

BLACK HOLES AND BLACK REGIONS, HORIZONS AND BARRIERS IN LORENTZIAN MANIFOLDS

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ABSTRACT. We prove that if S is a time-oriented null hypersurface of a Lorentzian n -manifold (M, g) , the causal world-lines, which intersect transversally S and are time-oriented in a compatible way, cross the hypersurface all in the same direction, the other being forbidden. Even if it is known that a smooth event horizon (in the sense of Penrose, Hawking and Ellis) is a null hypersurface and has the above semi-permeability property, at the best of our knowledge, it was not known so far that the latter is a mere consequence of the former. Our result leads to the concepts of barriers (= null hypersurfaces separating the space-time into disjoint regions) and black regions (= time-oriented regions bounded by barriers). These objects naturally include (smooth) event horizons and (smoothly bounded) black holes. Since barriers are defined by two simple properties – the merely local property of “nullity” combined with the global property of “separating the space-time” – we expect they may be used to simplify computations for locating static and/or dynamic horizons in numerical computations.

1. INTRODUCTION

The main result of this paper consists of a rigorous proof of the following property of the time-oriented null hypersurfaces: *any causal world-line of a Lorentzian manifold (M, g) , which intersects transversally a time-oriented null hypersurface S at a point x_o and is time-oriented in a compatible way with the time-orientation of the hypersurface, crosses S in just one of the two possible directions, the other being forbidden.* To the best of our knowledge, such a strong constraint on the possible ways of crossing the null hypersurfaces has not been stated so far in the literature, despite of the fact that it is quite reasonable to expect it, especially if one has in mind that the null hypersurfaces are somehow intermediate between the time-like and the space-like hypersurfaces, so that in certain regards they are similar to the former, in others similar to the latter. In order to make fully clear what we mean, we now very briefly review some basic facts on time-like and space-like hypersurfaces.

We recall that imposing a *time-orientation* on a Lorentzian manifold corresponds to selecting a (smoothly varying) family $\mathcal{T} = \{\mathcal{T}_x\}_{x \in M}$ of connected components $\mathcal{T}_x$ of the cones $C_x \subset T_x M \setminus \{0\}$ of causal (= time-like or null) non-zero vectors. To make such a choice one can just pick a nowhere vanishing smooth causal vector field Y and declare that, for any $x \in N$, the connected component $\mathcal{T}_x \subset C_x$ at x of the time-orientation is

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the one containing Y_x . There do exist Lorentzian manifolds on which no global time-orientation can be imposed, but on any sufficiently small open set there are always exactly two time-orientations which can be imposed, one opposite to the other.

If $\mathcal{S}$ is a *time-like hypersurface* in a Lorentzian n -manifold (M, g) , then $(\mathcal{S}, g|_{T\mathcal{S} \times T\mathcal{S}})$ is a Lorentzian $(n - 1)$ -manifold and any time-orientation on $\mathcal{S}$ (determined by a nowhere vanishing causal vector field Y in $T\mathcal{S}$) fixes a time-orientation on a neighbourhood $\mathcal{U} \subset M$ of $\mathcal{S}$. Indeed, it suffices to extend Y to a smooth vector field on M , so that one may consider a neighbourhood $\mathcal{U}$ of $\mathcal{S}$ where the extended Y is still causal and non-zero and use $Y|_{\mathcal{U}}$ to impose a time-orientation on $\mathcal{U}$. Nothing of this kind can be done if $\mathcal{S}$ is a *space-like hypersurface*, because no nowhere vanishing vector field Y in $T\mathcal{S}$ is causal and no time-orientation on neighbourhoods can be determined as above (Fig.1 and Fig.2).

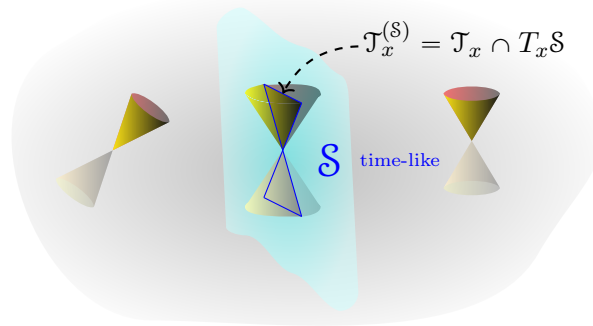


Fig.1 – Time-orientation nearby a time-oriented time-like hypersurface

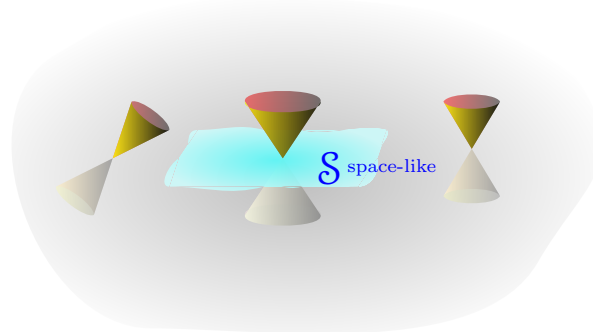


Fig.2 – One of the two freely choosable time-orientations on a neighbourhood of a space-like hypersurface

A similar contrast occurs in the directions of crossing a hypersurface by causal world-lines. Indeed, if $\mathcal{S}$ is a time-oriented, time-like hypersurface and $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathcal{U}}$ is the associated time-orientation of a neighbourhood $\mathcal{U}$ of $\mathcal{S}$, for any $x_o \in \mathcal{S}$ there exist causal world-lines, which are time-oriented in a compatible way with $\mathcal{T}$ and cross $\mathcal{S}$ at x_o in either one of the two possible ways of crossing. A simple way to check this consists in selecting a

time-like vector $v_o \in \mathcal{T}_{x_o} \cap T_{x_o}\mathcal{S}$, a space-like vector w_o in $T_{x_o}M \setminus T_{x_o}\mathcal{S}$ and two perturbations of v_o of the form $v_{\pm}^{(\varepsilon)} = v_o \pm \varepsilon w_o \in T_{x_o}M \setminus T_{x_o}\mathcal{S}$. If ε is sufficiently small, both perturbations are in $\mathcal{T}_{x_o} \setminus T_{x_o}\mathcal{S}$ and the traces of the time-like geodesics, passing through x_o and tangent to $v_{+}^{(\varepsilon)}$ and $v_{-}^{(\varepsilon)}$, respectively, are causal world-lines that are time-oriented in a compatible way and having the following property: one crosses $\mathcal{S}$ in one direction, the other in the opposite (Fig.3). Consider now a point x_o of a space-like hypersurface $\mathcal{S} \subset M$ and pick a time-orientation $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathcal{U}}$ on a neighbourhood $\mathcal{U}$ of $\mathcal{S}$ (it does not matter which of the two possible ones). In this situation there is only one possible way for a $\mathcal{T}$ -oriented causal world-line to cross $\mathcal{S}$ at x_o : All tangent vectors at x_o of such time-oriented world-lines have to be in $\mathcal{T}_{x_o}$ and hence all of them have to point towards the same side of $\mathcal{S}$ (Fig.4).

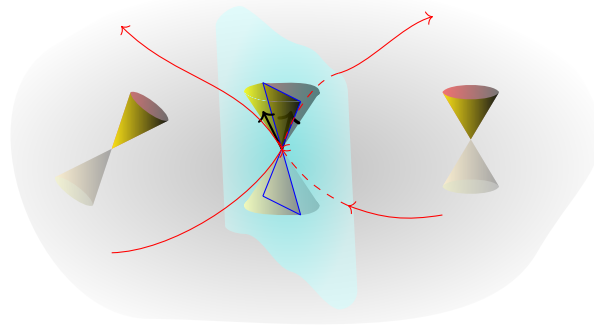
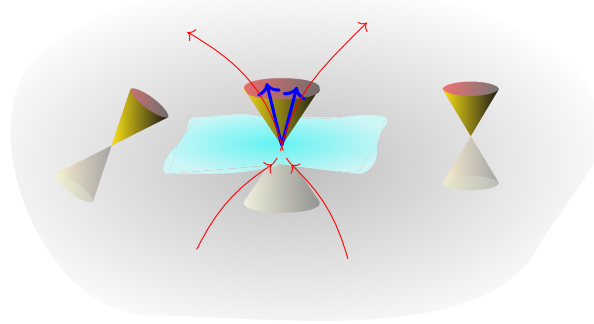


Fig.3 – The two ways of crossing a time-oriented time-like hypersurface



**Fig.4 – The only way of crossing a space-like hypersurface
(provided that a nearby time-orientation is prescribed)**

The *null* (or *light-like*) *hypersurfaces* of (M, g) constitute a sort of borderland between the classes of time-like and space-like hypersurfaces – indeed, any smooth homotopy of hypersurfaces, which begins with a space-like hypersurface and ends with a time-like one, must contain a null hypersurface at some intermediate step. The properties of null hypersurfaces have been intensively studied in the literatures and we refer the reader to [17, 21] and references therein. Here we just need to recall that any null hypersurface is equipped

with a rank one distribution $\mathcal{D} \subset TS$ consisting of light-like vectors. For any nowhere vanishing vector field E_o in $\mathcal{D}$, we may consider the unique family $\mathcal{T}^{(\mathcal{S})} = \{\mathcal{T}_x\}_{x \in \mathcal{S}}$ of connected components $\mathcal{T}_x$ of the causal cones $C_x \subset T_x M \setminus \{0\}$, $x \in \mathcal{S}$, containing the light-like vectors E_{ox} . By a small abuse of language, we call $\mathcal{T}^{(\mathcal{S})}$ the *time-orientation* of $\mathcal{S}$ determined by E_o . Note that any causal smooth extension of E_o to a neighbourhood $\mathcal{U}$ of $\mathcal{S}$ determines a time-orientation $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathcal{U}}$ on $\mathcal{U}$, which is also the unique time-orientation on $\mathcal{U}$ that induces the family of connected components $\mathcal{T}^{(\mathcal{S})}$ at the points of $\mathcal{S}$.

