LOGARITHMIC NEWTON POLYGONS AND POLYTOPES, AND THE FACTORIZATION OF DIRICHLET POLYNOMIALS

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Dedicated to Professor Mihai Cipu on the occasion of his retirement

ABSTRACT. To study a Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ by regarding it as a multivariate polynomial via the canonical map ϕ sending p_i^{-s} to an indeterminate X_i, with p_i the ith prime number, requires knowing the prime factorizations of all the integers in the support of f. We devise several methods to study the factorization of Dirichlet polynomials over unique factorization domains that circumvent the use of ϕ, and obtain irreducibility criteria that are analogous to the classical results of Schönemann, Eisenstein, Dumas, Stäckel, Ore and Weisner for polynomials, and to more recent results of Filaseta and Cavachi. Some of the proofs rely on logarithmic versions of the classical Newton polygons. Criteria that use two or more p-adic valuations by combining information from different logarithmic Newton polygons of f, as well as irreducibility conditions for Dirichlet polynomials that assume a prime or a prime power value are also obtained. We also find excluding intervals for the relative degrees of the factors of a Dirichlet polynomial, and upper bounds for the multiplicities of the irreducible factors, in particular square-free criteria, that use no derivatives. Criteria of absolute irreducibility analogous to results of Ostrowski, Gao and Stepanov-Schmidt are finally provided in the multivariate case by using logarithmic Newton polytopes and logarithmic upper Newton polygons.

CONTENTS

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1. Introduction

A Dirichlet polynomial $f(s)$ with complex coefficients a_i and exponents λ_i is a sum of the form $\sum_{i\in S} a_i e^{-\lambda_i s}$ with S a finite set of natural numbers and (λ_i) an increasing sequence of positive real numbers. Many fundamental results in analytic and multiplicative number theory that rely on their properties refer to the particular case when $\lambda_i = \log i$. Dirichlet polynomials are thus partial sums of corresponding Dirichlet series, and they can be used to approximate most Zeta-functions and L-functions, and their powers as well, and also their functional equations (Ivic [62], Titchmarsh [102], Bombieri and Friedlander [10]). Using the change of indeterminate $s \mapsto -s$, one may define the Dirichlet polynomials as sums of the form $\sum_{i\in S} a_i \cdot i^s$, which is a notation used by Lovász in [69], where such series are called finite Dirichlet series. Huxley [60] studied Dirichlet polynomials of the form $\sum_{n=N}^{2N} \frac{a(n)\chi(n)}{n^s}$ with χ a Dirichlet character and $s = \sigma + it$, $\sigma \geq 0$. In particular, the analytical properties of Dirichlet polynomials play an important role in the study of the Riemann zeta-function (Montgomery [73]). The study of almost-periodic functions relies too on the properties of Dirichlet polynomials (Besicovitch [5]), and some factorization properties of Dirichlet polynomials with integer coefficients have applications in group theory (Lucchini, Damian, Detomi and Morini [37], [38], [39], [40], [70], Brown [26], Hironaka [59], and Patassini [87], [88], [89], for instance). Dirichlet polynomials have also applications in lattice theory (Shala [100]). It is well-known that the ring of Dirichlet polynomials with coefficients in a unique factorization domain R is isomorphic to a ring of polynomials in countably many indeterminates over R, and that the ring of Dirichlet series with complex coefficients is isomorphic to a ring of formal power series in countably many indeterminates (Cashwell and Everett [31]).

In contrast to the intensive study of their analytical properties (see for instance Dickson [43], Montgomery [74], Halasz [57], Huxley [61], Heath-Brown [58], Jutila [66], Bourgain [22], Oliveira [78], Roy and Vatwani [94], Guth and Maynard [56], and the references therein), the study of the factorization properties of Dirichlet polynomials has received significantly less attention. In this respect, there seems to be a lack of effective results on the factorization properties of Dirichlet polynomials over unique factorization domains, such as irreducibility criteria, square-free criteria, or bounds for the number of their irreducible factors, or for their multiplicities, and also a lack of effective results on the factorization of multivariate Dirichlet polynomials over an arbitrary field. The aim of this paper is to provide a series of such results, some of them being analogous to classical results for polynomials, such as the irreducibility criteria of Schönemann-Eisenstein [97], [45], Dumas [44], Stäckel [101], Ore [82], Weisner [105], and Ostrowski [83], [84], or to more recent results of Filaseta [47], Cavachi [32] and Gao [50], for instance.

Besides being interesting on their own, a motivation to study Dirichlet polynomials over arbitrary unique factorization domains R comes from the fact that they offer an additional framework to study multivariate polynomials, with which they are intimately linked. When R is the ring of rational integers, another strong motivation to find and to study irreducible Dirichlet polynomials comes from their deep connections with arithmetic and number theory. For instance, we may hope that, under certain hypotheses, some of them might assume prime

values infinitely often when s runs in the set of negative integers. In this regard, one might easily formulate a conjecture similar to that of Bouniakovsky [6]:

Conjecture 1. *A primitive, irreducible Dirichlet polynomial $f(s)$ with positive leading coefficient and such that the set of values $f(\mathbb{Z}_-)$ has no common divisor greater than 1, assumes prime values infinitely often.*

However, such a conjecture should be considered with extreme caution, since two of its simplest cases, namely when $f_1(s) = -1 + \frac{1}{2^s}$ and $f_2(s) = 1 + \frac{1}{2^s}$, refer to Mersenne primes (primes of the form $2^p - 1$ with p prime) and to Fermat primes (primes of the form $2^{2^n} + 1$). While there is hope that there are infinitely many Mersenne primes, heuristics lead to a lot of skepticism on the existence of infinitely many Fermat primes. In this respect, such a conjecture might necessitate some additional assumptions on f. One might also consider an analogue for Dirichlet polynomials of Schinzel's Hypothesis H [99]:

Conjecture 2. *For every nonconstant irreducible Dirichlet polynomials $f_1(s), \dots, f_k(s)$ with positive leading coefficients, one of the following conditions holds:*

1) There are infinitely many negative integers n such that all of $f_1(n), \dots, f_k(n)$ are simultaneously prime numbers;

2) There exists an integer $m > 1$ that depends on $f_1, \dots, f_k$ which divides $f_1(n) \cdots f_k(n)$ for each negative integer n (i.e. there is a prime number p such that for every negative integer n there exists an index i such that $f_i(n)$ is divisible by p).

We mention that to avoid considering negative integers n in these conjectures, we may use the change of indeterminate $s \mapsto -s$, and define as in [69] the Dirichlet polynomials as sums of the form $a_m m^s + \cdots + a_n n^s$.

Some fundamental results concerning Schinzel's Hypothesis H have been recently proved in various settings. Skorobogatov and Sofos [98] proved that almost all polynomials of any fixed degree satisfy Schinzel's Hypothesis H. Bodin, Dèbes and Najib [8] proved that Schinzel's Hypothesis H holds if one replaces the ring $\mathbb{Z}$ of rational integers by the ring $R_m = R[x_1, \dots, x_m]$ with $m \geq 1$ and R is a unique factorization domain with quotient field K in which the product formula holds, and in case K has positive characteristic p, one has $K^p \neq K$ (and an extension of this result for multivariate polynomials). Improvements on a coprime version of Schinzel Hypothesis have been obtained by Bodin, Dèbes, König and Najib in [9]. We expect that some of the techniques used to prove these results might be adapted to the case of Dirichlet polynomials, to study the truthfulness of the two conjectures above.

For a third conjecture in the same vein, we state here the simplest form of an analogue for Dirichlet polynomials of Hilbert's irreducibility theorem:

Conjecture 3. *Let $f(s,t)$ be a bivariate irreducible Dirichlet polynomial with rational coefficients. Then f remains irreducible for infinitely many integer specializations of s.*

If we denote by $p_1, p_2, \dots$ the rational primes and use the map ϕ that sends p_i^{-s} to an indeterminate X_i, a term of a Dirichlet polynomial $f(s)$ of the form $\frac{a_n}{n^s}$ with n having the canonical decomposition $n = p_{i_1}^{k_1} \cdots p_{i_r}^{k_r}$, will be identified with the monomial $a_n X_{i_1}^{k_1} \cdots X_{i_r}^{k_r}$. Thus, one can regard f as a multivariate polynomial, say $F(X_{i_1}, \dots X_{i_t})$, and rephrase Conjecture 1, for instance, in terms of F, by asking whether F assumes prime values for infinitely many specializations of the form $(X_{i_1}, \dots, X_{i_t}) = (p_{i_1}^d, \dots, p_{i_t}^d)$, where the primes $p_{i_1}, \dots, p_{i_t}$ are fixed and d runs in the set of positive integers. Conversely, every multivariate polynomial $F(X_1, \dots, X_n)$ can be transformed into a Dirichlet polynomial if we replace in F the indeterminates X_i by p_i^{-s}.

Despite this intimate connection between Dirichlet polynomials and multivariate polynomials, there is no easy way to study them simultaneously, the main obstruction coming from arithmetic. The reason is that, in general, there is not enough "transparency" neither in factoring the integers in the support of f to see what monomials will F have, nor in computing and writing in ascending order the integers obtained by replacing the indeterminates of F by $p_1^{-s}, p_2^{-s}, \dots$ to find f. In this respect, given a Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$, to effectively use ϕ in order to find the associated multivariate polynomial F, requires knowing the prime factorizations of all the indices i with $a_i \neq 0$. Without knowing these prime factorizations, we cannot even find the number of indeterminates of F and its degree. To overcome this obstruction, we will devise some methods to study the factorization of Dirichlet polynomials over unique factorization domains that essentially circumvent the use of ϕ, or reduce the number of required prime factorizations to a minimum. This will be our main goal.

