

Turán number of four vertex-disjoint cliques

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Abstract. Given a graph H , the *Turán number* $\text{ex}(n, H)$ of H is the maximum number of edges of an n -vertex simple graph containing no H as a subgraph. Let kK_p denote the disjoint union of k copies of the complete graph K_p . In this paper, utilizing the idea of the proof of the Hajnal–Szemerédi Theorem and discharging, we determine the value $\text{ex}(n, 4K_p)$ for all n and $p \geq 3$.

Keywords. Turán number, Hajnal–Szemerédi Theorem, Discharging, Equitable Coloring.

1 Introduction

We use $|S|$ to denote the cardinality of S . Graphs in this paper are finite, undirected and simple. Terms and notation not defined here are from [4]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The number of edges of a graph G is denoted by $e(G)$. For a graph G , $v \in V(G)$ and $H \subseteq G$ (respectively, $S \subseteq V(G)$), the set of neighbors of v in H (respectively, S) is denoted by $N_H(v)$ (respectively, $N_S(v)$). We call $d_G(v) = |N_G(v)|$ the *degree of v in G* . For vertex subsets V_1 and V_2 of a graph G , we let $E[V_1, V_2]$ denote the set of edges of G with one endvertex in V_1 and the other in V_2 . Furthermore, let $e[V_1, V_2] = |E[V_1, V_2]|$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and maximum degree of a graph G . Let $\bar{G}$ denote the complement of a graph G . The *independence number* of a graph G is denoted by $\alpha(G)$. For a graph G and $V' \subseteq V(G)$, the subgraph of G induced by V' is denoted by $G[V']$. We use $G \cup H$ to denote the disjoint union of graphs G and H . We denote by $G \vee H$ the join operation of graphs G and H , which means adding an edge between each vertex in G and each vertex in H . We use K_p to denote the complete graph on p vertices. For given graphs G and H , G is H -free if G does not contain H as a subgraph.

The Turán number of a graph H , denoted by $\text{ex}(n, H)$, is the maximum number of edges in an H -free graph on n vertices. An n -vertex graph with $\text{ex}(n, H)$ edges not containing a copy of H is an *extremal* graph for H . Let $T_{n,p}$ denote the complete p -partite graph $K_{n_1, \dots, n_p}$, where $n_1 + \dots + n_p = n$ and $\lfloor \frac{n}{p} \rfloor \leq n_i \leq \lceil \frac{n}{p} \rceil$ for $1 \leq i \leq p$. Let $t_{n,p}$ be the number of edges of $T_{n,p}$. The famous result of Turán [24] is Theorem 1.1.

Theorem 1.1. [24]

$$\text{ex}(n, K_p) = t_{n,p-1}$$

and $T_{n,p-1}$ is the unique extremal graph.

Theorem 1.1 has the following corollary:

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Corollary 1.1.1 *Let G be a graph on n vertices with $\alpha(G) \leq p - 1$. Then $e(G) \geq \binom{n}{2} - t_{n,p-1}$ and $G = \overline{T_{n,p-1}}$ if $e(G) = \binom{n}{2} - t_{n,p-1}$.*

This extended the Mantel Theorem [21], which shows $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$. Since then, Turán number and extremal graphs of other graphs are widely studied in extremal graph theory. There are only a few graphs whose Turán number is determined exactly, see [10-12,17].

In 1959, Erdős and Gallai [10] determined the value $\text{ex}(n, kK_2)$ for all positive n and k .

Theorem 1.2. [10]

$$\text{ex}(n, kK_2) = \begin{cases} \binom{2k-1}{2}, & \text{if } 2k \leq n < \frac{5k}{2} - 1, \\ \binom{k-1}{2} + (k-1)(n-k+1), & \text{if } n \geq \frac{5k}{2} - 1. \end{cases}$$

Determining the Turán number of vertex-disjoint copies of cliques was studied by Erdős [9]. Some years later, Moon [22] and Simonovits [23] determined $\text{ex}(n, kK_p)$ for sufficiently large n .

Theorem 1.3. [22, 23] *For each fixed k and sufficiently large n ,*

$$\text{ex}(n, kK_p) = \binom{k-1}{2} + t_{n-k+1,p-1} + (k-1)(n-k+1),$$

and $K_{k-1} \vee T_{n-k+1,p-1}$ is the unique extremal graph for kK_p .

The remaining question is to determine the value of $\text{ex}(n, kK_p)$ for every n and k . There are very few cases when the Turán number $\text{ex}(n, kK_p)$ is known exactly for $n \geq kp$. In 2022, Chen, Lu and Yuan [8] determined the Turán number of two vertex-disjoint copies of K_p completely.

Theorem 1.4. [8] *If $p \geq 3$, then*

$$\text{ex}(n, 2K_p) = \begin{cases} \binom{n}{2} - 3(n-2p+1), & \text{if } 2p \leq n \leq 3p-2, \\ (n-1) + t_{n-1,p-1}, & \text{if } n \geq 3p-1. \end{cases}$$

Zhang and Yin [28] determined the value of $\text{ex}(n, K_p \cup K_q)$ for all n, q and $p = 2, 3$. Later, Hu [15] determined $\text{ex}(n, K_p \cup K_q)$ completely.

Theorem 1.5. [15] *Let n, p, q be positive integers with $q > p \geq 3$ and $n \geq p + q$. Then*

$$\text{ex}(n, K_p \cup K_q) = \begin{cases} \binom{n}{2} - 3(n-p-q+1), & \text{if } n \leq p+q + \max\{2p-q, \lfloor \frac{p}{2} \rfloor - 1\}, \\ t_{n,q-1}, & \text{if } n > p+q + \max\{2p-q, \lfloor \frac{p}{2} \rfloor - 1\}. \end{cases}$$

Brualdi and Mellendorf [3] and independently Zhang [27] determined $\text{ex}(kp, kK_p)$ for all k and p .

Theorem 1.6. [3, 27] *Let $p \geq 3, k \geq 1$ and $\overline{H}$ be an extremal graph for kK_p . Then*

$$\text{ex}(kp, kK_p) = \begin{cases} \binom{kp}{2} - \binom{k+1}{2}, & \text{if } k \leq 2p-2, \\ \binom{kp}{2} - (kp-p+1), & \text{if } k \geq 2p-1. \end{cases}$$

Moreover, $H \in \{K_{k+1} \cup \overline{K_{kp-k+1}}\}$ for $k \leq 2p-2$ and $H \in \{K_{1,x} \cup (kp-p-x+1)K_2 \cup (2p-kp+x-3)K_1 : kp-2p+3 \leq x \leq kp-p+1\}$ for $k \geq 2p-2$.

Recently, Zhang and Yin [29] and independently Zhang [27] determined the value of $\text{ex}(n, 3K_p)$ for all n .

Theorem 1.7. [27, 29] *If $p \geq 3$, then*

$$ex(n, 3K_p) = \begin{cases} \binom{n}{2} - 6, & \text{if } n = 3p, \\ \binom{n}{2} - 5(n - 3p + 1), & \text{if } 3p + 1 \leq n \leq 5p - 2, \\ 1 + 2(n - 2) + t_{n-2, p-1}, & \text{if } n \geq 5p - 1. \end{cases}$$

In this paper, we further determine $ex(n, 4K_p)$ for all $n \geq 4p$.

Theorem 1.8. *If $p \geq 3$, then*

$$(1) \quad ex(4p + 1, 4K_p) = \binom{n}{2} - 15,$$

$$(2) \quad ex(4p + 2, 4K_p) = \binom{n}{2} - 21,$$

$$(3) \quad ex(4p + 3, 4K_p) = \binom{n}{2} - 28,$$

$$(4) \quad ex(4p + 4, 4K_p) = \begin{cases} \binom{n}{2} - 35, & \text{if } p = 3, \\ \binom{n}{2} - 36, & \text{if } p \geq 4, \end{cases}$$

$$(5) \quad ex(n, 4K_p) = \begin{cases} \binom{n}{2} - 7(n - 4p + 1), & \text{if } 4p + 5 \leq n \leq 7p - 2, \\ 3 + 3(n - 1) + t_{n-3, p-1}, & \text{if } n \geq 7p - 1. \end{cases}$$

For some results on the Turán number for disjoint copies of other graphs, we refer the readers to [1, 2, 5, 6, 13, 18-20, 25, 26].

The structure of the paper is as follows. In the next section, we show sharpness of our bound, cite the known results and lemmas that we will use, and state Theorem 2.7, a slight sharpening of an important case of our main theorem. In Section 3, we prove Theorem 2.7, and in Section 4 use it to prove Theorem 1.8. We conclude the paper with some remarks in Section 5.

2 Preliminaries

Let $\overline{H}$ be a $4K_p$ -free graph. Then H contains no induced $4\overline{K_p}$. The idea of proof of Theorem 1.8 is to consider the complement of the extremal graph for $4K_p$. Let $f(n, G)$ denote the minimum number of edges in an n -vertex graph containing no induced G . To give an upper bound for the value of $f(n, 4\overline{K_p})$, we define $J(n)$ as follows:

For $p \geq 3$, $1 \leq s \leq 3p - 1$ and $n = 4p - 1 + s$,

$$J(n) = \begin{cases} \frac{s}{3}K_7 \cup \overline{K_{4p-1-\frac{4s}{3}}}, & \text{if } s \equiv 0 \pmod{3}, \\ \frac{s-4}{3}K_7 \cup K_8 \cup \overline{K_{4p-\frac{4s-1}{3}}}, & \text{if } s \equiv 1 \pmod{3}, \\ \frac{s-8}{3}K_7 \cup 2K_8 \cup \overline{K_{4p-\frac{4s-5}{3}}}, & \text{if } s \equiv 2 \pmod{3}. \end{cases}$$

Claim 2.1. $J(n)$ contains no induced $4\overline{K_p}$ and $e(J(n)) = 7s$.

Proof. If $s \equiv 0 \pmod{3}$, note that $\alpha(\frac{s}{3}K_7) = \frac{s}{3}$ and $\frac{s}{3}K_7$ contains 4 disjoint independent sets of size $\frac{s}{3}$, by $|V(\overline{K_{4p-1-\frac{4s}{3}}})| = 4p - 1 - \frac{4s}{3}$, then $J(n)$ contains no induced $4\overline{K_p}$. If $s \equiv 1 \pmod{3}$, noted that $\alpha(\frac{s-4}{3}K_7 \cup K_8) = \frac{s-1}{3}$ and $\frac{s-4}{3}K_7 \cup K_8$ contains 4 disjoint independent sets of size $\frac{s-1}{3}$, by $|V(\overline{K_{4p-\frac{4s-1}{3}}})| = 4p - \frac{4s-1}{3}$, then $J(n)$ contains no induced $4\overline{K_p}$. If $s \equiv 2 \pmod{3}$, noted that $\alpha(\frac{s-8}{3}K_7 \cup 2K_8) = \frac{s-2}{3}$

and $\frac{s-8}{3}K_7 \cup 2K_8$ contains 4 disjoint independent sets of size $\frac{s-2}{3}$, by $|V(\overline{K_{4p-\frac{4s-5}{3}}})| = 4p - \frac{4s-5}{3}$, then $J(n)$ contains no induced $4\overline{K_p}$. Therefore, $J(n)$ contains no induced $4\overline{K_p}$. Since $e(\frac{s}{3}K_7 \cup \overline{K_{4p-1-\frac{4s}{3}}}) = e(\frac{s-4}{3}K_7 \cup K_8 \cup \overline{K_{4p-\frac{4s-1}{3}}}) = e(\frac{s-8}{3}K_7 \cup 2K_8 \cup \overline{K_{4p-\frac{4s-5}{3}}}) = 7s$, $e(J(n)) = 7s$. $\square$

By Claim 2.1, $\overline{J(n)}$ is $4K_p$ -free and $e(J(n)) = 7s$. Thus $\text{ex}(n, 4K_p) \geq e(\overline{J(n)}) \geq \binom{n}{2} - 7s$. We want to prove that for $n \geq 4p + 5$ any graph with fewer edges than $J(n)$ has 4 disjoint independent sets of size p .