Being intermediate between the time-like and the space-like, it is sensible to expect that some aspects of the null hypersurfaces are similar to those of the time-like, while others are similar to those of the space-like. This expectation is fully confirmed by the following two properties: (a) On the one hand, as we just pointed out, *any time-orientation of a null hypersurface $\mathcal{S}$ admits a canonical extension to a time-like orientation of a neighbourhood $\mathcal{U}$, exactly as it occurs for the time-like and in contrast with the space-like* (Fig. 5); (b) On the other hand, our main result states that *any time-oriented null hypersurface can be crossed by appropriately time-oriented world-lines in just one of the two conceivable ways, exactly as it occurs for the space-like and in opposition to the time-like* (Fig. 6).

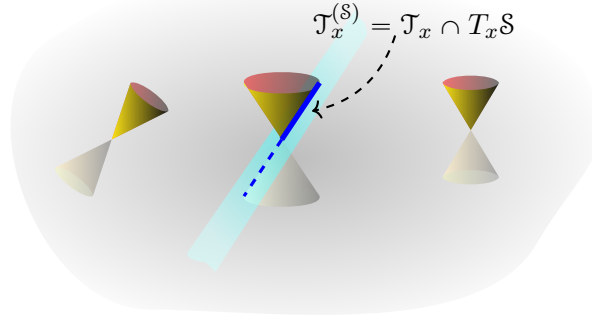


Fig. 5 – Time-orientation nearby a time-oriented light-like hypersurface

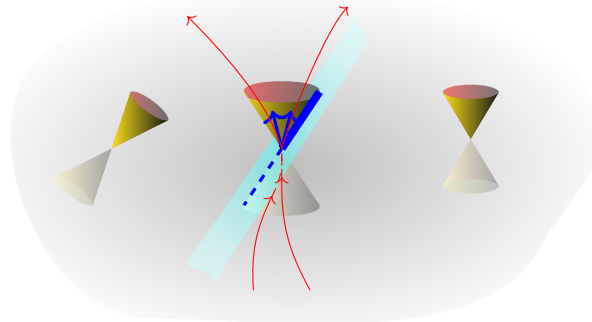


Fig. 6 – The only way of crossing a time-oriented light-like hypersurface

In order to get familiar with the relevance of (b), consider the following examples. A first and very easy one is given by the upper connected component of a light cone in the Minkowski space $\mathbb{R}^{1,3}$. Consider cylindrical coordinates $((\xi, \psi, r); t)$ for $\mathbb{R}^{1,3} \setminus \{x^1 = 0\}$, in

which the Minkowski metric takes the form $g_o = dt^2 - dr^2 - r^2 d\xi^2 - r^2 \sin^2 \xi d\psi^2$. The traces of the geodesics of $\mathbb{R}^{1,3}$ are segments and the traces of the *causal* geodesics are segments parallel to time-like or null vectors. The connected component $\mathcal{S} = \{t = r, t > 0\}$ of the light cone $\mathcal{V} = \{t = r\}$ is a light-like hypersurface with the following elementary property: *No trace of a causal geodesic of $\mathbb{R}^{1,3}$ (= causal segment), oriented towards increasing values of the time t , crosses $\mathcal{S}$ starting from a point of the region $\{t > r\}$ and ending in the region $\{t < r\}$.* This is because otherwise the geodesic would be parallel to a space-like vector, hence not causal. A simple argument easily extends this “no-crossing” property to any other causal world-line, not necessarily trace of a geodesic.

A second (and much less trivial) illustration of (b) is given by the horizon of a Schwarzschild black hole. Consider the manifold $M_S = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$, equipped with the Boyer-Lindquist coordinates $((\xi, \psi, r); u)$ and the Schwarzschild metric

$$g_S = \left(1 - \frac{2m}{r}\right) du^2 - 2du \vee dr - r^2 d\xi^2 - r^2 \sin^2 \xi d\psi^2. \quad (1.1)$$

It is well known that (M_S, g_S) has the following feature: *No trace of a causal geodesic, oriented towards increasing values for the coordinate u , starts from a point of the region $\{r < 2m\}$, crosses the null hypersurface $\mathcal{S} = \{r = 2m\}$ and ends in the region $\{r > 2m\}$.*

This is the main aspect of the event horizons of all known black holes metrics (Schwarzschild, Reissner–Nordström, Kerr, Kerr–Newman, Myers–Perry, topological black holes, etc – see e.g. [1, 2, 4, 6, 8, 9, 10, 11, 12, 14, 15, 23, 30, 22] and references therein). Even if it is known that any smooth event horizon (in the sense of Penrose, Hawking and Ellis) has to be a null hypersurface, it is remarkable that the traditional way to verify whether a null hypersurface has the above semi-permeability property is to consider explicit coordinate expressions for the equation of geodesics – as for instance using Boyer-Lindquist coordinates – and to check that no solution to the geodesic equations can cross the horizon in one of the two directions (see e.g. [7], [10, Appendix A], [23, Sect. 3.2]). In contrast with this habit, our result unveils that the semi-permeability property with respect to the causal geodesics is just a consequence of the fact that the horizons are null hypersurfaces: *No study of the geodesic equations is needed to establish this property!* Actually, our result *immediately* gives the following stronger property: *there is no causal world-line (not necessarily trace of a geodesic), which is appropriately time-oriented and crosses an event horizon $\mathcal{S}$ from the inner region to the outer one* ⁽¹⁾. Motivated by all this, we found convenient to consider the next two notions.

Given a Lorentzian n -manifold (M, g) , we say that a *hypersurface* $\mathcal{S} \subset M$ is a *semi-permeable barrier* (or just *barrier*, for brevity) if (i) it is time-oriented and null and (ii) it separates M into two regions, i.e. $M \setminus \mathcal{S} = M_+ \cup M_-$ with $M_{\pm}$ disjoint connected open sets, both equipped with a time-orientation which is compatible with the time-orientation of $\mathcal{S}$. By our main result *if $\mathcal{S}$ is a barrier for (M, g) , then either physical signals can go from M_+*

¹A direct proof of such stronger “no-crossing property” can be found in [13], but only for the Schwarzschild black holes. As far as we know, a simple and explicit proof for *any* black hole was so far missing.

to M_- but not from M_- to M_+ , or vice versa. Note that the only allowed direction for the crossing of signals is completely determined by the time-orientation of $\mathcal{S}$ (and hence of $M_\pm$ as well). The allowed direction is explicitly given in Theorem 3.2 below.

Consider now a time-oriented region (=connected open set) Ω_o of a Lorentzian n -manifold (M, g) and assume that its complement $\Omega_o^c = M \setminus \Omega_o$ consists of a countable union of time-oriented regions $\Omega_1, \Omega_2, \dots$ and of their boundaries $\mathcal{S}_1 = \partial\Omega_1, \mathcal{S}_2 = \partial\Omega_2, \dots$. Assume also that each $\mathcal{S}_i$ is a time-oriented barrier and that the following holds: for any index i , the time-orientations of Ω_o and Ω_i are both compatible with the time-orientation of $\mathcal{S}_i = \partial\Omega_i$ and, denoting by $M_{(i)+}, M_{(i)-}$ the two time-oriented components in which M is separated by $\mathcal{S}_i$, then either $\Omega_i = M_{(i)+}$ and $\Omega_o \subset M_{(i)-}$, or $\Omega_i = M_{(i)-}$ and $\Omega_o \subset M_{(i)+}$. Under these condition, we call a region $\Omega_i \subset \Omega_o^c$ *black* (resp. *white*) for Ω_o if no physical signal can go from Ω_i to Ω_o (resp. from Ω_o to Ω_i). By the properties of the barriers, each region Ω_i is either “black” or “white” for Ω_o .

