BOUNDS ON BASS NUMBERS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let $R = K[x_1, \dots, x_m]$ where K is an uncountable algebraically closed field of characteristic 0. For a prime ideal P of R, let $\mu_j(P, M)$ be the j-th Bass number of an R-module M with respect to the prime P. For $1 \leq g \leq m-1$, we construct a set $\mathcal{S}_g(t)$ such that $\mathcal{S}_g(t) \subseteq \mathcal{S}_g(t+1)$ for all $t \geq 1$ and $\bigcup_{t \geq 1} \mathcal{S}_g(t) = \operatorname{Spec}_g(R) = \{P \in \operatorname{Spec}(R) \mid \operatorname{height} P = g\}$. Let $\mathcal{T}$ be a Lyubeznik functor on $\operatorname{Mod}(R)$. We prove that there exists some function $\phi_i^g : \mathbb{N}^2 \to \mathbb{N}$ which is monotonic in both the variables such that $\mu_i(P, \mathcal{T}(R)) \leq \phi_i^g(e(\mathcal{T}(R)), t)$ for all $P \in \mathcal{S}_g(t)$. In particular, the result holds for composition of local cohomology functors of the form $H^{i_1}_{I_1}(H^{i_2}_{I_2}(\dots H^{i_r}_{I_r}(-)\dots))$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring. For a locally closed subscheme of $\operatorname{Spec}(R)$, let $H^i_Y(M)$ be the i-th local cohomology module of M supported in Y. $H^i_Y(M)$ is denoted by $H^i_I(M)$ when Y is closed in $\operatorname{Spec}(R)$, defined by an ideal I of R.

For a prime ideal P of R, the j-th Bass number of an R-module with respect to the prime P is defined as $\mu_j(P, M) = \dim_{\kappa(P)} \operatorname{Ext}^j_{R_P}(\kappa(P), M_P)$. Let the injective dimension of M be denoted by $\operatorname{injdim}_R M$. The local cohomology modules have been studied by a number of authors. Although these modules have been studied extensively, their structure remains poorly understood. Most of the time, we do not know when they vanish. Furthermore, when they do not vanish, they are rarely finitely generated even if M is. For example, if $(R, \mathfrak{m})$ is a local ring of dimension $d \geq 1$, then $H^d_{\mathfrak{m}}(R)$ is never finitely generated. Therefore, an important problem is to identify some finiteness properties of local cohomology modules for better understanding of their structure. In [6], Huneke and Sharp, for regular rings containing a field of characteristic $p > 0$ proved the following results:

(1) $H^j_{\mathfrak{m}}(H^i_I(R))$ is injective, where $\mathfrak{m}$ is any maximal ideal of R.
(2) $\operatorname{injdim} H^i_I(R) \leq \dim \operatorname{Supp} H^i_I(R)$.
(3) The set of associated primes of $H^i_I(R)$ is finite.
(4) All the Bass numbers of $H^i_I(R)$ are finite.

Following these results, Lyubeznik in his seminal paper [7] proved analogous finiteness properties of local cohomology modules in characteristic 0. He used the theory of $\mathcal{D}$-modules to prove the results for considerably large class of functors, known as Lyubeznik functor (See, 2.5 for the definition). The local cohomology and the composition of local cohomologies are examples of Lybeznik functors. The natural question is whether the results of Huneke and Sharp can be extended to the Lyubeznik functor. In [8], Lyubeznik developed the theory of F-modules over regular rings of characteristic $p > 0$ and proved similar results. He also used $\mathcal{D}$-modules and some new ideas to prove these results for unramified regular local

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ring (See, [9]). An ingenious blend of $\mathcal{D}$-module and F-module theory by Bhatt et al. in [2] was used to show that, for smooth $\mathbb{Z}$-algebras, local cohomology modules have only finitely many associated primes.

In this paper, we are mostly concerned with upper bounds of Bass numbers for local cohomology modules (more generally of $\mathcal{T}(R)$, where $\mathcal{T}$ is a Lyubeznik functor). Let $e(M)$ denote the multiplicity of an $A_m(K)$-module M where $A_m(K)$ is the Weyl algebra corresponding to the polynomial ring $K[x_1, \dots, x_m]$. We now state the results proved in this paper. The first result we prove is the following.

Theorem A. (See, Theorem 3.3) *Let $R = K[x_1, \dots, x_m]$ where K is a field of characteristic 0. Let $T(R) = H^{i_1}_{I_1}(H^{i_2}_{I_2}(\dots H^{i_r}_{I_r}(R)\dots)$. Suppose $\operatorname{injdim} T(R) = c$. Then for all $i = 0, 1, \dots, c$;*

$$\mu_i(\mathfrak{m}, T(R)) \leq \binom{m}{i}(1+i)^n e(T(R))$$

for all maximal ideal $\mathfrak{m}$ of R.

This improves an earlier result from [11] (See, Proposition 1.3), but for the composition of local cohomology functors. In the theorem mentioned above, we are able to remove the hypothesis that K is algebraically closed which was initially assumed in [11, Proposition 1.3].

One naturally asks, what happens when instead of maximal ideals, we consider prime ideals. The paper provides some partial answers to this question. In [1], Banerjee et al. proved the characteristic p version of the above theorem, where prime ideals are considered instead of maximal ideals. They proved that if M is a F-finite, F-module over a Noetherian regular ring S of finite Krull dimension, then there exists some $B > 0$ such that $\mu_i(P, M) \leq B$ for all $P \in \operatorname{Spec}(S)$, and $i \in \mathbb{N}$.

In the next few results, we study the Bass numbers of $T(R)$ with respect to prime ideals.

