generate the up- and down-type quark masses: $\hat{\mathcal{M}}_u = V^u_R\,\mathcal{M}_u\,V^{u\dagger}_L = \mathrm{diag}(m_u, m_c, m_t)$ and $\hat{\mathcal{M}}_d = V^d_R\,\mathcal{M}_d\,V^{d\dagger}_L = \mathrm{diag}(m_d, m_s, m_b)$ with the approximate relations

$$\begin{aligned} &m_u \simeq |\alpha_u|\Delta^{23}_\chi\,v_u\,, \quad m_c \simeq |\alpha_c|\Delta^{10}_\chi\,v_u\,, \quad m_t \simeq |\alpha_t|\,v_u\,, \\ &m_d \simeq |\alpha_d|\Delta^{20}_\chi\,v_d\,, \quad m_s \simeq |\alpha_s|\Delta^{10}_\chi\,v_d\,, \quad m_b \simeq |\alpha_b|\Delta^{4}_\chi\,v_d\,. \end{aligned} \tag{68}$$

Diagonalizing the matrices $\mathcal{M}^\dagger_f\mathcal{M}_f$ and $\mathcal{M}_f\mathcal{M}^\dagger_f$ (for $f = u, d$) determines the mixing matrices V^f_L and V^f_R, respectively [34], as in Eqs.(E1) and (E2). The left-handed quark mixing matrices V^u_L and V^d_L in Eq.(E1) enter into the CKM (Cabibbo-Kobayashi-Maskawa) matrix $V_{\rm CKM} = V^u_L V^{d\dagger}_L$. Redefining the quark fields via the transformations $u_L \to e^{-i(\alpha^u_1-2\alpha^u_2-\alpha^d_1+2\alpha^d_2)}u_L$, $c_L \to e^{i(2\alpha^d_3-\alpha^d_2)}c_L$, $t_L \to e^{i(2\alpha^d_3-\alpha^d_2+2\alpha^d_1-2\alpha^u_1)}t_L$, $s_L \to e^{i(2\alpha^d_3-\alpha^d_2)}s_L$, and using the Wolfenstein parametrization [35] (with high precision [36]), we obtain

$$\lambda \equiv \lambda_d - \theta^u_{12}\,, \qquad A\lambda^2 \equiv A_d\lambda_d^2\,, \qquad A\lambda^3(\rho + i\eta) \equiv B_d\lambda_d^3\, e^{-i\varphi_d} - A_d\lambda_d^2\,\theta^u_{12}\,,$$
$$A\lambda^3(1-\rho+i\eta) \equiv \lambda_d^3(A_d - B_d\, e^{i\varphi_d})\,, \qquad \alpha^u_1 - \alpha^u_2 - 2\alpha^u_3 = \alpha^d_1 - \alpha^d_2 - 2\alpha^d_3\,, \tag{69}$$

where $\varphi_d \equiv 2\alpha^d_3 - \alpha^d_2$. The quark masses and mixing parameters must be matched to the empirical values provided in Eqs.(E3) and (E4).

The charged-lepton mass matrix $\mathcal{M}_\ell$ of Eq.(64) generates the charged-lepton masses $\hat{\mathcal{M}}_\ell = V^\ell_R\,\mathcal{M}_\ell\, V^{\ell\dagger}_L = \text{diag}(m_e, m_\mu, m_\tau)$, with the approximate forms

$$m_e \simeq |\alpha_e|\,\Delta^{24}_\chi\, v_d\,, \qquad m_\mu \simeq |\alpha_\mu|\,\Delta^{11}_\chi\, v_d\,, \qquad m_\tau \simeq |\alpha_\tau|\,\Delta^{4}_\chi\, v_d\,. \tag{70}$$

These must be matched to the empirical PDG values [37], see above Eq.(E). The left-handed charged-lepton mixing matrix V^ℓ_L is one of the components of the PMNS (Pontecorvo-Maki-Nakagawa-Sakata) lepton mixing matrix, defined in Eq.(85), and reads in Eq.(E5).

After diagonalizing the quark mass matrices of Eqs.(61,62), the leading-order flavored-QCD axion interactions with quarks (up to $\mathcal{O}(\lambda^4)$) are [15]

$$\begin{aligned}
-\mathcal{L}^{aq} \simeq\; & -\frac{\partial_\mu a}{2f_A}\Big\{23\,\bar{u}\gamma^\mu\gamma_5 u + 10\,\bar{c}\gamma^\mu\gamma_5 c + 20\,\bar{d}\gamma^\mu\gamma_5 d + 10\,\bar{s}\gamma^\mu\gamma_5 s + 4\,\bar{b}\gamma^\mu\gamma_5 b\Big\}\\
& + \frac{\partial_\mu a}{2f_A}\Big\{12 A_d\lambda_d^2 e^{i\varphi_d}\,\bar{s}\gamma^\mu(1-\gamma_5)b + 12\lambda_d^3\big(B_d - A_d e^{i\varphi_d}\big)\bar{d}\gamma^\mu(1-\gamma_5)b + \text{h.c.}\Big\}\\
& + \sum_{q=d,s,b,u,c,t}\big(m_q\bar{q}q - \bar{q}i\partial\!\!\!/\, q\big)\,,
\end{aligned} \tag{71}$$

where $V^{u,d}_{L,R}$ in Appendix E encode quark mixing. These interactions are the result of a direct interaction of SM gauge-singlet scalars $\chi, \tilde{\chi}$ coupling to $U(1)_X$-charged quarks. Similarly, the flavored-QCD axion interactions with charged leptons (up to $\mathcal{O}(\lambda^4)$) are

$$-\mathcal{L}^{a\ell} \simeq -\frac{\partial_\mu a}{2f_A}\Big\{24\,\bar{e}\gamma^\mu\gamma_5 e - 11\,\bar{\mu}\gamma^\mu\gamma_5\mu - 4\,\bar{\tau}\gamma^\mu\gamma_5\tau\Big\} + \sum_{\ell=e,\mu,\tau}\big(m_\ell\bar{\ell}\ell - \bar{\ell}i\partial\!\!\!/\,\ell\big)\,. \tag{72}$$

The flavored-QCD axion a is produced by flavor-changing neutral Yukawa interactions in Eqs.(71) and (72), which leads to induced rare flavor-changing processes such as $b \to s + a$, $b \to d + a$ [38], and $\tau \to \mu + a$ [39, 40]. In our flavored-axion model, the bounds on f_A

[15] The left-handed quark and the right-handed charged-lepton mixing matrices, $V^{d(u)}_L$ and V^ℓ_R, are determined in Appendix E. These matrices govern the flavored-QCD axion couplings to down-type quarks, up-type quarks, and charged leptons, respectively.

arise from $B^\pm \to K^\pm + a$ [41–43], typically requiring $f_A \gtrsim 10^{5-6}$ GeV, rather than $K^+ \to \pi^+ + a$ [44–48]. Flavor-violating couplings of the flavored QCD axion to s, d-quarks and μ, e-leptons are suppressed to $\mathcal{O}(\lambda^4)$, as shown in Eqs.(71) and (72). This suppression is a direct consequence of the $U(1)_X$ charge assignments [16] and the following properties of the mixing matrices: the right-handed down-type quark matrix is approximately unitary up to $\mathcal{O}(\lambda^4)$ (for $K^+ \to \pi^+ + a$), and the right-handed charged-lepton matrix is approximately the identity up to $\mathcal{O}(\lambda^4)$ (for $\mu \to e + a$), as detailed in Appendix E.

