Characterizing all K_4-free well-edge-dominated graphs of girth 3

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November 11, 2025

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Abstract

Given a graph G, a set F of edges is an edge dominating set if all edges in G are either in F or adjacent to an edge in F. G is said to be well-edge-dominated if every minimal edge dominating set is also minimum. In 2022, it was proven that there are precisely three nonbipartite, well-edge-dominated graphs with girth at least four. Then in 2025, a characterization of all well-edge-dominated graphs containing exactly one triangle was found. In this paper, we characterize all well-edge-dominated graphs that contain a triangle and yet are K_4-free.

Keywords: well-edge-dominated, equimatchable, matching
AMS subject classification: 05C69, 05C76, 05C75

1 Introduction

Given a graph G, a set of edges F in G is called a *matching* if no pair of edges in F share a common vertex, and it is called an edge dominating set if each edge in G is adjacent to an edge in F. We typically want to maximize the cardinality of a matching, giving us the matching number $\alpha'(G)$, and minimize the cardinality of an edge dominating set, giving us the edge domination number $\gamma'(G)$. G is said to be *equimatchable* if all maximal matchings in G have the same cardinality. This property was first defined, independently, by Lewin [12] and Meng [13]. Later Lesk, Plummer, and Pulleyblank [11] presented a polynomial time algorithm for recognizing when a graph is equimatchable, and since then, a search for a structural characterization has been a running theme. G is said to be *well-edge-dominated* if all minimal edge dominating sets in G have the same cardinality. Seeing as how any maximal matching in G is also a minimal edge dominating set, it is no surprise that if G is well-edge-dominated, then it is also equimatchable.

The class of equimatchable graphs are well-studied (see [1, 4, 7, 10]). Notably, Frendrup, Hartnell, and Vestergaard [9] proved that a connected equimatchable graph with no cycles of length less than 5 is either C_5, C_7, or belongs to the family of bipartite graphs $\mathcal{C}$ that contains K_2 as well as those with a bipartition $A \cup B$ where A is the set of support vertices. Then in [4], Büyükçolak et al. characterized all nonbipartite equimatchable graphs containing 4-cycles. It is worth mentioning that although the equimatchable graphs were characterized in [11], no structural results were provided so it was unclear what the class of bipartite graphs containing 4-cycles look like. Since then, several papers have attempted to give a structural characterization of the bipartite equimatchable graphs containing 4-cycles (see [5, 6]). Frendrup, Hartnell, and Vestergaard [9] also proved that an equimatchable graph with girth at least 5 is well-edge-dominated. In [2] Anderson et al. showed that there are only three nonbipartite well-edge-dominated graphs with girth at least 4; namely, C_5, C_7, and C_7^* which is obtained from C_7 by adding a chord to C_7 between two vertices at distance 3. Then in [3], Berg et al. characterized all well-edge-dominated graphs containing exactly one triangle. Based on their construction, we characterize all K_4-free well-edge-dominated graphs of girth 3.

The remainder of the paper is organized as follows. In Section 1.1 we provide definitions and known results that we rely on in later sections. In Section 2 we provide an infinite class of nonbipartite K_4-free well-edge-dominated graphs which we refer to as $\mathcal{G}$, and we prove that any nonbipartite K_4-free well-edge-dominated graph is either in $\mathcal{G}$ or one of five exceptions.

1.1 Previous Results

Let $G = (V(G), E(G))$ be any graph where we write $n(G)$ to denote $|V(G)|$. Given any $v \in V(G)$, the open neighborhood of v is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ (or simply $N(v)$ when the context is clear). The closed neighborhood is defined to be $N_G[v] = N_G(v) \cup \{v\}$ (or simply $N[v]$ when the context is clear). For any set $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$. Similarly, given any edge $f \in E(G)$, the open edge neighborhood of f, denoted $N_e(f)$, is the set of all edges in G adjacent to f, and the closed edge neighborhood of f is $N_e[f]$. For any $v \in V(G)$, the degree of v is $\deg_G(v) = |N_G(v)|$ (or simply $\deg(v)$). Any vertex with degree one is called a leaf, and its lone neighbor in G is referred to as a support vertex in G. Given any two vertices $u, v \in V(G)$, the graph obtained by *identifying* u *and* v is found by removing u and v from G and replacing with a vertex w where w is adjacent to each vertex in $(N_G(u) \cup N_G(v)) - \{u, v\}$. We will refer to the operation of *identifying each vertex in* $S \subset V(G)$ *into a single vertex* to mean removing each vertex in S and replacing with a vertex w where w is adjacent to each vertex in $N_G(S) - S$. Given a graph F, G is said to F-free if G does not contain F as an induced subgraph.

An edge dominating set is a set $D \subseteq E(G)$ such that every edge in G is either in D or adjacent to an edge in D. The edge domination number of G, denoted $\gamma'(G)$, is the minimum cardinality among all edge dominating sets of G. Given $e \in D$, the private edge neighbors of e with respect to D is the set of edges f where $N_e[f] \cap D = \{e\}$. A matching in G is a set of independent edges in G, and the matching number, denoted $\alpha'(G)$, is the maximum cardinality of a matching in G. For any graph G, $\gamma'(G) \leq \alpha'(G)$. Recall that G is bipartite if we can partition $V(G)$ into two independent sets. If G is bipartite, we write $G = (A \cup B, E)$ to mean that A and B are the partite sets in G.

G is said to be equimatchable if every maximal matching in G has the same cardinality, namely, $\alpha'(G)$. Seeing as how our constructions for well-edge-dominated graphs rely on equimatchable bipartite graphs, one important result regarding equimatchable bipartite graphs is the following.

Lemma 1. *[5] Let $G = (A \cup B, E)$ with $|A| < |B|$ be a connected equimatchable bipartite graph. Then each vertex $u \in A$ satisfies at least one of the following.*

(i) u is a support vertex in G,

(ii) u is included in a subgraph $K_{2,2}$ in G.

The next question would be what can be said about bipartite equimatchable graphs $G = (A \cup B, E)$ where $|A| = |B|$? A graph G is defined to *randomly matchable* if it is an equimatchable graph admitting a perfect matching. Sumner characterized all randomly matchable graphs.

Theorem 1. *[14] A connected graph is randomly matchable if and only if it is isomorphic to K_{2n} or $K_{n,n}$ for $n \geq 1$.*

In this paper, we focus on graphs that are well-edge-dominated, i.e. graphs where every minimal edge dominating set is also minimum. There is a well-known chain of inequalities, namely

$$\gamma'(G) \leq \alpha'(G) \leq \Gamma'(G)$$

where $\Gamma'(G)$ is the maximum cardinality among all minimal dominating sets. Thus, if G is well-edge-dominated, then $\gamma'(G) = \Gamma'(G)$, implying that G is equimatchable. This together with the fact that all equimatchable graphs of girth 5 or more are characterized meant that the only well-edge-dominated graphs left to classify were those of girth 3 or 4. In 2022, Anderson et al. [2] proved that there are only three nonbipartite well-edge-dominated graphs of girth 4. We let C_7^* denote the graph obtained from C_7 by adding a chord between one pair of vertices at distance three.

Theorem 2. *[2] If G is well-edge-dominated with $g(G) \geq 4$, then either G is bipartite or $G \in \{C_5, C_7, C_7^*\}$.*

Then in [3], a characterization of well-edge-dominated graphs containing exactly one triangle was provided. It relied on two infinite families of graphs, each of which were constructed from well-edge-dominated bipartite graphs (of which we still do not have a nice structural description, but nevertheless). Given a well-edge-dominated bipartite graph $G = (A \cup B, E)$ such that $|A| < |B|$, we say that $v \in B$ is *detachable* if $G - v$ is well-edge-dominated. Note that in this case $\gamma'(G-v) = |A|$ as $G-v$ contains a matching of size $|A|$. Additionally, we say that $v \in B$ is *strongly detachable* if $G - v$ is well-edge-dominated and each vertex in $N_G(v)$ is a support vertex in $G - v$. The first class of well-edge-dominated graphs containing exactly one triangle is denoted $\mathcal{F}$ and defined as follows. We say that $G \in \mathcal{F}$ if G is obtained from the disjoint union of the house graph

$\mathcal{H}$, depicted in Figure 1(b), and a well-edge-dominated bipartite graph $G' = (A \cup B, E)$ where $|A| < |B|$, by identifying the vertex of degree two in $\mathcal{H}$ on the triangle with $w \in V(G')$ where w is strongly detachable. Examples of such graphs are W_7 and V_7, found in Figures 3 and 4, respectively. The second class of well-edge-dominated graphs containing exactly one triangle is denoted $\mathcal{T}$ and defined as follows. We say that $G \in \mathcal{T}$ if G is obtained from the disjoint union of K_3 and a well-edge-dominated bipartite graph $G' = (A \cup B, E)$ where $|A| < |B|$, by identifying $x \in V(K_3)$ with $w \in V(G')$ where w is detachable. Examples of such graphs are W_5, W_6, W_9, V_4, V_5, and V_6 in Figures 3 and 4. Letting Cr denote the "crystal" graph depicted in Figure 1(c) and $\mathcal{DH}$ denote the "dream house" graph depicted in Figure 1(a), the complete class of well-edge-dominated graphs containing exactly one triangle is the following.

Theorem 3. *[3] G is a well-edge-dominated graph with exactly one triangle if and only if $G \in \mathcal{T} \cup \mathcal{F} \cup \{K_3, Cr, \mathcal{H}, \mathcal{DH}\}$.*

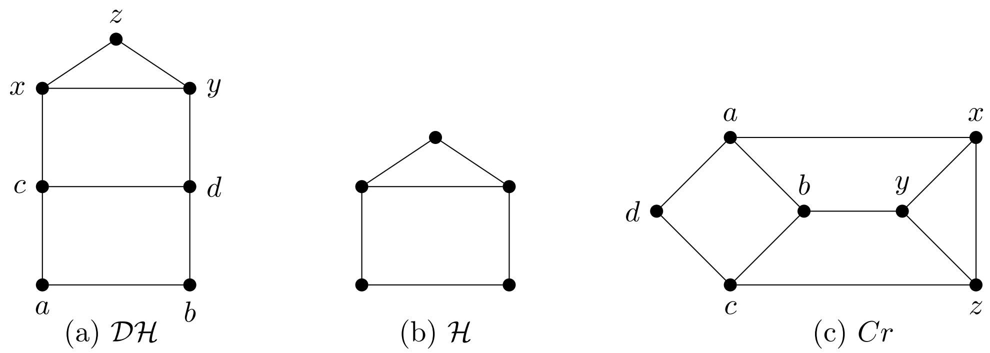

Figure 1: The dream house (a), the house graph (b) and the crystal graph (c)

In this paper, we characterize all K_4-free well-edge-dominated graphs containing a 3-cycle. In that vein, we will need the following preliminary results used in the next section.

Lemma 2. *[3] If G is a graph of order at least 4 obtained from the triangle xyz by appending leaves to x, y, or z, then G is not well-edge-dominated.*

Additionally, the "edge version" of a fact used by Finbow, Hartnell and Nowakowski in [8] is that provided G is well-edge-dominated, then $M \cup D_1$ and $M \cup D_2$ are both minimal edge dominating sets of G for any matching M and any pair D_1 and D_2 of minimal edge dominating sets of the graph $G - N_e[M]$. This was first observed in [2].

