

NOTIONS OF RANK AND INDEPENDENCE IN COUNTABLY CATEGORICAL THEORIES

VERA KOPONEN

ABSTRACT. For an ω -categorical theory T and model $\mathcal{M}$ of T we define a hierarchy of ranks, the n -ranks for $n < \omega$ which only care about imaginary elements “up to level n ”, where level n contains every element of M and every imaginary element that is an equivalence class of an $\emptyset$ -definable equivalence relation on n -tuples of elements from M . Using the n -rank we define the notion of n -independence. For all $n < \omega$, the n -independence relation restricted to M_n has all properties of an independence relation according to Kim and Pillay [15] with the *possible exception* of the symmetry property. We prove that, given any $n < \omega$, if $\mathcal{M} \models T$ and the algebraic closure in $\mathcal{M}^{\text{eq}}$ restricted to imaginary elements “up to level n ” which have n -rank 1 (over some set of parameters) satisfies the exchange property, then n -independence is symmetric and hence an independence relation when restricted to M_n . Then we show that if n -independence is symmetric for all $n < \omega$, then T is rosy. An application of this is that if T has weak elimination of imaginaries and the algebraic closure in $\mathcal{M}$ restricted to elements of M of 0-rank 1 (over some set of parameters from M^{eq}) satisfies the exchange property, then T is superrosy with finite U^b -rank.

1. INTRODUCTION

A variety of notions of rank and independence have played an important role in model theory at least since Morley’s influential work on uncountably categorical theories [19] in the 1960ies. Such notions have been central for developing more or less general (meta) theories which divide complete first-order theories into various classes. Shelah [24] used them in his development of stability theory by which theories can be classified into ω -stable, superstable, stable, or unstable. Later Shelah’s classification theory, and its notion of independence, was shown, by Kim and Pillay [15, 14], to make sense in a wider context and the classes of simple and supersimple first-order theories were introduced. Yet later, more general notions of independence, including thorn-independence, have been studied by Onshuus, Ealy and Adler and the classes of rosy and superrosy theories have been introduced [1, 7, 21]. The class of rosy theories is quite diverse and includes, for example, all simple theories (which exclude a linearly ordered universe) and all o-minimal theories (which assume a linearly ordered universe) [28]. It is also the largest class of theories for which there is an independence relation which satisfies certain basic and natural properties [7]. The work on NIP theories [25] also uses notions of rank and independence (and NIP theories and rosy theories partially overlap).

Another direction of model theoretic research, which has used notions of rank and independence as a crucial tool, has focused on understanding the *fine structure* of the models of more specific theories, and on finding “*nice*” *axiomatizations* of such theories. This line of research includes work on totally categorical theories, uncountably categorical theories [29, 2, 11, 12], theories of stable finitely homogeneous structures [18], ω -categorical ω -stable theories [4], theories of smoothly approximable structures [5, 13], theories of simple finitely homogeneous structures [3, 16], theories of Fraïssé limits of classes of finite structures with the free amalgamation property [6], ω -categorical NIP theories [26], and NIP finitely homogeneous rosy theories [22]. In the present context it

Date: 8 November, 2025.

is relevant that all finitely homogeneous structures (which are also called ultrahomogeneous, or simply homogeneous, and which include all Fraïssé limits with a finite relational vocabulary) and all smoothly approximable structures are ω -categorical.

In this study we consider (only) ω -categorical theories T and investigate a hierarchy of ranks, the n -ranks for $n < \omega$, where the n -rank is defined entirely in terms of the algebraic closure operator on M^{eq} (where M is a model of the theory) *restricted* to the set M_n containing every real element and every imaginary element that corresponds to an equivalence class of an $\emptyset$ -definable equivalence relation on $M^k = \underbrace{M \times \dots \times M}_{k \text{ times}}$

where $k \leq n$. With the n -rank we define a notion of n -independence. Without further assumptions than ω -categoricity, n -independence restricted to M_n has all the properties of an independence relation according to [15] (possibly) *except* for the symmetry property. The main technical contribution of this study is to isolate a property, parametrized by n , which implies (in fact, is equivalent to) that n -independence is symmetric. The property in question (Assumption 5.1) is the *exchange property* of algebraic closure restricted to elements of M_n with n -rank 1 (over some $C \subseteq M^{\text{eq}}$). More precisely, if, for any $C \subseteq M^{\text{eq}}$, the algebraic closure using parameters from C restricted to elements of M_n with n -rank 1 over C has the exchange property, then n -independence is symmetric, and hence it is an independence relation in the sense of [15] when restricted to M_n , which is stated by Theorem 5.15. Using this we can relatively easily show the statement of Theorem 6.10 that if, for *all* $n < \omega$, the algebraic closure restricted to elements of M_n with n -rank 1 has the exchange property, then the theory is rosy, which is proved by showing that thorn-independence has local character (see [7, Theorem 3.7]). Thus, an ω -categorical theory which is *not* rosy must, for some $n < \omega$, have a model M and $C \subseteq M^{\text{eq}}$ such that algebraic closure using parameters from C restricted to elements of M_n with n -rank 1 over C does *not* have the exchange property. This may be useful for finding a dividing line between ω -categorical rosy theories and ω -categorical non-rosy theories. Actually, I am not aware of any example of an ω -categorical theory that is *not* rosy. Hence the results presented here may help to either find an ω -categorical non-rosy theory, or to rule out the existence of such a theory.

We then elaborate on the mentioned results by showing that if the ω -categorical theory T has “soft” elimination of imaginaries (Definition 2.9), which is a consequence of the more familiar concept of weak elimination of imaginaries [10, 9, 20, 6], and, for any $M \models T$ and $C \subseteq M^{\text{eq}}$, algebraic closure using parameters from C restricted to elements of M (i.e. “real” elements) that have 0-rank 1 over C has the exchange property, then T is superrosy with finite U^b -rank; this is Theorem 6.14 below. It follows that if T is ω -categorical with weak elimination of imaginaries, $M \models T$ and, for any $C \subseteq M^{\text{eq}}$ the algebraic closure restricted to elements of M that have 0-rank 1 over C is *trivial* (Definition 6.11), then T is superrosy with finite U^b -rank.

Conant [6] has proved that the theory of the Fraïssé limit of any class of finite (relational) structures with *free* amalgamation has weak elimination of imaginaries and is superrosy with U^b -rank 1. The class of finite boolean algebras does *not* have the *free* amalgamation but it has the amalgamation property, so it has a Fraïssé limit, namely the (unique up to isomorphism) infinite countable atomless boolean algebra, with complete theory T_B say. Since T_B has elimination of quantifiers it is straightforward to see that the algebraic closure on the real elements of any model of T_B is trivial. By work of Newelski and Wencel [20], T_B has weak elimination of imaginaries and it follows from Theorem 6.14 below that T_B is superrosy with finite U^b -rank, which may be a new result. In any case, it follows that Theorem 6.14 below applies to a larger class of theories than the class of theories considered in [6]. It may also be the case that Theorem 6.10 or Theorem 6.14 can be applied to ω -categorical theories obtained by Hrushovski’s method

of construction of a “generic structure” for a class of finite structures (see e.g. [8]), but I have not investigated this.

2. PRELIMINARIES

We assume familiarity with basic model theory as can be found in for example [9, 24, 27]. A first-order structure will be denoted by $\mathcal{M}$ and its universe (also called domain) by M . Subsets of the universe (of a first-order structure) will be denoted by A, B, C, D and *finite* sequences (also called tuples) of elements of the universe will be denoted by $\bar{a}, \bar{b}, \bar{c}$, etcetera. If S is a set then $|S|$ denotes its cardinality, and if $\bar{s}$ is a finite sequence then $|\bar{s}|$ denotes its length and $\text{rng}(\bar{s})$ denotes the set of elements occurring in $\bar{s}$. By $\text{Th}(\mathcal{M})$ we denote the complete first-order theory of the structure $\mathcal{M}$. We sometimes write AB to denote the union $A \cup B$ of the sets A and B , and if $\bar{a} = (a_1, \dots, a_k)$ we sometimes write $\bar{a}B$ to denote the set $\{a_1, \dots, a_k\} \cup B$.

We assume familiarity with the structure $\mathcal{M}^{\text{eq}}$ “with imaginaries” which is constructed from any structure $\mathcal{M}$ as explained in (for example) [24] and [9]. In this study we will be concerned with “ n -level approximations” of M^{eq} (the universe of $\mathcal{M}^{\text{eq}}$) as follows:

Definition 2.1. Let V be a vocabulary and let $\mathcal{M}$ be a V -structure with universe M . Let $V \cup W$ be the vocabulary of $\mathcal{M}^{\text{eq}}$. Define $M_0 = M$ and, for all $n < \omega$, $M_{n+1} = M_n \cup X_{n+1}$ where X_{n+1} is the set of all imaginary elements $a \in M^{\text{eq}}$ such that a is an equivalence class of a ($\emptyset$ -definable) equivalence relation on $M^{n+1} = \underbrace{M \times \dots \times M}_{n+1 \text{ times}}$. For each $n < \omega$,

let $\mathcal{M}'_n$ be the reduct of $\mathcal{M}^{\text{eq}}$ to the subvocabulary of $V \cup W$ that contains V and every symbol from W that is associated to a $\emptyset$ -definable equivalence relation on M^k for some $1 \leq k \leq n$. Then let $\mathcal{M}_n$ be the substructure of $\mathcal{M}'_n$ with universe M_n .

Definition 2.2. Let $\mathcal{M}$ be a structure. For every $A \subseteq M^{\text{eq}}$, $\text{acl}^{\text{eq}}(A)$ denotes the algebraic closure of A computed in $\mathcal{M}^{\text{eq}}$. For all $n < \omega$ and $A \subseteq M^{\text{eq}}$, we define $\text{acl}^n(A) = \text{acl}^{\text{eq}}(A) \cap M_n$.

The following follows directly from the definition above:

Lemma 2.3. Suppose that $m < n < \omega$ and $C \subseteq M^{\text{eq}}$.

- (i) If $a \in \text{acl}^m(C)$ then $a \in \text{acl}^n(C)$.
- (ii) $a \notin \text{acl}^m(C)$ then $a \notin \text{acl}^n(C)$.

Since each $\mathcal{M}_n$ is interpretable in $\mathcal{M}$ the following result follows from [9, Theorem 7.3.8]:

Fact 2.4. If T is ω -categorical and $\mathcal{M} \models T$ then, for every $n < \omega$, $\text{Th}(\mathcal{M}_n)$ is ω -categorical.

We will use the following facts about ω -categorical theories (briefly explained below):

- Fact 2.5.** (i) If T is ω -categorical then, for all $0 < n < \omega$, there are only finitely many complete types over $\emptyset$ in the free variables $x_1, \dots, x_n$ and each one of them is implied (modulo T) by a single formula (the formula that isolates the type). Hence, there are only finitely many nonequivalent (modulo T) formulas in the same free variables.
- (ii) If T is ω -categorical, $\mathcal{M} \models T$, $n < \omega$, and $B \subseteq M^{\text{eq}}$ is finite, then $\text{acl}^n(B)$ is finite.

The first part above follows from the well-known theorem of Engeler, Ryll-Nardzewski, and Svenonius [9, Theorem 7.3.1]. The second part is a consequence of the fact that, since B is finite, there is $n \leq m < \omega$ such that $B \subseteq M_m$ and $\text{Th}(\mathcal{M}_m)$ is ω -categorical.

In the rest of the article we make the following assumptions.

- (1) T is a complete ω -categorical theory in a countable language.

- (2) $\mathcal{M} \models T$ is κ -saturated where κ is an infinite cardinal (which can be chosen as large as we like). It follows that $\mathcal{M}^{\text{eq}}$ is κ -saturated and we let T^{eq} be the complete theory of M^{eq} .
- (3) All subsets of M^{eq} that are mentioned have cardinality less than κ .

Notation 2.6. If $a_1, \dots, a_n \in M^{\text{eq}}$ and $B \subseteq M^{\text{eq}}$, then $\text{tp}(a_1, \dots, a_n/B)$ denotes the complete type of $a_1, \dots, a_n$ over B computed in $\mathcal{M}^{\text{eq}}$, and $\text{tp}(a_1, \dots, a_n)$ is an abbreviation of $\text{tp}(a_1, \dots, a_n/\emptyset)$.

Definition 2.7. (i) Let us call a *bijective* function σ from a subset $A \subseteq M^{\text{eq}}$ to a subset of M^{eq} *elementary* if for all $n < \omega$ and all $a_1, \dots, a_n \in A$, $\text{tp}(a_1, \dots, a_n) = \text{tp}(\sigma(a_1), \dots, \sigma(a_n))$.

(ii) Following [15] we say that a collection Γ of triples (A, B, C) , where $A, B, C \subseteq M^{\text{eq}}$ and A is *finite*, is an *independence relation* if the following hold, where $A \downarrow_C B$ means that $(A, B, C) \in \Gamma$:

- (1) (invariance) If σ is an elementary function from some subset of M^{eq} that includes ABC to some subset of M^{eq} , then $A \downarrow_C B$ if and only if $\sigma(A) \downarrow_{\sigma(C)} \sigma(B)$.
- (2) (local character) For all A and B (where A is finite) there is a countable $C \subseteq B$ such that $A \downarrow_C B$.
- (3) (finite character) $A \downarrow_C B$ if and only if for all finite $B' \subseteq B$, $A \downarrow_C B'$.
- (4) (extension) For all $n < \omega$, $A = \{a_1, \dots, a_n\}$, B , and C , such that $C \subseteq B$, there is $A' = \{a'_1, \dots, a'_n\}$ such that $\text{tp}(a'_1, \dots, a'_n/C) = \text{tp}(a_1, \dots, a_n/C)$ and $A' \downarrow_C B$.
- (5) (monotonicity)¹ If $B \subseteq C \subseteq D$ and $A \downarrow_B D$ then $A \downarrow_B C$ and $A \downarrow_C D$.
- (6) (transitivity) If $B \subseteq C \subseteq D$, $A \downarrow_B C$ and $A \downarrow_C D$, then $A \downarrow_B D$.
- (7) (symmetry) For all finite A and B and any C , $A \downarrow_C B$ if and only if $B \downarrow_C A$.

(iii) We call Γ an *independence relation restricted to M_n* if (1) – (7) hold whenever $A, B, C, D \subseteq M_n$.

Remark 2.8. The local character of $\downarrow$ (property (2) above) is (assuming that T is countable) equivalent to the following: There is do not exist finite $A \subseteq M^{\text{eq}}$ and finite B_α , for all $\alpha < \aleph_1$, such that $A \not\downarrow_{\bigcup_{i < \alpha} B_i} B_\alpha$ for all $\alpha < \aleph_1$. This formulation (roughly)

is used in e.g. [7], but I have not found a clear statement in the literature of the equivalence of the two versions of local character. However, the argument in the proof of Proposition 2.3.7 in [14] (the part showing that conditions (1) and (2) of that proposition are equivalent) can easily be adapted to prove the equivalence of the two versions of local character.

Definition 2.9. We say that T has *soft elimination of imaginaries* if for all $a \in M^{\text{eq}}$ there is a finite sequence $\bar{b}$ of elements from M such that $a \in \text{acl}^{\text{eq}}(\bar{b})$ and $\text{rng}(\bar{b}) \subseteq \text{acl}^{\text{eq}}(a)$, or equivalently, $\text{acl}^{\text{eq}}(a) = \text{acl}^{\text{eq}}(\bar{b})$.

The more commonly used notion of *weak elimination of imaginaries* (where, with the notation of the definition above, it is required that a belongs to the definable closure of $\bar{b}$) implies soft elimination of imaginaries. Weak elimination of imaginaries has been studied, for example, by Hodges, Hodkinson, and Macpherson, in [10] where they demonstrated, among other things, that the complete theories of dense linear order, the random (or Rado) graph, and the Fraïssé limit of the set of finite K_n -free graphs have weak elimination of imaginaries. Conditions that apply to ω -categorical theories and imply weak

¹In the formulation of [15] this condition is a part of the transitivity property.

elimination of imaginaries are given by [10, Lemma 6.5 and Proposition 8.2]. Newelski and Wencel [20] have proved that the complete theory of an infinite boolean algebra with only finitely many atoms has weak elimination of imaginaries. The above mentioned result about K_n -free graphs has been generalized by Conant [6] to a result saying that the theory of every Fraïssé limit of a class of finite structures with free amalgamation has weak elimination of imaginaries.