To prove this, we need some lemmas and known results of equitable coloring. An *equitable k -coloring* of a graph G is a proper k -coloring in which any two color classes differ in size by at most one. In 1970, Hajnal and Szemerédi [14] proved the following well-known result.

Theorem 2.2. [14] *Every graph with maximum degree at most r has an equitable $(r+1)$ -coloring.*

Chen, Lih and Wu [7] further proposed the following conjecture and confirmed the conjecture for $\Delta \leq 3$.

Conjecture 2.3. [7] *If G is an r -colorable graph with $\Delta(G) \leq r$, then either G has an equitable r -coloring, or r is odd and $K_{r,r} \subseteq G$.*

Kierstead and Kostochka [16] confirmed the conjecture for $\Delta = 4$.

Theorem 2.4. [16] *Let $r \leq 4$ and G be an r -colorable graph with $\Delta(G) \leq r$. Then either G has an equitable r -coloring or r is odd and G contains $K_{r,r}$.*

Based on Theorem 2.2 and the idea in [8], Zhang [27] obtained the following lemma.

Lemma 2.5. [27] *Let $G_p(n) = K_{k-1} \vee T_{n-k+1, p-1}$ and $n_0 \geq kp$, where $p \geq 3$ and $k \geq 2$. If $\text{ex}(n_0, kK_p) = e(G_p(n_0))$, then $\text{ex}(n, kK_p) = e(G_p(n))$ for every $n \geq n_0$.*

Using Theorem 2.4, Zhang and Yin [29] obtained following result.

Lemma 2.6. [29] *Let $p \geq 3$, $1 \leq s \leq 2p-1$ and $n = 3p-1+s$. Then*

$$f(n, 3\overline{K_p}) = \begin{cases} 6, & \text{if } s = 1, \\ 5s, & \text{if } 2 \leq s \leq 2p-1. \end{cases}$$

In the spirit of this lemma, we prove a theorem which solves an important part of our main result, Theorem 1.8.

Theorem 2.7. *Let $p \geq 3$ and G be a graph on $n = 4p-1+s$ vertices, where $1 \leq s \leq 3p-1$. If $|E(G)| \leq 7s$ and $\Delta(G) \leq 6$, then G contains a copy of $4K_p$ or $3|s$ and $G = \frac{s}{3}K_7 \cup \overline{K_{n-\frac{7s}{3}}}$.*

After proving Theorem 2.7, we consider several special cases for small n , and use induction and case analysis together with Theorem 2.7 to complete the proof of Theorem 1.8.

3 Proof of Theorem 2.7

We need some definitions. Given a partition $V_1, \dots, V_k$ of $V(G)$, define an auxiliary digraph D with vertices $V_1, \dots, V_k$, so that $V_i V_j$ ($1 \leq i, j \leq k$) is a directed edge if and only if some vertex $x \in V_i$ has no neighbors in V_j . In this case, we say that x is *movable* to V_j .

Let $V_{i_1}, V_{i_2}, \dots, V_{i_s} \in V(D)$ and $v_{i_\ell} \in V_{i_\ell}$ for each $\ell \in \{1, 2, \dots, s-1\}$, we define a vertex shifting:

$(v_{i_1}, V_{i_1}) \rightarrow (v_{i_2}, V_{i_2}) \rightarrow \dots \rightarrow V_s$ to denote moving the vertex $v_{i_\ell} \in V_{i_\ell}$ to $V_{i_{\ell+1}}$ for each $1 \leq \ell \leq s-1$. The sequence $V_{i_1} V_{i_2} \dots V_{i_s}$ is called an *accessible path from V_{i_1} to V_{i_s}* if we can find a vertex shifting as above, where for each $\ell \in \{1, 2, \dots, s-1\}$, v_{i_ℓ} is movable to $V_{i_{\ell+1}}$. We also say that V_j is inaccessible for V_i if no accessible path exists from V_j to V_i .

Set a target set V_k . Call $V_i \in V(D)$ *accessible* if there is an accessible path from V_i to V_k in D . Note that V_k is trivially accessible. Let $\mathcal{A}$ be the set of accessible classes, $\mathcal{B} = V(D) - \mathcal{A}$, $A = \bigcup \mathcal{A}$ and $B = \bigcup \mathcal{B}$.

For $v_i \in V_i$ and $v_j \in V_j$, the edge $v_i v_j$ between two classes is called a *solo edge* for v_i if it is the only edge from v_i to the class V_j . In this case, v_j is called a *solo neighbor* of v_i .

Claim 3.1. *For each $V_i \in \mathcal{B}$ and each $V_j \in \mathcal{A}$, every $v_i \in V_i$ has a neighbor in V_j .*

Proof of Claim 3.1. If there are $V_i \in \mathcal{B}$, $V_j \in \mathcal{A}$ and $v_i \in V_i$ such that $N_{V_j}(v_i) = \emptyset$, then $V_i V_j \in E(D)$. Since $V_j \in \mathcal{A}$, there is a directed path $V_j \dots V_k$ in D . Thus $V_i V_j \dots V_k$ is a directed path in D . This means that $V_i \in \mathcal{A}$, a contradiction. $\square$

Let G be a minimum counter example to Theorem 2.7 that satisfies all conditions without containing an induced copy of $4\overline{K_p}$. This implies that for any $uv \in E(G)$, $\overline{G - uv}$ contains a copy of $4K_p$, and one of these K_p contains both u and v . When we move u out of $V(4\overline{K_p})$, $V(G)$ can be divided into 5 classes V_1, V_2, V_3, V_4, V_5 , such that V_2, V_3, V_4, V_5 are four independent sets with $|V_2| = |V_3| = |V_4| = p$, $|V_5| = p-1$ and V_1 is the set of remaining vertices. Notice that $|V_1| = s$.

In the auxiliary digraph D , we let class V_5 be our destination set, $\mathcal{A}$ be composed of accessible sets and $\mathcal{B}$ be composed of inaccessible sets. By definition, $V_5 \in \mathcal{A}$. Note that class V_1 is always in $\mathcal{B}$, since otherwise there is a directed path from V_1 to V_5 . This directed path provides a vertex shifting which makes V_2, V_3, V_4, V_5 be 4 independent sets of size p .

Claim 3.2. *If $v \in V_i \in \mathcal{A} \setminus \{V_5\}$ is a solo neighbor of a vertex $x \in V_1$ and D has a directed path from V_j to V_5 that avoids V_i , then $N_{V_j}(v) \neq \emptyset$.*

Proof of Claim 3.2. If $N_{V_j}(v) = \emptyset$, then v is movable to V_j . We can move x to V_i and move v to V_j . The directed path from V_j to V_5 which avoids V_i in D provides a vertex shifting from V_j to V_5 that avoids V_i . By Claim 3.1, it also avoids V_1 . Hence $\overline{G}$ contains a copy of $4K_p$, a contradiction. $\square$

Claim 3.3. *No vertex $v \in V_i \in \mathcal{A}$, v is a solo neighbor of two non-adjacent vertices in V_1 .*

Proof of Claim 3.3. Assume that $v \in V_i \in \mathcal{A}$ is a solo neighbor of two non-adjacent vertices $v_1, v'_1 \in V_1$. Since $V_i \in \mathcal{A}$, there is a directed path $P = V_i \dots V_5$, where $V(P) \subseteq \mathcal{A}$. Let $v' \in V_i$ be movable to the successor of V_i . This directed path P provides a vertex shifting $(v', V_i) \rightarrow \dots \rightarrow V_5$. If $v' = v$, then we move v_1 to V_i ; and if $v' \neq v$, then we move v_1, v'_1 to V_i and move v to V_1 . In both cases, $\overline{G}$ contains a copy of $4K_p$, a contradiction. $\square$

Claim 3.4. *If $\Delta(G[V_1]) \leq 2$, then $1 \leq s \leq 3(p-1)$ and $e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}$.*

Proof of Claim 3.4 Suppose $\Delta(G[V_1]) \leq 2$. Then $G[V_1]$ is 3-colorable. Thus if $|V_1| = s \geq 3(p-1) + 1$, then $G[V_1]$ contains an induced $\overline{K_p}$. In this case, $G[V_1 \cup V_2 \cup V_3 \cup V_4]$ contains an induced copy of $4\overline{K_p}$, a contradiction. Thus $1 \leq s \leq 3(p-1)$. Note that $\alpha(G[V_1]) \leq p-1$, otherwise $G[V_1]$ contains an induced copy of $\overline{K_p}$, and so $\overline{G}$ contains a copy of $4K_p$, a contradiction. By Corollary 1.1.1, $e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}$. $\square$

Among all partitions as above, choose one with the smallest $|\mathcal{B}|$. (1)

In the next four subsections, we consider the four different possibilities for $|\mathcal{B}|$.

3.1 $|\mathcal{B}| = 1$.

By Claim 3.1, $e[V_1, V_2 \cup V_3 \cup V_4 \cup V_5] \geq 4s$. Since $V_1 \in \mathcal{B}$, $\Delta(G) \leq 6$ and every vertex in V_1 has a neighbor in each class in $\mathcal{A}$, $\Delta(G[V_1]) \leq 2$. We first prove some properties of the solo neighbors of vertices in V_1 , and then will use discharging to prove Theorem 2.7 in this case.

Claim 3.5. *We can move vertices between classes in $\mathcal{A}$ so that $|\mathcal{B}| = 1$ and at least two classes in $\mathcal{A} \setminus \{V_5\}$ are inneighbors of V_5 .*

Proof of Claim 3.5 Assume that there is exactly one class in $\mathcal{A} \setminus \{V_5\}$ that is an inneighbor of V_5 , say $V_2V_5 \in E(D)$, $V_3V_5 \notin E(D)$ and $V_4V_5 \notin E(D)$. This implies that any directed path from V_3 or V_4 to V_5 must go through V_2 . Without loss of generality, we assume $V_3V_2 \in E(D)$. Then we move a movable vertex in V_2 to V_5 , and let V_2 be a new destination set. Now every class in $\mathcal{A} \setminus \{V_2\}$ has a directed path to V_2 and $V_5V_2, V_3V_2 \in E(D)$. $\square$

Claim 3.6. *Every vertex in $V_i \in \mathcal{A} \setminus \{V_5\}$ that is the solo neighbor of a vertex in V_1 has neighbors in each class in $\mathcal{A} \setminus \{V_i\}$.*

Proof of Claim 3.6. Let $i \in \{2, 3, 4\}$, and $v_i \in V_i$ be a solo neighbor of $v_1 \in V_1$. By Claim 3.2, v_i has neighbors in V_5 . By Claim 3.5, it is enough to consider the following two cases.

