

The Livšic equation on differential forms over Anosov flows and applications

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Abstract

The goal of this paper is to explore the relationship between the geometric properties of an Anosov flow on a closed manifold M and the analytic properties of its infinitesimal generator X as a linear operator on the space of smooth differential forms of all degrees. In particular, we study the solvability of the Livšic equation $L_X \xi = \eta$ on the space of differential forms and show, for instance, that if the Anosov flow is *asymmetric*, then the equation has a unique solution in the continuous category in degrees $2 \leq k \leq n - 2$, where $n = \dim M$. Intuitively, an Anosov flow is asymmetric if in negative time it shrinks the volume of any $(n - 2)$ -dimensional parallelepiped exponentially fast when at least one side of it is in the strong unstable direction. As an application, we show that for volume-preserving asymmetric Anosov flows, the following result holds: the L^2 -closure of the image of L_X restricted to differential forms of degree $(n - 1)$ contains the space of L^2 -exact $(n - 1)$ -forms if and only if the sum of the strong bundles of the flow is uniquely integrable, in which case the flow is therefore topologically conjugate to a suspension of an Anosov diffeomorphism.

1 Introduction

Let X be a smooth¹ vector field on a smooth closed (i.e., compact and without boundary) connected manifold M . It is natural to ask: how are various properties of the flow Φ generated by X related to the properties of the differential operator X (or L_X , the Lie derivative) acting on some space of functions, distributions, or differential forms? Much work has been done on this question. Some notable results are those of Livšic [Liv71, Liv72] in the 1970's on the equation $X\varphi = f$, nowadays known as the Livšic equation. See Section 3.

There have been numerous recent results (cf., e.g., [FS11, GLP13, DZ16]; see also [Lef25] and the sources listed therein) relating the properties of the spectrum of the differential operator X (or a related transfer operator) acting on suitable spaces to the *statistical* properties of Φ . The goal of this paper is to explore what properties of X as a differential operator can tell us about the *geometric* properties of Φ . More precisely, we look at the properties of the Lie derivative L_X acting on the space of differential forms, and ask the following natural question:

¹We use the terms smooth and C^∞ interchangeably.

Given a continuous differential form η , does the Livšic equation $L_X \xi = \eta$ have a continuous solution ξ ?

We show that the answer is affirmative in intermediate degrees (i.e., $2 \leq k \leq n - 2$, where $n = \dim M$) for Anosov flows we call asymmetric. In other degrees, we characterize the image of the operator L_X .

Definition 1.1. *We call an Anosov flow on M **asymmetric** if for every $x \in M$ and every $(n - 2)$ -dimensional parallelepiped Π in the tangent space $T_x M$ with least one side of Π in the strong unstable space E_x^{uu} , then in negative time, the action of derivative of the flow on Π shrinks its volume exponentially fast.*

That is, if $\Phi = \{f_t\}$, then

$$\text{vol}(T_x f_{-t}(\Pi)) \leq C e^{-\lambda t} \text{vol}(\Pi),$$

for some $C, \lambda > 0$ and all $t \geq 0$. In other words, in negative time, the rate of contraction along the strong unstable direction dominates the joint rate of expansion in the remaining directions (including those in the strong stable bundle). It is clear that this notion makes sense only if $n \geq 4$.

Observe that if Φ is asymmetric, then the exponential shrinking of the volume also holds for all *lower dimensional* parallelepipeds with at least one side in the strong unstable bundle.

The structure of the paper is the following. In Section 2 we review some basic facts about Anosov flows and the space of differential forms as an inner product space. In Section 3 we prove the Livšic theorem for differential forms in intermediate degrees and related properties of the Lie derivative in other degrees. Our main result is an application of these properties; the proof is given in Section 4.

Main Theorem. *Let Φ be an asymmetric Anosov flow with infinitesimal generator X on a closed Riemannian manifold M of dimension $n \geq 4$. Then: the L^2 -closure of the image of L_X on $(n - 1)$ -forms² contains the space of L^2 -exact forms, i.e.,*

$$L^2 B^{n-1}(M) \subset \overline{\text{image}(L_X)},$$

if and only if the sum of the strong bundles of Φ is uniquely integrable and the flow is therefore topologically conjugate to a suspension of an Anosov diffeomorphism.

The closure is taken relative to the L^2 -norm (see Section 2).

2 Preliminaries

Anosov flows. Fix a non-singular smooth flow $\Phi = \{f_t\}$ on a closed Riemannian manifold M . Recall that Φ is called **Anosov** if there exists a Tf_t -invariant splitting of the tangent bundle into the strong unstable, center, and strong stable bundle,

$$TM = E^{uu} \oplus E^c \oplus E^{ss},$$

such that for all $t \geq 0$, $v \in E^{ss}$ and $w \in E^{uu}$, we have:

$$\|Tf_t(v)\| \leq c e^{-\nu t} \|v\| \quad \text{and} \quad \|Tf_t(w)\| \geq c e^{\lambda t} \|w\|, \quad (\spadesuit)$$

²We consider the domain of L_X to be the space of continuous differential forms with a continuous L_X -derivative.

where c, ν , and λ are fixed positive constants, and E^c is spanned by the infinitesimal generator X of the flow. The Anosov property is independent of the Riemannian metric, since on a compact manifold the Finsler structures defined by any two continuous Riemannian metrics are equivalent.

An Anosov flow is of **codimension one** if $\dim E^{uu} = 1$ or $\dim E^{ss} = 1$. We will always assume the former. It is **volume-preserving** if there exists a C^∞ volume form Ω such that $f_t^* \Omega = \Omega$, for all $t \in \mathbb{R}$.

It is well-known that the invariant bundles E^{ss} , E^{uu} , $E^{cs} = E^c \oplus E^{ss}$, and $E^{cu} = E^c \oplus E^{uu}$ are uniquely integrable, giving rise to Hölder continuous invariant foliations [HPS77, PSW97] denoted by $W^{cs}, W^{cu}, W^{ss}, W^{uu}$, respectively.