The paper is structured as follows. Section 2 contains notations, definitions, and some basic facts on the irreducibility of Dirichlet polynomials, mostly of arithmetical nature, followed by an analogue for Dirichlet polynomials of the Schönemann-Eisenstein irreducibility criterion. Section 3 presents the construction of the Newton log-polygon, a logarithmic version of the classical Newton polygon, which is most suitable for studying Dirichlet polynomials, and also provides results that are analogous to Dumas's irreducibility criterion for polynomials [44]. Using Newton log-polygon method, we also obtain irreducibility criteria for linear combinations of relatively prime Dirichlet polynomials of the form $f(s) + p^k g(s)$ with p prime, that are analogous to Cavachi's criterion for polynomials [32]. Other irreducibility criteria for linear combinations of Dirichlet polynomials, analogous to Schönemann's criterion for polynomials and its variations, will be given in Appendix A. In Section 4 we will adapt an elegant method of Filaseta that was crucial for proving the irreducibility of all but finitely many Bessel polynomials [47], and obtain explicit intervals in which no relative degrees of the factors of some Dirichlet polynomials can lie. We also obtain conditions that allow a Dirichlet polynomial to remain irreducible after multiplication of some of its coefficients by arbitrarily chosen elements belonging to the ring of coefficients. In Section 5 we obtain a series of irreducibility conditions that use two or more p-adic valuations, by combining information from two or more Newton log-polygons of a given Dirichlet polynomial with respect to different primes. In Section 6 we provide several irreducibility criteria for Dirichlet polynomials that assume a prime value, or a prime power value,

that are analogous to the irreducibility criteria of Stäckel, Ore, and Weisner for polynomials. These results rely on Gelfond's inequality for the product of the heights of the factors of a multivariate polynomial, and also on some suitable estimates for the prime counting function $\pi(n)$. In Section 7 we will first show that in some cases the multiplicities of the irreducible factors of a Dirichlet polynomial f are bounded by expressions that depend only on information about the indices i in the support of f, not on the values of the coefficients. We will then devise a general method to study the multiplicities of the irreducible factors of Dirichlet polynomials, that requires no derivatives. This method holds over arbitrary unique factorization domains R, that do not necessarily include the logarithms of positive rational integers, required to define $f'(s)$. In particular, we will obtain a square-free criterion that requires no derivatives, and which reduces in case R has positive characteristic to asking certain matrices associated to f to have full rank. Conditions for two Dirichlet polynomials f and g to have a common factor will be then given in terms of the rank of some matrices associated to f and g, which are analogous to the Sylvester matrix of two univariate polynomials. Section 8 is devoted to the study of the factorization of multivariate Dirichlet polynomials over an arbitrary field K, which requires using logarithmic versions of the classical Newton polytopes. Here we obtain a logarithmic analogue of Ostrowski's Theorem on the decomposability of the Newton polytope of a product of two multivariate polynomials [83], [84], and applications to the study of the absolute irreducibility of multivariate Dirichlet polynomials, that rely on the concept of *log-integrally indecomposable* polytopes. We also provide a criterion of log-integrally indecomposability of polytopes, which is a logarithmic analogue of Gao's criterion for integrally indecomposability of polytopes [50]. In Section 9 we use a logarithmic analogue for bivariate Dirichlet polynomials of the *upper* Newton polygon for bivariate polynomials [50], and obtain irreducibility conditions, analogous to those in the Stepanov-Schmidt criterion for bivariate polynomials, that are expressed in terms of the degrees of the coefficients with respect to one of the indeterminates. Several examples will be given in the last section of the paper.

2. Definitions, notations, some basic facts, and an analogue of the Schönemann-Eisenstein irreducibility criterion

A Dirichlet polynomial $f(s)$ with coefficients in a unique factorization domain R is a finite sum of the form $\frac{a_1}{i_1^s}+\frac{a_2}{i_2^s}+\cdots+\frac{a_n}{i_n^s}$, with $i_1<i_2<\cdots<i_n$ positive integers, $a_1,\ldots,a_n\in R$, $a_1\cdots a_n\neq 0$, and s regarded as an indeterminate. A simpler notation for f that we will mostly use is $\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$, with m,n arbitrary integers with $1\leq m\leq n$, and $a_m a_n\neq 0$. One can obviously add and multiply in a natural way two such objects, of finite arbitrary length, to obtain other finite sums of this type. As is known, unlike in the polynomial case, here we will use for multiplication the Dirichlet product. Thus, if $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ and $g(s)=\frac{b_u}{u^s}+\cdots+\frac{b_v}{v^s}$, then the product $f\cdot g(s)$ will be the Dirichlet polynomial $\frac{c_{mu}}{(mu)^s}+\cdots+\frac{c_{nv}}{(nv)^s}$, with coefficients

$$c_k=\sum_{i\cdot j=k}a_ib_j,\quad k=mu,\ldots,nv.$$

One may easily check that with respect to the usual addition and to the multiplication defined above, the set $DP(R)[s]$ of Dirichlet polynomials with coefficients in R is a commutative ring with unity. Using the map ϕ described before, the ring $DP(R)[s]$ of Dirichlet polynomials is easily seen to be isomorphic to a ring of polynomials in countably many indeterminates over R, thus being a unique factorization domain.

We will proceed now with some definitions and notations.

Definition 2.1. Let R be a unique factorization domain, $DP(R)[s]$ the ring of Dirichlet polynomial with coefficients in R, and let $f \in DP(R)[s]$. The *degree* of a nonzero term of f of the form $\frac{a_i}{i^s}$ will be by definition i, so for a Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ with $a_m a_n \neq 0$ we will refer to m as the *min-degree* of f and we will denote it $\deg_{\min} f$, while n will be called the *degree* of f, and will be denoted by $\deg f$. Also, by convention, the zero Dirichlet polynomial will have degree zero, and if $f \neq 0$, the rational number $\frac{n}{m} \geq 1$ will be referred to as the *relative degree* of f. Moreover, a_n will be referred to as the *leading coefficient* of f, a_m as the *min-degree coefficient* of f, a_1 as the *constant term* of f, and if $a_n = 1$, f will be called *monic*. The *support* of f, denoted by $Supp(f)$, is the set of indices i with $a_i \neq 0$.

We mention here that, unless $a_1 \neq 0$, we will try to avoid writing f as $\frac{a_1}{1^s} + \cdots + \frac{a_n}{n^s}$, since many factorization problems rely on knowing $\deg_{\min} f$, as we will see later.

A Dirichlet polynomial f is called *primitive* if its coefficients are coprime. An analogue of Gauss's lemma holds in this case too, so a product of primitive Dirichlet polynomials is easily seen to be primitive too.

A Dirichlet polynomial f is called *algebraically primitive* if the integers in the support of f are relatively prime. If $Supp(f) = \{i_1, \ldots, i_k\}$ and $\gcd(i_1, \ldots, i_k) = d > 1$, then the Dirichlet polynomial obtained from f by dividing each of the indices $i_1, \ldots, i_k$ by d will be called the *algebraically primitive part of* f. We note here that the product of two algebraically primitive Dirichlet polynomials is also algebraically primitive.

A nonconstant Dirichlet polynomial f is called *irreducible* if it cannot be expressed as a product of two nonconstant Dirichlet polynomials $g, h \in DP(R)[s]$, otherwise being called *reducible*. We note that in this definition we don't ask an irreducible Dirichlet polynomial to necessarily be primitive. Thus, just like in the case of polynomials, a non-primitive irreducible f will be *irreducible over* $Q(R)$, the quotient field of R, while a primitive irreducible f will be referred to as *irreducible over* R.

Let $P_f = \{p_1, \ldots, p_r\}$ be the set of all the prime numbers that appear in at least one of the the prime factorizations of the indices $i \in Supp(f)$. We will refer to the prime numbers $p \in P_f$ as the *relevant* primes of f.

A Dirichlet polynomial f is *square-free* if it has no repeated irreducible factors (including irreducible elements in R). Thus, a primitive Dirichlet polynomial is square-free if it has no repeated nonconstant irreducible factors. More generally, if $k \geq 2$ is an integer, we will say that (a primitive) f is *k-power-free* if the multiplicities of its (nonconstant) irreducible factors are at most $k - 1$.

Throughout the paper, proving the irreducibility of a Dirichlet polynomial f without explicitly assuming that f is primitive, will mean that f is irreducible over $Q(R)$, the quotient field of the ring R of coefficients. Moreover, in some of our results, where there is no risk of confusion, to simplify the statements, we will no longer specify the ring R of coefficients.

The simplest examples of infinite families of irreducible Dirichlet polynomials are easily obtained directly from Definition 2.1:

Proposition 2.2. *All Dirichlet polynomials of prime degree are irreducible.*

We note here that unlike univariate polynomials, where for instance, X^2+1 is irreducible over $\mathbb{Z}$, being reducible over $\mathbb{Z}[i]$, a Dirichlet polynomial of prime degree will remain irreducible even if we embed its ring of coefficients into a larger one, with enriched factorization properties. This is the simplest example of an *absolutely irreducible* Dirichlet polynomial, meaning that it remains irreducible over an algebraic closure of the quotient field of R.