The backbone of the theory that will be developed does not use the assumption that T has soft elimination of imaginaries, but, as we will see, the “hierarchy” of ranks and independence relations that we will study collapses to the bottom level under the assumption of soft elimination of imaginaries.

In Section 5 we will need the following concept [9, 23, 27]:

Definition 2.10. A *pregeometry*, also called *matroid*, consists of a set X and a function cl from the powerset of X to the powerset of X which has the following properties:

- (1) For all $A \subseteq X$, $A \subseteq \text{cl}(A)$.
- (2) For all $A \subseteq X$, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.
- (3) For all $A \subseteq X$, if $a \in \text{cl}(A)$ then $a \in \text{cl}(A')$ for some *finite* $A' \subseteq A$.
- (4) (Exchange property) For all $A \subseteq X$ and all $b, c \in X$, if $b \in \text{cl}(A \cup \{c\}) \setminus \text{cl}(A)$, then $c \in \text{cl}(A \cup \{b\})$.

Fact 2.11. Let (X, cl) be a pregeometry and let $A \subseteq X$.

- (i) Then there is $B \subseteq A$ such that $A \subseteq \text{cl}(B)$ (“ B spans A ”) and for every $b \in B$, $b \notin \text{cl}(B \setminus \{b\})$ (“ B is independent”). We call such B a *basis* of A .
- (ii) All bases of A have the same cardinality which we call the *dimension* of A .

3. n -RANK

In this section we develop, for an arbitrary $n < \omega$, a theory of a notion of “ n -rank” which will be used, in the next section, to define a notion of “ n -independence”.

Lemma 3.1. Let $n < \omega$, let $A \subseteq M_n$ be finite and let $B \subseteq M^{\text{eq}}$. Then there is $r < \omega$, depending only on n and $|A|$, such that if $a_1, \dots, a_m \in \text{acl}^n(A)$ and, for all $k = 1, \dots, m$, $a_k \notin \text{acl}^{\text{eq}}(a_1, \dots, a_{k-1}, B)$, then $m \leq r$.

Proof. Let $n < \omega$, let $A \subseteq M_n$ be finite, with cardinality $s < \omega$, say, and let $B \subseteq M^{\text{eq}}$. Suppose that $a_1, \dots, a_m \in \text{acl}^n(A)$ and, for all $k = 1, \dots, m$, $a_k \notin \text{acl}^{\text{eq}}(a_1, \dots, a_{k-1}, B)$. Then $a_i \neq a_j$ if $i \neq j$ so $|\text{acl}^n(A)| \geq m$. Since $\text{Th}(\mathcal{M}_n)$ is ω -categorical (by Fact 2.4) it follows there is $r < \omega$ such that whenever $A' \subseteq M^n$ and $|A'| \leq s$, then $|\text{acl}^n(A')| \leq r$. Hence $m \leq |\text{acl}^n(A)| \leq r$. $\square$

Definition 3.2. Let $n < \omega$. For all $A, B \subseteq M^{\text{eq}}$ we define the n -rank of A over B , denoted $\text{rk}^n(A/B)$, as follows:

- (1) $\text{rk}^n(A/B) \geq 0$.
- (2) For any ordinal α , $\text{rk}^n(A/B) \geq \alpha + 1$ if there is $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ such that $\text{rk}^n(A/\{a\} \cup B) \geq \alpha$.
- (3) For a limit ordinal α , $\text{rk}^n(A/B) \geq \alpha$ if $\text{rk}^n(A/B) \geq \beta$ for all $\beta < \alpha$.

Finally, $\text{rk}^n(A/B) = \alpha$ if $\text{rk}^n(A/B) \geq \alpha$ and $\text{rk}^n(A/B) \not\geq \alpha + 1$. If $\text{rk}^n(A/B) \geq \alpha$ for all ordinals α we say that the n -rank of A over B is *undefined*. We define $\text{rk}^n(A) = \text{rk}^n(A/\emptyset)$. If $\bar{a}$ is a sequence of elements (from M^{eq}) then $\text{rk}^n(\bar{a}/B)$ means the same as $\text{rk}^n(\text{rng}(\bar{a})/B)$, and if $\bar{b}$ is a sequence of elements then $\text{rk}^n(\bar{a}/\bar{b})$ means the same as $\text{rk}^n(\text{rng}(\bar{a})/\text{rng}(\bar{b}))$.

Note that it follows that $\text{rk}^n(A/B) \geq 1$ if and only if $\text{acl}^n(A) \not\subseteq \text{acl}^n(B)$.

For the rest of this section we fix an arbitrary $n < \omega$.

Example 3.3. Let T be the theory which expresses that E is an equivalence relation with infinitely many equivalence classes all of which are infinite. Let $\mathcal{M} \models T$ and let $a \in M$. It is well-known (and easy to show) that T is ω -categorical with elimination of quantifiers. From this it easily follows that, for all $A \subseteq M$, $\text{acl}^0(A) = A$. Therefore $\text{rk}^0(a) = 1$. Let $[a]_E$ denote the equivalence class of a with respect to E as a element of M^{eq} , so $[a]_E \in M_1$. Then $\text{rk}^1(a/\{a, [a]_E\}) \geq 0$, and as $a \in \text{acl}^1(a) \setminus \text{acl}^1(\emptyset)$ we get $\text{rk}^1(a/\{[a]_E\}) \geq 1$. Since $[a]_E \in \text{acl}^1(a) \setminus \text{acl}^1(\emptyset)$ we get $\text{rk}^1(a) \geq 2$. As T has elimination of quantifiers it follows that E is the only $\emptyset$ -definable equivalence relation on M . Hence $\text{acl}^1(a) = \{a, [a]_E\}$ and consequently $\text{rk}^1(a) = 2$. Let $b \in M$ be such that $b \neq a$ and $[b]_E = [a]_E$. By arguing similarly as above it follows that $\text{rk}^0(a/b) = \text{rk}^0(a) = 1$ and $\text{rk}^1(a/b) = 1 < 2 = \text{rk}^1(a)$.

Lemma 3.4. *Let $A, B \subseteq M^{\text{eq}}$ and suppose that $\text{rk}^n(A/B)$ and $\text{rk}^{n+1}(A/B)$ are defined. Then $\text{rk}^n(A/B) \leq \text{rk}^{n+1}(A/B)$.*

Proof. We prove by induction that if $\text{rk}^n(A/B) \geq \alpha$ then $\text{rk}^{n+1}(A/B) \geq \alpha$. This is clear for $\alpha = 0$. So suppose that $\text{rk}^n(A/B) \geq \alpha + 1$. Then there is $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ such that $\text{rk}^n(A/aB) \geq \alpha$. By the induction hypothesis we get $\text{rk}^{n+1}(A/aB) \geq \alpha$. We have $a \in \text{acl}^n(A) \subseteq \text{acl}^{n+1}(A)$ and $a \notin \text{acl}^n(B)$ so (by Lemma 2.3) $a \notin \text{acl}^{n+1}(B)$. Hence $\text{rk}^{n+1}(A/B) \geq \alpha + 1$. If α is a limit ordinal and $\text{rk}^n(A/B) \geq \alpha$ then $\text{rk}^n(A/B) \geq \beta$ for all $\beta < \alpha$, so by the induction hypothesis $\text{rk}^{n+1}(A/B) \geq \beta$ for all $\beta < \alpha$, hence $\text{rk}^{n+1}(A/B) \geq \alpha$. $\square$

Lemma 3.5. *Let $\alpha < \omega$ and $A, B \subseteq M^{\text{eq}}$.*

- (i) $\text{rk}^n(A/B) \geq \alpha$ if and only if there are $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$.
- (ii) If $\text{rk}^n(A/B) = \alpha$ then α is maximal such that there are $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$.
- (iii) Suppose that $\text{rk}^n(A/B) = \alpha$, $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ and, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$. Then, for all $k = 1, \dots, \alpha$,

- (1) $\text{rk}^n(a_k/\{a_1, \dots, a_{k-1}\} \cup B) = 1$,
- (2) $\text{rk}^n(a_1, \dots, a_k/B) = k$, and
- (3) $\text{rk}^n(A/\{a_1, \dots, a_k\} \cup B) = \alpha - k$.

In particular we have $\text{rk}^n(A/\{a_1, \dots, a_\alpha\} \cup B) = 0$ so $\text{acl}^n(A) \subseteq \text{acl}^n(\{a_1, \dots, a_\alpha\} \cup B)$, and $\text{rk}^n(a_1, \dots, a_\alpha/B) = \alpha = \text{rk}^n(A/B)$.

Proof. Let $\alpha < \omega$ and $A, B \subseteq M^{\text{eq}}$. We prove (i) by induction on α . For $\alpha = 0$ the statement is vacuous. Suppose that $\text{rk}^n(A/B) \geq \alpha + 1$. Then there is $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ such that $\text{rk}^n(A/aB) \geq \alpha$. By the induction hypothesis there are $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a, a_1, \dots, a_{k-1}\} \cup B)$. If we rename a_i by a_{i+1} for $i = 1, \dots, \alpha$ and then rename a by a_1 we get $a_i \notin \text{acl}^n(\{a_1, \dots, a_{i-1}\} \cup B)$ for all $i = 1, \dots, \alpha + 1$.

Now suppose that $a_1, \dots, a_{\alpha+1} \in \text{acl}^n(A)$ and, for all $k = 1, \dots, \alpha + 1$,

$$a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B).$$

By the induction hypothesis we have $\text{rk}^n(A/a_1B) \geq \alpha$. By the choice of $a_1, \dots, a_{\alpha+1}$ we have $a_1 \in \text{acl}^n(A) \setminus \text{acl}^n(B)$, so $\text{rk}^n(A/B) \geq \alpha + 1$. Now we have proved part (i). Part (ii) follows directly from part (i).

(iii) Suppose that $\text{rk}^n(A/B) = \alpha$ and that $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ are such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$. Then $\text{rk}^n(a_k/\{a_1, \dots, a_{k-1}\} \cup B) \geq 1$ for all $k = 1, \dots, \alpha$. Suppose, for a contradiction, that for some k , $\text{rk}^n(a_k/\{a_1, \dots, a_{k-1}\} \cup B) \geq 2$. Choose the least such k . Then there is

$$a \in \text{acl}^n(a_k) \setminus \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$$

such that $\text{rk}^n(a_k/\{a, a_1, \dots, a_{k-1}\} \cup B) \geq 1$ which in particular means that

$$a_k \notin \text{acl}^n(\{a, a_1, \dots, a_{k-1}\} \cup B).$$

Since $a \in \text{acl}^n(a_k)$ it follows from the choice of $a_1, \dots, a_\alpha$ that, for all $i = k+1, \dots, \alpha$, $a_i \notin \text{acl}^n(\{a, a_1, \dots, a_{i-1}\} \cup B)$. Since $a \in \text{acl}^n(a_k) \subseteq \text{acl}^n(A)$ it now follows from part (i) that $\text{rk}^n(A/B) \geq \alpha + 1$, contradicting the assumption.

By assumption and part (ii), for every $k = 1, \dots, \alpha$, the number $\alpha - k$ is maximal such that there are $a'_{k+1}, \dots, a'_\alpha \in \text{acl}^n(A)$ such that, for all $i = k+1, \dots, \alpha$,

$$a'_i \notin \text{acl}^{\text{eq}}(\{a_1, \dots, a_k, a'_{k+1}, \dots, a'_{i-1}\} \cup B).$$

Hence $\text{rk}(A/\{a_1, \dots, a_k\} \cup B) = \alpha - k$ for all $k = 1, \dots, \alpha$.

Let $k \in \{1, \dots, \alpha\}$. By part (i), the sequence $a_1, \dots, a_k$ witnesses that $\text{rk}(a_1, \dots, a_k/B) \geq k$. Suppose, for a contradiction, that $\text{rk}(a_1, \dots, a_k/B) \geq k+1$. Then (by part (i)) we find

$$a'_0, \dots, a'_k \in \text{acl}^n(a_1, \dots, a_k) \subseteq \text{acl}^n(A)$$

such that, for all $i = 0, \dots, k$, $a'_i \notin \text{acl}^n(\{a'_1, \dots, a'_{i-1}\} \cup B)$. Then $\text{acl}^n(\{a'_0, \dots, a'_k\} \cup B) \subseteq \text{acl}^n(\{a_1, \dots, a_k\} \cup B)$ so, for all $i = k+1, \dots, \alpha$, $a_i \notin \text{acl}^n(\{a'_0, \dots, a'_k, a_{k+1}, \dots, a_{i-1}\} \cup B)$. By part (i), the sequence $a'_0, \dots, a'_k, a_{k+1}, \dots, a_r$ witnesses that $\text{rk}(A/B) \geq \alpha + 1$ which contradicts the assumption. $\square$

Definition 3.6. Let $A, B \subseteq M^{\text{eq}}$ and suppose that $\text{rk}^n(A/B) = \alpha < \omega$. Then every sequence $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$ will be called an *n-coordinatization sequence* (*n-cs*) for A/B (“*A* over *B*”).

Lemma 3.7. If $A, B \subseteq M^{\text{eq}}$ and A is finite then $\text{rk}^n(A/B)$ is defined and finite.

Proof. Let $A, B \subseteq M^{\text{eq}}$ where A is finite. Then $|\text{acl}^n(A)| = \alpha$ for some $\alpha < \omega$. If $\text{rk}^n(A/B) \geq \alpha + 1$ then, by Lemma 3.5 (i), there are $a_1, \dots, a_{\alpha+1} \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha + 1$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$. Then $a_i \neq a_j$ if $i \neq j$ so $|\text{acl}^n(A)| \geq \alpha + 1$, a contradiction. $\square$

Lemma 3.8. Suppose that T has soft elimination of imaginaries. Let $\alpha < \omega$ and $A, B \subseteq M^{\text{eq}}$.

- (i) If $\text{rk}^n(A/B) \geq \alpha$ then $\text{rk}^0(A/B) \geq \alpha$.
- (ii) If A is finite then $\text{rk}^n(A/B) = \text{rk}^0(A/B)$.

Proof. (i) Suppose that $\text{rk}^n(A/B) \geq \alpha$. By Lemma 3.5, there are $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup B)$. By induction on α we prove that there are $a'_1, \dots, a'_\alpha \in \text{acl}^0(A)$ such that $a'_k \notin \text{acl}^0(\{a'_1, \dots, a'_{k-1}\} \cup B)$ and $a'_k \in \text{acl}^0(a_k)$ for all $k = 1, \dots, \alpha$, and it follows from Lemma 3.5 that $\text{rk}^0(A/B) \geq \alpha$. If $\alpha = 0$ there is nothing to prove, so suppose that $\alpha > 0$. By the induction hypothesis there are $a'_1, \dots, a'_{\alpha-1} \in \text{acl}^0(A)$ such that $a'_k \notin \text{acl}^0(\{a'_1, \dots, a'_{k-1}\} \cup B)$ and $a'_k \in \text{acl}^0(a_k)$ for all $k = 1, \dots, \alpha - 1$. Since T is assumed to have soft elimination of imaginaries, there is a finite sequence $\bar{c}$ of elements from $M_0 = M$ such that $\text{acl}^{\text{eq}}(a_\alpha) = \text{acl}^{\text{eq}}(\bar{c})$. Since $a_\alpha \notin \text{acl}(\{a_1, \dots, a_{\alpha-1}\} \cup B)$ we have $\text{rng}(\bar{c}) \not\subseteq \text{acl}(\{a_1, \dots, a_{\alpha-1}\} \cup B)$. As $a'_k \in \text{acl}^0(a_k)$ for all $k = 1, \dots, \alpha - 1$ it follows that $\text{rng}(\bar{c}) \not\subseteq \text{acl}^0(\{a'_1, \dots, a'_{\alpha-1}\} \cup B)$. Thus we can choose some element from $\bar{c}$, which we call a'_α , such that $a'_\alpha \notin \text{acl}^0(\{a'_1, \dots, a'_{\alpha-1}\} \cup B)$. Now Lemma 3.5 implies that $\text{rk}^0(A/B) \geq \alpha$.