Theorem B. (See, Theorem 4.2) *Let K be a field of characteristic 0 and R be a standard graded polynomial ring $K[x_1, \dots, x_m]$, i.e., $\deg(x_i) = 1$ for all i. Let $I_1, I_2, \dots, I_r$ be homogeneous ideals of R and $T(R) = H^{i_1}_{I_1}(H^{i_2}_{I_2}(\dots H^{i_r}_{I_r}(R)\dots)$. Then, there exists some $B > 0$ (depending on $T(R)$) such that*

$$\mu_i(P, T(R)) \leq B$$

for all graded prime ideal P of R of height $m-1$.

When we are not in a graded setup, the situation is different. We state few results that answer this question. For the remaining of the results we assume K to be an uncountable algebraically closed field of characteristic 0.

Lemma C. (See, Lemma 5.2) *Let K be an uncountable algebraically closed field of characteristic 0 and $R = K[x_1, \dots, x_m]$. Let $\mathcal{T}$ be a Lyubeznik functor on $\operatorname{Mod}(R)$. Then,*

$$\mu_0(P, \mathcal{T}(R)) \leq 2^m m \; e(\mathcal{T}(R))$$

for all prime ideal P of R of height $m-1$.

Eisenbud and Evans proved that every ideal in $K[x_1, \dots, x_m]$ can be generated by m elements up to radical; see [4, Theorem 1]. Therefore, if P is a prime ideal in $K[x_1, \dots, x_m]$, then there exists some ideal J such that $P = \sqrt{J}$ where $J = (f_1, \dots, f_m)$. Hence, we naturally define the following set.

Definition 1.1. *For a fixed number t,*

$$H_g(t) := \{P \in \operatorname{Spec}(R) \mid \text{ height}(P) = g, \; P = \sqrt{J} \textit{ where } J = (f_1, \dots, f_m), \deg(f_j) \leq t \; \}.$$

It is easy to see that $H_g(t) \subseteq H_g(t+1)$ for all $t \geq 1$ and $\bigcup_{t\geq 1} H_g(t) = \operatorname{Spec}_g(R) := \{P \in \operatorname{Spec}(R) \mid \operatorname{height} P = g\}$. Since $\operatorname{Spec}_g(R)$ is uncountable, there exists some t_0 such that $H_g(t_0)$ is uncountable. Hence, $H_g(t)$ is uncountable for all $t \geq t_0$.

As a consequence of Lemma C, we prove the following result which deals with the higher Bass numbers of $\mathcal{T}(R)$ for primes in $H_{m-1}(t)$.

Theorem D. (See, Theorem 5.3) *Let $R = K[x_1, \ldots, x_m]$ where K is an uncountable algebraically closed field of characteristic 0. Let $\mathcal{T}$ be a Lyubeznik functor on $\operatorname{Mod}(R)$. Then, for all $P \in H_{m-1}(t)$,*

$$\mu_i(P, \mathcal{T}(R)) \leq \phi_i^{m-1}(e(\mathcal{T}(R)), t)$$

where

$$\phi_i^{m-1}(u, v) = 2^m m \binom{m}{i} u(1 + iv)^m.$$

Next, we define $\mathcal{S}_g(t)$ for $1 \leq g \leq m-1$ inductively.

Definition 1.2. *Define $\mathcal{S}_{m-1}(t) := H_{m-1}(t)$. If $S_{g+1}(t)$ is defined for some $1 \leq g \leq m-2$, then we define $\mathcal{S}_g(t)$ in the following way:*

$$\mathcal{S}_g(t) := \{P \in H_g(t) \mid \ C_g^P(t) \text{ is uncountable}\}.$$

where

$$C_g^P(t) := \{Q \in S_{g+1}(t) \mid Q \supsetneq P\}$$

The following properties of the sets $\mathcal{S}_g(t)$ for $1 \leq g \leq m-2$ justify the way they are defined.

Proposition E. (See, Proposition 6.2) *Let $1 \leq g \leq m-2$. Then,*

(1) *Let $P \in \mathcal{S}_g(t)$. Then, $C_g^P(t) \subseteq C_g^P(t+1)$ for all $t \geq 1$ and $\bigcup_{t\geq 1} C_g^P(t) = \operatorname{Spec}_{g+1}(R) \cap \operatorname{V}(P)$;*
(2) *$\mathcal{S}_g(t) \subseteq \mathcal{S}_g(t+1)$ for all $t \geq 1$ and $\bigcup_{t\geq 1} \mathcal{S}_g(t) = \operatorname{Spec}_g(R)$.*

The final and main result result of this paper is the following.

Theorem F. (See, Theorem 6.3) *Let $1 \leq g \leq m-1$ and let $R = K[x_1, \ldots, x_m]$ where K is an uncountable algebraically closed field of characteristic 0. Let $M = \mathcal{T}(R)$ for some Lyubeznik functor $\mathcal{T}$. Then, there exists some function $\phi_i^g : \mathbb{N}^2 \to \mathbb{N}$ which is monotonic in both the variables such that $\mu_i(P, M) \leq \phi_i^g(e(M), t)$ for all $P \in \mathcal{S}_g(t)$.*

We provide a brief overview of the organization of the article. Section 2 introduces the necessary preliminaries - including definitions, assumptions, and supporting results needed to prove our theorems. In Section 3, we present the proof of Theorem A, followed by the proof of Theorem B in Section 4. In Section 5, we prove Lemma C and Theorem D. Finally, Section 6 completes the paper with the proof of Theorem F.

2. Preliminaries

Let R be a Noetherian commutative ring and G be a finite subgroup of $\operatorname{Aut}(R)$; the group of automorphism of R. Further assume that $|G|$ is invertible in R. Let R^G be the ring of invariants of G. By $R * G$, we denote the skew group ring. For basic properties of skew group ring we refer [13]. Let us recall few results from [13] that we need in this paper.

