

Interscale energy transfer in turbulent channels

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Abstract. We investigate the energy cascade in wall-bounded turbulence by analysing the interscale transfer between streamwise and spanwise length scales in periodic channels. This transfer originates from the nonlinear interactions in the advective term of the Navier-Stokes equations, which satisfy the classical triadic compatibility relations. Each triadic interaction is examined individually, and its corresponding nonlinear momentum and energy transfer are mapped to assess its relative importance in sustaining turbulence. Motivated by the anisotropy of the flow, we interpret each contribution $\partial_i(u_i u_j)$ to the advection term as carrying distinct physical information, and therefore analyse them separately. Time-averaged maps of the energy transfer across all length scales and wall-normal positions for a channel flow at $Re_\tau \approx 180$ are used to explore the mechanisms underlying the cascade process. As a proof of concept, reduced-order simulations are performed by retaining only the interactions identified as responsible for significant energy transfer based on our framework. Turbulent dynamics are successfully reproduced when 30% or more of the total interactions are included, while noticeable deviations emerge in the near-wall region when this proportion is further reduced.

1 Introduction

The turbulent energy cascade is central to our understanding of turbulence. In turbulent flows, energy is typically injected at large lengthscales and transferred through inter-scale interactions to smaller scales, ultimately being dissipated at viscous scales. In periodic channels, we can leverage the spatial periodicity of the flow to consider the governing equations in their streamwise- (x) and spanwise- (z) Fourier form. The inter-scale transfer is then inherently in the non-linear terms, as linear operators, though responsible for critical turbulent processes (Lozano-Durán et al. 2021), preserve wavenumber. In the Navier-Stokes equations, the non-linear term responsible for this transfer is the advection term, $\nabla \cdot (\mathbf{u}\mathbf{u})$, which enables the triadic wavenumber interactions: a catalyst velocity component with wavenumber $\mathbf{k}_c$ advects a donor velocity component with wavenumber $\mathbf{k}_d$ to transfer momentum and energy to a recipient mode with wavenumber $\mathbf{k}_r = \mathbf{k}_c + \mathbf{k}_d$. The cascade proceeds through a hierarchy of such triadic interactions, as illustrated in Figure 1, where different lengthscales can participate in the overall transfer with all three roles simultaneously, i.e. wavelength k_3 serves as the recipient for interaction I but is also the catalyst for interaction III, etc. The proposed framework emphasises the two-into-one directionality in each triadic interaction, i.e. catalyst and donor combining to cause an effect in a recipient, but makes no *a priori* distinction between interactions contributing to the direct cascade, energy transfer from large to small scales, and those contributing to the inverse cascade or backscatter.

The evolution of turbulent kinetic energy through inter-scale triadic interactions has been the subject of extensive research (Domaradzki 1992, Domaradzki & Rogallo 1990, Waleffe 1992). Previous studies have mapped triadic interactions in wall-bounded turbulence in various different forms. These include: maps of the bispectrum and biphasic of streamwise wave triads (Cui & Jacobi 2021); spanwise triadic

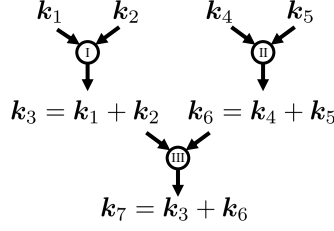


Figure 1: Sketch of interscale interactions, adapted from De Salis Young et al. (2024). Interaction I involves a recipient mode $\mathbf{k}_3$ receiving energy from an interaction between mode $\mathbf{k}_1$ and $\mathbf{k}_2$; $\mathbf{k}_3$ in turn interacts with $\mathbf{k}_6$ in interaction III to transfer energy into $\mathbf{k}_7$, and so on to viscosity.

wave interactions involved in turbulent energy transfer (Cho et al. 2018); spanwise and streamwise triadic wave interactions involved in turbulent energy transfer (Lee & Moser 2019); spanwise and streamwise wave dyad interactions into the principal resolvent mode (Bae et al. 2021); causal maps of inter-scale flux of information between signals of different wavelength (Lozano-Durán et al. 2019); interscale triadic causality maps (De Salis Young et al. 2024); or the overall transfer from a donor mode to a recipient mode (Ding et al. 2025). Inter-scale transfer has also been analysed in the spatial domain through the use of structure functions and the Kármán-Howarth-Monin-Hill equation (Chiarini et al. 2022, Yao et al. 2022), and in the context of resolvent analysis when trying to characterise the non-linear, forcing term (Morra et al. 2020, Symon et al. 2021, Zare et al. 2017). Kawata & Tsukahara (2021) considered the interscale energy transfer in minimal channels, with streak breakdown and vortex regeneration viewed as instances of forward and reverse energy cascade.

De Salis Young et al. (2024) drew a distinction between the nine individual contributions to the advective term in the Navier-Stokes equations, $\partial_i(u_i u_j)$. They argued that, due to the anisotropy of wall turbulence, each of these contributions could be interpreted as representing a distinct physical mechanism. They proposed a framework to map separately each of these contributions for each triadic-compatible set of modes. We extend their preliminary exploration by generating maps that represent flow dynamics throughout the channel. We identify the key interactions responsible for significant energy transfer and, as a proof of concept, perform reduced-order simulations in which only these interactions are retained in the nonlinear advective term. This allows us to assess whether turbulent flow dynamics can be reproduced, thereby verifying the significance of the selected modes. As a benchmark, we use a channel of size $L_x = 2\pi$, $L_z = \pi$, $L_y = 2$ at low friction Reynolds number, $\text{Re}_\tau \approx 180$, leaving the consideration of larger and more complex flows for future work.

The ultimate objectives of the proposed framework are threefold. First, to gain insights into the flow physics through the identification of the most significant nonlinear interactions. Second, to provide a roadmap for developing reduced-order models in which only a small number of key interactions are retained, thus reducing the computational cost of simulations and proposing new approach to turbulence modelling. Third, to identify potential bottleneck regions in the interscale transfer maps that may play a critical role in the energy cascade. If such regions exist, they could be strategically targeted for optimal disruption of turbulence in flow-control applications.

The paper is organised as follows. §2 describes the methods used to map the inter-scale transfer in an x - z -periodic channel and the details of the reduced order simulations. §3.1 examines the maps of the inter-scale energy transfer to two different scales, and §3.2 compares the from the reduced order simulation to those of full DNSs. §4 summarises and concludes the paper.

2 Methods

2.1 Nonlinear energy transfer maps

The Navier-Stokes and continuity equations in Fourier space for any given recipient mode $\mathbf{k}_r$ can be obtained by applying a streamwise (x)spanwise (z) Fourier transform to the equations in physical space,

$$\frac{\partial \hat{\mathbf{u}}}{\partial t}(\mathbf{k}_r, y, t) + \sum_{\mathbf{k}_c + \mathbf{k}_d = \mathbf{k}_r} \hat{\nabla} \cdot (\hat{\mathbf{u}}(\mathbf{k}_c, y, t) \hat{\mathbf{u}}(\mathbf{k}_d, y, t)) = -\hat{\nabla} \hat{p}(\mathbf{k}_r, y, t) + \nu \hat{\nabla}^2 \hat{\mathbf{u}}(\mathbf{k}_r, y, t), \quad (1)$$

$$\hat{\nabla} \cdot \hat{\mathbf{u}}(\mathbf{k}_r, y, t) = 0, \quad (2)$$

where $\mathbf{u} = [u, v, w]^T$ is the velocity vector, ν is the kinematic viscosity and p kinematic pressure. Fourier-transformed quantities are denoted with $\hat{(\cdot)}$ and $\hat{\nabla} = [ik_x \partial/\partial y \ ik_z]^T$ is the Fourier transform of the

del operator. The linear operators $\partial/\partial t$, ∇ and ∇^2 , along with the linear part of the advection $\hat{\nabla} \cdot (\hat{\mathbf{u}}(\mathbf{k}_r)\hat{\mathbf{U}}) + \hat{\nabla} \cdot (\hat{\mathbf{U}}\hat{\mathbf{u}}(\mathbf{k}_r))$, where $\mathbf{U}$ represents the mean flow ($\mathbf{k}_d$ or $\mathbf{k}_c = 0$), are Fourier transformed trivially. However, the non-linear part of the advection ($\hat{\mathbf{n}}$) becomes a convolution in the wavenumber domain, coupling velocity modes through triadic interactions, where

$$\hat{\mathbf{n}}(\mathbf{k}_r) = \sum_{\substack{\mathbf{k}_c + \mathbf{k}_d = \mathbf{k}_r \\ \mathbf{k}_c, \mathbf{k}_d \neq 0}} -\hat{\nabla}(\hat{\mathbf{u}}(\mathbf{k}_c)\hat{\mathbf{u}}(\mathbf{k}_d)). \quad (3)$$