(ii) Suppose that A is finite. By Lemma 3.7, $\text{rk}^n(A/B) = \alpha$ for some $\alpha < \omega$. By Lemma 3.4, $\text{rk}^n(A/B) \geq \text{rk}^0(A/B)$, and by part (i) $\text{rk}^n(A/B) \leq \text{rk}^0(A/B)$. $\square$

Lemma 3.9. Let $A, B, C, D \subseteq M^{\text{eq}}$ where $A \supseteq B$, $C \subseteq D$ and A is finite. Then

- (a) $\text{rk}^n(A/A) = 0$,
- (b) $\text{rk}^n(A/C) \geq \text{rk}^n(B/C)$,

- (c) $\text{rk}^n(A/C) \geq \text{rk}^n(A/D)$, and
 (d) $\text{rk}^n(A/C) \geq \text{rk}^n(B/C) + \text{rk}^n(A/BC)$.

Proof. Suppose that $A, B, C, D \subseteq M^{\text{eq}}$ where $A \supseteq B$, $C \subseteq D$ and A is finite, so all mentioned n -ranks in (a) – (d) are finite. Part (a) is obvious from the definition of n -rank.

For part (b), suppose that $\text{rk}^n(B/C) \geq \alpha$ (where $\alpha < \omega$). By Lemma 3.5, there are $a_1, \dots, a_\alpha \in \text{acl}^n(B)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup C)$. Since $A \supseteq B$ it follows that $a_1, \dots, a_\alpha \in \text{acl}^n(A)$, so, by Lemma 3.5 again, $\text{rk}^n(A/C) \geq \alpha$.

Now consider part (c). Suppose that $\text{rk}^n(A/D) \geq \alpha$. Then there are $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \alpha$, $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup D)$. Since $C \subseteq D$ we get $a_k \notin \text{acl}^n(\{a_1, \dots, a_{k-1}\} \cup C)$ for all $k = 1, \dots, \alpha$, so $\text{rk}^n(A/C) \geq \alpha$.

Finally we consider part (d). Let $\beta = \text{rk}^n(B/C)$ and $\alpha = \text{rk}^n(A/B)$, so $\alpha, \beta < \omega$. By Lemma 3.5, there are $b_1, \dots, b_\beta \in \text{acl}^n(B)$ and $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ such that, for all $k = 1, \dots, \beta$, $b_k \notin \text{acl}^n(\{b_1, \dots, b_{k-1}\} \cup C)$, and for all $l = 1, \dots, \alpha$, $a_l \notin \text{acl}^n(\{a_1, \dots, a_{l-1}\} \cup B)$. Moreover, β and α are maximal such that such sequences exist. By the same lemma we have $\text{acl}^n(B) \subseteq \text{acl}^n(\{b_1, \dots, b_\beta\} \cup C)$ and hence $a_l \notin \text{acl}^n(\{b_1, \dots, b_\beta, a_1, \dots, a_{l-1}\} \cup C)$, for all $l = 1, \dots, \alpha$. Since $B \subseteq A$ we have $\text{acl}^n(B) \subseteq \text{acl}^n(A)$ and the same lemma again implies that $\text{rk}(A/C) \geq \beta + \alpha$. $\square$

Lemma 3.10. *Let $A, B \subseteq M^{\text{eq}}$. If $\text{acl}^n(A) \setminus \text{acl}^n(B) \neq \emptyset$ then there is $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ such that $\text{rk}^n(a/B) = 1$.*

Proof. Note that (by Lemma 3.7), if $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ then $\text{rk}^n(a/B) < \omega$. We first prove the following claim:

Claim. If $a' \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ and $\text{rk}^n(a'/B) = \alpha + 1$ where $\alpha \geq 1$, then there is $a \in \text{acl}^n(a') \subseteq \text{acl}^n(A)$ such that $1 \leq \text{rk}^n(a/B) \leq \alpha$.

Proof of the claim. Suppose that $a' \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ and $\text{rk}^n(a'/B) = \alpha + 1$ where $\alpha \geq 1$. By the definition of rk^n there is $a \in \text{acl}^n(a') \setminus \text{acl}^n(B)$ such that $\text{rk}^n(a'/aB) = \alpha$. If $a' \in \text{acl}^n(aB)$ then $\text{rk}^n(a'/aB) = 0$, so $\alpha = 0$, which contradicts that $\alpha \geq 1$. Hence $a' \notin \text{acl}^n(aB)$. Let $\beta = \text{rk}^n(a/B)$, so $\beta < \omega$ and (by Lemma 3.5) there are $a_1, \dots, a_\beta \in \text{acl}^n(a)$ such that, for all $i = 1, \dots, \beta$, $a_i \notin \text{acl}^n(\{a_1, \dots, a_{i-1}\} \cup B)$. Since $a_1, \dots, a_\beta \in \text{acl}^n(a) \subseteq \text{acl}^n(a')$ and $a' \notin \text{acl}^n(aB)$ it follows that $a \notin \text{acl}^n(\{a_1, \dots, a_\beta\} \cup B)$. Hence the sequence $a_1, \dots, a_\beta, a$ witnesses that $\text{rk}^n(a'/B) \geq \beta + 1 > \beta = \text{rk}^n(a/B)$. As $\text{rk}^n(a'/B) = \alpha + 1$ we must have $\beta \leq \alpha$, so $\text{rk}^n(a/B) \leq \alpha$. As $a \notin \text{acl}^n(B)$ we also have $\text{rk}^n(a/B) \geq 1$. This concludes the proof of the claim.

Now suppose that $a' \in \text{acl}^n(A) \setminus \text{acl}^n(B)$. Let $\alpha = \text{rk}^n(a'/B)$. Since $a' \notin \text{acl}^n(B)$ we have $\alpha \geq 1$. If $\alpha = 1$ then we are done (by letting $a = a'$). So suppose that $\alpha \geq 2$. By repeatedly using the claim we eventually find $a \in \text{acl}^n(a') \subseteq \text{acl}^n(A)$ such that $\text{rk}^n(a/B) = 1$. This ends the proof of the lemma. $\square$

Lemma 3.11. (i) *Suppose that $A \subseteq M^{\text{eq}}$ is finite and $B \subseteq C \subseteq M^{\text{eq}}$. If $\text{rk}^n(A/C) < \text{rk}^n(A/B)$ then there is a finite $C' \subseteq C$ such that $\text{rk}^n(A/B \cup C') < \text{rk}^n(A/B)$.*

(ii) *Suppose that $A \subseteq M^{\text{eq}}$ is finite and $B \subseteq M^{\text{eq}}$. Then there is finite $B' \subseteq B$ such that $\text{rk}^n(A/B') = \text{rk}^n(A/B)$.*

Proof. (i) Let $A \subseteq M^{\text{eq}}$ be finite and let $B \subseteq C \subseteq M^{\text{eq}}$. We prove the claim by induction on $\text{rk}^n(A/B)$. If $\text{rk}^n(A/B) = 0$ then there is nothing to prove (as we cannot have $\text{rk}^n(A/C) < \text{rk}^n(A/B)$). So suppose that $\text{rk}^n(A/B) = \alpha + 1$. Then there is $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ such that $\text{rk}^n(A/\{a\} \cup B) \geq \alpha$. In fact we must have $\text{rk}^n(A/\{a\} \cup B) = \alpha$ because if $\text{rk}^n(A/\{a\} \cup B) \geq \alpha + 1$ then $\text{rk}^n(A/B) \geq \alpha + 2$ which contradicts the assumption. Since A is finite it follows that $\text{acl}^n(A)$ is finite and therefore there is finite $C_1 \subseteq C$ such that

$$\text{acl}^n(A) \cap \text{acl}^n(C) = \text{acl}^n(A) \cap \text{acl}^n(C_1).$$

By the induction hypothesis, if $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ is such that $\text{rk}^n(A/\{a\} \cup C) < \text{rk}^n(A/\{a\} \cup B) = \alpha$, then there is finite $C_a \subseteq C$ such that $\text{rk}^n(A/\{a\} \cup B \cup C_a) < \text{rk}^n(A/\{a\} \cup B)$. Let C_2 be the union of all $C_a \subseteq C$ where a ranges over the members of the finite set $\text{acl}^n(A)$ such that $\text{rk}^n(A/\{a\} \cup C) < \text{rk}^n(A/\{a\} \cup B)$. Let $C' = C_1 \cup C_2$ so C' is finite.

Now suppose that $\text{rk}^n(A/C) < \text{rk}^n(A/B)$. Then for every $a \in \text{acl}^n(A) \setminus \text{acl}^n(B)$ such that $\text{rk}^n(A/\{a\} \cup B) = \alpha$ we have either $a \in \text{acl}^{\text{eq}}(C)$ or $\text{rk}^n(A/\{a\} \cup C) < \alpha$ which in turn implies that either $a \in \text{acl}^n(C')$ or $\text{rk}^n(A/\{a\} \cup B \cup C') < \text{rk}^n(A/\{a\} \cup B)$. It follows that $\text{rk}^n(A/B \cup C') < \alpha + 1 = \text{rk}^n(A/B)$.

(ii) Suppose that $\text{rk}^n(A/B) = \alpha$ (where A is finite) and $\text{rk}^n(A/\emptyset) = \beta$ so $\alpha \leq \beta$ (by Lemma 3.9). If $\alpha = \beta$ we are done. So suppose that $\alpha < \beta$. Let $B_0 = \emptyset$. By part (i) there is finite $B_1 \subseteq B$ such that $\text{rk}^n(A/B_1) < \text{rk}^n(A/B_0)$. If $\text{rk}^n(A/B) < \text{rk}^n(A/B_1)$ then we use part (i) again and get a finite $B_2 \subseteq B$ such that $\text{rk}^n(A/B_0 \cup B_1 \cup B_2) < \text{rk}^n(A/B_0 \cup B_1)$. We can repeat this procedure as long as $\text{rk}^n(A/B) < \text{rk}^n(A/B_0 \cup \dots \cup B_m)$. But as there is no infinite decreasing sequence of natural numbers we eventually find $m < \omega$ and finite $B_k \subseteq B$ for $k \leq m$ such that $\text{rk}^n(A/B_0 \cup \dots \cup B_m) = \text{rk}^n(A/B)$. $\square$

Lemma 3.12. *Suppose that $A \subseteq M^{\text{eq}}$ is finite and $B \subseteq M^{\text{eq}}$. Then there is countable $C \subseteq B$ such that $\text{rk}^n(A/C) = \text{rk}^n(A/B)$ for all $n < \omega$.*

Proof. Suppose that $A \subseteq M^{\text{eq}}$ is finite and $B \subseteq M^{\text{eq}}$. Lemma 3.11 says that for every $n < \omega$ there is a finite $B_n \subseteq B$ such that $\text{rk}^n(A/B_n) = \text{rk}^n(A/B)$. Let $C = \bigcup_{n < \omega} B_n$, so $C \subseteq B$ is countable. By Lemma 3.9, for all $n < \omega$, $\text{rk}^n(A/B_n) \geq \text{rk}^n(A/C) \geq \text{rk}^n(A/B) = \text{rk}^n(A/B_n)$, and hence $\text{rk}^n(A/C) = \text{rk}^n(A/B_n) = \text{rk}^n(A/B)$. $\square$

Lemma 3.13. (i) *Let $m, r, k, l < \omega$ where $n \leq m$. The $(k+l)$ -ary relation on M_m which holds for $(a_1, \dots, a_k, b_1, \dots, b_l) \in (M_m)^{k+l}$ if and only if $\text{rk}^n(a_1, \dots, a_k/b_1, \dots, b_l) = r$ is $\emptyset$ -definable in $\mathcal{M}_m$ and in $\mathcal{M}^{\text{eq}}$.*

(ii) *Let $\bar{a}, \bar{a}', \bar{b}$ and $\bar{b}'$ be finite sequences of elements from M^{eq} . If $\text{tp}(\bar{a}, \bar{b}) = \text{tp}(\bar{a}', \bar{b}')$, then $\text{rk}^n(\bar{a}/\bar{b}) = \text{rk}^n(\bar{a}'/\bar{b}')$.*

Proof. (i) Suppose that $n \leq m < \omega$. Since T is ω -categorical it follows that $Th(\mathcal{M}_m)$ is ω -categorical and from this it follows that, for all $k < \omega$, the $(k+1)$ -ary relation on M_m which holds for $(b, a_1, \dots, a_k) \in (M_m)^{k+1}$ if and only if $b \in \text{acl}^n(a_1, \dots, a_k)$ is $\emptyset$ -definable in $\mathcal{M}_m$. It follows that for all $r, k, l < \omega$, the $(k+l)$ -ary relation on M_m which holds for $(a_1, \dots, a_k, b_1, \dots, b_l) \in (M_m)^{k+l}$ if and only if r is maximal such that

$$(3.1) \quad \begin{aligned} &\text{there are } a'_1, \dots, a'_r \in \text{acl}^n(a_1, \dots, a_k) \text{ such that,} \\ &\text{for all } i = 1, \dots, r, a'_i \notin \text{acl}^n(a'_1, \dots, a'_{i-1}, b_1, \dots, b_l) \end{aligned}$$

is $\emptyset$ -definable in $\mathcal{M}_m$, by $\varphi_r^m(x_1, \dots, x_k, y_1, \dots, y_l)$ say. Then the formula $\psi_r^m(x_1, \dots, x_k, y_1, \dots, y_l)$ which expresses that “all $x_1, \dots, x_k, y_1, \dots, y_l$ belong to M_m and $\varphi_r^m(x_1, \dots, x_k, y_1, \dots, y_l)$ holds” defines the same relation in $\mathcal{M}^{\text{eq}}$.

By Lemma 3.5, $\text{rk}^n(a_1, \dots, a_k/b_1, \dots, b_l) = r$ if and only if r is maximal such that (3.1) holds. It follows that, for every $r < \omega$, the $(k+l)$ -ary relation on $(M_m)^{k+l}$ which holds for $(a_1, \dots, a_k, b_1, \dots, b_l) \in (M_m)^{k+l}$ if and only if $\text{rk}^n(a_1, \dots, a_k/b_1, \dots, b_l) = r$ is $\emptyset$ -definable in $\mathcal{M}_m$ by $\varphi_r^m(x_1, \dots, x_k, y_1, \dots, y_l)$. The same relation is definable in $\mathcal{M}^{\text{eq}}$ by $\psi_r^m(x_1, \dots, x_k, y_1, \dots, y_l)$.

(ii) Suppose that $\bar{a}, \bar{a}' \in (M^{\text{eq}})^k$, $\bar{b}, \bar{b}' \in (M^{\text{eq}})^l$ and that $\text{rk}^n(\bar{a}/\bar{b}) = r$. Then there is $m < \omega$ such that $n \leq m$, $\bar{a}, \bar{a}' \in (M_m)^k$, and $\bar{b}, \bar{b}' \in (M_m)^l$. Then $\mathcal{M}^{\text{eq}} \models \psi_r^m(\bar{a}, \bar{b})$, and if $\text{tp}(\bar{a}, \bar{b}) = \text{tp}(\bar{a}', \bar{b}')$ then also $\mathcal{M}^{\text{eq}} \models \psi_r^m(\bar{a}', \bar{b}')$, so $\text{rk}^n(\bar{a}'/\bar{b}') = r$. $\square$

Lemma 3.14. *Let $B \subseteq C \subseteq M^{\text{eq}}$ be finite and $a \in M^{\text{eq}}$. If $a \notin \text{acl}^{\text{eq}}(B)$ then there is $a' \in M^{\text{eq}}$ such that $\text{tp}(a'/B) = \text{tp}(a/B)$ and $a' \notin \text{acl}^{\text{eq}}(C)$.*

Proof. Let S be the sort of a . Since $a \notin \text{acl}^{\text{eq}}(B)$ it follows that $\text{tp}(a/B)$ has infinitely many realizations. As C is finite it follows (from ω -categoricity) that $\text{acl}^{\text{eq}}(C) \cap S$ is finite and hence there is $a' \notin \text{acl}^{\text{eq}}(C)$ such that $\text{tp}(a'/B) = \text{tp}(a/B)$. $\square$

Lemma 3.15. *Suppose that $\bar{d}$ is a finite sequence of elements from M_n , $B, C \subseteq M^{\text{eq}}$ are finite and $B \subseteq C$. Let $\alpha = \text{rk}^n(A/B)$ (so $\alpha < \omega$). Then there is a finite sequence $\bar{d}'$ of elements from M_n such that $\text{tp}(\bar{d}'/B) = \text{tp}(\bar{d}/B)$ and $\text{rk}^n(\bar{d}'/C) = \alpha$.*

Proof. Let $\bar{d}$ be a finite sequence of elements from M_n . Suppose that $\alpha = \text{rk}^n(\bar{d}/B)$. By Lemma 3.5 there are $a_1, \dots, a_\alpha \in \text{acl}^n(\bar{d})$ such that,

$$(3.2) \quad \begin{aligned} \text{for all } k = 1, \dots, \alpha, \quad & \text{rk}^n(a_k/\{a_1, \dots, a_{k-1}\} \cup B) = 1, \\ & \text{rk}^n(a_1, \dots, a_k/B) = k, \text{ and} \\ & \text{rk}^n(\bar{d}/\{a_1, \dots, a_k\} \cup B) = \alpha - k. \end{aligned}$$

In particular, $\text{rk}^n(a_1, \dots, a_\alpha/B) = \alpha$ and $\text{rk}^n(\bar{d}/\{a_1, \dots, a_\alpha\} \cup B) = 0$, so (as $\text{rng}(\bar{d}) \subseteq M_n$) $\text{rng}(\bar{d}) \subseteq \text{acl}^n(A) \subseteq \text{acl}^n(\{a_1, \dots, a_\alpha\} \cup B)$ and hence $\text{acl}^n(\bar{d}B) = \text{acl}^n(\{a_1, \dots, a_\alpha\} \cup B)$.