A smooth compact codimension one submanifold Σ of M is called a **global cross section** for a flow if it intersects every orbit transversely. If a flow admits a global cross section Σ , then every point $p \in \Sigma$ returns to Σ , defining the Poincaré or first-return map $g : \Sigma \rightarrow \Sigma$ of the flow. The flow can be reconstructed by **suspending** g under the roof function equal to the first-return time (cf., e.g., [KH95]).

Notation, standing assumptions, and facts. Below we recall some basic facts, and fix the notation and terminology used in this paper.

1. $\Phi = \{f_t\}$ denotes a C^∞ Anosov flow on a closed connected C^∞ Riemannian manifold M of dimension n .
2. X denotes the associated infinitesimal generator of Φ ; L_X is the corresponding Lie derivative on tensor fields. The restriction of L_X to differential forms of degree k will be denoted by $L_X^{(k)}$. When there is little chance of confusion, the superscript k will be dropped.
3. C_X^1 will denote any space of continuous objects whose L_X -derivative is continuous. Thus $C_X^1(M)$ is the space of continuous function with a continuous X -derivative. $C_X^1 \Lambda^k(M)$ will denote the space of continuous k -forms ω such that $L_X \omega$ is continuous.
4. Ω is a C^∞ volume form invariant under the flow. Without loss we assume that $\int_M \Omega = 1$.
5. A standing assumption is that all invariant bundles are orientable; otherwise we can pass to a double cover of M .
6. $E^{su} = E^{ss} \oplus E^{uu}$; E^{su} is a Hölder continuous bundle (cf., [Has94, Has97, HPS77]).
7. If the flow is of codimension one and $n \geq 4$, E^{uu} and E^{cs} are both known to be C^1 (in fact, $C^{1+\theta}$, for some $0 < \theta < 1$); cf., [Has94, Has97, HPS77].
8. We denote by α the canonical invariant 1-form defined by:

$$\ker(\alpha) = E^{su}, \quad \alpha(X) = 1.$$

The regularity of α is the same as that of E^{su} , i.e., Hölder continuous.

9. We will call a continuous Riemannian metric g on M an **Anosov metric** associated with a fixed Anosov flow Φ if relative to g , X is orthogonal to E^{su} and $g(X, X) = 1$.
10. For a Riemannian metric g , its Riemannian volume form is denoted by $\text{vol}(g)$.

11. For an arbitrary continuous Riemannian metric g with $\text{vol}(g) = \Omega$ and vectors $v_1, \dots, v_k$ tangent to M at the same point, we write $\|v_1 \wedge \dots \wedge v_k\|_g$ for the k -dimensional volume with respect to g of the parallelepiped with sides $v_1, \dots, v_k$. Thus $\|v_1 \wedge \dots \wedge v_n\|_g = |\Omega(v_1, \dots, v_n)|$.
12. Recall that a continuous 1-form ω on M is said to have an exterior differential in the Stokes sense if there exists a continuous 2-form ξ such that

$$\int_{\partial D} \omega = \int_D \xi,$$

for every C^1 -immersed 2-disk D such that ∂D is piecewise C^1 . In that case we write $\xi = d\omega$, specifying that this holds in the Stokes sense. The Hartman-Frobenius theorem (i.e., P. Hartman's generalization of the classical theorem of Frobenius on integrability of plane fields; see [Pla72] and [Har02]) states that a continuous 1-form ω is integrable if and only if ω has a continuous exterior differential $d\omega$ in the Stokes sense and $\omega \wedge d\omega = 0$. Recall that a continuous 1-form ω on M is said to be integrable if the kernel of ω as a subbundle of TM is integrable.

The case of codimension one Anosov flows. We will show that codimension one Anosov flows in dimensions $n \geq 4$ are asymmetric.

Proposition 2.1. *Let $\Phi = \{f_t\}$ be a volume preserving codimension one Anosov flow on a closed manifold M of dimension $n \geq 4$. Assume, without loss, that E^{uu} is 1-dimensional and orientable, and let Y be a non-vanishing section of E^{uu} . Assume as before that with respect to a fixed Riemannian metric g and the associated Finsler structure on M :*

$$\|Tf_t(v)\| \leq ce^{-\nu t} \|v\|,$$

for all $t \geq 0$ and $v \in E^{ss}$, where $c, \nu > 0$ are as in $(\spadesuit)$. Then for every $p \in M$, all linearly independent unit vectors $v_1, \dots, v_{n-3} \in T_p M$, and $t > 0$, we have

$$\|T_p f_{-t}(v_1 \wedge \dots \wedge v_{n-3} \wedge Y_p)\|_g \leq Ce^{-\nu t},$$

where $C > 0$ is independent of p , v_i 's, and t .

Proof. We first assume that g is an Anosov metric for Φ . Let us deal with the worst-case scenario, i.e., when $v_1, \dots, v_{n-3}$ are all in E_p^{ss} .

Fix $t > 0$. Let $w_t \in E_p^{ss}$ be a unit vector such that $T_p f_{-t}(w_t)$ is orthogonal to the subspace spanned by $Y_{f_{-t}(p)}$ and $T_p f_{-t}(v_i)$, for $i = 1, \dots, n-3$, and $(v_1, \dots, v_{n-3}, Y_p, w_t, X_p)$ is a positively oriented basis of $T_p M$. Note that $\|Tf_{-t}(w_t)\| \geq c^{-1}e^{\nu t}$. Since Φ leaves Ω invariant, we have:

$$\begin{aligned} \|T_p f_{-t}(v_1 \wedge \dots \wedge v_{n-3} \wedge Y_p \wedge w_t \wedge X_p)\|_g &= (f_{-t}^* \Omega)(v_1, \dots, v_{n-3}, Y_p, w_t, X_p) \\ &= \Omega(v_1, \dots, v_{n-3}, Y_p, w_t, X_p) \\ &= \|v_1 \wedge \dots \wedge v_{n-3} \wedge Y_p \wedge w_t \wedge X_p\|_g \\ &\leq \|Y\|_\infty, \end{aligned}$$