Corollary 2.1. [13, Corollary 3.3] *Let $I, I_1, \ldots, I_r$ be ideals in R^G. Then:*

(1) $H^i_{IR}(R)$ *is an* $R * G$*-module and* $H^i_{IR}(R)^G \cong H^i_I(R^G)$ *for each* $i \geq 0$.
(2) *For all* $i_j \geq 0$, *where* $j = 1, \ldots, r$, $H^{i_1}_{I_1}(H^{i_2}_{I_2}(\ldots H^{i_r}_{I_r}(R)\ldots)$ *is an* $R * G$*-module and*

$$H^{i_1}_{I_1R}(H^{i_2}_{I_2R}(\ldots H^{i_r}_{I_rR}(R)\ldots)^G \cong H^{i_1}_{I_1}(H^{i_2}_{I_2}(\ldots H^{i_r}_{I_r}(R^G)\ldots).$$

The following result is crucial.

Theorem 2.2. [13, Theorem 6.1] *Let* K *be a field and* R *be a regular domain containing* K. *Let* G *be a finite subgroup of the group of automorphism of* R. *Let* $|G|$ *be invertible in* K. *Assume that* R^G *is Gorenstein and* $I_1, I_2, \ldots, I_r$ *be ideals in* R^G. *Set* $T(R^G) = H^{i_1}_{I_1}(H^{i_2}_{I_2}(\ldots H^{i_r}_{I_r}(R^G)\ldots)$ *and* $T(R) = H^{i_1}_{I_1R}(H^{i_2}_{I_2R}(\ldots H^{i_r}_{I_rR}(R)\ldots)$. *Let* P *be a prime ideal in* R^G. *Then* $\mu_j(P, T(R^G)) = \mu_j(P', T(R))$ *where* P' *is any prime in* R *lying over* P.

Throughout the article K is assumed to be a field of characteristic 0 unless explicitly stated otherwise. Let $R = K[x_1, \ldots, x_m]$ be the ring of polynomials in m variables over K and $A_m(K)$ be the corresponding Weyl algebra. We mainly follow [3, Chapter 1] for the theory of finitely generated left $A_m(K)$-modules. For finite generated left $A_m(K)$-modules M, let $l_{A_m(K)}(M)$ and $e_{A_m(K)}(M)$ denote the length and multiplicity respectively. When the ring $A_m(K)$ is clear from the context, we omit the subscript $A_m(K)$ and simply write $l(M)$ and $e(M)$. Let $\mathcal{B}_m$ be the class of holonomic $A_m(K)$-modules. In Björk's book [3], holonomic modules are referred to as modules in the Bernstein class.

2.3. Some properties of elements of $\mathcal{B}_m$: Let $M \in \mathcal{B}_m$ and $R = K[x_1, \ldots, x_m]$. Then,

(1) M has finite length as left $A_m(K)$-modules. In fact, every strictly increasing sequence of $A_m(K)$-modules in M contains at most $e(M)$ elements; see [3, Chapter 1, Proposition 5.3].
(2) $\mathcal{B}_m$ is stable under extension. That is, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of left $A_m(K)$-modules, then $M_2 \in \mathcal{B}_m$ if and only if M_1 and M_3 both $\in \mathcal{B}_m$; see [3, Chapter 1, Proposition 5.2].
(3) Let $f \in R$. Then R_f is a left $A_m(K)$-module and $R_f \in \mathcal{B}_m$; see [3, Chapter 1, Theorem 5.5].
(4) The natural localization map $R \to R_f$ is a morphism of left $A_m(K)$-modules. Let $I \subseteq R$ be an ideal of R generated by $\underline{f} = f_1, f_2, \ldots, f_r \in R$. Then the Čech complex of R with respect to $\underline{f}$ is defined by

$$\check{C}^\bullet(\underline{f}, R): \ 0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i,j} R_{f_if_j} \to \ldots \to R_{f_1\ldots f_r} \to 0$$

where the maps on every summand are localization map up to a sign. The local cohomology module of R with support on I is defined by

$$H^i_I(R) = H^i(\check{C}^\bullet(\underline{f}, R)).$$

Since $\mathcal{B}_m$ is stable under extension, using point 2 and 3, we see that $H^i_I(R) \in \mathcal{B}_m$.

Let us recall the following multiplicity bound on the multiplicity of localization of a holonomic $A_m(K)$-modules.

Theorem 2.4. *Let* M *be a holonomic* $A_m(K)$*-module. If* $f \in R = k[x_1, \ldots, x_m]$, *then*

$$e(M_f) \leq e(M)(1 + \deg f)^m.$$

Proof. The proof follows from [3, Chapter 1, Theorem 5.19]. □

Now we define Lyubeznik functors.

2.5. Lyubeznik functors: As before assume that R is a commutative Noetherian ring and let $X = \operatorname{Spec}(R)$. Let Y be a locally closed subset of X. If M is an R-module and Y is a locally closed subscheme of $\operatorname{Spec}(R)$, we denote by $H^i_Y(M)$ the i-th local cohomology module of M with support in Y. Suppose $Y = Y_1 \setminus Y_2$ where $Y_2 \subseteq Y_1$ are two closed subsets of X, then we have an exact sequence of functors

$$\cdots \to H^i_{Y_1}(-) \to H^i_{Y_2}(-) \to H^i_Y(-) \to H^{i+1}_{Y_1}(-) \to \cdots .$$

A Lyubeznik functor $\mathcal{T}$ is any functor of the form $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_m$ where every functor $\mathcal{T}_j$ is either $H^i_Y(-)$ for some locally closed subset of X or the kernel, image or cokernel of some arrow in the previous long exact sequence for closed subsets Y_1, Y_2 of X such that $Y_2 \subseteq Y_1$.