Each non-linear term for the recipient mode $\mathbf{k}_r$ is made up of all possible combinations of $\mathbf{k}_c$ and $\mathbf{k}_d$ such that $\mathbf{k}_c + \mathbf{k}_d = \mathbf{k}_r$. For each triadic combination of wavenumbers, the nonlinear term has nine contributions corresponding to the pairs of donor and catalyst velocity components,

$$\hat{\mathbf{n}}(\mathbf{k}_c, \mathbf{k}_d) = \begin{bmatrix} \hat{n}_{uu} + \hat{n}_{vu} + \hat{n}_{wu} \\ \hat{n}_{uv} + \hat{n}_{vv} + \hat{n}_{wv} \\ \hat{n}_{uw} + \hat{n}_{vw} + \hat{n}_{ww} \end{bmatrix} = \begin{bmatrix} -ik_{t,x}\hat{u}(\mathbf{k}_c)\hat{u}(\mathbf{k}_d) - \frac{\partial}{\partial y}(\hat{v}(\mathbf{k}_c)\hat{u}(\mathbf{k}_d)) - ik_{t,z}\hat{w}(\mathbf{k}_c)\hat{u}(\mathbf{k}_d) \\ -ik_{t,x}\hat{u}(\mathbf{k}_c)\hat{v}(\mathbf{k}_d) - \frac{\partial}{\partial y}(\hat{v}(\mathbf{k}_c)\hat{v}(\mathbf{k}_d)) - ik_{t,z}\hat{w}(\mathbf{k}_c)\hat{v}(\mathbf{k}_d) \\ -ik_{t,x}\hat{u}(\mathbf{k}_c)\hat{w}(\mathbf{k}_d) - \frac{\partial}{\partial y}(\hat{v}(\mathbf{k}_c)\hat{w}(\mathbf{k}_d)) - ik_{t,z}\hat{w}(\mathbf{k}_c)\hat{w}(\mathbf{k}_d) \end{bmatrix}. \quad (4)$$

De Salis Young et al. (2024) considered different metrics to quantify the nonlinear transfer. They showed that the mean-squared value is unable to account for the misalignment and cancellation between the phasors of $\hat{\mathbf{n}}$ from different interactions into the same recipient velocity. Thus, they considered time-averaged momentum and energy transfer metrics, making use of the time-averaged projection $\mathcal{M}$ of phasor b onto phasor a ,

$$\mathcal{M}(a, b) = \Re\langle a^* b \rangle, \quad (5)$$

where $\Re$ denotes the real part. We can thus define the time-averaged change in momentum of the recipient wave,

$$\left\langle \frac{\partial}{\partial t} |\hat{\mathbf{u}}(\mathbf{k}_r)| \right\rangle = \mathcal{M} \left(e_u(\mathbf{k}_r), \frac{\partial \hat{\mathbf{u}}(\mathbf{k}_r)}{\partial t} \right), \quad (6)$$

where $e_u = \hat{\mathbf{u}}(\mathbf{k}_r)/|\hat{\mathbf{u}}(\mathbf{k}_r)|$, and its time-averaged change in energy,

$$\left\langle \frac{\partial}{\partial t} \left(\frac{1}{2} |\hat{\mathbf{u}}(\mathbf{k}_r)|^2 \right) \right\rangle = \mathcal{M} \left(\hat{\mathbf{u}}(\mathbf{k}_r), \frac{\partial \hat{\mathbf{u}}(\mathbf{k}_r)}{\partial t} \right). \quad (7)$$

The latter gives rise to the standard energy metric stemming from energy budget analysis, while the former is intrinsically connected to the momentum equation 1. For a fixed recipient mode, the two metrics produce the same results with only a scaling factor $|\hat{\mathbf{u}}(\mathbf{k}_r)|$. However, when comparing across different recipient modes, only the energy metric provides a meaningful scale for comparison. As such, we use the energy metric for the present analysis. The nonlinear energy transfer due to a triad where velocity component u_i advects u_j can be computed with $\mathcal{M}(\hat{u}_j(\mathbf{k}_r), \hat{n}_{ij}(\mathbf{k}_c, \mathbf{k}_d))$ using flow fields from existing DNS datasets.

For a given recipient wavenumber of interest, De Salis Young et al. (2024) generated 2D maps of inter-scale transfer which can be portrayed in terms of donor wavenumber $\mathbf{k}_d = [k_{d,x}, k_{d,z}]^T$, with the catalyst wavenumber given implicitly by $\mathbf{k}_c = \mathbf{k}_r - \mathbf{k}_d$, or vice versa. In wavenumber space, there is no practical difference between expressing the maps in terms of $\mathbf{k}_c$ or $\mathbf{k}_d$, as the $\mathbf{k}_c$ map is simply the $\mathbf{k}_d$ map flipped across both axes and shifted by $\mathbf{k}_r$. An example of these maps for two of the nine components, $\hat{n}_{uu}$ and $\hat{n}_{wu}$, presented in terms of catalyst and donor, is shown in Figure 2.

In contrast, when representing the maps in log-wavelength space, as commonly done for spectra and cospectra, care must be taken in the treatment of positive and negative wavenumbers. For the pair $\pm k$, spectra typically combine their contributions into a single wavelength $\lambda = 2\pi/k$, which is appropriate since physical waves arise from the combination of both wavenumbers. However, as shown in Figure 2, the four quadrants of the interscale map are not symmetric, and such merging may lead to a loss of distinction in value and even cancellation between asymmetric contributions, obscuring the significance of contributions in different quadrants.