where $\|Y\|_\infty = \max\{\|Y_x\| : x \in M\}$; here we used $g(X_p, X_p) = 1$, for all $p \in M$. Our choice of w_t implies

$$\begin{aligned} \|T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge Y_p \wedge w_t \wedge X_p)\|_g &= \|T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge Y_p \wedge w_t) \wedge X_{f_{-t}p}\|_g \\ &= \|T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge w_t \wedge Y_p)\|_g \|X_{f_{-t}(p)}\| \\ &= \|T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge Y_p)\|_g \|T_p f_{-t}(w_t)\| \\ &\geq c^{-1} e^{\nu t} \|T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge Y_p)\|_g, \end{aligned}$$

since $T_p f_{-t}(w_t)$ is orthogonal to the subspace containing the parallelepiped $T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge Y_p)$. Combining the last two inequalities, we obtain for all $t \geq 0$:

$$\|T_p f_{-t}(v_1 \wedge \cdots \wedge v_{n-3} \wedge Y_p)\|_g \leq c \|Y\|_\infty e^{-\nu t}.$$

If g is not Anosov, then the Finsler norms defined by g and any fixed Anosov metric g_0 are equivalent, yielding an analogous inequality. $\square$

Therefore, we have:

Corollary 2.2. *Volume-preserving codimension one Anosov flows in dimensions $n \geq 4$ are asymmetric.*

The L^2 -structure on the space of differential forms. For $0 \leq r \leq \infty$ and $0 \leq k \leq n$, $C^r \Lambda^k(M)$ will denote the space of C^r exterior differential forms of degree k on M . The space of exact C^r k -forms will be denoted by $C^r B^k(M)$ and the space of closed C^r k -forms by $C^r Z^k(M)$.

On any oriented inner product space V with inner product $\langle \cdot, \cdot \rangle$ and corresponding volume form ω , one can uniquely define an inner product on exterior forms so that, in particular, $\langle \xi, \eta \rangle = \langle u, v \rangle$, for all exterior 1-forms ξ, η , where u, v are the vectors dual to ξ, η , respectively, relative to $\langle \cdot, \cdot \rangle$. See [Lee13, War83].

Recall also that for each $0 \leq k \leq n$ (where $n = \dim V$) there is a unique isomorphism $\star : \Lambda^k(V^*) \rightarrow \Lambda^{n-k}(V^*)$, called the **Hodge-star operator**, between the spaces of exterior k - and $(n-k)$ -forms on V such that

$$\xi \wedge \star \eta = \langle \xi, \eta \rangle \omega, \tag{1}$$

for any $\xi, \eta \in \Lambda^k(V^*)$.

The following lemma will be needed later in the paper. The proof is elementary (it is an exercise in [Lee13]) and therefore omitted.

Lemma 2.3. *If $V, \langle \cdot, \cdot \rangle$, and ω are as above, then for every $v \in V$, we have*

$$\star(i_v \omega) = (-1)^{n-1} \theta_v,$$

where $\theta_v = \langle v, \cdot \rangle$ is the exterior 1-form dual to v relative to $\langle \cdot, \cdot \rangle$.

If g is a C^r ($0 \leq r \leq \infty$) Riemannian metric on M with $\text{vol}(g) = \Omega$, we will denote by $\star_g$ the associated Hodge-star operator [Lee13, War83], defined pointwise as in (1). The inner product on $C^0 \Lambda^k(M)$ (for each $0 \leq k \leq n$) induced by g is defined by

$$\langle \xi, \eta \rangle_g = \int_M \xi \wedge \star_g \eta.$$

The metric g defines a Finsler structure on M , which we denote by $|\cdot|_g = \langle \cdot, \cdot \rangle_g^{1/2}$. For any form $\omega \in C^0\Lambda^k(M)$, with $1 \leq k \leq n$ and $p \in M$, we will denote by $|\omega_p|_g$ the operator norm of $\omega_p : (T_x M)^k \rightarrow \mathbb{R}$ as a k -linear map relative to this Finsler structure:

$$|\omega_p|_g = \max\{|\omega_p(u_1, \dots, u_k)|_g : u_j \in T_x M, |u_j|_g = 1\}.$$

The C^0 -norm of ω is defined by

$$\|\omega\|_\infty = \sup\{|\omega_p|_g : p \in M\}.$$

This is to be distinguished from the L^2 -norm $\|\omega\|_g = \langle \omega, \omega \rangle_g^{1/2}$. The completion of $C^0\Lambda^k(M)$ relative to this norm is the space $L^2\Lambda^k(M)$.

In an analogous way we can define an inner product on the space of continuous vector fields on M by setting

$$\langle Y, Z \rangle_g = \int_M g(Y, Z) \Omega.$$

The corresponding L^2 -norm is denoted by $\|Z\|_g = \langle Z, Z \rangle_g^{1/2}$.

As a direct consequence of Lemma 2.3, we have:

Corollary 2.4. *If Ω is a volume form on M , Z a non-vanishing vector field, and g a Riemannian metric with $\text{vol}(g) = \Omega$, then*

$$\star_g(i_Z \Omega) = (-1)^{n-1} \theta_Z,$$

where $\theta_Z = g(Z, \cdot)$. If X, Ω , and α are defined as before, and g is an Anosov metric for the flow, then

$$\star_g(i_X \Omega) = (-1)^{n-1} \alpha.$$

Consider now the unbounded linear operator

$$L_X^{(k)} : L^2\Lambda^k(M) \rightarrow L^2\Lambda^k(M),$$

with dense domain $C_X^1\Lambda^k(M)$. The underlying Riemannian metric g (used to define the L^2 -inner product on the space of differential forms) is assumed to be at least of class C_X^1 (i.e., continuous with a continuous L_X -derivative); note that this includes Anosov metrics. We have:

Proposition 2.5. (a) *The adjoint of $L_X^{(k)}$ is*

$$\left(L_X^{(k)}\right)^* = (-1)^{k(n-k)+1} \star_g L_X^{(n-k)} \star_g.$$

(b) *$L_X^{(k)}$ is a closed operator and $\left(L_X^{(k)}\right)^{**} = L_X^{(k)}$, for all $0 \leq k \leq n$.*

(c) *We have*

$$[\text{image}(L_X^{(k)})]^\perp_g = \ker(\star_g L_X^{(n-k)} \star_g) \quad \text{and} \quad [\text{image}(\star_g L_X^{(n-k)} \star_g)]^\perp_g = \ker(L_X^{(k)}).$$

Here $S^\perp_g$ denotes the orthogonal complement of a set S relative to the L^2 -inner product defined by the Riemannian metric g .