The axion-electron coupling, given by $g_{aee} = 24 m_e/f_A$, is constrained by red giant branch (RGB) stars [49]:

$$|g_{aee}| < 4.3 \times 10^{-13} \quad (95\%\,\text{CL}) \quad \Leftrightarrow \quad \delta^G_X F_a \gtrsim 2.85 \times 10^{10}\,\text{GeV}. \tag{73}$$

White dwarf (WD) cooling further limits $|g_{aee}| \lesssim 2.8 \times 10^{-13}$ [50], though theoretical uncertainties persist.

The QCD axion mass m_a in terms of the pion mass and pion decay constant reads [3, 31]

$$m_a^2 F_a^2 = m_{\pi^0}^2 f_\pi^2 F(z, w)\,, \tag{74}$$

where $f_\pi \simeq 92.1$ MeV [37] and $F(z, w) = z/(1+z)(1+z+w)$ with $\omega = 0.315\,z$. Here the Weinberg value lies in $z \equiv m_u^{\overline{\text{MS}}}(2\,\text{GeV})/m_d^{\overline{\text{MS}}}(2\,\text{GeV}) = 0.47^{+0.06}_{-0.07}$ [37]. After integrating out the heavy π^0 and η at low energies, there is an effective low energy Lagrangian with an axion-photon coupling $g_{a\gamma\gamma}$: $\mathcal{L}_{a\gamma\gamma} = -g_{a\gamma\gamma}\, a\, \vec{E} \cdot \vec{B}$ where $\vec{E}$ and $\vec{B}$ are the electromagnetic field components. The axion-photon coupling is expressed in terms of the QCD axion mass, pion mass, pion decay constant, z and w,

$$g_{a\gamma\gamma} = \frac{\alpha_{\text{em}}}{2\pi} \frac{m_a}{f_\pi m_{\pi^0}} \frac{1}{\sqrt{F(z, w)}} \left(\frac{E}{\delta^G_X} - \frac{2}{3} \frac{4+z+w}{1+z+w} \right). \tag{75}$$

The upper bound on the axion-photon coupling, derived from the recent analysis of the horizontal branch stars in galactic globular clusters [53], can be translated to

$$|g_{a\gamma\gamma}| < 6.6 \times 10^{-11}\,\text{GeV}^{-1}\,(95\%\,\text{CL}) \Leftrightarrow F_a \gtrsim 1.76 \times 10^7 \left| \frac{E}{\delta^G_X} - 1.903 \right| \text{GeV}\,, \tag{76}$$

where $z = 0.47$ is used. From Eq.(73) with the consideration of Eq.(82), we obtain a $U(1)_X$ breakdown scale or seesaw scale

$$f_A = 4 \times 10^{10}\,\text{GeV}\,, \tag{77}$$

[16] See C^d_L in Eq.(63).

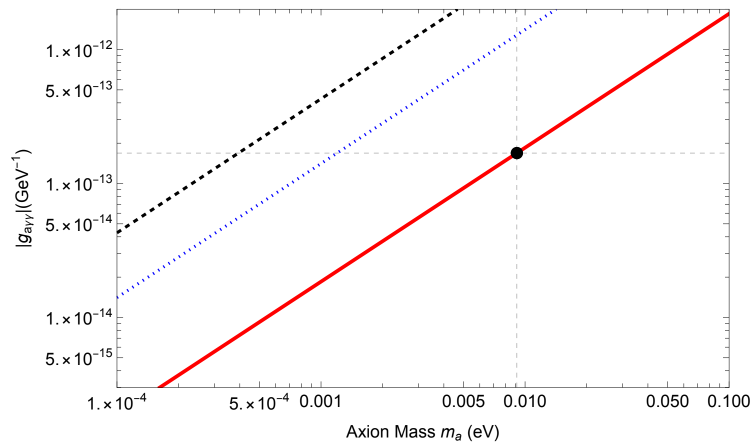

FIG. 2: Plot of $|g_{a\gamma\gamma}|$ versus m_a for Kim-Shifman-Vainshtein-Zakharov (KSVZ) [51] (black dashed line), Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) [52] (blue dotted line), and the model (red solid line) in terms of $E/\delta_X^G = 0, 8/3$, and $386/201$, respectively. With $f_A = 4 \times 10^{10}$ GeV of Eq.(82) and $z = 0.47$ our model predicts $m_a = 9.12 \times 10^{-3}$ eV and $|g_{a\gamma\gamma}| = 1.69 \times 10^{-13}\,\mathrm{GeV}^{-1}$ (black point).

for $\langle\chi\rangle = 2\times 10^{10}$ GeV in Eq.(82). The flavored-QCD axion mass and axion-photon coupling are predicted, as depicted in Fig.2, as

$$m_a = 9.12 \times 10^{-3}\,\mathrm{eV}\,, \qquad |g_{a\gamma\gamma}| = 1.69 \times 10^{-13}\,\mathrm{GeV}^{-1}\,. \tag{78}$$

Numerical simulation for quark mass and mixing: To simulate and match experimental quark data from Eqs.(E3) and (E4), we use linear algebra tools from Ref.[54]. The quark Yukawa matrices in Eqs.(61) and (62) are defined at the $U(1)_X$ symmetry breakdown scale, where their parameters receive quantum corrections. These matrices are then evolved down to the top quark mass scale (m_t) and diagonalized. We assume that the Yukawa matrices at the scale of $U(1)_X$ breakdown are the same as those at the scale m_t, since the one-loop renormalization group running effect on observables for hierarchical mass spectra is expected to be negligible. The low-energy Yukawa couplings required for experimental values are obtained from the physical masses and mixing angles compiled by the PDG [37] and CKMfitter [55]. Taking Eq.(41) and

$$\Delta_\chi = 0.634\,, \qquad \tan\beta = 6.4\,, \tag{79}$$

with effective Yukawa coefficients in the range $0.33 \lesssim |\alpha_i| \lesssim 1.67$ from Eq.(51), we obtain, for the quantum numbers listed in Table-I, the follwoing set of reference inputs

$$\begin{aligned}
&\alpha_u = 0.540, \alpha_c = 0.669, \alpha_{c1} = 0.565, \alpha_t = 1.010, \alpha_{t1} = 0.932, \alpha_{t2} = 0.978, \\
&\arg(\alpha_u) = 6.234\,, \arg(\alpha_c) = 1.619\,, \arg(\alpha_{c1}) = 1.761\,, \arg(\alpha_{t1}) = 0.989\,, \arg(\alpha_{t2}) = 5.125 \\
&\alpha_d = 1.652, \alpha_s = 0.346, \alpha_{s1} = 0.353, \alpha_b = 0.963, \alpha_{b1} = 0.373, \alpha_{b2} = 0.659, \\
&\arg(\alpha_d) = 3.706, \arg(\alpha_s) = 1.024, \arg(\alpha_{s1}) = 1.899, \arg(\alpha_{b1}) = 4.184, \arg(\alpha_{b2}) = 3.903,
\end{aligned} \tag{80}$$

which satisfy both the empirical constraints of Eqs.(E3) and (E4). This leads to the following physical observables $\theta^q_{23} = 2.323°, \theta^q_{13} = 0.211°, \theta^q_{12} = 13.017°$, $\delta^q_{CP} = 63.436°$; $m_d = 4.577$ MeV, $m_s = 103.800$ MeV, $m_b = 4.183$ GeV, $m_u = 2.488$ MeV, $m_c = 1.262$ GeV, $m_t = 173.582$ GeV.