We should remark that a significant example of a Lyubeznik functor $\mathcal{T}$ is the repeated local cohomology functor $H^{i_1}_{I_1}(H^{i_2}_{I_2}(\dots H^{i_r}_{I_r}(R)\dots)$.

2.6. Countable generation: We say that an R-module M is countably generated if there exists a countable set of elements that generates M as an R-module. The following results are crucial for proving our results.

(1) [15, Lemma 5.2] Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Then, M_2 is countably generated if and only if M_1 and M_3 are countably generated.
(2) [15, Lemma 5.2] Let $\mathcal{T}$ be a Lyubeznik functor on $\operatorname{Mod}(R)$. Then, $\mathcal{T}(M)$ is a countably generated R-module for any countably generated R-module M. In particular, $H^i_I(R)$ is a countably generated R-module for all ideals I of R and for all $i \geq 0$.
(3) [12, Lemma 2.3] Let M be a countably generated R-module. Then, $\operatorname{Ass}_R M$ is a countable set.

3. Proof of theorem A

In this section, we prove Theorem A. First, we prove a small lemma that we believe is already known in the literature. However, we do not have any reference, so we include the proof here.

Lemma 3.1. *Let K be a field of characteristic 0 and let $\mathfrak{m}$ be a maximal ideal of $R = K[x_1, \dots, x_m]$. Then there exists a finite Galois extension $\widetilde{K}$ (depending on $\mathfrak{m}$) of K such that if $\mathfrak{n}$ is a maximal ideal of $\widetilde{R}$ lying over $\mathfrak{m}$, then $\mathfrak{n}$ is of the form $(x_1 - c_1, \dots, x_m - c_m)$ for some $c_i \in \widetilde{K}$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $K[x_1, \dots, x_m]$. Then, by [10, Theorem 5.1], we can write $\mathfrak{m} = (f_1(x_1), f_2(x_1, x_2), \dots, f_m(x_1, \dots, x_m))$ where $f_i(x_1, \dots, x_i)$ is monic in x_i.

Let S_1 be the set of zeroes of the polynomial $f_1(x_1)$. For $a \in S_1$, let S^a_2 be the zero set of $f_2(a, x_2)$. Inductively for $3 \leq i \leq m$ and for $a_1 \in S_1$, $a_j \in S^{a_1,\dots,a_{j-1}}_j$, we define $S^{a_1,\dots,a_{i-1}}_i$ to be the set of zeroes of the polynomial $f_i(a_1, \dots, a_{i-1}, x_m)$. A careful look at the proof of [10, Theorem 5.1] shows that the polynomials $f_i(x_1, \dots, x_i)$ are chosen such a way that these $f_i(a_1, \dots, a_{i-1}, x_i)$ are irreducible.

We claim that our required field is $\widetilde{K}$ which is obtained by adjoining the elements of the set $S_1 \cup (\bigcup_{i=2}^m S^{a_1,\dots,a_{i-1}}_i)$ with K.

Now $\mathfrak{m}\widetilde{K}[x_1, \dots, x_m] = (f_1, \dots, f_m)\widetilde{K}[x_1, \dots, x_m]$. Let $\mathfrak{n}$ be a maximal in $\widetilde{K}[x_1, \dots, x_m]$ lying over $\mathfrak{m}$. Let $\mathfrak{n} = (g_1(x_1), g_2(x_1, x_2), \dots, g_m(x_1, \dots, x_m))$.

We note that $\mathfrak{n} \supseteq \mathfrak{m}\widetilde{K}[x_1, \dots, x_m]$. Similarly, using the roots of g_i we can form a field, say $\widetilde{\widetilde{K}}$ which is a finite extension of $\widetilde{K}$. Let $\mathfrak{n}'$ be a maximal ideal of $\widetilde{\widetilde{K}}[x_1, \dots, x_m]$ lying over $\mathfrak{n}$. Now $\mathfrak{n}' \supseteq \mathfrak{n}\widetilde{\widetilde{K}}[x_1, \dots, x_m] \supseteq \mathfrak{m}\widetilde{\widetilde{K}}[x_1, \dots, x_m]$. Therefore, $Z(\mathfrak{n}') \subseteq Z(\mathfrak{n}\widetilde{\widetilde{K}}[x_1, \dots, x_m]) \subseteq Z(\mathfrak{m}\widetilde{\widetilde{K}}[x_1, \dots, x_m])$. Let $(\alpha_1, \dots, \alpha_m) \in Z(\mathfrak{n}\widetilde{\widetilde{K}}[x_1, \dots, x_m]) \subseteq Z(\mathfrak{m}\widetilde{\widetilde{K}}[x_1, \dots, x_m])$. By construction, $\alpha_1 \in S_1$ and $\alpha_j \in S^{\alpha_1,\dots,\alpha_{j-1}}_j$

for $j \geq 2$. We note that $\alpha_i \in \widetilde{K}$ for all $1 \leq i \leq m$. Since for all $1 \leq i \leq m$, $g_i(\alpha_1, \ldots, \alpha_{i-1}, x_i)$ is irreducible, we can write $g_i(\alpha_1, \ldots, \alpha_{i-1}, x_i) = r_i(x_i - \alpha_i)$. Also when $i \geq 2$, division algorithm implies $g_i(x_1, \ldots, x_i) = (x_1 - \alpha_1) \cdots (x_{i-1} - \alpha_{i-1})Q_i(x_1 \ldots, x_i) + g_i(\alpha_1, \ldots, \alpha_{i-1}, x_i) = (x_1 - \alpha_1) \cdots (x_{i-1} - \alpha_{i-1})Q_i(x_1 \ldots, x_i) + r_i(x_i - \alpha_i)$. Therefore, $\mathfrak{n} = (x_1 - \alpha_1, \ldots, x_m - \alpha_m)$. This proves our result. $\square$

The following remark is important.