Case 1. *Each class in $\mathcal{A} \setminus \{V_5\}$ is an inneighbor of V_5 .* Since $V_2V_5, V_3V_5, V_4V_5 \in E(D)$, we are done by Claim 3.2.

Case 2. *There are exactly two classes in $\mathcal{A} \setminus \{V_5\}$ that are inneighbors of V_5 .* We may let $V_2V_5 \notin E(D)$ and $V_3V_5, V_4V_5 \in E(D)$. Since $V_2 \in \mathcal{A}$, without loss of generality, we assume that $V_2V_3 \in E(D)$. If $V_4V_3 \in E(D)$, let $v'_3 \in V_3$ with $N_{V_5}(v'_3) = \emptyset$. Then we move v'_3 to V_5 , and let V_3 be a new destination set. Now $V_2V_3, V_4V_3, V_5V_3 \in E(D)$, and we have Case 1. So, assume that $V_4V_3 \notin E(D)$.

For any $v_2 \in V_2$, by $V_3V_5, V_4V_5 \in E(D)$ and Claim 3.2, $N_{V_3}(v_2) \neq \emptyset$ and $N_{V_4}(v_2) \neq \emptyset$. For any $v_4 \in V_4$, by $V_3V_5 \in E(D)$, $V_2V_3V_5 \subseteq D$ and Claim 3.2, $N_{V_2}(v_4) \neq \emptyset$ and $N_{V_3}(v_4) \neq \emptyset$. Next, we consider $v_3 \in V_3$. Since $V_4V_5 \in E(D)$ and Claim 3.2, $N_{V_4}(v_3) \neq \emptyset$.

Then we only need to prove that $N_{V_2}(v_3) \neq \emptyset$ for each $v_3 \in V_3$. If $V_2V_4 \in E(D)$, then $V_2V_4V_5 \subseteq D$, and by Claim 3.2, we have $N_{V_2}(v_3) \neq \emptyset$. So we assume that $V_2V_4 \notin E(D)$. Then $e[V_2, V_4] \geq p$. Recall that $V_2V_5, V_4V_3 \notin E(D)$, so $e[V_2, V_5] \geq p$ and $e[V_4, V_3] \geq p$.

By Claim 3.1, every vertex in V_1 has at least one neighbor in each class in $\mathcal{A}$. Assume that there are x vertices in V_1 that have a solo neighbor in V_2 , and $s - x$ vertices in V_1 that have at least 2 neighbors in V_2 . Recall that $\Delta(G[V_1]) \leq 2$. It follows from Theorem 2.2 and Claim 3.3 that there are at least $\lceil \frac{x}{3} \rceil$ vertices in V_1 that have the distinct solo neighbors in V_2 . Since $N_{V_3}(v_2) \neq \emptyset$,

$$e[V_1, V_2] + e[V_2, V_3] \geq x + 2(s - x) + \lceil \frac{x}{3} \rceil = 2s - \lfloor \frac{2x}{3} \rfloor \geq 2s - \frac{2x}{3}.$$

Clearly, $x \leq s$. Then $e[V_1, V_2] + e[V_2, V_3] \geq \frac{4s}{3}$. Similarly, it follows from $N_{V_5}(v_3) \neq \emptyset$ and $N_{V_5}(v_4) \neq \emptyset$ that $e[V_1, V_3] + e[V_3, V_5] \geq \frac{4s}{3}$ and $e[V_1, V_4] + e[V_4, V_5] \geq \frac{4s}{3}$. Moreover, Since $\Delta(G[V_1]) \leq 2$, by Claim 3.4, $1 \leq s \leq 3(p - 1)$ and $e(G[V_1]) \geq \binom{s}{2} - t_{s, p-1}$. In summary, we have the following:

$$\begin{cases} e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}, \\ e[V_1, V_2] + e[V_2, V_3] \geq \frac{4s}{3}, \quad e[V_1, V_3] + e[V_3, V_5] \geq \frac{4s}{3}, \quad e[V_1, V_4] + e[V_4, V_5] \geq \frac{4s}{3}, \\ e[V_1 \cup V_2, V_5] \geq s + p, \quad e[V_4, V_3] \geq p, \quad e[V_2, V_4] \geq p. \end{cases}$$

Summing these inequalities we get $e(G) \geq 5s + 3p + \binom{s}{2} - t_{s,p-1}$.

Since $e(\overline{T_{s,p-1}}) = \binom{s}{2} - t_{s,p-1}$, $\binom{s}{2} - t_{s,p-1} = s - (p-1)$ for $p \leq s < 2(p-1)$ and $\binom{s}{2} - t_{s,p-1} = 2s - 3(p-1)$ for $2(p-1) \leq s \leq 3(p-1)$. Then

$$e(G) \geq 5s + 3p + \binom{s}{2} - t_{s,p-1} = \begin{cases} 5s + 3p > 7s, & \text{if } 0 \leq s < p, \\ 5s + 3p + s - (p-1) = 6s + 2p + 1 > 7s, & \text{if } p \leq s < 2(p-1), \\ 5s + 3p + 2s - 3(p-1) = 7s + 3 > 7s, & \text{if } 2(p-1) \leq s \leq 3(p-1), \end{cases}$$

a contradiction. Finally we get that $N_{V_2}(v_3) \neq \emptyset$. $\square$

Consider the following discharging procedure. At the start, each $v \in V(G)$ has charge $ch(v) = 0$ and each $e \in E(G)$ has charge $ch(e) = 1$. So, $\sum_{x \in V(G) \cup E(G)} ch(x) = |E(G)| \leq 7s$. Now we will move the charges between edges and vertices without changing the total sum as follows.

Step 1. Every edge $xy \in E(G)$ such that $x \in V_1, y \notin V_1$ gives charge 1 to x .

Step 2. Every edge $xy \in E(G)$ such that $x \in V_1, y \in V_1$ gives charge $1/2$ to x and $1/2$ to y .

Step 3. Every edge $xy \in E(G)$ such that x is in $A \setminus V_5$ and is a solo neighbor of a vertex in V_1 , y is in V_5 or is not a solo neighbor of any vertices in V_1 , gives charge 1 to x .

Step 4. Every edge $xy \in E(G)$ such that x and y is in $A \setminus V_5$ and both are solo neighbors of some vertices in V_1 , gives charge $1/2$ to x and $1/2$ to y .

Step 5. Every vertex v in $V_i \in \mathcal{A}$ distributes its charge equally between the vertices in V_1 for which v is the solo neighbor in V_i .

Let the charge of each $x \in V(G) \cup E(G)$ after Step j be denoted by $ch_j(x)$. Then $ch_4(e) \geq 0$ for every $e \in E(G)$ and $ch_5(v) \geq 0$ for every $v \in A$. Therefore

$$e(G) \geq \sum_{v \in V_1} ch_5(v),$$

and the equality holds if and only if each edge of G either is incident to a vertex in V_1 or is incident to a solo neighbor (in $A \setminus V_5$) of a vertex in V_1 .

Claim 3.7. *For each $v \in V_1$, $ch_5(v) \geq 7$. Moreover, the equality holds if and only if $d_{V_1}(v) = 2$ and v has common solo neighbors in V_2, V_3, V_4 with its two neighbors in V_1 .*

Proof of Claim 3.7. Recall that $\Delta(G[V_1]) \leq 2$. Thus, we consider the following cases.

Case 1. $d_{V_1}(v) = 0$. By Claim 3.1, v has neighbors in each of V_2, V_3, V_4, V_5 . By Claim 3.3, solo neighbors of v are not solo neighbors of other vertices in V_1 . Assume that v has exactly x solo neighbors in $A \setminus V_5$. Since $\Delta(G) \leq 6$, $x \geq 1$. Then $ch_1(v) \geq x + (3-x) \cdot 2 + 1$. Let $V_i \in \mathcal{A} \setminus \{V_5\}$ and $v_i \in V_i$ be a solo neighbor of v . By Claim 3.6, v_i has neighbors in each class in $\mathcal{A} \setminus \{V_i\}$. Hence $ch_4(v_i) \geq \frac{1}{2} + \frac{1}{2} + 1 = 2$. On Step 5, each solo neighbor will send all its charge to v . Therefore,

$$ch_5(v) \geq x \cdot (1 + 2) + (3 - x) \cdot 2 + 1 = 7 + x \geq 8.$$

Case 2. $d_{V_1}(v) = 1$. Let $N_{V_1}(v) = \{v'\}$. By Claim 3.3, each solo neighbor of v in A can be a solo neighbor only of v and v' . Since $d_G(v) - d_{V_1}(v) \leq 6 - 1 = 5$, v has solo neighbors in at least 3 classes of $\mathcal{A}$.

If v has 4 solo neighbors in A , then $|N(v)| = 5$. In this case, let $N(v) = \{v', a_2, a_3, a_4, a_5\}$, where $a_i \in V_i$ for $2 \leq i \leq 5$. Then $ch_1(v) = 4$. At Step 2, v receives charge $\frac{1}{2}$ from edge vv' . By Claim 3.6, $ch_4(a_i) \geq \frac{1}{2} + \frac{1}{2} + 1 = 2$ for $2 \leq i \leq 4$. At Step 5, a_i ($2 \leq i \leq 4$) sends at least half of its charge to v . Therefore,

$$ch_5(v) \geq 4 + \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot 2 = 7.5 > 7.$$

If v has 3 solo neighbors, then v has 5 neighbors in A . Without loss of generality, we may assume that $a'_2 \in V_2$ and $a'_3 \in V_3$ are solo neighbors of v . Then $ch_1(v) = 5$. At Step 2, v receives charge $\frac{1}{2}$ from edge vv' . By Claim 3.6, $ch_4(a'_i) \geq \frac{1}{2} + \frac{1}{2} + 1 = 2$ for $2 \leq i \leq 3$. At Step 5, a_i ($2 \leq i \leq 4$) sends at least half of its charge to v . Therefore,

$$ch_5(v) \geq 5 + \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot 2 = 7.5 > 7.$$

Case 3. $d_{V_1}(v) = 2$. Let $N_{V_1}(v) = \{a, b\}$. For $2 \leq i \leq 5$, let v_i be a neighbor of v in $V_i \in \mathcal{A}$. Since $\Delta(G) \leq 6$ and $d_{V_1}(v) = 2$, v_i is the solo neighbor of v . Hence $ch_1(v) = 4$. At Step 2, v receives charge $\frac{1}{2}$ from each of the edges va and vb . By Claim 3.6, $ch_4(v_i) \geq \frac{1}{2} + \frac{1}{2} + 1 = 2$ for $2 \leq i \leq 4$. By Claim 3.3, v_i ($2 \leq i \leq 4$) can be a solo neighbor of only v, a and b . At Step 5, each v'_i ($2 \leq i \leq 4$) sends at least $\frac{1}{3}$ of its charge to v . Therefore,

$$ch_5(v) \geq 4 + 1 + 3 \cdot \frac{1}{3} \cdot 2 = 7,$$

and equality holds if and only if v_i ($2 \leq i \leq 4$) is the common solo neighbor of v, a and b . $\square$

By Claim 3.7, $ch_5(v) \geq 7$ for all $v \in V_1$. If there is a vertex $v' \in V_1$ with $ch_5(v') > 7$, then $e(G) \geq \sum_{v \in V_1} ch_5(v) > 7s$, a contradiction. Thus $ch_5(v) = 7$ for all $v \in V_1$. By Claim 3.7, then every $v \in V_1$ has two neighbors in V_1 that are adjacent to each other and to the solo neighbors of v in V_2, V_3 and V_4 . Then all vertices in V_1 are divided into triples $\{v, v', v''\}$ of mutually adjacent vertices. Thus $3|s$.