Suppose that $k < \alpha$ and that there are $a'_1, \dots, a'_k \in M_n$ such that

$$(3.3) \quad \text{tp}(a'_1, \dots, a'_k/B) = \text{tp}(a_1, \dots, a_k/B), \text{ and}$$

$$(3.4) \quad \text{for all } i = 1, \dots, k, \quad a'_i \notin \text{acl}^n(\{a'_1, \dots, a'_{i-1}\} \cup C).$$

We will find a'_{k+1} so that the above holds also with k replaced by $k+1$. Let

$$p(x, y_1, \dots, y_k) = \text{tp}(a_{k+1}, a_1, \dots, a_k/B).$$

By (3.2), $p(x, a_1, \dots, a_k)$ is non-algebraic. By (3.3), also $p(x, a'_1, \dots, a'_k)$ is non-algebraic, so by Lemma 3.14 there is $a'_{k+1} \in M_n \setminus \text{acl}^n(\{a'_1, \dots, a'_k\} \cup C)$ which realizes $p(x, a'_1, \dots, a'_k)$, that is, $\text{tp}(a'_1, \dots, a'_k, a'_{k+1}/B) = \text{tp}(a_1, \dots, a_k, a_{k+1}/B)$. Now (3.3) and (3.4) hold if k is replaced by $k+1$. By induction it follows that there are $a'_1, \dots, a'_\alpha \in M_n$ such that

$$(3.5) \quad \text{tp}(a'_1, \dots, a'_\alpha/B) = \text{tp}(a_1, \dots, a_\alpha/B), \text{ and}$$

$$(3.6) \quad \text{for all } i = 1, \dots, \alpha, \quad a'_i \notin \text{acl}^n(\{a'_1, \dots, a'_{i-1}\} \cup C).$$

From (3.5) it follows that there is $\bar{d}'$ such that

$$(3.7) \quad \text{tp}(\bar{d}', a'_1, \dots, a'_\alpha/B) = \text{tp}(\bar{d}, a_1, \dots, a_\alpha/B).$$

Then $a'_1, \dots, a'_\alpha \in \text{acl}^n(\bar{d}')$. This together with (3.6) implies that $\text{rk}^n(\bar{d}'/C) \geq \alpha$. From (3.7) and Lemma 3.13 we get $\text{rk}^n(\bar{d}'/B) = \text{rk}^n(\bar{d}/B) = \alpha$. Since we must (by Lemma 3.9) have $\text{rk}^n(\bar{d}'/C) \leq \text{rk}^n(\bar{d}'/B)$ we get $\text{rk}^n(\bar{d}'/C) = \alpha$. From (3.7) we get $\text{tp}(\bar{d}'/B) = \text{tp}(\bar{d}/B)$ so the proof is completed. $\square$

Lemma 3.16. *Suppose that $\bar{d}$ is a finite sequence of elements from M_n , $B, C \subseteq M^{\text{eq}}$ and $B \subseteq C$. Then there is a finite sequence $\bar{d}'$ of elements from M_n such that $\text{tp}(\bar{d}'/B) = \text{tp}(\bar{d}/B)$ and $\text{rk}^n(\bar{d}'/C) = \text{rk}^n(\bar{d}/B)$.*

Proof. Let $\bar{d}$ be a finite sequence of elements from M_n and let $\alpha = \text{rk}^n(\bar{d}/B)$. Hence $\text{rk}^n(\bar{d}/B') \geq \alpha$ for every finite $B' \subseteq B$. By Lemma 3.13, for all finite $B' \subseteq B$ and finite $C' \subseteq C$ there is a formula $\varphi_{B'C'}(\bar{x})$ with parameters from $B' \cup C'$ that expresses that $\text{tp}(\bar{x}/B') = \text{tp}(\bar{d}/B')$ and $\text{rk}^n(\bar{x}/B'C') \geq \alpha$. Let $\Phi(\bar{x})$ be the set of all such formulas $\varphi_{B'C'}(\bar{x})$ as B' and C' varies over finite subsets of B and C , respectively. From Lemma 3.15 it follows that every finite subset of $\Phi(\bar{x})$ is consistent. From compactness it follows that $\Phi(\bar{x})$ is consistent. Let $\bar{d}'$ be a realization of the type $\Phi(\bar{x})$. It follows that $\text{tp}(\bar{d}'/B) = \text{tp}(\bar{d}/B)$ and, by Lemma 3.11, that $\text{rk}^n(\bar{d}'/C) \geq \alpha$. Since we must (by Lemmas 3.9 and 3.13) have $\text{rk}^n(\bar{d}'/C) \leq \text{rk}^n(\bar{d}'/B) = \text{rk}^n(\bar{d}/B) = \alpha$ it follows that $\text{rk}^n(\bar{d}'/C) = \alpha$. $\square$

4. n -INDEPENDENCE

Now we are ready to define a hierarchy of independence relations, the notions of n -independence, for all $n < \omega$. In this section we will prove that, for all $n < \omega$, n -independence has all properties of an independence relation restricted to M_n with the *possible exception* of the symmetry property. Actually the restriction to M_n is only needed for the extension property. We also show that if the theory T has soft elimination of imaginaries, then the hierarchy of n -independence relations collapses to the bottom level of 0-independence.

Definition 4.1. Let $n < \omega$. For $A, B, C \subseteq M^{\text{eq}}$ define $A \downarrow_C^n B$ if and only if, for all finite $A' \subseteq A$, $\text{rk}^n(A'/BC) = \text{rk}^n(A'/C)$. If $\bar{a}$ and $\bar{b}$ are sequences then $\bar{a} \downarrow_C^n \bar{b}$ means the same as $\text{rng}(\bar{a}) \downarrow_C^n \text{rng}(\bar{b})$.

Example 4.2. Let T be as in Example 3.3, $\mathcal{M} \models T$ and let $a, b \in M$ be distinct and such that $[a]_E = [b]_E$. By similar arguments as in that example we get $\text{rk}^0(a/b) = \text{rk}^0(a)$ and $\text{rk}^1(a/b) = 1 < 2 = \text{rk}^1(a)$. Hence $a \downarrow_\emptyset^0 b$ and $a \not\downarrow_\emptyset^1 b$.

For the rest of this section we fix an arbitrary $n < \omega$.

Lemma 4.3. (Invariance) Suppose that σ is an elementary function from a subset of M^{eq} to a subset of M^{eq} and that A, B , and C are subsets of the domain of σ . If $A \downarrow_C^n B$ then $\sigma(A) \downarrow_{\sigma(C)}^n \sigma(B)$.

Proof. Suppose that σ is an elementary function and that A, B , and C are subsets of the domain of σ . Suppose that $A \downarrow_C^n B$, so by Definition 4.1 of $\downarrow^n$ we have $\text{rk}^n(A'/BC) = \text{rk}^n(A'/C)$ for all finite $A' \subseteq A$. It suffices to show that for every finite $A' \subseteq A$, $\text{rk}^n(\sigma(A')/\sigma(B)\sigma(C)) = \text{rk}^n(\sigma(A')/\sigma(C))$. This follows if we can show that for all $A, B \subseteq M^{\text{eq}}$ where A is finite, $\text{rk}^n(A/B) = \text{rk}^n(\sigma(A)/\sigma(B))$. Since also the inverse of σ is an elementary function, it actually suffices to show that $\text{rk}^n(A/B) \geq \text{rk}^n(\sigma(A)/\sigma(B))$.

So let $A = \{a_1, \dots, a_k\} \subseteq A$ be finite. Lemma 3.11 tells that there is finite $B' = \{b_1, \dots, b_l\} \subseteq B$ such that $\text{rk}^n(A/B') = \text{rk}^n(A/B)$. Since $\text{tp}(a_1, \dots, a_k, b_1, \dots, b_l) = \text{tp}(\sigma(a_1), \dots, \sigma(a_k), \sigma(b_1), \dots, \sigma(b_l))$ it follows from Lemmas 3.9 and 3.13 that

$$\text{rk}^n(\sigma(A)/\sigma(B)) \leq \text{rk}^n(\sigma(a_1), \dots, \sigma(a_k)/\sigma(b_1), \dots, \sigma(b_l)) = \text{rk}^n(A/B') = \text{rk}^n(A/B).$$

□

Lemma 4.4. (Monotonicity) Let $A, B, C, D \subseteq M^{\text{eq}}$ where $B \subseteq C \subseteq D$. If $A \downarrow_B^n D$, then $A \downarrow_B^n C$ and $A \downarrow_C^n D$.

Proof. Let $A, B, C, D \subseteq M^{\text{eq}}$ where $B \subseteq C \subseteq D$. Suppose that $A \downarrow_B^n D$ and let $A' \subseteq A$ be finite. Then $\text{rk}^n(A'/D) = \text{rk}^n(A'/B)$. By Lemma 3.9, $\text{rk}^n(A'/B) = \text{rk}^n(A'/D) \leq \text{rk}^n(A'/C) \leq \text{rk}^n(A'/B)$. Hence $\text{rk}^n(A'/D) = \text{rk}^n(A'/C) = \text{rk}^n(A'/B)$ and thus $A \downarrow_B^n C$ and $A \downarrow_C^n D$. □

Lemma 4.5. (Transitivity) Suppose that $A, B, C, D \subseteq M^{\text{eq}}$ and $B \subseteq C \subseteq D$. Then $A \downarrow_B^n D$ if and only if $A \downarrow_B^n C$ and $A \downarrow_C^n D$.

Proof. Suppose that $A, B, C, D \subseteq M^{\text{eq}}$ and $B \subseteq C \subseteq D$. By the definition of n -independence it suffices to prove that for all finite $A' \subseteq A$, $A' \downarrow_B^n D$ if and only if $A' \downarrow_B^n C$ and $A' \downarrow_C^n D$. Therefore we can assume that A is finite. It follows from Lemma 4.4 that if $A \downarrow_B^n D$ then $A \downarrow_B^n C$ and $A \downarrow_C^n D$.

So suppose that $A \downarrow_B^n C$ and $A \downarrow_C^n D$. Then $\text{rk}^n(A/B) = \text{rk}^n(A/C) = \text{rk}^n(A/D)$, so $A \downarrow_B^n D$. $\square$

Lemma 4.6. (Finite character) *Let $A, B, C \subseteq M^{\text{eq}}$ and suppose that $A \not\downarrow_C^n B$. Then there are finite $A' \subseteq A$ and finite $B' \subseteq B$ such that $A' \not\downarrow_C^n B'$.*

Proof. If $A \not\downarrow_C^n B$ then (by definition of $\not\downarrow$) $A' \not\downarrow_C^n B$ for some finite $A' \subseteq A$. Then $\text{rk}^n(A'/BC) < \text{rk}^n(A'/C)$. By Lemma 3.11 (i) there is finite $B' \subseteq B$ such that $\text{rk}^n(A'/B'C) < \text{rk}^n(A'/C)$. Hence $A' \not\downarrow_C^n B'$. $\square$

Lemma 4.7. (Locality) *If $A, B \subseteq M^{\text{eq}}$ and A is finite then there is finite $C \subseteq B$ such that $A \downarrow_C^n B$.*

Proof. Let $A, B \subseteq M^{\text{eq}}$ and suppose that A is finite. From Lemma 3.11 (ii) it follows that there is a finite $C \subseteq B$ such that $\text{rk}^n(A/C) = \text{rk}^n(A/B)$. Hence $A \downarrow_C^n B$. $\square$

Lemma 4.8. (Extension) *Let $\bar{a}$ be a finite sequence of elements from M_n , and let $B, C \subseteq M^{\text{eq}}$ where $B \subseteq C$. Then there is $\bar{d}' \subseteq M^{\text{eq}}$ such that $\text{tp}(\bar{d}'/B) = \text{tp}(\bar{d}/B)$ and $\bar{d}' \downarrow_B^n C$.*

Proof. Under the given assumptions, Lemma 3.16 implies that there is a finite sequence $\bar{a}'$ of elements from M_n such that $\text{tp}(\bar{a}'/B) = \text{tp}(\bar{a}/B)$ and $\text{rk}^n(\bar{a}'/C) = \text{rk}^n(\bar{a}/B)$. Hence $\text{rk}^n(\bar{a}'/B) = \text{rk}^n(\bar{a}/B)$, so $\bar{a}' \downarrow_B^n C$. $\square$

Lemma 4.9. *Suppose that T has soft elimination of imaginaries. For all $n < \omega$ and all $A, B, C \subseteq M^{\text{eq}}$ we have $A \downarrow_C^n B$ if and only if $A \downarrow_C^0 B$.*

Proof. By definition of $\downarrow^n$, $A \downarrow_C^n B$ if and only if $\text{rk}^n(A'/BC) = \text{rk}^n(A'/C)$ for all finite $A' \subseteq A$. Under the assumption that T has soft elimination of imaginaries, Lemma 3.8 implies that $\text{rk}^n(A'/BC) = \text{rk}^0(A'/BC)$ and $\text{rk}^n(A'/C) = \text{rk}^0(A'/C)$ for all finite $A' \subseteq A$. Hence $A \downarrow_C^n B$ if and only if $A \downarrow_C^0 B$. $\square$

5. THE EXCHANGE PROPERTY AND SYMMETRY

According to the results in Section 4, for each $n < \omega$, $\downarrow^n$ has all properties of an independence relation restricted to M_n with the *possible exception* of the symmetry property. In this section we prove that, for any $n < \omega$, the symmetry of $\downarrow^n$ is a consequence of a more restricted symmetry property, namely that the following assumption holds:

Assumption 5.1. (Exchange property with respect to n) In this section we fix an arbitrary $n < \omega$. We assume that if $C \subseteq M^{\text{eq}}$, $2 \leq k < \omega$, $a_1, \dots, a_k \in M_n$, $\text{rk}^n(a_i/C) = 1$ for all $i = 1, \dots, k$, and $a_k \in \text{acl}^{\text{eq}}(\{a_1, \dots, a_{k-1}\} \cup C) \setminus \text{acl}^{\text{eq}}(\{a_2, \dots, a_{k-1}\} \cup C)$, then $a_1 \in \text{acl}^{\text{eq}}(\{a_2, \dots, a_k\} \cup C)$.