Among the Dirichlet polynomials $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ of composite degree n, or having the min-degree $m>1$, one may easily identify a class of irreducible ones, consisting of those having nonzero terms of degree sufficiently close to n or m. In this respect we have:

Proposition 2.3. *Let f be a Dirichlet polynomial of composite degree n, and let p_n be the smallest prime divisor of n. If f has a nonzero coefficient a_i with $n-p_n<i<n$, then f is irreducible.*

Proof: Let us assume to the contrary that $f(s)=g(s)\cdot h(s)$ for two nonconstant Dirichlet polynomials $g(s)=\frac{b_{d_1}}{d_1^s}+\cdots+\frac{b_{d_2}}{d_2^s}$ and $h(s)=\frac{c_{e_1}}{e_1^s}+\cdots+\frac{c_{e_2}}{e_2^s}$, say, with $d_2>1$, $e_2>1$, $b_{d_2}\neq 0$, $c_{e_2}\neq 0$, and $d_2\cdot e_2=n$. By the multiplication rule, the coefficient a_i is given by

$$a_i=\sum_{j\cdot k=i} b_j c_k,$$

and in this sum we must have $b_jc_k\neq 0$ for at least one pair (j,k) with $j\cdot k=i$, as $a_i\neq 0$. Consider such a pair (j,k). Since $d_2\cdot e_2=n$ and $i<n$, at least one of the inequalities $j\leq d_2$ and $k\leq e_2$ must be a strict one, so without loss of generality we will assume that $j<d_2$. Since e_2 is a divisor of n and $e_2\neq 1$, we must have $e_2\geq p_n$. This and the fact that $j<d_2$ allow us to successively deduce that

$$i=j\cdot k\leq (d_2-1)k\leq (d_2-1)e_2=n-e_2\leq n-p_n,$$

which contradicts our assumption that $n-p_n<i$, and completes the proof. □

Proposition 2.4. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be a Dirichlet polynomial with $m>1$ and $a_ma_n\neq 0$, and let p_m and p_n be the smallest prime divisors of m and n, respectively. If $m>n/p_n$ and f has a nonzero coefficient a_i with $m<i<m+p_m$, then f is irreducible.*

Proof: We may obviously assume that n is composite. We will use the same notations as in the proof of Proposition 2.3. So let us assume again that f is reducible. We first observe that

our condition $m > n/p_n$ prevents d_1 and e_1 to be equal to 1. Indeed, if $e_1 = 1$, say, then d_1 must be equal to m, so

$$d_2 \geq m. \tag{1}$$

On the other hand, as g and h are assumed to be nonconstant factors of f, e_2 must be greater than 1, so d_2 must be a proper divisor of n, and hence $d_2 \leq n/p_n$, which by (1) yields $m \leq n/p_n$, a contradiction. In a similar way one can prove that $d_1 > 1$.

We next see that in the sum $\sum_{j\cdot k=i} b_j c_k$ defining a_i, for any pair (j,k) with $b_j c_k \neq 0$ and $j \cdot k = i$, at least one of the inequalities $j \geq d_1$ and $k \geq e_1$ must be a strict one, as $i > m = d_1 e_1$. Without loss of generality we may assume that $j > d_1$. This allows us to deduce that

$$i = j \cdot k \geq (d_1 + 1)k \geq (d_1 + 1)e_1 = m + e_1 \geq m + p_m,$$

as $e_1 \mid m$ and $e_1 \neq 1$ (by the argument above). This contradicts our assumption that $i < m + p_m$, and completes the proof. □

In particular, from Propositions 2.2, 2.3 and 2.4 we obtain:

Corollary 2.5. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be a Dirichlet polynomial with $a_m a_n \neq 0$. If $a_{n-1} \neq 0$, or, if $m > n/2$ and $a_{m+1} \neq 0$, then f is irreducible.*

These results also show that in the case of Dirichlet polynomials with integer coefficients, Turán's conjecture on the irreducibility of neighboring polynomials is trivially settled in the affirmative, for if $f(s)$ has no nonzero terms $\frac{a_i}{i^s}$ with $n - p_n < i < n$, we may pick such an index i, and the Dirichlet polynomial $f(s) + \frac{1}{i^s}$ will be irreducible. Thus, for a nontrivial analogue of Turán's conjecture, one may for instance consider the following one (with possible refinements that take into account Proposition 2.4).

Conjecture 4. *There is an absolute constant $C > 0$ such that, if $f(s) = \sum_{i=1}^{n} \frac{a_i}{i^s}$ is a Dirichlet polynomial of degree n with integer coefficients, and p_n is the smallest prime factor of n, then there is a Dirichlet polynomial $g(s) = \sum_{i=1}^{n-p_n} \frac{b_i}{i^s}$ with integer coefficients such that $\sum_{i=1}^{n-p_n} |b_i| \leq C$ and $f(s) + g(s)$ is irreducible over $\mathbb{Q}$.*

Unlike univariate polynomials, a Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ is subject to more restrictive conditions on the possible values of the degrees and relative degrees of its factors, and these inherent restrictions are imposed by the prime factorizations of m and n. So m and n alone might dictate much of the factorization properties of f, sometimes diminishing the role of the coefficients a_i. To see this, we need to introduce some arithmetical objects, which will be used here and in some of the following sections.

Definition 2.6. For a pair of fixed, arbitrarily chosen integers (m,n) with $1 \leq m < n$, and any integer $k > 0$ we define

$$\begin{aligned} S_{rd}(m,n) &:= \left\{ \frac{d}{c} \ : \ d \mid n,\ c \mid m, \ \text{ and } \ 1 < \frac{d}{c} < \frac{n}{m} \right\}, \\ S^k_{rd}(m,n) &:= \left\{ \frac{d}{c} \ : \ d \mid n,\ c \mid m, \ \text{ and } \ 1 < \frac{d}{c} \leq \sqrt[k+1]{\frac{n}{m}} \right\}, \\ \rho(m,n) &:= \max \left\{ \frac{d}{c} \ : \ d \mid n,\ c \mid m, \ \text{ and } \ \frac{d}{c} \leq \sqrt{\frac{n}{m}} \right\}, \quad \text{and} \\ \delta(m,n) &:= \min \left\{ \frac{d}{c} \ : \ d \mid n,\ c \mid m, \ \text{ and } \ d > c > 0 \right\}. \end{aligned}$$

We may regard $S_{rd}(m,n)$ as the set of all the possible relative degrees of the non-trivial proper factors of an arbitrary algebraically primitive Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ with $a_m a_n \neq 0$, and $S^k_{rd}(m,n)$ as the set of all such possible relative degrees that do not exceed $\sqrt[k+1]{\frac{n}{m}}$. We will call $\rho(m,n)$ the *rational square root* of the pair (m,n), and $\delta(m,n)$ the *rational floor* of the pair (m,n).

Remark 2.7. i) We note that $\rho(m,n) \geq 1$, as $c = d = 1$ satisfy trivially this definition, and also that if $\rho(m,n) > 1$, then $\delta(m,n) \leq \rho(m,n)$. It should be stressed that $\rho(m,n)$ and $\delta(m,n)$ depend on both m and n, not only on their ratio n/m. To see this, take for instance $(m_1,n_1) = (3,7)$ and $(m_2,n_2) = (15,35)$. Even if $7/3 = 35/15$, we have $\rho(3,7) = 1$ and $\delta(3,7) = 7/3$, while $\rho(15,35) = \delta(15,35) = 7/5$.

ii) Let c and d be positive divisors of m and n, respectively, and let $c' = \frac{m}{c}$ and $d' = \frac{n}{d}$ be their complementary divisors. It is easy to check that $1 < \frac{d}{c} \leq \sqrt{\frac{n}{m}}$ if and only if $\sqrt{\frac{n}{m}} \leq \frac{d'}{c'} < \frac{n}{m}$. This allows us to conclude that

$$S_{rd}(m,n) = \emptyset \quad \Leftrightarrow \quad S^1_{rd}(m,n) = \emptyset \quad \Leftrightarrow \quad \rho(m,n) = 1. \tag{2}$$

iii) Let c and d be positive divisors of m and n, respectively. For $\frac{d}{c}$ to belong to $S^1_{rd}(m,n)$, d must be a proper divisor of n, for otherwise n would be forced to satisfy the inequality $n \leq \frac{c^2}{m}$, which cannot hold, as $n > m \geq c$.

iv) Although it is a trivial fact, we will also mention here that given an algebraically primitive Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ with $a_m a_n \neq 0$, and two positive divisors c and d of m and n respectively, with $c < d$, for d/c to be the relative degree of a hypothetical non-trivial factor g of f, we also have to ask $d/c < n/m$ (with equality possible if one drops the condition that f is algebraically primitive) and in this case the complementary factor of g will have relative degree $(nc)/(md)$, which will also belong to $S_{rd}(m,n)$. Analyzing the sets $S_{rd}(m,n)$ and $S^k_{rd}(m,n)$ might provide us valuable information on the canonical decomposition of f, obtained without even looking at its coefficients.

v) Rather surprisingly, $\rho(m,n)$ also appears in the study of the irreducibility of integer polynomials $f(X)$ whose roots lie in the Apollonius circle defined as the locus of points P with $d(P,(b,0)) = \rho(|f(a)|,|f(b)|) \cdot d(P,(a,0))$, and a,b integers with $|f(b)| > |f(a)|$ (see [13]).

If one of the equivalent conditions in (2) is satisfied, then an algebraically primitive Dirichlet polynomial $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ with $a_ma_n\neq 0$ must be irreducible, as seen in the following simple result.

Proposition 2.8. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with $a_ma_n\neq 0$. If $\rho(m,n)=1$, then f must be irreducible.*

Proof: Assume to the contrary that we can write f as a product of two nonconstant Dirichlet polynomials, say $g(s)=\frac{\alpha_{c_1}}{c_1^s}+\cdots+\frac{\alpha_{d_1}}{d_1^s}$ and $h(s)=\frac{\beta_{c_2}}{c_2^s}+\cdots+\frac{\beta_{d_2}}{d_2^s}$. Since

$$\frac{n}{m}=\frac{d_1}{c_1}\cdot\frac{d_2}{c_2},$$

one of the relative degrees of the factors g and h, say d_1/c_1, must be less than or equal to $\sqrt{n/m}$, so $d_1/c_1\leq\rho(m,n)=1$. This forces $c_1=d_1>1$, so g must have a single term, which obviously cannot hold, as f was assumed to be algebraically primitive. Therefore f must be irreducible, and this completes the proof. □

We notice here that the conclusion in the statement of Proposition 2.8 fails if one drops the condition that f is algebraically primitive. Reasoning in a similar way, one can actually prove the following more general result.

Proposition 2.9. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with $a_ma_n\neq 0$. If $S^k_{rd}(m,n)=\emptyset$, then f is a product of at most k irreducible factors.*

Definition 2.10. If $\rho(m,n)=1$, then an algebraically primitive Dirichlet polynomial $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ with $a_ma_n\neq 0$ will be called *arithmetically irreducible*.

