COPRODUCTS INTERNAL TO PROFINITE SPACES

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Abstract. We give a categorical explanation for many properties of profinite coproducts of profinite groups, which were previously proven on a case-by-case basis. All of these properties take the form "certain functors preserve profinite coproducts". We give various examples to show how our framework can be applied. We also point out connections to internal category theory and profinite Bass–Serre theory.

1. Introduction

The category **PGrp** of profinite groups has all small coproducts, but they are poorly behaved and difficult to use in practice. On the other hand, [1], [2] and [3] develop a notion of coproducts of profinite groups (or modules) which are indexed not by a set, but by a profinite space itself. This has numerous applications, as illustrated in the previously cited papers. There has been recent work, such as [4], [5] and [6], which presents more categorical approaches to study this notion of topological coproducts. However, there are still many results on these topological coproducts in the literature which are proven on a case-by-case basis and do not yet seem to have any satisfactory categorical explanation. To demonstrate our point, consider the following. Let $\coprod_{x\in X} G_x$ be the "topological coproduct" of profinite groups G_x indexed by a profinite space X, and let $\widehat{\bigoplus}_{x\in X} M_x$ be the corresponding notion for profinite abelian groups, or more generally, profinite modules M_x over a fixed profinite ring R. These will be defined properly later (see Example 2.1). We now state a few facts about these coproducts and put what the right categorical explanation should be in brackets at the end.

- (Example 3.11(iv)) If G^{ab} denotes the abelianisation of a profinite group G, then
$$\left(\coprod_{x\in X} G_x\right)^{\mathrm{ab}} = \widehat{\bigoplus}_{x\in X}(G_x^{\mathrm{ab}}).$$
In particular, if I is a set, then the abelianisation of the free profinite group on I (converging to 1) is the free proabelian group on I (converging to 1).
(Abelianisation commutes with coproducts.)
- ([3, Theorem 9.1.1]) If N is a profinite left R-module, then
$$\mathrm{Tor}_i^R\left(\widehat{\bigoplus}_{x\in X} M_x, N\right) = \widehat{\bigoplus}_{x\in X}\mathrm{Tor}_i^R(M_x, N).$$
(Tor commutes with coproducts.)
- ([7, Proposition A.14]) If $f\colon Y \twoheadrightarrow X$ is a surjection of profinite spaces, then
$$R[\![Y]\!] = \widehat{\bigoplus}_{x\in X} R[\![f^{-1}(x)]\!].$$
(The free profinite R-modules functor $R[\![-]\!]$ commutes with coproducts.)

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Main Aim. The aim of this paper is to provide, in Theorem 3.8, a general categorical framework that explains all of the above and more. In fact, [7, Proposition A.13] already suggests that categorical explanations should exist for the above facts, and our main theorem can be viewed as a generalisation of [7, Proposition A.13]. Informally, we will show that many naturally occurring functors in the profinite setting (many of which are automatically left adjoints) commute with these topological coproducts. These coproducts have various existing applications already: [3, Theorem 9.4.1] uses them to construct a Mayer–Vietoris sequence for profinite groups acting on profinite trees, [6, Theorem 2.19] uses them to prove Mackey's Formula for general profinite groups, and so on. All of the stated results mimic classical results involving ordinary groups and modules, but use this notion of topological coproducts in place of ordinary coproducts. The generality of our main theorem (Theorem 3.8), which we will illustrate in Example 3.11, thus provides a concrete strategy to transport classical results of categorical flavours into the profinite world.

Outline of Paper. The main body of this paper is divided into Sections 2 and 3, which are somewhat disjoint. Readers who are only interested in the main theorem mentioned above could therefore skip Section 2 altogether. In Section 2.1, we recall the definition of an internal category (specifically internal to **Pro**, the category of profinite spaces), as it turns out that the topological coproducts mentioned above are, in a precise sense, a kind of internal coproducts (Example 2.1). More generally, we can define colimits internal to **Pro**, which are constructed from coproducts and coequalisers (Theorem 2.3). We will see, in Section 2.2, that internal pushouts of profinite groups in our sense are the profinite fundamental groups of certain graphs of groups, in exactly the same way as classical Bass–Serre theory.

In Section 3, we focus on internal coproducts (rather than all colimits) and study when they are preserved by functors. The conditions we need are mild (Proposition 3.1) and are often automatic, since many naturally occurring functors in the profinite setting arise from *relative adjunctions* (Lemma 3.3 and Example 3.4). In this case, we get more than just the preservation of coproducts (Theorem 3.8). We will end the paper with Example 3.11, which shows that there is an abundance of situations where our theorem applies. In the appendix, we will briefly discuss Pontryagin duality in our context and show that the (seemingly different) perspectives of [1], [2], [4], [5] and ours are equivalent.

We will assume basic knowledge of category theory (refer to [8]) and profinite groups (refer to [9]). Section 2 will require knowledge of internal category theory (refer to [10]) and profinite graphs (refer to [3]), while Section 3 will require knowledge of pro-completions (refer to [11]).

Remark: When we write "=" in this paper, we usually mean canonically isomorphic (or equivalent in the case of categories).

Convention: Unless otherwise specified, all rings are associative with a 1 but not necessarily commutative.

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2. Categories Internal to **Pro**

In this section, we will recall the concept of internal categories, following mostly the notations of [10, Chapter 8]. We will see that the topological coproducts of profinite groups appearing in the introduction can be described as coproducts internal to **Pro** (Example 2.1). More generally, colimits internal to **Pro** can be constructed from coproducts and coequalisers (Theorem 2.3), as one would expect. In Section 2.2, we will show that our notion of colimits internal to **Pro** in fact occurs naturally in the context of profinite Bass–Serre theory.

2.1. **Internal Categories and Colimits.** We recall the following special case of [10, Chapter 8]. Let **Pro** denote the category of profinite spaces (and continuous maps) and let A be a category internal to **Pro**. Spelt out, this means that there are two spaces $A_0, A_1 \in \mathbf{Pro}$ and maps $d_0, d_1 \colon A_1 \to A_0$, $i \colon A_0 \to A_1$, $c \colon A_1 \times_{A_0} A_1 \to A_1$ in **Pro**, such that the underlying sets and functions describe an ordinary category. One can then define functors and natural transformations internal to **Pro**. Let $\mathbf{Cat}(\mathbf{Pro})$ denote the ordinary 1-category of categories and functors internal to **Pro**.

Even though **Pro** is not a category internal to itself, it makes sense to talk about internal functors from an internal category A to **Pro**. Spelt out, an internal functor $P \colon A \to \mathbf{Pro}$ consists of a space $P_0 \in \mathbf{Pro}$ and maps $p_0 \colon P_0 \to A_0$, $p_1 \colon A_1 \times_{A_0} P_0 \to P_0$, such that the underlying sets and functions describe an ordinary functor $A \to \mathbf{Set}$ which sends an object $a \in A_0$ to the set $P(a) = p_0^{-1}(\{a\})$, and which sends a map $f \colon a \to b$ in A_1 to the function of sets $P(f) \colon P(a) \to P(b)$, $x \mapsto p_1(f, x)$. Given two such functors $P, Q \colon A \to \mathbf{Pro}$, an internal natural transformation is a map $\alpha \colon P_0 \to Q_0$ in **Pro**, such that the underlying sets and functions describe an ordinary natural transformation. Internal functors from A to **Pro** and their natural transformations form an ordinary category, which we denote by $\mathrm{IFun}(A, \mathbf{Pro})$. The reader should consult [10] for the omitted details.

There is an ordinary functor $\Delta_A \colon \mathbf{Pro} \to \mathrm{IFun}(A, \mathbf{Pro})$ which sends $X \in \mathbf{Pro}$ to the constant internal functor $\Delta_A(X)_0 = X \times A_0 \to A_0$. As **Pro** has finite limits and coequalisers, it is easy to show that Δ_A has a left adjoint for every $A \in \mathbf{Cat}(\mathbf{Pro})$ i.e. that **Pro** has all internal colimits (see [10, Proposition 8.3.2]). We point out that **Pro** does not have all internal limits, since it is not Cartesian closed (see [10, Proposition 8.3.5]).

Let us forget that we are dealing with internal categories for a moment and think about **Set**. Given any $A \in \mathbf{Cat}$, the functor $\Delta_A \colon \mathbf{Set} \to \mathrm{Fun}(A, \mathbf{Set})$ has a left adjoint i.e. **Set** has all (small) colimits. Since both **Set** and $\mathrm{Fun}(A, \mathbf{Set})$ have finite products, we can look at group objects (or ring objects, or R-module objects...) in them. The functor Δ_A preserves finite products, so induces a functor on group objects (also denoted by Δ_A), which is simply the functor $\Delta_A \colon \mathbf{Grp} \to \mathrm{Fun}(A, \mathbf{Grp})$ that sends $G \in \mathbf{Grp}$ to the constant functor with value G. This new functor Δ_A on **Grp** also has a left adjoint for any $A \in \mathbf{Cat}$ i.e. **Grp** has all (small) colimits. Note that this does not follow automatically from the fact that **Set** has all colimits and is genuinely a theorem about **Grp**.

Now, let us return to categories internal to **Pro**. It is easy to check that given $A \in \mathbf{Cat}(\mathbf{Pro})$, the internal functor category $\mathrm{IFun}(A, \mathbf{Pro})$ has all finite products. The terminal object is given by the space A_0. Given two internal functors $P, Q \in \mathrm{IFun}(A, \mathbf{Pro})$, their products $P \times Q$ is the internal functor with space $(P \times Q)_0 = P_0 \times_{A_0} Q_0$, and where $(p \times q)_0 \colon P_0 \times_{A_0} Q_0 \to A_0$ is the obvious map, and $(p \times q)_1 \colon A_1 \times_{A_0} (P_0 \times_{A_0} Q_0) \to P_0 \times_{A_0} Q_0$ is the unique map induced by

$$\begin{tikzcd}
A_1 \times_{A_0} P_0 \times_{A_0} Q_0 \arrow[dr, dashed] \arrow[drr, bend left, "q_1 \circ \pi"] \arrow[ddr, bend right, "p_1 \circ \pi"'] & & \\
& P_0 \times_{A_0} Q_0 \arrow[r, "\pi"'] \arrow[d, "\pi"] & Q_0 \arrow[d, "q_0"'] \\
& P_0 \arrow[r, "p_0"] & A_0
\end{tikzcd}$$

(Note that we are using the same letter π to denote different projection maps, but there should be no confusion.) The careful reader might wish to check that it is the binary product in $\mathrm{IFun}(A, \mathbf{Pro})$, which is not difficult.