B. Neutrino mass and mixing

After integrating out the right-handed heavy Majorana neutrinos, the effective neutrino mass matrix $\mathcal{M}_\nu$ is given at leading order by

$$\mathcal{M}_\nu \simeq -m_D^T M_R^{-1} m_D = U_\nu^* \,\text{diag.}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})\, U_\nu^\dagger\,, \tag{81}$$

where U_ν is the unitary matrix diagonalizing $\mathcal{M}_\nu$, and m_{ν_i} $(i = 1, 2, 3)$ are the light neutrino masses. Considering the effective Yukawa coefficients Eq.(51), the seesaw scale $\langle\chi\rangle$ can be estimated from Eqs.(65) and (66) as

$$\langle\chi\rangle \sim 2 \times 10^{10}\,\text{GeV}\,, \tag{82}$$

for $\Delta_\chi = 0.634$ and $\text{Im}\,\tau \simeq 1$, and $m_{\nu_3} \sim 0.05$ eV. Then the neutrino masses are obtained by

$$U_\nu^T\,\mathcal{M}_\nu\,U_\nu = \text{diag.}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})\,. \tag{83}$$

Here m_{ν_i} $(i = 1, 2, 3)$ are the light neutrino masses. The observed hierarchy $|\Delta m^2_{\rm Atm}| = |m^2_{\nu_3} - (m^2_{\nu_1} + m^2_{\nu_2})/2| \gg \Delta m^2_{\rm Sol} \equiv m^2_{\nu_2} - m^2_{\nu_1} > 0$ and the requirement of a Mikheyev-Smirnov-Wolfenstein resonance [56] for solar neutrinos lead to two possible neutrino mass spectra: normal mass ordering (NO) $m^2_{\nu_1} < m^2_{\nu_2} < m^2_{\nu_3}$ and inverted mass ordering (IO) $m^2_{\nu_3} < m^2_{\nu_1} < m^2_{\nu_2}$.

Then from Eq.(59) the PMNS mixing matrix becomes

$$U_{\rm PMNS} = V_L^\ell\, U_\nu \,, \tag{84}$$

where the left-handed charged-lepton mixing matrix V_L^ℓ is given by Eq.(E5). The matrix $U_{\rm PMNS}$ is expressed in terms of three mixing angles, $\theta_{12}, \theta_{13}, \theta_{23}$, and a Dirac type CP violaitng phase δ_{CP} and two additional CP violating phases $\varphi_{1,2}$ if light neutrinos are Majorana particle as [37]

$$U_{\rm PMNS} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta_{CP}} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta_{CP}} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta_{CP}} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix} Q_\nu \,, \tag{85}$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$ and $Q_\nu = {\rm diag.}(e^{-i\varphi_1/2}, e^{-i\varphi_2/2}, 1)$. Nine physical observables can be derived from Eqs.(85) and (83): θ_{23}, θ_{13}, θ_{12}, δ_{CP}, φ_1, φ_2, m_{ν_1}, m_{ν_2}, and m_{ν_3}. Recent

TABLE III: The global fit of three-flavor oscillation parameters at the best-fit and 3σ level with Super-Kamiokande atmospheric data [60]. NO = normal neutrino mass ordering; IO = inverted mass ordering. And $\Delta m^2_{\rm Sol} \equiv m^2_{\nu_2} - m^2_{\nu_1}$, $\Delta m^2_{\rm Atm} \equiv m^2_{\nu_3} - m^2_{\nu_1}$ for NO, and $\Delta m^2_{\rm Atm} \equiv m^2_{\nu_2} - m^2_{\nu_3}$ for IO.

	$\theta_{13}[^\circ]$	$\delta_{CP}[^\circ]$	$\theta_{12}[^\circ]$	$\theta_{23}[^\circ]$	$\Delta m^2_{\rm Sol}[10^{-5}{\rm eV}^2]$	$\Delta m^2_{\rm Atm}[10^{-3}{\rm eV}^2]$
NO	$8.58^{+0.33}_{-0.35}$	232^{+118}_{-88}	$33.41^{+2.33}_{-2.10}$	$42.2^{+8.8}_{-2.5}$	$7.41^{+0.62}_{-0.59}$	$2.507^{+0.083}_{-0.080}$
IO	$8.57^{+0.37}_{-0.34}$	276^{+68}_{-82}		$49.0^{+2.5}_{-9.1}$		$2.486^{+0.084}_{-0.080}$

global fits [57–59] of neutrino oscillations have enabled a more precise determination of the mixing angles and mass squared differences, with large uncertainties remaining for θ_{23} and δ_{CP} at 3σ. The most recent analysis [60] lists global fit values and 3σ intervals for these parameters in Table-III. Furthermore, recent constraints on the rate of $0\nu\beta\beta$ decay have added to these findings. Specifically, the most tight upper bounds for the effective Majorana mass $(\mathcal{M}_\nu)_{ee}$, which is the modulus of the ee-entry of the effective neutrino mass matrix, are given by

$$(\mathcal{M}_\nu)_{ee} < 0.036 - 0.156\,{\rm eV} \;\; (^{136}{\rm Xe\text{-}based\ experiment\ [61]}) \tag{86}$$

at 90% CL.

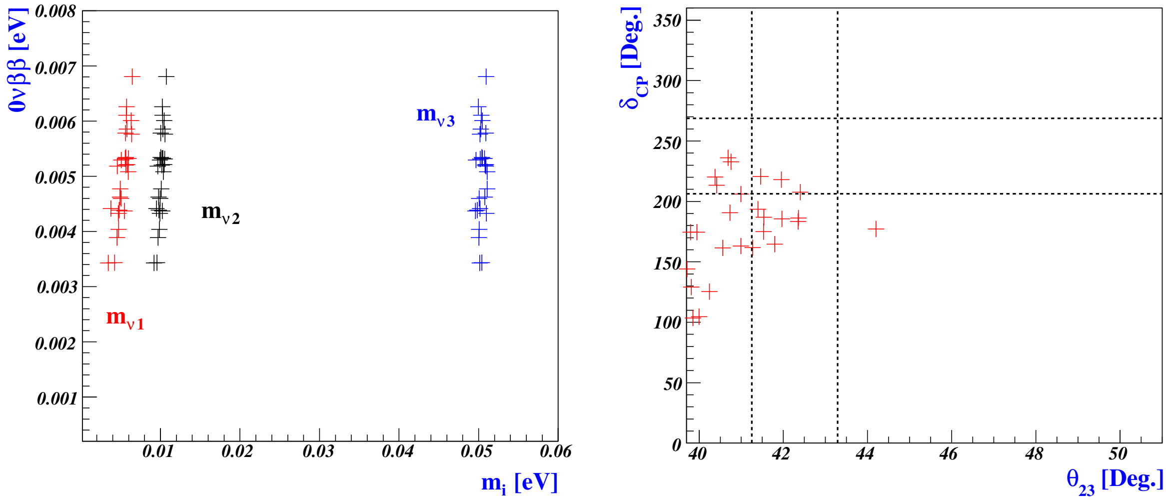

FIG. 3: Plots for $0\nu\beta\beta$-decay rate as a function of the neutrino masses m_{ν_i} (left) and leptonic Dirac CP phase δ_{CP} as a function of the atmospheric mixing angle θ_{23} (right). Vertical and horizontal dashed lines represent the 1σ bounds for θ_{23} and δ_{CP}, respectively, in Table-III.

Numerical simulation for lepton mass and mixing: Similar to the quark sector, to simulate and match experimental results for charged-leptons and neutrinos, Eqs.(E3) and (E4), we use linear algebra tools from Ref.[54].