Even if the proofs so far have been quite simple, at this point we may already state the following seemingly difficult problem:

Problem 2.11. Find all the pairs of integers $n>m\geq 1$ with $\rho(m,n)=1$.

As we shall see in the following result, our condition $\rho(m,n)=1$ forces m and n to be "rather close".

Proposition 2.12. *Let m and n be integers with $1\leq m<n$, let p be the smallest prime factor of n, and q the largest divisor of m smaller than p. If $\rho(m,n)=1$, then $1<\frac{n}{m}<\frac{p^2}{q^2}$. In particular, if $\rho(m,n)=1$ and n is even, then $1<\frac{n}{m}<4$.*

Proof: If we assume to the contrary that $\frac{n}{m}\geq\frac{p^2}{q^2}$, then $\frac{p}{q}\leq\sqrt{\frac{n}{m}}$, which further implies that $\rho(m,n)\geq\frac{p}{q}>1$, a contradiction. For an even n we have $p=2$, so q is equal to 1. □

Our hopes to solve Problem 2.11 in its entire generality are somehow dashed by the fact that, even if we fix the primes dividing m and n and let their multiplicities vary, computing $\rho(m,n)$ will require testing inequations between products of prime powers, which can only be done on a case by case basis. However, we present in the following result some simple cases where one can easily prove that $\rho(m,n)$ is equal to 1.

Proposition 2.13. *$\rho(m,n)=1$ in each one of the following cases:*
i) n is a prime number greater than m;
ii) $m=p^kq$, $n=p^{k+1}$ with p a prime number and integers k,q with $k\geq 0$ and $0<q<p$;
iii) $m=p^k$, $n=p^{k+1}$ for some prime number p and some integer $k\geq 0$;
iv) $m=p^k$, $n=p^kq$ for some prime numbers p,q with $q<p$ and some integer $k\geq 1$.

Proof: i) In our first case, as $n=p$ with p a prime greater than m, any positive quotient $\frac{d}{c}$ in the definition of $\rho(m,n)$ either has the form $\frac{1}{c}$ with $c\mid m$, and hence is at most 1, or is equal to $\frac{p}{c}$, which obviously exceeds $\sqrt{\frac{p}{m}}$. Therefore $\rho(m,n)$ must be equal to 1.

ii) In this case any positive quotient $\frac{d}{c}$ with $d\mid n$ and $c\mid m$ has the form $\frac{p^i}{r}$ with i an integer satisfying $-k\leq i\leq k+1$, and r a divisor of q. For $i\leq 0$ these quotients will be at most 1, while for positive i all the corresponding quotients will exceed $\sqrt{\frac{p}{q}}$, since $\frac{p}{r}>\sqrt{\frac{p}{q}}$.

iii) Here we observe that if $m=p^k$ and $n=p^{k+1}$, then any positive quotient d/c with $c\mid m$ and $d\mid n$ has the form p^i with i an integer satisfying $-k\leq i\leq k+1$, and the maximal such power of p that is less than or equal to $\sqrt{p}$ is 1, so in this case too we have $\rho(m,n)=1$.

iv) In this case, if $m=p^k$ and $n=p^kq$, any positive quotient d/c with $c\mid m$ and $d\mid n$ has the form p^iq^j with i an integer satisfying $-k\leq i\leq k$ and $j\in\{0,1\}$. For $j=0$, no such quotient other than 1 belongs to the interval $[1,\sqrt{q}]$, since $p>\sqrt{q}$. Finally, we observe that for $j=1$ no integer i can satisfy the condition $1<p^iq<\sqrt{q}$, since $p>q$. □

As we shall see later in Theorems 4.2 and 4.8, when $\rho(m,n)>1$, both $\rho(m,n)$ and $\delta(m,n)$ play an important role in some irreducibility tests that rely on p-adic information about the coefficients of Dirichlet polynomials, where p is a prime element of the ring of coefficients.

We will proceed now with irreducibility conditions that use information about the prime factorization of the coefficients of a Dirichlet polynomial. For polynomials, the first such irreducibility conditions appeared in the Schönemann-Eisenstein irreducibility criterion (see Cox [36] for the history of this criterion). Over the time, this criterion was generalized by various authors, of which we will only mention Königsberger [67], Netto [77], Bauer [7], Perron [90], Kurshach [68], Ore [79], [80], [81], Rella [93], MacLane [71], and in more recent years Panaitopol and Ştefănescu [85], [86], Mott [75], Brown [25], Bush and Hajir [30], Weintraub [104] and Jakhar [63], [64]. A natural question is whether for Dirichlet polynomials there exists an analogue of the Schönemann-Eisenstein criterion. The answer is affirmative, and the simplest such result that we can prove by ignoring the conditions in Propositions 2.3 and 2.4 is the following one.

Theorem 2.14. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_ma_n\neq 0$. If for a prime element p of R we either have*
i) $p\mid a_i$ for each $i=m,\dots,n-1$, $p\nmid a_n$ and $p^2\nmid a_m$, or
ii) $p\mid a_i$ for each $i=m+1,\dots,n$, $p\nmid a_m$ and $p^2\nmid a_n$,
then f is irreducible over $Q(R)$, the quotient field of R.

Proof: i) Assume to the contrary that $f(s)=g(s)h(s)$ with $g(s)=\frac{b_r}{r^s}+\cdots+\frac{b_t}{t^s}$ and $h(s)=\frac{c_u}{u^s}+\cdots+\frac{c_v}{v^s}$, so $b_rc_u=a_m$ and $b_tc_v=a_n$. Besides, since f is algebraically primitive, we must have $r<t$ and $u<v$. Since $p^2\nmid a_m$, only one of b_r and c_u is divisible by p, say $p\mid b_r$ and $p\nmid c_u$. On the other hand, as $p\nmid a_n$, none of b_t and c_v is divisible by p, so it makes sense to consider the least index k such that $p\nmid b_k$. This index k should therefore satisfy the inequalities $r<k\leq t$. Let us consider now the coefficient $a_{k\cdot u}$. By the multiplication rule we have

$$a_{k\cdot u}=\sum_{i\cdot j=k\cdot u}b_ic_j.$$

We observe that for a term b_ic_j in this sum, we must have $i\geq r$, $j\geq u$ and $i\cdot j=k\cdot u$, so i cannot exceed k. Therefore, except for b_kc_u, which is not divisible by p, in all the remaining terms b_ic_j (if any) the index i is less than k, so b_i must be divisible by p. This shows that $p\nmid a_{k\cdot u}$, which contradicts our hypothesis that $p\mid a_i$ for each $i=m,\ldots,n-1$, as $ku<kv\leq tv=n$;

ii) In this case, since $p^2\nmid a_n$, only one of b_t and c_v is divisible by p, say $p\mid b_t$ and $p\nmid c_v$. On the other hand, as $p\nmid a_m$, none of b_r and c_u is divisible by p, so it makes sense to consider this time the largest index k such that $p\nmid b_k$, which will satisfy the inequalities $r\leq k<t$. We will consider this time the coefficient $a_{k\cdot v}$, given by

$$a_{k\cdot v}=\sum_{i\cdot j=k\cdot v}b_ic_j.$$

For a term b_ic_j in the above sum, we must have $i\leq t$, $j\leq v$ and $i\cdot j=k\cdot v$, so $i\geq k$. Therefore, except for b_kc_v, which is not divisible by p, in all the remaining terms b_ic_j (if any) the index i must exceed k, so b_i must be divisible by p. This shows that $p\nmid a_{k\cdot v}$, which contradicts our hypothesis that $p\mid a_i$ for each $i=m+1,\ldots,n$, as $kv>ku\geq ru=m$. This completes the proof of the theorem. □

As we shall see in the following section, Theorem 2.14 is a corollary of a more general result obtained by using logarithmic Newton polygons, namely Theorem 3.4, which is an analogue for Dirichlet polynomials of the classical irreducibility criterion of Dumas for polynomials.

By combining Theorem 2.14 and Propositions 2.3 and 2.4, one can find similar irreducibility criteria that avoid imposing divisibility conditions to terms a_i with i close to m and n.

3. Logarithmic Newton polygons and an analogue of Dumas's irreducibility criterion

In this section we will construct a logarithmic version of the Newton polygon, which is an analogue for algebraically primitive Dirichlet polynomials of the classical Newton polygon for polynomials, and we will refer to it as the *logarithmic Newton polygon*, or briefly the *Newton log-polygon*. First of all, we will fix an arbitrarily chosen real number $\mathfrak{b}$ greater than 1, which will be the base our logarithms will be considered to (say $\mathfrak{b}=2$, or the base e of natural logarithms, or 10). For a given base $\mathfrak{b}$, we will call *log-integral* any point having coordinates $(\log_{\mathfrak{b}}x,\ y)$ with x a positive integer, and y an integer. As we shall see later, the base $\mathfrak{b}$ will be in some sense

irrelevant, so unless for some reason there is a need to use a particular base, to keep notations as simple as possible, we choose to work with natural logarithms. It is however worth mentioning that the construction of the Newton log-polygon below can be done using any base $\mathfrak{b} > 1$ for our logarithms.