Similar to our previous discussion for sets and groups, let us now consider group objects in $\mathbf{Pro}$ and $\mathrm{IFun}(A, \mathbf{Pro})$. The functor $\Delta_A \colon \mathbf{Pro} \to \mathrm{IFun}(A, \mathbf{Pro})$ again preserves finite products, so induces a functor $\Delta_A \colon \mathbf{PGrp} \to \mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$, where $\mathbf{PGrp} = \mathbf{Grp}(\mathbf{Pro})$ is the category of profinite groups. It sends $G \in \mathbf{PGrp}$ to the constant functor $(\Delta_A(G))_0 = G \times A_0 \to A_0$. If $P \in \mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$ has a reflection along Δ_A, then we call the reflection the *colimit of P (internal to* $\mathbf{Pro}$*)*. If such a colimit exists for a fixed A and for every P i.e. if Δ_A has a left adjoint, then we say the category $\mathbf{PGrp}$ *has all colimits of shape A internal to* $\mathbf{Pro}$. If such a left adjoint exists for every A, then we say that the category $\mathbf{PGrp}$ *has all colimits internal to* $\mathbf{Pro}$. We will soon show that this is in fact the case (see Theorem 2.3). We have chosen to work specifically with group objects here for concreteness, but our main theorem (Theorem 3.8) will be stated more generally.

Before we look at some examples of these colimits, let us make explicit what the objects and maps in $\mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$ look like. An object $P \in \mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$ consists of:

- a profinite space P_0, which as a set is a disjoint union $\coprod_{a \in A_0} P(a)$;
- a map $p_0 \colon P_0 \to A_0$ in $\mathbf{Pro}$ which sends $P(a)$ to a;
- a map $p_1 \colon A_1 \times_{A_0} P_0 \to P_0$ in $\mathbf{Pro}$ which sends $(f, x) \in A_1 \times_{A_0} P_0$ (where $d_0(f) = a$ and $x \in P(a)$) to an element $P(f)(x) \in P(b)$ (where $d_1(f) = b$);
- maps $P_0 \times_{A_0} P_0 \to P_0$, $P_0 \to P_0$ and $A_0 \to P_0$ in the slice category $\mathbf{Pro}_{/A_0}$ (which we think of as multiplication, inversion and identity);
- such that when restricted to each $P(a) = p_0^{-1}(\{a\})$, these three maps make $P(a)$ a group;
- and such that the assignment $a \mapsto P(a)$, $f \mapsto P(f)$ describes a functor $A \to \mathbf{Grp}$.

Given two such objects P and Q, a map $\alpha \colon P \to Q$ in $\mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$ consists of:

- a map $\alpha \colon P_0 \to Q_0$ in $\mathbf{Pro}_{/A_0}$;
- such that the restrictions $\alpha_a \colon P(a) \to Q(a)$ describe a natural transformation from P to Q, viewed as functors $A \to \mathbf{Grp}$.

Example 2.1. (i) Let A be an ordinary finite category i.e. A_0 and A_1 are both finite. Then A is canonically a category internal to $\mathbf{Pro}$ by equipping A_0 and A_1 each with the discrete topology.

In this case, an internal functor $A \to \mathbf{Pro}$ is just an ordinary functor etc., so the functor $\Delta_A \colon \mathbf{PGrp} \to \mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro})) = \mathrm{Fun}(A, \mathbf{PGrp})$ does have a left adjoint, since the category $\mathbf{PGrp}$ has all (ordinary) finite colimits.

(ii) Let $X \in \mathbf{Pro}$ be a profinite space. Then we canonically obtain a category internal to $\mathbf{Pro}$, which we denote by X^δ, called the *discrete internal category associated to X*. It is given by $X^\delta_0 = X$, $X^\delta_1 = X$, and all the maps d_0, d_1, i, c are the identity map. (Note that "discrete" here is referring not to the topology of X, but to the fact that the only morphisms in X^δ are the identity maps on objects.)

In this case, the internal functor category $\mathrm{IFun}(X^\delta, \mathbf{Pro})$ is equivalent to the slice category $\mathbf{Pro}_{/X}$ (see [10, Proposition 8.2.5]). An object $P \in \mathbf{Grp}(\mathrm{IFun}(X^\delta, \mathbf{Pro}))$ is exactly what [3, Section 5.1] calls a *sheaf of profinite groups over X*. The functor $\Delta_{X^\delta} \colon \mathbf{PGrp} \to \mathbf{Grp}(\mathrm{IFun}(X^\delta, \mathbf{Pro}))$ does have a left adjoint by [3, Proposition 5.1.2]. Following [3], we will call the image of $P \in \mathbf{Grp}(\mathrm{IFun}(X^\delta, \mathbf{Pro}))$ under the left adjoint the *free (profinite) product of P* and denote it by $\coprod_{x \in X} P(x) \in \mathbf{PGrp}$, where the topology on $P_0 = \coprod_{x \in X} P(x)$ is implicit. Note that this is the "topological coproduct" mentioned in the introduction (except we now write $P(x)$ instead of P_x). Notice that, while the free product of ordinary groups is the coproduct in **Grp**, the free profinite product of profinite groups, as defined above, is really a "coproduct internal to **Pro**", rather than the categorical coproduct in **PGrp**.

Remark 2.2. Let R be a profinite ring. If we replace groups with R-modules in (ii) above, interpreted appropriately (see below), then we get the *profinite direct sum* $\widehat{\bigoplus}$ of profinite modules, which is defined in [3, Section 9.1] and is the main object of study in [6]. This is the "topological coproduct" of profinite modules mentioned in the introduction.

To elaborate, note that the functor $\Delta_A\colon \mathbf{Pro} \to \mathrm{IFun}(A, \mathbf{Pro})$ induces a functor on ring objects. Given a profinite ring R, let $R \times A_0$ denote its image under Δ_A. Now observe that Δ_A induces a functor from the category of R-module objects in **Pro** (i.e. profinite R-modules) to the category of $(R \times A_0)$-module objects in $\mathrm{IFun}(A, \mathbf{Pro})$. This functor, still denoted by Δ_A, sends a profinite R-module M to the constant functor $M \times A_0$. If $X \in \mathbf{Pro}$, then the left adjoint to Δ_{X^δ} is the profinite direct sum $\widehat{\bigoplus}_X$.

As we saw in (i) and (ii) of the example above, the category **PGrp** has finite colimits (so in particular coequalisers) and all coproducts internal to **Pro**. The obvious next question to ask is whether this implies **PGrp** has all colimits internal to **Pro**, and the answer is yes, as we shall now show.

Theorem 2.3. *Let $A \in \mathbf{Cat}(\mathbf{Pro})$. If $P \in \mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$, then its colimit (internal to* **Pro***) exists and can be constructed using coequalisers and coproducts of shapes A_0^δ and A_1^δ. In particular,* **PGrp** *has all colimits internal to* **Pro**.

Proof. This is similar to the proof of the classical statement (cf. [8, Theorem V.2.1]), but we have to work in the internal language of **Pro**. In fact, we will see below (Remark 2.4) that this is just a special case of [12, Lemma 7.4.8], but we include a proof anyway for completeness[1].

Recall that A_i^δ is the discrete internal category and that $\mathrm{IFun}(A_i^\delta, \mathbf{Pro}) = \mathbf{Pro}_{/A_i}$. Define an object $Q \in \mathbf{Grp}(\mathbf{Pro}_{/A_0})$, where $Q_0 = P_0$ and $q_0 = p_0\colon Q_0 \to A_0$, and another object $R \in \mathbf{Grp}(\mathbf{Pro}_{/A_1})$, where $R_0 = A_1 \times_{A_0} P_0$ (the pullback of d_0 and p_0) and $r_0 = \pi_{A_1}\colon R_0 \to A_1$ is the canonical projection. Note that each fibre $R(f) = r_0^{-1}(\{f\}) = P(d_0(f))$ (where $f \in A_1$) is indeed a group. Let $\coprod_{a\in A_0} P(a)$ and $\coprod_{f\in A_1} P(d_0 f)$ be the free products of Q and R respectively.

We shall now define two maps $\phi, \psi\colon \coprod_{f\in A_1} P(d_0 f) \to \coprod_{a\in A_0} P(a)$ in **PGrp**, which is the same as defining maps $R_0 = A_1 \times_{A_0} P_0 \to \coprod_{a\in A_0} P(a)$ in **Pro** such that their restriction to each fibre is a group homomorphism $R(f) = P(d_0 f) \to \coprod_{a\in A_0} P(a)$ (where $f \in A_1$). The map ϕ is given by

$$\phi\colon R_0 = A_1 \times_{A_0} P_0 \xrightarrow{\pi} P_0 = Q_0 \xrightarrow{\eta_Q} \left(\coprod_{a\in A_0} P(a)\right) \times A_0 \xrightarrow{\pi} \coprod_{a\in A_0} P(a),$$

where η is the unit of the adjunction involving $\Delta_{A_0^\delta}$. The restriction of ϕ to the fibre above $f \in A_1$ is the canonical map $P(d_0 f) \to \coprod_{a\in A_0} P(a)$. On the other hand, the map ψ is given by

$$\psi\colon R_0 = A_1 \times_{A_0} P_0 \xrightarrow{p_1} P_0 = Q_0 \xrightarrow{\eta_Q} \left(\coprod_{a\in A_0} P(a)\right) \times A_0 \xrightarrow{\pi} \coprod_{a\in A_0} P(a).$$

The restriction of ψ to the fibre above $f \in A_1$ is the map $P(d_0 f) \xrightarrow{P(f)} P(d_1 f) \to \coprod_{a\in A_0} P(a)$.

[1]In fact, cocompleteness.

Let $C \in \mathbf{PGrp}$ be the coequaliser of ϕ and ψ. We leave it to the reader to check that C is the colimit of P (internal to $\mathbf{Pro}$). □

Remark 2.4. The above theorem is simply a special case of [12, Lemma 7.4.8]. The reader should refer to the cited book for the following discussion.