We now recall that (see e.g. [28, 20, 31, 11, 12]), given an asymptotically flat Lorentzian manifold (M, g) , an *event horizon* of M is a connected component of the boundary of a distinguished time-oriented region $\Omega_o \subset M$, named *domain of outer communication* by Carter in [11]. It essentially consists of the collection of all points of M that are causally joinable to the infinite future boundary of the observers. The *black holes* are the connected components of the complement $\Omega^c = M \setminus \bar{\Omega}_o$. Note that, by a lemma of Penrose on achronal boundaries ([28, Lemma I, p. 214]), any event horizon is generated by segments of null geodesics so that, in the cases in which it is a smooth hypersurface, such hypersurface is a null one. Due to this, if the domain of outer communication $\Omega_o \subset M$ has a smooth boundary, then each connected component of its boundary is a barrier and the black holes for Ω_o are black regions as we defined above.

In other words, the class of black (or possibly white) regions naturally include all black (or white) holes with smooth boundaries. It is however important to keep in mind that the notion of black/white region is strictly more general than what is usually understood by the term “(smoothly bounded) black/white hole”. For instance, the complement of a Schwarzschild black hole is a white region for the inner part of the black hole, while the observers that are located outside the black hole have never been described as “being sitting inside a white hole” ⁽²⁾. Another example of a black region, which cannot be taken as a black hole, can be easily built considering the half space $M_o = \{t > 0\}$ of the Minkowski space $\mathbb{R}^{3,1}$, equipped with the flat metric g_o and the time-orientation corresponding to increasing of time-coordinate t . It suffices to observe that, for such a time-oriented Lorentzian manifold, the (correspondingly time-oriented) half light-cone $\mathcal{S} = \{t = r, t > 0\}$ ⁽³⁾ is a barrier and the region $\Omega_1 = \{t > r\}$ is a black region for $\Omega_o = M \setminus (\Omega_1 \cup \mathcal{S}) = \{0 < t < r\}$.

²Nonetheless, such a term might be meaningful if the space-time is considered under the view point of an observer sitting *inside* a black hole. About the possibility of the existence of an observer of this kind, see the nice paper [16]

³We recall that, in Penrose’s terminology, this hypersurface is a *particle horizon* [28].

Due to the fact that the class of barriers include all smooth event horizons, the task of locating the (smooth or piecewise smooth) event horizons for space-times, equipped with numerically determined metrics, might be reduced to locating null hypersurfaces which separate the space-time. In principle this is a simpler problem than the previous. In fact, determining event horizons requires to detect *all* null geodesics that will neither reach future null infinity nor hit a singularity of the space-time (see e.g. [5, § 6.2] and references therein). This requires to study systems of ode's that in principle might be quite involved. The problem of finding barriers is simpler, because a null hypersurface is locally a level set $\{F(x) = R\}$ for a smooth function $F : \mathcal{U} \subset M \rightarrow \mathbb{R}$ with null differential at such level set. This condition consists of the single equation $g_{\{F=R\}}^{-1}(dF, dF) = 0$, which in general might be numerically solved either via integrations (see e.g. [21, §2]; see also [3, 24] for interesting tricks to quickly integrate null hypersurfaces) or minimising the functional $\mathcal{F}(F, R) = \int_{\mathcal{U}} [g_{\{F(x)=R\}}^{-1}(dF, dF)]^2 \text{vol}_{g_o}$ for a Riemannian volume form vol_{g_o} in the class of pairs, determined by non-constant functions $F : \mathcal{U} \rightarrow \mathbb{R}$ on a fixed open set $\mathcal{U}$ and real constants $R \in \mathbb{R}$.

We have to admit that the strategy of determining event horizons by looking for them among the null hypersurfaces is not new – it has been frequently used in the literature. For instance, in [7] Boyer and Lindquist describe the event horizons as follows: “in stationary space-times they may *provisionally* be defined as stationary null hypersurfaces”. Much more recently, in [19], Gürses and Tekin locate dynamic horizons for a Vaidya type dynamic black hole through discussions of certain types of null hypersurfaces. However, at the best of our knowledge, such a strategy has been always considered as a sort of a heuristic approach to be later confirmed by a posteriori checks of either the semi-permeability feature for causal geodesics or the second law of black hole thermodynamics. By our results, no further check is needed to locate an event horizon (in the sense of Penrose, Hawking and Ellis) as soon as one succeeds in determining an outmost barrier.

The structure of this paper is as follows. In §2 we review in great detail various definitions and properties concerning orientations and time-orientations of hypersurfaces and curves in a Lorentzian manifold. Then we prove our main Theorem 3.5 in §3. In §4 we present illustrative examples on how barriers and event horizons can be determined using our Theorem 3.5. For the benefit of the reader, in Appendix A we briefly outline the classical definitions of “event horizon” and “black hole”, following presentations of a few well known textbooks.

2. PRELIMINARIES ON ORIENTATIONS, TIME-ORIENTATIONS AND CROSSING

Here we review basic definitions and facts that are exploited in our main result.

2.1. Oriented curves, oriented hypersurfaces and crossing. Let M be an n dimensional manifold. In this paper, we use the term *open curve* to indicate a 1-dimensional submanifold $\mathcal{C} \subset M$, which is homeomorphic to $\mathbb{R}$. Due to this, an open curve always admits a

global regular parameterisation (i.e. a smooth homeomorphism $\gamma : (a, b) \subset \mathbb{R} \longrightarrow \mathcal{C} \subset M$ with $\dot{\gamma}(s) := \frac{d\gamma}{ds}|_s \neq 0$ for any $s \in (a, b)$) ⁽⁴⁾.

Two global regular parameterisations $\gamma : (a, b) \rightarrow \mathcal{C}$, $\gamma' : (\alpha, \beta) \rightarrow \mathcal{C}$ of the same open curve are called *consonant* if there is a smooth change of parameter $h : (a, b) \rightarrow (\alpha, \beta)$ such that $\gamma(s) = \gamma'(h(s))$ with $\dot{h}(s) > 0$ at all s . The consonance relation is manifestly an equivalence relation between regular parameterisations and there are exactly two equivalence classes for such relation: If $\gamma : (a, b) \rightarrow \mathcal{C}$ is in one class, the other class is the one of $\gamma(-t)$, $t \in (-b, -a)$. Each of these equivalence classes is named *orientation* of $\mathcal{C}$ and are denoted by $[\gamma]$ and $[-\gamma]$, respectively. An *oriented (open) curve* is an open curve $\mathcal{C} \subset M$ equipped with an orientation.

Remark 2.1. The above definition of “orientation of an open curve $\mathcal{C}$ ” is perfectly equivalent to the most common one, namely to the choice of a smooth family of non-zero tangent vectors, each of them fixed up to multiplication by a positive scalar.

A hypersurface $\mathcal{S} \subset M$ is called *orientable* if there exists at least one nowhere vanishing vector field $N : \mathcal{S} \rightarrow TM|_{\mathcal{S}}$ on $\mathcal{S}$ with $N_x \in T_x M \setminus T_x \mathcal{S}$ for any x . A vector field N of this kind is called *orienting vector field* for $\mathcal{S}$. Two orienting vector fields N , N' are said to be *consonant* if there is a smooth positive real function $\lambda : \mathcal{S} \rightarrow (0, +\infty)$ and a vector field $X : \mathcal{S} \rightarrow TM|_{\mathcal{S}}$, which is tangent to $\mathcal{S}$ at all points, such that $N' = \lambda N + X$. As for the open curves, such a consonance relation is an equivalence relation, with only two equivalence classes per each orientable hypersurface $\mathcal{S}$: If N is an orienting vector field, one of the two classes is the equivalence class $[N]$ of N , the other is the class $[-N]$ containing $-N$. Each of these classes is an *orientation* of $\mathcal{S}$. An *oriented hypersurface* is an orientable hypersurface $\mathcal{S}$ equipped with an orientation $[N]$.

Remark 2.2. As before, this notion of “oriented hypersurface” is equivalent to all other common definitions, which can be found in textbooks on Differential Geometry, as e.g. [29, 32]. We adopt this one simply because it allows shorter proofs for our results.

Consider now an open curve $\mathcal{C}$ and an orientable hypersurface $\mathcal{S}$ of M . We say that $\mathcal{C}$ *intersects transversally* $\mathcal{S}$ at x_o if $x_o \in \mathcal{C} \cap \mathcal{S}$ and $T_{x_o} \mathcal{C} \cap T_{x_o} \mathcal{S} = \{0\}$. This is equivalent to require that, given a global regular parameterisation $\gamma : (a, b) \rightarrow \mathcal{C}$, if $s_o \in (a, b)$ is such that $\gamma(s_o) = x_o$, then $\dot{\gamma}(s_o) \notin T_{x_o} \mathcal{S}$. In case $\mathcal{C}$ is equipped with the orientation $[\gamma]$ and if $\mathcal{S}$ is equipped with the orientation $[N]$, we say that *the oriented curve $\mathcal{C}$ crosses $\mathcal{S}$ at x_o in the direction of $[N]$* (resp. *of $[-N]$*) if

$$\dot{\gamma}(s_o) = \lambda N_{x_o} + v \quad \text{for some } \lambda > 0 \text{ (resp. } \lambda < 0 \text{) and some } v \in T_{x_o} \mathcal{S} .$$

⁴In some textbooks – as e.g. in [27] – our “open curves” are called “world-lines”. Here we call them differently because we are going to use the term “world-line” with a more restricted meaning.