Now let p be a fixed, prime element of R, and let $f(s) = \frac{A_m}{m^s} + \cdots + \frac{A_n}{n^s}$ be a Dirichlet polynomial with coefficients in R, $A_m A_n \neq 0$, $1 \leq m < n$. Let us represent the nonzero coefficients of f as $A_i = a_i p^{\alpha_i}$ with $a_i \in R$ and $p \nmid a_i$. We will sometimes denote α_i by $\nu_p(A_i)$. To each nonzero coefficient $a_i p^{\alpha_i}$ we will associate a point in the plane with coordinates $(\log i, \alpha_i)$. The lower convex hull of the points $(\log m, \alpha_m), \ldots, (\log n, \alpha_n)$ is an analogue for f of the classical Newton polygon, and is constructed as follows: Let $i_1 = m$, $P_1 = (\log i_1, \alpha_{i_1}) = (\log m, \alpha_m)$, and let $P_2 = (\log i_2, \alpha_{i_2})$ with i_2 the largest integer with the property that there are no points $(\log i, \alpha_i)$ lying below the line passing through P_1 and P_2. Then let $P_3 = (\log i_3, \alpha_{i_3})$, with i_3 the largest integer such that there are no points $(\log i, \alpha_i)$ lying below the line passing through P_2 and P_3, and so on. The last line segment built in this way will be $P_{r-1}P_r$, with $P_r = (\log n, \alpha_n)$. The broken line $P_1 P_2 \ldots P_r$ thus constructed will be called the *Newton log-polygon of* f with respect to the prime element p. We will refer to the log-integral points $P_1, \ldots, P_r$ as the *vertices* of the Newton log-polygon of f, and the Newton log-polygon of f is the lower convex hull of its vertices $P_1, \ldots, P_r$. The line segments $P_l P_{l+1}$ will be called *edges*. An edge $P_l P_{l+1}$ might pass through some log-integral points, other than its endpoints P_l and P_{l+1}. These points have coordinates of the form $(\log i, x_i)$ with $i \in \mathbb{Z}_+$ and $x_i \in \mathbb{N}$, with x_i not necessarily equal to α_i. Let us denote all these intermediate points by $Q_1, \ldots, Q_k$, say. Then $P_l Q_1$, $Q_1 Q_2$, $\ldots$, $Q_{k-1} Q_k$ and $Q_k P_{l+1}$ will be called *segments* of the Newton log-polygon. Thus, a segment will have no log-integral points other than its endpoints, and an edge will consist of a number of segments, possibly just one, in which case the edge itself will be called a segment. Besides, any two edges must have different slopes, while two segments that belong to the same edge will obviously have the same slope. The vectors $\overline{P_i P_{i+1}}$ will be called *vectors* of the edges of the Newton log-polygon, and *the vector system* of the Newton log-polygon will be the union of all the vectors of its edges.

Recall that $\frac{n}{m}$ is called the *relative degree* of f. Since there is no risc of confusion, for an edge (segment) AB of the Newton log-polygon of f having endpoints $A = (\log i, x_i)$ and $B = (\log j, x_j)$ with $i < j$, the rational number $\frac{j}{i} > 1$ will be called the *relative degree* of the edge (segment) AB, with $\log \frac{j}{i}$ being the *width* of AB. Thus, the relative degree of an edge is the product of the relative degrees of all of its segments, while the relative degree of f is the product of the relative degrees of all the edges appearing in the Newton log-polygon of f.

One can also use this construction for Dirichlet polynomials f that are not algebraically primitive, but doing so will only result in a translation on the x-axis of the Newton log-polygon of the algebraically primitive part of f by some log-integral value.

The reader may naturally wonder how difficult will be to plot the lower convex hull of some set of log-integral points. Fortunately the Newton log-polygon is in some sense a virtual construction, that requires no precision at all when plotting log-integral points in the plane, not even

approximations of the logarithms that we use. The reason is that, actually, the construction of the vertices $P_1, \ldots, P_r$ depends only on some diophantine inequalities, and the identification of all the intermediate log-integral points lying on the edges $P_1P_2, \ldots, P_{r-1}P_r$ only requires solving some diophantine equations. Indeed, if we choose two indices i, j with $i < j$, we see that a log-integral point $(\log k, \nu_p(A_k))$ lies over or on the line passing through the log-integral points $(\log i, \nu_p(A_i))$ and $(\log j, \nu_p(A_j))$ if and only if

$$j^{\nu_p(A_k)-\nu_p(A_i)} \geq k^{\nu_p(A_j)-\nu_p(A_i)} \cdot i^{\nu_p(A_k)-\nu_p(A_j)}, \tag{3}$$

while a log-integral point $(\log k, y)$ with y and k integers with $i < k < j$, lies on this line if and only if we have the equality

$$j^{y-\nu_p(A_i)} = k^{\nu_p(A_j)-\nu_p(A_i)} \cdot i^{y-\nu_p(A_j)}. \tag{4}$$

The Newton log-polygon will help us visualize and gain geometric insight on some factorization properties, which are arithmetic in nature, and often reduce to conditions such as (3) and (4) that do not depend on the base of the logarithms that we use.

The total number of log-integral points lying on a line segment with endpoints that are log-integral too will be given later in Lemma 3.3, which will be fundamental in establishing effective irreducibility conditions for our Dirichlet polynomials.

Let us consider an example for the case that $p = \mathfrak{b} = 2$. So let us take for instance $f(s) = \frac{2^2}{4^s} + \frac{2^2}{6^s} + \frac{2}{8^s} + \frac{1}{9^s} + \frac{2^2}{10^s} + \frac{1}{12^s} + \frac{2}{15^s}$, and observe that $f = g \cdot h$ with $g(s) = \frac{2}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{2}{5^s}$ and $h(s) = \frac{2}{2^s} + \frac{1}{3^s}$. Now let us plot in Figure 1 the Newton log-polygon of f, and in Figure 2 the Newton log-polygons of its factors g and h. We note that the point $Q_1 = (\log_2 6, 1)$ is a log-integral point that belongs to the line segment P_1P_2. We also observe that the edges of the Newton log-polygon of f are obtained by translates of the edges of the Newton polygons of g and h, so their slopes and lengths are not altered, and the translates of the edges with the same slope (which are the edges with negative slope $\log_2 \frac{2}{3}$ in the Newton log-polygons of g and h) are "glued" together in Q_1. As we shall see immediately, this is not just a coincidence.

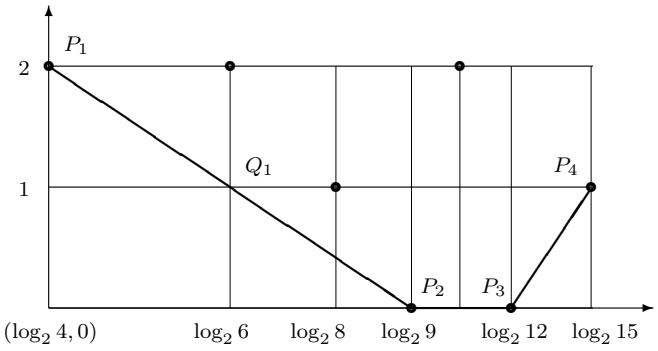

Figure 1. The Newton log-polygon for $f(s) = \frac{2^2}{4^s} + \frac{2^2}{6^s} + \frac{2}{8^s} + \frac{1}{9^s} + \frac{2^2}{10^s} + \frac{1}{12^s} + \frac{2}{15^s}$ with respect to the prime number $p = 2$

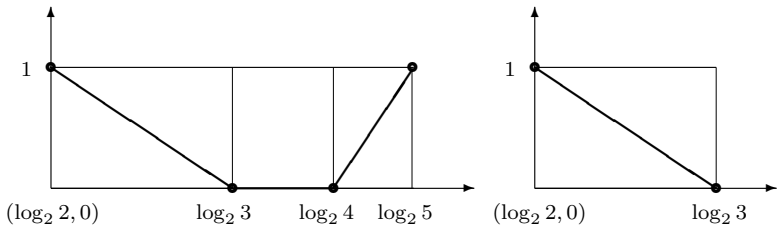

Figure 2. The Newton log-polygons for $g(s) = \frac{2}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{2}{5^s}$ and $h(s) = \frac{2}{2^s} + \frac{1}{3^s}$ with respect to the prime number $p = 2$

The following result, which will be fundamental to the theory of Newton log-polygons, is an analogue for Dirichlet polynomials of the famous Theorem of Dumas [44] for polynomials (see also Prasolov [92, Th. 2.2.1] for a short proof of Dumas's Theorem).

Theorem 3.1. *Let $p \in R$ be a prime element, g, h nonconstant algebraically primitive Dirichlet polynomials with coefficients in R, and let $f = g \cdot h$. Then the Newton log-polygon of f with respect to p is obtained by using translates of the edges of the Newton log-polygons of g and h with respect to p, using precisely one translate for each edge, so as to form a polygonal line with increasing slopes, when considered from left to the right.*

In other words, this theorem says that the vector system of the Newton log-polygon of f is obtained by considering the union of the vector systems of the Newton log-polygons of g and h, repetitions allowed, and then replacing any two vectors having the same orientation (slope) by their sum.

Proof: We will adapt here some of the notations and ideas in the proof of Dumas's Theorem given in [92], and the proof will be given in full detail.

Let $f(s) = \sum_{i=n'}^{n} \frac{a_i p^{\alpha_i}}{i^s}$, $g(s) = \sum_{j=d'}^{d} \frac{b_j p^{\beta_j}}{j^s}$ and $h(s) = \sum_{k=n'/d'}^{n/d} \frac{c_k p^{\gamma_k}}{k^s}$ with a_i, b_j and c_k not divisible by p. Let us consider an edge of the Newton log-polygon of f, say $P_l P_{l+1}$ (that may consist of more than a single segment), and let us assume that the points P_l and P_{l+1} have coordinates $(\log i_m, \alpha_{i_m})$, and $(\log i_M, \alpha_{i_M})$, respectively, with $i_m < i_M$. The slope of $P_l P_{l+1}$ is

$$S = \frac{\alpha_{i_M} - \alpha_{i_m}}{\log i_M - \log i_m}.$$

Let now $\alpha_{i_M} - \alpha_{i_m} = A$ and $\log i_M - \log i_m = I$, so $S = \frac{A}{I}$. The edge $P_l P_{l+1}$ of the Newton log-polygon of f belongs to the line $Iy - Ax = F$, where

$$F = I\alpha_{i_M} - A \log i_M = I\alpha_{i_m} - A \log i_m.$$

According to the definition, all the points $(\log i, \alpha_i)$, $i = n', \ldots, n$, either lie on this line, or above it, so $I\alpha_i - A \log i \geq F$ for all i, with a strict inequality for $i < i_m$ and $i > i_M$, and we have equality in the endpoints corresponding to i_m and i_M (and for the points $(\log i, \alpha_i)$ with $i_m < i < i_M$, if any, both cases ">" and "=" being possible).