Consider the category $\mathbf{BPGrp}$ whose objects are pairs (P, X) with $X \in \mathbf{Pro}$ and $P \in \mathbf{Grp}(\mathbf{Pro}_{/X})$. A morphism from (P, X) to (Q, Y) in $\mathbf{BPGrp}$ consists of continuous maps $f\colon P \to Q$ and $X \to Y$ making the obvious square commute, such that the restriction of f to each fibre above X is a group homomorphism (see [3, Section 5.1]; the notation $\mathbf{BPGrp}$ stands for "bundles of profinite groups".). There is a functor $\mathrm{Cod}\colon \mathbf{BPGrp} \to \mathbf{Pro}$, called the *codomain functor*, which sends an object $(P, X) \in \mathbf{BPGrp}$ to its codomain X, and it is easy to see that Cod is a Cartesian fibration (cf. [12, Proposition 1.1.6]).

As pointed out in [12, Definition 7.4.1], an object of $\mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$ in our sense is exactly what [12] would call an *internal diagram of type A in* Cod. In fact, $\mathbf{Grp}(\mathrm{IFun}(A, \mathbf{Pro}))$ is exactly the full subcategory of the category $\mathbf{BPGrp}^A$ in [12, Definition 7.4.5] consisting of objects with $I = *$. As such, the above theorem is just (the dual of) [12, Lemma 7.4.8] and its proof restricted to $I = *$.

2.2. **Connections to Profinite Bass–Serre Theory.** Free profinite products and finite colimits of profinite groups are used extensively in profinite Bass–Serre theory, similar to the abstract case. For instance, the profinite fundamental group of a graph of groups over a single edge is an amalgamated free product i.e. finite pushout (see [3, Example 6.2.3(d)]), whilst the fundamental group over a "cone graph" with trivial edge groups and cone-vertex group is a free product (see [3, Example 6.2.3(b)]). However, as far as the author is aware, there has not been much work that studies infinite amalgamated free products of profinite groups. Let us now see that in the context of Bass–Serre theory (at least), our notion of colimits internal to $\mathbf{Pro}$ gives the correct notion of infinite amalgamated free products of profinite groups. We remark that these are in general not infinite categorical pushouts in $\mathbf{PGrp}$: those do exist[2] but may not be well-behaved (cf. [14, Remark 2]). The reader should consult [3] for the terminology and facts that we are about to use.

Let $X \in \mathbf{Pro}$ and $\Gamma = C(X, *)$ be the profinite cone graph of X, which is connected (see [3, Example 3.4.2]). That is, Γ has vertices $V = X \sqcup \{*\}$ and edges $E = \overline{X} \cong X$, with incidence relations saying that an edge $\overline{x} \in E$ has source $* \in V$ and target $x \in V$. Suppose that $\mathcal{G} = (\mathcal{G}, \varpi, \Gamma)$ is a profinite graph of groups over Γ, where all edges groups, as well as the vertex group above $*$, are (isomorphic to) a fixed profinite group H, such that the restriction of ∂_0 to each edge group is an isomorphism of H. We shall identify all these copies of H. In particular, we can think of H as a common subgroup of every vertex group $\mathcal{G}(x)$, $x \in V \backslash \{*\}$. Let us write $\theta_x\colon H \hookrightarrow \mathcal{G}(x)$ for the restriction of ∂_1 to the edge group $\mathcal{G}(\overline{x}) = H$. For ordinary groups (and the ordinary version of the graph of groups we just defined), the fundamental group is the amalgamated free product $\circledast_H \mathcal{G}(x)$ (cf. [15, Section I.4.4]). Let us think about what the profinite fundamental group of $\mathcal{G}$ looks like.

Note that Γ, being an inverse limit of finite trees, is a profinite tree and is simply connected ([3, Proposition 3.3.3]). Moreover, the set of edges E is a closed subspace of Γ. Therefore, as pointed out at the end of [3, page 178], the fundamental group $\Pi_1(\mathcal{G})$ of $\mathcal{G}$ is rather easy to define: it is a profinite group $\Pi = \Pi_1(\mathcal{G}) \in \mathbf{PGrp}$ with the universal property of having

[2] Warning: In [13] and [14], the authors say that a pushout *exists* if the canonical maps into it are injective, which is not what we mean here. In [9] and [3], the authors say that a pushout is *proper* if the canonical maps are injective.

a map $\beta\colon \mathcal{G}_V = \varpi^{-1}(V) \to \Pi$ in **Pro**, such that the restriction of β to each fibre is a group homomorphism $\beta_v\colon \mathcal{G}(v) \to \Pi$ (where $v \in V$), and such that $\beta_x \circ \theta_x = \beta_*$ for each $x \in X$.

We now construct an internal category $A = A_\Gamma$ associated to Γ. It has $A_0 = X \sqcup \{*\} = V$ and $A_1 = \overline{X} \sqcup X \sqcup \{*\} = E \sqcup V$, with the obvious composition rule etc.: we think of A_Γ as Γ with an identity map added to each vertex. We also define an internal functor $P \in \mathbf{Grp}(\mathrm{IFun}(A_\Gamma, \mathbf{Pro}))$ as follows. It has $P_0 = \mathcal{G}_V = \bigsqcup_{v\in V} \mathcal{G}(v)$, $p_0 = \varpi|_{\mathcal{G}_V}$ and $p_1 = \partial_1\colon A_1 \times_{A_0} \mathcal{G}_V = \mathcal{G} \to \mathcal{G}_V$. Note that $P(\overline{x}) = \theta_x$ (where $\overline{x} \in E$) and that P is in fact a group object in the internal functor category.

Consider the colimit of P, which exists by Theorem 2.3. By definition, it is a profinite group C with the universal property of having a map $P \to \Delta_{A_\Gamma}(C)$ in $\mathbf{Grp}(\mathrm{IFun}(A_\Gamma, \mathbf{Pro}))$. It is easy to see that a map $P \to \Delta_{A_\Gamma}(C)$ is the same as a map $\alpha\colon P_0 = \mathcal{G}_V \to C$ in **Pro** such that the restriction of α to each fibre is a group homomorphism $\alpha_v\colon \mathcal{G}(v) \to C$ (where $v \in V$) and such that all the α_v collectively define a natural transformation from P to $\Delta_{A_\Gamma}(C)$, viewed as functors $A \to \mathbf{Grp}$. Moreover, to say that the α_v satisfy the naturality condition is the same as to say that for each $\overline{x} \in E = \overline{X}$, we have $\alpha_x \circ \theta_x = \alpha_*$. But we have just written down precisely the universal property of the fundamental group $\Pi_1(\mathcal{G})$!

Thus, the fundamental group $\Pi_1(\mathcal{G})$ of the specific graph of groups $\mathcal{G}$ over Γ we defined is an infinite *internal pushout* i.e. a colimit of shape A_Γ internal to **Pro**. We will denote this group by $\coprod_H \mathcal{G}(x)$ (not to be confused with the free profinite product) and should think of it as the *free (profinite) product of the $\mathcal{G}(x)$ amalgamating H along θ_x*, where the topology on $\mathcal{G}$ is implicit. Since we have established $\coprod_H \mathcal{G}(x)$ as a fundamental group, we could use the results of [3] to study this group if we wish. As an example, the Mayer–Vietoris sequence ([3, Theorem 9.4.1]) easily implies the following.

Proposition 2.5. *Let $G = \coprod_H \mathcal{G}(x)$ be as above and assume that it is proper i.e. the canonical maps $\mathcal{G}(x) \to G$ are injective. Let R be a profinite quotient ring of $\widehat{\mathbb{Z}}$ and B be a profinite $R[\![G]\!]$-module which is projective when restricted to H. Moreover, write $H_i(G, B)$ for the ith homology group of G with coefficients in B. Then for every i, there is an isomorphism of profinite R-modules*

$$H_i(G, B) = \widehat{\bigoplus}_{x\in X} H_i(\mathcal{G}(x), B).^3$$

Remark 2.6. The construction $\Gamma \mapsto A_\Gamma$ above might seem familiar. Recall that the forgetful functor $\mathbf{Cat} \to \mathbf{Graph}$, where **Graph** is the category of directed multigraphs, has a left adjoint which takes a graph Γ and freely constructs a category A_Γ from it (see [8, Section II.7]). Let **PGraph** be the category of profinite graphs, where by a profinite graph[4] here we mean a pair of profinite spaces (V, E) together with continuous maps $d_0, d_1\colon E \to V$, and morphisms are required to send edges to edges. Then there is clearly a forgetful functor $\mathbf{Cat}(\mathbf{Pro}) \to \mathbf{PGraph}$. Given a profinite graph $\Gamma = (V, E) \in \mathbf{PGraph}$, if every directed path in Γ has length bounded by a fixed constant, then we can easily construct a category $A = A_\Gamma$ internal to **Pro** by following the argument of [8, Section II.7]. By a directed path, we just mean a sequence of composable arrows, possibly with repetition. Briefly, we have $A_0 = V$ and $A_1 = \bigsqcup_{n=0}^{\infty} E_n$, where E_n is the space of directed paths in Γ of length n. We need to bound the lengths of all paths in Γ so that A_1 is in fact a finite disjoint union. It is then obvious how to make A_Γ a category internal to **Pro** and that A_Γ is the reflection of Γ along the forgetful functor.

---

[3]Recall that $\widehat{\oplus}$ refers to the coproduct of profinite R-modules internal to **Pro** (i.e. the "profinite direct sum" in [6]), analogous to what we have already done for groups. The symbol $\oplus$ (resp. $\boxplus$) is used in [3] (resp. [4]) instead.

[4]This is less general than the profinite graphs appearing in [3]. In fact, the profinite graphs defined here are exactly those profinite graphs in [3] whose set of edges is closed.

However, if we allow Γ to have arbitrarily long paths, the theory quickly becomes tricky. This already happens when Γ consists of just a single vertex and a single loop—then A_Γ should be the one-object category corresponding to the "free profinite monoid of rank 1"!

Remark 2.7. Another important construction that appears in Bass–Serre theory is that of HNN-extensions. It is well-known that these are not 1-colimits of groups, but 2-coinserters (a type of finite 2-colimits). We can develop a similar theory here, but since this is not the focus of the current writing, we will sketch the basic ideas below and leave it to the interested reader to fill the gaps.