Lemma 5.2. *Let $C \subseteq M^{\text{eq}}$, $X = \{d \in M_n : \text{rk}^n(d/C) = 1\}$, and for every $A \subseteq X$, let $\text{cl}(A) = \text{acl}^n(AC) \cap X$. Then (X, cl) is a pregeometry.*

Proof. It is well-known that for all $A \subseteq M^{\text{eq}}$, $A \subseteq \text{acl}^{\text{eq}}(A)$, $\text{acl}^{\text{eq}}(\text{acl}^{\text{eq}}(A)) = \text{acl}^{\text{eq}}(A)$, and if $a \in \text{acl}^{\text{eq}}(A)$ then $a \in \text{acl}^{\text{eq}}(A')$ for some finite $A' \subseteq A$. It follows straightforwardly that cl has properties (1) – (3) in Definition 2.10, even without Assumption 5.1. Suppose that $A \subseteq X$, $b, c \in X$, and $b \in \text{cl}(A \cup \{c\}) \setminus \text{cl}(A)$. Then there is finite $A' \subseteq A$ such that $b \in \text{cl}(A' \cup \{c\}) \setminus \text{cl}(A)$. By Assumption 5.1, we get $c \in \text{cl}(A \cup \{b\})$. Hence (X, cl) is a pregeometry. $\square$

Due to Lemma 5.2, whenever $C \subseteq M^{\text{eq}}$, $X = \{d \in M_n : \text{rk}^n(d/C) = 1\}$, and $A \subseteq X$ it makes sense to talk about a *basis* of A and the *dimension* of A with respect to (X, cl) where cl is defined as in Lemma 5.2.

Definition 5.3. Let $A, B \subseteq M^{\text{eq}}$ and suppose that A is finite. An *n-canonical coordinatization sequence* (*n-ccs*) for A/B (“ A over B ”) is a (finite) sequence $a_1, \dots, a_\alpha \in \text{acl}^n(A)$ together with a *core sequence of indices* $0 = k_0 < k_1 < \dots < k_m = \alpha$ such that

- (1) $\text{acl}^n(A) \subseteq \text{acl}^n(\{a_1, \dots, a_\alpha\} \cup B)$ and,
- (2) for all $j = 0, \dots, m-1$, $\{a_{k_j+1}, \dots, a_{k_{j+1}}\}$ is a basis of

$$\text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_j}\} \cup B) = 1\}.$$

Lemma 5.4. Let $A, B \subseteq M^{\text{eq}}$ where A is finite.

- (i) There is an *n-ccs* for A/B .
- (ii) Suppose that $a_1, \dots, a_\alpha$ and $a'_1, \dots, a'_{\alpha'}$ are two *n-ccs* for A/B with core sequences $0 = k_0 < k_1 < \dots < k_s = \alpha$ and $0 = l_0 < l_1 < \dots < l_t = \alpha'$, respectively. Then $s = t$ and, for all $i = 0, \dots, s$, $k_i = l_i$ (so $\alpha = \alpha'$). Moreover, for all $m = 1, \dots, s$, $\text{acl}^{\text{eq}}(\{a_1, \dots, a_{k_m}\} \cup B) = \text{acl}^{\text{eq}}(\{a'_1, \dots, a'_{k_m}\} \cup B)$.

Proof. We prove part (i) and point out during the argument why the uniqueness properties of part (ii) follow. Suppose that $\text{acl}^n(A) \not\subseteq \text{acl}^n(B)$ for otherwise the empty sequence has the required properties. By convention let $k_0 = 0$. By Lemma 3.10 the following set is nonempty:

$$X_1 = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/B) = 1\}.$$

By Lemma 5.2 $(\{d \in M_n : \text{rk}^n(d/B) = 1\}, \text{cl})$ with $\text{cl}(Y) = \text{acl}^n(YB) \cap \{d \in M_n : \text{rk}^n(d/B) = 1\}$ for all $Y \subseteq \{d \in M_n : \text{rk}^n(d/B) = 1\}$ is a pregeometry. Let k_1 be the dimension of X_1 and let $\{a_1, \dots, a_{k_1}\} \subseteq X_1$ be a basis of X_1 . Note that k_1 is determined only by A , B and n . Also observe that for any other basis $\{a'_1, \dots, a'_{k_1}\} \subseteq X_1$ we have $\text{acl}^{\text{eq}}(\{a'_1, \dots, a'_{k_1}\} \cup B) = \text{acl}^{\text{eq}}(\{a_1, \dots, a_{k_1}\} \cup B)$.

Now suppose that, for some m , $k_1 < \dots < k_m \leq \alpha$ and $a_1, \dots, a_{k_m} \in \text{acl}^n(A)$ have been defined in such a way that, for all $j = 0, \dots, m-1$, $\{a_{k_j+1}, \dots, a_{k_{j+1}}\}$ is a basis of

$$X_j = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_j}\} \cup B) = 1\}.$$

Also suppose that the uniqueness properties of part (ii) hold for $k_1, \dots, k_m$ and $\text{acl}^{\text{eq}}(\{a_1, \dots, a_{k_i}\} \cup B)$ for $i \leq m$. If $\text{acl}^n(A) \subseteq \text{acl}^n(\{a_1, \dots, a_{k_m}\} \cup B)$ then we let $\alpha = k_m$ and then $a_1, \dots, a_\alpha$ is an *n-ccs* for A/B .

Suppose that $A \not\subseteq \text{acl}^n(\{a_1, \dots, a_{k_m}\} \cup B)$. By Lemma 3.10 the set

$$X_{m+1} = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_m}\} \cup B) = 1\}$$

is nonempty. Let d be the dimension of X_{m+1} , let $k_{m+1} = k_m + d$, and let

$$\{a_{k_m+1}, \dots, a_{k_{m+1}}\} \subseteq X_{m+1}$$

be a basis of X_{m+1} . Note that k_{m+1} depends only on k_m and the dimension of X_{m+1} where the later depends only on A , B , $a_1, \dots, a_{k_m}$ and n . The inductive assumption that for any choice of $a'_1, \dots, a'_{k_m}$ with the same properties as $a_1, \dots, a_{k_m}$ we have $\text{acl}^{\text{eq}}(\{a'_1, \dots, a'_{k_m}\} \cup B) = \text{acl}^{\text{eq}}(\{a_1, \dots, a_{k_m}\} \cup B)$ implies that $\text{acl}^{\text{eq}}(\{a'_1, \dots, a'_{k_{m+1}}\} \cup B) = \text{acl}^{\text{eq}}(\{a_1, \dots, a_{k_{m+1}}\} \cup B)$ for any basis $\{a'_{k_m+1}, \dots, a'_{k_{m+1}}\} \subseteq X_{m+1}$. It also implies that, ultimately, k_{m+1} depends only on A , B and n .

Observe that for all $i = 1, \dots, k_{m+1}$, $a_i \notin \text{acl}^n(\{a_1, \dots, a_{i-1}\} \cup B)$ so by Lemma 3.5 we have $\text{rk}^n(A/B) \geq k_{m+1}$. Since $\text{rk}^n(A/B)$ is finite (as A is finite) it follows that the process of extending the current sequence $a_1, \dots, a_{k_{m+1}}$ in the described way will terminate after finitely many steps and then we have an *n-ccs* for A/B . $\square$

Lemma 5.5. *Let $A, B \subseteq M^{\text{eq}}$ and suppose that $\text{rk}^n(A/B) = \alpha < \omega$. If $a_1, \dots, a_\alpha$ is an n -ccs for A/B then it is also an n -cs of A/B .*

Proof. Suppose that $a_1, \dots, a_\alpha$ is an n -ccs for A/B with core sequence of indices $0 = k_0 < k_1 < \dots < k_m = \alpha$. From Definition 5.3 of an n -ccs it follows that, for all $i = 1, \dots, \alpha$, $a_i \notin \text{acl}^n(\{a_1, \dots, a_{i-1}\} \cup B)$. Since $\alpha = \text{rk}^n(A/B)$ it follows from Definition 3.6 of an n -cs that $a_1, \dots, a_\alpha$ is an n -cs for A/B . $\square$

Lemma 5.6. *Let $A, B \subseteq M^{\text{eq}}$ where A is finite. If $\text{rk}^n(A/B) = \alpha$ then every n -ccs for A/B has length α .*

Proof. Suppose that $\text{rk}^n(A/B) = \alpha$. By Lemma 5.4, it suffices to prove that there is at least one n -ccs for A/B that has length α . By Lemma 3.5 and Definition 3.6 of n -cs, there is an n -cs $a_1, \dots, a_\alpha$ for A/B . By the same definition and lemma we have

$$(5.1) \quad \begin{aligned} a_i &\notin \text{acl}^n(a_i/\{a_1, \dots, a_{i-1}\} \cup B) \quad \text{and} \\ \text{rk}^n(a_i/\{a_1, \dots, a_{i-1}\} \cup B) &= 1 \quad \text{for all } i = 1, \dots, \alpha. \end{aligned}$$

The rest of the proof will show how to transform $a_1, \dots, a_\alpha$ into an n -ccs for A/B with length α . This will be done step by step via a sequence of claims.

Claim 5.7. *Suppose that $0 \leq k_1 < k_2 < \alpha$ and that $\{a_{k_1+1}, \dots, a_{k_2}\}$ is an independent subset of*

$$X = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_1}\} \cup B) = 1\}.$$

If $\{a_{k_1+1}, \dots, a_{k_2}\}$ is not a basis of X then there are $d \in X$ and $k_2 < l \leq \alpha$ such that $\{a_{k_1+1}, \dots, a_{k_2}, d\}$ is an independent subset of X and $a_1, \dots, a_{l-1}, d, a_{l+1}, \dots, a_\alpha$ is an n -cs for A/B .

Proof. Suppose that $\{a_{k_1+1}, \dots, a_{k_2}\}$ is not a basis of X . There is $d \in X$ such that $d \notin \text{acl}^n(\{a_1, \dots, a_{k_2}\} \cup B)$. If $d \notin \text{acl}^n(\{a_1, \dots, a_\alpha\} \cup B)$ then it follows (by (5.1) and Lemma 3.5) that $\text{rk}^n(A/B) \geq \alpha + 1$, which contradicts the assumption. Hence there is a minimal $1 \leq l \leq \alpha$ such that $d \in \text{acl}^n(\{a_1, \dots, a_l\} \cup B)$. Since $d \notin \text{acl}^n(\{a_1, \dots, a_{k_2}\} \cup B)$ we must have $l > k_2$. Then $d \notin \text{acl}^n(\{a_1, \dots, a_{l-1}\} \cup B)$. As also $d \in X$ it follows (using Lemma 3.9) that $\text{rk}^n(d/\{a_1, \dots, a_{l-1}\} \cup B) = 1$. By (5.1), we have $\text{rk}^n(a_l/\{a_1, \dots, a_{l-1}\} \cup B) = 1$. Now Assumption 5.1 implies that $a_l \in \text{acl}^n(\{a_1, \dots, a_{l-1}, d\} \cup B)$.

Suppose, for a contradiction, that there is $i > l$ such that

$$a_i \in \text{acl}^n(\{a_1, \dots, a_{l-1}, d, a_{l+1}, \dots, a_{i-1}\} \cup B).$$

Since $d \in \text{acl}^n(\{a_1, \dots, a_l\} \cup B)$ it follows that $a_i \in \text{acl}^n(\{a_1, \dots, a_{i-1}\} \cup B)$ which contradicts (5.1). Hence, no such $i > l$ exists and it follows that $a_1, \dots, a_{l-1}, d, a_{l+1}, \dots, a_\alpha$ is an n -cs for A/B . $\square$

Claim 5.8. *Suppose that, for some $0 \leq k \leq \alpha$ and $1 \leq s \leq \alpha - k$, we have*

$$\text{rk}^n(a_{k+i}/\{a_1, \dots, a_k\} \cup B) = 1 \quad \text{for all } i = 1, \dots, s.$$

If $k + s < l \leq \alpha$ and $\text{rk}^n(a_l/\{a_1, \dots, a_k\} \cup B) = 1$ then

$$a_1, \dots, a_{k+s}, a_l, a_{k+s+1}, \dots, a_{l-1}, a_{l+1}, \dots, a_\alpha \quad \text{is an } n\text{-cs for } A/B.$$

Proof. Let k and s be as assumed. Suppose that $k + s < l \leq \alpha$ and $\text{rk}^n(a_l/\{a_1, \dots, a_k\} \cup B) = 1$. Suppose, for a contradiction, that there is $k + s < j < l$ such that $a_j \in \text{acl}^n(\{a_1, \dots, a_{k+s}, a_l, a_{k+s+1}, \dots, a_{j-1}\} \cup B)$. By (5.1) and Lemma 3.9,

$$\begin{aligned} 1 &= \text{rk}^n(a_l/\{a_1, \dots, a_k\} \cup B) \\ &\geq \text{rk}^n(a_l/\{a_1, \dots, a_{j-1}\} \cup B) \\ &\geq \text{rk}^n(a_l/\{a_1, \dots, a_{l-1}\} \cup B) = 1. \end{aligned}$$

Hence $\text{rk}^n(a_l/\{a_1, \dots, a_{j-1}\} \cup B) = 1$. By (5.1) we also have $\text{rk}^n(a_j/\{a_1, \dots, a_{j-1}\} \cup B) = 1$. Now Assumption 5.1 implies that $a_l \in \text{acl}^n(\{a_1, \dots, a_j\} \cup B)$ and since $j < l$ this contradicts (5.1).

For a contradiction, suppose that there is $l < j \leq \alpha$ such that

$$a_j \in \text{acl}^n(\{a_1, \dots, a_{k+s}, a_l, a_{k+s+1}, \dots, a_{l-1}, a_{l+1}, \dots, a_{j-1}\} \cup B).$$

Since $\{a_1, \dots, a_{k+s}, a_l, a_{k+s+1}, \dots, a_{l-1}, a_{l+1}, \dots, a_{j-1}\} = \{a_1, \dots, a_{j-1}\}$ we have a contradiction to (5.1). It follows that $a_1, \dots, a_{k+s}, a_l, a_{k+s+1}, \dots, a_{l-1}, a_{l+1}, \dots, a_\alpha$ is an n -cs for A/B . $\square$

Claim 5.9. *Suppose that $0 \leq k_1 < k_2 < \alpha$ and that $\{a_{k_1+1}, \dots, a_{k_2}\}$ is an independent subset of*

$$X = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_1}\} \cup B) = 1\}.$$

If $\{a_{k_1+1}, \dots, a_{k_2}\}$ is not a basis of X then there is $d \in X$ such that $\{a_{k_1+1}, \dots, a_{k_2}, d\}$ is an independent subset of X and, for some $k_2 < i \leq \alpha$,

$$a_1, \dots, a_{k_2}, d, a_{k_2+1}, \dots, a_{i-1}, a_{i+1}, \dots, a_\alpha \text{ is an } n\text{-cs for } A/B.$$

Proof. Let us make the assumptions of the claim. Moreover, suppose that $\{a_{k_1+1}, \dots, a_{k_2}\}$ is not a basis of X . Now we first apply Claim 5.7 to find $d \in X$ and $k_2 < i \leq \alpha$ such that $\{a_{k_1+1}, \dots, a_{k_2}, d\}$ is an independent subset of X and $a_1, \dots, a_{i-1}, d, a_{i+1}, \dots, a_\alpha$ is an n -cs for A/B . Then Claim 5.8 implies that also $a_1, \dots, a_{k_2}, d, a_{k_2+1}, \dots, a_{i-1}, a_{i+1}, \dots, a_\alpha$ is an n -cs for A/B . $\square$

Claim 5.10. *Suppose that $0 \leq k_1 < k_2 < \alpha$ and that $\{a_{k_1+1}, \dots, a_{k_2}\}$ is an independent subset of*

$$X = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_1}\} \cup B) = 1\}.$$

If $\{a_{k_1+1}, \dots, a_{k_2}\}$ is not a basis of X then there are $1 \leq s \leq \alpha - k_2$, $d_1, \dots, d_s \in X$, and $a'_{\alpha-k_2-s}, \dots, a'_\alpha \in M_n$ such that $\{a_{k_1+1}, \dots, a_{k_2}, d_1, \dots, d_s\}$ is a basis of X and $a_1, \dots, a_{k_2}, d_1, \dots, d_s, a'_{\alpha-k_2-s}, \dots, a'_\alpha$ is an n -cs for A/B .

Proof. The claim follows from repeated uses of Claim 5.9 and the fact that all bases of X have the same finite cardinality. $\square$

Claim 5.11. *Suppose that there are $m < \alpha$ and $0 = k_0 < k_1 < \dots < k_m < \alpha$ such that, for all $j = 0, \dots, m-1$, $\{a_{k_j+1}, \dots, a_{k_{j+1}}\}$ is a basis of*

$$X_j = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_j}\} \cup B) = 1\}.$$

Then there are $k_m < k_{m+1} \leq \alpha$, a basis $\{a'_{k_m+1}, \dots, a'_{k_{m+1}}\}$ of

$$X_{m+1} = \text{acl}^n(A) \cap \{d \in M_n : \text{rk}^n(d/\{a_1, \dots, a_{k_m}\} \cup B) = 1\},$$

and $a'_{k_{m+1}+1}, \dots, a'_\alpha \in M_n$ such that $a_1, \dots, a_{k_m}, a'_{k_m+1}, \dots, a'_\alpha$ is an n -cs for A/B .