To each term $\frac{a_i p^{\alpha_i}}{i^s}$ we will associate a real number

$$w\Big(\frac{a_i p^{\alpha_i}}{i^s}\Big) := I\alpha_i - A\log i,$$

called its *weight* with respect to the edge P_lP_{l+1}. We may then regard i_m and i_M as the least and the largest i such that the corresponding term $\frac{a_i p^{\alpha_i}}{i^s}$ of f has minimum weight (which is F) with respect to the edge P_lP_{l+1}, which shows that i_m and i_M are in this way uniquely determined. For the Dirichlet polynomial g we will define now

$$G := \min_{d'\le j\le d}\{I\beta_j - A\log j\}$$

and we will also define j_m and j_M as the least and the largest index, respectively, such that

$$G = I\beta_{j_m} - A\log j_m = I\beta_{j_M} - A\log j_M. \tag{5}$$

We notice that $j_m \le j_M$ (while $i_m < i_M$, here we don't necessarily have a strict inequality). Similarly, for h we define

$$H := \min_{n'/d'\le k\le n/d}\{I\gamma_k - A\log k\}$$

and then we denote by k_m and k_M the least and the largest index, respectively, such that

$$H = I\gamma_{k_m} - A\log k_m = I\gamma_{k_M} - A\log k_M. \tag{6}$$

Here too, we have $k_m \le k_M$ (not necessarily a strict inequality). By the multiplication rule for Dirichlet series, we have

$$\frac{a_{j_mk_m}p^{\alpha_{j_mk_m}}}{(j_mk_m)^s} = \sum_{j\cdot k = j_m\cdot k_m}\frac{b_jp^{\beta_j}}{j^s}\cdot\frac{c_kp^{\gamma_k}}{k^s}. \tag{7}$$

We observe that for two terms that are multiplied in (7) we have

$$\begin{aligned} w\Big(\frac{b_jp^{\beta_j}}{j^s}\cdot\frac{c_kp^{\gamma_k}}{k^s}\Big) &= w\Big(\frac{b_jc_kp^{\beta_j+\gamma_k}}{(jk)^s}\Big) = I(\beta_j+\gamma_k) - A\log(jk) \\ &= I(\beta_j+\gamma_k) - A(\log j + \log k) = w\Big(\frac{b_jp^{\beta_j}}{j^s}\Big) + w\Big(\frac{c_kp^{\gamma_k}}{k^s}\Big), \end{aligned}$$

so the weight of the summand with $j = j_m$ and $k = k_m$ in (7) is $G + H$, while the weights of the rest of the summands all exceed $G + H$, since for these ones we either have $j < j_m$, or $k < k_m$. Indeed, let us assume that $j < j_m$ and $k > k_m$, for instance. Then $w(\frac{b_jp^{\beta_j}}{j^s}) > G$ and $w(\frac{c_kp^{\gamma_k}}{k^s}) \ge H$, so

$$w\Big(\frac{b_jp^{\beta_j}}{j^s}\cdot\frac{c_kp^{\gamma_k}}{k^s}\Big) > G + H$$

(and similarly for $k < k_m$). Besides, we notice that when $j\cdot k$ is constant, the weight of a product $\frac{b_jp^{\beta_j}}{j^s}\cdot\frac{c_kp^{\gamma_k}}{k^s}$ regarded as a function on $\beta_j+\gamma_k$ is a strictly increasing function, as $I > 0$. Therefore, since in (7) we have $j\cdot k = j_m\cdot k_m$, hence $j\cdot k$ is constant, the sum $\beta_j+\gamma_k$ attains its minimum for $j = j_m$ and $k = k_m$, and is strictly larger for any other pair (j,k). Thanks to (7) we see now that

$$w\Big(\frac{a_{j_mk_m}p^{\alpha_{j_mk_m}}}{(j_mk_m)^s}\Big) = G + H. \tag{8}$$

Moreover, reasoning similarly with (7) replaced by

$$\frac{a_i p^{\alpha_i}}{i^s} = \sum_{j\cdot k=i} \frac{b_j p^{\beta_j}}{j^s} \cdot \frac{c_k p^{\gamma_k}}{k^s}, \tag{9}$$

we will prove that

$$w\Big(\frac{a_i p^{\alpha_i}}{i^s}\Big) \;>\; G+H \qquad \text{for } i < j_m \cdot k_m, \tag{10}$$

$$w\Big(\frac{a_i p^{\alpha_i}}{i^s}\Big) \;\geq\; G+H \qquad \text{for } i \geq j_m \cdot k_m. \tag{11}$$

Indeed, in view of (9) we see that

$$\alpha_i \geq \min_{j\cdot k=i}(\beta_j + \gamma_k), \tag{12}$$

and let us assume that this minimum is attained for a pair (j_0, k_0) with $j_0 \cdot k_0 = i < j_m \cdot k_m$. Then at least one of the inequalities $j_0 < j_m$ and $k_0 < k_m$ must hold, say $j_0 < j_m$. In this case we have $w(\frac{b_{j_0} p^{\beta_{j_0}}}{j_0^s}) > G$ and $w(\frac{c_{k_0} p^{\gamma_{k_0}}}{k_0^s}) \geq H$, so $w(\frac{b_{j_0} p^{\beta_{j_0}}}{j_0^s} \cdot \frac{c_{k_0} p^{\gamma_{k_0}}}{k_0^s}) > G+H$. According to (12) we then deduce that

$$\begin{aligned} w\Big(\frac{a_i p^{\alpha_i}}{i^s}\Big) &= I\alpha_i - A\log i \geq I(\beta_{j_0}+\gamma_{k_0}) - A\log(j_0\cdot k_0) \\ &= w\Big(\frac{b_{j_0} c_{k_0} p^{\beta_{j_0}+\gamma_{k_0}}}{(j_0\cdot k_0)^s}\Big) > G+H. \end{aligned}$$

Assume now that the minimum in (12) is attained for a pair (j_0, k_0) with $j_0 \cdot k_0 = i \geq j_m \cdot k_m$. In this case we might have $j_0 = j_m$ and $k_0 = k_m$, so we only get

$$w\Big(\frac{a_i p^{\alpha_i}}{i^s}\Big) = I\alpha_i - A\log i \geq w\Big(\frac{b_{j_0} c_{k_0} p^{\beta_{j_0}+\gamma_{k_0}}}{i^s}\Big) \geq G+H.$$

Summarizing, by (8), (10) and (11), we must have

$$G+H = F \qquad \text{and} \qquad j_m \cdot k_m = i_m.$$

Similarly, we also obtain $j_M \cdot k_M = i_M$. Taking the logaritm, one obtains

$$\log i_M - \log i_m = (\log j_M - \log j_m) + (\log k_M - \log k_m). \tag{13}$$

In particular, we see that at least one of the numbers $\log j_M - \log j_m$ and $\log k_M - \log k_m$ must be nonzero (hence positive). If both $\log j_M - \log j_m$ and $\log k_M - \log k_m$ are nonzero, then the geometric segment with endpoints $(\log j_m, \beta_{j_m})$ and $(\log j_M, \beta_{j_M})$ must be an edge of the Newton log-polygon of g, and the geometric segment with endpoints $(\log k_m, \gamma_{k_m})$ and $(\log k_M, \gamma_{k_M})$ must be an edge of the Newton log-polygon of h, and moreover, the slopes of these two edges will be the same, namely $S = \frac{A}{I}$, since from (5) and (6) one obtains

$$\frac{\beta_{j_M} - \beta_{j_m}}{\log j_M - \log j_m} = \frac{A}{I} = \frac{\gamma_{k_M} - \gamma_{k_m}}{\log k_M - \log k_m}. \tag{14}$$

Indeed, the fact that the geometric segment with endpoints $(\log j_m, \beta_{j_m})$ and $(\log j_M, \beta_{j_M})$ is an edge of the Newton log-polygon of g follows since, according to the definition of G and of the indices j_m and j_M, for $j < j_m$ and $j > j_M$ we have $I\beta_j - A\log j > G$, which shows that there

are no points $(\log j, \beta_j)$ lying under the line determined by this geometric segment. Moreover, the slope of the geometric segment with endpoints $(\log j_m, \beta_{j_m})$ and $(\log j_M, \beta_{j_M})$ is precisely $\frac{A}{I}$. Absolutely similar, one proves that the geometric segment with endpoints $(\log k_m, \gamma_{k_m})$ and $(\log k_M, \gamma_{k_M})$ is an edge of the Newton log-polygon of h. Moreover, if one of the numbers $\log j_M - \log j_m$ and $\log k_M - \log k_m$ is zero, say $\log k_M - \log k_m = 0$, in relation (14) only the first equality will make sense, but this will still be sufficient to argue in the same way that the segment with endpoints $(\log j_m, \beta_{j_m})$ and $(\log j_M, \beta_{j_M})$ is an edge of the Newton log-polygon of g. Thus the conclusion remains valid for g in this case too, the only difference being the fact that in this case there no longer exists an edge of slope $\frac{A}{I}$ in the Newton log-polygon of h.