Lemma 3. *[2] Let M be a matching in a graph G. If G is well-edge-dominated, then $G - N_e[M]$ is well-edge-dominated. If G is equimatchable, then $G - N_e[M]$ is equimatchable.*

We also need the following result regarding well-edge-dominated bipartite graphs.

Lemma 4. *Let $G = (A \cup B, E)$ be a well-edge-dominated bipartite graph where $|A| < |B|$. If $y \in A$ is not a support vertex in G, then there exists a minimal edge dominating set F of G that does not contain an edge incident to y.*

Proof. To see this, we construct such an edge dominating set F. First, for each $b \in N_G(y)$, choose an edge incident to b and a vertex in $A - \{y\}$ and call the resulting set F_0. Let $A' = \{a_1, \ldots, a_k\}$ be the vertices in $A - \{y\}$ that are not incident to an edge in F_0. We construct sets F_i for each $i \in [k]$ in the following way. If there exists an edge a_1b' such that b' is not incident to an edge in F_0, set $F_1 = F_0 \cup \{a_1b'\}$. Otherwise, set $F_1 = F_0$. Continue this process for each $2 \le i \le k$ until we arrive at the set F_k. We claim that F_k is a minimal edge dominating set of G. Note that if $x \in A - \{y\}$ is not incident to an edge in F_k, then every neighbor of x in B is incident to an edge in F_k by construction. So F_k is indeed an edge dominating set of G. Moreover, each edge $e \in F_k - F_0$ is its own private edge neighbor with respect to F_k. Finally, each edge in F_0 is of the form bx where $x \in A - \{y\}$ and $b \in N_G(y)$. Thus, the private edge neighbor of bx is by. □

Finally, we introduce two other graphs that will show up throughout Section 2. The graph F_5 is the fan on 5 vertices. A *diamond* is the graph $K_4 - e$ for any edge e. Given a diamond with vertices $\{w, x, y, z\}$ where w and z have degree 2, we refer to w and z as the *exterior vertices* of the diamond, and we refer to x and y as the *interior vertices* of the diamond.

2 K_4-free well-edge-dominated graphs

Our first step in finding all K_4-free well-edge-dominated graphs was to have Sage find all such graphs up to order 8. Figures 3 and 4 are a complete list of all connected, well-edge-dominated graphs of orders 7 and 8, respectively. We are now ready to define three infinite classes of graphs, each of which whose members are well-edge-dominated. First, we define the class $\mathcal{P}$ (referred to as the propellers) and the class $\mathcal{W}$ (referred to as the windmills). Let $T_1, \ldots, T_k$ be k disjoint triangles and for each $i \in [k]$ label one vertex of triangle T_i by v_i. $G \in \mathcal{P}$ if it is obtained from $T_1, \ldots, T_k$ by identifying each vertex in $\{v_1, \ldots, v_k\}$ into a single vertex (see W_{13} in Figure 3). To define $\mathcal{W}$, we let $\mathcal{H}$ be the house graph, depicted in Figure 1(b), and label the vertex on the triangle with degree two x. $G \in \mathcal{W}$ if it is obtained from $T_1, \ldots, T_k$, and $\mathcal{H}$ by identifying each vertex in $\{v_1, \ldots, v_k, x\}$ into a single vertex (see W_8 in Figure 3). In either case, the identified vertex in G will be referred to as the "nose" of the windmill/propeller. We will also consider K_3 as a propeller and $\mathcal{H}$ as a windmill. Before defining our broader class, we show that every graph in $\mathcal{P} \cup \mathcal{W}$ is indeed well-edge-dominated.

Lemma 5. *If $G \in \mathcal{P} \cup \mathcal{W}$, then G is well-edge-dominated.*

Proof. Suppose first that $G \in \mathcal{P}$ and is obtained from the triangles $T_1, \ldots, T_k$. Any minimal edge dominating set will contain exactly one edge from each of the k edge disjoint triangles in G, meaning that $\gamma'(G) = k = \Gamma'(G)$.

Next, let $G \in \mathcal{W}$. Note that if $G \cong \mathcal{H}$, then G is well-edge-dominated. Thus, we may assume $G \ncong \mathcal{H}$. In addition, assume we have obtained G from the triangles $T_1, \ldots, T_k$ and the house graph $\mathcal{H}$ where the vertex in $\mathcal{H}$ on the triangle with degree two is x by identifying each vertex in $\{v_1, \ldots, v_k, x\}$ (for any v_i in T_i, $i \in [k]$) into a single vertex w. Any minimal edge dominating set D of G will contain exactly one edge from each of the k triangles in G originally found in $T_1, \ldots, T_k$. Moreover, no matter which k edges are chosen from $T_1, \ldots, T_k$, the vertices in $\mathcal{H} - x$ induce a C_4. So D must contain at least two edges from $\mathcal{H}$. To see that D contains exactly two edges from $\mathcal{H}$, suppose to the contrary that D contains three edges from $\mathcal{H}$. Then all three edges must be incident to a vertex with degree three in $\mathcal{H}$, say y. However, in this case, xy must have a private edge neighbor which is not in $\mathcal{H}$ and since all edges outside $\mathcal{H}$ are dominated by an edge in its respective triangle, D is not minimal. Therefore, D contains exactly two edges from $\mathcal{H}$. It follows that G is well-edge-dominated. $\square$

Now, we are ready to define our broader class $\mathcal{G}$, built from the classes $\mathcal{P} \cup \mathcal{W}$, as follows. Consider the graph obtained from the disjoint union of a diamond and K_2 by identifying a vertex of K_2 with an exterior vertex of the diamond. Call the graph K^* and put K^* in $\mathcal{G}$. Additionally, if $G \in \mathcal{P} \cup \mathcal{W}$, then put G in $\mathcal{G}$. All other graphs $G \in \mathcal{G}$ can be constructed in the following way. Take the disjoint union of

$$G', W_1, \ldots, W_k, P_1, \ldots, P_r, D_1, \ldots, D_\ell$$

where $G' = (A \cup B, E')$ is a connected, nontrivial, bipartite and well-edge-dominated graph with $|A| < |B|$, $W_i \in \mathcal{W}$ is a windmill, $P_i \in \mathcal{P}$ is a propeller, and each D_i is a diamond. For each diamond, label one of the vertices of degree two as the "nose". Let $B' \subset B$ be a set such that $G' - B'$ is a well-edge-dominated graph with $\gamma'(G' - B') = \gamma'(G')$, $G' - B'$ has no trivial components, and $|A| \leq |B - B'|$. Each vertex v in B' is called detachable and it is called strongly detachable if each neighbor of it in G' is a support vertex in $G' - B'$. Enumerate the vertices of $B' = \{s_1, \ldots, s_k, x_1, \ldots, x_r\}$ such that each s_i is strongly detachable. Then choose a set $\{y_1, \ldots, y_\ell\} \subset A$ where y_i is a support vertex in $G' - B'$. We obtain G from $G', W_1, \ldots, W_k, P_1, \ldots, P_r, D_1, \ldots, D_\ell$ under the following rules:

(a) The nose of W_i is identified with the strongly detachable vertex s_i of B'.

(b) The nose of P_i is identified with the detachable vertex x_i of B'.

(c) The nose of D_i is identified with the support vertex y_i of G' (which stays a support vertex in G).

Note that of all the graphs shown in Figures 3 and 4, only W_1, W_2, W_3, W_{10}, and W_{12} are not in $\mathcal{G}$. However, $W_{10} \cong \mathcal{DH}$ and $W_{12} \cong Cr$. So there are only three exceptions of connected, well-edge-dominated graphs with order 7 or 8 that both contain at least two triangles (and are therefore not in $\mathcal{T} \cup \mathcal{F}$) and are not in $\mathcal{G}$. We next prove that each graph in $\mathcal{G}$ is in fact well-edge-dominated. First, we prove a more general result about constructing well-edge-dominated graphs from two smaller well-edge-dominated graphs.

Lemma 6. *Let G_1 and G_2 be two nontrivial well-edge-dominated graphs with $x \in V(G_1)$ and $y \in V(G_2)$ such that $G_1 - x$ is well-edge-dominated with $\gamma'(G_1 - x) = \gamma'(G_1)$ and $G_2 - y$ is well-edge-dominated with $\gamma'(G_2 - y) = \gamma'(G_2)$. Then the graph H obtained from G_1 and G_2 by identifying x and y is well-edge-dominated with $\gamma'(H) = \gamma'(G_1) + \gamma'(G_2)$.*

Proof. Let w be the identified vertex in H and let F be a minimal edge dominating set of H. We claim that $F \cap E(G_1 - x)$ is a minimal edge dominating set of $G_1 - x$. Indeed, if not, then some edge $e \in F \cap E(G_1 - x)$ has no private edge neighbor in $G_1 - x$ with respect to $F \cap E(G_1 - x)$. However, e does have a private edge neighbor in H with respect to F, meaning e has a private edge neighbor incident to w. This in turn implies that no edge in F is incident to w and $F \cap E(G_1 - x)$ is a minimal edge dominating set of G_1. Hence, $|F \cap E(G_1 - x)| = \gamma'(G_1) = \gamma'(G_1 - x)$, a contradiction. It follows that $F \cap E(G_1 - x)$ is a minimal edge dominating set of $G_1 - x$, and as $G_1 - x$ is well-edge-dominated, $|F \cap E(G_1 - x)| = \gamma'(G_1 - x) = \gamma'(G_1)$. Similarly, $|F \cap E(G_2 - y)| = \gamma'(G_2 - y) = \gamma'(G_2)$ from which it follows that $|F| = \gamma'(G_1) + \gamma'(G_2)$. As F is an arbitrary minimal edge dominating set of H, H is well-edge-dominated. $\square$

Note that each graph in $\mathcal{G}$ can be viewed as performing a sequence of identifications between two well-edge-dominated graphs. Next, we show every graph in $\mathcal{G}$ is in fact well-edge-dominated.

Corollary 1. *If $G \in \mathcal{G}$, then G is well-edge-dominated.*

Proof. Note that the graph obtained by identifying a vertex of K_2 with an exterior vertex of a diamond is well-edge-dominated. Moreover, if $G \in \mathcal{P} \cup \mathcal{W}$, then Lemma 5 shows G is well-edge-dominated. Therefore, we shall assume that G was created from the disjoint union of

$$G', W_1, \ldots, W_k, P_1, \ldots, P_r, D_1, \ldots, D_\ell$$

where $G' = (A \cup B, E')$ is connected, nontrivial, bipartite and well-edge-dominated with $|A| < |B|$, $W_i \in \mathcal{W}$ is a windmill, $P_i \in \mathcal{P}$ is a propeller, and each D_i is a diamond. Further, we shall assume that $B' \subset B$ was chosen to create G where $B' = \{s_1, \ldots, s_k, x_1, \ldots, x_r\}$ such that each s_i is strongly detachable, and we chose $Y = \{y_1, \ldots, y_\ell\} \subset A$ where y_i is a support vertex in $G' - B'$. In G, we give the vertex representing an identification with a vertex in $B' \cup Y$ the same label as the vertex in $B' \cup Y$.