The category $\mathbf{Cat}(\mathbf{Pro})$ is naturally a 2-category if we take internal natural transformations as 2-morphisms. Observe that one-object internal categories are precisely profinite monoids i.e. monoid objects internal to $\mathbf{Pro}$ (viewed as one-object categories), and internal functors between them are precisely continuous monoid homomorphisms. Thus, the 1-category $\mathbf{PGrp}$ can be viewed as a full 1-subcategory of $\mathbf{Cat}(\mathbf{Pro})$ and hence $\mathbf{PGrp}$ inherits a 2-categorical structure. It is easy to see that given two profinite groups G, H and two maps $\alpha, \beta \colon G \to H$, an internal natural transformation $t \colon \alpha \to \beta$ is the same as an element $t \in H$ such that $\alpha(g) = t\beta(g)t^{-1}$ for all $g \in G$. Thus, given α, β as above and assuming they are injective, the profinite HNN-extension of α and β is exactly a profinite group K with the universal property of having a 1-morphism $\gamma \colon H \to K$ and a 2-morphism $\gamma \circ \alpha \to \gamma \circ \beta$ i.e. it is a 2-coinserter.

3. Left Adjoints and Coproducts

Now we shall study how left adjoints and other functors interact with coproducts internal to $\mathbf{Pro}$. This will eventually lead to our main theorem (Theorem 3.8) and main examples (Example 3.11). Let us once again forget about internal categories for a moment and think about $\mathbf{Set}$ and $\mathbf{Grp}$. The forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ has a left adjoint F, namely the free groups functor. The fact that left adjoints preserve colimits in this situation can be expressed as follows: for any $A \in \mathbf{Cat}$, we have a commutative diagram, up to natural isomorphism, of (large) categories

$$\begin{tikzcd}
\mathrm{Fun}(A, \mathbf{Set}) \arrow[r, "\coprod_A"] \arrow[d, "F^* = F \circ (-)"'] & \mathbf{Set} \arrow[d, "F"] \\
\mathrm{Fun}(A, \mathbf{Grp}) \arrow[r, "\coprod_A"'] & \mathbf{Grp}
\end{tikzcd}$$

What would the corresponding diagram in the context of internal categories be? The top row should be $\mathrm{IFun}(A, \mathbf{Pro}) \xrightarrow{\coprod_A} \mathbf{Pro}$, where $A \in \mathbf{Cat}(\mathbf{Pro})$ and $\coprod_A$ is the left adjoint of Δ_A, and similarly for the bottom row but for group objects. The left vertical map, however, presents difficulties, as it seems like we would need to view the free profinite groups functor $F \colon \mathbf{Pro} \to \mathbf{PGrp}$ as an internal functor. We can circumvent this problem by considering only coproducts rather than all colimits internal to $\mathbf{Pro}$.

3.1. Preservation of Coproducts Internal to Pro.

Let $A = X^\delta$ be the discrete internal category associated to $X \in \mathbf{Pro}$. Then we would like to have a commutative diagram, up to natural isomorphism, of ordinary categories

$$\begin{tikzcd}
\mathbf{Pro}_{/X} \arrow[r] \arrow[d, "F^*"'] & \mathbf{Pro} \arrow[d, "F"] \\
\mathbf{Grp}(\mathbf{Pro}_{/X}) \arrow[r, "\coprod_X"'] & \mathbf{PGrp}
\end{tikzcd} \tag{1}$$

where F is any functor (not necessarily a left adjoint for now) and the top arrow simply sends an object $(P, X) = (P \to X)$ to P. Next, we need to construct the functor F^*, which should be induced by F. A first idea is that for an object $p\colon P \to X$ in $\mathbf{Pro}_{/X}$, the object $F^*(P)$ should have fibres $F^*(P)(x) = F(P(x))$, $x \in X$, where $P(x) = p^{-1}(\{x\})$. The issue is that we still need to give $F^*(P) = \bigsqcup_{x\in X} F(P(x))$ an appropriate profinite topology. We will do so by using the fact that all the categories above are subcategories of certain pro-completions (refer to [11] for the necessary background). What follows should be viewed as a more general version of [7, Proposition A.13].

Let $\mathbf{BPro} = \mathrm{Fun}(\mathbf{2}, \mathbf{Pro})$ be the category of bundles of profinite spaces, where $\mathbf{2}$ is the interval category, and let $\mathbf{BSet}_{\mathrm{fin}}$ be the full subcategory of bundles $P \to Y$ where both P and Y are finite. It is a classical fact (see [16, Proposition 8.8.5]) that $\mathbf{BPro} = \mathbf{Pro}(\mathbf{BSet}_{\mathrm{fin}})$.[5] Similarly, let $\mathbf{BPGrp}$ be the category of bundles of profinite groups (see Remark 2.4) and let $\mathbf{BGrp}_{\mathrm{fin}}$ be the full subcategory of bundles where both spaces are finite. It follows from the group analogue of [6, Corollary 2.13] that $\mathbf{BPGrp} = \mathbf{Pro}(\mathbf{BGrp}_{\mathrm{fin}})$. One important fact we are going to constantly use is that inverse limits in $\mathbf{BPGrp}$ "commute with taking fibres" (cf. [6, Proposition 2.7]).

Note that the internal coproduct functor $\coprod\colon \mathbf{BPro} \to \mathbf{Pro}\colon (P, Y) \mapsto P$ is left adjoint to $Z \mapsto (Z, *)$ and similarly $\coprod\colon \mathbf{BPGrp} \to \mathbf{PGrp}$ is left adjoint to $G \mapsto (G, *)$. These facts are crucial for what we are about to do.

Let $F\colon \mathbf{Pro} \to \mathbf{PGrp}$ be any functor (not necessarily a left adjoint). Then there is a functor $F^*\colon \mathbf{BSet}_{\mathrm{fin}} \to \mathbf{BPGrp}$ which sends a finite bundle $P \to Y$ to the bundle $\bigsqcup_{y\in Y} F(P(y)) \to Y$, where the finite disjoint union is given the disjoint union topology, and sends morphisms in the obvious way. By the universal property of pro-completions, this uniquely extends to a functor $\mathbf{BPro} \to \mathbf{BPGrp}$, also denoted by F^*, by requiring that it commutes with inverse limits. Note that $F^*\colon \mathbf{BPro} \to \mathbf{BPGrp}$, by construction, commutes with taking fibres i.e. $F^*(P)(y) = F^*(P(y))$, which is also isomorphic to $F(P(y))$ if F commutes with inverse limits. We then obtain the following analogue of [7, Proposition A.13].

Proposition 3.1. *Suppose the functor $F\colon \mathbf{Pro} \to \mathbf{PGrp}$ commutes with inverse limits and finite coproducts. Then the following diagram commutes:*

$$\begin{tikzcd} \mathbf{BPro} \arrow[r] \arrow[d, "F^*"'] & \mathbf{Pro} \arrow[d, "F"] \\ \mathbf{BPGrp} \arrow[r, "\coprod"'] & \mathbf{PGrp} \end{tikzcd} \tag{2}$$

Proof. Note that all the functors in the above diagram commute with inverse limits: F does by assumption, F^* does by definition, and for the other two see [3, Proposition 5.1.7] or [6, Proposition 2.7]. By the universal property of pro-completions, it suffices to check that the diagram commutes when restricted to $\mathbf{BSet}_{\mathrm{fin}}$. In this case, the top-right composition sends a finite bundle (P, Y) to $F(\bigsqcup_{y\in Y} P(y))$ and the bottom-left composition sends it to $\coprod_{y\in Y} F(P(y))$. These are the same since F commutes with finite coproducts. We remind the reader that finite coproducts internal to $\mathbf{Pro}$ are just ordinary categorical coproducts (see Example 2.1). □

Remark: It is clear that this works in greater generality than just $\mathbf{Pro}$ and $\mathbf{PGrp}$: we will state a more general version of the above proposition in Theorem 3.8.

Now, let us see how we can obtain diagram (1). The diagrams (1) and (2) are different but are clearly related. It is tempting to say that (1) is just the restriction of (2), but we have to be

[5] Here, $\mathbf{Pro}(\mathbf{C})$ denotes the pro-completion of $\mathbf{C}$. Recall that we also write $\mathbf{Pro}$ for the category $\mathbf{Pro}(\mathbf{Set}_{\mathrm{fin}})$ of profinite spaces, which hopefully doesn't cause any confusion.

careful: $\mathbf{Pro}_{/X}$ (resp. $\mathbf{Grp}(\mathbf{Pro}_{/X})$) is a *non-full* subcategory of $\mathbf{BPro}$ (resp. $\mathbf{BPGrp}$), so we have to check that the restriction of F^* lives in the correct place.

Proposition 3.2. *The restriction of diagram (2) gives a well-defined diagram (1) which commutes if (2) does.*

Proof. As mentioned in the previous paragraph, let's check that the restriction of $F^*\colon \mathbf{BPro} \to \mathbf{BPGrp}$ indeed gives a functor $\mathbf{Pro}_{/X} \to \mathbf{Grp}(\mathbf{Pro}_{/X})$. This is clear for objects. Given a morphism $f\colon P \to Q$ in $\mathbf{Pro}_{/X}$, decompose it into an inverse limit of

$$\begin{tikzcd} P_i \arrow[rr, "f_i"] \arrow[dr] & & Q_i \arrow[dl] \\ & X_i & \end{tikzcd}$$

where the spaces are finite. By the definition of F^*, the morphism $F^*(f) \in \mathbf{BPGrp}$ is given by the inverse limit of

$$\begin{tikzcd} F^*(P_i) \arrow[rr, "F^*(f_i)"] \arrow[dr] & & F^*(Q_i) \arrow[dl] \\ & X_i & \end{tikzcd}$$

where $F^*(P_i) \to X_i$ is the obvious map $\coprod_{x \in X_i} F(P_i(x)) \to X_i$ and similarly for Q_i. It is thus clear that the inverse limit of the above diagrams in fact lives in $\mathbf{Grp}(\mathbf{Pro}_{/X})$. $\square$

3.2. **Relative Adjunctions.** In Proposition 3.1, we required F to commute with inverse limits and finite coproducts, not that it be a left adjoint. This is the case for tensor products or more generally Tor (cf. [3, Theorem 9.1.1]). However, it turns out that many such functors F which occur naturally in the profinite setting also happen to be left adjoints (see Examples 3.4 and 3.11). In this case, we can say more about diagrams (1) and (2). To explain why left adjoints are common in this setting, we need the concept of *relative adjoints*.