The asymmetry occurs because points on each of the quadrants of the map represent a distinct triadic interaction, as illustrated in figure 3(a). For example, the interaction of $\mathbf{k}_d = [2, 3]$ into $\mathbf{k}_r = [6, 4]$ requires a catalyst of $\mathbf{k}_c = [4, 1]$, while if the donor mode is $\mathbf{k}_d = [-2, 3]$, the catalyst is $\mathbf{k}_c = [8, 1]$. These two interactions, along with the corresponding ones in the two other quadrants, have donors and

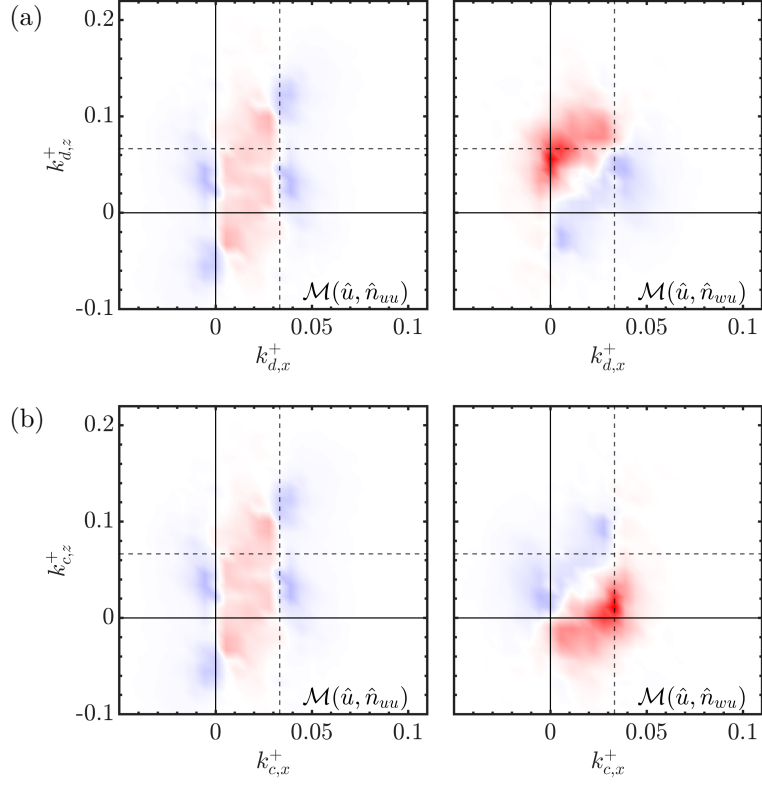


Figure 2: Maps of inter-scale energy transfer in (a) donor and (b) catalyst wavenumbers to recipient lengthscale $\lambda_{t,x}^+ = 188, \lambda_{t,z}^+ = 94$ at $y^+ = 15$ from the uu and wu contributions to the advection term for an $Re_\tau = 180$ channel flow. The colour scale in wall units is from -0.1 (blue) to 0.1 (red). The black dashed lines indicate the recipient wavenumber k_r .

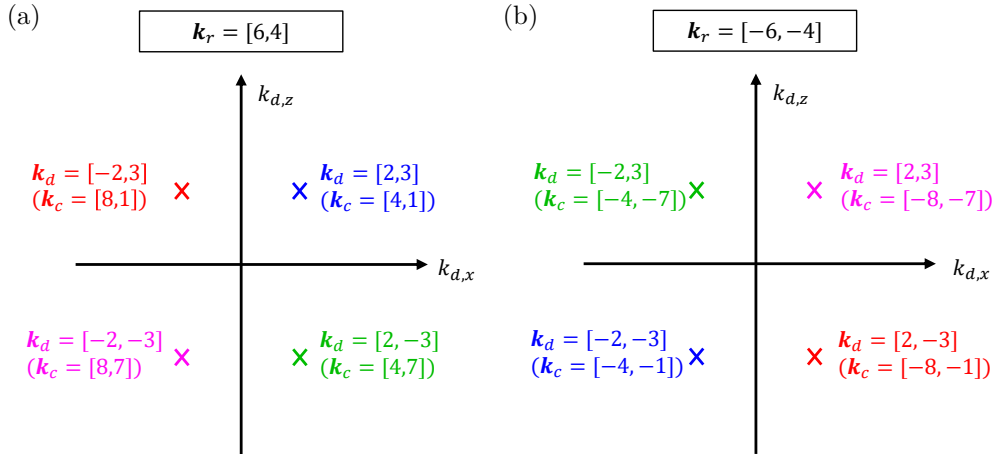


Figure 3: Illustration of triadic interactions represented in each quadrant of the maps for two recipient modes with opposite signs. Modes of the same colour represent interactions between the same lengthscales, with the only difference being the sign of the wavenumber.

recipients of equal wavelengths, but involve catalysts of different wavelengths, thus corresponding to four physically different mechanisms. Interactions between physical waves, comprising both positive and negative wavenumbers, which justifies the combination of the quadrants in a typical spectrum, arise when the sign change is applied consistently to both $\mathbf{k}_d$ and $\mathbf{k}_r$. Comparing between figures 3(a) and (b), which have recipient modes of opposite sign, we notice interactions that include modes of the same wavelengths (coloured the same) but shown in different quadrants. These two maps can thus be combined after rotating one by 180° without the loss of information, and this can similarly be done for maps of $\mathbf{k}_r = [-6, 4]$ and $[6, -4]$. Thus, only one map is required for each recipient lengthscale, but it must retain all four quadrants, with each converted separately into log-wavelength space. In our maps, we omit the negative signs in wavelengths for simplicity, and the axis directions are reversed to preserve the visual layout of four quadrants, as illustrated in figure 4. For maps given in terms of catalyst wavelengths $\lambda_{c,i}$, like the ones in figure 4(b), the corresponding $\lambda_{d,i}$ is given by

$$\lambda_{d,i} = \frac{1}{|1 \mp \lambda_{r,i}/\lambda_{c,i}|} \lambda_{r,i}, \quad (8)$$

where i refers to either x or z and the sign in the denominator depends on the quadrant for $\lambda_{c,i}$. In turn, for maps given in terms of donor wavelengths the corresponding $\lambda_{c,i}$ is given by

$$\lambda_{c,i} = \frac{1}{|1 \mp \lambda_{r,i}/\lambda_{d,i}|} \lambda_{r,i}. \quad (9)$$

The construction of log-wavelength maps also involves premultiplication by $\mathbf{k}_c$ or $\mathbf{k}_d$, which makes maps expressed in terms of the catalyst or the donor wavelength provide different information. Circled regions in the figure correspond to the same interactions, and illustrate how the different pre-multiplications undergone can result in very different magnitudes. We note in particular that the maps for the same velocity component in catalyst and donor, $\partial(u_i u_i)/\partial x_i$, are visually the same, since $\hat{\nabla}(\hat{u}(\mathbf{k}_c)\hat{u}(\mathbf{k}_d)) = \hat{\nabla}(\hat{u}(\mathbf{k}_d)\hat{u}(\mathbf{k}_c))$. However, the same points on the maps do not correspond to the same interaction; instead, there is a swap in the roles of donor and catalyst. Altogether, the maps in figure 4(a) and (b) can be considered to contain complementary information.

2.2 Reduction of data size with spectral binning

To identify the key interactions throughout the channel, maps are required for recipient wavenumbers spanning the full DNS-resolved range and extending across the entire channel height. However, this full set for a channel with $Re_\tau \approx 180$ would produce a dataset roughly 20,000 times the size of one instantaneous flow field for one velocity component. To make the analysis plausible, we apply the spectral binning procedure of Jiménez et al. (2010), in which the information for groups of high-wavenumber modes, forming the densely populated region at short wavelengths in a logwavelength spectrum, is combined as illustrated in figure 5. The values within each bin are averaged and represented at the geometric centre of the interval, defined as $k = \sqrt{k_1 k_2}$ in one dimension, where k_1 and k_2 denote the start and end wavenumbers of the bin, respectively.

Similar procedures are applied to reduce the number of recipient modes for which maps are constructed. Since all modes that act as catalyst or donor modes in a map may also serve as recipient modes, the black points in figure 5 also represent the full range of recipient modes considered. We select the recipient modes closest to the geometric centre of each bin for map construction and assume that the nonlinear interactions into all recipient modes within a bin are comparable to that of the selected mode. In the wall-normal direction, rather than computing maps across the entire DNS y grid, a reduced subset of y^+ locations is chosen to generate representative maps that capture the flow dynamics in the key regions of the channel: $y^+ = 5, 10, 15, 25, 40, 60, 90, 130$ and 180 .

Overall, this procedure reduces the dataset size for the maps to roughly the equivalent of 10 flow fields, making the analysis far more tractable. This number scales approximately with Re_τ^2 , depending on the number of wavenumbers after binning, which does not scale linearly with total wavenumbers in each direction. For $Re_\tau \approx 550$, for example, the full dataset would be around 200,000 times the size of one flow field, and the reduced one based on the above binning method would be around 100 times the size of one flow field.