Suppose first that W_i was used to create G where W_i is a windmill that was created from n_i triangles and the house graph. We claim that $|E(W_i) \cap F| = n_i + 2$. Note first that each of the n_i triangles must contain an edge from F. Moreover, the house graph contains a 4-cycle whose vertices are not the nose of the W_i. This means that F contains at least two edges from the house graph in W_i. To see that F does not contain three edges from the house, label the vertices on the house graph in W_i as $\{x_i, y_i, s_i, c_i, d_i\}$ where s_i is the nose of W_i, $x_iy_is_i$ is the triangle, and c_ix_i and d_iy_i are edges in W_i. Note that the only way for F to contain 3 edges from the house graph is if F contains $\{x_is_i, x_iy_i, a_ix_i\}$ or F contains $\{y_is_i, y_ix_i, y_ib_i\}$. Without loss of generality, we may assume F contains $\{x_is_i, x_iy_i, a_ix_i\}$. It follows that the private edge neighbor of x_is_i must be an edge incident to s_i, but not y_is_i. Moreover, since each of the n_i triangles used to create W_i contain an edge of F, the private edge neighbor of x_is_i is s_iw_i where s_iw_i was originally an edge in G'. Moreover, by the construction of G, s_i is a strongly detachable vertex in B', meaning that s_i is a support vertex in $G' - B'$ so that F contains an edge incident to s_i. Therefore, x_is_i indeed has no private edge neighbor and $|E(W_i) \cap F| = n_i + 2$.

Next, suppose for some $i \in [r]$ that P_i was used to create G where P_i is a propeller that was created from n_i triangles. It is clear that $|E(P_i) \cap F| = n_i$. Lastly, suppose for some $i \in [\ell]$ that D_i was used to create G where D_i is a diamond with vertices x_i, y_i, z_i, w_i where y_i is the nose of the diamond and w_i is the other exterior vertex of the diamond. Note that $|E(D_i) \cap F| \leq 2$. Suppose that $|E(D_i)| \cap F| = 1$. It follows that $x_iz_i \in F$ for otherwise all edges of the triangle $w_ix_iz_i$ are not edge dominated. Note further that by construction of G, y_i is a support vertex in $G' - B'$ so that F contains an edge of the form y_ia where y_ia is an edge in G'. We may weakly partition the diamonds $D_1, \dots, D_\ell$ as $D_1, \dots, D_\alpha, D_{\alpha+1}, \dots, D_\ell$ where $|E(D_i) \cap F| = 2$ for $i \in [\alpha]$ and $|E(D_i) \cap F| = 1$ otherwise.

Next, note that if $|E(D_i) \cap F| = 2$, then at least one of the edges in $E(D_i) \cap F$ is incident to y_i. This is because in $G' - B'$, y_i is a support vertex. Therefore, if neither edge in $E(D_i) \cap F$ is incident to y_i, then $E(G') \cap F$ contains an edge incident to y_i which would imply that for some edge in $E(D_i) \cap F$, say e, $F - \{e\}$ is also an edge dominating set of G, contradicting the minimality of F. Now for each $i \in [\alpha]$, let a_i be a leaf of y_i in $G' - B'$, Z be the set of all edges of the form a_iy_i for $i \in [\alpha]$, and let $F' \subseteq F$ be a matching that saturates all $y_i \in Y$ where $\alpha + 1 \leq i \leq \ell$. As G' is well-edge-dominated, $G' - N_e[F' \cup Z]$ is well-edge-dominated by Lemma 3. Thus, any minimal edge dominating set of $G' - N_e[F' \cup Z]$ has cardinality $|A| - \ell$. Therefore, there are exactly $|A| - \ell$ edges in $E(G') \cap F$ that are not incident to a vertex in Y. It follows that

$$|E(G') \cap F| + \sum_{i=1}^{\ell} |E(D_i) \cap F| \geq |A| - \ell + 2\ell = |A| + \ell.$$

Seeing as how F is arbitrary, we have that

$$|F| = |E(G') \cap F| + \sum_{i=1}^{k}(n_i + 2) + \sum_{i=1}^{r} n_i + \sum_{i=1}^{\ell} |E(D_i) \cap F| = |A| + \ell + \sum_{i=1}^{k}(n_i + 2) + \sum_{i=1}^{r} n_i$$

and G is well-edge-dominated.

□

The remainder of the paper is now focused on showing that we have found all K_4-free well-edge-dominated graphs of girth 3. We note that our proof will rely heavily on considering when a well-edge-dominated graph contains a diamond or not. For this reason, we point out the following observation about the graphs depicted in Figures 3 and 4.

Observation 1. *Assume G has order 7 and contains a diamond as an induced subgraph. G is a K_4-free well-edge-dominated graph with girth 3 if and only if $G \in \{W_1, W_2, W_3, W_4\}$, where W_4 is the graph in $\mathcal{G}$ that contains a diamond as well as three pendent edges.*

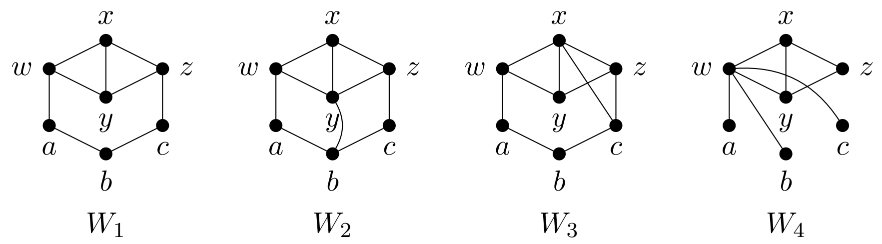

Figure 2: The graphs W_1, W_2, W_3, and W_4

Before we prove the general result, we consider well-edge-dominated graphs containing triangles that have small diameter. We refer to any graph G obtained by appending $n \geq 1$ leaves to exactly one exterior vertex of a diamond as a *kite*. We let $\mathcal{K}$ be the set of all kites. We also let $\mathcal{R}$ be the set of all graphs G obtained from the disjoint union of a propeller or windmill and $k \geq 1$ copies of P_3 by identifying the nose of the propeller or windmill with exactly one leaf from each of the P_3s. Note that $\mathcal{K} \cup \mathcal{R} \subset \mathcal{G}$.

Lemma 7. *Let G be a connected K_4-free well-edge-dominated graph built from an independent set I, a set S, and a triangle xyz such that no vertex $t \in I$ is adjacent to a vertex in $\{x, y, z\} \cup (I - \{t\})$, and each vertex in S has a neighbor in $\{x, y, z\}$. Then $G \in \{Cr, \mathcal{H}\} \cup \mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$.*

Proof. Suppose to the contrary that a counterexample exists and among all such counterexamples, choose one with minimum cardinality. We may assume that $n(G) \geq 9$. Furthermore, if G contains only one triangle, then by Theorem 3 and the assumptions on G, $G \in \mathcal{F} \cup \mathcal{T} \cup \{\mathcal{H}, K_3, Cr\}$ where $K_3 \in \mathcal{P}$ and $\mathcal{H} \in \mathcal{W}$. Thus, we may assume that G contains at least two triangles. We proceed by considering whether $I = \emptyset$ or $I \neq \emptyset$.

Case 1: Suppose $I \neq \emptyset$.

Let $st \in E(G)$ where $t \in I$ and $s \in S$. As G is assumed to be well-edge-dominated, $H = G - N_e[st]$ is well-edge-dominated. Moreover, H satisfies the assumptions in the statement of the theorem so we may conclude that $H \in \{Cr, \mathcal{H}\} \cup \mathcal{K} \cup \mathcal{P} \cup \mathcal{W} \cup \mathcal{R}$.

Assume first that at least two vertices of $\{x, y, z\}$ have degree at least three in H. Without loss of generality, we may assume $\deg_H(x) \geq 3$ and $\deg_H(y) \geq 3$. It follows that $H \in \{Cr, \mathcal{H}\} \cup \mathcal{K} \cup \mathcal{W} \cup \mathcal{R}$. However, we have assumed $n(H) \geq 7$ meaning that $H \cong Cr$ or $H \in \mathcal{K} \cup \mathcal{W} \cup \mathcal{R}$. Let us assume that $H \cong Cr$ and label the vertices as in Figure 1 (c). It follows that $\{s, a, b, c\} = S$ and $\{t, d\} = I$. Without loss of generality, we may assume that s is adjacent x or y. If $sx \in E(G)$, then $H' = G - N_e[cd]$ is a well-edge-dominated graph containing a triangle with order 7 and yet by choice of minimal counterexample,

$H' \in \{Cr, \mathcal{H}\} \cup \mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$. However, this cannot be as $\deg_{H'}(x) \geq 4$, $\deg_{H'}(y) \geq 3$, and $\{x, y, z, a, b\}$ induces the house graph and no such graph in $\mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$ satisfies these conditions. If $xs \notin E(G)$, then we may also assume that $zs \notin E(G)$ and $sy \in E(G)$. In this case, H' contains the house graph induced by $\{x, y, z, a, b\}$ and yet $\deg_{H'}(y) \geq 4$, which cannot be. Therefore, we may assume that $H \in \mathcal{K} \cup \mathcal{W} \cup \mathcal{R}$. Assume first $H \in \mathcal{K}$. First, label the remaining vertex on the diamond as w and assume that both x and y are the interior vertices of the diamond and that we have appended leaves $b_1, \ldots, b_n$ to w. It follows that $\{s, w\} = S$ and $\{b_1, \ldots, b_n, t\} = I$. Further, we may assume that s is adjacent to at least one of x or z. If $sx \in E(G)$, then both $\{sx, wy\}$ and $\{st, b_1w, yz\}$ are minimal edge dominating sets. Otherwise, if $sx \notin E(G)$ and $sz \in E(G)$, then $\{sz, wy\}$ and $\{st, yz, wb_1\}$ are minimal edge dominating sets. Note that if z is the exterior vertex of the diamond with appended leaves, then a similar argument as above works as well. Finally, assume $H \in \mathcal{W} \cup \mathcal{R}$. Note that if $H \in \mathcal{R}$, then H is constructed from a windmill as x and y are assumed to have degree at least 3 in H. Therefore, whether $H \in \mathcal{W}$ or $H \in \mathcal{R}$, we may assume that H is constructed from a windmill W which contains the house graph with vertices $\{x, y, z, a, b\}$ where xa and yb are edges, and $k \geq 0$ triangles where the nose of the windmill is z. If $k \geq 1$, we label the remaining vertices of the windmill as $\{c_i, d_i : i \in [k]\}$ where zc_id_i is a triangle in H for $i \in [k]$. Moreover, if $H \in \mathcal{R}$, then said windmill has $n \geq 1$ P_3s attached to the windmill by identifying z with one leaf from each of the P_3s. In this case, we label each of the remaining vertices of the P_3s as $\{e_i, f_i : i \in [n]\}$ where e_if_iz is a path in H. Note that if s is adjacent to some vertex $u \in \{x, y\} \cup \{c_i, d_i : i \in [k]\}$, then $G - N_e[ab]$ contains the propeller with nose z and yet u does not have degree 2, making it a smaller counterexample. Hence, we shall assume that the only neighbor of s which is on a triangle is z. Consider $\widetilde{H} = G - N_e[ab]$ which we know is either Cr or in $\mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$. $\widetilde{H}$ contains the propeller $W - \{a, b\}$ meaning that $\widetilde{H} \ncong Cr$ and $\widetilde{H} \notin \mathcal{K} \cup \mathcal{P}$. Moreover, $\widetilde{H} \notin \mathcal{W}$ as t is not adjacent to z. Thus, $\widetilde{H} \in \mathcal{R}$, t is a leaf in $\widetilde{H}$, $N_G(s) \subseteq \{z, a, b, t\}$, and $N_G(t) \subseteq \{z, a, b, s\}$. If $W - \{a, b\}$ contains the triangle c_1d_1z, then $G - N_e[c_1d_1] \in \mathcal{R}$ and we may conclude that $G \in \mathcal{R}$. Similarly, if $\widetilde{H}$ contains the pendent edge e_1f_1, then $G - N_e[e_1f_1] \in \mathcal{R}$ and we may conclude $G \in \mathcal{R}$. However, now we have $V(G) = \{x, y, z, a, b, s, t\}$ contradicting that our chosen counterexample has order at least 9.