Proof. From (5.1) we get $\text{rk}^n(a_{k_m+1}/\{a_1, \dots, a_{k_m}\} \cup B) = 1$. If $\{a_{k_m+1}\}$ is a basis of X_{m+1} then let $a'_i = a_i$ for all $i = k_m + 1, \dots, \alpha$ and we are done. Otherwise we get the conclusion of the claim by Claim 5.10. $\square$

Observe that the assumptions of Claim 5.11 are vacuously satisfied if $m = 0$. Therefore Lemma 5.6 follows by induction on m where Claim 5.11 serves as the inductive step. This completes the proof of Lemma 5.6. $\square$

Lemma 5.12. *Let $A, B, C \subseteq M^{\text{eq}}$ where B is finite. Suppose that $\text{rk}^n(B/AC) = \text{rk}^n(B/C) = \beta$ (so $\beta < \omega$).*

- (i) *If $b_1, \dots, b_\beta$ is an n -ccs for B/AC with core sequence of indices $0 = k_0 < k_1 < \dots < k_t = \beta$, then $b_1, \dots, b_\beta$ is also an n -ccs for B/C with the same core sequence of indices.*
- (ii) *If $b_1, \dots, b_\beta$ is an n -ccs for B/C with core sequence of indices $0 = k_0 < k_1 < \dots < k_t = \beta$, then $b_1, \dots, b_\beta$ is also an n -ccs for B/AC with the same core sequence of indices.*

Proof. Suppose that B is finite and $\text{rk}^n(B/AC) = \text{rk}^n(B/C) = \beta$.

(i) Let $b_1, \dots, b_\beta$ be an n -ccs for B/AC with core sequence of indices $0 = k_0 < k_1 < \dots < k_t = \beta$. Then, for all $m = 0, \dots, t-1$,

$$(5.2) \quad \begin{aligned} &\{b_{k_m+1}, \dots, b_{k_{m+1}}\} \text{ is a basis of} \\ &X_m^{AC} = \text{acl}^n(B) \cap \{d \in M_n : \text{rk}^n(d/\{b_1, \dots, b_{k_m}\} \cup AC) = 1\} \text{ and,} \\ &\text{for all } i = k_m + 1, \dots, k_{m+1}, \text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup AC) = 1. \end{aligned}$$

This implies that

$$(5.3) \quad \begin{aligned} &\text{for all } i = 1, \dots, \beta, b_i \notin \text{acl}^n(\{b_1, \dots, b_{i-1}\} \cup AC) \text{ and hence} \\ &b_i \notin \text{acl}^n(\{b_1, \dots, b_{i-1}\} \cup C). \end{aligned}$$

It also follows (using Lemma 3.9) that, for all $m = 0, \dots, t-1$ and all $i = k_m+1, \dots, k_{m+1}$,

$$\text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup C) \geq 1.$$

Towards a contradiction, suppose that there are $m \in \{0, \dots, t-1\}$ and $i \in \{k_m+1, \dots, k_{m+1}\}$ such that $\text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup C) \geq 2$. Then there is $d \in \text{acl}^n(b_i)$ such that $d \notin \text{acl}^n(\{b_1, \dots, b_{k_m}\} \cup C)$ and $\text{rk}^n(b_i/\{d, b_1, \dots, b_{k_m}\} \cup C) \geq 1$. Since $d \in \text{acl}^n(b_i)$ it follows from (5.3) that for all $k_m < j \leq \alpha$,

$$b_j \notin \text{acl}^n(\{b_1, \dots, b_{k_m}, d, b_{k_m+1}, \dots, b_{j-1}\} \cup C).$$

This implies (via Lemma 3.5) that $\text{rk}^n(B/C) \geq \beta + 1$, contradicting the assumption. Thus we conclude that

$$(5.4) \quad \begin{aligned} &\text{for all } m = 0, \dots, t-1 \text{ and all } i = k_m + 1, \dots, k_{m+1}, \\ &\text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup C) = 1. \end{aligned}$$

By (5.3) and (5.4), for all $m = 0, \dots, t-1$, $\{b_{k_m+1}, \dots, b_{k_{m+1}}\}$ is an independent subset of

$$X_m^C = \text{acl}^n(B) \cap \{d \in M_n : \text{rk}^n(d/\{b_1, \dots, b_{k_m}\} \cup C) = 1\}.$$

Towards a contradiction, suppose that for some $m \in \{0, \dots, t-1\}$ there is $d \in X_m^C$ such that $d \notin \text{acl}^n(\{b_1, \dots, b_{k_m}\} \cup C)$. Since $\{b_1, \dots, b_{k_m}\}$ is a basis of X_m^{AC} it follows that $d \in \text{acl}^n(\{b_1, \dots, b_{k_m}\} \cup AC)$. If, for some $i > k_m$, we would have that $b_i \in \text{acl}^n(\{b_1, \dots, b_{k_m}, d, b_{k_m+1}, \dots, b_{i-1}\} \cup C)$, then we would get

$$b_i \in \text{acl}^n(\{b_1, \dots, b_{k_m}, b_{k_m+1}, \dots, b_{i-1}\} \cup AC)$$

which would contradict (5.3). Hence we conclude that for all $i > k_m$,

$$b_i \notin \text{acl}^n(\{b_1, \dots, b_{k_m}, d, b_{k_m+1}, \dots, b_{i-1}\} \cup C).$$

But this implies that $\text{rk}^n(B/C) \geq \beta + 1$ which contradicts the assumption. Hence there cannot be any $m \in \{0, \dots, t-1\}$ and $d \in X_m^C$ such that $d \notin \text{acl}^n(\{b_1, \dots, b_{k_m}\} \cup C)$, and therefore $\{b_1, \dots, b_{k_m}\}$ is a basis of X_m^C . It follows that $b_1, \dots, b_\beta$ is an n -ccs for B/C with core sequence of indices $0 = k_0 < k_1 < \dots < k_t = \beta$.

(ii) Now suppose that $b_1, \dots, b_\beta$ is an n -ccs for B/C with core sequence of indices $0 = k_0 < k_1 < \dots < k_\delta = \beta$. It follows that for all $m = 0, \dots, \delta-1$ and all $i = k_m + 1, \dots, k_{m+1}$,

$$(5.5) \quad \begin{aligned} &\text{rk}^n(b_i/\{b_1, \dots, b_{i-1}\} \cup C) = \text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup C) = 1, \text{ and} \\ &b_i \notin \text{acl}^n(\{b_1, \dots, b_{i-1}\} \cup C). \end{aligned}$$

By Lemmas 5.4 and 5.6, there is an n -ccs $b'_1, \dots, b'_\beta$ for B/AC with core sequence of indices $0 = l_0 < l_1 < \dots < l_\gamma = \beta$. By part (i), $b'_1, \dots, b'_\beta$ is also an n -ccs for B/C with the same core sequence of indices $0 = l_0 < l_1 < \dots < l_\gamma = \beta$. By Lemma 5.4, we then

have $\gamma = \delta$ and $l_m = k_m$ for all $m = 0, \dots, \delta$. The same lemma also tells that, for all $m = 1, \dots, \delta$,

$$(5.6) \quad \text{acl}^{\text{eq}}(\{b_1, \dots, b_{k_m}\} \cup C) = \text{acl}^{\text{eq}}(\{b'_1, \dots, b'_{k_m}\} \cup C)$$

and hence

$$(5.7) \quad \text{acl}^{\text{eq}}(\{b_1, \dots, b_{k_m}\} \cup AC) = \text{acl}^{\text{eq}}(\{b'_1, \dots, b'_{k_m}\} \cup AC).$$

For all $m = 0, \dots, \delta - 1$, let

$$X_m^{AC} = \text{acl}^n(B) \cap \{d \in M_n : \text{rk}^n(d/\{b_1, \dots, b_{k_m}\} \cup AC) = 1\}$$

and note that by (5.7) we also have

$$(5.8) \quad X_m^{AC} = \text{acl}^n(B) \cap \{d \in M_n : \text{rk}^n(d/\{b'_1, \dots, b'_{k_m}\} \cup AC) = 1\}.$$

Since $b'_1, \dots, b'_\beta$ is an n -ccs for B/AC with core sequence of indices $0 = k_0 < k_1 < \dots < k_\delta = \beta$ it follows that, for all $m = 0, \dots, \delta - 1$, $\{b'_{k_m+1}, \dots, b'_{k_{m+1}}\}$ is a basis of X_m^{AC} , so X_m^{AC} has dimension $k_{m+1} - k_m$. From (5.7) it follows that

$$X_m^{AC} \subseteq \text{acl}^n(\{b_1, \dots, b_{k_{m+1}}\} \cup AC).$$

Fix any $m \in \{0, \dots, \delta - 1\}$. From (5.5), we get

$$\text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup AC) \leq 1 \quad \text{for all } i = k_m + 1, \dots, k_{m+1}.$$

Suppose, for a contradiction, that there is $i \in \{k_m + 1, \dots, k_{m+1}\}$ such that

- $\text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup AC) = 0$ (equivalently, $b_i \in \text{acl}^n(\{b_1, \dots, b_{k_m}\} \cup AC)$),
- or $b_i \in \text{acl}^n(B_i AC)$ where $B_i = \{b_{k_m+1}, \dots, b_{k_{m+1}}\} \setminus \{b_i\}$.

Then some proper subset of $\{b_{k_m+1}, \dots, b_{k_{m+1}}\}$ is a basis of X_m^{AC} which contradicts our previous conclusion that X_m^{AC} has dimension $k_{m+1} - k_m$. Hence we conclude that $\text{rk}^n(b_i/\{b_1, \dots, b_{k_m}\} \cup AC) = 1$ for all $i = k_m + 1, \dots, k_{m+1}$ and that $\{b_{k_m+1}, \dots, b_{k_{m+1}}\}$ is independent (in the pregeometry $\{d \in M_n : \text{rk}^n(d/\{b_1, \dots, b_{k_m}\} \cup AC) = 1\}$) and hence a basis of X_m^{AC} . Since the argument holds for all $m = 0, \dots, \delta - 1$ it follows that $b_1, \dots, b_\beta$ is an n -ccs for B/AC with core sequence of indices $0 = k_0 < k_1 < \dots < k_\delta = \beta$. $\square$

Proposition 5.13. *Let $A, C \subseteq M^{\text{eq}}$ and $B \subseteq M_n$ where A and B are finite. If $\text{rk}^n(A/BC) < \text{rk}^n(A/C)$ then $\text{rk}^n(B/AC) < \text{rk}^n(B/C)$.*

Proof. Let $A, C \subseteq M^{\text{eq}}$ and $B \subseteq M_n$ where A and B are finite. We prove the result by induction on $\text{rk}^n(B/C)$. If $\text{rk}^n(B/C) = 0$ then, since $B \subseteq M_n$ and by the definition of rk^n , $B \subseteq \text{acl}^n(B) \subseteq \text{acl}^n(C)$ so $\text{acl}^n(BC) = \text{acl}^n(C)$. Hence (by the definition of rk^n) $\text{rk}^n(A/BC) = \text{rk}^n(A/C)$ and the statement is vacuously satisfied.

The induction step remains. The induction hypothesis will be:

- (IH) For every finite $B' \subseteq M^{\text{eq}}$, if $\text{rk}^n(B'/C) < \text{rk}^n(B/C)$ and $\text{rk}^n(A/B'C) < \text{rk}^n(A/C)$, then $\text{rk}^n(B'/AC) < \text{rk}^n(B'/C)$.

Suppose that $\text{rk}^n(A/BC) < \text{rk}^n(A/C)$. Also let $\beta = \text{rk}^n(B/C)$. (We will show that $\text{rk}^n(B/AC) < \beta$.) Then there is an n -ccs $b_1, \dots, b_\beta$ for B/C with core sequence of indices $0 = k_0 < k_1 < \dots < k_\delta = \beta$. In particular, $b_1, \dots, b_\beta \in \text{acl}^n(B)$. We divide the argument into two main cases.

Case 1. Suppose that there is $s < \beta$ such that $\text{rk}^n(A/\{b_1, \dots, b_s\} \cup C) < \text{rk}^n(A/C)$.

We may assume that s is minimal such that the above holds. As $b_1, \dots, b_\beta$ is an n -ccs for B/C , hence (by Lemma 5.5) an n -cs for B/C , it follows from Lemma 3.5 that

$$\text{rk}^n(b_1, \dots, b_s/C) = s < \beta = \text{rk}^n(B/C).$$

Now the induction hypothesis (IH) implies that

$$\text{rk}^n(b_1, \dots, b_s/AC) < \text{rk}^n(b_1, \dots, b_s/C) = s.$$

By Lemma 3.5 there is $t \leq s$ such that

$$(5.9) \quad b_t \in \text{acl}^n(\{b_1, \dots, b_{t-1}\} \cup AC).$$

Let t be minimal such that the above holds. Towards a contradiction, suppose that $\text{rk}^n(B/AC) = \text{rk}^n(B/C) = \beta$. Then there is an n -ccs $b'_1, \dots, b'_\beta$ for B/AC with core sequence of indices $0 = l_0 < l_1 < \dots < l_\gamma = \beta$. By Lemma 5.12, $b_1, \dots, b_\beta$ is also an n -ccs for B/AC with core sequence of indices $0 = k_0 < k_1 < \dots < k_\gamma = \beta$. By Lemma 5.4 we have $\gamma = \delta$ and $l_i = k_i$ for all $i = 0, \dots, \delta$. From Lemma 5.4 it follows that if, for any $m \in \{0, \dots, \delta - 1\}$, we define

$$X_m^{AC} = \text{acl}^n(B) \cap \{d \in M_n : \text{rk}^n(d/\{b_1, \dots, b_{k_m}\} \cup AC) = 1\}$$

then

$$X_m^{AC} = \text{acl}^n(B) \cap \{d \in M_n : \text{rk}^n(d/\{b'_1, \dots, b'_{k_m}\} \cup AC) = 1\}.$$

Also, $b_{k_m+1}, \dots, b_{k_{m+1}}, b'_{k_m+1}, \dots, b'_{k_{m+1}} \in X_m^{AC}$. Let m be such that $k_m < t \leq k_{m+1}$. Now (5.9) implies that some proper subset of $\{b_{k_m+1}, \dots, b_{k_{m+1}}\}$ is a basis of X_m^{AC} . But since $b'_1, \dots, b'_\beta$ is an n -ccs for B/AC with core sequence of indices $0 = k_0 < k_1 < \dots < k_\delta = \beta$ it follows that $\{b'_{k_m+1}, \dots, b'_{k_{m+1}}\}$ is a basis of X_m^{AC} . So there are two bases of X_m^{AC} with different cardinalities, which is impossible. Thus we conclude that $\text{rk}^n(B/AC) < \beta$.

Case 2. Suppose that Case 1 does not hold, that is, suppose that for all $s < \beta$, $\text{rk}^n(A/\{b_1, \dots, b_s\} \cup C) = \text{rk}^n(A/C)$.