2.3 Scaling across recipient modes and heights

To enable meaningful comparison between maps corresponding to different recipient wavenumbers at a given wall-normal plane, an appropriate overarching scaling is required. The energy metric of equation

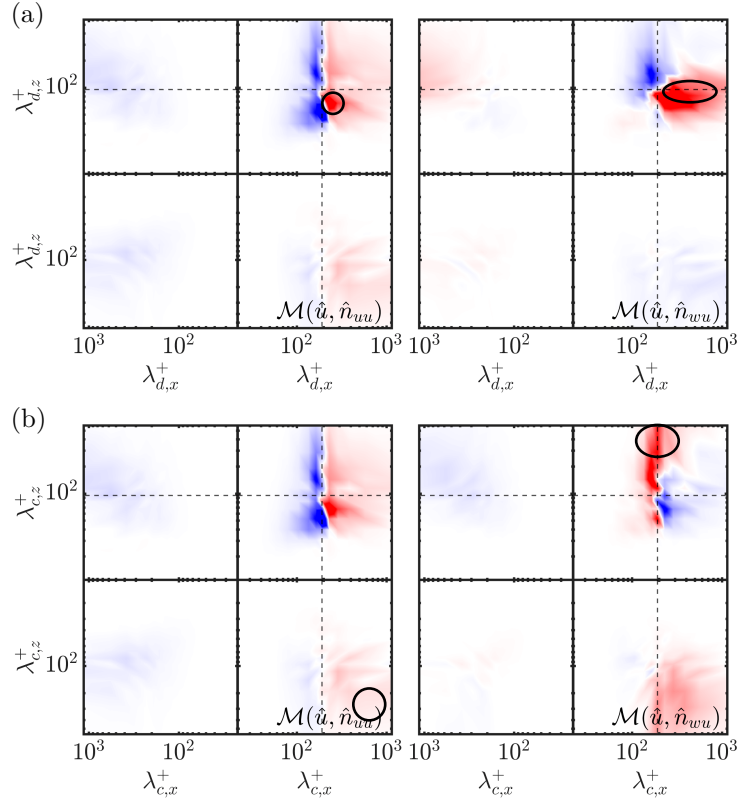


Figure 4: Log- λ maps of inter-scale energy transfer in (a) donor and (b) catalyst modes to recipient lengthscale $\lambda_{t,x}^+ = 188, \lambda_{t,z}^+ = 94$ at $y^+ = 15$ from the uu and wu contributions to the advection term for an $Re_\tau = 180$ channel flow. The colour scale in wall units is from -2×10^{-5} (blue) to 2×10^{-5} (red), scales based on methods detailed in §2.3. The black dashed lines indicate the recipient wavenumber k_r . Circled regions in black in the corresponding maps of (a) and (b) represent the same interactions.

6 enables integration over all nine contributions ($\partial_i(u_i u_j)$) of each map to yield the total nonlinear energy transfer into the corresponding recipient wavenumber – note that a similar summation property is not available using the momentum metric of equation 7. Figure 6(a) shows, for $y^+ = 15$, the energy transferred into each recipient lengthscale, where each data point represents the integral of the maps for one given recipient wavenumber. Since this spectral map is itself presented in logwavelength space, each data point is premultiplied by its associated recipient wavenumber. This premultiplication provides the necessary scaling for comparing results across maps for different target wavelengths. Overall, each map then requires two stages of premultiplication: first by the catalyst or donor wavenumbers of the map axes, to portray the map in logwavelength space, and second by the recipient wavenumber, for consistent scaling across the whole set of recipients. We note that the latter does not alter the appearance of the maps, e.g. those portrayed in figure 4, but allows for quantitative comparison of maps for different recipient wavelengths. We also note that the procedure is equivalent to that used in Ding et al. (2025) to produce maps of energy transfer into each recipient wavelength integrated in y across the whole channel. Our results for e.g. figure 6(a) collapse with theirs once integrated in y , as shown in figure 6(b), validating the present mapping procedure.

Integrating maps like that in figure 6(a) across all recipient wavelengths yields the total nonlinear energy transfer at any given height y , and performing the same procedure across the entire y range produces figure 7. The intensity of these net nonlinear interactions decreases markedly with distance from the wall. The relative importance of advection and nonlinear transfer, however, increases far away from the wall, where viscous effects are negligible. Consequently, y -scaling is necessary to ensure that the significance of interscale transfer in the channel core is accurately represented. The total nonlinear energy transfer at each height cannot be used directly for this scaling, as it reverses sign and more importantly becomes zero at some heights. We therefore use the total positive energy transfer, obtained by summing only the positive contributions to the energy transferred into the recipient modes, such as the region in

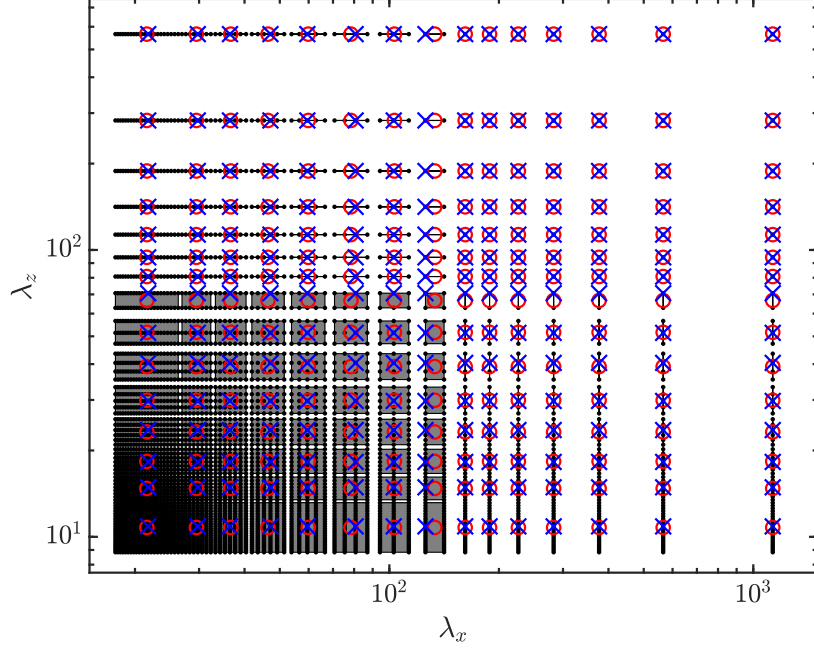


Figure 5: Spectral binning procedure illustrated in $\log \lambda$ space. Black points indicate the computed modes, and grey regions represent the selected bins. Red circles mark the geometric centres where the averaged data are stored, while the blue cross denotes the mode closest to each geometric centre.