Next, assume that exactly one vertex of $\{x, y, z\}$ has degree at least three in H. Without loss of generality, we may assume $\deg_H(x) \geq 3$. This implies that H does not contain the house graph as an induced subgraph nor does it contain a diamond. Therefore, $H \in \mathcal{P} \cup \mathcal{R}$ where x is the nose of the propeller with $\deg_H(x) \geq 4$. Label the vertices of the remaining triangles other than xyz (if they exist) as xc_id_i for $i \in [k]$ and label the remaining P_3s (if they exist) as xe_jf_j for $j \in [m]$. Recall that G contains at least two triangles. Therefore, if H contains only one triangle, then we can interchange the roles of st and e_1f_1 so that H contains at least two triangles and $k \geq 1$. If e_1f_1 exists, then consider $H'' = G - N_e[e_1f_1]$ where $\deg_{H''}(x) \geq 3$. If $\deg_{H''}(y) \geq 3$ or $\deg_{H''}(z) \geq 3$, then we have already considered this case above. So we may assume that $\deg_{H''}(y) = \deg_{H''}(z) = 2$ meaning that $N_G(s) \cap \{x, y, z\} = \{x\}$. On the other hand, if e_1f_1 does not exist, then $k \geq 2$ and we consider $G - N_e[c_1d_1]$ which again will lead us to

$N_G(s) \cap \{x, y, z\} = \{x\}$. Lastly, $G - N_e[yz]$ contains the triangle xc_1d_1 and is therefore in $\mathcal{P} \cup \mathcal{R}$ where x is the nose of the propeller as it is adjacent to c_1, d_1, and s. It follows that $G \in \mathcal{R}$.

Case 2: Suppose $I = \emptyset$.

We proceed by considering whether S is an independent set or not. Suppose first that S is independent. As we have assumed that G contains at least two triangles, it follows that there is some vertex in S that is adjacent to two vertices in $\{x, y, z\}$. Therefore, we assume that $s \in S$ is adjacent to both x and y. If $\deg_G(z) = 2$, then every vertex in S is adjacent to either x or y or both. In this case, xy is an edge dominating set of G and yet $\{xz, sy\}$ is a matching in G, which cannot be. Therefore, we may assume $\deg_G(z) \geq 3$. Let $s_z \in S - \{s\}$ be a neighbor of z (s is not adjacent to z as G is K_4-free). If there exists $s' \in S - \{s, s_z\}$ adjacent to y, then both $\{sx, yz\}$ and $\{sx, s'y, s_z z\}$ are maximal matchings in G. So we may assume $N_G(y) \subseteq \{x, z, s, s_z\}$. A similar argument shows that we may assume $N_G(x) \subseteq \{y, z, s, s_z\}$. If both x and y have degree 3 in G, then $G \in \mathcal{K}$ and we are done. Thus, we shall assume either $xs_z \in E(G)$ or $ys_z \in E(G)$. Without loss of generality, we shall assume $xs_z \in E(G)$. As G is K_4-free, it follows that $ys_z \notin E(G)$. Further, $|S - \{s, s_z\}| \geq 4$ as $n(G) \geq 9$. Let $s' \in S - \{s, s_z\}$ which is adjacent to z. In this case, both $\{xs_z, ys, s'z\}$ and $\{xz, sy\}$ are maximal matchings in G. Hence, this case cannot occur and S is not an independent set.

Therefore, we may assume that S is not an independent set. Let $s_1, s_2 \in S$ be adjacent vertices in G. We consider $H'' = G - N_e[s_1s_2]$. Thus, $H'' \in \{Cr\} \cup \mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$. Note that we may assume $H'' \notin \mathcal{R}$ as this would imply $I \neq \emptyset$. If $H'' \cong Cr$, then we label the remaining vertices of the induced P_4 as a, b, c, and d where xa, yb, and zc are in $E(G)$. However, this implies that $d \in I$ which is a contradiction. Next, suppose $H'' \in \mathcal{K}$. This implies that H'' is built from the diamond induced by $\{w, x, y, z\}$ where $w \in S$. Without loss of generality, we may assume x and y are the interior vertices of the diamond. Furthermore, since $w \in S$, it must be that H'' is built by attaching leaves $b_1, \ldots, b_n$ to z for otherwise $I \neq \emptyset$. If both s_1 and s_2 are adjacent to a vertex in $\{w, x, y\}$, then we could instead view G as built from the triangle wxy so that z, s_1, and s_2 are in S and $I = \{b_1, \ldots, b_n\}$, which is a case we have already considered. Therefore, we shall assume that $N_G(s_1) \cap \{x, y, z\} = \{z\}$ and $N_G(s_2) \cap \{x, y, z\} = \{z\}$. In this case, we could instead view G as built from the triangle zs_1s_2 where $I = \{w\}$, yet another case that we have already considered. Therefore, $H'' \notin \mathcal{K}$.

Lastly, suppose $H'' \in \mathcal{P} \cup \mathcal{W}$ where z is the nose of the propeller or windmill. Since $n(H'') \geq 7$, we may assume that H'' is either built from the house graph where xyz is the triangle of the house graph and k triangles of the form zr_it_i for $i \in [k]$, or H'' is built from triangles xyz and zr_it_i for $i \in [k]$. In either case, $k \geq 1$. If $N_G(s_i) = \{z, s_j\}$ where $\{i, j\} = \{1, 2\}$, then $G \in \mathcal{P} \cup \mathcal{W}$. Thus, we may assume, up to relabeling, that s_1 is adjacent to either x, r_1, or perhaps a vertex of degree two on the house graph. If $k \geq 2$, then $G - N_e[r_2t_2]$ contains the propeller or windmill induced by $V(G) - \{s_1, s_2, r_2, t_2\}$, meaning that $G - N_e[r_2t_2] \in \mathcal{P} \cup \mathcal{W}$ and $N_G(s_i) = \{s_j, z\}$ for $\{i, j\} = \{1, 2\}$, which is a contradiction. Thus, we shall assume that $V(H'') = \{x, y, z, a, b, r_1, t_1\}$ where a and

b are the vertices of degree two on the house graph. If s_1 is adjacent to a vertex on the house graph other than z, then $G - N_e[r_1t_1]$ contains the house graph induced by $\{x, y, z, a, b\}$ and yet either x or a is adjacent to s_1, which does not fit the description of any graph in $\{Cr, \mathcal{H}\} \cup \mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$. Thus, we may assume that s_1 (and similarly, s_2) is not adjacent to any vertex on the house graph. In this case, $G - N_e[ab]$ contains an induced propeller and it must be that $N_G(s_i) = \{s_j, z\}$ for $\{i, j\} = \{1, 2\}$, and $G \in \mathcal{W}$. Having exhausted all possibilities, no such counterexample exists.

□

We are now ready to prove we have completely characterized all well-edge-dominated graphs with girth 3. Recall that F_5 is the fan of order 5.

Theorem 4. *Let G be K_4-free. G is well-edge-dominated with girth 3 if and only if $G \in \{\mathcal{DH}, F_5, Cr, W_1, W_2, W_3\}$ or $G \in \mathcal{G}$.*

Proof. Based on Corollary 1, we only need to show that if G is well-edge-dominated with girth 3, then $G \in \mathcal{G} \cup \{\mathcal{DH}, F_5, Cr, W_1, W_2, W_3\}$. Moreover, if G contains only one triangle, then we know by Theorem 3 that $G \in \mathcal{T} \cup \mathcal{F} \cup \{K_3, Cr, \mathcal{H}, \mathcal{DH}\}$ and $\{\mathcal{H}, K_3\} \cup \mathcal{T} \cup \mathcal{F} \subset \mathcal{G}$. So we only consider when G contains at least two triangles. We first note that if $n(G) \le 8$, then Sage verified that $G \in \mathcal{G} \cup \{W_1, W_2, W_3\}$. Therefore, we may assume that $n(G) \ge 9$. Assuming a counterexample exists, we choose the counterexample G with the smallest order. We may choose a triangle xyz and vertex t in G such that if we contract x, y, and z into a single vertex T, then $\text{dist}(T, t)$ is greatest among all choices of triangles xyz and vertices t. Then among all neighbors of t in G, choose s so that $\text{dist}(T, s)$ is maximum. Consider $G' = G - N_e[st]$. We may write $G' = G_0 \cup I$ where I is an independent set in G and G_0 is connected for otherwise st was chosen incorrectly. We can partition $I = I_s \cup I_{st} \cup I_t$ where each vertex in I_s is adjacent to s and not t, each vertex in I_t is adjacent to t and not s, and each vertex in I_{st} is adjacent to both s and t. We claim that we may assume $I_t \cup I_{st} = \emptyset$. Suppose to the contrary that $\ell \in I_t \cup I_{st}$. If $\text{dist}(T, s) < \text{dist}(T, t)$, then $\text{dist}(T, \ell) \ge \text{dist}(T, t)$ which would imply that we should have chosen $s = \ell$. On the other hand, if $\text{dist}(T, s) = \text{dist}(T, t)$, then $\text{dist}(T, \ell) > \text{dist}(T, t)$ and we should have chosen $s = t$. In either case, we reach a contradiction. So we may assume that $I_t \cup I_{st} = \emptyset$. Furthermore, we may assume that $|I_s| \le 1$ for otherwise for some $\ell \in I_s$, $G - \ell$ is also well-edge-dominated, contradicting our choice of minimal counterexample. Thus, if $\text{dist}(T, s) < \text{dist}(T, t)$, then it is possible ℓ exists and is a leaf in G. Otherwise, if $\text{dist}(T, s) = \text{dist}(T, t)$, we should have chosen $\ell = t$ so this case cannot occur. It follows that $G' = G_0 \cup I_s$ where $|I_s| \le 1$ and $n(G_0) \ge 6$. We proceed by considering which graph G_0 is isomorphic to. Note that since G is well-edge-dominated, G_0 is well-edge-dominated. Since G is the smallest counterexample, $G_0 \in \mathcal{G} \cup \{\mathcal{DH}, F_5, Cr, W_1, W_2, W_3\}$. Since $n(G) \ge 9$, $G_0 \ncong F_5$. We consider when $G_0 \in \{W_1, W_2, W_3\}$ (Case 1), $G_0 \in \{\mathcal{DH}, Cr\}$ (Case 2), and $G_0 \in \mathcal{G}$ (Case 3).