Using the reference values Eqs.(41) and (79), we obtain the charged lepton masses, which agree well with the empirical values shown [17] below Eq.(70), and the left-handed charged-lepton mixing matrix of Eq.(E5), by putting the inputs in Eq.(64):

$$\begin{aligned}
&\alpha_e = [1.1477, 1.1486]\,, \quad \alpha_{e2} = [1.0532, 1.0633]\,, \quad \alpha_\mu = [0.6091, 0.6095]\,, \\
&\alpha_\tau = [0.3703, 0.3707]\,, \quad \alpha_{\tau 1} = [0.3954, 0.3975]\,, \quad \alpha_{\tau 2} = [0.7390, 0.7419]\,, \\
&\arg(\alpha_e) = [1.26, 1.46]\,, \quad \arg(\alpha_{e2}) = [0.44, 0.95]\,, \quad \arg(\alpha_\mu) = [0.71, 1.65]\,, \\
&\arg(\alpha_{\tau 1}) = [4.90, 5.97]\,, \quad \arg(\alpha_{\tau 2}) = [3.15, 4.13]\,.
\end{aligned} \tag{87}$$

The seesaw mechanism in Eq.(81) operates at the $U(1)_X$ symmetry breakdown scale, while its implications are measured by experiments below the electroweak scale. Therefore, quantum corrections to neutrino masses and mixing angles can be crucial, especially for degenerate neutrino masses [54]. However, based on our numerical calculation that the neutrino

[17] In our numerical analysis, we have employed the constraints $0.5110 \lesssim m_e[\text{eV}] \lesssim 0.5112$ for the electron mass and $105.65 \lesssim m_\mu[\text{MeV}] \lesssim 105.69$ for the muon mass.

mass spectra exhibit normal mass hierarchy (NH) at the scale of $U(1)_X$ breakdown (as depicted in the left panel of Fig.3), we can safely assume that the renormalization group running effect on observables can be ignored. To show the viable parameter space, we scan the precision constraints $\{\theta_{13},\, \theta_{23},\, \theta_{12},\, \Delta m^2_{\rm Sol},\, \Delta m^2_{\rm Atm}\}$ at 3σ from Table-III. Using the reference values from Eqs.(41,79) and taking $\langle\chi\rangle = 2\times 10^{10}$ GeV of Eq.(77), see also Eq.(82), we determine the input parameter spaces in Eqs.(65) and (66) at the $U(1)_X$ breaking scale :

$$\begin{gathered}
\beta_{1e} = [0.642, 0.668], \quad \tilde\beta_{1e} = [1.000, 1.129], \quad \beta_{2\mu} = [1.470, 1.479], \quad \tilde\beta_{2\mu} = [0.514, 0.535], \\
\beta_{3\tau} = [0.998, 1.122], \quad \tilde\beta_{3\tau} = [0.335, 0.374], \quad \beta_{1\mu} = [1.428, 1.464], \quad \tilde\beta_{1\mu} = [0.629, 0.666], \\
\beta_{1\tau} = [1.184, 1.230], \quad \tilde\beta_{1\tau} = [1.530, 1.566], \quad \beta_{2e} = [1.204, 1.230], \quad \tilde\beta_{2e} = [0.723, 0.759]\,, \\
\beta_{2\tau} = [1.394, 1.416], \quad \tilde\beta_{2\tau} = [0.671, 0.697], \quad \beta_{3e} = [0.658, 0.696], \quad \tilde\beta_{3e} = [0.946, 1.135], \\
\beta_{3\mu} = [1.340, 1.386], \quad \tilde\beta_{3\mu} = [1.274, 1.338], \\
\arg(\beta_{1e}) = [1.20, 1.71], \quad \arg(\beta_{2\mu}) = [5.50, 2\pi], \quad \arg(\beta_{3\tau}) = [0.69, 1.17], \\
\arg(\beta_{1\mu}) = [2.61, 3.75], \quad \arg(\beta_{1\tau}) = [2.60, 3.20], \quad \arg(\beta_{2e}) = [3.46, 4.20], \\
\arg(\beta_{2\tau}) = [5.57, 2\pi], \quad \arg(\beta_{3e}) = [5.71, 2\pi], \quad \arg(\beta_{3\mu}) = [2.50, 3.56],
\end{gathered} \tag{88}$$

for the Dirac neutrino of Eq.(66);

$$\begin{gathered}
\gamma_{11} = [0.616, 0.659], \quad \tilde\gamma_{11} = [0.876, 0.924], \quad \gamma_{12} = [1.331, 1.379], \quad \tilde\gamma_{12} = [0.946, 0.967], \\
\gamma_{13} = [0.768, 0.828], \quad \tilde\gamma_{13} = [0.760, 0.816], \quad \gamma_{22} = [1.426, 1.468], \quad \tilde\gamma_{22} = [0.652, 0.715], \\
\gamma_{23} = [1.365, 1.409], \quad \tilde\gamma_{23} = [0.930, 0.966], \quad \gamma_{33} = [1.343, 1.413], \quad \tilde\gamma_{33} = [0.800, 0.874], \\
\arg(\gamma_{11}) = [1.70, 2.40], \quad \arg(\gamma_{12}) = [4.41, 5.11], \quad \arg(\gamma_{13}) = [3.70, 4.40], \\
\arg(\gamma_{22}) = [4.10, 4.99], \quad \arg(\gamma_{23}) = [5.34, 6.10], \quad \arg(\gamma_{33}) = [0.18, 0.81],
\end{gathered} \tag{89}$$

for the heavy neutrino of Eq.(65). Neutrino oscillation experiments currently aim to make precise measurements of the Dirac CP-violating phase δ_{CP} and atmospheric mixing angle θ_{23}. For the parameter regions explored in our model, referring to the two-dimensional allowed regions at 3σ presented in Ref.[60], we note that there are no favored regions with respect to θ_{23} and δ_{CP}, as shown in the right panel of Fig.3. Ongoing experiments like DUNE [65], as well as proposed next-generation experiments such as Hyper-K [66], are poised to greatly reduce uncertainties in the values of θ_{23} and δ_{CP}, providing appropriate data set for our proposed model. As shown in the left panel of Fig.4, the predicted $0\nu\beta\beta$-decay rate, plotted

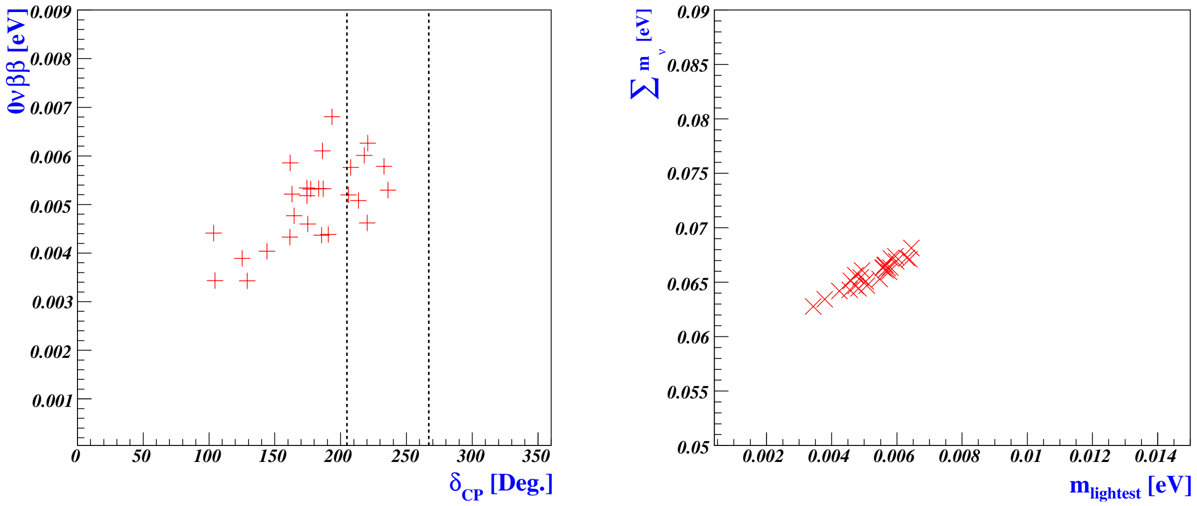

FIG. 4: Plots for $0\nu\beta\beta$-decay rate as a function of leptonic Dirac CP phase δ_{CP} (left) and $\sum m_\nu = m_{\nu_1} + m_{\nu_2} + m_{\nu_3}$ as a function of $m_{\rm lightest}$ (right). Vertical dashed lines indicate the 1σ bound for δ_{CP} listed in Table-III.