2.2. Causal world-lines and time-orientations in Lorentzian manifolds. Let (M, g) be a Lorentzian manifold of dimension $n \geq 2$ with a mostly minus signature. We recall that a subspace $\{0\} \neq V \subset T_{x_o}M$ of a tangent space $T_{x_o}M$ is called *time-like* (resp. *light-like* or *null*, *space-like*) if the restricted scalar product $g_{x_o}|_{V \times V}$ is non-degenerate and such that $g_{x_o}(v, v) > 0$ for some $v \in V$, thus Lorentzian whenever $\dim V \geq 2$ (resp. semi-negative degenerate, negative definite). A non-zero vector $0 \neq v \in T_{x_o}M$ is said to be *time-like* (resp. *light-like* or *null*, *space-like*) if it generates a time-like (resp. light-like, space-like) one dimensional subspace. A vector $v \in T_{x_o}M$ is called *causal* if it is either time-like or null.

According to this terminology, an open curve $\mathcal{C}$ (resp. a hypersurface $\mathcal{S}$) is called *time-like* (resp. *light-like* or *null*, *space-like*) if its tangent spaces are all time-like (resp. null, space-like). A curve is called *causal* if all of its non-zero tangent vectors are causal.

We recall that if $\mathcal{S} \subset M$ is a null hypersurface, then the collection of its null tangent vectors determine a distribution $\mathcal{D} \subset T_x\mathcal{S}$ of rank 1. The integral leaves of this distribution are traces of geodesics and constitute a geodesic congruence on $\mathcal{S}$ ([21]).

As we mentioned in the Introduction, a *time-orientation* of an open subset $\mathcal{U} \subset M$ is a smooth family $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathcal{U}}$ of connected components (= half cones) $\mathcal{T}_x \subset T_xM \setminus \{0\}$, $x \in \mathcal{U}$, of the cones of causal vectors $C_x = \{v \in T_xM \setminus \{0\} : g_x(v, v) \geq 0\}$. By “smooth family” we mean that for any $x \in \mathcal{U}$, there exists a smooth causal vector field Y on a neighbourhood of x , such that, at all points y on which Y is defined, the half-cone $\mathcal{T}_y$ coincides with the connected component of C_y to which Y_y belongs. The notions of time-orientability and time-orientation immediately extend to time-like hypersurfaces, because they are Lorentzian if equipped with the induced metric. As we already remarked, they extend to null hypersurfaces too, provided that the following definition is adopted.

Definition 2.3. Let $\mathcal{S} \subset M$ be a null hypersurface and denote by $\mathcal{D} \subset T\mathcal{S}$ its 1-distribution of null tangent vectors. We say that $\mathcal{S}$ is *time-orientable* if there exists a nowhere vanishing vector field E in $\mathcal{D}$. In this case, we call *time-orientation for $\mathcal{S}$ determined by E* the family $\mathcal{T}^{(\mathcal{S})} = \{\mathcal{T}_x\}_{x \in \mathcal{S}}$ of the half-cones $\mathcal{T}_x$ of the causal cones $C_x \subset T_xM \setminus \{0\}$, $x \in \mathcal{S}$, which contain the vectors E_x .

Note that if $\mathcal{S}$ is null and time-orientable, then it admits exactly two time-orientations.

Remark 2.4. We stress the fact that *a null hypersurface $\mathcal{S}$ is time-orientable if and only if it is orientable as a hypersurface*. Indeed, if there exists a nowhere vanishing smooth vector field $E \in \mathcal{D}$, at each $x \in \mathcal{S}$ we may consider a null vector $E_x^\perp \in T_xM \setminus T_x\mathcal{S}$ that satisfies $g(E_x, E_x^\perp) = 1$. Since this vector can be chosen smoothly depending on x , this gives an orienting vector field $N = E^\perp$ for the hypersurface $\mathcal{S}$. Conversely, if $\mathcal{S}$ admits an orienting vector field $N \in TM|_{\mathcal{S}} \setminus T\mathcal{S}$, there exists a space-like vector field $X \in T\mathcal{S}$ such that $E^\perp = N - X$ is null (this can be checked working with the expansion of N in terms of locally defined frame field $(e_i)_{i=1, \dots, n}$ with vector fields $e_3, \dots, e_n$ space-like and tangent to $\mathcal{S}$ at the points of the hypersurface). Thus also $E^\perp|_{\mathcal{S}}$ is an orienting vector field for $\mathcal{S}$ and there is a nowhere vanishing null vector field E in $\mathcal{D} \subset T\mathcal{S}$ with $g(E_x^\perp, E_x) = 1$ at any $x \in \mathcal{S}$. This means that $\mathcal{S}$ is time-orientable.

We conclude this preliminary section with the next crucial definition.

Definition 2.5. A *causal world-line* (or just *world-line*, for short) in an open set $\mathcal{U} \subset M$ is a causal open curve $\mathcal{C} \subset \mathcal{U}$. If a world-line $\mathcal{C}$ is equipped with an orientation $[\gamma]$ and if there is a time orientation $\mathcal{T}$ on an open subset $\mathcal{U}' \subset \mathcal{U}$ with non-trivial intersection with $\mathcal{C}$, we say that $\mathcal{C}$ is $\mathcal{T}$ -*time oriented* (resp. $-\mathcal{T}$ -*time oriented*) if for one (and hence for any) point $x = \gamma(s) \in \mathcal{C} \cap \mathcal{U}'$, the non-zero vector $\dot{\gamma}(s)$ (resp. $-\dot{\gamma}(s)$) is in $\mathcal{T}_{x=\gamma(s)}$.

3. THE SEMI-PERMEABILITY PROPERTY OF NULL HYPERSURFACES

We start with the following useful notion.

Definition 3.1. Let (M, g) be a Lorentzian n -manifold with mostly negative signature. A *dressed barrier* of M is a pair $(\mathcal{S}, \mathbb{T})$ given by

- a time-orientable null hypersurface $\mathcal{S} \subset M$ and
- a smooth vector field $\mathbb{T}$ on M

satisfying the following conditions:

- (a) $M \setminus \mathcal{S}$ consists of exactly two connected components, say M_+ and M_- ;
- (b) for any $y \in \mathcal{S}$ the vector $\mathbb{T}_y$ is non-zero, null and tangent to $\mathcal{S}$;
- (c) for any $y \in M_+$ the vector $\mathbb{T}_y$ is non-zero and time-like.

The vector field $\mathbb{T}$ is *the type switching field* of the dressed barrier, while the component M_+ , on which $\mathbb{T}$ is time-like, is called the *time-like region* for $\mathbb{T}$.

The relevance of this notion comes from the fact that it allows to prove our Theorem 3.5 in two steps. At first we show that if $(\mathcal{S}, \mathbb{T})$ is a dressed barrier, then any oriented causal world-line, which has positive time-orientation with respect to the time-orientation $\mathcal{T}$ on M_+ determined by $\mathbb{T}|_{M_+}$, may cross $\mathcal{S}$ in just one of the two possible directions. In the second step, we prove that if $\mathcal{S}$ is a null hypersurface (not necessarily a dressed barrier), then for any $x_o \in \mathcal{S}$ there exist a neighbourhood $\mathcal{U}$ and a vector field $\mathbb{T}$ on $\mathcal{U}$, which makes the pair $(\mathcal{S} \cap \mathcal{U}, \mathbb{T})$ a dressed barrier for the Lorentzian manifold $(\mathcal{U}, g|_{\mathcal{U}})$. From the result of the first step, it immediately follows that any appropriately oriented causal world-line may cross $\mathcal{S}$ at x_o in only one direction, the other being forbidden.

3.1. The semi-permeability of the dressed barriers. The semi-permeability property of dressed barriers is a direct consequence of the following

Lemma 3.2. Let $(\mathcal{S}, \mathbb{T})$ be a dressed barrier in a Lorentzian n -manifold (M, g) , $n \geq 2$ with mostly minus signature. Let $\mathcal{T}$ be the time orientation of M_+ determined by $\mathbb{T}|_{M_+}$ and denote by $[E^\perp]$ the orientation of $\mathcal{S}$, given by a vector field $E^\perp$ on a neighbourhood $\mathcal{V} \subset \mathcal{S}$ of a point $x_o \in \mathcal{S}$ satisfying the following pair of conditions:

$$g_{x_o}(E_{x_o}^\perp, E_{x_o}^\perp) = 0, \quad g_{x_o}(\mathbb{T}_{x_o}, E_{x_o}^\perp) = -1. \quad (3.1)$$

Then there is no time-like world-line $\mathcal{C}$ in M , which is $\mathcal{T}$ -time oriented in the region M_+ and crosses $\mathcal{S}$ at x_o in the direction of $[E^\perp]$.