Remark 3.2. Let $K_1 \subseteq K_2$ be two fields. Let M be $A_m(K_1)$-module. Then $M \otimes_{K_1} K_2$ is $A_m(K_2)$-module and $\dim(M) = \dim(M \otimes_{K_1} K_2)$. Also if $U_1 \subsetneq U_2$ are $A_m(K_1)$-submodules of M, then $U_1 \otimes_{K_1} K_2 \subsetneq U_2 \otimes_{K_1} K_2$ are $A_m(K_2)$-submodules of $M \otimes_{K_1} K_2$. So $l_{A_m(K_1)}(M) \leq l_{A_m(K_2)}(M \otimes_{K_1} K_2)$. Also, if Ω is a good filtration on M, then $\Omega \otimes K_2$ is a good filtration on $M \otimes K_2$ and hence $e_{A_m(K_1)}(M) = e_{A_m(K_2)}(M \otimes_{K_1} K_2)$.

We now prove Theorem A, which we restate for the reader's convenience.

Theorem 3.3. *Let $R = K[x_1, \ldots, x_m]$ where K is a field of characteristic 0. Assume that $T(R) = H^{i_1}_{I_1}(H^{i_2}_{I_2}(\ldots H^{i_r}_{I_r}(R) \ldots))$. Suppose* $\operatorname{injdim} T(R) = c$. *Then for all $i = 0, 1, \ldots, c$;*

$$\mu_i(\mathfrak{m}, T(R)) \leq \binom{m}{i}(1+i)^n e_{A_m(K)} T(R)$$

for all maximal ideal $\mathfrak{m}$ of R.

Proof. Let $\widetilde{K}$ be a finite Galois extension of K as in 3.1. Assume that the Galois group is G. Then G acts on $\widetilde{R} = \widetilde{K}[x_1, \ldots, x_m]$ and $\widetilde{R}^G = R$. Since $T(R) = T(\widetilde{R}^G)$ so by Theorem 2.2, $\mu_i(\mathfrak{m}, T(R)) = \mu_i(\mathfrak{m}, T(\widetilde{R}^G)) = \mu_i(\mathfrak{n}, T(\widetilde{R}))$ where $\mathfrak{n}$ is any maximal in $\widetilde{R}$ lying over $\mathfrak{m}$. By 3.1, $\mathfrak{n}$ is of the form $(x_1 - c_1, \ldots, x_m - c_m)$ for some $c_i \in \widetilde{K}$. From the proof of [11, Proposition 1.3],

$$\mu_i(\mathfrak{n}, T(\widetilde{R})) \leq \binom{m}{i}(1+i)^n e_{A_m(\widetilde{K})}(T(\widetilde{R})). \tag{1}$$

Note that $T(\widetilde{R}) = T(R) \otimes_K \widetilde{K}$. Using Remark 3.2, $e_{A_m(\widetilde{K})}(T(\widetilde{R})) = e_{A_m(K)} T(R)$. Hence, from equation 1, $\mu_i(\mathfrak{n}, T(\widetilde{R})) \leq \binom{m}{i}(1+i)^n e_{A_m(K)} T(R)$. This proves our result. $\square$

4. PROOF OF THEOREM B

Before proving our result, we recall a result due to Goto and Watanabe from [5].

Theorem 4.1. [5, Theorem 1.1.2] *Let M be a graded R-module and let P be a non-graded prime ideal of R. Then $\mu_0(P, M) = 0$ and $\mu_{i+1}(P, M) = \mu_i(P^*, M)$ for every integer $i \geq 0$. Here, P^* is the largest homogeneous prime ideal of R contained in P.*

Now we prove Theorem B.

Theorem 4.2. *Let K be a field of characteristic 0 and $R = K[x_1, \ldots, x_m]$ be standard graded i.e.,* $\deg(x_i) = 1$ *for all i. Let $T(R) = H^{i_1}_{I_1}(H^{i_2}_{I_2}(\ldots H^{i_r}_{I_r}(R) \ldots)$ for some homogeneous ideals $I_1, I_2, \ldots, I_r$ of R. Then, there exists some $B > 0$ (depending on $T(R)$) such that*

$$\mu_i(P, T(R)) \leq B$$

for all graded prime ideal P of R of height $m - 1$.

Proof. Let $\bar{\mathfrak{m}}$ be a maximal ideal of R/P which is not graded. By Theorem 4.1, we get $\mu_i(P, T(R)) = \mu_{i+1}(\mathfrak{m}, T(R))$. Hence, the result follows from Theorem 3.3. $\square$

5. Proof of Lemma C and Theorem D

In this section, we prove Lemma C, and as an immediate consequence, we obtain Theorem D. Let us first highlight the following important remark before proceeding to the proof.

Remark 5.1. Let $R = K[x_1, \ldots, x_m]$ be a polynomial ring over a field of characteristic 0 and let P be a prime ideal of R of height g where $1 \le g \le m-1$. Let Q be a prime ideal of R containing P such that $\operatorname{height} Q = g+1$. Note R_Q/PR_Q is Cohen-Macaulay ring of dimension 1. Therefore, from [11, Theorem 1.2, Proposition 2.2], we get $\mu_1(Q, H^g_P(R)) = \mu_1(QR_Q, H^g_{PR_Q}(R_Q)) = 1$.

We now present the proof.

Lemma 5.2. *Let K be an uncountable algebraically closed field of characteristic 0 and $R = K[x_1, \ldots, x_m]$. Let $\mathcal{T}$ be a Lyubeznik functor on* $\operatorname{Mod}(R)$ *. Then,*

$$\mu_0(P, \mathcal{T}(R)) \le 2^m m\ e(\mathcal{T}(R))$$

for all prime ideal P of R of height $m-1$.