Relation (13) is an equality of lengths of projections on the x-axis of some line segments having the same slope (according to (14)), so this relation also shows us that the sum of the lengths of the edges with slope S in the Newton log-polygons of g and h is precisely the length of the edge with slope S in the Newton log-polygon of f. If one of the numbers $\log j_M - \log j_m$ and $\log k_M - \log k_m$ is zero, then the Newton log-poligon of one of g and h has an edge with slope S of the same length as the edge with slope S in the Newton log-polygon of f, while the Newton log-polygon of the other factor will not contain any edge with slope S. Thus, the vector of the edge with slope S in the Newton log-polygon of f is the sum of the vectors of the edges with slope S in the Newton log-polygons of g and h. Besides, the fact that $\log n = \log d + \log n/d$ and $\log n' = \log d' + \log n'/d'$ combined with (13) and (14) shows us that if one of the factors g and h has an edge with slope S in its Newton log-polygon, then S will necessarily appear among the slopes of the edges in the Newton log-polygon of f. Indeed, let us denote by $Lf_1, \ldots, Lf_r$ the lengths of the projections on the x-axis of the edges of the Newton log-polygon of f, and with $S_1, \ldots, S_r$ the slopes of these edges, respectively. Similarly, let $Lg_1, \ldots, Lg_r$ and $Lh_1, \ldots, Lh_r$ be the lengths of the projections on the x-axis of the edges with slopes $S_1, \ldots, S_r$, respectively, in the Newton log-polygons of g and h (some of these lengths possibly being zero). We then have

$$Lf_i = Lg_i + Lh_i, \quad i = 1, \ldots, r, \tag{15}$$

where for a fixed i we have $Lg_i \geq 0$ and $Lh_i \geq 0$, with at least one of Lg_i and Lh_i nonzero. Denote now

$$W_f := \sum_{i=1}^{r} Lf_i, \quad W_g := \sum_{i=1}^{r} Lg_i, \quad W_h := \sum_{i=1}^{r} Lh_i.$$

We have $W_f = \log n - \log n'$, and since the Newton log-polygons of g and h might in principle contain edges of slopes different from $S_1, \ldots, S_r$, we deduce that $W_g \leq \log d - \log d'$ and $W_h \leq \log n - \log n' - (\log d - \log d')$. On the other hand by (13) one obtains by summation that $W_f = W_g + W_h$, which forces the equalities $W_g = \log d - \log d'$ and $W_h = \log n - \log n' - (\log d - \log d')$. This shows that in fact, the Newton log-polygons of g and h cannot have edges with slopes other than $S_1, \ldots, S_r$. This completes the proof of the theorem. □

In particular, this theorem shows that each nonconstant factor of a Dirichlet polynomial f must contribute at least one edge to the Newton log-polygon of f. Therefore, as in the case of polynomials we have:

Corollary 3.2. *If with respect to a prime element p of R, the Newton log-polygon of an algebraically primitive Dirichlet polynomial f consists of a single edge, and this edge contains no log-integral points other than its endpoints, then f must be irreducible.*

Here it is crucial to make distinction between edges and segments. It is not sufficient to ask the Newton log-polygon of f to have a single edge, for an edge might in principle contain multiple segments coming from the Newton log-polygons of two or more of the nonconstant factors of f. That's why in Corollary 3.2 we had to ask the edge to contain no log-integral points other than its endpoints. In order to obtain an effective instance of this corollary, we will need the following lemma, that counts the log-integral points on a line segment having endpoints that are log-integral too, thus allowing one to see when an edge in the Newton log-polygon contains a single segment.

Lemma 3.3. *Let x_1, x_2 be positive integers with $x_1 < x_2$, and $p_1, \dots, p_k$ the distinct prime factors of $x_1 \cdot x_2$. Let also y_1, y_2 be two integers with $y_1 \neq y_2$. Then the log-integral points $(\log x, y)$ lying on the line segment with endpoints $P = (\log x_1, y_1)$ and $Q = (\log x_2, y_2)$ are*

$$\left(\big(1 - \frac{i}{\delta}\big) \log x_1 + \frac{i}{\delta} \log x_2,\ \big(1 - \frac{i}{\delta}\big) y_1 + \frac{i}{\delta} y_2 \right), \quad i = 0, \dots, \delta,$$

with $\delta = \gcd(y_2 - y_1, \nu_{p_1}(x_2) - \nu_{p_1}(x_1), \dots, \nu_{p_k}(x_2) - \nu_{p_k}(x_1))$. Thus, the number of line segments lying on PQ that contain no log-integral points other than their endpoints is precisely δ.

Proof: Without loss of generality we will assume that $y_1 < y_2$. The line passing through P and Q has equation

$$Y = \frac{y_2 - y_1}{\log x_2 - \log x_1} \cdot X + \frac{y_1 \log x_2 - y_2 \log x_1}{\log x_2 - \log x_1}. \tag{16}$$

Suppose that there exists an integer x with $x_1 < x < x_2$ and an integer y with $y_1 < y < y_2$ such that the log-integral point $(\log x, y)$ lies on the line with equation (16). Then x and y must satisfy the relation

$$x_2^{y - y_1} = x_1^{y - y_2} \cdot x^{y_2 - y_1}. \tag{17}$$

Let us consider all the prime numbers that divide at least one of x_1 and x_2, say $p_1, \dots, p_k$. In view of (17), the prime factors of x are forced to belong to the set $\{p_1, \dots, p_k\}$. We may therefore assume without loss of generality that

$$x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \quad x_1 = p_1^{\beta_1} \cdots p_k^{\beta_k}, \quad x_2 = p_1^{\gamma_1} \cdots p_k^{\gamma_k},$$

with multiplicities $\alpha_i \geq 0$, $\beta_i \geq 0$ and $\gamma_i \geq 0$, such that for any given i, at most one of α_i, β_i and γ_i might be zero, so any prime $p_1, \dots, p_k$ might be missing in at most one of the prime decompositions of x, x_1 and x_2. Now, equating the multiplicities in (17), we deduce that

$$\gamma_i(y - y_1) = \beta_i(y - y_2) + \alpha_i(y_2 - y_1), \quad \text{for } i = 1, \dots, k,$$

or equivalently, that

$$(\gamma_i - \beta_i)y = (\gamma_i - \beta_i)y_1 + (\alpha_i - \beta_i)(y_2 - y_1), \quad \text{for } i = 1, \dots, k. \tag{18}$$

We will analyse the possible values for α_i in (18). We first observe that if for an index i we have $\gamma_i = \beta_i$, by (18) we must also have $\alpha_i = \beta_i$, so in this case we have only one possible value for α_i. As $x_1 < x_2$, we cannot have $\gamma_i = \beta_i$ for all i, so there exists a non-empty subset of $\{p_1, \dots, p_k\}$, say $\{p_1, \dots, p_l\}$ after relabelling, with $\gamma_1 \neq \beta_1, \dots, \gamma_l \neq \beta_l$. This will allow us to divide by $\gamma_1 - \beta_1$, …, $\gamma_l - \beta_l$ in the first l equations in (18), respectively, to obtain

$$y = y_1 + \frac{\alpha_i - \beta_i}{\gamma_i - \beta_i} \cdot (y_2 - y_1), \quad \text{for } i = 1, \dots, l. \tag{19}$$

By (19), the rational number $\frac{\alpha_i - \beta_i}{\gamma_i - \beta_i}$ must belong to the interval $(0, 1)$, and moreover, it must be the same for each i. If we write it in lowest terms, say

$$\frac{\alpha_i - \beta_i}{\gamma_i - \beta_i} = \frac{a}{b} \quad \text{with} \quad 0 < a < b \quad \text{and} \quad \gcd(a, b) = 1,$$

we see that the multiplicities α_i must be of the form

$$\alpha_i = \beta_i + \frac{a}{b}(\gamma_i - \beta_i) \quad \text{for } i = 1, \dots, l. \tag{20}$$

For these multiplicities to be integers, b must divide $\gamma_i - \beta_i$ for each i. On the other hand, y must be an integer too, so by (19) b must also divide $y_2 - y_1$. This forces b to be a divisor of

$$\delta := \gcd(y_2 - y_1, \gamma_1 - \beta_1, \dots, \gamma_l - \beta_l). \tag{21}$$

Thus, the solutions (x, y) of (17) with $x_1 < x < x_2$ and $y_1 < y < y_2$ are in a one-to-one correspondence with the rational numbers $\frac{a}{b}$ with $0 < a < b$, $\gcd(a, b) = 1$ and $b \mid \delta$. The number of these rationals is obviously equal to $\delta - 1$, as they can only be obtained by cancelling the common factors of the numerators and denominators in each of the quotients $\frac{d}{\delta}$ with $d = 1, \dots, \delta - 1$. We recall that if for an index j we have $\gamma_j = \beta_j$, then α_j will also be equal to β_j, thus satisfying (20) trivially, and we may obviously add $\gamma_j - \beta_j$ in the list of terms in the gcd in (21) without affecting the value of δ. The number of log-integral points lying on the line segment PQ (including P and Q) is thus equal to

$$\gcd(y_2 - y_1, \gamma_1 - \beta_1, \dots, \gamma_k - \beta_k) + 1,$$

and their coordinates are

$$\left(\log p_1^{\beta_1 + \frac{i}{\delta}(\gamma_1 - \beta_1)} \cdots p_k^{\beta_k + \frac{i}{\delta}(\gamma_k - \beta_k)}, \; y_1 + \frac{i}{\delta} \cdot (y_2 - y_1) \right), \quad i = 0, \dots, \delta,$$

or, equivalently,

$$\left(\log(x_1^{\frac{\delta - i}{\delta}} x_2^{\frac{i}{\delta}}), \; y_1 \frac{\delta - i}{\delta} + y_2 \frac{i}{\delta} \right), \quad i = 0, \dots, \delta,$$

with $\delta = \gcd(y_2 - y_1, \gamma_1 - \beta_1, \dots, \gamma_k - \beta_k)$. This completes the proof of the lemma. □

In [44], Dumas proved his famous irreducibility criterion for polynomials, imposing explicit conditions for the Newton polygon to consist of a single edge having no lattice points other than its endpoints:

Theorem (Dumas, 1906) *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be a polynomial with coefficients in a unique factorization domain R, and let p be a prime element of R. If*

i) $\frac{\nu_p(a_i)}{i} > \frac{\nu_p(a_n)}{n}$ *for* $i = 1, \ldots, n-1$,
ii) $\nu_p(a_0) = 0$,
iii) $\gcd(\nu_p(a_n), n) = 1$,

then f is irreducible over $Q(R)$, the quotient field of R.

We can now prove the following effective instance of Corollary 3.2, which may be regarded as the analogous for Dirichlet polynomials of Dumas's Theorem.