Let $\alpha = \text{rk}^n(A/C)$, so by assumption,

$$\text{rk}^n(A/\{b_1, \dots, b_{\beta-1}\} \cup C) = \text{rk}^n(A/C) = \alpha.$$

Since $b_1, \dots, b_\beta$ is an n -ccs for B/C , hence an n -cs for B/C , it follows (by Lemma 3.5 and the assumption that $B \subseteq M_n$) that $B \subseteq \text{acl}^n(B) \subseteq \text{acl}^n(\{b_1, \dots, b_\beta\} \cup C)$ and hence $\text{rk}^n(A/BC) = \text{rk}^n(A/\{b_1, \dots, b_\beta\} \cup C)$. From the assumption that $\text{rk}^n(A/BC) < \text{rk}^n(A/C)$ we now get

$$(5.10) \quad \text{rk}^n(A/\{b_1, \dots, b_\beta\} \cup C) < \text{rk}^n(A/\{b_1, \dots, b_{\beta-1}\} \cup C) = \alpha.$$

Let $a_1, \dots, a_\alpha$ be an n -ccs for A/C with core sequence of indices $0 = l_0 < l_1 < \dots < l_\gamma = \alpha$. By Lemma 5.12, $a_1, \dots, a_\alpha$ is also an n -ccs for $A/\{b_1, \dots, b_{\beta-1}\} \cup C$ with the same core sequence of indices $0 = l_0 < l_1 < \dots < l_\gamma = \alpha$. This implies that,

$$\text{for all } i = 1, \dots, \alpha, \quad \text{rk}^n(a_i/\{a_1, \dots, a_{i-1}, b_1, \dots, b_{\beta-1}\} \cup C) = 1.$$

Since $b_1, \dots, b_\beta$ is an n -ccs for B/C also have that

$$(5.11) \quad \text{for all } i = 1, \dots, \beta, \quad \text{rk}^n(b_i/\{b_1, \dots, b_{i-1}\} \cup C) = 1.$$

By (5.10) (and Lemma 3.5) there is $s \leq \alpha$ such that

$$(5.12) \quad a_s \in \text{acl}^n(\{a_1, \dots, a_{s-1}, b_1, \dots, b_\beta\} \cup C).$$

Let s be minimal such that the above holds.

If $b_\beta \in \text{acl}^n(\{b_1, \dots, b_{\beta-1}\} \cup AC)$ then $b_1, \dots, b_\beta$ is not an n -ccs for B/AC so by Lemma 5.12 we get $\text{rk}^n(B/AC) < \text{rk}^n(B/C)$.

Now suppose that $b_\beta \notin \text{acl}^n(\{b_1, \dots, b_{\beta-1}\} \cup AC)$. As $a_1, \dots, a_{s-1} \in \text{acl}^n(A)$ we get $b_\beta \notin \text{acl}^n(\{b_1, \dots, b_{\beta-1}, a_1, \dots, a_{s-1}\} \cup C)$ and hence $\text{rk}^n(b_\beta/\{b_1, \dots, b_{\beta-1}, a_1, \dots, a_{s-1}\} \cup C) \geq 1$. This and (5.11) (together with Lemma 3.9) gives

$$(5.13) \quad \text{rk}^n(b_\beta/\{b_1, \dots, b_{\beta-1}, a_1, \dots, a_{s-1}\} \cup C) = 1.$$

Since $a_1, \dots, a_\alpha$ is an n -ccs for $A/\{b_1, \dots, b_{\beta-1}\} \cup C$ we get

$$(5.14) \quad \text{rk}^n(a_s/\{b_1, \dots, b_{\beta-1}, a_1, \dots, a_{s-1}\} \cup C) = 1.$$

Now (5.12), (5.13), (5.14) and Assumption 5.1 imply that

$$b_\beta \in \text{acl}^n(\{b_1, \dots, b_{\beta-1}, a_1, \dots, a_{s-1}\} \cup C)$$

and therefore $b_\beta \in \text{acl}^n(\{b_1, \dots, b_{\beta-1}\} \cup AC)$. Then $b_1, \dots, b_\beta$ is not an n -ccs for B/AC , so by Lemma 5.12 it follows that $\text{rk}^n(B/AC) < \text{rk}^n(B/C)$. This completes the proof of Proposition 5.13. $\square$

Proposition 5.14. (Symmetry of $\downarrow^n$ restricted to M_n) *Let $A, C \subseteq M^{\text{eq}}$ and $B \subseteq M_n$. If $A \not\downarrow_C^n B$ then $B \not\downarrow_C^n A$.*

Proof. Suppose that $A \not\downarrow_C^n B$ where $B \subseteq M_n$. Then there is finite $A' \subseteq A$ such that $\text{rk}^n(A'/BC) < \text{rk}^n(A'/C)$. By Lemma 3.11, there is finite $B' \subseteq B$ such that $\text{rk}^n(A'/B'C) < \text{rk}^n(A'/C)$. By Lemma 3.9 and Proposition 5.13,

$$\text{rk}^n(B'/AC) \leq \text{rk}^n(B'/A'C) < \text{rk}^n(B'/C)$$

and hence $B \not\downarrow_C^n A$. $\square$

In Section 4 we saw that, even without Assumption 5.1, $\downarrow^n$ has all the properties of an independence relation *except*, possibly, for the symmetry property. Now we have:

Theorem 5.15. *Let T be ω -categorical, let $n < \omega$, and suppose that Assumption 5.1 holds for this n . Then $\downarrow^n$ is an independence relation restricted to M_n .*

Proof. By Proposition 5.14, $\downarrow^n$ has the symmetry property with respect to subsets of M_n . By the results in Section 4 $\downarrow^n$ has all the other properties of an independence relation, with respect to any subsets of M^{eq} . $\square$

Under Assumption 5.1 we can strengthen one part of Lemma 3.9 as follows:

Lemma 5.16. *Suppose that $A, B, C \subseteq M^{\text{eq}}$ where A is finite and $B \subseteq A$. Then $\text{rk}^n(A/C) = \text{rk}^n(B/C) + \text{rk}^n(A/BC)$.*

Proof. Let $\beta = \text{rk}^n(B/C)$ and $\alpha = \text{rk}^n(A/BC)$. By Lemmas 5.4 and 5.6 there is an n -ccs $b_1, \dots, b_\beta$ for B/C with core sequence of indices $0 = k_0 < k_1 < \dots < k_s = \beta$. By the same lemmas there is also an n -ccs $a_1, \dots, a_\alpha$ for A/BC with core sequence of indices $0 = l_0 < l_1 < \dots < l_t = \alpha$. Since we must have $\text{acl}^n(B) \subseteq \text{acl}^n(\{b_1, \dots, b_\beta\} \cup C)$ and hence $\text{acl}^n(BC) = \text{acl}^n(\{b_1, \dots, b_\beta\} \cup C)$ it follows that $b_1, \dots, b_\beta, a_1, \dots, a_\alpha$ is an n -ccs for A/C with core sequence of indices $0 = k_0 < k_1 < \dots < k_s < k_s + l_1 < k_s + l_2 < \dots < k_s + l_t = \beta + \alpha$. By Lemma 5.6, we must have $\text{rk}^n(A/C) = \beta + \alpha = \text{rk}^n(B/C) + \text{rk}^n(A/BC)$. $\square$

Corollary 5.17. *If $a_1, \dots, a_k \in M^{\text{eq}}$ and $C \subseteq M^{\text{eq}}$, then*

$$\begin{aligned} \text{rk}^n(a_1, \dots, a_k/C) &= \text{rk}^n(a_1/C) + \text{rk}^n(a_2/\{a_1\} \cup C) + \dots + \text{rk}^n(a_k/\{a_1, \dots, a_{k-1}\} \cup C) \\ &\leq \text{rk}^n(a_1/C) + \dots + \text{rk}^n(a_k/C). \end{aligned}$$

6. CONNECTION TO ROSINESS

In the previous section we proved that if the algebraic closure satisfies the exchange property on elements of M_n with n -rank 1 (over some $C \subseteq M^{\text{eq}}$), then n -independence is an independence relation *restricted to M_n* (recall Definition 2.7). In this section we will use this result to make conclusions about (super)rosiness of ω -categorical theories. In order to do this we also need to involve the notion of thorn-independence.

Definition 6.1. [7, 21] Let $\bar{a}, \bar{b}$ be finite tuples of elements from M^{eq} and let $C \subseteq M^{\text{eq}}$.

- (1) A formula $\varphi(\bar{x}, \bar{a})$ (with all parameters listed by $\bar{a}$) *strongly divides* over C if $\text{tp}(\bar{a}/C)$ is nonalgebraic (i.e. has infinitely many realizations) and the set of formulas $\{\varphi(\bar{x}, \bar{a}') : \text{tp}(\bar{a}'/C) = \text{tp}(\bar{a}/C)\}$ is k -inconsistent (meaning that any set of k of these formulas is inconsistent, in a model of T^{eq}).
- (2) A formula $\varphi(\bar{x}, \bar{a})$ *thorn-divides* over C if there is a finite tuple $\bar{d}$ of elements from M^{eq} such that $\varphi(\bar{x}, \bar{a})$ strongly divides over $C\bar{d}$ ($= C \cup \text{rng}(\bar{d})$).
- (3) A formula $\varphi(\bar{x}, \bar{a})$ *thorn-forks* over C if it implies (modulo T^{eq}) a finite disjunction of formulas, all of which thorn-divide over C .
- (4) For $A \subseteq M^{\text{eq}}$, a complete type $p(\bar{x})$ over A *thorn-divides* (*thorn-forks*) over C if there is a formula in $p(\bar{x})$ which thorn-divides (thorn-forks) over C .
- (5) We say that $\bar{a}$ is *thorn-independent from $\bar{b}$ over C* , denoted $\bar{a} \downarrow_C^{\text{b}} \bar{b}$, if $\text{tp}(\bar{a}/C\bar{b})$ does not thorn-fork over C .

The following technical lemma appears in [7, Remark 3.2], but we prove it to make the arguments that follow self-contained.

Lemma 6.2. *Let $a, b \in M^{\text{eq}}$, $C \subseteq M^{\text{eq}}$ and let $\bar{c}$ be a sequence of elements from C . If $M^{\text{eq}} \models \varphi(a, b, \bar{c})$ and $\varphi(x, b, \bar{c})$ strongly divides over C , then $b \in \text{acl}^{\text{eq}}(aC) \setminus \text{acl}^{\text{eq}}(C)$.*

Proof. The assumption that $\varphi(x, b, \bar{c})$ strongly divides over C means that $\text{tp}(b/C)$ is nonalgebraic and, for some $k < \omega$, $\{\varphi(x, b', \bar{c}) : \text{tp}(b'/C) = \text{tp}(b/C)\}$ is k -inconsistent. As explained in [21, Remark 2.1.2], it follows from a compactness argument that there is $\theta(y, \bar{d}) \in \text{tp}(b/C)$ such that

$$\Phi = \{\varphi(x, b', \bar{c}) : M^{\text{eq}} \models \theta(b', \bar{d})\} \text{ is } k\text{-inconsistent.}$$

Since $\text{tp}(b/C)$ is nonalgebraic it follows that $b \notin \text{acl}^{\text{eq}}(C)$. For a contradiction, suppose that $b \in \text{acl}^{\text{eq}}(aC)$. Then there are distinct b_i , for $i < \omega$, such that $\text{tp}(a, b_i, \bar{c}, \bar{d}) = \text{tp}(a, b, \bar{c}, \bar{d})$ for all i . Then $M^{\text{eq}} \models \varphi(a, b_i, \bar{c}) \wedge \theta(b_i, \bar{d})$ for all $i < \omega$, and this contradicts that Φ is k -inconsistent. $\square$

Definition 6.3. We say that $\downarrow^n$ is *symmetric restricted to M_n* if for all $a, b \in M_n$, and $C \subseteq M^{\text{eq}}$, if $a \downarrow_C^n b$ then $b \downarrow_C^n a$.

Lemma 6.4. *Let $n < \omega$, $a, b \in M_n$, and $C \subseteq M^{\text{eq}}$. If $b \in \text{acl}^n(aC) \setminus \text{acl}^n(C)$ and $\downarrow^n$ is symmetric restricted to M_n , then $a \not\downarrow_C^n b$.*

Proof. Suppose that $n < \omega$, $a, b \in M_n$, $C \subseteq M^{\text{eq}}$, and $b \in \text{acl}^n(aC) \setminus \text{acl}^n(C)$. Then (by the definition of rk^n) $\text{rk}^n(b/C) \geq 1$ and $\text{rk}^n(b/aC) = 0$. It follows that $b \not\downarrow_C^n a$, so by the assumed symmetry of $\downarrow^n$ restricted to M_n we have $a \not\downarrow_C^n b$. $\square$

Below we repeat Assumption 5.1 which implies that $\downarrow^n$ is symmetric restricted to M_n .

Assumption 6.5. (Exchange property with respect to n) Let $n < \omega$ and suppose that if $C \subseteq M^{\text{eq}}$, $2 \leq k < \omega$, $a_1, \dots, a_k \in M_n$, $\text{rk}^n(a_i/C) = 1$ for all $i = 1, \dots, k$, and $a_k \in \text{acl}^{\text{eq}}(\{a_1, \dots, a_{k-1}\} \cup C) \setminus \text{acl}^{\text{eq}}(\{a_2, \dots, a_{k-1}\} \cup C)$, then $a_1 \in \text{acl}^{\text{eq}}(\{a_2, \dots, a_k\} \cup C)$.

The proof of the next lemma is an adaptation of the proof of Theorem 3.3 in [7].

Lemma 6.6. *Let $n < \omega$ and suppose that Assumption 6.5 holds for this n . Let $\bar{a}$ and $\bar{b}$ be a finite sequences of elements from M_n , and let $C \subseteq M^{\text{eq}}$. If $\bar{a} \not\downarrow_C^{\text{b}} \bar{b}$ then $\bar{a} \not\downarrow_C^{\text{b}} \bar{b}$.*

Proof. Let $n < \omega$, let $\bar{a}$ and $\bar{b}$ be finite sequences of elements from M_n , and let $C \subseteq M^{\text{eq}}$. Suppose that $\bar{a} \not\downarrow_C^{\text{b}} \bar{b}$. Suppose, for a contradiction, that $\bar{a} \downarrow_C^{\text{b}} \bar{b}$.

From $\bar{a} \downarrow_C^{\text{b}} \bar{b}$ it follows that $\text{tp}(\bar{a}/\bar{b}C)$ thorn forks over C which means that there is some $\varphi(\bar{x}, \bar{b}, \bar{c}) \in \text{tp}(\bar{a}/\bar{b}C)$ which thorn forks over C . This means that $\varphi(\bar{x}, \bar{b}, \bar{c})$ implies a

disjunction $\bigvee_{i=1}^s \psi_i(\bar{x}, \bar{b}, \bar{c})$ where each $\psi_i(\bar{x}, \bar{b}, \bar{c})$ thorn divides over C . Since $\text{tp}(\bar{a}/\bar{b}C)$ is a complete type there is some i such that $\psi_i(\bar{x}, \bar{b}, \bar{c})$ belongs to $\text{tp}(\bar{a}/\bar{b}C)$. Let $\psi(\bar{x}, \bar{b}, \bar{c}) = \psi_i(\bar{x}, \bar{b}, \bar{c})$. Then there is some $D \supseteq C$ such that $\psi(\bar{x}, \bar{b}, \bar{c})$ strongly divides over D . Since $\mathcal{M}^{\text{eq}} \models \psi(\bar{a}, \bar{b}, \bar{c})$ it follows from Lemma 6.2 that $\bar{b} \in \text{acl}^{\text{eq}}(\bar{a}D) \setminus \text{acl}^{\text{eq}}(D)$.

The assumptions that $\bar{a} \downarrow_C^n \bar{b}$ and that $\text{rng}(\bar{a}) \subseteq M_n$ together with the extension property of $\downarrow^n$ (Lemma 4.8) implies that there is $\bar{a}'$ such that $\text{tp}(\bar{a}'/\bar{b}C) = \text{tp}(\bar{a}/\bar{b}C)$ and $\bar{a}' \downarrow_{\bar{b}C}^n D$, so in particular $\text{rng}(\bar{a}') \subseteq M_n$. The assumption that $\bar{a} \downarrow_C^n \bar{b}$ (and automorphism invariance of $\downarrow^n$) gives $\bar{a}' \downarrow_C^n \bar{b}$. By transitivity of $\downarrow^n$ we then get $\bar{a}' \downarrow_C^n \bar{b}D$. The assumption that $\bar{a} \not\downarrow_C^n \bar{b}$ together with $\text{tp}(\bar{a}'/\bar{b}C) = \text{tp}(\bar{a}/\bar{b}C)$ gives $\bar{a}' \not\downarrow_C^n \bar{b}$. We also have $\psi(\bar{x}, \bar{b}, \bar{c}) \in \text{tp}(\bar{a}'/\bar{b}C)$. From $\bar{a}' \downarrow_C^n \bar{b}D$ and monotonicity of $\downarrow^n$ we get $\bar{a}' \downarrow_D^n \bar{b}D$ which means the same as $\bar{a}' \downarrow_D^n \bar{b}$.