Proof. Let $M = \mathcal{T}(R)$ and P be a prime ideal of R height $m-1$. By [7, Lemma 1.4, Theorem 3.4(b)], $\mu_0(P, M) = \mu_0(P, H^0_P(M))$. We note that $e(H^0_P(M)) \le e(M)$ and $\operatorname{Ass}_R(H^0_P(M))$ is a finite set. If $P \notin \operatorname{Ass} H^0_P(M)$, then there is nothing to prove since $\mu_0(P, M) = 0$. Hence, assume that

$$\operatorname{Ass}_R(H^0_P(M)) = \{P, \mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$$

where $\mathfrak{m}_i$ are maximal ideals. Consider the exact sequence

$$0 \to \Gamma_{\mathfrak{m}_1 \cdots \mathfrak{m}_s}(H^0_P(M)) \to H^0_P(M) \to N \to 0.$$

Clearly $e(N) \le e(H^0_P(M))$ and $N = \mathcal{F}(R)$ for some Lyubeznik functor $\mathcal{F}$. By [7, Theorem 3.4(b)], there exists some finite $l > 0$ such that $N_P = H^{m-1}_{PR_P}(R_P)^l = E(\kappa(P))^l$ as R_P-module and hence as R-module. Note that $l = \mu_0(P, N) = \mu_0(P, M)$. Let $C = N \cap H^{m-1}_P(R)^l$. Therefore, $C_P = N_P \cap H^{m-1}_{PR_P}(R_P)^l = N_P = H^{m-1}_{PR_P}(R_P)^l$. Consider the exact sequence

$$0 \to C \to H^{m-1}_P(R)^l \to W_1 \to 0. \tag{2}$$

Note that W_1 is P-torsion and $(W_1)_P = 0$ so $\operatorname{Supp}_R W_1 \subseteq \operatorname{MaxSpec} R\ \cap \mathrm{V}(P)$. Since $H^{m-1}_P(R)$ is countably generated, W_1 is also countably generated (See, 2.6). Hence, $\operatorname{Ass}_R W_1 = \operatorname{Supp}_R W_1$ is a countable subset of $\operatorname{MaxSpec} R\ \cap \mathrm{V}(P)$.

Similarly, the exact sequence

$$0 \to C \to N \to W_2 \to 0 \tag{3}$$

implies that $\operatorname{Ass}_R W_2 = \operatorname{Supp}_R W_2$ is a countable subset of $\operatorname{MaxSpec} R\ \cap \mathrm{V}(P)$.

We note that $\operatorname{MaxSpec} R\ \cap \mathrm{V}(P)$ is an uncountable set. This is because by Noether normalization $K[z] \hookrightarrow R/P$ is an integral extension and since $K[z]$ has uncountably many maximal ideals, R/P also has uncountably many maximal ideals. Therefore, there is some $\mathfrak{n} \in \operatorname{MaxSpec} R\ \cap \mathrm{V}(P)$ such that $(W_1)_\mathfrak{n} = (W_2)_\mathfrak{n} = 0$. Hence, from equation 2, we get $C_\mathfrak{n} = H^{m-1}_P(R)^l_\mathfrak{n}$. Remark 5.1 implies that $\mu_1(\mathfrak{n}, C) = \mu_1(\mathfrak{n}R_\mathfrak{n}, C_\mathfrak{n}) = l$. Also since $(W_2)_\mathfrak{n} = 0$, from equation 3 we get $C_\mathfrak{n} = N_\mathfrak{n}$. By [11, Proposition 1.3], $l = \mu_1(\mathfrak{n}, C) = \mu_1(\mathfrak{n}R_\mathfrak{n}, C_\mathfrak{n}) = \mu_1(\mathfrak{n}R_\mathfrak{n}, N_\mathfrak{n}) = \mu_1(\mathfrak{n}, N) \le 2^m m\ e(N) \le 2^m m\ e(H^0_P(M)) \le 2^m m\ e(M)$. This completes our proof. $\square$

As a consequence of the preceding lemma, we next examine the higher Bass numbers of $\mathcal{T}(R)$. We reiterate that, as discussed in the introduction, there exists a value of t such that $H_{m-1}(t)$ is an uncountable set. The next result states that there is a bound of Bass numbers of $\mathcal{T}(R)$ for the primes on the uncountable set $H_{m-1}(t)$.

Theorem 5.3. *Let $R = K[x_1, \ldots, x_m]$ where K is an uncountable algebraically closed field of characteristic 0. Let $\mathcal{T}$ be a Lyubeznik functor on $\operatorname{Mod}(R)$. Then, for all $P \in H_{m-1}(t)$,*

$$\mu_i(P, \mathcal{T}(R)) \leq \phi_i^{m-1}(e(\mathcal{T}(R)), t)$$

where

$$\phi_i^{m-1}(u, v) = 2^m m \binom{m}{i} u(1 + iv)^m.$$

Proof. Let $M = \mathcal{T}(R)$. By [7, Lemma 1.4, Theorem 3.4(b)], $\mu_i(P, M) = \mu_0(P, H^i(M))$. By Lemma 5.2, $\mu_0(P, H^i(M)) \leq 2^m m\ e(H^i_P(M))$. Note that $H^i_P(M) = H^i_J(M)$ where $J = (f_1, \ldots, f_m)$ with $\deg f_j \leq t$ for all j. $H^i_J(M)$ is the cohomology of the following Cech complex

$$\check{C}^\bullet(\underline{f}, M) = \check{C}^\bullet :\ 0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i,j} M_{f_i f_j} \to \ldots \to M_{f_1 \ldots f_m} \to 0.$$

Hence, $e(H^i_P(M)) \leq e(\check{C}^i) \leq \binom{m}{i} e(M)(1+it)^m$. The last inequality follows from Theorem 2.4. Therefore, $\mu_i(P, M) \leq 2^m m \binom{m}{i} e(M)(1 + it)^m$. This completes the proof of our theorem. $\square$

6. Proof of Proposition E and Theorem F

In this section, our goal is to prove Theorem F. Before that, let us prove Proposition E. For the reader's convenience, we again recall the definition of the sets $\mathcal{S}_g(t)$ as stated below.