Let $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ be ordinary categories and $L\colon \mathbf{C} \to \mathbf{E}$, $R\colon \mathbf{D} \to \mathbf{C}$, $J\colon \mathbf{D} \to \mathbf{E}$ functors. Recall from (the dual of) [17, Definition 2.2] that L is *J-left coadjoint to* R if for every $c \in \mathbf{C}$ and $d \in \mathbf{D}$, there is a natural isomorphism $\mathrm{Hom}_{\mathbf{E}}(Lc, Jd) = \mathrm{Hom}_{\mathbf{C}}(c, Rd)$. If J is the identity functor, then we recover the usual definition of a left adjoint. The following simple observation is why relative adjoints appear often in the study of profinite groups.

Lemma 3.3. *(cf.* [18]*) Let $\mathbf{C}$ and $\mathbf{D}$ be small categories, $J\colon \mathbf{D} \hookrightarrow \mathbf{Pro}(\mathbf{D})$ be the canonical embedding, and $L\colon \mathbf{C} \to \mathbf{Pro}(\mathbf{D})$, $R\colon \mathbf{D} \to \mathbf{C}$ be functors such that L is J-left coadjoint to R. Let $L'\colon \mathbf{Pro}(\mathbf{C}) \to \mathbf{Pro}(\mathbf{D})$ and $\mathbf{Pro}(R)\colon \mathbf{Pro}(\mathbf{D}) \to \mathbf{Pro}(\mathbf{C})$ be the canonical extensions. Then L' is left adjoint to $\mathbf{Pro}(R)$.*

Proof. Let $c = \varprojlim c_i \in \mathbf{Pro}(\mathbf{C})$ and $d = \varprojlim d_j \in \mathbf{Pro}(\mathbf{D})$. Then we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Pro}(\mathbf{D})}(L'c, d) &= \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathbf{Pro}(\mathbf{D})}(Lc_i, Jd_j) \\ &= \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathbf{C}}(c_i, Rd_j) \\ &= \mathrm{Hom}_{\mathbf{Pro}(\mathbf{C})}(c, \varprojlim Rd_j) \\ &= \mathrm{Hom}_{\mathbf{Pro}(\mathbf{C})}(c, \mathbf{Pro}(R)d). \end{aligned}$$

$\square$

Example 3.4. Take $\mathbf{C} = \mathbf{Set}_{\text{fin}}$ and $\mathbf{D} = \mathbf{Grp}_{\text{fin}}$. Let $L\colon \mathbf{Set}_{\text{fin}} \to \mathbf{PGrp}$ be the free profinite groups functor, i.e. $L(X)$ is the free profinite group on the finite set X. This can be constructed as the profinite completion of the abstract free group on X. Let $R\colon \mathbf{Grp}_{\text{fin}} \to \mathbf{Set}_{\text{fin}}$ be the forgetful functor. It is immediate that L is J-left coadjoint to R (where $J\colon \mathbf{Grp}_{\text{fin}} \hookrightarrow \mathbf{PGrp}$), so by the above lemma, we get the free profinite groups functor $L'\colon \mathbf{Pro} \to \mathbf{PGrp}$ which is left adjoint to the forgetful functor $\mathbf{PGrp} \to \mathbf{Pro}$. We used here the fact that the forgetful functor $\mathbf{PGrp} \to \mathbf{Pro}$ commutes with inverse limits, so that it is indeed $\mathbf{Pro}(R)$. By construction, we have $L'(\varprojlim X_i) = \varprojlim L(X_i)$ (X_i finite). The point is that it's easy to check L is J-left coadjoint to R because only finite bases are involved, and then we automatically get an actual adjunction between the pro-completions.

This works in various other situations in the profinite setting. To name two, notice that the free modules functor $R[\![-]\!]\colon \mathbf{Pro} \to \mathbf{PMod}(R)$ (where R is a fixed profinite ring and $\mathbf{PMod}(R)$ is the category of profinite R-modules) and induction $\mathrm{Ind}_H^G(-)\colon \mathbf{PMod}(k[\![H]\!]) \to \mathbf{PMod}(k[\![G]\!])$ both arise this way.

In the situation of bundles above, suppose $F\colon \mathbf{Pro} \to \mathbf{PGrp}$ is actually a left adjoint which arises from Lemma 3.3. That is, suppose F is the canonical extension of a functor $\mathbf{Set}_{\text{fin}} \to \mathbf{PGrp}$ which is J-left coadjoint to a functor $U\colon \mathbf{Grp}_{\text{fin}} \to \mathbf{Set}_{\text{fin}}$, where $J\colon \mathbf{Grp}_{\text{fin}} \hookrightarrow \mathbf{PGrp}$. Let $J^*\colon \mathbf{BGrp}_{\text{fin}} \hookrightarrow \mathbf{BPGrp}$ be the canonical embedding and $U^*\colon \mathbf{BGrp}_{\text{fin}} \to \mathbf{BSet}_{\text{fin}}$ be constructed from U in the same way as F^* from F. Then it's easy to see that $F^*\colon \mathbf{BSet}_{\text{fin}} \to \mathbf{BPGrp}$ is J^*-left coadjoint to $U^*\colon \mathbf{BGrp}_{\text{fin}} \to \mathbf{BSet}_{\text{fin}}$. By Lemma 3.3, the extension $F^*\colon \mathbf{BPro} \to \mathbf{BPGrp}$ is left adjoint to $\mathbf{Pro}(U^*)\colon \mathbf{BPGrp} \to \mathbf{BPro}$.

Moreover, F automatically commutes with inverse limits (because it is an extension from $\mathbf{Set}_{\text{fin}}$ to $\mathbf{Pro}$) and with finite coproducts (because it is a left adjoint). By Proposition 3.1, we obtain the following commutative diagram consisting of four left adjoints:

$$\begin{tikzcd} \mathbf{BPro} \arrow[r] \arrow[d, "F^*"'] & \mathbf{Pro} \arrow[d, "F"] \\ \mathbf{BPGrp} \arrow[r, "\coprod"'] & \mathbf{PGrp} \end{tikzcd} \tag{2}$$

In fact, the top and bottom maps also arise from Lemma 3.3, just like F and F^*. For example, the bottom map is the extension of $\coprod\colon \mathbf{BGrp}_{\text{fin}} \to \mathbf{PGrp}$, which is left coadjoint to the functor $\mathbf{Grp}_{\text{fin}} \to \mathbf{BGrp}_{\text{fin}}\colon G \mapsto (G, *)$.

We already knew that (2) is commutative, but this also follows from the easy observation that the "dual" diagram involving the four right adjoints is commutative:

$$\begin{tikzcd} \mathbf{BPro} & \mathbf{Pro} \arrow[l] \\ \mathbf{BPGrp} \arrow[u] & \mathbf{PGrp} \arrow[l] \arrow[u] \end{tikzcd}$$

More generally, note that this diagram is simply the pro-completion of

$$\begin{tikzcd} \mathbf{BSet}_{\text{fin}} & \mathbf{Set}_{\text{fin}} \arrow[l] \\ \mathbf{BGrp}_{\text{fin}} \arrow[u] & \mathbf{Grp}_{\text{fin}} \arrow[l] \arrow[u] \end{tikzcd}$$

Thus, in situations more general than ours, it is often still easy to check whether a diagram like (2) commutes as long as all four functors arise from Lemma 3.3. Let us make this a formal observation.

Lemma 3.5. *Let $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ be small categories and let $J_i\colon \mathbf{C}_i \hookrightarrow \mathbf{Pro}(\mathbf{C}_i)$ be the canonical embeddings. Suppose that there are some functors $R_{ij}\colon \mathbf{C}_i \to \mathbf{C}_j$ making the following square commute*

$$\begin{tikzcd} \mathbf{C}_1 & \mathbf{C}_2 \arrow[l, "R_{21}"'] \\ \mathbf{C}_3 \arrow[u, "R_{31}"] & \mathbf{C}_4 \arrow[l, "R_{43}"] \arrow[u, "R_{42}"'] \end{tikzcd}$$

and that, for each (i,j) appearing above, there is a functor $L_{ji}\colon \mathbf{C}_j \to \mathbf{Pro}(\mathbf{C}_i)$ which is J_i-left coadjoint to R_{ij}. Let $L'_{ji}\colon \mathbf{Pro}(\mathbf{C}_j) \to \mathbf{Pro}(\mathbf{C}_i)$ be the (unique) extension of L_{ji}. Then the following squares commute, where each map in the left square is left adjoint to the corresponding map in the right square.

$$\begin{tikzcd} \mathbf{Pro}(\mathbf{C}_1) \arrow[r, "L'_{12}"] \arrow[d, "L'_{13}"'] & \mathbf{Pro}(\mathbf{C}_2) \arrow[d, "L'_{24}"] \\ \mathbf{Pro}(\mathbf{C}_3) \arrow[r, "L'_{34}"'] & \mathbf{Pro}(\mathbf{C}_4) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Pro}(\mathbf{C}_1) & \mathbf{Pro}(\mathbf{C}_2) \arrow[l, "\mathbf{Pro}(R_{21})"'] \\ \mathbf{Pro}(\mathbf{C}_3) \arrow[u, "\mathbf{Pro}(R_{31})"] & \mathbf{Pro}(\mathbf{C}_4) \arrow[l, "\mathbf{Pro}(R_{43})"] \arrow[u, "\mathbf{Pro}(R_{42})"'] \end{tikzcd}$$

Once again, we can restrict diagram (2) to get a commutative diagram (1), but we have to be careful as we are dealing with non-full subcategories. Moreover, we point out that although the left adjoints, i.e. the maps in diagram (1), are just restrictions of the maps in diagram (2), the horizontal right adjoints are genuinely different.

Proposition 3.6. *Suppose the functor $F\colon \mathbf{Pro} \to \mathbf{PGrp}$ is a left adjoint which arises from Lemma 3.3. Then the restriction of the commutative diagram (2) gives a well-defined commutative diagram (1) where all the maps are left adjoints.*

Proof. In view of the discussion above, as well as Proposition 3.2, we only need to check that the right adjoint $\mathbf{Pro}(U^*)\colon \mathbf{BPGrp} \to \mathbf{BPro}$ of $F^*\colon \mathbf{BPro} \to \mathbf{BPGrp}$ restricts correctly and is still right adjoint. The exact same argument of Proposition 3.2 shows that $\mathbf{Pro}(U^*)$ restricts to the correct subcategories.