Proof. Aiming to a contradiction, let us assume that there is a time-like world-line $\mathcal{C}$ crossing $\mathcal{S}$ at x_o in the direction of $[E^\perp]$ and $\mathcal{T}$ -time oriented at the points of M_+ . Let $\gamma : (a, b) \rightarrow \mathcal{C}$ be a global regular parameterisation of $\mathcal{C}$, which is consonant with its orientation, and denote by $\mathcal{B} = (e_0, e_1, e_2, \dots, e_{n-1})$ a linear frame for $T_{x_o}M$ satisfying the following conditions:

- (a) the vectors $e_2, \dots, e_{n-1}$ span a space-like subspace W of $T_{x_o}\mathcal{S}$ and are g_{x_o} -orthonormal;
- (b) $e_1 := \mathbb{T}_{x_o}$ (and hence it is a generator for the kernel of $g_{x_o}|_{T_{x_o}\mathcal{S} \times T_{x_o}\mathcal{S}}$);
- (c) e_0 is null, g_{x_o} -orthogonal to W and such that $g_{x_o}(e_0, e_1) = -1$.

By (3.1), the vector

$$w := E_{x_o}^\perp - e_0$$

is g_{x_o} -orthogonal to $e_1 = \mathbb{T}_{x_o}$ and hence it is in the space $\mathbb{T}_{x_o}^\perp = \text{span}\{e_1, e_2, \dots, e_{n-1}\}$. It follows that, if we consider a smooth vector field $\tilde{E}^\perp : \mathcal{V} \subset \mathcal{S} \rightarrow TM$ on a neighbourhood $\tilde{\mathcal{V}}$ of x_o which satisfies $\tilde{E}_{x_o}^\perp = e_0$, then at the points of a sufficiently small $\mathcal{V}$ the vector fields $E^\perp$ and $\tilde{E}^\perp$ are consonant and giving the same orientation $[E^\perp] = [\tilde{E}^\perp]$ for $\mathcal{S} \cap \mathcal{V}$.

Let us now denote by s_o the value of the parameter of γ , corresponding to $x_o = \gamma(s_o)$, and let $\dot{\gamma}(s_o) = v^0 e_0 + v^1 e_1 + \sum_{j=2}^{n-1} v^j e_j$ be the expansion of $\dot{\gamma}(s_o)$ in terms of the frame $\mathcal{B}$. From the fact that $\dot{\gamma}(s_o)$ is time-like and from the assumptions on $\mathcal{B} = (e_i)$, we have that

$$0 < g(\dot{\gamma}(s_o), \dot{\gamma}(s_o)) = -2v^0 v^1 + \sum_{j=2}^{n-1} (v^j)^2 g_{x_o}(e_j, e_j) \leq -2v^0 v^1. \quad (3.2)$$

We claim that the component v^1 is positive. In order to check this, we first observe that, due to the $\mathcal{T}$ -time orientation of $\mathcal{C}$, for any $s \in (a, b)$ that correspond to a point $y = \gamma(s)$ in M_+ , the tangent vector $\dot{\gamma}(s) \in T_y M$ is in the same causal half-cone containing $\mathbb{T}_y$. On the other hand, for each $s \in (s_o - \varepsilon, s_o + \varepsilon)$ in a sufficiently small neighbourhood of s_o , we may consider an n -tuple $\tilde{\mathcal{B}}_y = (f_0(y), f_1(y), f_2(y), \dots, f_{n-1}(y))$ of vectors in $T_{y=\gamma(s)}M$ constructed as follows:

- the vectors $f_0(y), f_2(y), \dots, f_{n-1}(y)$ are obtained by parallel transport along the parameterised curve γ from the vectors $e_0, e_2, \dots, e_{n-1} \in T_{x_o}M$;
- $f_1(y) := \mathbb{T}_y$.

Since parallel transport preserves scalar products, we have that

- $g_y(f_0(y), f_0(y)) = 0$,
- $g_y(f_i(y), f_j(y)) = -\delta_{ij}$ for any $2 \leq i, j \leq n-1$;
- $g_y(f_0(y), f_j(y)) = 0$ for any $2 \leq j \leq n-1$.

Moreover, when s tends to s_o , each vector of the n -tuple $\tilde{\mathcal{B}}_{y=\gamma(s)}$ tends to the corresponding element of $\mathcal{B} = (e_0, e_1, \dots, e_{n-1})$. Thus, if $\varepsilon > 0$ is sufficiently small, each n -tuple $\tilde{\mathcal{B}}_{y=\gamma(s)}$, $s \in (s_o - \varepsilon, s_o + \varepsilon)$, is linearly independent and is therefore a linear frame for $T_y M$. Due to this, a vector w of a tangent space $T_y M$ at some $y = \gamma(s) \in M_+ \cap \gamma((s_o - \varepsilon, s_o + \varepsilon))$, is time-like if and only if it has the form $w = w^0 f_0(y) + w^1 \mathbb{T}_y + w^2 f_2(y) + \dots + w^{n-1} f_{n-1}(y)$

with components w^i satisfying the inequality

$$(w^1)^2 g(\mathbb{T}_y, \mathbb{T}_y) + 2w^1 \left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right) - \sum_{j=2}^{n-1} (w^j)^2 > 0 .$$

Since $g(\mathbb{T}_y, \mathbb{T}_y) > 0$ for any $y \in M_+$, this is equivalent to

$$\begin{aligned} & \left(g(\mathbb{T}_y, \mathbb{T}_y)w^1 + \left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right) \right)^2 > \\ & \quad > \left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right)^2 + g(\mathbb{T}_y, \mathbb{T}_y) \sum_{j=2}^{n-1} (w^j)^2 . \end{aligned}$$

It follows that the two connected components of the set of non-zero time-like vectors in such tangent spaces are characterised by either one of the following conditions:

$$\begin{aligned} g(\mathbb{T}_y, \mathbb{T}_y)w^1 &> - \left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right) + \\ &+ \sqrt{\left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right)^2 + g(\mathbb{T}_y, \mathbb{T}_y) \sum_{j=2}^{n-1} (w^j)^2} , \\ &\text{or} \end{aligned} \tag{3.3}$$

$$\begin{aligned} g(\mathbb{T}_y, \mathbb{T}_y)w^1 &< - \left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right) - \\ &- \sqrt{\left(g_y(\mathbb{T}_y, f_0(y))w^0 + \sum_{k=2}^{n-1} g_y(\mathbb{T}_y, f_k(y))w^k \right)^2 + g(\mathbb{T}_y, \mathbb{T}_y) \sum_{j=2}^{n-1} (w^j)^2} . \end{aligned} \tag{3.4}$$

Since at the points $y \in M_+$ the right hand side of (3.3) is non-negative, while the right hand side of (3.4) is non-positive, the half-cones of the time-like vectors are distinguished by the condition $w^1 > 0$ or the condition $w^1 < 0$. This implies that, for any $s \neq s_o$ in a sufficiently small neighbourhood of s_o and with $y = \gamma(s)$ in M_+ , the condition that the vector $\dot{\gamma}(s)$ is in the same causal half-cone of $\mathbb{T}_{y=\gamma(s)}$ is equivalent to demanding that its component w^1 in the direction of the vector $f_0(y) = \mathbb{T}_y$ is strictly positive.

We are now ready to conclude the proof that the component v^1 of $\dot{\gamma}(s_o)$ is positive. Indeed, if $v^1 < 0$, by continuity, the same should hold for the component w^1 of any tangent vector $\dot{\gamma}(s)$, with $s \neq s_o$ sufficiently close to s_o . But this would contradict the above remarks and the hypothesis on the time orientation of $\mathcal{C}$. Combining this with the fact that $v^1 \neq 0$ by (3.2), we get that $v^1 > 0$.

We may now conclude the proof. Indeed, since $v^1 > 0$, (3.2) implies that $v^0 < 0$ and thus that $\mathcal{C}$ crosses $\mathcal{S}$ in the direction $[-\tilde{E}^\perp]$, contradicting the hypothesis that $\mathcal{C}$ crosses $\mathcal{S}$ at x_o in the direction of $[\tilde{E}^\perp] = [E^\perp]$. $\square$

This result has the following corollary, which concludes the first step.