Next, we prove that $G = \frac{8}{3}K_7 \cup \overline{K_{n-\frac{7s}{3}}}$. We claim $G[\{v, v', v''\}, v_2, v_3, v_4] = K_6$. Assume that $G[\{v_2, v_3, v_4\}] \neq K_3$. By Claim 3.6 and $\Delta(G) \leq 6$, we may let $N_{V_3}(v_2) = \{u_3\}$ and $u_3 \neq v_3$. We switch v' with v_2 , then $\{v', v_3, v_4, v_5, u_3\} \subseteq N_A(\{v, v_2, v''\})$. By Claim 3.7, $ch_5(v) > 7$, a contradiction. Thus $G[\{v_2, v_3, v_4\}] = K_3$. This implies $G[\{v, v', v'', v_2, v_3, v_4\}] = K_6$. Recall that $N_{V_5}(v) = \{v_5\}$. By $\Delta(G) \leq 6$, $|N_{V_5}(v')| = |N_{V_5}(v'')| = |N_{V_5}(v_2)| = |N_{V_5}(v_3)| = |N_{V_5}(v_4)| = 1$.

Thus, for each triangle in $G[V_1]$, we can find the corresponding K_6 in $G[V_1 \cup V_2 \cup V_3 \cup V_4]$. For each K_6 , their neighbors in V_5 will not intersect with other K_6 's neighbors in V_5 . Otherwise we can move the vertices on K_6 so that this neighbor will be the solo neighbor of two non-adjacent vertices in V_1 , which contradicts Claim 3.3. Thus, we have $G = \bigcup_{i=1}^{\frac{8}{3}s} T_i \cup \overline{K_j}$, where $\sum_{i=1}^{\frac{8}{3}s} V(T_i) + V(\overline{K_j}) = n = 4p + s - 1$ and T_i is the induced subgraph of the vertex set consisting of a K_6 and its neighbors in V_5 .

Claim 3.8. *If $T_i \neq K_7$, then there are two vertices $a, a' \in V(T_i)$ such that $T_i \setminus \{a, a'\}$ has an equitable 4-coloring.*

Proof of Claim 3.8. Let $\{a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq V(T_i)$ and $T_i[\{a_1, a_2, a_3, a_4, a_5, a_6\}] = K_6$. Then there are two vertices $a, a' \in \{a_1, a_2, a_3, a_4, a_5, a_6\}$ so that $T_i \setminus \{a, a'\}$ contains no K_5 . Otherwise, for all $\{a, a'\} \subseteq \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $T_i \setminus \{a, a'\}$ contains a copy of K_5 . This implies $T_i = K_7$, a contradiction. Clearly, $\Delta(T_i \setminus \{a, a'\}) \leq 4$, thus $T_i \setminus \{a, a'\}$ is 4-colorable. By Theorem 2.4, $T_i \setminus \{a, a'\}$ has an equitable 4-coloring. $\square$

Let I_1, I_2, I_3, I_4 be four empty sets. We will put vertices into I_1, I_2, I_3 and I_4 by keeping them independent sets. If $T_i = K_7$, then we can put 4 vertices into I_1, I_2, I_3 and I_4 such that I_1, I_2, I_3, I_4 is equitable. If $T_i \neq K_7$, by Claim 3.8, then we can put $|V(T_i)| - 2$ vertices into I_1, I_2, I_3 and I_4 such that I_1, I_2, I_3, I_4 is equitable. After work on all T_i , we assign remaining isolated vertices in $\overline{K_j}$ to I_1, I_2, I_3, I_4 and always keep I_1, I_2, I_3 and I_4 equitable. Let the number of K_7 be x . Then

$$|I_i| \geq \frac{1}{4} \left(n - 3x - 2 \left(\frac{s}{3} - x \right) \right) = \frac{1}{4} \left(4p - 1 + \frac{s}{3} - x \right)$$

for $1 \leq i \leq 4$. Note that $0 \leq x \leq \frac{s}{3}$. If $0 \leq x < \frac{s}{3}$, then $|I_i| \geq p$ for $1 \leq i \leq 4$. This implies that $\overline{G}$ contains a copy of $4K_p$, a contradiction. Thus $x = \frac{s}{3}$. Then $G = \frac{s}{3}K_7 \cup \overline{K_{n-\frac{7s}{3}}}$.

3.2 $|\mathcal{B}| = 2$

We may let $\mathcal{B} = \{V_1, V_2\}$. By Claim 3.1, $e[V_1, V_3 \cup V_4 \cup V_5] \geq 3s$ and $e[V_2, V_3 \cup V_4 \cup V_5] \geq 3p$. Moreover, we can assume that $V_3V_5, V_4V_5 \in E(D)$. Otherwise there is an directed path $V_3V_4V_5$ (or $V_4V_3V_5$). Then we can move the movable vertex in V_4 (or V_3) to V_5 , and let V_4 (or V_3) be the new destination set. The new case is equivalent to the previous case where $V_3V_5, V_4V_5 \in E(D)$.

Case 1. $V_1V_2 \notin E(D)$. Since $V_1V_3, V_1V_4, V_1V_5 \notin E(D)$, $\Delta(G[V_1]) \leq 2$. Consider the following discharging procedure. At the start, each $v \in V(G) \setminus V_2$ has charge $ch(v) = 0$ and each $e \in E[V_1, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ has charge $ch(e) = 1$. So, $\sum_{x \in V(G) \setminus V_2 \cup E[V_1, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])} ch(x) = e[V_1, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5])$. Now we will move the charges between edges and vertices without changing the total sum as follows.

Step 1. Every edge $xy \in E[V_1, V_3 \cup V_4 \cup V_5]$ such that $x \in V_1, y \notin V_1$ gives charge 1 to x .

Step 2. Every edge $xy \in E(G[V_3 \cup V_4 \cup V_5])$ such that x is in $A \setminus V_5$ and is a solo neighbor of a vertex in V_1 , y is in V_5 or is not a solo neighbor of any vertices in V_1 , gives charge 1 to x .

Step 3. Every edge $xy \in E(G[V_3 \cup V_4 \cup V_5])$ such that x and y are in $A \setminus V_5$ and both are solo neighbors of some vertices in V_1 , gives charge $1/2$ to x and $1/2$ to y .

Step 4. Every vertex v in $V_i \in \mathcal{A}$ distributes its charge equally between the vertices in V_1 for which v is the solo neighbor in V_i .

Let the charge of each $x \in V(G) \setminus V_2 \cup E[V_1, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ after Step j be denoted by $ch_j(x)$. Then $ch_3(e) \geq 0$ for every $e \in E[V_1, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ and $ch_4(v) \geq 0$ for every $v \in A$. Therefore

$$e[V_1, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]) \geq \sum_{v \in V_1} ch_4(v),$$

and the equality holds if and only if each edge of $E[V_1, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ either is incident to a vertex in V_1 or is incident to a solo neighbor in $A \setminus V_5$ of a vertex in V_1 .

Claim 3.9. For each $v \in V_1$, $ch_4(v) \geq 4$.

Proof of Claim 3.9. Recall that $\Delta(G[V_1]) \leq 2$. Then we have following cases.

Case i. $d_{V_1}(v) = 0$. By Claim 3.1, v has neighbors in each of V_3, V_4, V_5 . By Claim 3.3, those solo neighbors cannot be the solo neighbors of another vertex in V_1 . Assume that v has exactly x solo neighbors in $A \setminus V_5$. Since $\Delta(G) \leq 6$, $ch_1(v) \geq x + (2 - x) \cdot 2 + 1$.

Let $V_i \in \{V_3, V_4\}$ and $v_i \in V_i$ be a solo neighbor of v . By $V_3V_5, V_4V_5 \in E(D)$ and Claim 3.2, v_i has neighbors in each class in $\{V_3, V_4, V_5\} \setminus \{V_i\}$. Hence $ch_3(v_i) \geq \frac{1}{2} + 1$. On Step 4, each solo neighbor will send all its charge to v . Therefore,

$$ch_4(v) \geq x \cdot \left(\frac{1}{2} + 2\right) + (2 - x) \cdot 2 + 1 = 5 + \frac{x}{2} > 4.$$

Case ii. $d_{V_1}(v) = 1$. Let $N_{V_1}(v) = \{v'\}$. By Claim 3.3, each solo neighbor of v in A can be a solo neighbor only of v and v' . Since $d_G(v) - d_{V_1}(v) \leq 6 - 1 = 5$ and $V_1V_2 \notin E(D)$, v has solo neighbors in at least two classes of $\mathcal{A}$.

If v has 3 solo neighbors in A , then $|N_A(v)| = 3$. Thus $ch_1(v) = 3$. In this case, let $N_A(v) = \{a_3, a_4, a_5\}$, where $a_i \in V_i$ for $3 \leq i \leq 5$. By $V_3V_5, V_4V_5 \in E(D)$ and Claim 3.2, $ch_3(a_i) \geq \frac{1}{2} + 1 = \frac{3}{2}$ for $3 \leq i \leq 4$. At Step 5, a_i ($3 \leq i \leq 4$) sends at least half of its charge to v . Therefore,

$$ch_4(v) \geq 3 + 2 \cdot \frac{1}{2} \cdot \frac{3}{2} = 4.5 > 4.$$

If v has 2 solo neighbors, then $|N_A(v)| \geq 4$. Thus $ch_4(v) \geq ch_1(v) \geq 4$.