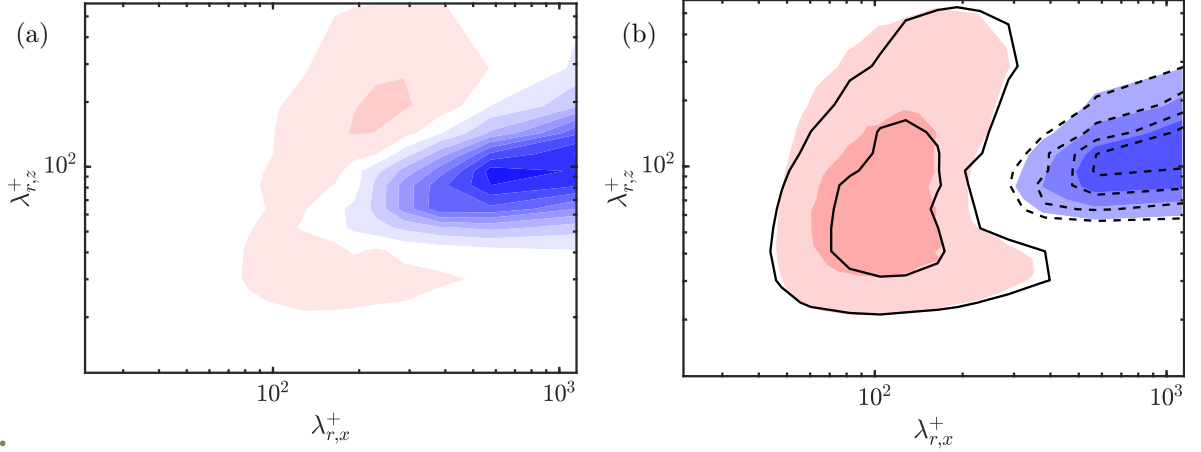


Figure 6: (a) Total nonlinear energy transferred ($\hat{N}$) into each recipient wavelength at $y^+ = 15$. Coloured contours from blue to red are from -0.05 to 0.05, with increments of 0.005, in viscous units. (b) Total nonlinear energy transferred into each recipient wavelength integrated in y across the channel. Solid and dashed contours are results for the present channel, and coloured ones data from Ding et al. (2025). Contour levels are from -1.4 to 1.4 with increments of 0.2.

red in figure 6(a).

2.4 Reduced order simulations

We next conduct reduced-order simulations in which only the nonlinear interactions identified as important from our maps are retained. If turbulence is sustained and remains essentially the same as in a full DNS, this would suggest that the selected interactions are indeed physically meaningful.

We conduct reduced order simulations at $\text{Re}_\tau \approx 180$ by applying a constant mean pressure gradient.

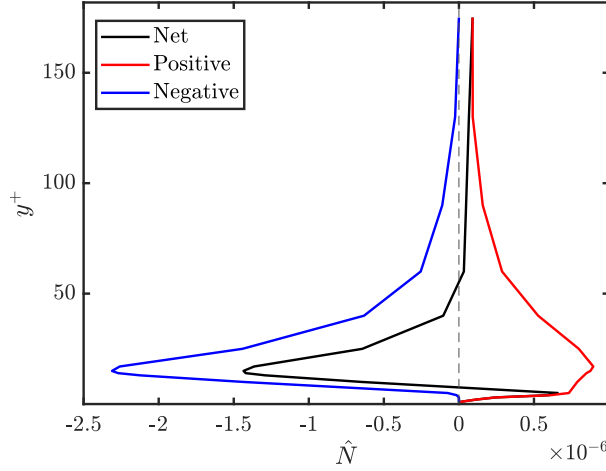


Figure 7: Total nonlinear energy transferred at each y^+ . Red and blue lines are the total positive and negative energy transferred from integrating only the red or blue parts of 6(a).

The channel is of size $2\pi\delta \times \pi\delta \times \delta$ in the streamwise, spanwise and wall-normal direction. The numerical code is adapted from Fairhall & García-Mayoral (2018), in which the three-dimensional incompressible NavierStokes equations are solved using a spectral discretisation in the streamwise and spanwise directions, while the wall-normal direction is discretised using second-order finite differences on a staggered grid. The wall normal grid is stretched with $\Delta y_{min}^+ \approx 0.3$ near the wall and $\Delta y_{max}^+ \approx 3$ in the centre of the channel. The wall-parallel resolutions are $\Delta x^+ \approx 6$ and $\Delta z^+ \approx 3$. The temporal integration uses a RungeKutta discretisation, where pressure is corrected to reinforce incompressibility. Every time step is divided into three substeps, each of which uses a semi-implicit scheme for the viscous terms and an explicit scheme for the advective terms. Once the flow reaches a statistically steady state, statistics are collected for at least $10\delta/u_\tau$.

In a standard DNS simulation, the nonlinear term is computed by transforming the velocity fields to physical space via Fourier transforms, multiplying the components, and transforming back. To manage each triadic interaction individually, we instead compute the nonlinear term in Fourier space using a convolution, which allows calculation of only a selected set of wavenumber interactions from the maps obtained above.

The set of interactions is chosen by applying a threshold on the nonlinear energy transferred, such that only interactions with energy transfer above this threshold are retained. Since maps based on catalyst and donor wavelengths contain different information, all interactions above the threshold in either map are included. By varying the selection threshold, we can vary the percentage of interactions retained relative to a full DNS. In this study, simulations were conducted with 15%, 30%, 45% and 60% of all interactions.

The premultiplication of the modes with $k_x = 0$ or $k_z = 0$, i.e., the modes with infinite wavelength in either the streamwise or spanwise direction, requires careful consideration. Due to the limited size of our domain, $L_x = 2\pi$ and $L_z = \pi$, energy in wavelengths larger than the domain is transferred to the corresponding $k_x = 0$ or $k_z = 0$ modes (Lozano-Durán & Jiménez 2014). In a larger channel, where these modes were resolved, the interscale maps would include their wavelengths and the corresponding premultiplication factors would be straightforward. However, in the present domain, all interactions involving the zero modes are, in principle, premultiplied by zero, effectively rendering their contribution negligible. Some of these interactions, however, may be critical for sustaining turbulence.

We leave for future work to perform this analysis in larger domains, where all relevant modes are sufficiently resolved and all relevant interactions have non-zero premultiplication. As a preliminary approach, we instead premultiply these modes by $k_1/2$, where k_1 is the first non-zero mode, rather than by 0. Our reasoning is that, considering a channel size twice as large as the present one, some of the energy that is in $k = 0$ in the present channel would become resolved into wavenumber $k = k_1/2$, while some remained in $k = 0$. Premultiplying all this energy by $k_1/2$ could then result in us keeping unimportant interactions, but not in us neglecting some important ones by accident. We note, however, that a more conservative approach would entail premultiplying by k_1 as, for channels of any size larger than the present one, k_1 would be the overall upper bound for the wavenumber of modes with wavelengths larger than λ_1 .