Definition 6.1. *Define $\mathcal{S}_{m-1}(t) := H_{m-1}(t)$. If $S_{g+1}(t)$ is defined for some $1 \leq g \leq m-2$, then we define $\mathcal{S}_g(t)$ in the following way:*

$$\mathcal{S}_g(t) := \{P \in H_g(t) \mid\ C^P_g(t) \text{ is uncountable}\}.$$

where

$$C^P_g(t) := \{Q \in S_{g+1}(t) \mid Q \supsetneq P\}$$

We note that $H_{m-1}(t) \subseteq H_{m-1}(t+1)$ for all $t \geq 1$ and $\bigcup_{t \geq 1} H_{m-1}(t) = \operatorname{Spec}_{m-1}(R) = \{P \in \operatorname{Spec}(R) \mid \operatorname{height} P = m-1\}$. We now show via the following proposition that for $1 \leq g \leq m-2$, the sets $\mathcal{S}_g(t)$ have same properties.

Proposition 6.2. *Let $1 \leq g \leq m-2$. Then,*

(1) *Let $P \in \mathcal{S}_g(t)$. Then, $C^P_g(t) \subseteq C^P_g(t+1)$ for all $t \geq 1$ and $\bigcup_{t \geq 1} C^P_g(t) = \operatorname{Spec}_{g+1}(R) \cap \operatorname{V}(P)$;*
(2) *$\mathcal{S}_g(t) \subseteq \mathcal{S}_g(t+1)$ for all $t \geq 1$ and $\bigcup_{t \geq 1} \mathcal{S}_g(t) = \operatorname{Spec}_g(R)$.*

Proof. Let $g = m-2$. Then $C^P_{m-2}(t) = \{Q \in H_{m-1}(t) \mid Q \supsetneq P\}$. Clearly, $C^P_{m-2}(t) \subseteq C^P_{m-2}(t+1)$ for all $t \geq 1$ and $\bigcup_{t \geq 1} C^P_{m-2}(t) = \operatorname{Spec}_{m-1}(R) \cap \operatorname{V}(P)$. This proves (1) and we now prove (2). Since $\operatorname{Spec}_{m-1}(R) \cap \operatorname{V}(P)$ is uncountable, there exists some $t_0 \geq 1$ such that $C^P_{m-2}(t_0)$ is uncountable and therefore $C^P_{m-2}(t)$ is uncountable for all $t \geq t_0$. We claim that $\bigcup_{t \geq 1} \mathcal{S}_{m-2}(t) = \bigcup_{t \geq 1} H_{m-2}(t) = \operatorname{Spec}_{m-2}(R)$. This is because if $P \in \bigcup_{t \geq 1} H_{m-2}(t)$, then $P \in H_{m-2}(r_0)$ for some r_0. Choose a sufficiently large s_0 (bigger than r_0) so that $C^P_{m-2}(s_0)$ is uncountable. Then, $P \in H_{m-2}(r_0) \subseteq H_{m-2}(s_0)$ and this implies $P \in \mathcal{S}_{m-2}(s_0)$. Hence, $\bigcup_{t \geq 1} \mathcal{S}_{m-2}(t) = \bigcup_{t \geq 1} H_{m-2}(t)$.

Now assume that the result holds for some $2 \leq g+1 \leq m-2$ and we prove it for g. We have

$$C_g^P(t) = \{Q \in S_{g+1}(t) \mid Q \supsetneq P\}$$

and

$$\mathcal{S}_g(t) = \{P \in H_g(t) \mid \ C_g^P(t) \text{ is uncountable}\}.$$

Since for all $t \geq 1$, $\mathcal{S}_{g+1}(t) \subseteq \mathcal{S}_{g+1}(t+1)$, we get that $C_g^P(t) \subseteq C_g^P(t+1)$. Also, $\bigcup_{t\geq 1} C_g^P(t) = \bigcup_{t\geq 1} \mathcal{S}_{g+1}(t) \cap \mathrm{V}(P) = \mathrm{Spec}_{g+1}(R) \cap \mathrm{V}(P)$. This proves (1) of the proposition, and we proceed to prove (2). Since $\mathrm{Spec}_{g+1}(R) \cap \mathrm{V}(P)$ is uncountable, there exists some $t_1 \geq 1$ such that $C_g^P(t_1)$ is uncountable and therefore $C_g^P(t)$ is uncountable for all $t \geq t_1$. Now a similar argument as in $m-2$ case proves that $\bigcup_{t\geq 1} \mathcal{S}_g(t) = \bigcup_{t\geq 1} H_g(t)$. In fact, if $P \in \bigcup_{t\geq 1} H_g(t)$, then $P \in H_g(r_1)$ for some r_1. Choose a sufficiently large s_1 (bigger than r_1) so that $C_g^P(s_1)$ is uncountable. Then, $P \in H_g(r_1) \subseteq H_g(s_1)$ and this implies $P \in \mathcal{S}_g(s_1)$. Hence, $\bigcup_{t\geq 1} \mathcal{S}_g(t) = \bigcup_{t\geq 1} H_g(t) = \mathrm{Spec}_g(R)$. This completes the proof of the proposition. □

Finally, we prove the main result of this paper.