Case iii. $d_{V_1}(v) = 2$. Let $N_{V_1}(v) = \{a, b\}$. For $3 \leq i \leq 5$, let v_i be a neighbor of v in $V_i \in \mathcal{A}$. Since $\Delta(G) \leq 6$, $d_{V_1}(v) = 2$ and $V_1V_2 \notin E(D)$, v_i is the solo neighbor of v . Hence $ch_1(v) = 3$. By $V_3V_5, V_4V_5 \in E(D)$ and Claim 3.2, $ch_3(v_i) \geq \frac{1}{2} + 1 = \frac{3}{2}$ for $3 \leq i \leq 4$. By Claim 3.3, v_i ($3 \leq i \leq 4$) can be a solo neighbor of only v, a and b . At Step 4, each v'_i ($3 \leq i \leq 4$) sends at least $\frac{1}{3}$ of its charge to v . Therefore,

$$ch_4(v) \geq 3 + 2 \cdot \frac{1}{3} \cdot \frac{3}{2} = 4.$$

□

It follows from Claim 3.9 that $e[V_1, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]) \geq 4s$. By $\Delta(G[V_1]) \leq 2$ and Claim 3.4, $1 \leq s \leq 3(p-1)$ and $e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}$. Since $V_1V_2 \notin E(D)$, $e[V_1, V_2] \geq s$. In summary, we have the following:

$$\begin{cases} e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}, \\ e[V_1, V_2] \geq s, \\ e[V_1, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]) \geq 4s, \\ e[V_2, V_3 \cup V_4 \cup V_5] \geq 3p. \end{cases}$$

Then $e(G) \geq 5s + 3p + \binom{s}{2} - t_{s,p-1} \geq \begin{cases} 5s + 3p > 7s, & \text{if } 0 \leq s < p, \\ 5s + 3p + s - (p-1) = 6s + 2p + 1 > 7s, & \text{if } p \leq s < 2(p-1), \\ 5s + 3p + 2s - 3(p-1) = 7s + 3 > 7s, & \text{if } 2(p-1) \leq s \leq 3(p-1), \end{cases}$
a contradiction.

Case 2. $V_1V_2 \in E(D)$. Since $\mathcal{B} = \{V_1, V_2\}$, $\Delta(G[V_1 \cup V_2]) \leq 3$. Let $v_1 \in V_1$ be movable to V_2 .

Claim 3.10. For $i = 3, 4$, let $v_i \in V_i$ be a solo neighbor of $v \in V_1 \cup V_2$. Then v_i has neighbors in each class of $\{V_3, V_4, V_5\} \setminus \{V_i\}$.

Proof of Claim 3.10. If $v \in V_2$, then we move v_1 to V_2 and move v to V_1 . Thus we may let $v \in V_1$. By $V_3V_5, V_4V_5 \in E(D)$ and Claim 3.2, then $N_{\{V_3, V_4, V_5\} \setminus \{V_i\}}(v_i) \neq \emptyset$. □

Claim 3.11. For every vertex $v \in V_i \in \mathcal{A} \setminus \{V_5\}$, v is not a solo neighbor of 2 non-adjacent vertices in $V_1 \cup V_2$.

Proof of Claim 3.11. Assume that $v \in V_i \in \mathcal{A} \setminus \{V_5\}$ is a solo neighbor of 2 vertices $v_x, v_y \in V_1 \cup V_2$. If $v_x, v_y \in V_1$, then by Claim 3.3, we are done. Thus at least one of v_x, v_y is in V_2 . We may let $v_y \in V_2$.

Recall that $V_i V_5 \in E(D)$ for $i = 3, 4$. Let $v' \in V_i$ be movable to V_5 . By Claim 3.10, $v' \neq v$. Then there is a vertex shifting $(v_x, V_1 \cup V_2) \rightarrow (v, V_i) \rightarrow V_1$. After the vertex shifting, v_y is movable to V_i and $V_i V_5 \in E(D)$ still holds for $i = 3, 4$. If $v_x \in V_1$, then $|\mathcal{B}| \leq 1$. This contradicts (1). If $v_x \in V_2$, then we move v_1 to V_2 . Again $|\mathcal{B}| \leq 1$. $\square$

Consider the following discharging procedure. At the start, each $v \in V(G)$ has charge $ch(v) = 0$ and each $e \in E[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ has charge $ch(e) = 1$. So,

$$\sum_{x \in V(G) \cup E[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])} ch(x) = e[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]).$$

Now we will move the charges between edges and vertices without changing the total sum as follows.

Step 1. Every edge $xy \in E[V_1 \cup V_2, V_3 \cup V_4 \cup V_5]$ such that $x \in V_1 \cup V_2, y \notin V_1 \cup V_2$ gives charge 1 to x .

Step 2. Every edge $xy \in E(G[V_3 \cup V_4 \cup V_5])$ such that x is in $A \setminus V_5$ and is a solo neighbor of a vertex in $V_1 \cup V_2$, y is in V_5 or is not a solo neighbor of any vertices in $V_1 \cup V_2$, gives charge 1 to x .

Step 3. Every edge $xy \in E(G[V_3 \cup V_4 \cup V_5])$ such that x and y is in $A \setminus V_5$ and both are solo neighbors of some vertices in $V_1 \cup V_2$, gives charge $1/2$ to x and $1/2$ to y .

Step 4. Every vertex v in $V_i \in \mathcal{A}$ distributes its charge equally between the vertices in $V_1 \cup V_2$ for which v is the solo neighbor in V_i .

Let the charge of each $x \in V(G) \cup E[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ after Step j be denoted by $ch_j(x)$. Then $ch_3(e) \geq 0$ for every $e \in E[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ and $ch_4(v) \geq 0$ for every $v \in A$. Therefore

$$e[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]) \geq \sum_{v \in V_1 \cup V_2} ch_4(v),$$

and the equality holds if and only if each edge of $E[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] \cup E(G[V_3 \cup V_4 \cup V_5])$ either is incident to a vertex in $V_1 \cup V_2$ or is incident to a solo neighbor (in $A \setminus V_5$) of a vertex in $V_1 \cup V_2$.

Claim 3.12. For each $v \in V_1 \cup V_2$, $ch(v) \geq \frac{15}{4}$.

Proof of Claim 3.12. By Claim 3.1, v has neighbors in each of V_3, V_4, V_5 . If v has a solo neighbor $v_3 \in V_3$ and a solo neighbor $v_4 \in V_4$, then $|N_A(v)| \geq 3$. Thus $ch_1(v) \geq 3$. By Claim 3.10, v_i has neighbors in each class in $\{V_3, V_4, V_5\} \setminus \{V_i\}$ for $i = 3, 4$. Hence $ch_3(v_i) \geq \frac{1}{2} + 1 = \frac{3}{2}$. Note that $\Delta(G[V_1 \cup V_2]) \leq 3$. By Claim 3.11, v_i ($3 \leq i \leq 4$) can be a solo neighbor of only v and the vertices in $N_{V_1 \cup V_2}(v)$. On Step 4, each solo neighbor will send at least $\frac{1}{4}$ of its charge to v . Therefore,

$$ch_4(v) \geq 3 + 2 \cdot \frac{1}{4} \cdot \frac{3}{2} = \frac{15}{4}.$$

Assume that v has the solo neighbor in exactly one class of $\{V_3, V_4\}$. We may let $v'_3 \in V_3$ be a solo neighbor of v and v has at least 2 neighbors in V_4 . Then $|N_A(v)| \geq 4$. Thus $ch_4(v) \geq ch_1(v) \geq 4 > \frac{15}{4}$.

If v has at least 2 neighbors in V_3 and V_4 , then $|N_A(v)| \geq 5$. Thus $ch_4(v) \geq ch_1(v) \geq 5 > \frac{15}{4}$. $\square$

If $s < p$, by Claim 3.12, $e(G) \geq e[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]) \geq \frac{15}{4}(s+p) > 7s$, a contradiction. Assume that $p \leq s \leq 3p - 1$. If $G[V_1 \cup V_2]$ contains an induced copy of $2\overline{K}_p$, then $G[V_1 \cup V_2 \cup V_3 \cup V_4]$

contains an induced copy of $4\overline{K_p}$, a contradiction. If $G[V_1 \cup V_2]$ contains no induced $2\overline{K_p}$, then by Theorem 1.4, $e[V_1 \cup V_2] \geq 3(s - p + 1)$. Thus $e(G) \geq e[V_1 \cup V_2] + e[V_1 \cup V_2, V_3 \cup V_4 \cup V_5] + e(G[V_3 \cup V_4 \cup V_5]) \geq 3(s - p + 1) + \frac{15}{4}(s + p) = \frac{27}{4}s + \frac{3}{4}p + 3 > 7s$, a contradiction.

3.3 $|\mathcal{B}| = 3$.

We may let $\mathcal{B} = \{V_1, V_2, V_3\}$. Then $\mathcal{A} = \{V_4, V_5\}$. This implies $V_4V_5 \in E(D)$. By Claim 3.1, $e[V_1 \cup V_2 \cup V_3, V_4 \cup V_5] \geq 2s + 4p$.

Case 1. $V_1V_2 \in E(D)$ or $V_1V_3 \in E(D)$. We may let $V_1V_2 \in E(D)$. Let $v_1 \in V_1$ be movable to V_2 .

Claim 3.13. *No vertex in V_4 is a solo neighbor of 2 vertices in V_2 .*

Proof of Claim 3.13. Assume that $v \in V_4$ is a solo neighbor of 2 vertices $v_2, v'_2 \in V_2$. Then v has a neighbor in V_5 . Otherwise we can obtain an induced copy of $4\overline{K_p}$ by vertex shifting $(v_1, V_1) \rightarrow (v_2, V_2) \rightarrow (v, V_4) \rightarrow V_5$. Let $v' \in V_4$ be movable to V_5 . Then $v' \neq v$. Note that there is a vertex shifting $(v_1, V_1) \rightarrow (v_2, V_2) \rightarrow (v, V_4) \rightarrow V_1$. After the vertex shifting, v'_2 is movable to V_4 and $V_4V_5 \in E(D)$ still holds. Then $|\mathcal{B}| \leq 2$. This contradicts (1). $\square$

Consider $e[V_2, V_4] + e[V_4, V_5]$. By Claim 3.1, every $v \in V_2$ has at least one neighbor in V_4 . If v has a solo neighbor $v_4 \in V_4$, then v_4 has neighbor in V_5 . Otherwise we can obtain an induced copy of $4\overline{K_p}$ by vertex shifting $(v_1, V_1) \rightarrow (v, V_2) \rightarrow (v_4, V_4) \rightarrow V_5$. Let there be x vertices in V_2 having a solo neighbor in V_4 . Then $e[V_2, V_4] + e[V_4, V_5] \geq 2x + 2(p - x) = 2p$.

If $s < p$, then

$$e(G) \geq e[V_1, V_4 \cup V_5] + (e[V_2, V_4] + e[V_4, V_5]) + e[V_2, V_5] + e[V_3, V_4 \cup V_5] \geq 2s + 2p + p + 2p = 2s + 5p > 7s,$$

a contradiction. Assume that $p \leq s \leq 3p - 1$. If $G[V_1 \cup V_2 \cup V_3]$ contains an induced copy of $3\overline{K_p}$, then $G[V_1 \cup V_2 \cup V_3 \cup V_4]$ contains an induced copy of $4\overline{K_p}$, a contradiction. If $G[V_1 \cup V_2 \cup V_3]$ contains no induced $3\overline{K_p}$, by Theorem 1.7, then $e[V_1 \cup V_2 \cup V_3] \geq 5(s - p + 1)$. In summary, we have the following:

$$\begin{cases} e[V_1 \cup V_2 \cup V_3] \geq 5(s - p + 1), \\ e[V_1, V_4 \cup V_5] \geq 2s, & e[V_3, V_4 \cup V_5] \geq 2p, \\ e[V_2, V_4] + e[V_4, V_5] \geq 2p, & e[V_2, V_5] \geq p. \end{cases}$$

Then $e(G) \geq 7s + 5$, a contradiction.