Case 1: Suppose first that $G_0 \in \{W_1, W_2, W_3\}$.

Label the vertices as in Figure 2. Suppose first that ℓ exists and G has order 10. Note that $H = G - N_e[ab]$ is well-edge-dominated with order 8 and contains both a diamond and a leaf, namely ℓ. Either H is not connected or $H \in \{V_1, V_2, V_3\}$ in Figure 4. Therefore, s (and t) is not adjacent to any vertex in $\{w, x, y\}$. Similarly, $G - N_e[bc]$ is well-edge-dominated with order 8 and therefore s (and t) is not adjacent to any vertex in $\{x, y, z\}$. Thus, the only neighbors of s in G are in $\{a, b, c\}$. Consider $H' = G - N_e[cz]$ which contains the triangle wxy, no diamond, and leaf ℓ. Suppose first that H' is connected so that $H' \in \{V_4, V_5, \ldots, V_{12}\}$ in Figure 4. However, H' contains at most two leaves and if abs is a triangle, then wxy and abs (or abt) are vertex disjoint triangles with $wa \in E(G)$. Therefore, $H' \notin \{V_4, V_5, V_{10}, V_{12}\}$. Furthermore, $\deg_{H'}(x) = 2 = \deg_{H'}(y)$ and $\deg_{H'}(w) = 3$ so $H' \notin \{V_6, V_7, V_{11}\}$. Also, ℓ is not adjacent to a so $H' \ncong V_8$. It follows that $H' \cong V_9$ where $at \in E(G)$ and $bs \in E(G)$. However, this implies $G - N_e[at]$ is connected, with order 8, contains a diamond and one leaf (ℓ). The only graph in Figure 4 that satisfies these requirements is V_3, but ℓ is not adjacent to z. Therefore, this case cannot occur and we may assume that H' is not connected, meaning that s (and t) can only be adjacent to c in G_0. Thus, H is connected, with order 8, contains a diamond and at most two leaves. Therefore, $H \in \{V_2, V_3\}$. In either case, this would imply s is adjacent to z, which we have already ruled out as a possibility. Hence, we may assume that ℓ does not exist and G has order 9.

As before, setting $H = G - N_e[ab]$, we see that H contains a diamond, but now has order 7. Suppose first that H is not connected. It follows that s and t are not adjacent to any vertex in $\{w, x, z, c\}$. Consider $H' = G - N_e[cz]$, which is a connected, well-edge-dominated graph of order 7. By Figure 3, no connected, well-edge-dominated graph of order 7 contains three triangles except W_{13}. Thus, $\{a, b, s, t\}$ is not a diamond. If H' contains exactly one triangle, then it must be the case that $H' \cong W_9$ or $H' \cong W_{10}$. In either case, $H'' = G - N_e[bc]$ is a connected, well-edge-dominated graph of order 7 that contains a diamond; however, H'' is not W_1, W_2, W_3, or W_4, which is a contradiction. If H'' contains exactly two triangles, then those triangles do not share a common vertex, which cannot occur by Figure 3. Thus, we may assume that H is connected. By Observation 1, $H \in \{W_1, W_2, W_3, W_4\}$. It is not possible for H to have three leaves. Thus, $H \ncong W_4$ and we may assume that $H \in \{W_1, W_2, W_3\}$. Hence, w is adjacent to either s or t. In either case, $G - N_e[bc]$ is also in $\{W_1, W_2, W_3\}$ using the same reasoning that $H \in \{W_1, W_2, W_3\}$, and yet w has degree four in $G - N_e[bc]$. Therefore, this case cannot occur either and we may assume $G_0 \notin \{W_1, W_2, W_3\}$.

($\Box$)

Case 2: Suppose that $G_0 \in \{\mathcal{DH}, Cr\}$.

Suppose first that $G_0 \cong \mathcal{DH}$ and label the vertices as in Figure 1 (a). Since $\alpha'(\mathcal{DH}) = 3$, it follows that $\alpha'(G) = 4$. Note that by our choice of s and t, s cannot be adjacent to x, y, or z for otherwise we should have chosen ab to act as the edge st. Moreover, by our choice of s and t, this implies that we may assume t is not adjacent to any vertex in $\{x, y, z\}$. Therefore, we have only to consider when s is adjacent a vertex in $\{a, b, c, d\}$. Moreover, since G is symmetric, we need only consider the two cases where s is adjacent to a versus the case that s is adjacent to c.

We consider $H' = G - N_e[bd]$ which is connected, has order 7 or 8 and contains the triangle xyz where $\deg_{H'}(z) = \deg_{H'}(y) = 2$ so x is the nose of the triangle. If H' contains only one triangle, then by Theorem 3, $\{s, t, a, c, x\}$ (or $\{s, t, a, c, x, \ell\}$ if ℓ exists) induces a bipartite graph J in G with bipartition $A_J \cup B_J$ where $|A_J| < |B_J|$ and x is detachable. Suppose first that ℓ exists. This implies that $s \in A_J$, meaning $sa \in E(H')$ and $sc \notin E(H')$. However, this implies that x is not a detachable vertex in J. So we shall assume that ℓ does not exist. Moreover, we must have $s \in B_J$ meaning that $sc \in E(H')$ and $sa \notin E(H')$. If $ta \notin E(H')$, then again x is not detachable in J. Now when we consider $G - N_e[xy]$, we have a well-edge-dominated (and thus equimatchable) graph induced by $\{s, t, a, b, c, d\}$ containing a perfect matching. Thus, by Theorem 1, the graph induced by $\{s, t, a, b, c, d\}$ must be $K_{3,3}$ so sb and td are edges in G. Now $\{sc, st, sb, dy, xz\}$ is a minimal edge dominating set contradicting the fact that $\alpha'(G) = 4$. So we shall assume that H' contains at least two triangles, and perhaps ℓ exists. Note that H' is diamond-free for otherwise said diamond is induced by $\{s, t, a, c\}$ and the only graph in Figures 3 and 4 that have an edge between a diamond and a triangle is V_3. However, s is not adjacent to x and ℓ is not adjacent to c so this case cannot occur. Moreover, the only diamond-free graphs in Figures 3 and 4 that contain at least two triangles are in $\{W_8, W_{11}, W_{13}, V_{10}, V_{12}\}$. We know H' cannot contain an induced propeller so the only possibility is $H' \cong V_{12}$, yet this is impossible as it implies ℓ is adjacent to c, which cannot be. Thus, we may assume $G_0 \ncong \mathcal{DH}$.

Next, assume that $G_0 \cong Cr$ and label the vertices as in Figure 1 (c). If s is adjacent to a vertex in $\{x, y, z\}$ and $\{t, d\}$ (or $\{t, d, \ell\}$ if ℓ exists) is an independent set, then G satisfies the statement of Lemma 7. Thus, $G \in \{\mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}\} \subset \mathcal{G}$ since $\mathcal{K} \cup \mathcal{R} \subset \mathcal{G}$, and we are done. So we may assume that either s is not adjacent to any vertex in $\{x, y, z\}$ or $td \in E(G)$. Suppose first that s (and therefore t) is not adjacent to any vertex in $\{x, y, z\}$. As G contains at least two triangles, we may assume one is xyz and another contains s and two vertices from $\{a, b, c, d\}$ or contains s and t and one vertex from $\{a, b, c, d\}$. Suppose first that either abs, sta, or stb is a triangle in G. Then $G - N_e[cd]$ is connected, has order 7 or 8, contains two vertex disjoint triangles and yet xyz is not on a diamond and each of x and y have degree 3. No such graph exists in Figures 3 and 4 so this case cannot occur. Similarly, if bcs or stc is a triangle in G, then $G - N_e[ad]$ satisfies the above and yet is not depicted in Figures 3 and 4. So a second triangle of G must be one of std, sad, or scd. In this case, $G - N_e[by]$ contains a triangle and an induced C_5 ($\{a, d, c, x, z\}$) which is not depicted in Figures 3 and 4. Having exhausted all possibilities, we may assume that s is adjacent to some vertex in $\{x, y, z\}$ and $td \in E(G)$. In this case, $G - N_e[td]$ is connected, has order 7 or 8, contains the triangle xyz where each of x, y, and z have degree at least three and at least one of x, y, or z has degree 4, and contains at most one leaf. Based on the graphs in Figures 3 and 4, it follows that $G - N_e[cd] \in \{W_2, W_3, W_8, V_3\}$. However, $\{a, b, c\}$ induces a path so $G - N_e[cd] \notin \{W_8, V_3\}$. It follows that $G - N_e[cd] \in \{W_2, W_3\}$, meaning that $\{s, x, y, z\}$ induces a diamond. In this case, exactly two vertices in $\{x, y, z\}$ have degree 4 in $G - N_e[cd]$, meaning $G - N_e[cd] \notin \{W_2, W_3\}$, which is a contradiction. Therefore, $G_0 \ncong Cr$.

(□)

Case 3: Suppose that $G_0 \in \mathcal{G}$.

We may assume that of $\{x, y, z\}$, x has the maximum distance to s. Suppose first that s (and therefore t) is not adjacent to any vertex of $\{x, y, z\}$. We proceed by considering whether xyz is on a propeller, a windmill, or a diamond in G_0.

1. Suppose first that xyz is on a propeller in G_0. We may assume that z is the nose of the propeller. It follows that $\deg_{G_0}(x) = \deg_{G_0}(y) = 2$. Consider $H = G - N_e[xy]$ which has order at least 7. Since neither s nor t is adjacent to a vertex in $\{x, y, z\}$, we infer that $\deg_G(x) = \deg_G(y) = 2$. If $H \in \mathcal{G}$ where z is the nose of a propeller, then $G \in \mathcal{G}$ and we are done. Thus, we shall assume that either $H \in \{Cr, \mathcal{DH}, W_1, W_2, W_3\}$ or $H \in \mathcal{G}$ where z is not the nose of a propeller.