against δ_{CP}, lies entirely below the experimental upper bound in Eq.(86). Moreover, ongoing and future experiments on $0\nu\beta\beta$-decay like NEXT [67], SNO+ [68], KamLAND-Zen [61], Theia [69], SuperNEMO [70] may reach a sensitivity to test the NH of our model. Cosmological and astrophysical measurements provide powerful constraints on the sum of neutrino masses. The upper bound on the sum of the three active neutrino masses can be summarized as $\sum m_\nu = m_{\nu_1}+m_{\nu_2}+m_{\nu_3} < 0.120$ eV at 95% CL for TT, TE, EE+lowE+lensing+BAO [71]. The right panel of Fig.4 illustrates that the sum of neutrino masses lies in the range of 0.0634 to 0.0756 eV when $0.0034 \lesssim m_{\rm lightest} = m_{\nu_1} \lesssim 0.0065$ eV.

V. CONCLUSION

We have constructed a 4D effective model within a string-derived supergravity framework, based on the symmetry $G_{\rm SM} \times SL(2,\mathbb{Z}) \times U(1)_X$, where $G_{\rm SM}$ is the SM gauge group and $U(1)_X$ is a gauged symmetry. The string-theoretic origin of the $SL(2,\mathbb{Z})$ modular symmetry fundamentally constrains the superpotential, Kähler potential, and gauge kinetic function, while quantum anomaly-cancellation conditions impose stringent restrictions on the chiral matter content. To simultaneously resolve the strong CP problem and account for the observed fermion mass and mixing hierarchies, we introduced a flavored $U(1)_X$ symmetry. After decoupling of the $U(1)_X$ gauge boson, the residual global symmetry yields a flavored QCD axion, providing a solution to the strong CP problem and a well-motivated dark matter candidate. The combined $SL(2,\mathbb{Z}) \times U(1)_X$ flavor structure successfully generates realistic quark and lepton mass hierarchies and mixing patterns, with all Yukawa coefficients restricted to unit-magnitude complex numbers.

We have shown that $SL(2,\mathbb{Z})$- and $U(1)_X$-mixed anomalies should vanish, despite $SL(2,\mathbb{Z})$ being a global symmetry. The anomalies induced by Kähler transformations match those generated by the chiral rotation of gauginos. And, we have argued that despite the non-trivial transformation of SM fermions under $SL(2,\mathbb{Z})$, the strong CP phase remains unaffected, even in the global supersymmetry limit ($M_P \to \infty$), where we assume the symmetry-breaking scalar fields have modular weights zero – so their VEVs stay invariant under the modular transformations. The vanishing modular anomalies ($SL(2,\mathbb{Z}) \times [SU(3)_C]^2$ and $SL(2,\mathbb{Z}) \times [U(1)_{\rm EM}]^2$ guaranteed by modular- and SM gauge-invariance with non-negative weight modular forms) together with the anomaly-free conditions such as $SL(2,\mathbb{Z}) \times \left\{[SU(2)_L]^2,\, [U(1)_Y]^2,\, [U(1)_X]^2\right\}$ can determine the flavor structure in both quark and lepton sectors and set the scales for $U(1)_X$ symmetry breaking.

We have constructed a simple moduli superpotential to determine Yukawa couplings, gauge couplings, SUSY-breaking scale, and cosmological constant. We also demonstrated that the modulus VEV stabilizes near a fixed point (particularly $\tau \approx i$). Although $SL(2,\mathbb{Z})$–interpreted as T-duality–is treated as an exact discrete gauge symmetry, it becomes spontaneously broken when τ develops a VEV. Notably, at $\langle\tau\rangle \approx i$, no non-trivial subgroup of the modular group survives at low energies. Supersymmetry breaking is driven by the dilaton S, the modulus τ, and the complex structure modulus U_X.

The flavored $U(1)_X$ plays a crucial role in QCD axion solution to the strong CP problem, as a candidate for dark matter, and in explaining the SM fermion mass hierarchies. Its gauge transformation requires the GS parameter $\delta_X^{\rm GS}$ to be directly related to the mixed anomaly coefficients δ_X^Q of $U(1)_X \times \left\{[SU(3)_C]^2,\ [SU(2)_L]^2,\ [U(1)_Y]^2,\ [U(1)_X]^2\right\}$. These anomaly coefficients, determined by the $U(1)_X$ charges of SM fermions, can either vanish or remain finite depending on the specific charge assignments across SM fermions. While $\delta_X^{\rm GS}$ is generically non-zero, the specific case where the $U(1)_X$ anomaly coefficients vanish leads to decoupling of the $U(1)_X$ gauge boson at low energies, leaving a non-anomalous global $U(1)_X$

symmetry without a massless NG mode. Remarkably, with charge assignments $X_{Q_i} = \frac{1}{3}$, $X_{U_i^c} = X_{D_i^c} = -\frac{1}{3}$, $X_{L_i} = -1$, $X_{\ell_i^c} = 1$, and $X_{N_i^c} = 1$, this non-anomalous $U(1)_X$ reproduces $U(1)_{B-L}$ without leaving a NG mode.

A phenomenologically viable model has been presented, in which $U(1)_X$ charges and modular weights k_I are systematically assigned to all quark and lepton fields. We have shown that, unlike in ordinary cases, flavor-violating interactions of the flavored QCD axion to s, d-quarks and μ, e-leptons are suppressed to $\mathcal{O}(\lambda^4)$. This suppression makes rare decay $B^\pm \to K^\pm + a$ the dominant processes constraining the axion decay constant f_A, while kaon decays ($K^+ \to \pi^+ + a$) remain subdominant. So, the strongest experimental bounds in our model arise from the axion-electron coupling, which is tightly constrained by red giant branch star cooling observations.

Numerically, the model is consistent with current experimental data on flavor mixing and mass spectra. It successfully accounts for the observed quark Dirac CP phase and predicts a neutrino mass spectrum with a normal hierarchy. Future reductions in the uncertainties of atmospheric mixing angle θ_{23} and lepton Dirac CP phase δ_{CP} will provide appropriate data set for our proposed model. Additionally, it provides specific predictions for a flavored QCD axion, including a mass $m_a = 9.12 \times 10^{-3}$ eV and a photon coupling strength $|g_{a\gamma\gamma}| = 1.69 \times 10^{-13}$ GeV^{-1}, which are consistent with all known flavor constraints.