Corollary 3.3. *Under the hypotheses of Theorem 3.2, there is no causal world-line $\mathcal{C} \subset \mathcal{U}$, which is $\mathcal{T}$ -time oriented and crosses $\mathcal{S}$ at x_o in the direction of $[E^\perp]$.*

Proof. We only need to show that there is no causal world-line $\mathcal{C}$ which is null at x_o , $\mathcal{T}$ -time oriented and crossing $\mathcal{S}$ in the direction of $[E^\perp]$. Assume on the contrary that such a curve does exist and pick a regular parameterisation $\gamma : (s_o - \varepsilon, s_o + \varepsilon) \rightarrow \mathcal{J} \subset \mathcal{C}$ for a neighbourhood $\mathcal{J}$ of $x_o = \gamma(s_o)$. Denote $v_o := \dot{\gamma}(s_o) \neq 0$. Since v_o is null, there exists a sequence of time-like vectors $v_k \in T_{x_o}M$ with $\lim_{k \rightarrow \infty} v_k = v_o$. Using the same frame $\mathcal{B} = (e_0, \dots, e_{n-1})$ for $T_{x_o}M$ of the proof of Lemma 3.2, we get that v_o and the v_k 's uniquely decompose as $v_o = \lambda_o E_{x_o}^\perp + w$ and $v_k = \lambda_k E_{x_o}^\perp + w_k$, respectively, for some $w, w_k \in T_{x_o}\mathcal{S}$. Since $\lambda_o > 0$ by hypothesis, we also have that $\lambda_k > 0$ for any k sufficiently large. Working in coordinates, one can construct a sequence of smooth parameterised curves $\gamma_k : (s_o - \varepsilon, s_o + \varepsilon) \rightarrow \mathcal{U}$, which converge uniformly on closed intervals to the map $\gamma : (s_o - \varepsilon, s_o + \varepsilon) \rightarrow \mathcal{U}$ and such that $\gamma_k(s_o) = x_o$, $\dot{\gamma}_k(s_o) = v_k$. By construction, for k sufficiently large, there are neighbourhoods $(s_o - \varepsilon_k, s_o + \varepsilon_k)$ of s_o where the restrictions $\gamma_k|_{(s_o - \varepsilon_k, s_o + \varepsilon_k)}$ are regular parameterisations of time-like world-lines and with the property that such parameterisations are consonant with the $\mathcal{T}$ -time orientation of $\mathcal{U}_+$. Thus, all such curves are time-like world-lines, which are $\mathcal{T}$ -time oriented and crossing $\mathcal{S}$ in the direction $[E^\perp]$. This cannot be by Lemma 3.2. $\square$

3.2. The semi-permeability property of time-oriented null hypersurfaces. The second step of our proof basically consists of the following

Lemma 3.4. *Let (M, g) be a Lorentzian n -manifold with mostly negative signature and $\mathcal{S} \subset M$ a null hypersurface, which is time-oriented by a nowhere vanishing null vector field $E \in T\mathcal{S}$. For any $x_o \in \mathcal{S}$, there exist a neighbourhood $\mathcal{U} \subset M$ of x_o and a vector field $\mathbb{T}$ on $\mathcal{U}$, such that $\mathbb{T}|_{\mathcal{U} \cap \mathcal{S}} = E|_{\mathcal{U}}$ and $(\mathcal{S} \cap \mathcal{U}, \mathbb{T})$ is a dressed barrier for the Lorentzian manifold $(\mathcal{U}, g|_{\mathcal{U}})$.*

Proof. Consider a neighbourhood $\mathcal{U}$ of x_o , on which there is a nowhere vanishing null vector field $\tilde{E}$ which coincides with $E|_{\mathcal{U} \cap \mathcal{S}}$ at the points of $\mathcal{U} \cap \mathcal{S}$. We may also assume that on $\mathcal{U}$ there is a null vector field $\tilde{E}^\perp$ satisfying the condition $g(\tilde{E}, \tilde{E}^\perp) \equiv -1$ and that there is a system of coordinates $(x^1, \dots, x^{n-2}, u, v)$ on $\mathcal{U}$, such that $\mathcal{S} \cap \mathcal{U} = \{v = 0\}$. Let

$$\mathcal{U}_+ = \{x \in \mathcal{U} : v > 0\}, \quad \mathcal{U}_- = \{x \in \mathcal{U} : v < 0\}$$

and denote by $\mathbb{T}$ the vector field on $\mathcal{U}$ defined by

$$\mathbb{T} = \tilde{E} - v\tilde{E}^\perp. \quad (3.5)$$

By construction, for any $x \in \mathcal{S} \cap \mathcal{U}$, the vector $\tilde{E}_x = E_x$ is tangent to $\mathcal{S}$ and for any $y \in (\mathcal{S} \cap \mathcal{U}) \cup \mathcal{U}_+$ we have that

$$g_y(\mathbb{T}, \mathbb{T}) = -2v(y)g_y(\tilde{E}, \tilde{E}^\perp) = 2v(y) \geq 0$$

with $g_y(\mathbb{T}, \mathbb{T}) = 0$ if and only if $y \in \mathcal{S} \cap \mathcal{U}$.

It follows that $(\mathcal{S} \cap \mathcal{U}, \mathbb{T})$ satisfies all conditions to be a dressed barrier for $(\mathcal{U}, g|_{\mathcal{U}})$. $\square$

We are now able to prove our main result.

Theorem 3.5. *Let (M, g) be a Lorentzian n -manifold with mostly negative signature and $\mathcal{S} \subset M$ an orientable null hypersurface, equipped with the time-orientation $\mathcal{T}^{(\mathcal{S})}$ determined by a nowhere vanishing null vector field $E \in T\mathcal{S}$. Denote by $E^\perp \in TM|_{\mathcal{S}} \setminus T\mathcal{S}$ a null vector field such that*

$$g_x(E^\perp, E) = -1 \quad \text{for any } x \in \mathcal{S}. \quad (3.6)$$

Then no oriented causal world-line $\mathcal{C} \subset M$ satisfies all of the following three conditions:

- (a) *it intersects transversally $\mathcal{S}$ at the point $x_o = \mathcal{C} \cap \mathcal{S}$;*
- (b) *the velocity $v = \dot{\gamma}(s_o)$ at x_o of a parameterisation γ of its orientation $[\gamma]$ is in $\mathcal{T}_{x_o}^{(\mathcal{S})}$;*
- (c) *it crosses $\mathcal{S}$ at x_o in the direction of $[E^\perp]$.*

Proof. Assume that there is an oriented causal world-line $\mathcal{C}$ satisfying (a) – (c) and consider a neighbourhood $\mathcal{U}$ of x_o and a vector field $\mathbb{T}$ on $\mathcal{U}$ as in Lemma 3.4. Denote also by $\mathcal{T}$ the time-orientation of $\mathcal{U}_+$ determined by $\mathbb{T}|_{\mathcal{U}_+}$. Then the oriented arc $\mathcal{C}' = \mathcal{C} \cap \mathcal{U}$ is an oriented causal world-line for the Lorentzian manifold $(\mathcal{U}, g|_{\mathcal{U}})$. From (b) and the fact that $\mathbb{T}|_{\mathcal{U} \cap \mathcal{S}} = E|_{\mathcal{U} \cap \mathcal{S}}$, the orientation of $\mathcal{C}'$ gives a $\mathcal{T}$ -time orientation on $\mathcal{C}' \cap \mathcal{U}_+$, while by (a) and (c) $\mathcal{C}'$ crosses $\mathcal{S} \cap \mathcal{U}$ in the direction of $[E^\perp]$. This contradicts Corollary 3.3. $\square$

4. A FEW (CLASSICAL AND NOT) EXAMPLES OF BLACK HOLES AND BARRIERS

As we defined in the Introduction, a null hypersurface of a Lorentzian n -manifold (M, g) is called *(semi-permeable) barrier* if (a) it is equipped with a time-orientation $\mathcal{T}^{(\mathcal{S})}$ determined by a no-where vanishing null vector field E in $T\mathcal{S}$, and (b) separates M into two connected components, i.e. $M \setminus \mathcal{S} = M_- \cup M_+$ where M_+ and M_- are disjoint connected regions.

By Theorem 3.5, *if the signature of g is mostly minus and if $E^\perp$ is a null vector field at the points of a barrier $\mathcal{S}$, which takes values in $TM|_{\mathcal{S}} \setminus T\mathcal{S}$ and satisfies (3.6), then no appropriately oriented causal world-line can cross the barrier $\mathcal{S}$ in the direction of $[E^\perp]$.* On the other hand, *in case the signature is mostly plus, by just reversing the sign of the metric, one immediately gets that the forbidden direction is $[-E^\perp]$ (instead of $[E^\perp]$).*

Note that these two rules give an immediate way to determine whether a smoothly bounded region Ω in the complement $M \setminus \Omega_o$ of a time-oriented region Ω_o is *black* or *white* for Ω_o according to the definitions given in the Introduction.

As we mentioned above, the semi-permeability property of the barriers is the well known crucial property of Penrose, Hawking and Ellis' event horizons (see Appendix A). Indeed,

our main result provides an extremely simple method to determine the event horizons for all known examples of static and Vaidya-type black holes. In this section we illustrate it in the cases of Kerr-Newman black holes and a certain kind of Myers-Perry black holes. One can directly check that the method works equally well for any other known models of black holes. Finally, we will use the same method to determine very quickly the barriers for the large two new classes of Einstein metric recently determined in [18] and which naturally generalise the event horizons of the classical Kerr metrics of rotating black holes.