Write $V = \mathbf{Pro}(U^*)$ for simplicity. To show that the restrictions $F^*\colon \mathbf{Pro}_{/X} \to \mathbf{Grp}(\mathbf{Pro}_{/X})$ and $V\colon \mathbf{Grp}(\mathbf{Pro}_{/X}) \to \mathbf{Pro}_{/X}$ are still adjoints, we need to check that given $P \in \mathbf{Pro}_{/X}$, $Q \in \mathbf{Grp}(\mathbf{Pro}_{/X})$ and a map $P \to V(Q)$ in $\mathbf{Pro}_{/X}$, the unique induced map $F^*(P) \to Q$ in $\mathbf{BPGrp}$ in fact lives in $\mathbf{Grp}(\mathbf{Pro}_{/X})$. Let $\epsilon\colon F^* \circ V \to \mathrm{id}$ be the counit of the adjunction between $\mathbf{BPro}$ and $\mathbf{BPGrp}$, so given $Q \in \mathbf{BPGrp}$, the component ϵ_Q is the image of the identity map under $\mathrm{Hom}_{\mathbf{BPro}}(V(Q), V(Q)) \to \mathrm{Hom}_{\mathbf{BPGrp}}(F^*V(Q), Q)$. Observe that if Q is in $\mathbf{Grp}(\mathbf{Pro}_{/X})$, then ϵ_Q in fact lives in $\mathrm{Hom}_{\mathbf{Grp}(\mathbf{Pro}_{/X})}(F^*V(Q), Q)$ (cf. the isomorphisms in Lemma 3.3), which completes the proof. ☐

3.3. **Main Theorem and Examples.** Although we have focused on **Pro** and **PGrp** for concreteness, everything obviously works in greater generality. For example, we could replace **Pro** and **PGrp** each with any of **Pro**, **PGrp**, **PAb** or $\mathbf{PMod}(R)$, where $\mathbf{PAb} = \mathbf{PMod}(\widehat{\mathbb{Z}})$ is the category of profinite abelian groups and R is a fixed profinite ring. We also write **BPro**, **BPGrp**,

$\mathbf{BPAb} = \mathbf{Pro}(\mathbf{BAb}_{\mathrm{fin}})$, $\mathbf{BPMod}(R) = \mathbf{Pro}(\mathbf{BMod}(R)_{\mathrm{fin}})$[6] for the corresponding category of bundles. By $\mathbf{Mod}(R)_{\mathrm{fin}}$ we of course mean finite discrete *topological* R-modules.

To simplify notation, let us write $\mathbf{Pro}_{/X}$, $\mathbf{PGrp}_{/X} = \mathbf{Grp}(\mathbf{Pro}_{/X})$, $\mathbf{PAb}_{/X}$, $\mathbf{PMod}(R)_{/X}$ for the corresponding "slice category" over $X \in \mathbf{Pro}$. Note that $\mathbf{PGrp}_{/X}$ is certainly not the slice category over a profinite group X. For coproducts of profinite modules internal to $\mathbf{Pro}$ in our sense, the reader could refer to [7, Appendix A] or [6] (see also Remark 2.2). We will use the symbol $\widehat{\bigoplus}$ instead of $\coprod$ if we are referring specifically to the (internal) coproduct in $\mathbf{PAb}$ or $\mathbf{PMod}(R)$.

Remark 3.7. There are good reasons for studying either diagram (1) or diagram (2). As we have already seen, one major advantage of diagram (2) is that the categories involved are pro-completions. On the other hand, the categories in diagram (1) have good exactness properties. Indeed, as $\mathbf{Pro}$ is a regular category, so is $\mathbf{Pro}_{/X}$ by [19, Example A.5.5], and hence so are $\mathbf{PGrp}_{/X}$ and $\mathbf{PMod}(R)_{/X}$ by [20, Theorem 5.11]. More importantly, $\mathbf{Pro}$ is not exact (see [21, Example 5.6]), but $\mathbf{PGrp}_{/X}$ and $\mathbf{PMod}(R)_{/X}$ are exact. This essentially follows from [3, Proposition 5.5.8]. In particular, this means that $\mathbf{PMod}(R)_{/X}$ is abelian (and in fact satisfies (AB5*) since limits and cokernels are computed on fibres); cf. [6, Remark 2.16(ii)].

To make it easier for the reader to apply this framework, we now write down a general theorem for what we have done. We will omit details of the proof, since it is exactly the same as before.

Theorem 3.8. *Let $\mathbf{C}$ and $\mathbf{D}$ be any of $\mathbf{Set}_{\mathrm{fin}}$, $\mathbf{Grp}_{\mathrm{fin}}$, $\mathbf{Ab}_{\mathrm{fin}}$ or $\mathbf{Mod}(R)_{\mathrm{fin}}$, where R is a profinite ring*[7]*. Write $\mathbf{BC}$ and $\mathbf{BD}$ for the corresponding category of bundles, $\mathbf{Pro}(\mathbf{BC}) = \mathbf{BPro}(\mathbf{C})$ and $\mathbf{Pro}(\mathbf{BD}) = \mathbf{BPro}(\mathbf{D})$ for their pro-completions, and $\mathbf{Pro}(\mathbf{C})_{/X}$ and $\mathbf{Pro}(\mathbf{D})_{/X}$ for the corresponding "slice category" over $X \in \mathbf{Pro}$. Let $J\colon \mathbf{D} \hookrightarrow \mathbf{Pro}(\mathbf{D})$ and $J^*\colon \mathbf{BD} \hookrightarrow \mathbf{Pro}(\mathbf{BD})$ be the canonical embeddings.*

Let $L\colon \mathbf{C} \to \mathbf{Pro}(\mathbf{D})$ be any functor and let $L'\colon \mathbf{Pro}(\mathbf{C}) \to \mathbf{Pro}(\mathbf{D})$ be its extension. Then:

(i) The functor L induces a functor $L^*\colon \mathbf{BC} \to \mathbf{Pro}(\mathbf{BD}) = \mathbf{BPro}(\mathbf{D})$ by sending a finite bundle $(P, X) = \bigsqcup_{x\in X} P(x) \in \mathbf{BC}$ to the disjoint union $\bigsqcup_{x\in X} LP(x) \in \mathbf{BPro}(\mathbf{D})$, which then extends to a functor $L^{*\prime}\colon \mathbf{BPro}(\mathbf{C}) \to \mathbf{BPro}(\mathbf{D})$. Suppose further that L' commutes with finite coproducts. Then we obtain a commutative diagram (up to natural isomorphism)

$$\begin{array}{ccc} \mathbf{BPro}(\mathbf{C}) & \xrightarrow{\coprod} & \mathbf{Pro}(\mathbf{C}) \\ {\scriptstyle L^{*\prime}}\big\downarrow & & \big\downarrow{\scriptstyle L'} \\ \mathbf{BPro}(\mathbf{D}) & \xrightarrow[\coprod]{} & \mathbf{Pro}(\mathbf{D}) \end{array} \tag{2}$$

Moreover, the restriction of the above diagram gives a well-defined commutative diagram

$$\begin{array}{ccc} \mathbf{Pro}(\mathbf{C})_{/X} & \xrightarrow{\coprod_X} & \mathbf{Pro}(\mathbf{C}) \\ {\scriptstyle L^{*\prime}}\big\downarrow & & \big\downarrow{\scriptstyle L'} \\ \mathbf{Pro}(\mathbf{D})_{/X} & \xrightarrow[\coprod_X]{} & \mathbf{Pro}(\mathbf{D}) \end{array} \tag{1}$$

(ii) Suppose L' arises from Lemma 3.3. That is, suppose L is J-left coadjoint to a functor $R\colon \mathbf{D} \to \mathbf{C}$, so L' is left adjoint to $\mathbf{Pro}(R)$ and automatically commutes with finite

[6]This category is denoted by $\mathbf{CoSh}(R)$ in [6].

[7]We allow $\mathbf{C} = \mathbf{Mod}(R)_{\mathrm{fin}}$ and $\mathbf{D} = \mathbf{Mod}(S)_{\mathrm{fin}}$ for different profinite rings R and S.

coproducts. Let $R^*\colon \mathbf{BD} \to \mathbf{BC}$ be constructed from R in the same way as L^* from L. Then L^* is J^*-left coadjoint to R^*, $L^{*\prime}$ is left adjoint to $\mathbf{Pro}(R^*)$, and the corresponding diagrams (1) and (2) above have all their maps be left adjoints.

Proof. (i) See Propositions 3.1 and 3.2.
(ii) See Proposition 3.6 and the paragraphs below Example 3.4.

□

Remark 3.9. Recall that Proposition 3.1 only asks for the functor F to commute with inverse limits and finite coproducts, not that it be a left adjoint. So if $U\colon \mathbf{Pro}(\mathbf{D}) \to \mathbf{Pro}(\mathbf{C})$ is any right adjoint, then the diagram

$$\begin{tikzcd} \mathbf{BPro}(\mathbf{C}) \arrow[r, "\coprod"] & \mathbf{Pro}(\mathbf{C}) \\ \mathbf{BPro}(\mathbf{D}) \arrow[u, "U^*"] \arrow[r, "\coprod"'] & \mathbf{Pro}(\mathbf{D}) \arrow[u, "U"'] \end{tikzcd}$$

commutes if U commutes with finite coproducts, since it always commutes with inverse limits (being a right adjoint). Thus, if $\mathbf{C} = \mathbf{Mod}(R)_{\text{fin}}$ and $\mathbf{D} = \mathbf{Mod}(S)_{\text{fin}}$, so that finite products and coproducts agree in $\mathbf{PMod}(R)$ and in $\mathbf{PMod}(S)$, then we get the somewhat unintuitive statement that right adjoints *always* preserve (internal) coproducts. This should sound a lot more reasonable once we realise that the internal coproduct $\widehat{\bigoplus}$ of profinite modules, despite being defined as a left adjoint, actually generalises categorical *products* of modules (see [3, Example 5.6.4(c)] or Example 3.11(iv) below).

Remark 3.10. In fact, we can allow more general algebraic theories with coproducts internal to $\mathbf{Pro}$ and with the property that "bundles commute with pro-completions". To make this precise, let $\mathbb{T}$ be a (finitary) Lawvere theory and write $\mathbf{Alg}_{\mathbb{T}}(\mathbf{C})$ for the category of $\mathbb{T}$-algebras in $\mathbf{C}$, where $\mathbf{C}$ is a category with finite products. For background on Lawvere theories, the reader may refer to [22].