Proof. (a) Let $\xi, \eta \in C_X^1 \Lambda^k(M)$. Then:

$$\begin{aligned}
\langle L_X^{(k)} \xi, \eta \rangle_g &= \int_M L_X^{(k)} \xi \wedge \star_g \eta \\
&= - \int_M \xi \wedge L_X^{(n-k)} \star_g \eta \\
&= (-1)^{k(n-k)+1} \int_M \xi \wedge \star_g [\star_g L_X^{(n-k)} \star_g \eta] \\
&= (-1)^{k(n-k)+1} \langle \xi, \star_g L_X^{(n-k)} \star_g \eta \rangle_g,
\end{aligned}$$

which proves (a). We used the fact that on k -forms, $\star_g \star_g = (-1)^{k(n-k)} \text{id}$.

(b) Recall (see, e.g. [Con07]) that a densely defined unbounded operator is closable if its adjoint is densely defined. Since $\star_g$ maps C_X^1 -forms to C_X^1 -forms, the domain of $\star_g L_X^{(n-k)} \star_g$ is $C_X^1 \Lambda^k(M)$, which is dense in $L^2 \Lambda^k(M)$, so $L_X^{(k)}$ is closable. To compute its second adjoint, we have:

$$\begin{aligned}
\langle (L_X^{(k)})^* \xi, \eta \rangle_g &= \langle \eta, (L_X^{(k)})^* \xi \rangle_g \\
&= \int_M \eta \wedge \star_g (L_X^{(k)})^* \xi \\
&= - \int_M \eta \wedge L_X^{(n-k)} \star_g \xi \\
&= \int_M L_X^{(k)} \eta \wedge \star_g \xi \\
&= \langle L_X^{(k)} \eta, \xi \rangle_g \\
&= \langle \xi, L_X^{(k)} \eta \rangle_g.
\end{aligned}$$

Thus $(L_X^{(k)})^{**} = L_X^{(k)}$, so $L_X^{(k)}$ is in fact closed, being the adjoint of another operator (see [Con07]).

(c) A direct consequence of the general theory of unbounded linear operators (see [Con07], Proposition X.1.13) and (a). $\square$

The Gol'dshtein-Troyanov complex To make the paper as self-contained as possible, we briefly review a result from [GT06] we will need later. In [GT06], Gol'dshtein and Troyanov define the following spaces:

$$\Omega_{p,q}^k(M) = \{\omega \in L^q \Lambda^k(M) : d\omega \in L^p \Lambda^{k+1}(M)\},$$

where (M, g) is a Riemannian manifold (which we assume to be compact), $1 \leq p, q \leq \infty$, and d denotes the *weak* exterior differential. For each p and q , this is a Banach space with the graph norm

$$\|\omega\|_{\Omega_{p,q}^k} = \|\omega\|_{L^q} + \|d\omega\|_{L^p}.$$

The spaces $\Omega_{p,q}^k(M)$ are used to define the so called $L_{p,q}$ -cohomology of M , which we do not need here. We will however use some parts of the following result (Theorem 12.5 in [GT06]):

Theorem 2.6 (The regularization and homotopy operators). *There exists a family of regularization operators R_ε and homotopy operators A_ε (with $\varepsilon > 0$) satisfying the following properties:*

- (a) For every $\omega \in L^1 \Lambda^k(M)$, the form $R_\varepsilon \omega$ is smooth.
- (b) For any $\omega \in \Omega_{q,p}^k(M)$, we have $dR_\varepsilon \omega = R_\varepsilon d\omega$.
- (c) For any $1 \leq p, q < \infty$ and $\varepsilon > 0$, $R_\varepsilon : \Omega_{q,p}^k(M) \rightarrow \Omega_{q,p}^k(M)$ is a bounded linear operator such that $\|R_\varepsilon\|_{q,p} \rightarrow 1$, as $\varepsilon \rightarrow 0$.
- (d) For any $1 \leq p, q < \infty$ and $\omega \in \Omega_{q,p}^k(M)$, we have $\|R_\varepsilon \omega - \omega\|_p \rightarrow 0$, as $\varepsilon \rightarrow 0$. Thus smooth forms are dense in $\Omega_{q,p}^k(M)$ (if p, q are finite).
- (e) The homotopy operator $A_\varepsilon : \Omega_{p,r}^k(M) \rightarrow \Omega_{q,p}^{k-1}(M)$ (where $1 \leq k \leq n$) is bounded in the following cases:
- (i) If $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$ and $\frac{1}{r} - \frac{1}{p} < \frac{1}{n}$;
 - (ii) If $1 < p, q, r \leq \infty$ satisfy $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$ and $\frac{1}{r} - \frac{1}{p} \leq \frac{1}{n}$.
- (f) The following homotopy formula holds:

$$\omega - R_\varepsilon \omega = dA_\varepsilon \omega + A_\varepsilon d\omega.$$

Recall that a continuous 1-form ω on M is said to be closed in the Stokes sense if

$$\int_{\partial D} \omega = 0,$$

for every C^1 -immersed 2-disk D with piecewise C^1 boundary. It is closed in the weak sense if its weak differential is zero, i.e.,

$$\int_M \omega \wedge d\eta = 0,$$

for every smooth $(n-2)$ -form η .