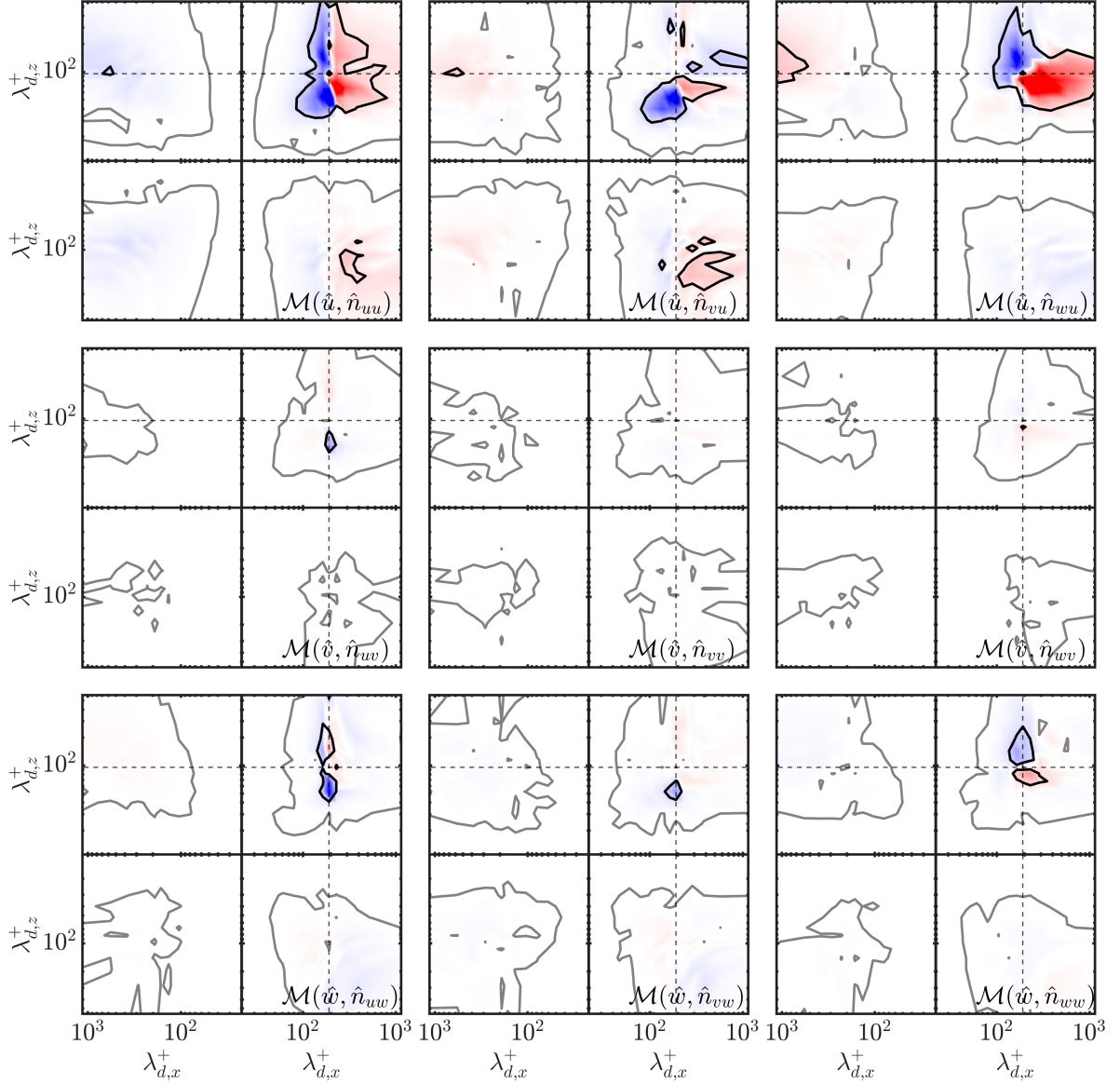


Figure 8: Maps of inter-scale energy transfer in donor and log-wavelength space to recipient lengthscale $\lambda_{t,x}^+ = 188, \lambda_{t,z}^+ = 94$ at $y^+ = 15$ from the nine contributions to the advection term for an $Re_\tau = 180$ channel flow. The colour scale in wall units is from -5×10^{-5} (blue) to 5×10^{-5} (red). The black dashed lines indicate the recipient wavenumber $\mathbf{k}_r$. Black and grey contour lines represent interactions selected if 0.5% and 15% of all interactions are kept, respectively.

3 Results

3.1 Nonlinear energy transfer maps

De Salis Young et al. (2024) investigated the significant interactions into the recipient lengthscale of $\lambda_{r,x}^+ = 188, \lambda_{r,z}^+ = 94$ at $y^+ = 15$ as a representative scale of the near-wall cycle, using maps in wavenumber space such as those in figure 2. Their results suggested that the dominating mechanism of interscale transfer was the streak meandering, where spanwise-velocity (w) structures with a streamwise length similar to the size of the recipient mode but much longer in span advected streamwise-velocity (u) structures with a spanwise length similar to the size of the recipient mode but much longer in x , i.e. the streaks, resulting in a transfer of energy into u in the recipient mode and causing the initially- x -elongated streaks to become sinuous. This interaction also appears significant in the present premultiplied maps in log-wavelength space, i.e. in the panel for $\mathcal{M}(\hat{u}, \hat{n}_{wu})$ in figures 8 and 9. However, the new maps

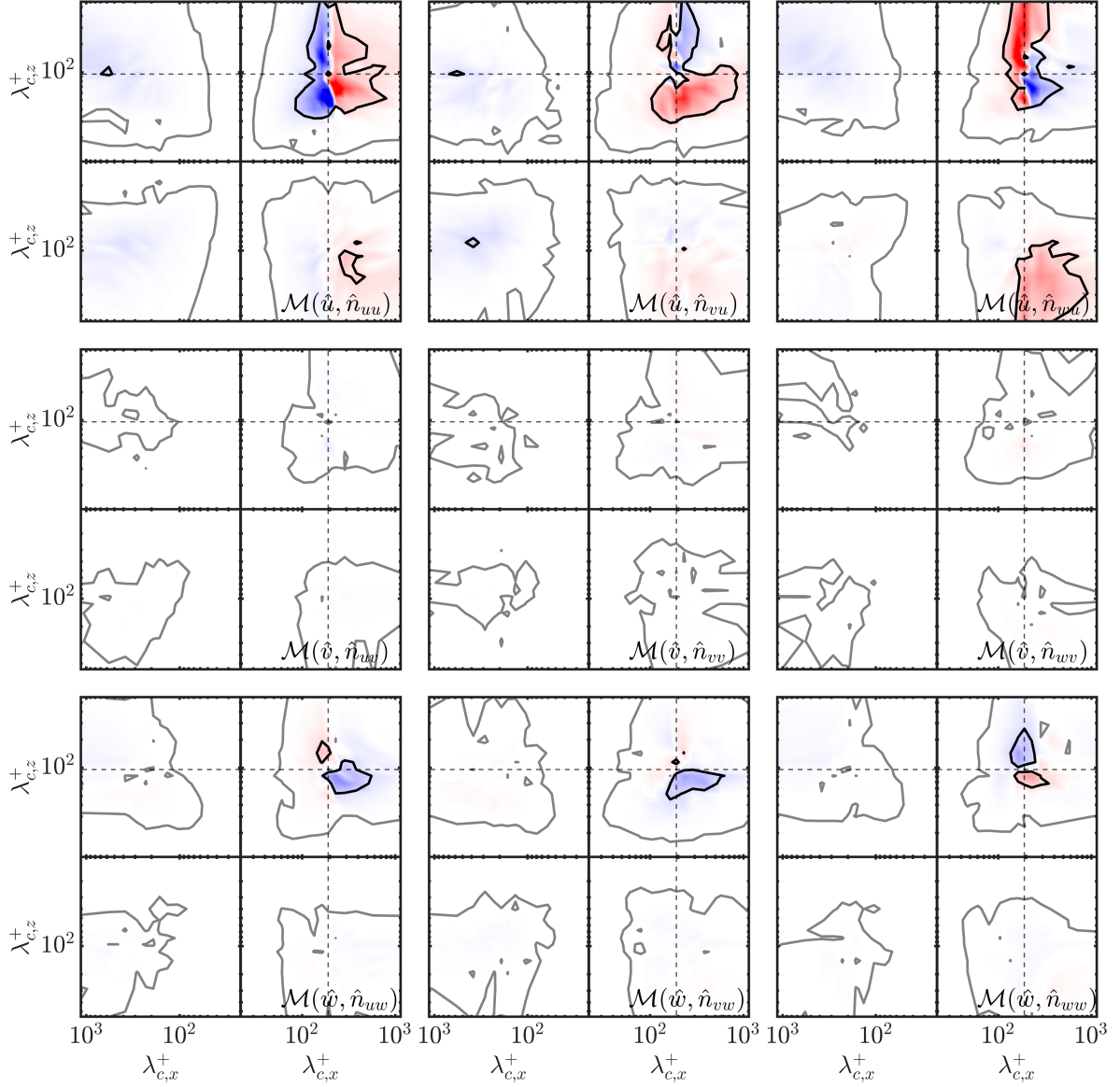


Figure 9: Maps of inter-scale energy transfer in catalyst log-wavelength space, parameters of the graph are the same as those in Figure 8.

highlight additional interactions that are similarly significant, such as the intense regions in the panels for $\mathcal{M}(\hat{u}, \hat{n}_{uu})$ and $\mathcal{M}(\hat{u}, \hat{n}_{vu})$.