Let us write $\mathbf{BAlg}_{\mathbb{T}}(\mathbf{Pro})$ for the category whose objects are pairs (P, X) with $X \in \mathbf{Pro}$ and $P \in \mathbf{Alg}_{\mathbb{T}}(\mathbf{Pro}_{/X})$. A morphism from (P, X) to (Q, Y) consists of continuous maps $f\colon P \to Q$ and $X \to Y$ making the obvious square commute, such that the restriction of f to each fibre above X is a $\mathbb{T}$-map. Observe that $\mathbf{BAlg}_{\mathbb{T}}(\mathbf{Pro})$ has all limits which are computed on fibres. Let $\mathbf{BAlg}_{\mathbb{T}}(\mathbf{Set}_{\text{fin}})$ be the full subcategory of $\mathbf{BAlg}_{\mathbb{T}}(\mathbf{Pro})$ where both spaces P and X are finite.

One can check that if the following two conditions on $\mathbb{T}$ hold, then we can replace $\mathbf{C}$ (and similarly for $\mathbf{D}$) by $\mathbf{Alg}_{\mathbb{T}}(\mathbf{Set}_{\text{fin}})$ in Theorem 3.8 and everything still works:

(a) For every $X \in \mathbf{Pro}$, the diagonal functor $\Delta_{X^\delta}\colon \mathbf{Alg}_{\mathbb{T}}(\mathbf{Pro}) \to \mathbf{Alg}_{\mathbb{T}}(\mathbf{Pro}_{/X})$ has a left adjoint, i.e. the theory $\mathbb{T}$ has all coproducts internal to $\mathbf{Pro}$. In particular, $\mathbf{Alg}_{\mathbb{T}}(\mathbf{Pro})$ has ordinary finite coproducts.
(b) The canonical map $\mathbf{Pro}(\mathbf{BAlg}_{\mathbb{T}}(\mathbf{Set}_{\text{fin}})) \to \mathbf{BAlg}_{\mathbb{T}}(\mathbf{Pro})$ is an equivalence. In particular, the canonical map $\mathbf{Pro}(\mathbf{Alg}_{\mathbb{T}}(\mathbf{Set}_{\text{fin}})) \to \mathbf{Alg}_{\mathbb{T}}(\mathbf{Pro})$ is an equivalence, i.e. pro-(finite $\mathbb{T}$-algebras) are exactly $\mathbb{T}$-algebras internal to $\mathbf{Pro}$.

Potential candidates for this include the theory of sets, groups, abelian groups, rings, semigroups, monoids, and so on. **Warning**: The standard single-sorted Lawvere theory of R-modules does not carry any topology on R, so we do not obtain the theory of profinite R-modules from this (for a fixed profinite ring R), at least not in the standard way (cf. Remark 2.2).

Let us now see various examples, both old and new, where diagram (1) or (2) appears. We could call some of these "corollaries", but we want to emphasise how easy it is to obtain the following from Theorem 3.8.

Example 3.11. (i) Consider the original diagram (1) as written on page 8 and take F to be the free profinite groups functor. This does arise from Lemma 3.3, as we saw in Example 3.4. If $(Y, X) = (Y \to X) \in \mathbf{Pro}_{/X}$, then we conclude from diagram (1) that $F(Y) = \coprod_X F^*(Y) = \coprod_X F(Y(x))$, where we recall that $Y(x)$ is the fibre of Y above $x \in X$, and that a topology is implicitly present in the final expression. This is saying that "the free group over a disjoint union $Y = \bigsqcup_X Y(x)$ is the free product $\coprod_X F(Y(x))$".

If we replace $\mathbf{PGrp}$ with $\mathbf{PMod}(R)$, $\mathbf{PGrp}_{/X} = \mathbf{Grp}(\mathbf{Pro}_{/X})$ with $\mathbf{PMod}(R)_{/X}$, and the free profinite groups functor with the free profinite R-modules functor $R[\![-]\!]$, then we get exactly [7, Proposition A.14], whose notations are different from ours.

(ii) Let k be a profinite ring and $H \leqslant G$ be profinite groups. Consider diagram (1) or (2) where the right vertical map is the induction functor $\mathrm{Ind}_H^G(-)\colon \mathbf{PMod}(k[\![H]\!]) \to \mathbf{PMod}(k[\![G]\!])$, which is left adjoint to restriction $\mathrm{Res}_H^G(-)$. As we noted at the end of Example 3.4, Lemma 3.3 applies here. In particular, we have managed to define induction and restriction in the abelian categories $\mathbf{PMod}(k[\![H]\!])_{/X}$ and $\mathbf{PMod}(k[\![G]\!])_{/X}$ (see Remark 3.7). We conclude from diagram (1) that given $(P, X) \in \mathbf{PMod}(k[\![H]\!])_{/X}$, we have $\mathrm{Ind}_H^G(\widehat{\bigoplus}_X P(x)) = \widehat{\bigoplus}_X(\mathrm{Ind}_H^G P(x))$, so "induction preserves coproducts (internal to $\mathbf{Pro}$)".

More generally, let R, S be profinite rings and $R \to S$ be a map. Then extension of scalars $(-) \mathbin{\widehat{\otimes}}_R S\colon \mathbf{PMod}(R) \to \mathbf{PMod}(S)$ is left adjoint to restriction $\mathbf{PMod}(S) \to \mathbf{PMod}(R)$, and Lemma 3.3 applies here. By Remark 3.9, restriction of scalars preserves $\widehat{\bigoplus}$ (cf. [6, Proposition 2.8]).

(iii) Let R, S be profinite rings and M be a profinite R-S-bimodule. Consider diagram (1) or (2) where the right vertical map is the profinite tensor product $(-) \mathbin{\widehat{\otimes}}_R M\colon \mathbf{PMod}(R) \to \mathbf{PMod}(S)$. This might seem more general than the last part of (ii), but we cannot in general apply Lemma 3.3 here. However, we can still apply Theorem 3.8(i). In particular, we have managed to define a tensor product $\mathbf{BPMod}(R) \to \mathbf{BPMod}(S)$. This is a special case of the construction in [6, Appendix A] (see also [5, Section 3.2]). To recover the full construction in [6, Appendix A], we will need to study products of bundle categories, which we will not carry out in the current paper. In addition, restricting to diagram (1) defines a tensor product $(-) \mathbin{\widehat{\otimes}}_R M\colon \mathbf{PMod}(R)_{/X} \to \mathbf{PMod}(S)_{/X}$ where both categories are abelian. We conclude from diagram (1) that "tensor products commute with coproducts".

If k is a profinite ring, $H \leqslant G$ are profinite groups, $R = k[\![H]\!]$, $S = k[\![G]\!]$, and $M = k[\![G]\!]$, then we leave it to the reader to check that the functor $\mathbf{PMod}(k[\![H]\!])_{/X} \to \mathbf{PMod}(k[\![G]\!])_{/X}$ obtained from (iii) is exactly the induction functor obtained from (ii). Perhaps surprisingly, using the universal property of pro-completions, we have managed to define induction of bundles (which is a left adjoint) *before* defining the tensor product of bundles (which is generally not a left adjoint).

Similarly, for a profinite ring R and $M \in \mathbf{PMod}(R)$, we can consider diagram (1) or (2) where the right vertical map is the Tor functor $\mathrm{Tor}(-, M)\colon \mathbf{PMod}(R) \to \mathbf{PAb}$. This extends to a Tor functor on bundles and the commutativity of diagram (1) is precisely [3, Theorem 9.1.1]. Some applications of this fact can be found in [6].

(iv) Consider diagram (1) or (2) where the right vertical map is the abelianisation functor $(-)^{\mathrm{ab}}\colon \mathbf{PGrp} \to \mathbf{PAb}$, which is left adjoint to the forgetful functor $\mathbf{PAb} \to \mathbf{PGrp}$. Lemma 3.3 trivially applies here, since the abelianisation of a finite group is still finite (unlike in (i) where a free profinite group/module with finite basis can very much be infinite, and similarly for (ii)). We used here the fact that the functor $G \mapsto G^{\mathrm{ab}} = G/\overline{[G,G]}$ preserves inverse limits (see [23, Section IV.2, Exercise 6/Section IV.1, Exercise 5]), so it is indeed the (unique) extension of the functor $(-)^{\mathrm{ab}}\colon \mathbf{Grp}_{\mathrm{fin}} \to \mathbf{Ab}_{\mathrm{fin}}$. We conclude from diagram (1) that given $(P,X) \in \mathbf{PGrp}_{/X}$, we have $(\coprod_X P(x))^{\mathrm{ab}} = \widehat{\bigoplus}_X (P(x)^{\mathrm{ab}})$.

To give a concrete example, let $I \in \mathbf{Set}$ and let $X = I \sqcup \{*\}$ be the one-point compactification of I. For $i \in I$, let G_i be a profinite group. Then we can form a bundle $(P,X) \in \mathbf{PGrp}_{/X}$, where $P(i) = G_i$ and $P(*) = 1$, as follows (see [3, Example 5.1.1(c)]). A local base around $p \in P(i) = G_i$ $(i \neq *)$ is given by the open subsets of G_i containing p, whereas a local base around $1 \in P(*)$ is given by subsets of the form $\bigsqcup_{u \in U} P(u)$, where U is an open neighbourhood of $* \in X$, i.e. U contains $*$ and almost all elements of I. This is called a *bundle of groups over I converging to 1*, and the corresponding coproduct $\coprod_X P(x) \in \mathbf{PGrp}$ is called the *restricted free product of the G_i* and denoted by $\coprod_{i\in I}^r G_i$ (see [3, Example 5.1.9]).

We claim that the abelianisation $(P^{\mathrm{ab}}, X) \in \mathbf{PAb}_{/X}$ of the above $(P,X) \in \mathbf{PGrp}_{/X}$ is also a bundle of (abelian) groups converging to 1. Let us recall how P^{ab} is constructed. First write $(P,X) = \varprojlim (P_i, X_i)$, where $(P_i, X_i) \in \mathbf{BGrp}_{\mathrm{fin}}$. Then, by definition, $P^{\mathrm{ab}} \to X$ is the inverse limit of the right maps in the diagrams

$$\begin{tikzcd} P_i \arrow[rr, two heads] \arrow[dr, two heads] & & P_i^{\mathrm{ab}} \arrow[dl, two heads] \\ & X_i & \end{tikzcd}$$

where $P_i = \bigsqcup_{x \in X_i} P_i(x)$ and $P_i^{\mathrm{ab}} = \bigsqcup_{x\in X_i} (P_i(x)^{\mathrm{ab}})$. Thus, the top map in the inverse limit

$$\begin{tikzcd} P \arrow[rr, two heads] \arrow[dr, two heads] & & P^{\mathrm{ab}} \arrow[dl, two heads] \\ & X & \end{tikzcd}$$

when restricted to each fibre is just the usual abelianisation $P(x) \twoheadrightarrow P^{\mathrm{ab}}(x) = P(x)^{\mathrm{ab}}$. In particular, the top map is a quotient map (which also follows from the simple fact that $P \twoheadrightarrow P^{\mathrm{ab}}$ is surjective). We leave it to the reader to verify that (P^{ab}, X) is therefore also a bundle converging to 1.