   Suppose first that $H \in \{W_1, W_2, W_3\}$. This implies that $n(G) = 9$ and ℓ does not exist. Label the vertices of the diamond as $\{a, b, c, d\}$ where a and c are the interior vertices. Further, if $H \cong W_2$, then assume c is the interior vertex with degree 4 and if $H \cong W_3$, then a is the interior vertex with degree 4 and b is adjacent to the vertex with degree 3 that is not on the diamond. Label the remaining vertices of H as $\{j, k, m\}$ where j is adjacent to d and m is adjacent to b. Note that s, t and z are all vertices in H. We proceed by considering each of the possibilities for z. Note that $\alpha'(G) = 4$. If $z = j$, then $\{jx, ac, bm\}$ is an edge dominating set of G, which is a contradiction. If $z = k$, then $\{kx, ad, bc\}$ is an edge dominating set of G. If $z = m$, then $\{xm, ac, dj\}$ is an edge dominating set of G. In each case, we reach a contradiction. So we may assume that z is on the diamond in H. If $z = d$, then $\{xd, km, ac\}$ is an edge dominating set of G. If $z = a$, then $\{xa, cd, km\}$ is an edge dominating set of G. If $z = c$, then $\{cx, ab, jk\}$ is an edge dominating set of G. If $z = b$, then $\{bx, km, cd\}$ is an edge dominating set of G. In each case, we reach a contradiction so we may assume that $H \notin \{W_1, W_2, W_3\}$.

   Next, suppose that $H \cong \mathcal{DH}$. Again we may assume that ℓ does not exist. Label the vertices of the triangle in H as a, b, and c where a has degree 2 in H. Label the remaining vertices in H with j, k, m, and r where $jkmr$ induces a path, b is adjacent to j, and r is adjacent to c. Note that s, t, and z are vertices in H. We consider the different possibilities for z in H. Since $\mathcal{DH}$ is symmetric, we need only to consider when $z \in \{a, b, j, k\}$. Note that $\alpha'(G) = 4$. If $z = a$, then $st = km$ by choice of s and t. This implies that $\{jk, cr, xa\}$ is an edge dominating set of G. If $z = b$, then $\{bx, cr, km\}$ is an edge dominating set of G. If $z = j$, then $\{jx, mr, bc\}$ is an edge dominating set of G. If $z = k$, then $\{kx, cr, ab\}$ is an edge dominating set of G. In each case, we reach a contradiction so we may assume $H \ncong \mathcal{DH}$.

   Next, assume $H \cong Cr$. As above, we may assume that ℓ does not exist. Label the vertices of the triangle in H as a, b, c and the remaining vertices in H as j, k, m, and r where $jkmr$ induces a path in H, a is adjacent to j, b is adjacent to k and c is adjacent to m. As above, we consider the different possibilities for z in H. Further, as Cr is symmetric, we only need to consider when $z \in \{a, b, j, k, r\}$. Note that $\alpha'(G) = 4$. If $z = a$, then $\{ax, bk, mr\}$ is an edge dominating set of G. If $z = b$, then $\{bx, aj, cm\}$ is an edge dominating set of G. If $z = j$, then $\{xj, km, ac\}$

is an edge dominating set of G. If $z = k$, then $\{xk, ac, mr\}$ is an edge dominating set of G. If $z = r$, then $\{rx, ac, bk\}$ is an edge dominating set of G. In each case, we reach a contradiction so this case cannot occur either.

Finally, assume that $H \in \mathcal{G}$ yet z is not the nose of a propeller in H. This implies that the induced propeller containing the triangle xyz in G_0 is K_3. Suppose first that $H \in \mathcal{P}$ and built from k triangles $T_1, \dots, T_k$ with $V(T_i) = \{a_i, b_i, c_i\}$ for $i \in [k]$. We may assume that H is obtained from the disjoint union $T_1 \cup \cdots \cup T_k$ by identifying the set $\{a_i : i \in [k]\}$ into a single vertex, call it a. Since we have assumed z is not the nose of P, we may assume with no loss of generality that $z = b_1$. Moreover, H contains s and t and $G_0 \in \mathcal{G}$. It follows that $a \notin \{s, t\}$ and $st = b_i c_i$ for some $2 \le i \le k$. Further, since $G_0 \in \mathcal{G}$, $k = 2$ as z is the nose of the propeller containing xyz in G_0. However, this implies $n(G) = 7$ which is a contradiction. Hence, we may assume $H \notin \mathcal{P}$.

Since xyz is on a propeller in G_0, it is also the case that $H \notin \mathcal{W}$ and $H \ncong K^*$. Therefore, we can construct H from the disjoint union of graphs

$$J, P_1, \dots, P_r, W_1, \dots, W_k, D_1, \dots, D_\ell$$

where $J = (A_J \cup B_J, E_J)$ is connected, nontrivial, bipartite well-edge-dominated with $|A_J| < |B_J|$ and we have chosen $B'_J \subset B_J$ such that $J - B'_J$ is well-edge-dominated with no trivial components. Further, when we write

$$B'_J = \{x_1, \dots, x_r, s_1, \dots, s_k\},$$

every neighbor of s_i in A_J is a support vertex in $J - B'_J$. So we know that $|A_J| + |B_J| \ge 3$. If we can show that $B''_J = \{x_0, x_1, \dots, x_r, s_1, \dots, s_k\}$ is such that $J - B''_J$ is well-edge-dominated with no trivial components where x_0 is the vertex in J that we identified z with to create G_0 and every neighbor of s_i in A_J is a support vertex in $J - B''_J$, then $G \in \mathcal{G}$. Consider $H' = G - N_e[xz]$. Since y is on a propeller of G_0 and it is assumed that y is not adjacent to s or t, it follows that y is an isolate in H'. Moreover, H' contains a triangle as G contains at least two triangles. We consider the cardinality of J. Suppose first that J is a star. It follows that there is some vertex $v \in B_J - B'_J$ for otherwise $J - B'_J$ contains a trivial component. Therefore, $x_0 \in B_J - B'_J$ and if $|B_J - B'_J| \ge 2$, then $G \in \mathcal{G}$. So we shall assume that $B_J - B'_J = \{x_0\}$. Thus, $B_J = \{x_0, x_1, \dots, x_r, s_1, \dots, s_k\}$ and $A_J = \{a\}$. If P_1 exists with triangle jkm such that j is the nose of P_1, then the component $\widetilde{G}$ of $G - N_e[jk]$ containing a is not well-edge-dominated as it is not in $\{Cr, \mathcal{DH}, W_1, W_2, W_3\}$ nor is it in $\mathcal{G}$. We also reach a similar contradiction if W_1 exists. However, this would imply that $J = K_2$ and $|B_J| = |A_J|$ which is another contradiction.

Thus, we may assume that J is not a star. This implies $|A_J| \ge 2$ and $|B_J| \ge 3$. If $H' = G - N_e[xz]$ contains an induced windmill, then $n(H') \ge 8$ and we have $H' \in \mathcal{G}$. Thus, H' is obtained from

$$\widetilde{J}, P_1, \dots, P_r, W_1, \dots, W_k, D_1, \dots, D_\ell$$

where $\widetilde{J} = J - \{x_0\}$, $B_{\widetilde{J}} = B_J - \{x_0\}$ and $B'_{\widetilde{J}} = B'_J$ so that $\widetilde{J} - B'_{\widetilde{J}}$ is well-edge-dominated. Therefore, $J - B''_J$ is well-edge-dominated. Moreover, each neighbor of s_i in A_J is a support vertex in $\widetilde{J} - B'_{\widetilde{J}}$, and $G \in \mathcal{G}$. Similarly, if H' contains two induced diamonds, or H' contains an induced diamond and an induced propeller, then $n(H') \geq 8$ and both H' and G are in $\mathcal{G}$. On the other hand, since $H \in \mathcal{G}$, every triangle in H (and therefore G and H') is either on a diamond where an exterior vertex has degree two in H, or has exactly one vertex with degree at least three in H. Hence, since $n(H') \geq 6$ and $H' \notin \{W_1, W_2, W_3, \mathcal{DH}, Cr\}$, it must be that $H' \in \mathcal{G}$. Using the same argument as above, we have that $G \in \mathcal{G}$.

2. Next, suppose xyz is on an induced windmill W in G_0 where z is the nose of the windmill. Suppose first that $G_0 \in \mathcal{W}$ built from $\mathcal{H}$ with vertices $\{a, b_1, c_1, d, e\}$ where ab_1c_1 is the triangle, b_1d is an edge, and all remaining triangles in G_0 are of the form ab_ic_i for $2 \leq i \leq k$ (note that $k \geq 2$ as $n(G_0) \geq 7$). Since neither s (nor t) is adjacent to any vertex in $\{x, y, z\}$, we may assume that s is not adjacent to a in G. Suppose first that s is adjacent to b_i (or c_i) for some $i \in [k]$. Then $G - N_e[de]$ is not well-edge-dominated as there is no graph in $\mathcal{G} \cup \{Cr, \mathcal{DH}, W_1, W_2, W_3\}$ containing an induced propeller where a vertex other than the nose of the propeller has degree at least three in the graph. So the only neighbors of s (and therefore t) in G are in $\{d, e\}$. Without loss of generality, assume that $sd \in E(G)$. Let $H' = G - N_e[b_2c_2]$ which is connected and $n(H') \geq 7$. As H' contains the house graph as an induced subgraph (with vertices $\{a, b_1, c_1, d, e\}$) and $\deg_{H'}(c_1) = \deg_{H'}(b_1) = 3$ and $\deg_{H'}(d) \geq 3$, it follows that $H' \cong \mathcal{DH}$. In this case, both $\{st, de\} \cup \{b_ic_i : i \in [k]\}$ and $\{et, b_1d\} \cup \{ab_i : 2 \leq i \leq k\}$ are minimal edge dominating sets in G, another contradiction. Thus, we may assume $G_0 \notin \mathcal{W}$.

   Therefore, we can construct G_0 from the disjoint union of graphs

$$J, P_1, \ldots, P_r, W_1, \ldots, W_k, D_1, \ldots, D_\ell$$

   where $W_1 = W$, $J = (A_J \cup B_J, E_J)$ is connected, nontrivial, bipartite well-edge-dominated with $|A_J| < |B_J|$ and we have chosen $B'_J \subset B_J$ such that $J - B'_J$ is well-edge-dominated with no trivial components. Further, when we write $B'_J = \{x_1, \ldots, x_r, s_1, \ldots, s_k\}$, every neighbor of s_i in A_J is a support vertex in $J - B'_J$. So we know that $|A_J| + |B_J| \geq 3$ as there is some neighbor $u \in A_J$ adjacent to s_1 which is a support vertex in $J - B'_J$. We let v be a leaf of u in $J - B'_J$. Regardless of whether xyz is on the house graph in W_1 or not, we label the vertices of the house graph in W_1 as $\{z, a, b, c, d\}$ where zcd is the triangle and $da \in E(G)$. We argue that $\deg_G(a) = \deg_G(b) = 2$. Suppose to the contrary this is not the case. Since $\deg_{G_0}(a) = \deg_{G_0}(b) = 2$, it must be that s is adjacent to either a or b in G. Without loss of generality, assume $sa \in E(G)$. Let $\widetilde{G}$ be the component of $G - N_e[uv]$ containing z. Therefore, $\widetilde{G}$ contains an induced windmill with nose z yet the degree of a is at least three. Since z is not adjacent to s or t, it follows that $\widetilde{G} \cong \mathcal{DH}$ and $W_1 \cong \mathcal{H}$. Thus, $zcd = zxy$, $\mathrm{dist}(T, s) = 2$ and since $\widetilde{G} \cong \mathcal{DH}$, neither s not t is adjacent to any vertex in G other than potentially $\{u, v, a, b\}$.