Case 2. $V_1V_2, V_1V_3 \notin E(D)$. Note that $\Delta(G[V_1]) \leq 2$. Let v and v' be 2 non-adjacent vertices in V_1 . If $v_i \in V_2$ (or V_3) is a common solo neighbor of v, v' , then we move v to V_2 (or V_3) and move v_i to V_1 . Thus we obtain $V_1V_2 \in E(D)$ (or $V_1V_3 \in E(D)$) and accordingly turn to Case 1, which is done. Therefore, we can assume that any 2 non-adjacent vertices $v, v' \in V_1$ have no common solo neighbors in V_2 or V_3 .

Let $v_2 \in V_2$ be a solo neighbor of $v \in V_1$. If $N_{V_3}(v_2) = \emptyset$, then we can switch v and v_2 to turn into Case 1 (by $V_1V_3 \in E(D)$), which is done. Thus we can assume that v_2 has neighbors in V_3 . Consider $e[V_1, V_2] + e[V_2, V_3]$. Since $V_1V_2 \notin E(D)$, we may assume that there are x vertices in V_1 that have a solo neighbor in V_2 , and $s - x$ vertices in V_1 that have at least 2 neighbors in V_2 . It follows from $\Delta(G[V_1]) \leq 2$ and Theorem 2.2 that there are at least $\lceil \frac{x}{3} \rceil$ vertices in V_1 that all have distinct solo neighbors in V_2 . Thus $e[V_1, V_2] + e[V_2, V_3] \geq x + 2(s - x) + \lceil \frac{x}{3} \rceil = 2s - \lfloor \frac{2x}{3} \rfloor \geq 2s - \frac{2x}{3}$. Clearly, $x \leq s$. Then $e[V_1, V_2] + e[V_2, V_3] \geq \frac{4s}{3}$.

Now consider $e[V_1, V_4] + e[V_4, V_5]$. By Claim 3.3, no two non-adjacent vertices $v_1, v'_1 \in V_1$ have a common solo neighbor in V_4 . Let $v_4 \in V_4$ be a solo neighbor of $v_1 \in V_1$. By Claim 3.2, v_4 has neighbors in V_5 . Similar

to the above discussion, $e[V_1, V_4] + e[V_4, V_5] \geq \frac{4s}{3}$.

By $\Delta(G[V_1]) \leq 2$ and Claim 3.4, $1 \leq s \leq 3(p-1)$ and $e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}$. In summary, we have the following:

$$\begin{cases} e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}, \\ e[V_1, V_2] + e[V_2, V_3] \geq \frac{4s}{3}, & e[V_1, V_3] \geq s, \\ e[V_1, V_4] + e[V_4, V_5] \geq \frac{4s}{3}, & e[V_1, V_5] \geq s, \\ e[V_2, V_4 \cup V_5] \geq 2p, & e[V_3, V_4 \cup V_5] \geq 2p. \end{cases}$$

Then

$$e(G) \geq \frac{14s}{3} + 4p + \binom{s}{2} - t_{s,p-1} = \begin{cases} \frac{14s}{3} + 4p > 7s, & \text{if } 0 \leq s < p, \\ \frac{14s}{3} + 4p + s - (p-1) = \frac{17s}{3} + 3p + 1 > 7s, & \text{if } p \leq s < 2(p-1), \\ \frac{14s}{3} + 4p + 2s - 3(p-1) = \frac{20s}{3} + p + 3 > 7s, & \text{if } 2(p-1) \leq s \leq 3(p-1), \end{cases}$$

a contradiction.

3.4 $|\mathcal{B}| = 4$

Then $\mathcal{B} = \{V_1, V_2, V_3, V_4\}$. By Claim 3.1, $e[V_1, V_5] \geq s$ and $e[V_i, V_5] \geq p$ for $2 \leq i \leq 4$. Define $\mathcal{B}'$ as the set of all color classes V_i in $\mathcal{B}$ such that there are no accessible paths from V_1 to V_i .

Choose our partition with minimum $\sum_{v \in V_5} d(v)$, and modulo this, with minimum $|\mathcal{B}'|$. (2)

Claim 3.14. *If $v \in V_5$ is the solo neighbor of $u \in V_1 \cup V_2 \cup V_3 \cup V_4$, then $d_G(u) \geq d_G(v)$.*

Proof of Claim 3.14. Assume that $d_G(u) < d_G(v)$. It suffices to prove that after moving u to V_5 and v to V_1 , we can switch vertices in $\mathcal{B}$ so that V_2, V_3 and V_4 are independent sets of size p .

If $u \in V_1$, then we move u to V_5 and move v to V_1 . Since $d_G(u) < d_G(v)$, this contradicts (2). Assume that $u \in V_i$, where $i \in \{2, 3, 4\}$. Clearly, $0 \leq |\mathcal{B}'| \leq 3$.

Case 1. $|\mathcal{B}'| = 0$. Since $V_i \notin \mathcal{B}'$, there is an accessible path P from V_1 to V_i . We move u to V_5 , move v to V_1 and move vertices along the accessible path P . Since $d_G(u) < d_G(v)$, this contradicts (2).

Case 2. $|\mathcal{B}'| = 1$. We may let $\mathcal{B}' = \{V_2\}$. Then $e[V_1, V_2] \geq s$ and V_3, V_4 are reachable from V_1 . This implies $V_3V_2, V_4V_2 \notin E(D)$. Thus $e[V_3 \cup V_4, V_2] \geq 2p$. In summary, we have the following:

$$e[V_1, V_2] \geq s, \quad e[V_1, V_5] \geq s, \quad e[V_3 \cup V_4, V_2] \geq 2p, \quad e[V_2 \cup V_3 \cup V_4, V_5] \geq 3p.$$

Hence $e[V_1 \cup V_3 \cup V_4, V_2 \cup V_5] + e(G[V_2 \cup V_5]) \geq (2s + 4p) + p = 2s + 5p$. If $s < p$, then $e(G) \geq 2s + 5p > 7s$, a contradiction. Assume that $p \leq s \leq 3p - 1$. If $G[V_1 \cup V_3 \cup V_4]$ contains an induced copy of $3\overline{K_p}$, then $G[V_1 \cup V_2 \cup V_3 \cup V_4]$ contains an induced copy of $4\overline{K_p}$, a contradiction. If $G[V_1 \cup V_3 \cup V_4]$ contains no induced $3\overline{K_p}$, then by Theorem 1.7, $e(G[V_1 \cup V_3 \cup V_4]) \geq 5(s - p + 1)$. Thus

$$e(G) = e(G[V_1 \cup V_3 \cup V_4]) + e[V_1 \cup V_3 \cup V_4, V_2 \cup V_5] + e(G[V_2 \cup V_5]) \geq 5(s - p + 1) + 2s + 5p = 7s + 5 > 7s,$$

a contradiction.

Case 3. $|\mathcal{B}'| = 2$. We may let $\mathcal{B}' = \{V_2, V_3\}$. Then by Claim 3.1, $e[V_1 \cup V_4, V_2 \cup V_3 \cup V_5] \geq 3(s + p)$ and $e[V_2 \cup V_3, V_5] \geq 2p$. If there are at most 4 vertices in V_1 that are movable to V_4 , then $e[V_1, V_4] \geq s - 4$. In this

case, $e(G) \geq 4s + 5p - 4$. If $s < p$, then $4s + 5p - 4 > 7s$, a contradiction. Assume that $p \leq s \leq 3p - 1$. If $G[V_1 \cup V_2 \cup V_3]$ contains an induced copy of $3\overline{K_p}$, then $G[V_1 \cup V_2 \cup V_3 \cup V_4]$ contains an induced copy of $4\overline{K_p}$, a contradiction. If $G[V_1 \cup V_2 \cup V_3]$ contains no induced $3\overline{K_p}$, then by Theorem 1.7, $e[V_1 \cup V_2 \cup V_3] \geq 5(s - p + 1)$. By Claim 3.1,

$$e[V_1 \cup V_2 \cup V_3, V_4 \cup V_5] + e(G[V_4 \cup V_5]) = e[V_1 \cup V_2 \cup V_3 \cup V_4, V_5] + e[V_4, V_2 \cup V_3] + e[V_1, V_4] \geq 2s + 5p - 4.$$

Thus

$$e(G) = e(G[V_1 \cup V_2 \cup V_3]) + e[V_1 \cup V_2 \cup V_3, V_4 \cup V_5] + e(G[V_4 \cup V_5]) \geq 5(s - p + 1) + 2s + 5p - 4 = 7s + 1 > 7s,$$

a contradiction.

Assume now that there are at least 5 vertices that are movable to V_4 . Since $|\mathcal{B}'| = 2$ and $\Delta(G) \leq 6$, $\Delta(G[V_1]) \leq 3$. Then there are 2 non-adjacent vertices $v_1, v'_1 \in V_1$ that are movable to V_4 . If $v_2 \in V_2$ is a solo neighbor of vertices $v_4, v'_4 \in V_4$, then we move there is a vertex shifting $(v_1, V_1) \rightarrow (v_4, V_4) \rightarrow (v_2, V_2) \rightarrow V_1$. After the vertex shifting, v'_4 is movable to V_2 . Since v'_1 is movable to V_4 , there is an accessible path $V_1 V_4 V_2$. Then $|\mathcal{B}'| \leq 1$, contradicting (2).

So, we may assume that no $x \in V_2$ is a solo neighbor of 2 vertices in V_4 . Consider $e[V_4, V_2] + e[V_2, V_3]$. For every vertex $a_4 \in V_4$, a_4 has at least one neighbor in V_2 . If a_4 has a solo neighbor $a_2 \in V_2$, then a_2 has neighbors in V_3 . Otherwise let $v_1, v'_1 \in V_1$ be two non-adjacent vertices that is movable to V_4 . We can have a vertex shifting by $(v_1, V_1) \rightarrow (a_4, V_4) \rightarrow (a_2, V_2) \rightarrow V_1$. Now v'_1 is movable to V_4 and a_2 is movable to V_3 . This implies $V_3 \notin \mathcal{B}'$. So $|\mathcal{B}'| \leq 1$. This contradicts (2).