Theorem 6.3. *Let $1 \leq g \leq m-1$ and let $R = K[x_1, \ldots, x_m]$ where K is an uncountable algebraically closed field of characteristic 0. Let $M = \mathcal{T}(R)$ for some Lyubeznik functor $\mathcal{T}$. Then, there exists some function $\phi_i^g : \mathbb{N}^2 \to \mathbb{N}$ which is monotonic in both the variables such that $\mu_i(P, M) \leq \phi_i^g(e(M), t)$ for all $P \in \mathcal{S}_g(t)$.*

Proof. If $g = m-1$, the result follows from Theorem 5.3. Next, we proceed in two steps to prove the theorem.

Step 1: Let $g = m-2$.

Let $i = 0$. By [7, Lemma 1.4, Theorem 3.4(b)], $\mu_0(P, M) = \mu_0(P, H_P^0(M))$. We note that $e(H_P^0(M)) \leq e(M)$ and $\mathrm{Ass}_R(H_P^0(M))$ is a finite set. If $P \notin \mathrm{Ass}\, H_P^0(M)$, then there is nothing to prove since $\mu_0(P, M) = 0$. Hence, assume that

$$\mathrm{Ass}_R(H_P^0(M)) = \{P, Q_1 \ldots, Q_r\}$$

where $\mathrm{height}\, Q_i \geq m-1$. Consider the exact sequence

$$0 \to \Gamma_{Q_1 \cdots Q_r}(H_P^0(M)) \to H_P^0(M) \to N \to 0.$$

Clearly, $e(N) \leq e(H_P^0(M))$ and $N = \mathcal{F}(R)$ for some Lyubeznik functor $\mathcal{F}$. By [7, Theorem 3.4(b)], there exists some finite $l > 0$ such that $N_P = H_{PR_P}^{m-2}(R_P)^l = E(\kappa(P))^l$ as R_P-module and hence as R-module. Note that $l = \mu_0(P, N) = \mu_0(P, M)$. Let $C = N \cap H_P^{m-2}(R)^l$. Therefore, $C_P = N_P \cap H_{PR_P}^{m-2}(R_P)^l = N_P = H_{PR_P}^{m-2}(R_P)^l$. Consider the exact sequence

$$0 \to C \to H_P^{m-2}(R)^l \to W_1 \to 0. \tag{4}$$

Since $H_P^{m-2}(R)$ is countably generated, W_1 is also countably generated (See, 2.6). We note that $\mathrm{Supp}_R\, W_1 \subseteq \{Q : \mathrm{height}\, Q \geq m-1, P \subseteq Q\}$. Let $L(W_1) := \{Q : \mathrm{height}\, Q = m-1, Q \supseteq P,\ Q \in \mathrm{Supp}_R\, W_1\} \subseteq \mathrm{Ass}_R\, W_1$ is a countable set.

Similarly, the exact sequence

$$0 \to C \to N \to W_2 \to 0 \tag{5}$$

implies $L(W_2) := \{Q : \mathrm{height}\, Q = m-1, Q \supseteq P,\ Q \in \mathrm{Supp}_R\, W_2\} \subseteq \mathrm{Ass}_R\, W_2$ is a countable set. Since $C_{m-2}^P(t)$ is uncountable, there is some $Q \in C_{m-2}^P(t)$ such that $(W_1)_Q = (W_2)_Q = 0$. Hence, equations 4 and 5 imply $C_Q = H^{m-2}_P(R)^l_Q = N_Q$. By Remark 5.1, $l = \mu_1(Q, C) = \mu_1(Q, N) \leq 2^m m^2 e(N)(1 + t)^m \leq 2^m m^2 e(M)(1 + t)^m$. The second last inequality follows from Theorem 5.3. Let $\phi^{m-2}_0(u, v) := 2^m m^2 u(1 + v)^m$.

Now suppose $i \geq 1$. Then, $\mu_i(P, M) = \mu_0(P, H^i_P(M))$. Therefore, from the $i = 0$ case we get $\mu_i(P, M) \leq \phi^{m-2}_0(e(H^i_P(M), t)$. Again, from the proof of 5.3, we have $e(H^i_P(M)) \leq h(e(M), t)$ where $h(u, v) = \binom{m}{i}u(1 + iv)^m$. Hence, $\mu_i(P, M) \leq \phi^{m-2}_i(e(M), t)$ where $\phi^{m-2}_i(u, v) = \phi^{m-2}_0(h(u, v), v)$. This completes the proof of $m - 2$ case.

Step 2: Now assume that the result holds for some $2 \leq g + 1 \leq m - 2$ and we prove it for g.

Let $i = 0$. An argument similar to that in the case $m - 2$ gives us that there is some $Q \in C^P_g(t)$ such that $\mu_0(P, M) = \mu_1(Q, N) \leq \phi^{g+1}_1(e(N), t) \leq \phi^{g+1}_1(e(M), t)$ where ϕ^{g+1}_1 is some monotonic function in both variables. Choose ϕ^g_0 to be ϕ^{g+1}_1.

Now suppose $i \geq 1$. Then, $\mu_i(P, M) = \mu_0(P, H^i_P(M))$. Therefore, from the $i = 0$ case we get $\mu_i(P, M) \leq \phi^g_0(e(H^i_P(M), t)$. Again, from the proof of 5.3, we have $e(H^i_P(M)) \leq h(e(M), t)$ where $h(u, v) = \binom{m}{i}u(1 + iv)^m$. Hence, $\mu_i(P, M) \leq \phi^g_i(e(M), t)$ where $\phi^g_i(u, v) = \phi^g_0(h(u, v), v)$.

This completes the proof of the theorem. □

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