Lemma 2.7. *A continuous 1-form ω on M is closed in the weak sense if and only if it is closed in the Stokes sense.*

Proof. ($\Rightarrow$) Assume $d\omega = 0$ in the weak sense. Fix $\varepsilon > 0$. Since ω is continuous and weakly closed, it follows that $\omega \in \Omega_{\infty,\infty}^1(M)$. By Theorem 2.6 (f) we have:

$$\omega - R_\varepsilon \omega = dA_\varepsilon \omega. \tag{2}$$

Furthermore, by Theorem 2.6 (e), it follows that $u_\varepsilon := A_\varepsilon \omega \in \Omega_{\infty,\infty}^0(M)$, i.e., u_ε is Lipschitz. Thus du_ε exists a.e. in the Fréchet sense (and a.e. equals the weak differential of u_ε). Moreover, by (2) $du_\varepsilon = \omega - R_\varepsilon \omega$, so du_ε coincides a.e. with a continuous 1-form. Thus u_ε can be chosen to be C^1 . If D is a C^1 -immersed 2-disk with piecewise C^1 boundary, then:

$$\int_{\partial D} \omega = \int_{\partial D} (R_\varepsilon \omega + du_\varepsilon) = 0,$$

since $dR_\varepsilon \omega = 0$. Therefore, ω is closed in the Stokes sense.

($\Leftarrow$) Assume now $d\omega = 0$ in the Stokes sense. If U is a sufficiently small simply connected set in M , then on U we have $\omega = dg$, for some C^1 function $g : U \rightarrow \mathbb{R}$. Let $d\eta$ be an arbitrary smooth exact

$(n-1)$ -form. Let $\{(U_i, \psi_i)\}$ be a smooth partition of unity on M , where U_i is a sufficiently small disk such that there exists a C^1 -function g_i with $\omega = dg_i$ on U_i . Then $\eta = \sum_i \eta_i$, where $\eta_i = \psi_i \eta$ is supported in U_i . It follows that

$$\begin{aligned} \int_M \omega \wedge d\eta &= \sum_i \int_{U_i} \omega \wedge d\eta_i \\ &= \sum_i \int_{U_i} dg_i \wedge d\eta_i \\ &= - \sum_i \int_{U_i} d(dg_i \wedge \eta_i) \\ &= - \sum_i \int_{\partial U_i} dg_i \wedge \eta_i \\ &= 0, \end{aligned}$$

since $\eta_i = 0$ on ∂U_i . Therefore, ω is closed in the weak sense. $\square$

3 A Livšic theorem on the space of differential forms

In its basic form, the classical Livšic equation over an Anosov flow is an equation of the form $X\varphi = f$, where f and φ are real-valued functions on M . (An analogous cohomological equation has also been studied over Anosov diffeomorphisms, partially hyperbolic diffeomorphisms, and other types of dynamical systems.) In the category of Hölder continuous functions, the original proof of the existence of solutions was established in the seminal work of Livšic [Liv71, Liv72]. In the smooth case the result was proved by de la Llave, Marco, and Moriyón [dLMM86], and the Sobolev regularity case was treated in [dL01]. A proof of the classical (as well as the smooth one, assuming volume-preservation) result using microlocal analysis was done in [Gui17]. A Livšic theorem for sections of vector bundles also using microlocal analysis was recently established in [CL25]. See also Lefeuvre's book [Lef25] for a more comprehensive (and readable) survey of results and references.

The main goal of this paper is to investigate the obstacles to the solvability of the Livšic equation on the space of differential forms of different degrees using somewhat elementary means (i.e., without the use of microlocal analysis).³

Invariant forms. We will first describe the set of invariant differential forms in all degrees. We set

$$\text{Inv}^k(M, X) = \{\omega \in C_X^1 \Lambda^k(M) : L_X \omega = 0\}.$$

Some of the results in the following Proposition are well-known and elementary, but we include them for completeness.

Proposition 3.1. *Let Φ be a smooth Anosov flow with infinitesimal generator X , preserving a smooth volume form Ω . Then:*

(a) $\text{Inv}^0(M, X)$ consists of constant functions.

³However, we do hope that in the near future using the heavy machinery of microlocal analysis may lead to results stronger than the ones in this paper.

$$(b) \operatorname{Inv}^1(M, X) = \mathbb{R}\alpha.$$

$$(c) \text{ If the flow is asymmetric and } 2 \leq k \leq n-2, \text{ then } \operatorname{Inv}^k(M, X) = \{\mathbf{0}\}.$$

$$(d) \operatorname{Inv}^{n-1}(M, X) = \mathbb{R} i_X \Omega.$$

$$(e) \operatorname{Inv}^n(M, X) = \mathbb{R}\Omega.$$

Proof. (a) and (e) are clear. To prove (b), assume $L_X \omega = 0$, for some $\omega \in C_X^1 \Lambda^1(M)$. Then $f_t^* \omega = \omega$, for all t , which clearly implies that $\omega(v) = 0$, for all $v \in E^{ss} \oplus E^{uu}$. Thus $\omega = \psi \alpha$, for some continuous $\psi : M \rightarrow \mathbb{R}$. Since

$$0 = L_X \omega = (X\psi)\alpha = \psi L_X \alpha = (X\psi)\alpha$$

it follows that ψ is flow invariant. Since the flow is ergodic (being volume preserving), ψ is constant a.e., hence constant by continuity.

(c) Assume the flow is asymmetric, $2 \leq k \leq n-2$, and $L_X \eta = 0$, for a continuous k -form η . We again have $f_t^* \eta = \eta$, for all t . Let $v_1, \dots, v_k$ be arbitrary linearly independent vectors in the same tangent space of M . We claim that $\eta(v_1, \dots, v_k) = 0$. Since $TM = E^{cs} \oplus E^{uu}$ and η is multilinear, it is sufficient to show $\eta(v_1, \dots, v_k) = 0$ in the following two cases:

Case 1: $\{v_1, \dots, v_k\} \subset E^{cs}$.

Case 2: $\{v_1, \dots, v_k\} \subset E^{uu} \cup E^{cs}$ and at least one vector v_j is in E^{uu} .