Let there be x vertices in V_4 having the solo neighbor in V_2 . Then $e[V_4, V_2] + e[V_2, V_3] \geq 2x + 2(p - x) = 2p$. In summary, we have the following:

$$e[V_1, V_2 \cup V_3 \cup V_5] \geq 3s, \quad e[V_4, V_2] + e[V_2, V_3] \geq 2p, \quad e[V_4, V_3] \geq p, \quad e[V_2 \cup V_3 \cup V_4, V_5] \geq 3p.$$

If $s < p$, then $e(G) \geq 3s + 6p > 7s$, a contradiction. Assume that $p \leq s \leq 3p - 1$. If $G[V_1 \cup V_4]$ contains an induced copy of $2\overline{K_p}$, then $G[V_1 \cup V_2 \cup V_3 \cup V_4]$ contains an induced copy of $4\overline{K_p}$, a contradiction. If $G[V_1 \cup V_4]$ contains no induced $2\overline{K_p}$, by Theorem 1.4, then $e[V_1 \cup V_4] \geq 3(s - p + 1)$. Note that

$$\begin{aligned} & e[V_1 \cup V_4, V_2 \cup V_3 \cup V_5] + e(G[V_2 \cup V_3 \cup V_5]) \\ &= (e[V_1, V_2] + e[V_1, V_3] + e[V_1, V_5] + e[V_4, V_2] + e[V_4, V_3] + e[V_4, V_5]) + (e[V_2, V_3] + e[V_2, V_5] + e[V_3, V_5]) \\ &\geq 3s + 6p. \end{aligned}$$

Thus

$$e(G) = e(G[V_1 \cup V_4]) + e[V_1 \cup V_4, V_2 \cup V_3 \cup V_5] + e(G[V_2 \cup V_3 \cup V_5]) \geq 3(s - p + 1) + 3s + 6p = 6s + 3p + 3 > 7s,$$

a contradiction.

Case 4. $|\mathcal{B}'| = 3$. Clearly, $\mathcal{B}' = \{V_2, V_3, V_4\}$. Then $V_1 V_i \notin E(D)$ for $2 \leq i \leq 4$. Since $V_1 \in \mathcal{B}$ and $\Delta(G) \leq 6$, $\Delta(G[V_1]) \leq 2$. So, by Claim 3.4, $1 \leq s \leq 3p - 3$ and $e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}$.

If for some $2 \leq i \leq 4$, $v_i \in V_i$ is a solo neighbor of 2 non-adjacent vertices $v_1, v'_1 \in V_1$, then there is a vertex shifting $(v_1, V_1) \rightarrow (v_i, V_i) \rightarrow V_1$. After this shifting, v'_1 is movable to V_i . This implies $|\mathcal{B}'| \leq 2$, contradicting (2).

Thus we may assume that no $w \in V_2 \cup V_3 \cup V_4$ is a solo neighbor of 2 non-adjacent vertices in V_1 . Consider $e[V_1, V_i] + e[V_i, V_j]$, where $2 \leq i, j \leq 4$. Note that every $a_1 \in V_1$ has at least one neighbor in V_i .

If a solo neighbor $a_i \in V_i$ of some $a_1 \in V_1$ has no neighbors in $V_j \in \mathcal{B}' - V_i$, then after moving a_i to V_1 and a_1 to V_i , $V_j \notin \mathcal{B}'$, since now a_i is movable to it. This contradiction shows that for each $2 \leq i \leq 4$, every $a_i \in V_i$ that is a solo neighbor of some $a_1 \in V_1$ has neighbors in each class in $\mathcal{B}' - V_i$.

Assume that V_1 contains x vertices that have a solo neighbor in V_i and $s - x$ vertices that have at least 2 neighbors in V_i . Recall that $\Delta(G[V_1]) \leq 2$. By Theorem 2.2, there are at least $\lceil \frac{x}{3} \rceil$ vertices in V_1 that all have distinct solo neighbors in V_i . Then for any $j \in \{2, 3, 4\} - \{i\}$,

$$e[V_1, V_i] + e[V_i, V_j] \geq x + 2(s - x) + \lceil \frac{x}{3} \rceil = 2s - \lfloor \frac{2x}{3} \rfloor \geq 2s - \frac{2x}{3} \geq \frac{4s}{3}. \quad (3)$$

Applying (3) for pairs $(i, j) \in \{(2, 3), (3, 4), (4, 2)\}$, we get $e[V_1, V_2 \cup V_3 \cup V_4] + e(G[V_2 \cup V_3 \cup V_4]) \geq 3 \cdot \frac{4s}{3} = 4s$. Together with $e(G[V_1]) \geq \binom{s}{2} - t_{s,p-1}$ and $e[V_1 \cup V_2 \cup V_3 \cup V_4, V_5] \geq s + 3p$, this yields

$$e(G) \geq 5s + 3p + \binom{s}{2} - t_{s,p-1} = \begin{cases} 5s + 3p > 7s, & \text{if } 0 \leq s < p, \\ 5s + 3p + s - (p - 1) = 6s + 2p + 1 > 7s, & \text{if } p \leq s < 2(p - 1), \\ 5s + 3p + 2s - 3(p - 1) = 7s + 3 > 7s, & \text{if } 2(p - 1) \leq s \leq 3(p - 1), \end{cases}$$

a contradiction. $\square$

Consider the following discharging procedure. At the start, each $v \in V_1 \cup V_2 \cup V_3 \cup V_4$ has charge $ch(v) = 1$, each $v \in V_5$ has charge $ch(v) = 0$ and each $e \in E(G)$ has charge $ch(e) = 0$. So, $\sum_{x \in V(G) \cup E(G)} ch(x) = s + 3p$. Now we will move the charges between edges and vertices without changing the total sum as follows.

Step 1. Every vertex v in $V_1 \cup V_2 \cup V_3 \cup V_4$ gives charge $\frac{1}{10}$ to each edge incident to v that is in $G[V_1 \cup V_2 \cup V_3 \cup V_4]$.

Step 2. Every vertex v in $V_1 \cup V_2 \cup V_3 \cup V_4$ distributes its charge equally to its neighbors in V_5 .

Let the charge of each $x \in V(G) \cup E(G)$ after Step j be denoted by $ch_j(x)$. Then $ch_2(e) \geq 0$ for every $e \in E(G[V_1 \cup V_2 \cup V_3 \cup V_4])$ and $ch_2(v) \geq 0$ for every $v \in V_5$. Therefore

$$s + 3p = \sum_{e \in E(G[V_1 \cup V_2 \cup V_3 \cup V_4])} ch_2(e) + \sum_{v \in V_5} ch_2(v).$$

Claim 3.15. For each $v \in V_5$, $ch_2(v) \leq 3$.

Proof of Claim 3.15. Note that $\Delta(G) \leq 6$. If $d_G(v) \leq 3$, then $ch_2(v) \leq 3$. Assume that $4 \leq d_G(v) \leq 6$. Let $N(v) = \{u_1, u_2, \dots, u_{d_G(v)}\}$. If v is the solo neighbor of $u_i \in N(v)$, by Claim 3.14, then $d_G(u_i) \geq d_G(v)$. Hence $ch_2(v) \leq d_G(v) \cdot [1 - (d_G(v) - 1) \cdot \frac{1}{10}] = d_G(v) \cdot \frac{11 - d_G(v)}{10}$. If v is not the solo neighbor of $u_i \in N(v)$, then $ch_2(v) \leq d_G(v) \cdot \frac{1}{2}$. Therefore, $ch_2(v) \leq d_G(v) \cdot \max\{\frac{11 - d_G(v)}{10}, \frac{1}{2}\} \leq 3$. $\square$

By Claim 3.15, $\sum_{v \in V_5} ch_2(v) \leq 3(p - 1)$. For each edge $e \in E(G[V_1 \cup V_2 \cup V_3 \cup V_4])$, $ch_2(e) \leq \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$. Since

$$s + 3p = \sum_{e \in E(G[V_1 \cup V_2 \cup V_3 \cup V_4])} ch_2(e) + \sum_{v \in V_5} ch_2(v) \leq \frac{1}{5}e(G[V_1 \cup V_2 \cup V_3 \cup V_4]) + 3(p - 1),$$

we have $e(G[V_1 \cup V_2 \cup V_3 \cup V_4]) \geq 5(s + 3)$. Thus

$$e(G) = e(G[V_1 \cup V_2 \cup V_3 \cup V_4]) + e[V_1 \cup V_2 \cup V_3 \cup V_4, V_5] \geq 5(s + 3) + (s + 3p) = 6s + 3p + 15 > 7s,$$

a contradiction. $\square$

4 Proof of Theorem 1.8

We prove Theorem 1.8 using Theorem 2.7, first proving Parts (1)–(4) of the theorem and then (5) by induction.

Proof of Theorem 1.8. For $n \geq 4p + 1$, we let $\overline{H}_n$ be an extremal graph for $4K_p$ on n vertices.

Proof of Part (1). Let $G_1 = K_6 \cup \overline{K_{4p-5}}$. Since $\overline{G}_1$ is $4K_p$ -free, $e(\overline{H_{4p+1}}) \geq e(\overline{G}_1)$, which implies that $e(H_{4p+1}) \leq e(G_1) = 15$. Assume that $e(H_{4p+1}) \leq 14$. If $\Delta(H_{4p+1}) \leq 6$, then by Theorem 2.7 for $s = 2$, H_{4p+1} contains an induced copy of $4\overline{K_p}$. Thus $\overline{H_{4p+1}}$ contains a copy of $4K_p$, a contradiction. If $\Delta(H_{4p+1}) \geq 7$, let $v_1 \in V(H_{4p+1})$ be a vertex with $d_{H_{4p+1}}(v_1) = \Delta(H_{4p+1})$ and $G'_1 = H_{4p+1} \setminus \{v_1\}$. Then $e(G'_1) \leq 14 - 7 = 7$ and $|V(G'_1)| = 4p$. By Theorem 1.6, $\text{ex}(4p, 4K_p) = \binom{4p}{2} - 10$. But $e(\overline{G'_1}) \geq \binom{4p}{2} - 7$. Thus $\overline{G'_1}$ contains a copy of $4K_p$, and hence $\overline{H_{4p+1}}$ contains a copy of $4K_p$, a contradiction. Thus $e(H_{4p+1}) = 15$. This implies that $\text{ex}(4p + 1, 4K_p) = \binom{n}{2} - 15$.

Proof of Part (2). Let $G_2 = K_7 \cup \overline{K_{4p-5}}$. Since $\overline{G}_2$ is $4K_p$ -free, $e(\overline{H_{4p+2}}) \geq e(\overline{G}_2)$, which implies that $e(H_{4p+2}) \leq e(G_2) = 21$. Assume that $e(H_{4p+2}) \leq 20$. If $\Delta(H_{4p+2}) \leq 6$, then by Theorem 2.7 for $s = 3$, H_{4p+2} contains an induced copy of $4\overline{K_p}$, and so $\overline{H_{4p+2}}$ contains a copy of $4K_p$, a contradiction. If $\Delta(H_{4p+2}) \geq 7$, let $v \in V(H_{4p+2})$ be a vertex with $d_{H_{4p+2}}(v) = \Delta(H_{4p+2})$ and $G'_2 = H_{4p+2} \setminus \{v\}$. Then $e(G'_2) \leq 20 - 7 = 13$ and $|V(G'_2)| = 4p + 1$. By Part (1), $\text{ex}(4p + 1, 4K_p) = \binom{n}{2} - 15 < e(\overline{G'_2})$. Thus $\overline{G'_2}$ contains a copy of $4K_p$, and hence $\overline{H_{4p+2}}$ contains a copy of $4K_p$, a contradiction. Therefore, $e(H_{4p+2}) = 21$, and so $\text{ex}(4p + 2, 4K_p) = \binom{n}{2} - 21$.