Before introducing these various examples, we just need to recall that one of the simplest way to check whether a given hypersurfaces $\mathcal{S}$ is time-like (resp. space-like, null) is to verify whether, for any $x \in \mathcal{S}$, there exists a locally defined real function $F : \mathcal{U} \subset M \longrightarrow \mathbb{R}$ on a neighbourhood $\mathcal{U}$ of x satisfying the following two conditions

- $\mathcal{S} \cap \mathcal{U} = \{F = R\}$ for some constant R and
- the 1-form dF is time-like (resp. space-like, null) at the points of $\mathcal{S}$ i.e.

$$g^{-1}(dF, dF)|_{\{F=R\}} > 0 \quad \left(\text{resp. } < 0, = 0 \right). \quad (4.1)$$

4.1. The inner and outer barriers of Kerr-Newman black holes. The Kerr-Newman metrics ([25, 26, 10, 15]) can be considered as the Lorentzian metrics g_{KN} on the manifold $M = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$, which take the following form in cylindrical *Kerr coordinates* $((\xi, \psi, \rho); v)$ (see e.g. [15, eq. (7.4)]) ⁽⁵⁾.

$$\begin{aligned} g_{KN} := & -\frac{1}{2}(\rho^2 + a^2 \cos^2 \xi) (d\xi^2 + \sin^2 \xi d\psi^2) - \\ & - (dv - a \sin^2 \xi d\psi) \vee (d\rho - a \sin^2 \xi d\psi) + \frac{1}{2} \left(1 - \frac{2m\rho - e^2}{\rho^2 + a^2 \cos^2 \xi} \right) (dv - a \sin^2 \xi d\psi)^2, \end{aligned} \quad (4.2)$$

where m , a and e are three constants, which are physically interpreted as mass, angular momentum per mass unit and electric charge, respectively, of the black hole. We discuss just the case in which m , a and e satisfy the inequality $a^2 + e^2 < m^2$. Under this assumption, let us look for a null hypersurface of the form $\mathcal{S}_R = \{\rho = R\}$ for some real constant R . Such a hypersurface (if it exists) is clearly diffeomorphic to $S^2 \times \mathbb{R}$ (and thus orientable and time-orientable) and separates $M = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ into two connected components. In particular *any such hypersurface, equipped with a time-orientation, is a barrier*.

A simple computation shows that the real function $F(\xi, \psi, \rho, v) := \rho$ is such that for any real constant R

$$g_{KN}^{-1}(dF, dF)|_{\{F=R\}} = 2 \frac{-a^2 - R^2 + 2mR - e^2}{R^2 + a^2 \cos^2 \xi}.$$

⁵Note that (4.2) is not exactly the metric appearing in [15], but it is scaled by the factor $-\frac{1}{2}$. In this way the signature changes from mostly plus to mostly minus and the metric is normalised in a way that allows an easier comparison with the metrics in [18], which are discussed later in this section.

Thus, the hypersurface $\mathcal{S}_R = \{F = R\}$ is null if and only if R makes the numerator equal to zero, i.e. if and only if $R = R_{\pm}$ with

$$\text{either} \quad R_+ = m + \sqrt{m^2 - a^2 - e^2} \quad \text{or} \quad R_- = m - \sqrt{m^2 - a^2 - e^2} .$$

By the above remarks $\mathcal{S}_{R_+}$ and $\mathcal{S}_{R_-}$ are barriers with null distributions generated by the vector fields

$$\begin{aligned} E_{(+)} &:= -\frac{\partial}{\partial\psi} + \frac{2}{a}(m + \sqrt{m^2 - a^2 - Q^2})\frac{\partial}{\partial v}\Big|_{\mathcal{S}_{R_+}} , \\ E_{(-)} &:= -\frac{\partial}{\partial\psi} + \frac{2}{a}(m - \sqrt{m^2 - a^2 - Q^2})\frac{\partial}{\partial v}\Big|_{\mathcal{S}_{R_-}} , \end{aligned} \quad (4.3)$$

respectively. Since $\frac{\partial}{\partial\rho}$ is transversal to $\mathcal{S}_{R_+}$ and such that $g\left(\frac{\partial}{\partial\rho}, E_{(+)}\right) < 0$, it follows that $\frac{\partial}{\partial\rho}|_{\mathcal{S}_{(+)}}$ and the vector field $E_{(+)}^\perp \in TM|_{\mathcal{S}_{(+)}} \setminus T\mathcal{S}_{(+)}$ defined in (3.6) point towards the same side of $\mathcal{S}_{(+)}$. Hence if we time-orient $\mathcal{S}_{(+)}$ using the vector field $E_{(+)}$, by Theorem 3.5 we get that $\mathcal{S}_{(+)}$ is a barrier for (M, g_{KN}) with the property that no signal can pass from the region $\Omega_1 := \{\rho < R_+\}$ into the region $\Omega_o := \{\rho > R_+\}$. In fact, $\mathcal{S}_{(+)}$ is *nothing but the well known outer event horizon of the Kerr-Newman black hole*.

A similar discussion shows that $\mathcal{S}_{(-)}$, equipped with either one of its two possible time-orientations, is a barrier. In fact it is what is usually called *inner horizon*.

4.2. Barriers and horizons of Myers and Perry's n-dimensional black holes. One of the simplest version of a Myers and Perry's spinning uncharged n -dimensional black holes are given by the Lorentzian metrics g_{MP} on the manifold $M = S^{n-4} \times (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$, which in coordinates $((\theta_1, \dots, \theta_{n-4}), (\xi, \psi, \rho), v)$, with θ_i coordinates for S^{n-4} and (ξ, ψ, ρ) spherical coordinates for $\mathbb{R}^3 \setminus \{0\}$, take the form (see [23, eq. (3.1)]) ⁽⁶⁾.

$$\begin{aligned} g_{MP} &= -\frac{1}{2}(\rho^2 + a^2 \cos^2 \xi)(dx^2 + \sin^2 \xi d\psi^2) + \rho^2 \cos^2 \xi g_{S^{n-4}} - \\ &- (dv - a \sin^2 \xi d\psi) \vee (d\rho - a \sin^2 \xi d\psi) + \frac{1}{2} \left(1 - \frac{m}{\rho^{n-5}(\rho^2 + a^2 \cos^2 \xi)} \right) (dv - a \sin^2 \xi d\psi)^2 , \end{aligned} \quad (4.4)$$

where m and a are constants, physically interpreted as mass and angular momentum per mass unit of the black hole, and $g_{S^{n-4}}$ is the standard round metric of S^{n-4} . The time-oriented null hypersurfaces of the form $\mathcal{S}_R = \{\rho = R\}$ for some real constant R are barriers by the same observations we made for the Kerr-Newman metric. These hypersurfaces are those for which the function $F(\theta_i, \xi, \psi, \rho, v) := \rho$ satisfies the condition

⁶As for (4.2), this is the expression given in [23] scaled by the factor $-\frac{1}{2}$.

$g_{MP}^{-1}(dF, dF)|_{\{F=R\}} \equiv 0$. A direct check shows that this condition has the form

$$0 = g_{MP}^{-1}(dF, dF)|_{\{F=0\}} = G(\theta_i, \xi, \psi, R)(R^{n-5}(R^2 + a^2) - m) \quad (4.5)$$

where $G(\theta_i, \xi, \psi, R)$ is an appropriate nowhere vanishing function on M .

Since $f(R) := R^{n-5}(R^2 + a^2) - m$ is monotone increasing on $[0, +\infty)$ and with $\lim_{R \rightarrow \infty} f(R) = +\infty$, there is either none or exactly one positive solution to the equation (4.5), depending on whether $\lim_{R \rightarrow 0^+} f(R)$ is non-negative or negative, respectively. If $n \geq 6$, this limit is surely negative (as it has been observed in [23]), while if $n = 5$, the limit is negative if and only if $a^2 < m$. In these cases, the unique solution R_o to (4.5) determines a barrier $\mathcal{S}_{R_o}$. Indeed it is the event horizon of a Myers and Perry's black hole (or of a Myers and Perry's white hole, depending on which time-orientation is considered).

Similar discussions allow to determine the event horizons of Myers and Perry's black holes for other metrics considered in [23, § 3.1].

4.3. Barriers in Ricci flat manifolds with Kerr type optical structures. Let M be a 4-dimensional manifold of the form $M = \mathcal{U} \times \mathbb{R}^2 \subset \mathbb{C} \times \mathbb{R}^2$, with $\mathcal{U} \subset \mathbb{C}$ open and simply connected, equipped with coordinates (x, y, v, ρ) among which x, y are the real and imaginary parts of the standard complex coordinate $z = x + iy$ of $\mathcal{U} \subset \mathbb{C}$ and (v, ρ) are the standard coordinates of $\mathbb{R}^2$. Denote also by κ an integer which can be either $+1$ or -1 and let $m \geq 0$ be a non-negative real constant. In [18] it was proved that, for any nowhere vanishing $\varphi : \mathcal{U} \rightarrow \mathbb{R}$ with $\kappa\varphi < 0$ and solution to the elliptic equation

$$\Delta\varphi + \kappa \frac{8\varphi}{(1 + \kappa(x^2 + y^2))^2} = 0, \quad (4.6)$$

the Lorentzian metric on M

$$g^{(\kappa, \varphi)} := -2\kappa \frac{\rho^2 + \varphi^2}{(1 + \kappa(x^2 + y^2))^2} (dx^2 + dy^2) + \left(dv - \frac{\partial\varphi}{\partial y} dx + \frac{\partial\varphi}{\partial x} dy \right) \vee \left(dv + d\rho + \frac{1}{2} \left(-1 + \frac{2m\rho}{\rho^2 + \varphi^2} \right) \left(dv - \frac{\partial\varphi}{\partial y} dx + \frac{\partial\varphi}{\partial x} dy \right) \right), \quad (4.7)$$

is Ricci flat.