Across both donor and catalyst maps, the most significant interactions are concentrated in the first quadrant, roughly centred around the recipient wavelength and, in some cases, extending towards infinitely large ones. In terms of wavenumbers, these dominant regions lie approximately at $\mathbf{k} = \mathbf{k}_r \pm \mathbf{k}_r$. Consequently, the third mode in the triad spans from $\mathbf{k} = 0$ to $\mathbf{k} = \pm \mathbf{k}_r$ ($\lambda_r \lesssim \lambda \lesssim \infty$), which explains the moderate energy levels observed in these regions. This indicates that interactions between scales comparable to and larger than the recipient, regardless of which acts as donor and which as catalyst, are generally the most significant contributors to energy transfer. Such behaviour is consistently observed across different recipient modes. This trend aligns with the general depiction of the energy cascade as going from larger to smaller scales, though the presence of substantial negative energy transfer suggests a non-negligible backscatter from the recipients to larger scales.

The black solid contour in the maps in figures 8 and 9 encloses the regions of most intense energy transfer. When this threshold is applied consistently across all maps at different wall-normal positions

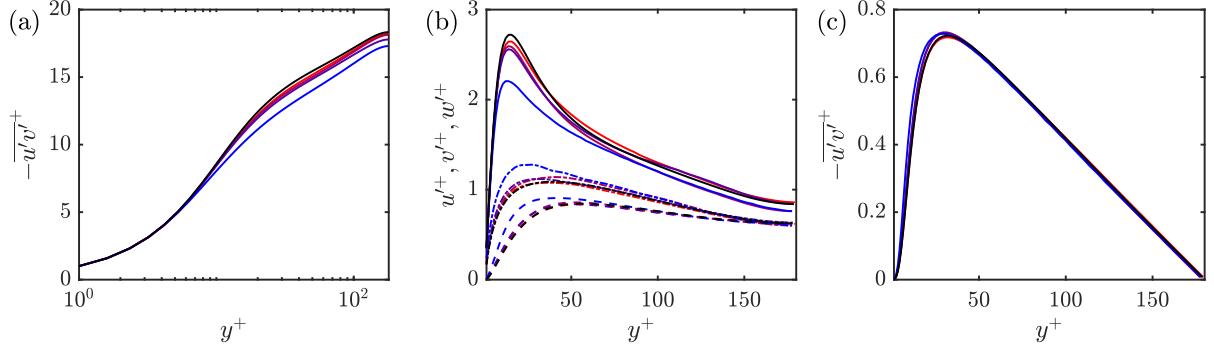


Figure 10: Comparison of (a) mean flow profile, (b) rms of velocity fluctuation and (c) Reynolds stress of reduced order simulations. Black: full DNS, blue to red: reduced order simulations computing 15% to 60% of all interactions.

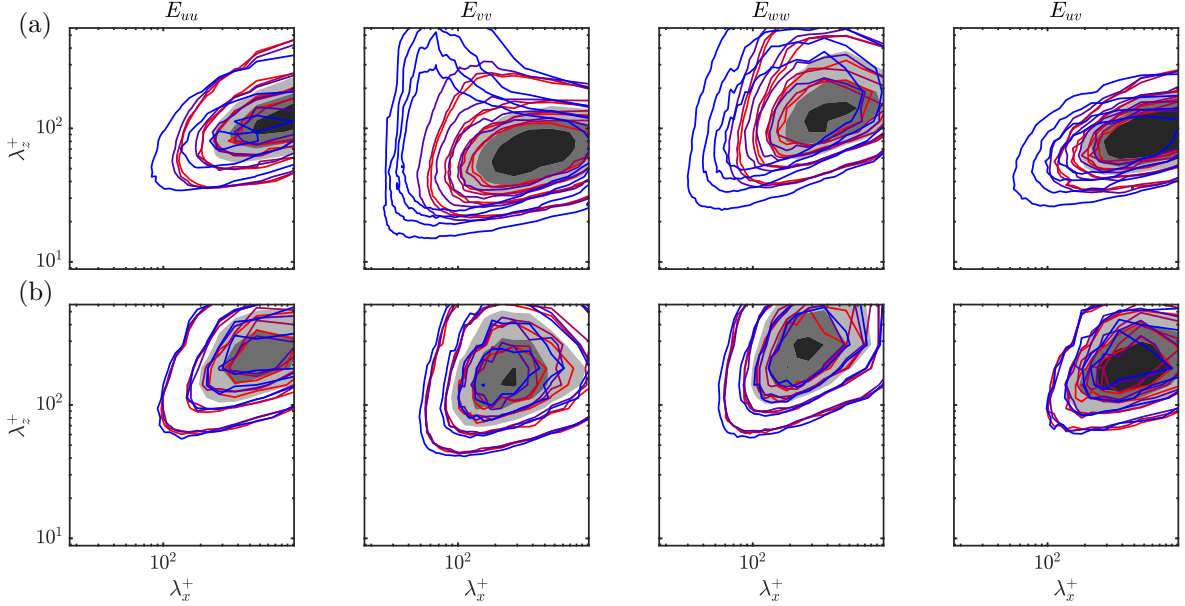


Figure 11: Comparison of the spectral energy densities of the three velocities and Reynolds stress at (a) $y^+ = 15$ and (b) $y^+ = 100$. Filled contours are from full DNS, contour line colours are as in figure 10. Contour levels are the same across the different cases.

and recipient scales as detailed in § 2.3, the enclosed region contains roughly 0.5% of all interactions. We would expect that retaining these modes alone would be sufficient to sustain turbulence, but as shown later, a substantially larger fraction is required. For illustration, the larger region delimited by the grey contour line contains roughly 15% of all interactions, which is not yet sufficient to sustain healthy turbulence. Note that the above 0.5% and 15% apply to the interactions across the whole channel; the percentage in any specific plane can vary. Note also that the percentage number of modal interactions retained is not necessarily the same as the percentage area enclosed in the map, as in logarithmic maps modes are more widely spaced for higher wavelengths.

3.2 Reduced order simulations

Results for the reduced order simulations, retaining from 15% to 60% of interactions compared to regular DNS, are presented below. The simulations are conducted following the methods detailed in § 2.4.

The turbulent statistics from the reduced order simulations are compared with the full DNS simulation in figure 10. The mean velocity profile and the Reynolds shear stress across all reduced-order simulations align well with the regular DNS results, suggesting that this modelling approach may be of use to predict