Proof of Part (3). Let $G_3 = K_8 \cup \overline{K_{4p-5}}$. Since $\overline{G}_3$ is $4K_p$ -free, $e(\overline{H_{4p+3}}) \geq e(\overline{G}_3)$, which implies that $e(H_{4p+3}) \leq e(G_3) = 28$. Assume that $e(H_{4p+3}) \leq 27$. If $\Delta(H_{4p+3}) \leq 6$, then by Theorem 2.7 for $s = 4$, H_{4p+3} contains an induced copy of $4\overline{K_p}$, and so $\overline{H_{4p+3}}$ contains a copy of $4K_p$, a contradiction. If $\Delta(H_{4p+3}) \geq 7$, let $v \in V(H_{4p+3})$ be a vertex with $d_{H_{4p+3}}(v) = \Delta(H_{4p+3})$ and $G'_3 = H_{4p+3} \setminus \{v\}$. Then $e(G'_3) \leq 27 - 7 = 20$ and $|V(G'_3)| = 4p + 2$. By Part (2), $\text{ex}(4p + 2, 4K_p) = \binom{n}{2} - 21 < e(\overline{G'_3})$. In this case, $\overline{G'_3}$ contains a copy of $4K_p$, and hence $\overline{H_{4p+3}}$ contains a copy of $4K_p$, a contradiction. Thus $e(H_{4p+3}) = 28$, and so $\text{ex}(4p + 3, 4K_p) = \binom{n}{2} - 28$.

Proof of Part (4). If $p = 3$, then $4p + 4 = 16$. Let $G_4 = K_8 \cup S_7$, where S_7 be a star on 8 vertices. Since $\overline{G}_4$ is $4K_p$ -free, $e(\overline{H_{4p+4}}) \geq e(\overline{G}_4)$, which implies that $e(H_{4p+4}) \leq e(G_4) = 35$. Assume that $e(H_{4p+4}) \leq 34$. If $\Delta(H_{4p+4}) \leq 6$, then by Theorem 2.7 for $s = 5$, H_{4p+4} contains an induced copy of $4\overline{K_p}$, and so $\overline{H_{4p+4}}$ contains a copy of $4K_p$, a contradiction. If $\Delta(H_{4p+4}) \geq 7$, let $v \in V(H_{4p+4})$ be a vertex with $d_{H_{4p+4}}(v) = \Delta(H_{4p+4})$ and $G'_4 = H_{4p+4} \setminus \{v\}$. Then $e(G'_4) \leq 34 - 7 = 27$ and $|V(G'_4)| = 4p + 3$. By Part (3), $\text{ex}(4p + 3, 4K_p) = \binom{n}{2} - 28 < e(\overline{G'_4})$. Thus, $\overline{G'_4}$ contains a copy of $4K_p$ and so $\overline{H_{4p+4}}$ contains a copy of $4K_p$, a contradiction. This implies that $\text{ex}(16, 4K_3) = \binom{16}{2} - 35 = 85$.

Assume now that $p \geq 4$. Let $G_5 = K_9 \cup \overline{K_{4p-5}}$. Since $\overline{G}_5$ is $4K_p$ -free, $e(\overline{H_{4p+4}}) \geq e(\overline{G}_5)$, which implies that $e(H_{4p+4}) \leq e(G_5) = 36$. Assume that $e(H_{4p+4}) \leq 35$. If $\Delta(H_{4p+4}) \leq 6$, then by Theorem 2.7 for $s = 5$, H_{4p+4} contains an induced copy of $4\overline{K_p}$, and so $\overline{H_{4p+4}}$ contains a copy of $4K_p$, a contradiction. If $\Delta(H_{4p+4}) \geq 7$, let $v \in V(H_{4p+4})$ be a vertex with $d_{H_{4p+4}}(v) = \Delta(H_{4p+4})$ and $G'_5 = H_{4p+4} \setminus \{v\}$.

If $d_{H_{4p+4}}(v) \geq 8$, then $e(G'_5) \leq 35 - 8 = 27$ and $|V(G'_5)| = 4p + 3$. By Part (3), $\text{ex}(4p + 3, 4K_p) = \binom{n}{2} - 28 < e(\overline{G'_5})$. Thus, $\overline{G'_5}$ contains a copy of $4K_p$, and so $\overline{H_{4p+4}}$ contains a copy of $4K_p$, a contradiction.

Suppose now that $d_{H_{4p+4}}(v) = 7$. Then $e(G'_5) \leq 35 - 7 = 28$. If $\Delta(G'_5) \leq 6$, then by Theorem 2.7 for $s = 4$, G'_5 contains an induced copy of $4\overline{K_p}$, and so $\overline{H_{4p+4}}$ contains a copy of $4K_p$, a contradiction. Therefore, we may assume that G'_5 has a vertex v' of degree 7. Let $G''_5 = G'_5 - \{v'\}$. If G''_5 has a vertex v''

of degree 7, then the graph $G_5''' = G_5'' - \{v''\}$ has $4p + 1$ vertices and 14 edges. In this case, by Part (1), $\text{ex}(4p + 1, 4K_p) = \binom{n}{2} - 15 < e(G_5''')$, and so $\overline{G_5''}$ contains a copy of $4K_p$, a contradiction. Hence we may assume $\Delta(G_5'') \leq 6$. Since $e(G_5'') = 21$ and G_5'' does not contain an induced copy of $4\overline{K_p}$, by Theorem 2.7 for $s = 3$, $G_5'' = K_7 \cup \overline{K_{4p-5}}$.

Suppose the set of vertices of G_5'' is $U \cup W$, where $G_5''[U] = K_7$, $U = \{u_1, \dots, u_7\}$ and $W = \{w_1, \dots, w_{4p-5}\}$. Since $\Delta(H_{4p+4}) = 7$, each vertex in U is adjacent to at most one of v, v' . So, we may assume that $u_7v \notin E(H_{4p+4})$. Since $d_{H_{4p+4}}(v) = 7$, it has at least $4p - 5 - 7 \geq p$ nonneighbors in W , say the set $W_4 = \{w_{3p-2}, w_{3p-1}, \dots, w_{4p-5}\}$ is disjoint from $N(v)$. Then H_{4p+4} contains the independent sets $V_1, \dots, V_4$, where for $1 \leq i \leq 3$, $V_i = \{w_{(i-1)(p-1)+1}, w_{(i-1)(p-1)+2}, \dots, w_{i(p-1)}, u_i\}$, and $V_4 = W_4 \cup \{u_7, v\}$. This contradicts the choice of H_{4p+4} .

Proof of Part (5). First, we show that $\text{ex}(n, 4K_p) = \binom{n}{2} - 7(n - 4p + 1)$ for $4p + 5 \leq n \leq 7p - 2$. In other words, we prove:

$$\text{For } 6 \leq s \leq 3p - 1 \text{ and } n = 4p + s - 1, e(H_n) = 7s. \quad (4)$$

For this, recall that the definition of the graph $J(n)$ is given in Section 2. By Claim 2.1, $J(n)$ contains no induced $4\overline{K_p}$ and $e(J(n)) = 7s$. Then $e(H_n) \leq e(J(n)) = 7s$. To prove the lower bound on $e(H_n)$, we use induction on s .

The base case is $s = 6$, i.e. $n = 4p + 5$. Assume that $e(H_{4p+5}) \leq 41$. If $\Delta(H_{4p+5}) \leq 6$, then by Theorem 2.7, H_{4p+5} contains an induced copy of $4\overline{K_p}$. Thus $\overline{H_{4p+5}}$ contains a copy of $4K_p$, a contradiction. If $\Delta(H_{4p+5}) \geq 7$, let $v \in V(H_{4p+5})$ be a vertex with $d_{H_{4p+5}}(v) = \Delta(H_{4p+5})$ and $G' = H_{4p+5} \setminus \{v\}$. Then $e(G') \leq 41 - 7 = 34$ and $|V(G')| = 4p + 4$. By Part (4), $\text{ex}(4p + 4, 4K_p) \leq \binom{n}{2} - 35 < e(\overline{G'})$. In this case, $\overline{G'}$ contains a copy of $4K_p$, and hence $\overline{H_{4p+5}}$ contains a copy of $4K_p$, a contradiction. Thus $e(H_{4p+5}) = 42$, and so $\text{ex}(4p + 5, 4K_p) = \binom{n}{2} - 42$.

Assume $s \geq 7$. If $\Delta(H_n) \leq 6$, then by Theorem 2.7, $e(H_n) \geq 7s$. Suppose $\Delta(H_n) \geq 7$ and $e(H_n) \leq 7s - 1$. Choose $v \in V(H_n)$ with $d_{H_n}(v) \geq 7$. Then $e(H_n \setminus \{v\}) \leq (7s - 1) - 7 = 7s - 8$. Since $H_n \setminus \{v\}$ is $4K_p$ -free, this contradicts the induction hypothesis.

By (4), $\text{ex}(n, 4K_p) = \binom{n}{2} - 7s = \binom{n}{2} - 7(n - 4p + 1)$ for $4p + 5 \leq n \leq 7p - 2$. Note that $\text{ex}(7p - 2, 4K_p) = \binom{7p-2}{2} - 7(3p - 1) = \binom{7p-2}{2} - 21p + 7$ and

$$e(K_3 \vee T_{7p-5, p-1}) = \binom{n}{2} - e(\overline{K_3 \vee T_{7p-5, p-1}}) = \binom{7p-2}{2} - 21(p - 3) - 56 = \binom{7p-2}{2} - 21p + 7.$$

Thus $\text{ex}(7p - 2, 4K_p) = e(K_3 \vee T_{7p-5, p-1})$. By Lemma 2.5, $\text{ex}(n, 4K_p) = e(K_3 \vee T_{n-3, p-1}) = 3 + 3(n - 1) + t_{n-3, p-1}$ for $n \geq 7p - 1$. This completes the proof of Theorem 1.8. $\square$

5 Concluding remarks

1. It would be interesting to describe all extremal graphs for $4K_p$.
2. We think that the following analog of Theorem 2.7 holds.

Conjecture 5.1. *Let $p \geq 3$, $k \geq 2$ and G be a graph on $n = kp - 1 + s$, where $1 \leq s \leq (k - 1)p - 1$. If $|E(G)| \leq (2k - 1)s$ and $\Delta(G) \leq 2k - 2$, then G contains an induced copy of $k\overline{K_p}$ or $(k - 1)s$ and $G = \frac{s}{k-1}K_{2k-1} \cup \overline{K_{n - \frac{(2k-1)s}{k-1}}}$ with $|E(G)| = (2k - 1)s$.*

3. It is likely that our main result can be extended to graphs with no k disjoint K_p as follows.

Conjecture 5.2. *Let $p \geq 3$, $k \geq 4$ and $(k-1)p - k^2 + 3k - 3 \geq 0$. Then*

$$\text{ex}(n, kK_p) = \begin{cases} \binom{n}{2} - (2k-1)(n - kp + 1), & \text{if } kp + k^2 - 3k + 1 \leq n \leq (2k-1)p - 2, \\ e(K_{k-1} \vee T_{n-k+1, p-1}), & \text{if } n \geq (2k-1)p - 1. \end{cases}$$

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