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Appendix A: The potentials $V_F^{(0)}$ and $V_F^{(1)}$

The potentials $V_F^{(0)}$ and $V_F^{(1)}$ are expressed, along the $\sigma = \mathrm{Re}[U_X]$ direction, as

$$\begin{aligned}
V_F^{(0)} &= e^{K/M_P^2}\frac{M_P^6}{|\eta(\tau)|^{4h}}\Big\{|D|^2\Big(K^{\tau\bar\tau}|H|^2 + K^{S\bar S}\frac{1}{y^2}\Big) - \frac{4\sigma^2}{3M_P^2}(-Aae^{-a\sigma}+Bbe^{-b\sigma}) \\
&\qquad + \frac{2\sigma}{M_P^2}(Aae^{-a\sigma}-Bbe^{-b\sigma})(D+\bar D)\Big\}\,, \\
V_F^{(1)} &= e^{K/M_P^2}\frac{C_0M_P^6}{|\eta(\tau)|^{4h}}\Big\{\Big(-D(\bar S+\frac{h}{2\pi^2}\ln\eta(\bar\tau)) - \mathrm{h.c.}\Big)\Big(K^{\tau\bar\tau}|H|^2 + \frac{K^{S\bar S}}{y^2}\Big) \\
&\qquad - \frac{2\sigma}{M_P^2}(Aae^{-a\sigma}-Bbe^{-b\sigma})(S+\bar S+\frac{h}{2\pi^2}\ln\eta(\bar\tau)\eta(\tau)) + \frac{K^{S\bar S}}{y}(D+\bar D) \\
&\qquad + \frac{h}{2\pi^2}K^{\tau\bar\tau}\Big(DH\frac{\eta'(\bar\tau)}{\eta(\bar\tau)} + \bar D\bar H\frac{\eta'(\tau)}{\eta(\tau)}\Big)\Big\}\,.
\end{aligned} \tag{A1}$$

where $D = C_0 + Ae^{-aU_X} - Be^{-bU_X}$.

Appendix B: SUSY breaking in the direction of U_X

As shown in Eq.(37), in the limit $\alpha \to 0$, the gravitino mass vanishes, while the mass squared of the field σ at the local minimum is given by $m_\sigma^2 = \frac{1}{2}K^{U_X\bar U_X}\partial_{U_X}\partial_{\bar U_X}V_F|_{\sigma_0,\tau_0,s_0}$, where $\partial_{U_X}\partial_{\bar U_X}V_F = e^{K/M_P^2}K^{U_X\bar U_X}|W_{U_XU_X}|^2$ with $W_{U_XU_X} = \partial^2W/(\partial U_X)^2$:

$$m_\sigma^2 = \frac{\sigma_0}{9\langle y\rangle(2\mathrm{Im}\tau_0)^3}\Big|A_0a^2\Big(\frac{aA_0}{bB_0}\Big)^{\frac{a}{b-a}} + B_0b^2\Big(\frac{aA_0}{bB_0}\Big)^{\frac{b}{b-a}}\Big|^2\frac{M_P^2}{|\eta(\tau_0)|^{12}}\,. \tag{B1}$$

For particularly simple parameter choices $A_0 = B_0 = 0.0005, (1)$, $a = 2\pi/100$, $b = 2\pi/101$, $s_0 = 1$, $|\eta(i)| \approx 0.768$, these values yield $C_0 = 1.83\times10^{-6}, (3.66\times10^{-3})$ and $\sigma_0 = 15.99$, with resulting scalar masses $m_\sigma \simeq 5.75\times10^{12}$ GeV (1.15×10^{16} GeV).

In this limit, analogous to the Kallosh-Linde (KL) model [28], there exists a deeper supersymmetric AdS minimum in the U_X direction. When a small weak-scale perturbation ΔW is introduced along U_X, the potential minimum shifts from zero to a slightly negative value $V_{\mathrm{AdS}} < 0$ at $(\tilde\sigma_0, \tilde\tau_0, s_0)$, with $\tilde\sigma_0 = \sigma_0 + \delta\sigma$. At this shifted minimum, supersymmetry is preserved, that is $D_{U_X}W(\sigma_0+\delta\sigma, \tau_0, s_0) = 0$. This implies $W_{U_X}(\sigma_0,\tau_0) = 0$, and the minimum shifts by $\delta\sigma \simeq 3\Delta W/(2\sigma_0W_{U_XU_X}(\sigma_0,\tau_0))$. The potential at this minimum, expressed in terms of $W(\sigma_0+\delta\sigma) = \Delta W + \mathcal{O}(\Delta W)^2$, becomes

$$V_{\mathrm{AdS}}(\Delta W) = -e^{\langle K/M_P^2\rangle}\frac{3}{M_P^2}\big|\langle W\rangle_{\mathrm{AdS}}\big|^2 = -\frac{1}{\langle y\rangle(2\mathrm{Im}\,\tau_0)^3(2\sigma_0)^3}\frac{3}{M_P^2}\big|\Delta W\big|^2\,, \tag{B2}$$

where $\Delta W = \langle W\rangle_{\rm AdS}$ is the value of the superpotential at the AdS minimum. By introducing an uplifting potential of the form $\Delta V \approx |V_{\rm AdS}|(\sigma_0/\sigma)^n$, where typical choices are $n=2$ (as in KKLT [62]) or $n=3$ (as in Ref. [30]), the AdS minimum turns into a dS minimum. This uplifting simultaneously breaks supersymmetry. The uplifting induces a shift $\Delta\sigma = \sigma - \sigma_0$ in the U_X field, see Eq.(32), determined by solving the minimization condition $\partial_\sigma(V+\Delta V)|_{\sigma_0+\delta\sigma} = 0$ along the $\sigma = {\rm Re}[U_X]$ direction. For the case of a D-term induced uplift parametrized as $\Delta V = \frac{1}{2}(\xi_X^{\rm FI})^2 g_X^2 \approx |V_{\rm AdS}|(\sigma_0/\sigma)^3$ [30], and using the approximation $(D_\sigma W)_\sigma \approx W_{\sigma\sigma}|_{\sigma_0}$, we obtain the displacement $\Delta\sigma \simeq \langle y\rangle(2{\rm Im}\,\tau_0)^3 18 M_P^2|V_{\rm AdS}|/(W_{\sigma\sigma})^2$. This leads to the SUSY-breaking F-term in the uplifted minimum $D_\sigma W \simeq (D_\sigma W)_\sigma\Delta\sigma \simeq W_{\sigma\sigma}(\sigma_0)\Delta\sigma$:

$$D_\sigma W \simeq \sqrt{\langle y\rangle\sigma_0(2{\rm Im}\,\tau_0)^3}\,\frac{6|V_{\rm AdS}|}{m_\sigma}\,, \tag{B3}$$

where the suppression factor $m_\sigma^2 \gg \sqrt{|V_{\rm AdS}|}$ is characteristic of the KL model framework [28], ensuring the smallness of SUSY breaking effects after uplifting.

Appendix C: Scalar potential for flavon

In the global SUSY limit, *i.e.* $M_P \to \infty$, the scalar potential is determined by the vanishing of the F- and D-terms for all fields. Then the relevant F-term from Eq.(47) and D-term in Eq.(29) scalar potentials reads

$$V_F^{\rm global} = \left|g_\chi \chi\tilde{\chi} - \mu_\chi^2\right|^2, \qquad V_D^{\rm global} = \frac{g_X^2}{2}\left(-\xi_X^{\rm FI} + |\chi|^2 - |\tilde{\chi}|^2\right)^2 . \tag{C1}$$

The scalar fields χ and $\tilde{\chi}$, with X-charges $+1$ and -1 respectively, transform under $U(1)_X$ as

$$\chi \to e^{+i\xi}\chi\,, \qquad \tilde{\chi} \to e^{-i\xi}\tilde{\chi}\,, \tag{C2}$$

with a constant ξ, ensuring that the potential $V_{\rm SUSY}$ respects $U(1)_X$ symmetry. Since supersymmetry is preserved after spontaneous breaking of $U(1)_X$, the scalar potential vanishes at its ground states, *i.e.* $\langle V_F^{\rm global}\rangle = 0$ and $\langle V_D^{\rm global}\rangle = 0$. A vanishing F-term implies a vanishing D-term as well.