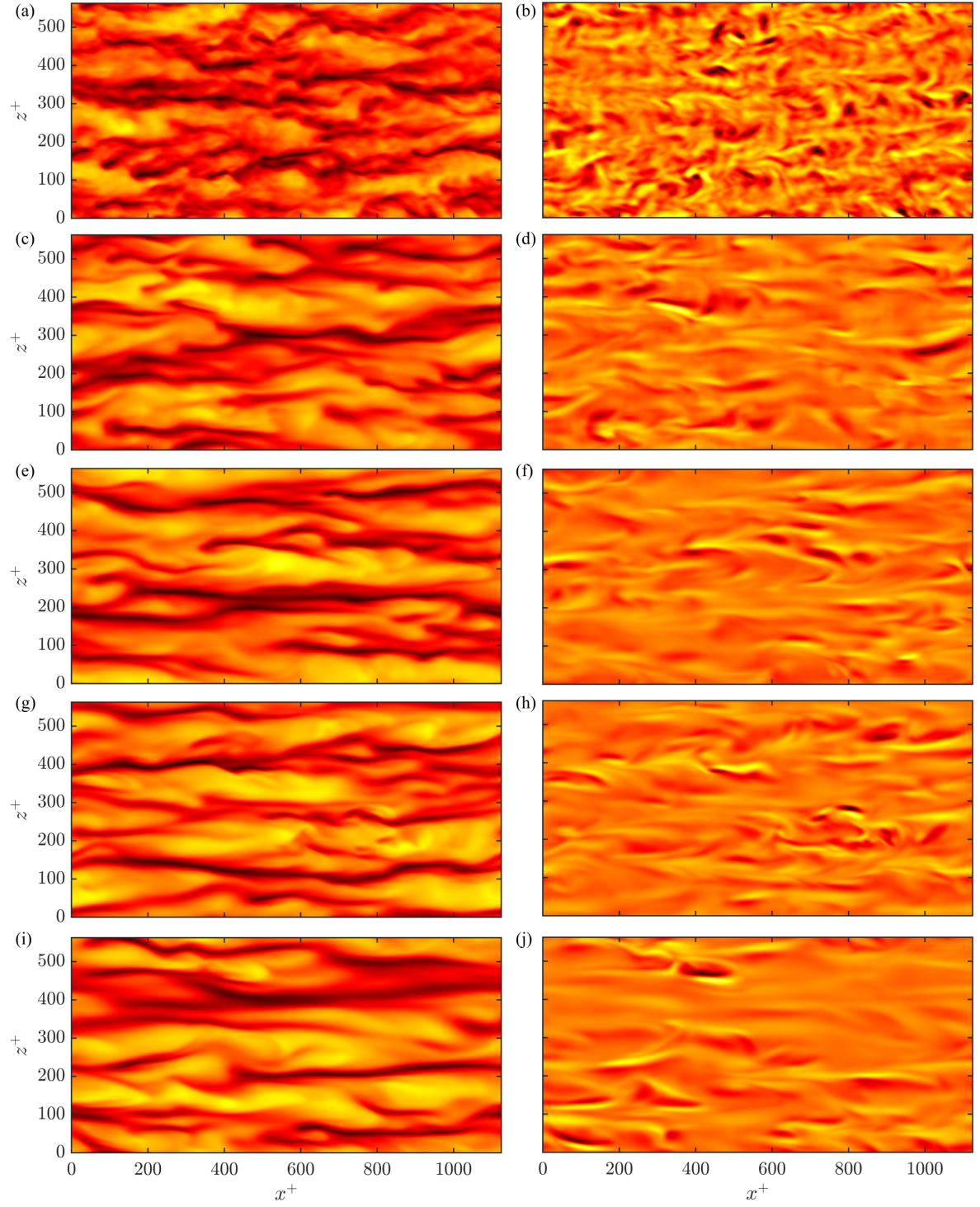


Figure 12: Instantaneous realisation of streamwise u^+ (left column) and wall-normal v^+ (right column) velocity at $y^+ = 15$. (a-h), reduced order simulations, from top to bottom with 15%, 30%, 45% and 60% of all interactions; (i-j), standard DNS. Colours from dark to clear are from 4 to 20 for u^+ and from -3 to 3 for v^+ .

bulk flow properties. For the velocity fluctuations, however, simulations retaining 30% of the interactions or more show good agreement with the full DNS, but for 15% significant deviations appear. Attempts to use a lower percentage, e.g. 5%, resulted in the simulations diverging, and are thus not presented in the figure.

Premultiplied energy spectra are portrayed at $y^+ = 15$ and $y^+ = 100$ in figure 11. Other than for the 15% case at $y^+ = 15$, there is a good collapse in the regions of high spectral density for all variables. The reduced-order simulations have however higher intensity than the full DNS, which appears not to be consistent with the rms values portrayed in figure 10. The reason is that the streamwise and spanwise infinite modes, which are omitted in the spectra, include significantly less energy than in the full DNS, resulting overall in a smaller change in the 1D statistics. This likely occurs due to our treatment of interactions involving modes with k_x or $k_z = 0$, as discussed in § 2.4; further investigation into the strategy for selecting which of these modes to retain is left for future work. For the simulation retaining 15% of the interactions, more significant deviations are observed at $y^+ = 15$ across all the spectra. There is an overall shift toward shorter λ_x , and for E_{vv} also a widening in the spanwise direction, a feature reminiscent of flows over textured surfaces such as riblets, rough or porous substrates (Abderrahaman-Elena et al. 2019, García-Mayoral & Jiménez 2011, Gómez-de-Segura & García-Mayoral 2019, Hao & García-Mayoral 2025). At $y^+ = 100$, in contrast, good agreement with the full DNS is observed across all reduced-order simulations in terms of characteristic lengthscales. The excess of small-scale energy near the wall may be associated with excessive nonlinear energy transfer into the overexcited modes, coupled with insufficient transfer away from them, resulting in spurious energy accumulation. This suggests that the omitted interactions would have a non-negligible net dissipative contribution into these modes, and that omitting them results in an energy imbalance for these recipient wavelengths. We leave for future investigation the task of ensuring net nonlinear energy balance for these modes.

The agreement or disagreement between reduced-order simulations and full DNS observed at $y^+ = 15$ in the spectra can also be observed in instantaneous flow realisations, as shown in figure 12. The flow fields of simulations retaining 30% or more of the interactions are visually similar to those of the full DNS. For the simulation retaining 15%, however, the typical streamwise-elongated signatures in u and v are disrupted and shortened, particularly for v , for which they also become somewhat spanwise elongated at small λ_x , in agreement with the observation on the E_{vv} spectrum in figure 11(a).

Overall, the simulations with reduced interactions successfully reproduce the key dynamics of turbulent flow. However, even at 15%, a substantial number of interactions that appear weak or negligible in the maps of figures 8 and 9 are still included. This suggests that the current thresholding method is far from yielding the truly minimal set of essential interactions. The present work supports the hypothesis that only a fraction of all the possible interscale interactions need to be retained to sustain healthy turbulence, but further investigation is needed to systematically reduce the percentage of interactions retained, and to better assess the efficiency and usefulness of the proposed framework for modelling purposes.

4 Conclusions

In this work, we have proposed a framework for the analysis of interscale interactions in wall turbulence by focusing on how information is transferred between x - and z -lengthscales in the Navier-Stokes momentum equations. Inter-scale transfer is possible only through the non-linear advection terms, in which velocity fluctuations of lengthscale $\mathbf{k}_c$ advect velocities of lengthscale $\mathbf{k}_d$ to cause a transfer of energy to a recipient lengthscale $\mathbf{k}_r$. There are three catalyst and donor velocity components, which correspond to nine contributions to the advection term. Given the high anisotropy of wall turbulence, the proposed framework discriminates between these nine contributions as representing different physical mechanisms even within the same triadic set. We propose to use this framework to identify the dominant advection mechanisms through which energy is transferred between different lengthscales.

To identify and quantify the intensity of the dominant interactions governing the turbulent energy cascade, we have generated maps for the time-averaged interscale energy transfer across the entire channel. The validity of this framework was assessed through reduced-order simulations, in which the nonlinear term was computed in Fourier space via convolutions but restricted to only the selected interactions. For cases retaining 30% or more of the total number of interactions, the resulting turbulent statistics match well those from full DNS. Noticeable deviations appear near the wall when only 15% of the interactions are retained, and further reduction leads to numerical instability. The 15% simulation exhibited an accumulation of energy in short streamwise wavelengths, suggesting that the reduced set of interactions causes a substantial alteration in the nonlinear energy transfer into those modes.

While the good representation of turbulence with 30% of the interactions is a promising result, many apparently weak interactions are still included, which suggests that efforts are needed for further minimising the retained interactions and isolating the essential mechanisms of energy transfer in wall-bounded

turbulence.

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