We know that G contains a second triangle and if that triangle is on P_1 or W_2 in G_0, then st was chosen incorrectly. Therefore, $B'_J = \{s_1\}$. Similarly, st was chosen incorrectly unless J is a star and we can write $A_J = \{u\}$ and $B_J = \{v_1, \ldots, v_m, s_1\}$ where $v_1 = v$. If D_1 exists with vertices $\{u, w_1, w_2, w_3\}$ where w_3 is an exterior vertex, then we should have chosen $xyz = w_1w_2w_3$ as neither s nor t is adjacent to any vertex on this triangle. Hence, D_1 does not exist either and the second triangle in G involves st. In this case, $G - N_e[ab]$ is connected and in $\mathcal{G}$ with triangle stv_1 and v_2 exists. But then $\{as, tb, ad, cb, uv_2\}$ is a minimal edge dominating set of G as well as $\{st, ab, cd, uv_2\}$, the final contradiction. Thus, $\deg_G(a) = \deg_G(b) = 2$.

Now consider $H = G - N_e[ab]$ where xyz is on a propeller with nose z and H is built from

$$\widetilde{J}, P_1, \ldots, P_{r+1}, W_2, \ldots, W_k, D_1, \ldots, D_\ell$$

where $\widetilde{J} = (A_{\widetilde{J}} \cup B_{\widetilde{J}}, E_{\widetilde{J}})$ with $|A_{\widetilde{J}}| < |B_{\widetilde{J}}|$ which is nontrivial, connected, well-edge-dominated and $B'_{\widetilde{J}} \subset B_{\widetilde{J}}$ is chosen so that $\widetilde{J} - B'_{\widetilde{J}}$ is well-edge-dominated with no trivial components. Further, $P_{r+1} = W_1 - \{a, b\}$ and we may write $B'_{\widetilde{J}} = (x_1, \ldots, x_{r+1}, s_2, \ldots, s_k\}$ where the nose z of P_{r+1} is identified with x_{r+1}. All we need to show is that every neighbor of x_{r+1} in $\widetilde{J}$ is a support vertex in $\widetilde{J} - B'_{\widetilde{J}}$. Suppose that there is some $u' \in N_{\widetilde{J}}(x_{r+1})$ that is not a support vertex in $\widetilde{J} - B'_{\widetilde{J}}$. Note that $u' \notin \{s, t\}$ as neither s nor t is adjacent to z. As u' was a support vertex in $J - B'_J$ with leaf v', this means that in H v' is adjacent to either s or t. Without loss of generality, let us assume $sv' \in E(G)$. Let F be any minimal edge dominating set of G_0. We can also assume that the remaining vertices in W_1 other than those on the house graph induced by $\{z, a, b, c, d\}$ are of the form b_i, c_i for $i \in [k]$ where zb_ic_i is a triangle. Note that F contains an edge incident to u' so $|F \cap E(W_1)| = k + 2$. Let $F' = F \cap E(W_1)$ together with the edge in F incident to u'. One can easily verify that

$$(F - F') \cup \{sv'\} \cup \{b_ic_i : i \in [k]\} \cup \{zc, ad\}$$

is a minimal edge dominating set of G with cardinality

$$|F - F'| + 1 + k + 2 = |F|.$$

On the other hand, $F \cup \{st\}$ is also a minimal edge dominating set of G, which is a contradiction. It follows that no such u' exists and $G \in \mathcal{G}$.

3. Suppose that xyz is on a diamond and w is the fourth vertex of the diamond. Without loss of generality, we may assume x and y are interior vertices of the diamond. We shall assume first that w is the nose of the diamond and consider $H = G - N_e[yz]$ which is connected and has order at least 7 as s is not adjacent to either y or z. Moreover, x is a leaf of w in H so that $H \in \mathcal{G}$ and $H \notin \mathcal{P} \cup \mathcal{W}$. This implies that H is built from

$$J, P_1, \ldots, P_r, W_1, \ldots, W_k, D_1, \ldots, D_\ell$$

where $J = (A_J \cup B_J, E_J)$ is a connected, well-edge-dominated bipartite graph with $|A_J| < |B_J|$ and we have chosen $B'_J \subset B_J$ such that $J - B'_J$ is well-edge-dominated with no trivial components. Furthermore, $x \in B_J$. Set $J' = J - \{x\}$, $A_{J'} = A_J$, $B_{J'} = B_J - \{x\}$, and $B'_{J'} = B'_J$. If we can show that $J' = (A_{J'} \cup B_{J'}, E_{J'})$ is a connected, well-edge-dominated bipartite graph with $|A_{J'}| < |B_{J'}|$ and $J' - B'_{J'} = J' - B'_J$ is well-edge-dominated with no trivial components such that w is still a support vertex in $J' - B'_J$, then we have shown that $G \in \mathcal{G}$. First, consider $H' = G - N_e[xy]$ which is also connected, but has order at least 6 as z is an isolate in H'. However, since $H \notin \{Cr, \mathcal{DH}, W_1, W_2, W_3\}$, $H' \notin \{Cr, \mathcal{DH}, W_1, W_2, W_3\}$ so $H' \in \mathcal{G}$. Thus, since H and H' differ only by a leaf, this means that we can build H' from

$$J', P_1, \ldots, P_r, W_1, \ldots, W_k, D_1, \ldots, D_\ell$$

such that J' is well-edge-dominated where $|A_{J'}| < |B_{J'}|$. Moreover, letting $B'_{J'} = B'_J$, we see that $J' - B'_J$ is well-edge-dominated with no trivial components. We only need to show that w is a support vertex in $J' - B'_J$. Suppose for the sake of contradiction that w is not a support vertex in $J' - B'_J$. We construct two different minimal edge dominating sets for G. For each P_i, let E_i be the set of edges which are not incident to the nose of P_i. For each W_j, let F_j be any maximal matching of W_j with edges that are not incident to the nose of W_j. For each D_i, let e_i be the edge between two interior vertices of D_i. Since w is not a support vertex in J', by Lemma 4, there exists a minimal edge dominating set $\widetilde{F}$ of J' that does not contain an edge incident to w. Then both

$$\widetilde{F} \cup \bigcup_{i=1}^{r} E_i \cup \bigcup_{j=1}^{k} F_j \cup \{e_1, \ldots, e_\ell\} \cup \{xy\}$$

and

$$\widetilde{F} \cup \bigcup_{i=1}^{r} E_i \cup \bigcup_{j=1}^{k} F_j \cup \{e_1, \ldots, e_\ell\} \cup \{xz, yz\}$$

are minimal edge dominating sets of G, which cannot be. Thus, w is a support vertex in J' and $G \in \mathcal{G}$.

Lastly, assume that z is the nose of the diamond containing xyz and $\deg_{G_0}(w) = 2$. If w is not adjacent to s or t, then we should have chosen wxy to act as the triangle where $\operatorname{dist}(T, t)$ is maximum. Therefore, we shall assume that $sw \in E(G)$. It follows that $\operatorname{dist}(T, s) = 2$ and $\operatorname{dist}(T, t) \leq 3$. Moreover, G_0 is built from

$$\widetilde{J}, P_1, \ldots, P_r, W_1, \ldots, W_k, D_1, \ldots, D_\ell$$

where D_1 has vertices $\{w, x, y, z\}$, $\widetilde{J} = (A_{\widetilde{J}} \cup B_{\widetilde{J}}, E_{\widetilde{J}})$ is a connected, well-edge-dominated bipartite graph with $|A_{\widetilde{J}}| < |B_{\widetilde{J}}|$ and we have chosen $B'_{\widetilde{J}} \subset B_{\widetilde{J}}$ such that $\widetilde{J} - B'_{\widetilde{J}}$ is well-edge-dominated with no trivial components. Furthermore, we have identified z with $a \in A_{\widetilde{J}}$ which is a support vertex in $\widetilde{J} - B'_{\widetilde{J}}$ with leaf b_z.

Suppose there exists $a' \in A_{\widetilde{J}} - \{a\}$. As $\widetilde{J}$ is connected, we may assume that a and a' share a common neighbors, say b', in $B_{\widetilde{J}}$. It follows that the component $\widetilde{H}$ in $G - N_e[a'b']$ containing w contains the vertices $\{w, x, y, z, s, t, b_z\}$. Thus, $\widetilde{H} \in \{Cr, \mathcal{DH}, W_1, W_2, W_3\} \cup \mathcal{G}$. However, $\widetilde{H}$ contains a diamond and therefore $\widetilde{H} \notin \{Cr, \mathcal{DH}\}$. On the other hand, $\widetilde{H}$ contains a diamond where both exterior vertices have degree at least three and neither s nor t is adjacent to x or y. It follows that $\widetilde{H} \cong W_1$ and $tb_z \in E(G)$. Moreover, ℓ does not exist. And now when we consider $G - N_e[tb_z]$, we have a connected graph with order at least 7 and a diamond where both exterior vertices have degree at least three. Thus, $G - N_e[tb_z] \cong W_1$ meaning that $\widetilde{J} \cong P_4$, which is not a well-edge-dominated bipartite graph. It follows that no such $a' \in A_{\widetilde{J}} - \{a\}$ exists and $\widetilde{J}$ is a star and we can write $B_{\widetilde{J}} = \{b_1, \ldots, b_n\}$ where $b_n = b_z$. If t is adjacent to some b_i, then $G - N_e[tb_i]$ is connected with order at least six and contains a diamond where both exterior vertices have degree at least three. Thus, $G - N_e[tb_i] \in \{W_1, W_2, W_3\}$ and yet this cannot be as $B_{\widetilde{J}}$ is an independent set. Therefore, this case cannot occur either and we may conclude that $G \in \mathcal{G}$.

Finally, we assume that s is adjacent to at least one vertex of $\{x, y, z\}$. In this case, we can write $V(G) = \{x, y, z\} \cup S \cup T$ where each vertex in S has a neighbor in $\{x, y, z\}$, and each vertex in T has a neighbor in S, but no neighbor in $\{x, y, z\}$, and T is an independent set in G. However, now G satisfies the statement of Lemma 7, meaning that $G \in \{Cr, \mathcal{H}\} \cup \mathcal{K} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{W}$. Since $\mathcal{K} \cup \mathcal{R} \subset \mathcal{G}$, we are done.

□

3 Concluding Remarks

We now have a characterization for all non-bipartite, well-edge-dominated graphs that do not contain an induced K_4. Therefore, the final step to characterizing all non-bipartite, well-edge-dominated graphs would be to characterize those that contain an induced K_4. We believe that it is possible to characterize such graphs based on adding K_4s to graphs in $\mathcal{G}$.

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4 Appendix

4.1 Classes of Graphs

- **Class** $\mathcal{F}$: A graph $G \in \mathcal{F}$ if G is obtained from the disjoint union of the house graph $\mathcal{H}$, depicted in Figure 1(b), and a well-edge-dominated bipartite graph $G' = (A \cup B, E)$ where $|A| < |B|$, by identifying the vertex of degree two in $\mathcal{H}$ on the triangle with $w \in V(G')$ where w is strongly detachable.