In Case 1, by decomposing each v_j into the sum $v_j = v_j^c + v_j^s \in E^c \oplus E^{ss}$, using the flow invariance of η , and the fact that $k \geq 2$, we obtain

$$\eta(v_1, \dots, v_k) = \eta(f_{t*}(v_1), \dots, f_{t*}(v_k)) \rightarrow 0,$$

as $t \rightarrow +\infty$.

In Case 2, the asymmetry of the flow implies

$$\eta(v_1, \dots, v_k) = \eta(f_{t*}(v_1), \dots, f_{t*}(v_k)) \rightarrow 0,$$

as $t \rightarrow -\infty$. Thus $\eta = 0$, as desired.

To prove (d), assume $L_X \Theta = 0$, for some $\Theta \in C_X^1 \Lambda^{(n-1)}(M)$. Observe that since $L_X(i_X \Theta) = i_X L_X \Theta = 0$ ⁴, $i_X \Theta$ is a continuous invariant $(n-2)$ -form, hence zero by (c).

Consider the continuous n -form $\alpha \wedge \Theta$. Since $L_X(\alpha \wedge \Theta) = L_X \alpha \wedge \Theta + \alpha \wedge L_X \Theta = 0$, $\alpha \wedge \Theta$ is invariant, hence $\alpha \wedge \Theta = c \Omega$, for some constant c . It follows that

$$\Theta = i_X(\alpha \wedge \Theta) = i_X(c \Omega) = c i_X \Omega,$$

as desired. □

Theorem 3.2 (Livšic theorem for forms of intermediate degree). *Let Φ be an asymmetric Anosov flow on a closed manifold M , and let ξ be a continuous k -form on M , with $2 \leq k \leq n-2$. Then:*

⁴Note that i_X and L_X do commute on $C_X^1 \Lambda^*(M)$.

(a) There exists a unique continuous k -form η such that $L_X\eta = \xi$.

(b) If ξ, E^{uu} , and E^{cs} are C^1 (as in the case of volume-preserving codimension one Anosov flows in dimensions $n \geq 4$), then there exists a family $(\eta_t)_{t \geq 0}$ in $C^1\Lambda^k(M)$ such that

$$\eta_t \rightarrow \eta \quad \text{and} \quad L_X\eta_t \rightarrow L_X\eta = \xi,$$

as $t \rightarrow \infty$, both with respect to the C^0 -norm. Each $L_X\eta_t$ is also C^1 .

Proof. (a) (Uniqueness) Follows directly from Proposition 3.1 (c).

(Existence) To prove the existence of η , given a continuous k -form ξ , we need to define $\eta(v_1, \dots, v_k)$ for all vectors $v_1, \dots, v_k \in TM$. By the same argument as in the proof of part (c) of Proposition 3.1, it is enough to specify $\eta(v_1, \dots, v_k)$ in Cases 1 and 2 defined above, then extend η by multi-linearity and the alternating property.

For $t > 0$ define

$$\eta_t(v_1, \dots, v_k) = \begin{cases} -\int_0^t (f_s^*\xi)(v_1, \dots, v_k) ds & \text{in Case 1,} \\ \int_0^t (f_{-s}^*\xi)(v_1, \dots, v_k) ds & \text{in Case 2.} \end{cases}$$

The asymmetry of the flow guarantees that η_t converges, as $t \rightarrow \infty$, in the C^0 -sense to a continuous form η . It is clear that if ξ, E^{uu} , and E^{cs} are C^1 , then so is η_t , for every $t \geq 0$.

Let us show that $L_X\eta = \xi$, i.e., $L_X\eta(v_1, \dots, v_k) = \xi(v_1, \dots, v_k)$, for all $v_1, \dots, v_k \in TM$. As above, it suffices to prove this in each of the two cases above. In Case 1, we have:

$$\begin{aligned} (f_\tau^*\eta)(v_1, \dots, v_k) &= \eta(f_{\tau*}(v_1), \dots, f_{\tau*}(v_k)) \\ &= -\int_0^\infty f_s^*\xi(f_{\tau*}(v_1), \dots, f_{\tau*}(v_k)) ds \\ &= -\int_0^\infty f_{s+\tau}^*\xi(v_1, \dots, v_k) ds \\ &= -\int_0^\infty f_t^*\xi(v_1, \dots, v_k) ds + \int_0^\tau f_t^*\xi(v_1, \dots, v_k) ds \\ &= \eta + \int_0^\tau f_t^*\xi(v_1, \dots, v_k) ds, \end{aligned}$$

for all $\tau \geq 0$. Differentiating both sides with respect to τ at zero, we obtain $L_X\eta(v_1, \dots, v_k) = \xi(v_1, \dots, v_k)$. Case 2 is dealt with in a similar way. This proves that $L_X\eta = \xi$.

A similar calculation yields

$$L_X\eta_t = \begin{cases} \xi - f_t^*\xi & \text{in Case 1,} \\ \xi - f_{-t}^*\xi & \text{in Case 2.} \end{cases}$$

It follows that $L_X\eta_t \rightarrow \xi$, as $t \rightarrow \infty$, in the C^0 -sense; in Case 2 this follows again by asymmetry. Finally, observe that if ξ, E^{uu} , and E^{cs} are C^1 , then so is $L_X\eta_t$. $\square$

The action of L_X in all degrees. We now investigate the action of the Lie derivative L_X on differential forms of all degrees. The well-known results are included for completeness.