Appendix D: Modular functions

The Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q \equiv e^{i2\pi\tau} \text{ and } \mathrm{Im}(\tau) > 0\,. \tag{D1}$$

It satisfies the following modular transformation properties

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)\,, \qquad \eta(\tau+1) = e^{i\pi/12}\,\eta(\tau)\,. \tag{D2}$$

The q-expansions of the three linearly independent modular functions $Y_i(\tau)$ are given by

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + ... \\ Y_2(\tau) &= -6q^{1/3}(17q + 8q^2 + ...) \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + ...)\,. \end{aligned} \tag{D3}$$

The modular forms of weights 4, 8, 12, 16, 20, and 24 under $SL(2,\mathbb{Z})$ read [6, 33]

$$Y_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3 = E_4\,, \qquad Y_{\mathbf{1}}^{(8)} = (Y_1^2 + 2Y_2Y_3)^2 = E_8 = E_4^2\,. \tag{D4}$$

$$\begin{aligned} Y_{\mathbf{1}A}^{(12)} &= (Y_1^2 + 2Y_2Y_3)^3 = E_4^3\,,\ Y_{\mathbf{1}B}^{(12)} = (Y_3^2 + 2Y_1Y_2)^3 = E_6^2 - E_4^3\,, \\ Y_{\mathbf{1}A}^{(16)} &= (Y_1^2 + 2Y_2Y_3)^4 = E_4^4\,,\ Y_{\mathbf{1}B}^{(16)} = (Y_1^2 + 2Y_2Y_3)(Y_3^2 + 2Y_1Y_2)^3 = E_4(E_6^2 - E_4^3)\,, \\ Y_{\mathbf{1}A}^{(20)} &= (Y_1^2 + 2Y_2Y_3)^5 = E_4^5\,,\ Y_{\mathbf{1}B}^{(20)} = (Y_1^2 + 2Y_2Y_3)^2(Y_3^2 + 2Y_1Y_2)^3 = E_4^2(E_6^2 - E_4^3)\,, \\ Y_{\mathbf{1}A}^{(24)} &= (Y_1^2 + 2Y_2Y_3)^6 = E_4^6\,,\ Y_{\mathbf{1}B}^{(24)} = (Y_1^2 + 2Y_2Y_3)^3(Y_3^2 + 2Y_1Y_2)^3 = E_4^3(E_6^2 - E_4^3)\,, \\ Y_{\mathbf{1}A}^{(28)} &= (Y_1^2 + 2Y_2Y_3)^7 = E_4^7\,,\ Y_{\mathbf{1}B}^{(28)} = (Y_1^2 + 2Y_2Y_3)^4(Y_3^2 + 2Y_1Y_2)^3 = E_4^4(E_6^2 - E_4^3) \end{aligned} \tag{D5}$$

Appendix E: Quark and charged-lepton mixing matrices

The left-handed quark mixing matrices V_L^d and V_L^u are given by

$$\begin{aligned} V_L^d &= \tilde{C}_d \begin{pmatrix} 1 - \frac{1}{2}\lambda_d^2 & -\lambda_d\, e^{i(2\alpha_3^d - \alpha_2^d)} & \lambda_d^3(A_d\, e^{i(2\alpha_3^d - \alpha_2^d)} - B_d) \\ \lambda_d\, e^{-i(2\alpha_3^d - \alpha_2^d)} & 1 - \frac{1}{2}\lambda_d^2 & -A_d\lambda_d^2 \\ B_d\lambda_d^3 & A_d\lambda_d^2 & 1 \end{pmatrix} \tilde{K}_d + \mathcal{O}(\lambda^4)\,, \\ V_L^u &= \tilde{C}_u \begin{pmatrix} 1 - \frac{1}{2}\theta_{12}^{u2} & -\theta_{12}^u\, e^{i(2\alpha_3^u - \alpha_2^u)} & 0 \\ \theta_{12}^u\, e^{-i(2\alpha_3^u - \alpha_2^u)} & 1 - \frac{1}{2}\theta_{12}^{u2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{K}_u + \mathcal{O}(\lambda^4)\,, \end{aligned} \tag{E1}$$

where $\tilde{C}_f = \mathrm{diag}(e^{i(\alpha_2^f-\alpha_1^f-\alpha_3^f)}, e^{i(\alpha_3^f-\alpha_1^f)}, e^{i(\alpha_2^f-\alpha_1^f)})$, $\tilde{K}_f = \mathrm{diag}(e^{i(\alpha_1^f-2\alpha_2^f)}, 1, e^{i2\alpha_1^f})$ (for $f = d, u$), and the relevant parameters are given in a good approximation by

$$\begin{aligned}
\lambda_d &\simeq \frac{|\alpha_{s1}^*\alpha_s Y_{\mathbf{1}}^{(4)*} + \alpha_{b1}^*\alpha_{b2}Y_{\mathbf{1}}^{(8)*}Y_{\mathbf{1}}^{(4)}\Delta_\chi^{-4}(2\mathrm{Im}\,\tau)^{-4}|}{|\alpha_s|^2}(2\mathrm{Im}\,\tau)^{-2}\,,\\
B_d\lambda_d^3 &\simeq |\frac{\alpha_{b1}}{\alpha_b}Y_{\mathbf{1}}^{(8)}|\Delta_\chi^4(2\mathrm{Im}\,\tau)^{-4}\,,\\
A_d\lambda_d^2 &\simeq |\frac{\alpha_{b2}}{\alpha_b}Y_{\mathbf{1}}^{(4)}|\Delta_\chi^4(2\mathrm{Im}\,\tau)^{-2}\,, \qquad \theta_{12}^u \simeq \frac{|\alpha_{c1}Y_{\mathbf{1}}^{(4)}|}{|\alpha_c|}(2\mathrm{Im}\,\tau)^{-2}\,,
\end{aligned} \tag{E2}$$