By [3, Example 5.6.4(c)], the internal coproduct $\widehat{\bigoplus}_X (P(x)^{\mathrm{ab}})$ of $(P^{\mathrm{ab}}, X) \in \mathbf{PAb}_{/X}$ is the categorical *product* $\prod_{i\in I} P(i)^{\mathrm{ab}} = \prod_{i \in I} G_i^{\mathrm{ab}} \in \mathbf{PAb}$. We therefore obtain the seemingly wrong statement $(\coprod_{i\in I}^r G_i)^{\mathrm{ab}} = \prod_{i\in I} G_i^{\mathrm{ab}}$. In particular, the abelianisation of $F(X) = F(I \sqcup \{*\}) = \coprod_I^r \widehat{\mathbb{Z}}$, the free profinite group on I (converging to 1), is $F(I \sqcup \{*\})^{\mathrm{ab}} = \prod_I \widehat{\mathbb{Z}}$, the free proabelian group on I (converging to 1); see [3, Example 5.1.3(c)] and [9, Example 3.3.8(c)].

Appendix A. Pontryagin duality for bundles

This paper is perhaps incomplete without at least mentioning Pontryagin duality, which forms a significant part of [4] and [5] (see [4, Theorem 4.1] and [5, Section 2.6]). Both of these papers utilise sheaf theory to obtain Pontryagin duality for bundles, but we will directly describe Pontryagin duality in terms of bundles. The following is therefore not original and can be recovered by carefully studying the previously mentioned papers.

Let us first point out that the perspectives of [2], [4], [1], [5] (and ours) are in fact equivalent. Indeed, [4] exactly follows the perspective of [2] (through the book [3]) by studying sheaves of profinite modules, whereas the equivalence between the perspectives of [5] and [1], which study étale spaces of profinite modules, is pointed out in [5, Proposition 2.15]. We note that [2] already states in the addendum that its main definition is equivalent to that of [1]. To elaborate, [5, Proposition 2.15] shows that the category of étale spaces under consideration is the pro-completion of the category of finite ones. Hence, by [6, Corollary 2.13], this is just the category of sheaves of profinite modules studied in [2].

In fact, the Pontryagin duality functors defined in [4] and [5] are also equivalent (i.e. naturally isomorphic), though they may seem distinct at first glance. This will be clear once we state Pontryagin duality in terms of bundles and pro-completions.

Let R be a fixed profinite ring and consider the category $\mathbf{BMod}(R)_{\mathrm{fin}}$ of bundles of finite (discrete topological) R-modules and its pro-completion $\mathbf{BPMod}(R)$. Recall that an object of $\mathbf{BMod}(R)_{\mathrm{fin}}$ is a pair (P, X) of finite sets together with a function $p\colon P \to X$, such that each fibre $P(x) = p^{-1}(\{x\})$ is a finite R-module. A morphism from (P, X) to (Q, Y) consists of maps $a\colon X \to Y$ and $f\colon P \to Q$ making the obvious square commute, such that the restriction of f to each $x \in X$ is a module homomorphism $P(x) \to Q(a(x))$.

The dual category will be given by the category $\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R)_{\mathrm{fin}}$ of "opposite bundles of finite R-modules", which one obtains by making all the spaces finite in [4, Definition 2.1]. Spelt out, $\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R)_{\mathrm{fin}}$ has the same objects as $\mathbf{BMod}(R)_{\mathrm{fin}}$, but a morphism from (P, X) to (Q, Y) now consists of maps $b\colon Y \to X$ and $g\colon P \times_X Y \to Q$ making the triangle

$$\begin{tikzcd} P \times_X Y \arrow[rr, "g"] \arrow[dr, "\pi_Y"'] & & Q \arrow[dl] \\ & Y & \end{tikzcd}$$

commute, such that the restriction of g to each $y \in Y$ is a module homomorphism $P(b(y)) \to Q(y)$.

There is a functor $(-)^\vee = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})\colon \mathbf{BMod}(R)_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}}$, the *Pontryagin duality functor*, which sends $(P, X) \in \mathbf{BMod}(R)_{\mathrm{fin}}^{\mathrm{op}}$ to the "opposite bundle" $(P^\vee, X)$, where $P^\vee(x) = P(x)^\vee$. Given a morphism $(P, X) \to (Q, Y)$ in $\mathbf{BMod}(R)_{\mathrm{fin}}^{\mathrm{op}}$ (i.e. maps $b\colon Y \to X$ and $f\colon Q \to P$ making the obvious square commute, such that each $Q(y) \to P(b(y))$ is a module homomorphism), the functor $(-)^\vee$ sends it to the morphism $(P^\vee, X) \to (Q^\vee, Y)$ in $\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}}$, where $b\colon Y \to X$ is as above, and $g\colon P^\vee \times_X Y \to Q^\vee$ is given on $y \in Y$ by taking the usual Pontryagin dual of $Q(y) \to P(b(y))$.

Similarly, there is a functor $(-)^\vee\colon \mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}} \to \mathbf{BMod}(R)_{\mathrm{fin}}^{\mathrm{op}}$. The following is then easy to verify.

Lemma A.1. *The functor* $(-)^\vee = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})\colon \mathbf{BMod}(R)_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}}$ *is an equivalence, i.e. it establishes a duality between* $\mathbf{BMod}(R)_{\mathrm{fin}}$ *and* $\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}}$.

Taking pro-completions on both sides, we then get the equivalence

$$\mathbf{BPMod}(R) = \mathbf{Pro}(\mathbf{BMod}(R)_{\mathrm{fin}}) = \mathbf{Pro}(\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}}^{\mathrm{op}}) = \mathbf{Ind}(\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}})^{\mathrm{op}},$$

where $\mathbf{Ind}$ means ind-completion. This therefore establishes a (Pontryagin) duality between $\mathbf{BPMod}(R) = \mathbf{Pro}(\mathbf{BMod}(R)_{\mathrm{fin}})$ and $\mathbf{Ind}(\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}})$.

Next, let us prove that the Pontryagin duality functor we have defined is in fact equivalent to the ones in [4] and [5]. Since we have defined $(-)^\vee$ using the universal property of pro-completions, it suffices to show that the corresponding functors in [4] and [5] commute with inverse limits and agree with ours on finite bundles. But the functors in [4] and [5] obviously commute with inverse limits (i.e. send inverse limits in $\mathbf{BPMod}(R)$ to direct limits in $\mathbf{Ind}(\mathbf{B}^{\mathrm{op}}\mathbf{Mod}(R^{\mathrm{op}})_{\mathrm{fin}})$), by the respective Pontryagin duality estabilished in those papers. For finite bundles, it is easy to check directly that the Pontryagin duality functors in [4] and [5] both agree with ours. For example, we only need to know that the Pontryagin duality functors in [4] and [5] commute with fibres: see [4, Theorem 4.3] and [5, Proposition 2.7].

We should point out there is obviously a dual theory to what we have done for modules in Section 3, which is studied in [4], [5] and [6]. We leave it to the interested reader to figure out the details.

References

[1] Dan Haran. On closed subgroups of free products of profinite groups. *Proceedings of the London Mathematical Society*, 3(2):266–298, 1987.

[2] Oleg Vladimirovich Mel'nikov. Subgroups and the homology of free products of profinite groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):97–120, 1989.

[3] Luis Ribes. *Profinite graphs and groups*, volume 66. Springer, 2017.

[4] Gareth Wilkes. Pontryagin duality and sheaves of profinite modules. *arXiv preprint arXiv:2408.13059*, 2024.

[5] Marco Boggi. Lannes' T-functor and mod-p cohomology of profinite groups. *arXiv preprint arXiv:2408.12488*, 2024.

[6] Jiacheng Tang. Profinite direct sums with applications to profinite groups of type Φ_R. *arXiv preprint arXiv:2504.05182 (Accepted in Bull. Lond. Math. Soc.)*, 2025.

[7] Gareth Wilkes. Relative cohomology theory for profinite groups. *J. Pure Appl. Algebra*, 223(4):1617–1688, 2019.

[8] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[9] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Springer, 2000.

[10] Francis Borceux. *Handbook of categorical algebra: Basic category theory*, volume 1. Cambridge University Press, 1994.

[11] Peter T Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume 2. Oxford University Press, 2002.

[12] Bart Jacobs. *Categorical logic and type theory*, volume 141. Elsevier, 1999.

[13] Luis Ribes. On amalgamated products of profinite groups. *Mathematische Zeitschrift*, 123(4):357–364, 1971.

[14] Otmar Venjakob. Subgroup theorems for free profinite products with amalgamation. *Journal of the London Mathematical Society*, 68(1):12–24, 2003.

[15] Jean-Pierre Serre. *Trees*. Springer Science & Business Media, 2002.

[16] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. Théorie des topos et cohomologie étale des schémas. *Lecture notes in mathematics*, 1972.

[17] Friedrich Ulmer. Properties of dense and relative adjoint functors. *Journal of Algebra*, 8(1):77–95, 1968.

[18] nLab authors. "pro-left adjoint", 2021. URI: https://ncatlab.org/nlab/show/pro-left+adjoint.

[19] Francis Borceux and Dominique Bourn. *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566. Springer Science & Business Media, 2004.

[20] Michael Barr. *Exact categories*. Springer, 1971.

[21] Vincenzo Marra and Luca Reggio. A characterisation of the category of compact Hausdorff spaces. *Theory Appl. Categ.*, 35:Paper No. 51, 1871–1906, 2020.

[22] F William Lawvere. Functorial semantics of algebraic theories. *Proceedings of the National Academy of Sciences*, 50(5):869–872, 1963.

[23] Jürgen Neukirch. *Algebraic number theory*, volume 322. Springer Science & Business Media, 2013.