Theorem 3.3. *Let Φ be a smooth Anosov flow on a closed manifold M .*

- (a) *If Φ is transitive, then the image of $L_X^{(0)} : C_X^1(M) \rightarrow C^0(M)$ consists of continuous functions whose integral over all periodic orbits equals zero.*
- (b) *If Φ is transitive, then the image of $L_X^{(1)} : C_X^1\Lambda^1(M) \rightarrow C^0\Lambda^1(M)$ consists of continuous 1-forms ω such that $\int_\gamma \omega = 0$, for all periodic orbits γ of Φ .*
- (c) *If Φ is asymmetric and $2 \leq k \leq n-2$, then $L_X^{(k)} : C_X^1\Lambda^k(M) \rightarrow C^0\Lambda^k(M)$ is a bijection.*
- (d) *If Φ preserves a smooth volume form Ω , then for every C_X^1 -Riemannian metric g on M , we have a g -orthogonal decomposition:*

$$L^2\Lambda^{n-1}(M) = \overline{\text{image}(L_X^{(n-1)})} \oplus_g \mathbb{R}(\star_g \alpha),$$

where the closure is taken relative to the L^2 -topology. If g is an Anosov metric, then

$$L^2\Lambda^{n-1}(M) = \overline{\text{image}(L_X^{(n-1)})} \oplus_g \mathbb{R}(i_X \Omega).$$

- (e) *If Φ preserves a smooth volume form Ω , then the image of $L_X^{(n)} : C_X^1\Lambda^n(M) \rightarrow C^0\Lambda^n(M)$ consists of all n -forms of type $(X\psi)\Omega$, where $\psi \in C_X^1(M)$.*

Proof. Part (a) is just the classical Livšic theorem. Part (c) is a restatement of Theorem 3.2. Part (e) is easy to prove. To prove (b), let us first show that if $\omega = L_X\xi$, for some $\xi \in C_X^1\Lambda^1(M)$, then $\int_\gamma \omega = 0$, for every closed orbit γ . Indeed, for every periodic orbit γ , we have:

$$\begin{aligned} \int_\gamma \omega &= \int_\gamma L_X\xi \\ &= \int_\gamma \frac{d}{dt} \Big|_0 f_t^* \xi \\ &= \frac{d}{dt} \Big|_0 \int_\gamma f_t^* \xi \\ &= \frac{d}{dt} \Big|_0 \int_\gamma \omega \\ &= 0. \end{aligned}$$

Now assume that $\int_\gamma \omega = 0$, for every periodic orbit γ . Let $\varphi = \omega(X)$. Since the integral of φ over every periodic orbit γ is zero, the classical Livšic theorem yields a function $\psi \in C_X^1(M)$ such that $\varphi = X\psi$. Set $\beta = \alpha \wedge \omega$. It follows from (c) that $\beta = L_X\xi$, for some continuous 2-form ξ . Contracting $\alpha \wedge \omega = L_X\xi$ by X (i.e., applying i_X to both sides), we obtain

$$\omega - \varphi\alpha = i_X L_X\xi = L_X(i_X\xi).$$

Thus

$$\omega = (X\psi)\alpha + L_X(i_X\xi) = L_X(\psi\alpha + i_X\xi),$$

as desired.

Part (d) follows from Propositions 2.5 and 3.1. Indeed,

$$L^2\Lambda^{n-1}(M) = \overline{\text{image}(L_X^{(n-1)})} \oplus_g \text{image}(L_X^{(n-1)})^{\perp_g}.$$

By Proposition 2.5, $\text{image}(L_X^{(n-1)})^{\perp_g} = \ker(*_g L_X^{(1)} *_g)$. Since $*_g$ is an isomorphism and $\ker(L_X^{(1)}) = \mathbb{R}\alpha$ (Prop. 3.1 (b)), the result follows. Recall that if g is an Anosov metric, then $*_g(i_X\Omega) = (-1)^{n-1}\alpha$. $\square$

Remark. Observe that if $L^2B^{n-1}(M) \subset \overline{\text{image}(L_X^{(n-1)})}$, then part (d) of Theorem 3.3 implies that $i_X\Omega$ is g -orthogonal to exact forms, where g is any Anosov metric for the flow. Since $i_X\Omega$ is also closed, it follows that $i_X\Omega$ is harmonic with respect to g (at least formally speaking, since g is not smooth). Thus the main theorem is consistent with the result of [Sim23], which states that $i_X\Omega$ is intrinsically harmonic if and only the flow admits a global cross section (where X is allowed to be any non-singular smooth vector field which preserves a smooth volume form Ω).

Corollary 3.4. *We have:*

$$\overline{\text{image}(L_X^{(n-1)} \lrcorner_{C^1Z^{n-1}(M)})} = \overline{\text{image}(L_X^{(n-1)} \lrcorner_{C^1B^{n-1}(M)})},$$

where the closures are taken in $L^2\Lambda^{n-1}(M)$.

Proof. Let g be a smooth Riemannian metric on M . It suffices to show

$$\text{image}(L_X^{(n-1)} \lrcorner_{C^1Z^{n-1}(M)})^{\perp_g} = \text{image}(L_X^{(n-1)} \lrcorner_{C^1B^{n-1}(M)})^{\perp_g}.$$

The $\subset$ part of the proof is clear. Let us show the $\supset$ part. Let $\Theta \in \text{image}(L_X^{(n-1)} \lrcorner_{C^1B^{n-1}(M)})^{\perp_g}$ and $\omega \in C^1Z^{n-1}(M)$ be arbitrary. We will show that $\langle \Theta, L_X\omega \rangle_g = 0$.

First observe that $L_X\omega = di_X\omega + i_Xd\omega = di_X\omega$. Next, by Theorem 3.3 (d), we have

$$\omega = \lim_{j \rightarrow \infty} L_X\xi_j + c i_X\Omega,$$

for some $\xi_j \in C_X^1\Lambda^{n-1}(M)$ and $c \in \mathbb{R}$. (This decomposition is not orthogonal with respect to g , but that will not matter.) It follows that

$$i_X\omega = i_X(\lim_{j \rightarrow \infty} L_X\xi_j) = \lim_{j \rightarrow \infty} i_XL_X\xi_j = \lim_{j \rightarrow \infty} i_Xdi_X\xi_j.$$

Thus:

$$\begin{aligned} \langle L_X\omega, \Theta \rangle_g &= \int_M di_X\omega \wedge *_g\Theta \\ &= \int_M i_X\omega \wedge d(*_g\Theta) \\ &= \lim_{j \rightarrow \infty} \int_M i_Xdi_X\xi_j \wedge d(*_g\Theta) \\ &= \lim_{j \rightarrow \infty} \int_M di_Xdi_X\xi_j \wedge *_g\Theta \\ &= \lim_{j \rightarrow \infty} \langle di_Xdi_X\xi_j, \Theta \rangle_g \\ &= \lim_{j \rightarrow \infty} \langle L_X(di_X\xi_j), \Theta \rangle_g \\ &= 0, \end{aligned}$$

since $\Theta \perp_g \text{image}(L_X^{(n-1)} \upharpoonright_{C^1 B^{n-1}(M)})$. This completes the proof. $\square$

Remark. The Corollary remains true if C^1 is replaced by C^∞ on both sides.