with the phases defined as $\alpha_1^d = \frac{1}{2}\arg(\alpha_{b2}^*Y_{\mathbf{1}}^{(4)*})$, $\alpha_2^d \simeq \frac{1}{2}\arg(\alpha_{b1}^*Y_{\mathbf{1}}^{(8)*}) - \frac{1}{2}\alpha_1^d$, $\alpha_3^d \simeq \frac{1}{2}\arg(\alpha_{s1}^*\alpha_s Y_{\mathbf{1}}^{(4)*}) + \frac{1}{2}\alpha_1^d - \frac{1}{2}\alpha_2^d$; $\alpha_1^u = \frac{1}{2}\arg(\alpha_{t2}^*Y_{\mathbf{1}}^{(4)*})$, $\alpha_2^u \simeq \frac{1}{2}\arg(\alpha_{t1}^*Y_{\mathbf{1}}^{(8)*}) - \frac{1}{2}\alpha_1^u$, $\alpha_3^u \simeq \frac{1}{2}\arg(\alpha_{c1}^*\alpha_c Y_{\mathbf{1}}^{(4)*}) + \frac{1}{2}\alpha_1^u - \frac{1}{2}\alpha_2^u$. The right-handed quark mixing matrices V_R^d and V_R^u are approximately identity matrices up to $\mathcal{O}(\lambda^4)$, that is $V_R^{d(u)} = \mathbf{I} + \mathcal{O}(\lambda^4)$, with small mixing angles given by $\theta_{d23}^r \simeq \frac{|\alpha_s\alpha_{b2}^*Y_{\mathbf{1}}^{(4)*}|}{|\alpha_b|^2}\Delta_\chi^{10}(2\mathrm{Im}\,\tau)^{-2}$, $\theta_{d13}^r \simeq \frac{|\alpha_d\alpha_{b1}^*Y_{\mathbf{1}}^{(8)*}|}{|\alpha_b|^2}\Delta_\chi^{20}(2\mathrm{Im}\,\tau)^{-4}$, $\theta_{d12}^r \simeq \frac{|\alpha_d\alpha_{s1}^*Y_{\mathbf{1}}^{(4)*}|}{|\alpha_s|^2}\Delta_\chi^{10}(2\mathrm{Im}\,\tau)^{-2}$ for down-type quarks, and $\theta_{u23}^r \simeq \frac{|\alpha_c\alpha_{t2}^*Y_{\mathbf{1}}^{(4)*}|}{|\alpha_t|^2}\Delta_\chi^{22}(2\mathrm{Im}\,\tau)^{-2}$, $\theta_{u13}^r \simeq \frac{|\alpha_u\alpha_{t1}^*Y_{\mathbf{1}}^{(8)*}|}{|\alpha_t|^2}\Delta_\chi^{35}(2\mathrm{Im}\,\tau)^{-4}$, $\theta_{u12}^r \simeq \frac{|\alpha_u\alpha_{c1}^*Y_{\mathbf{1}}^{(4)*}|}{|\alpha_c|^2}\Delta_\chi^{13}(2\mathrm{Im}\,\tau)^{-2}$ for up-type quarks.

The current best-fit values of the CKM mixing angles in the standard parameterization [64] read in the 3σ range [55]

$$\theta_{23}^q[^\circ] = 2.376^{+0.054}_{-0.070}\,, \quad \theta_{13}^q[^\circ] = 0.210^{+0.016}_{-0.010}\,, \quad \theta_{12}^q[^\circ] = 13.003^{+0.048}_{-0.036}\,, \quad \delta_{CP}^q[^\circ] = 65.5^{+3.1}_{-4.9}\,. \tag{E3}$$

The physical structure of the up- and down-type quark Lagrangian should match up with the empirical results calculated from the Particle Data Group (PDG) [37]:

$$\begin{aligned}
m_d &= 4.67^{+0.48}_{-0.17}\,\mathrm{MeV}\,, \qquad m_s = 93^{+11}_{-5}\,\mathrm{MeV}\,, \qquad m_b = 4.18^{+0.03}_{-0.02}\,\mathrm{GeV}\,,\\
m_u &= 2.16^{+0.49}_{-0.29}\,\mathrm{MeV}\,, \qquad m_c = 1.27\pm 0.02\,\mathrm{GeV}\,, \qquad m_t = 173.1\pm 0.9\,\mathrm{GeV}\,.
\end{aligned} \tag{E4}$$

where t-quark mass is the pole mass, c- and b-quark masses are the running masses in the $\overline{\mathrm{MS}}$ scheme, and the light u-, d-, s-quark masses are the current quark masses in the $\overline{\mathrm{MS}}$ scheme at the momentum scale $\mu \approx 2$ GeV. Below the scale of spontaneous $SU(2)_L \times U(1)_Y$ gauge symmetry breaking, the running masses of c- and b-quark receive corrections from QCD and QED loops [37]. The top quark mass at scales below the pole mass is unphysical since the t-quark decouples at its scale, and its mass is determined more directly by experiments [37].

For the charged-leptons, Eq.(70) must be matched with the empirical values from the PDG [37] given by $m_e = 0.511\,\mathrm{MeV}$, $m_\mu = 105.658\,\mathrm{MeV}$, $m_\tau = 1776.86 \pm 0.12\,\mathrm{MeV}$. The left-handed charged-lepton mixing matrix V_L^ℓ reads

$$V_L^\ell = \tilde{C}_\ell \begin{pmatrix} 1-\frac{1}{2}\theta_2^2 & 0 & -\theta_2 \\ 0 & 1-\frac{1}{2}\theta_1^2 & -\theta_1 \\ \theta_2 & \theta_1(1-\frac{1}{2}\theta_2^2) & 1-\frac{1}{2}\theta_2^2 \end{pmatrix} \tilde{K}_\ell + \mathcal{O}(\lambda^4)\,, \tag{E5}$$

where $\varphi_\ell = 2\alpha_3^\ell - \alpha_2^\ell$, $\tilde{C}_\ell = \mathrm{diag}(e^{i(\alpha_2^\ell-\alpha_1^\ell-\alpha_3^\ell)}, e^{i(\alpha_3^\ell-\alpha_1^\ell)}, e^{i(\alpha_2^\ell-\alpha_1^\ell)})$, and $\tilde{K}_\ell = \mathrm{diag}(e^{i(\alpha_1^\ell-2\alpha_2^\ell)}, 1, e^{i2\alpha_1^\ell})$ with $\alpha_1^\ell = \frac{1}{2}\arg(\alpha_{\tau 2}^* Y_{\mathbf{1}}^{(8)*})$, $\alpha_2^\ell \simeq \frac{1}{2}\arg(\alpha_{\tau 1}^* Y_{\mathbf{1}}^{(4)*}) - \frac{1}{2}\alpha_1^\ell$, $\alpha_3^\ell \simeq \frac{1}{2}\arg(\alpha_e^*\alpha_{e2} Y_{\mathbf{1}}^{(4)}) + \frac{1}{2}\alpha_1^\ell - \frac{1}{2}\alpha_2^\ell$, and

$$\theta_1 \simeq \tfrac{|\alpha_{\tau 2} Y_{\mathbf{1}}^{(8)}|}{|\alpha_\tau|}(2\,\mathrm{Im}\,\tau)^{-4}\,, \theta_2 \simeq \tfrac{|\alpha_{\tau 1} Y_{\mathbf{1}}^{(4)}|}{|\alpha_\tau|}(2\,\mathrm{Im}\,\tau)^{-2} \sim \lambda\,, \theta_3 \simeq \tfrac{|\alpha_e\alpha_{e2} Y_{\mathbf{1}}^{(4)}|}{|\alpha_\mu|^2}(2\,\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{26}\,. \tag{E6}$$

Here $\theta_1 \sim \mathcal{O}(\lambda^2)$, $\theta_2 \sim \mathcal{O}(\lambda)$, and $\theta_3 \ll \mathcal{O}(\lambda^4)$ are roughly expected for $\Delta_\chi = 0.634$ and $\mathrm{Im}\,\tau \simeq 1$. The right-handed charged-lepton mixing matrix V_R^ℓ is approximately the unit matrix up to $\mathcal{O}(\lambda^4)$:

$$\begin{aligned} \theta_1^R &\simeq \frac{|\alpha_\mu\alpha_{\tau 2} Y_{\mathbf{1}}^{(8)}|}{|\alpha_\tau|^2}(2\,\mathrm{Im}\,\tau)^{-4}\Delta_\chi^7\,, \\ \theta_2^R &\simeq \frac{|\alpha_e\alpha_{\tau 1} Y_{\mathbf{1}}^{(4)}|}{|\alpha_\tau|^2}(2\,\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{20}\,, \\ \theta_3^R &\simeq \frac{|\alpha_{e2} Y_{\mathbf{1}}^{(4)}|}{|\alpha_\mu|}(2\,\mathrm{Im}\,\tau)^{-2}\Delta_\chi^{13}\,. \end{aligned} \tag{E7}$$

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