The Einstein metrics of this kind with $\kappa = +1$ constitute a very large class, which properly include the metrics of the Kerr black holes. Those with $\kappa = -1$ give an equally large class and can be taken as their “hyperbolic” analogs. Both classes have been characterised in [18] as the only Ricci flat metrics which are compatible with a special type of optical structures on M , called *of Kerr type*, and satisfy a natural set of conditions that encode some of the most important features of Kerr metrics (see [18] for details).

Explicit solutions to (4.6) – and, consequently, explicit expressions for the corresponding metrics $g^{(\kappa, \varphi)}$ – can be determined as follows. It was remarked in [18] that the solutions to (4.6) on a disk $\overline{\mathbb{D}(0, r_o)} := \{\sqrt{x^2 + y^2} \leq r_o\}$, $0 < r_o < 1$, are real analytic up to the boundary and in one-to-one correspondence with the Fourier series of real analytic

boundary data on $\partial\mathbb{D}(0, r_o)$. In fact, using polar coordinates (r, ψ) for the (x, y) -plane (so that $(x = r \cos \psi, y = r \sin \psi)$) and considering the Fourier expansion $f(\psi) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\psi) + b_n \sin(n\psi))$ of a non-negative real analytic function $f(\psi)$ on $\partial\mathbb{D}(0, r_o)$, $r_o < 1$, the corresponding solution φ to (4.6) with $\kappa = +1$ (resp. $\kappa = -1$) is the sum of the series

$$\varphi(r, \psi) = - \left(a_0 \frac{\varphi_0(r)}{\varphi_0(r_o)} + \sum_{n=1}^{\infty} \frac{\varphi_n(r)}{\varphi_n(r_o)} (a_n \cos(n\psi) + b_n \sin(n\psi)) \right), \quad (4.8)$$

with $\varphi_n(r) := \left(1 - \frac{2r^2}{(1+r^2)} \frac{1}{n+1} \right) r^n$ (resp. $\varphi_n(r) := \left(\frac{1+r^2}{1-r^2} + n \right) r^n$).

Up to isometries, the classical Kerr metrics are those of the family with $\kappa = +1$ and for which the function φ is as in (4.8) with $a_n = b_n = 0$ for any $n \geq 2$ ([18, §6.2.1]).

Inspired by how the event horizons of Kerr-Newman metrics can be determined, one can look for barriers of the form $\mathcal{S}_R = \{F = 0\}$ for some real F of the form

$$F(x, y, \rho, v) := \rho - R(x, y, v) \quad \text{for a smooth function of three variables } R(x, y, v).$$

These hypersurfaces are barriers if and only if $R(x, y, v)$ satisfies the first order differential constraint

$$\begin{aligned} g^{(\kappa, \varphi)-1} \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial v} dv, \frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial v} dv \right) \Big|_{\rho=R(x, y, v)} &= \\ &= g^{(\kappa, \varphi)-1}(d\rho, d\rho) \Big|_{\rho=R(x, y, v)}. \end{aligned} \quad (4.9)$$

This is a real analytic differential equation, which is quadratic in the partial derivatives $\frac{\partial R}{\partial x}$, $\frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial v}$ and admits at least two distinct solutions if $\kappa = +1$ and $\varphi(x, y)$ corresponds to a Kerr metric. It is therefore reasonable to expect that whenever $\varphi(x, y)$ is sufficiently close (in an appropriate norm) to a function for a Kerr metric, then (4.9) still admits two distinct solutions, possibly diverging at some points.

On the other hand, we recall that the definition of barriers in a Lorentzian manifold is completely coordinate-free. This yields that the class of barriers is preserved by any isometry of the space-time that contains them. Thus studies on the (local) existence and on the geometric properties of the solutions to (4.9) might determine useful information on the space-times $(M, g^{(\kappa, \varphi)})$ and how the event horizons of classical Kerr black holes might be deformed. Investigations of this kind are left to future work.

APPENDIX A. PENROSE, HAWKING AND ELLIS' DEFINITIONS OF BLACK HOLES AND EVENT HORIZONS

Here is a very short presentation of the notions of *black hole* and *event horizon*, as it can be found in classical papers and textbooks as e.g. [28, 20, 31, 11, 12]. At the end, we also give a direct proof of the fact that any smooth event horizon is necessarily a null hypersurface.

Let (M, g) be a Lorentzian n -manifold admitting a conformal embedding

$$\iota : (M, g) \longrightarrow (\mathcal{M}, \mathfrak{g})$$

into a larger Lorentzian manifold $(\mathcal{M}, \mathfrak{g})$ with a relatively compact image $\iota(M)$ ⁽⁷⁾. Assume that (M, g) satisfies conditions that guarantee that it is *asymptotically flat at null and spatial infinity*. For an explicit detailed lists of conditions of this kind see e.g. [31, p. 276] and [12, §II.2]. Let us now fix a time-oriented region $\tilde{\Omega}_o$ of M and denote by $\mathcal{I}^+(\tilde{\Omega}_o)$ the set of the points in $\overline{\iota(M)}$, which are final end-points of the (traces of the) time-oriented inextensible null geodesic of (M, g) starting from points of $\tilde{\Omega}_o$. Finally, let us denote by $\Omega_o := J^-(\mathcal{I}^+(\tilde{\Omega}_o))$ the *causal past* of $\mathcal{I}^+(\tilde{\Omega}_o)$, i.e. the subset of points of M that can be joined to points in $\mathcal{I}^+(\tilde{\Omega}_o)$ through polygonals of positively time-oriented causal geodesics.

Following Carter's terminology [11], we call such a region Ω_o the *domain of outer communication determined by $\tilde{\Omega}_o$* . We are now ready to state the following (very minor) variant of *Penrose, Hawking and Ellis' definitions for black holes and event horizons* (see e.g. [31, p.300]).

Definition A.1. The *black holes for the observers in $\tilde{\Omega}_o$* are the (interiors of the) connected components of the complement $\Omega_o^c := M \setminus \Omega_o$ of the domain of outer communication Ω_o determined by $\tilde{\Omega}_o$. An *event horizon for $\tilde{\Omega}_o$* is the boundary in M of a black hole.

By a lemma of Penrose on “achronal boundaries” ([28, Lemma I at p. 214] or [20, Lemma 6.3.2]), any event horizon is generated by segments of null geodesics. This immediately implies that *if an event horizon is a (smooth) hypersurface, then such hypersurface is null*.

Such a crucial property of smooth event horizons admits the following alternative, quite elementary proof. First of all, it is important to observe that, as an immediate consequence of Definition A.1, if an event horizon $\mathcal{S} \subset M$ is a smooth hypersurface which bounds a black hole Ω , then for any neighbourhood $\mathcal{U} \subset \mathcal{S}$ of a point $x_o \in \mathcal{S}$, no physical (causal) signal might cross $\mathcal{U}$ passing from the black hole to the domain of outer communication. This means that such $\mathcal{U} \subset \mathcal{S}$ has the semi-permeability property with respect to time-oriented causal world-lines described in Theorem 3.5. By the discussion in the Introduction, this implies that $\mathcal{U}$ cannot be time-like (Fig.3). We claim that $\mathcal{U} \subset \mathcal{S}$ cannot be space-like either. In fact, otherwise a sufficiently small smooth deformation $\mathcal{U}'$ of $\mathcal{U}$ would be part of a new space-like hypersurface $\mathcal{S}'$, with the following two properties: (a) it still has the semi-permeability property with respect to signals and (b) it bounds a region Ω' , which properly includes the black hole Ω and with no intersection with the domain of outer communication Ω_o . This would contradict the assumption that Ω is a connected component (= connected subset which is maximal with respect to inclusion) of $\Omega_o^c = M \setminus \Omega_o$. We conclude that $\mathcal{S}$ is necessarily a null hypersurface.

⁷The existence of such a conformal embedding with compact closure $\overline{\iota(M)}$ guarantees that each oriented causal world-line of (M, g) is mapped by the embedding ι into an oriented causal world-line $\mathcal{C}$ of $(\mathcal{M}, \mathfrak{g})$ with a well-defined upper limit end-point $p^{(\mathcal{C}, \infty)}$ in $\mathcal{M}$. Such upper limit $p^{(\mathcal{C}, \infty)}$ can be taken as a mathematical object that properly encodes the intuitive notion of the “infinite future point” of the causal world-line $\mathcal{C}$.

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