Consider the Lie algebra $\mathfrak{X}(M, \Omega)$ of smooth divergence-free vector fields on M (i.e., $X \in \mathfrak{X}(M, \Omega)$ if X is smooth and $L_X \Omega = 0$). Denote by $\text{Comm}(X, \Omega)$ its commutator subalgebra spanned by the Lie brackets $[Y, Z]$, where $Y, Z \in \mathfrak{X}(M, \Omega)$. It is well-known (cf., [Arn69, Lic74]) that there is a natural identification of $\mathfrak{X}(M, \Omega)$ with closed $(n-1)$ -forms on M and of $\text{Comm}(X, \Omega)$ with exact $(n-1)$ -forms via the map $Z \mapsto i_Z \Omega$.

Corollary 3.5. *For every $Z \in \mathfrak{X}(M, \Omega)$ there is a sequence (W_j) in $\text{Comm}(X, \Omega)$ such that*

$$[X, W_j] \rightarrow [X, Z],$$

as $j \rightarrow \infty$, in the L^2 -sense.

Proof. Let $Z \in \mathfrak{X}(M, \Omega)$ be arbitrary. Since

$$i_{[X, Z]} \Omega = di_X i_Z \Omega = L_X(i_Z \Omega)$$

the previous Corollary yields a sequence $(d\xi_j)$ in $C^\infty B^{n-1}(M)$ such that $L_X(d\xi_j) \rightarrow L_X(i_Z \Omega)$, as $j \rightarrow \infty$, in the L^2 -sense. Since $C^\infty B^{n-1}(M)$ corresponds to $\text{Comm}(X, \Omega)$ via the map $W \mapsto i_W \Omega$, there exists a sequence (W_j) in $\text{Comm}(X, \Omega)$ such that $d\xi_j = i_{W_j} \Omega$. Thus:

$$\begin{aligned} i_{[X, Z]} \Omega &= L_X(i_Z \Omega) \\ &= \lim_{j \rightarrow \infty} L_X(d\xi_j) \\ &= \lim_{j \rightarrow \infty} L_X(i_{W_j} \Omega) \\ &= \lim_{j \rightarrow \infty} di_X i_{W_j} \Omega \\ &= \lim_{j \rightarrow \infty} i_{[X, W_j]} \Omega. \end{aligned}$$

It follows that $[X, W_j] \rightarrow [X, Z]$, as $j \rightarrow \infty$, as desired. $\square$

4 Proof of the Main Theorem

($\Rightarrow$) Assume

$$L^2 B^{n-1}(M) \subset \overline{\text{image}(L_X)}.$$

Let $\omega \in C^\infty \Lambda^{n-2}(M)$ be arbitrary. Since $d\omega \in C^\infty B^{n-1}(M) \subset L^2 B^{n-1}(M)$, there exists a sequence (Θ_j) in $C_X^1 \Lambda^{n-1}(M)$ such that

$$L_X \Theta_j \rightarrow d\omega,$$

as $j \rightarrow \infty$, in the L^2 -sense. Let g be an arbitrary smooth Riemannian metric on M . Then:

$$\begin{aligned} \int_M d\omega \wedge \alpha &= (-1)^{n-1} \langle d\omega, \star_g \alpha \rangle_g \\ &= (-1)^{n-1} \lim_{j \rightarrow \infty} \langle L_X \Theta_j, \star_g \alpha \rangle_g \\ &= \lim_{j \rightarrow \infty} \int_M L_X \Theta_j \wedge \alpha \\ &= 0, \end{aligned}$$

by integration by parts, since $L_X\alpha = 0$. Thus α is weakly closed, hence closed in the Stokes sense, by Lemma 2.7. By the Hartman-Frobenius theorem, it follows that $E^{ss} \oplus E^{uu}$ is uniquely integrable, which, by [Pla72] implies that the flow is topologically conjugate to the suspension of an Anosov diffeomorphism.

($\Leftarrow$) Assume now that $E^{ss} \oplus E^{uu}$ is uniquely integrable. By the Hartman-Frobenius theorem, $d\alpha$ exists in the Stokes sense and is continuous. Since it is also invariant, it follows without difficulty that $d\alpha = 0$, also in the Stokes sense. By Lemma 2.7, $d\alpha = 0$ also in the weak sense.

Back to the proof of the Main Theorem, assume that $\Theta = d\xi$ is a smooth *exact* $(n-1)$ -form. Then by Theorem 3.3 we can write

$$\Theta = \hat{\Theta} + c i_X \Omega,$$

for some $\hat{\Theta} \in \overline{\text{image}(L_X)}$ and a constant c . It is enough to show $c = 0$.

Since $\hat{\Theta} \in \overline{\text{image}(L_X)}$, we have $\Theta = \lim_{j \rightarrow \infty} L_X \Theta_j$, for some sequence of smooth $(n-1)$ -forms (Θ_j) (the limit being in the L^2 -sense). Observe that

$$\int_M \alpha \wedge \Theta = \int_M \alpha \wedge d\xi = 0,$$

by integration by parts and the fact that α is weakly closed.

On the other hand,

$$\int_M \alpha \wedge \Theta = \lim_{j \rightarrow \infty} \int_M \alpha \wedge L_X \Theta_j + c \int_M \alpha \wedge i_X \Omega = c.$$

Thus $c = 0$, which implies $\Theta = \hat{\Theta} \in \overline{\text{image}(L_X)}$. Since $C^\infty B^{n-1}(M)$ is dense in $L^2 B^{n-1}(M)$ (cf., [GT06]), the desired conclusion follows. $\square$

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