Recall the conclusion above that $\bar{b} \in \text{acl}^{\text{eq}}(\bar{a}'D) \setminus \text{acl}^{\text{eq}}(D)$. Since $\bar{b} \in M_n$ it follows that $\bar{b} \in \text{acl}^n(\bar{a}'D) \setminus \text{acl}^n(D)$. Since Assumption 6.5 and Proposition 5.14 imply that $\downarrow^n$ has the symmetry property restricted to M_n it follows from Lemma 6.4 that $\bar{a}' \not\downarrow_D^n \bar{b}$ which contradicts the conclusion above that $\bar{a}' \downarrow_D^n \bar{b}$. Hence we conclude that $\bar{a} \not\downarrow_C^n \bar{b}$. $\square$

Corollary 6.7. *Suppose that Assumption 6.5 holds for all $n < \omega$. If $a, b \in M^{\text{eq}}$, $C \subseteq M^{\text{eq}}$ and $a \not\downarrow_C^n b$, then $a \not\downarrow_C^n b$ for all sufficiently large $n < \omega$.*

Proof. Let $a, b \in M^{\text{eq}}$ and $C \subseteq M^{\text{eq}}$. There is $n < \omega$ such that $a, b \in M_n$. Since $M_n \subseteq M_m$ whenever $n < m$ the corollary now follows from Lemma 6.6. $\square$

Proposition 6.8. *If T is ω -categorical and Assumption 6.5 holds for all $n < \omega$, then thorn-independence has local character.*

Proof. By Remark 2.8 it suffices to prove that there do *not* exist $a \in M^{\text{eq}}$ and $b_\alpha \in M^{\text{eq}}$, for $\alpha < \aleph_1$, such that $a \not\downarrow_{(b_i)_{i < \alpha}}^p b_\alpha$ for all $\alpha < \aleph_1$. Towards a contradiction, suppose that there are $a \in M^{\text{eq}}$ and $b_\alpha \in M^{\text{eq}}$, for $\alpha < \aleph_1$, such that $a \not\downarrow_{(b_i)_{i < \alpha}}^p b_\alpha$ for all $\alpha < \aleph_1$. Let $B = \{b_\alpha : \alpha < \aleph_1\}$. By Lemma 3.12, there is a countable $C \subseteq B$ such that $a \downarrow_C^n B$ for all $n < \omega$. Let $\beta = \sup\{\alpha : b_\alpha \in C\}$. As $\aleph_1$ is a regular cardinal and C is countable it follows that $\beta < \aleph_1$, so β is a countable ordinal. Let $B_\beta = \{b_\alpha : \alpha \leq \beta\}$ so B_β is countable and $C \subseteq B_\beta$. By the monotonicity of $\downarrow^n$ we get $a \downarrow_{B_\beta}^n B$ for all $n < \omega$. By assumption we have $a \not\downarrow_{B_\beta}^p b_{\beta+1}$, so Corollary 6.7 (which uses Assumption 6.5) implies that $a \not\downarrow_{B_\beta}^n b_{\beta+1}$ for all sufficiently large n . Since $b_{\beta+1} \in B$ it follows that $a \not\downarrow_{B_\beta}^n B$ for all sufficiently large n . But this contradicts the earlier conclusion that $a \downarrow_{B_\beta}^n B$ for all $n < \omega$. $\square$

The following result by Ealy and Onshuus [7] is crucial for the results that follow:

Theorem 6.9. [7, Theorem 3.7] *A theory is rosy if and only if thorn independence has local character.*

Theorem 6.10. *If T is ω -categorical and Assumption 6.5 holds for all $n < \omega$, then T is rosy.*

Proof. If T is ω -categorical and Assumption 6.5 holds for all $n < \omega$, then, by Proposition 6.8, thorn independence has local character. By Theorem 6.9, T is rosy. $\square$

Definition 6.11. (i) We say that algebraic closure is *trivial in T^{eq}* if whenever $\mathcal{M} \models T$, $a \in M^{\text{eq}}$, $B \subseteq M^{\text{eq}}$ and $a \in \text{acl}^{\text{eq}}(B)$, then $a \in \text{acl}^{\text{eq}}(b)$ for some $b \in B$.

(ii) Let $n < \omega$. We say that algebraic closure in T^{eq} is *trivial up to level n* if the following holds for every $\mathcal{M} \models T$: If $a \in M_n$, $B \subseteq M_n$, $C \subseteq M^{\text{eq}}$, and $a \in \text{acl}^{\text{eq}}(BC) \setminus \text{acl}^{\text{eq}}(C)$ then $a \in \text{acl}^{\text{eq}}(bC)$ for some $b \in B$.

Lemma 6.12. *Let T be ω -categorical and $m < \omega$. If algebraic closure in T^{eq} is trivial up to level m then Assumption 6.5 holds for all $n \leq m$.*

Proof. Suppose that T is ω -categorical and that algebraic closure in T^{eq} is trivial up to some $m < \omega$. Fix any $n \leq m$ and suppose that $C \subseteq M^{\text{eq}}$, $2 \leq k < \omega$, $a_1, \dots, a_k \in M_n$, $\text{rk}^n(a_i/C) = 1$ for all $i = 1, \dots, k$, and $a_k \in \text{acl}^{\text{eq}}(\{a_1, \dots, a_{k-1}\} \cup C) \setminus \text{acl}^{\text{eq}}(\{a_2, \dots, a_{k-1}\} \cup C)$. Since $M_n \subseteq M_m$ and the algebraic closure in T^{eq} is trivial up to m it follows that $a_k \in \text{acl}^{\text{eq}}(a_1)$.

For a contradiction, suppose that $a_1 \notin \text{acl}^{\text{eq}}(\{a_k\} \cup C)$. Then $\text{rk}^n(a_1/\{a_k\} \cup C) \geq 1$. Since $\text{rk}^n(a_k/C) = 1$ (and $a_k \in M_n$) we have $a_k \notin \text{acl}^{\text{eq}}(C)$. It now follows from the definition of rk^n that $\text{rk}^n(a_1/C) \geq 2$ which contradicts the assumption. Hence $a_1 \in \text{acl}^{\text{eq}}(\{a_k\} \cup C)$, which since $a_1 \notin \text{acl}^{\text{eq}}(C)$ (and algebraic closure in T^{eq} is trivial up to m) implies that $a_1 \in \text{acl}^{\text{eq}}(a_k)$. $\square$

Theorem 6.13. *If T is ω -categorical with trivial algebraic closure in T^{eq} then T is rosy.*

Proof. Suppose that T is ω -categorical with trivial algebraic closure in T^{eq} . Then, for every $n < \omega$, algebraic closure in T^{eq} is trivial up to n . Then Lemma 6.12 implies that Assumption 6.5 holds for all $n < \omega$. Now Theorem 6.10 implies that T is rosy. $\square$

For definitions of superrosy theory and U^{b} -rank see [7, 21].

Theorem 6.14. *If T is ω -categorical with soft elimination of imaginaries and Assumption 6.5 holds for $n = 0$, then T is superrosy with finite U^{b} -rank.*

Proof. Let T be ω -categorical with soft elimination of imaginaries and suppose that Assumption 6.5 holds for $n = 0$. It follows from Theorem 5.15 that $\downarrow^0$ is an independence relation. Since T has soft elimination of imaginaries Lemma 4.9 implies that, for all $n < \omega$ and all $A, B, C \subseteq M^{\text{eq}}$, $A \downarrow_C^n B$ if and only if $A \downarrow_C^0 B$. Hence, for all $n < \omega$, $\downarrow^n$ is an independence relation. It follows that we can argue just as in the proofs of Lemma 6.6 and its corollary and conclude that if $a \not\downarrow_C^{\text{b}} b$ then $a \not\downarrow_C^n b$ for all sufficiently large n .

To show that T is superrosy with finite U^{b} -rank it suffices to show that $U^{\text{b}}(p) < \omega$ for every complete type $p(x)$ over $\emptyset$. For a contradiction, suppose that $U^{\text{b}}(p) \geq \omega$. Let $\alpha = \text{rk}^0(a)$ where a is any realization of $p(x)$, so $\alpha < \omega$. Take any $\alpha < \beta < \omega$. As $U^{\text{b}}(p) \geq \beta$ there are $a \in M^{\text{eq}}$ realizing $p(x)$ and $b_k \in M^{\text{eq}}$, for $k \leq \beta$, such that $a \not\downarrow_{(b_i)_{i < k}}^{\text{b}} b_k$ for all $k \leq \beta$. From our conclusion above it follows that there is $n' < \omega$ such that if $n \geq n'$ then $a \not\downarrow_{(b_i)_{i < k}}^n b_k$ for all $k \leq \beta$. As $\downarrow^n$ coincides with $\downarrow^0$ it follows that $a \not\downarrow_{(b_i)_{i < k}}^0 b_k$ for all $k \leq \beta$. Hence

$$\text{rk}^0(a) > \text{rk}^0(a/b_0) > \text{rk}^0(a/b_0, b_1) > \dots > \text{rk}^0(a/b_0, \dots, b_\beta) \geq 0$$

so $\text{rk}^0(a) \geq \beta > \alpha$, contradicting the choice of α . $\square$

Remark 6.15. Due to Lemma 6.12, the conclusion of Theorem 6.14 still holds if the assumption that Assumption 6.5 holds for $n = 0$ is replaced by the assumption that algebraic closure in T^{eq} is trivial up to level 0.

Epilogue. The work resulting in this article began by considering a sequence $(\mathcal{B}_n : n < \omega)$ of finite structures $\mathcal{B}_n$ where $\lim_{n \rightarrow \infty} |\mathcal{B}_n| = \infty$ and where all $\mathcal{B}_n$ have a “uniformly well-behaved” closure operator, where “well-behaved” essentially means that the closure operator is uniformly definable and has the properties of the algebraic closure operator in an ω -categorical structure. The idea was that such a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ would

generalize the “base sequence” of finite trees with bounded height considered in [17] where a closure operator has a crucial role (the operator that collects all ancestors of a set of vertices of the tree). It turns out that if we have such a well-behaved closure operator and the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ has an additional property (not explained here), then the same sequence has an infinite (countable) limit structure (in a seemingly new sense), say $\mathcal{M}$, which is ω -categorical. However, only knowing that the limit $\mathcal{M}$ is ω -categorical did not seem to be of much interest. It was clear that the closure operator on the structures $\mathcal{B}_n$ gives rise to a definable closure operator cl on $\mathcal{M}$ which is such that $\text{cl}(A) \subseteq \text{acl}(A)$ for all $A \subseteq M$ where $\text{acl}(A)$ denotes the algebraic closure of A in $\mathcal{M}$. Since the “nice” properties of cl on all $\mathcal{B}_n$ meant that a notion of rank was possible to define on all $\mathcal{B}_n$, in a uniform way, it followed that a notion of rank (defined in terms of cl) existed in $\mathcal{M}$. However, if the notion of rank is going to be useful to define a “nice” independence relation in the sense of [15], then one has to consider “closure” not only on “real elements” of $\mathcal{M}$ but also on imaginary elements. Since, for all $A \subseteq M$, $\text{cl}(A) \subseteq \text{acl}(A)$ it seemed to be reasonable, at the present state of affairs at that time, to just forget about the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ and consider ranks and independence relations defined by the algebraic closure operator in an ω -categorical structure. This eventually led to the results presented here. Given these results it may now be more meaningful to go back to considerations of the sequence $(\mathcal{B}_n : n < \omega)$ and the connections between the closure operator on all $\mathcal{B}_n$ (or some generalization of it to take into account “imaginary elements in $\mathcal{B}_n$ ”) and the limit structure $\mathcal{M}$.

Acknowledgement. This work is partially supported by the Swedish Research Council, grant 2023-05238.

REFERENCES

- [1] H. Adler, A geometric introduction to forking and thorn-forking, *Journal of Mathematical Logic*, Vol. 9 (2009) 1–20.
- [2] G. Ahlbrandt, M. Ziegler, Quasi-finitely axiomatizable totally categorical theories, *Annals of Pure and Applied Logic*, Vol. 30 (1986), 63–82.
- [3] J. Baldwin, J. Freitag, S. Mutchnik, Simple homogeneous structures and indiscernible sequence invariants, <https://arxiv.org/abs/2405.08211>.
- [4] G. Cherlin, L. Harrington, A. H. Lachlan, ω -categorical ω -stable structures, *Annals of Pure and Applied Mathematics*, Vol. 28 (1986) 103–135.
- [5] G. Cherlin, E. Hrushovski, Finite Structures with Few Types, *Annals of Mathematics Studies*, Nr. 152, Princeton University Press (2003).
- [6] G. Conant, An axiomatic approach to free amalgamation, *The Journal of Symbolic Logic* Vol. 82, (2017), 648–671.
- [7] C. Ealy, A. Onshuus, Characterizing rosy theories, *The Journal of Symbolic Logic*, Vol. 72 (2007) 919–940.
- [8] D. Evans, $\aleph_0$ -categorical structures with a predimension, *Annals of Pure and Applied Logic*, Vol. 116 (2002) 157–186.
- [9] W. Hodges, *Model theory*, Cambridge University Press (1993).
- [10] W. Hodges, I.M. Hodkinson, D. Macpherson, Omega-categoricity, relative categoricity and coordinatisation, *Annals of Pure and Applied Logic*, Vol. 46 (1990) 169–199.
- [11] E. Hrushovski, Totally categorical structures *Transactions of the American Mathematical Society*, Vol. 313, (1989), 131–159.
- [12] E. Hrushovski, An new strongly minimal set, *Annals of Pure and Applied Logic*, Vol. 62 (1993) 147–166.
- [13] W. M. Kantor, M. W. Liebeck, H. D. Macpherson, $\aleph_0$ -categorical structures smoothly approximated by finite substructures, *Proceedings of the London Mathematical Society*, Vol. 59 (1989) 439–463.
- [14] B. Kim, *Simplicity Theory*, Oxford University Press (2014).
- [15] B. Kim, A. Pillay, Simple theories, *Annals of Pure and Applied Logic*, Vol. 88 (1997) 149–164.
- [16] V. Koponen, Binary simple homogeneous structures, *Annals of Pure and Applied Logic*, Vol. 169 (2018) 1335–1368.

- [17] V. Koponen, Y. Tousinejad, Random expansions of trees with bounded height, *Theoretical Computer Science*, Vol. 1040 (2025) 115201.
- [18] A. H. Lachlan, Stable finitely homogeneous structures: a survey, in B. T. Hart, A. H. Lachlan, M. A. Valeriote (editors), *Algebraic Model Theory*, Kluwer Academic Publishers (1997) 145–159.
- [19] M. Morley, Categoricity in power, *Transactions of the American Mathematical Society*, Vol. 114 (1965), 514–538.
- [20] L. Newelski, R. Wencel, Definable sets in Boolean-ordered o-minimal structures. I, *The Journal of Symbolic Logic*, Vol. 66 (2001) 1821–1836.
- [21] A. Onshuus, Properties and consequences of thorn-independence, *The Journal of Symbolic Logic*, Vol. 71 (2006) 1–21.
- [22] A. Onshuus, P. Simon, Dependent finitely homogeneous rosy theories, <https://arxiv.org/pdf/2107.02727>.
- [23] J. Oxley, *Matroid Theory (2nd edn)*, Oxford University Press (2011).
- [24] S. Shelah, *Classification Theory*, Revised Edition, North-Holland (1990).
- [25] P. Simon, *A guide to NIP theories*, Lecture Notes in Logic Vol. 44, Cambridge University Press (2015).
- [26] P. Simon, NIP ω -categorical structures: The rank 1 case, *Proceedings of the London Mathematical Society*, Vol. 125 (2022) 1253–1331.
- [27] K. Tent, M. Ziegler, *A Course in Model Theory*, Lecture Notes in Logic, Vol. 40, Cambridge University Press (2012).
- [28] L. van den Dries, Tame Topology and O-minimal Structures, *London Mathematical Society Lecture Note Series*, Vol. 248, Cambridge University Press (1998).
- [29] B. Zilber, *Uncountably Categorical Theories*, AMS Translations of Mathematical Monographs, Vol. 117 (1993).

VERA KOPONEN, DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, SWEDEN.
 Email address: vera.koponen@math.uu.se