- **Class** $\mathcal{T}$: A graph $G \in \mathcal{T}$ if G is obtained from the disjoint union of K_3 and a well-edge-dominated bipartite graph $G' = (A \cup B, E)$ where $|A| < |B|$, by identifying $x \in V(K_3)$ with $w \in V(G')$ where w is detachable.

- **Class** $\mathcal{P}$ **(Propellers)**: Let $T_1, \ldots, T_k$ be k disjoint triangles and for each $i \in [k]$ label one vertex of triangle T_i by v_i. A graph $G \in \mathcal{P}$ if it is obtained from $T_1, \ldots, T_k$ by identifying each vertex in $\{v_1, \ldots, v_k\}$ into a single vertex.

- **Class $\mathcal{W}$ (Windmills)**: Let $T_1, \dots, T_k$ be k disjoint triangles and for each $i \in [k]$ label one vertex of triangle T_i by v_i. Let $\mathcal{H}$ be the house graph, depicted in Figure 1(b), and label the vertex on the triangle with degree two x. A graph $G \in \mathcal{W}$ if it is obtained from $T_1, \dots, T_k$, and $\mathcal{H}$ by identifying each vertex in $\{v_1, \dots, v_k, x\}$ into a single vertex.

- **Class $\mathcal{G}$**: A graph $G \in \mathcal{G}$ if:

  - The graph $G \cong K^*$, where K^* is obtained from the disjoint union of a diamond and K_2 by identifying a vertex of K_2 with an exterior vertex of the diamond.
  - $G \in \mathcal{P} \cup \mathcal{W}$.
  - G can be constructed in the following way. Take the disjoint union of

    $$G', W_1, \dots, W_k, P_1, \dots, P_r, D_1, \dots, D_\ell$$

    where $G' = (A \cup B, E')$ is a connected, nontrivial, bipartite and well-edge-dominated graph with $|A| < |B|$, $W_i \in \mathcal{W}$ is a windmill, $P_i \in \mathcal{P}$ is a propeller, and each D_i is a diamond. For each diamond, label one of the vertices of degree two as the "nose". Let $B' \subset B$ be a set such that $G' - B'$ is a well-edge-dominated graph with $\gamma'(G' - B') = \gamma'(G')$, $G' - B'$ has no trivial components, and $|A| \leq |B - B'|$. Each vertex v in B' is called detachable and it is called strongly detachable if each neighbor of it in G' is a support vertex in $G' - B'$. Enumerate the vertices of $B' = \{s_1, \dots, s_k, x_1, \dots, x_r\}$ such that each s_i is strongly detachable. Then choose a set $\{y_1, \dots, y_\ell\} \subset A$ where y_i is a support vertex in $G' - B'$. We obtain G from $G', W_1, \dots, W_k, P_1, \dots, P_r, D_1, \dots, D_\ell$ under the following rules:

    (a) The nose of W_i is identified with the strongly detachable vertex s_i of B'.

    (b) The nose of P_i is identified with the detachable vertex x_i of B'.

    (c) The nose of D_i is identified with the support vertex y_i of G' (which stays a support vertex in G).

- **Class $\mathcal{K}$ (Kites)**: A graph $G \in \mathcal{K}$ if G is obtained by appending $n \geq 1$ leaves to exactly one exterior vertex of a diamond.

- **Class $\mathcal{R}$**: A graph $G \in \mathcal{R}$ if G is obtained from the disjoint union of a propeller or windmill and $k \geq 1$ copies of P_3 by identifying the nose of the propeller or windmill with exactly one leaf from each of the P_3s.

4.2 Connected Well-Edge-Dominated Graphs with 7 Vertices

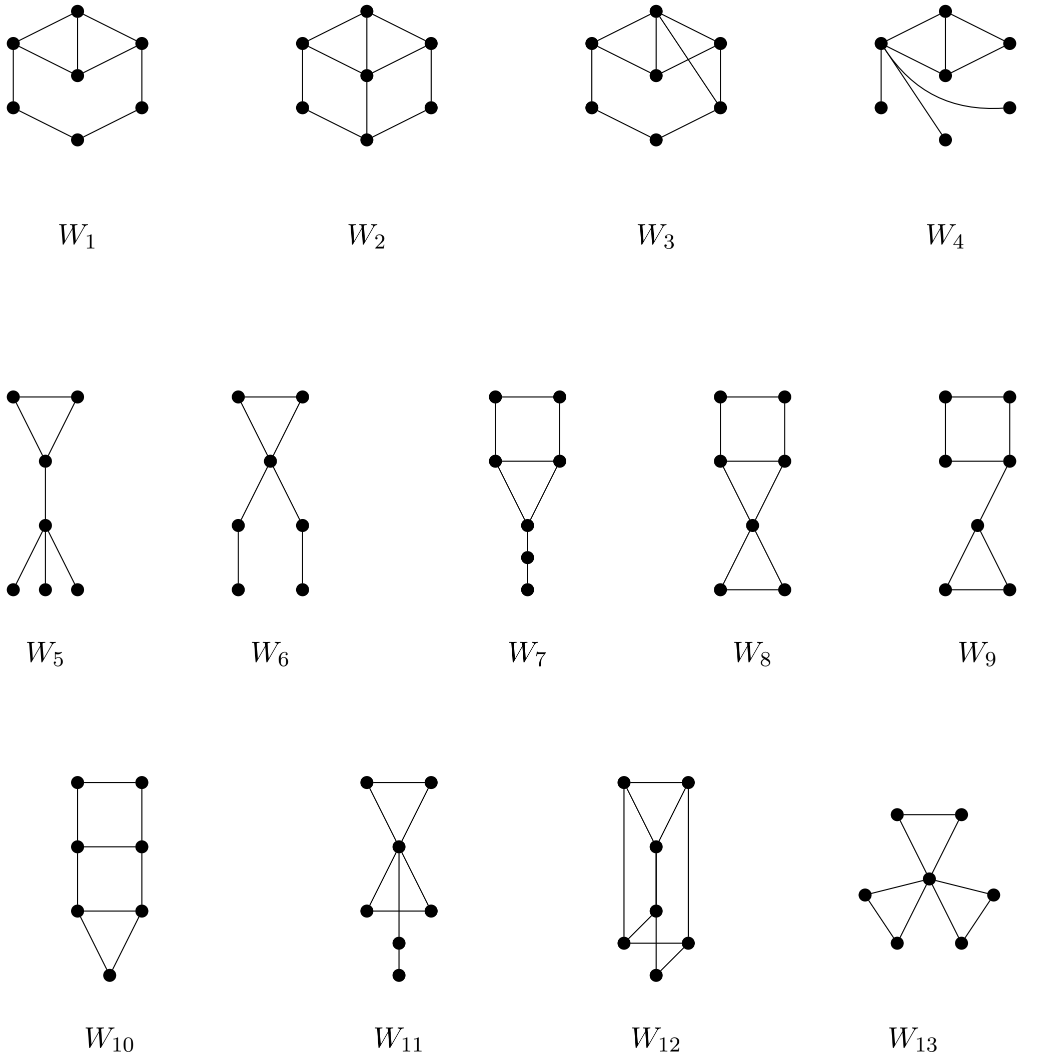

Figure 3: All well-edge dominated graphs of order 7, only W_1, W_2, W_3, and W_4 contain a diamond

4.3 Connected Well-Edge-Dominated Graphs with 8 Vertices

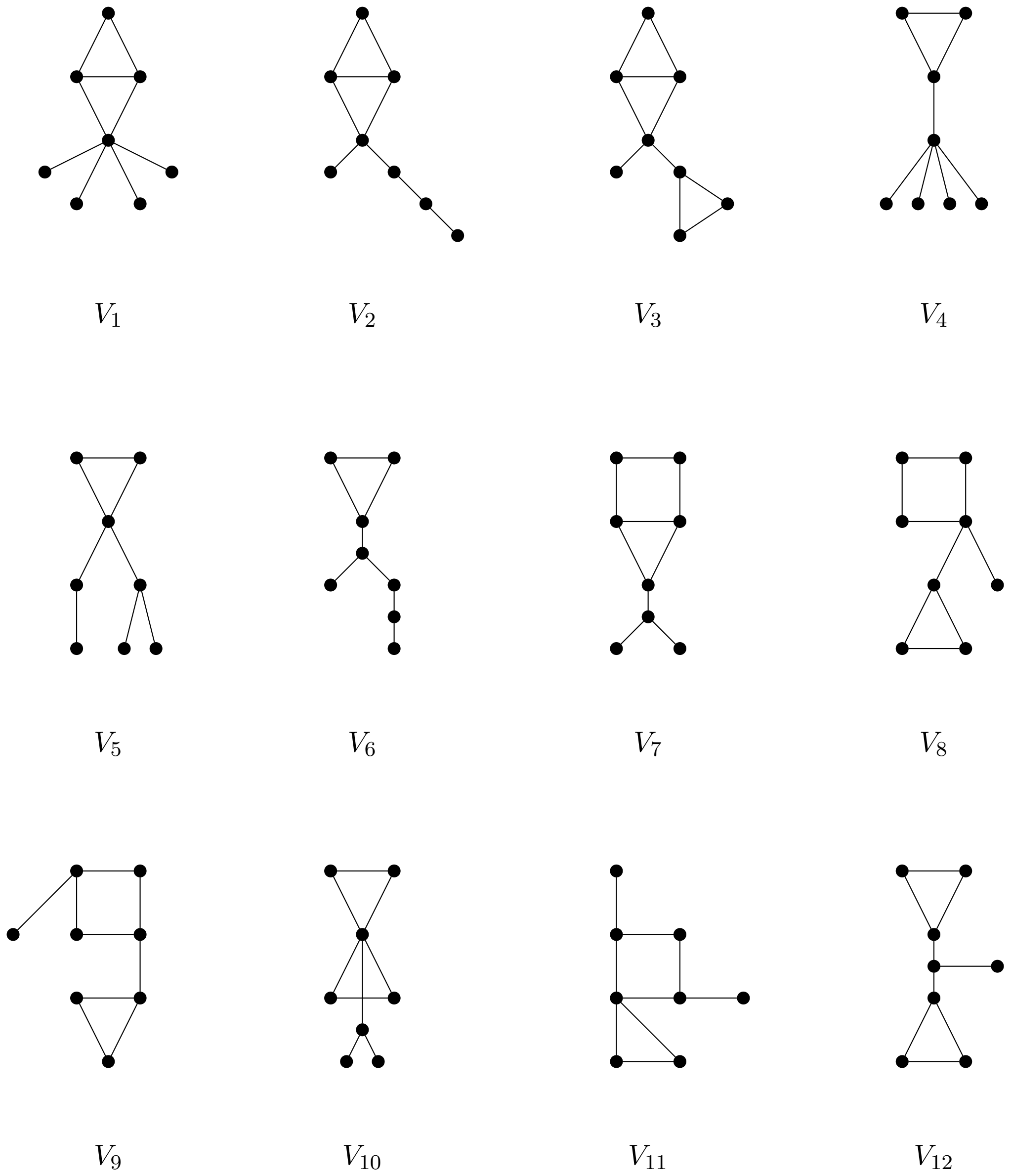

Figure 4: All well-edge dominated graphs of order 8 where only V_1, V_2, and V_3 contain a diamond