Theorem 3.4. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p be a prime element of R, and assume that $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$. If*

i) $\nu_p(a_n) \neq \nu_p(a_m)$,
ii) $\left(\frac{n}{i}\right)^{\nu_p(a_i)-\nu_p(a_m)} > \left(\frac{m}{i}\right)^{\nu_p(a_i)-\nu_p(a_n)}$ *for* $m < i < n$,
iii) $\gcd(\nu_p(a_n) - \nu_p(a_m), \nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m)) = 1$,

then f is irreducible over $Q(R)$, the quotient field of R.

Proof: We will prove that the Newton log-polygon of f consists of a single edge joining the points $P = (\log m, \nu_p(a_m))$ and $Q = (\log n, \nu_p(a_n))$. To do so, we first note that the line passing through P and Q has equation

$$y = \frac{\nu_p(a_n) - \nu_p(a_m)}{\log n - \log m} \cdot x + \frac{\nu_p(a_m)\log n - \nu_p(a_n)\log m}{\log n - \log m}.$$

One may easily check now that condition *ii)* forces all the points $(\log i, \nu_p(a_i))$ with $m < i < n$ to lie above this line. To prove that the edge PQ is actually a segment, we need to check that the only log-integral points lying on PQ are its endpoints P and Q. By Lemma 3.3 with $x_1 = m$, $x_2 = n$, $y_1 = \nu_p(a_m)$, $y_2 = \nu_p(a_n)$, the total number of log-integral points on the line segment PQ is $1 + \gcd(\nu_p(a_n) - \nu_p(a_m), \nu_{p_1}(n) - \nu_{p_1}(m), \ldots, \nu_{p_k}(n) - \nu_{p_k}(m))$, which by *iii)* is equal to 2, so PQ is indeed a segment, hence by Corollary (3.2), f must be irreducible. □

Theorem 2.14 is now easily seen to be the simplest instance of Theorem 3.4. In the first case of Theorem 2.14, $\left(\frac{n}{i}\right)^{\nu_p(a_i)-\nu_p(a_m)} \geq 1$ and $\left(\frac{m}{i}\right)^{\nu_p(a_i)-\nu_p(a_n)} < 1$, while in the second case $\left(\frac{n}{i}\right)^{\nu_p(a_i)-\nu_p(a_m)} > 1$ and $\left(\frac{m}{i}\right)^{\nu_p(a_i)-\nu_p(a_n)} \leq 1$, so condition ii) in Theorem 3.4 is satisfied.

Two simple instances of Theorem 3.4 are obtained when $m = 1$, like in the following two corollaries, corresponding to the cases that $p \nmid a_1$, $p \mid a_n$, and $p \mid a_1$, $p \nmid a_n$, respectively.

Corollary 3.5. *Let $f(s)=\frac{a_1}{1^s}+\frac{a_2}{2^s}+\cdots+\frac{a_n}{n^s}$ be a Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_1a_n\neq 0$, let p be a prime element of R, and assume that $q_1,\ldots,q_k$ are all the prime factors of n. If*

i) $p\nmid a_1$, $p\mid a_n$,
ii) $n^{\nu_p(a_i)}>i^{\nu_p(a_n)}$ *for* $1<i<n$,
iii) $\gcd(\nu_p(a_n),\nu_{q_1}(n),\ldots,\nu_{q_k}(n))=1$,

then f is irreducible over $Q(R)$.

Corollary 3.6. *Let $f(s)=\frac{a_1}{1^s}+\frac{a_2}{2^s}+\cdots+\frac{a_n}{n^s}$ be a Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_1a_n\neq 0$, let p be a prime element of R, and assume that $q_1,\ldots,q_k$ are all the prime factors of n. If*

i) $p\mid a_1$, $p\nmid a_n$,
ii) $n^{\nu_p(a_1)-\nu_p(a_i)}<i^{\nu_p(a_1)}$ *for* $1<i<n$,
iii) $\gcd(\nu_p(a_1),\nu_{q_1}(n),\ldots,\nu_{q_k}(n))=1$,

then f is irreducible over $Q(R)$.

Remark 3.7. Theorem 3.1 combined with Lemma 3.3 allows us to find an upper bound for the total number n_f of irreducible factors of f (counting multiplicities), by counting all the segments on all the edges $P_1Q_1,\ldots,P_\ell Q_\ell$ in the Newton log-polygon of f. Thus, we have $n_f\leq\delta_1+\cdots+\delta_\ell$, where for each i, δ_i is obtained by applying Lemma 3.3 to the edge P_iQ_i.

For Dirichlet polynomials having only two nonzero terms we obtain from Theorem 3.4 the following simple irreducibility conditions.

Corollary 3.8. *Let $f(s)=\frac{a_m}{m^s}+\frac{a_n}{n^s}$ be a Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_ma_n\neq 0$, $\gcd(m,n)=1$, and assume that a_ma_n has a prime factor of multiplicity one. Then f is irreducible over $Q(R)$.*

Proof: Note that f is algebraically primitive, as m and n are coprime. Since f has only two nonzero terms, condition ii) in Theorem 3.4 is trivially satisfied, and our hypothesis that $\nu_p(a_ma_n)=1$ for some prime element p of R shows that we must have $\nu_p(a_n)-\nu_p(a_m)=\pm 1$, so conditions i) and iii) in Theorem 3.4 are satisfied too. □

For Dirichlet polynomials with integer coefficients we can generalize Theorem 3.4 by using a change of indeterminate, as follows.

Theorem 3.9. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with integer coefficients, with $a_ma_n\neq 0$, $t\geq 0$ an integer, p a prime number, and assume that $q_1,\ldots,q_k$ are all the prime factors of $m\cdot n$. If*

i) $\nu_p(a_nn^t)\neq\nu_p(a_mm^t)$,
ii) $\left(\frac{n}{i}\right)^{\nu_p(a_ii^t)-\nu_p(a_mm^t)}>\left(\frac{m}{i}\right)^{\nu_p(a_ii^t)-\nu_p(a_nn^t)}$ *for* $m<i<n$,
iii) $\gcd(\nu_p(a_nn^t)-\nu_p(a_mm^t),\nu_{q_1}(n)-\nu_{q_1}(m),\ldots,\nu_{q_k}(n)-\nu_{q_k}(m))=1$,

then f is irreducible over $\mathbb{Q}$.

Proof: Consider the Dirichlet polynomial $f(s-t) = \frac{a_m \cdot m^t}{m^s} + \cdots + \frac{a_n \cdot n^t}{n^s}$, which has integer coefficients, is algebraically primitive, and according to i), ii) and iii), satisfies the hypotheses of Theorem 3.4, and hence is irreducible over $\mathbb{Q}$. Assume to the contrary that $f(s) = g(s)h(s)$ with g, h nonconstant Dirichlet polynomials with integer coefficients, say $g(s) = \frac{b_u}{u^s} + \cdots + \frac{b_v}{v^s}$ and $h(s) = \frac{c_l}{l^s} + \cdots + \frac{c_w}{w^s}$, with $a_i = \sum_{i=j\cdot k} b_j c_k$ for each $i \in \{m, \ldots, n\}$. Multiplication by i^t yields $a_i i^t = \sum_{i=j\cdot k} (b_j j^t)(c_k k^t)$ for each i, which shows that $f(s-t) = g(s-t)h(s-t)$, implying that $f(s-t)$ is reducible, a contradiction. □

We notice that for Dirichlet polynomials with $|a_m| = |a_n|$ we cannot apply Theorem 3.4, no matter what primes p we consider. For such Dirichlet polynomials f, we may at least try to test the conditions i), ii) and iii) in Theorem 3.9 for some of the relevant primes of f, as in the following result.

Corollary 3.10. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with integer coefficients, with $|a_m| = |a_n| \neq 0$, and assume that $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$. Let $p \in \{q_1, \ldots, q_k\}$ be a prime number for which $\nu_p(m) < \nu_p(n)$. If*

$$\nu_p(ia_i) \geq \nu_p(na_n) \quad for \quad m < i < n \quad and \quad \gcd(\nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m)) = 1,$$

then f is irreducible over $\mathbb{Q}$.

Proof: First of all we observe that since $m < n$, there exists at least one prime p among $q_1, \ldots, q_k$, such that $\nu_p(m) < \nu_p(n)$, so if we choose $t = 1$, condition i) in Theorem 3.9 will hold for our prime p, as $|a_m| = |a_n|$. We then observe that condition iii) in Theorem 3.9 reduces to $\gcd(\nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m)) = 1$, since for $t = 1$ we have

$$\nu_p(a_n n^t) - \nu_p(a_m m^t) = \nu_p(na_n) - \nu_p(ma_m) = \nu_p(n) - \nu_p(m),$$

which is one of the terms $\nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m)$. Finally, we observe that for $m < i < n$ we have $\frac{n}{i} > 1$ and $\frac{m}{i} < 1$, and our assumption that $\nu_p(ia_i) \geq \nu_p(na_n)$ together with the fact that $\nu_p(ma_m) < \nu_p(na_n)$ imply that

$$\nu_p(ia_i) - \nu_p(ma_m) > \nu_p(ia_i) - \nu_p(na_n) \geq 0.$$

Therefore for $t = 1$ condition ii) in Theorem 3.9 is satisfied too. □

We will also include in this section two lemmas that will be used to derive irreducibility conditions for linear combinations of the form $f(s) + p^k g(s)$, with f, g Dirichlet polynomials of coprime degrees, p a prime element of R, and k a positive integer, and whose proofs rely on Theorem 3.1 and Lemma 3.3. We mention here that linear combinations of relatively prime polynomials have been studied in both univariate and multivariate cases by Cavachi, Vâjâitu and Zaharescu [32], [33], [34], Zhang, Yuan and Zhou [106], and also in [21], [20] and [12], and the same techniques apply to compositions and multiplicative convolutions of polynomials [11], [14], [16], [19], [17] and [15].

Lemma 3.11. *Let f, g be two Dirichlet polynomials with coefficients in a unique factorization domain R, with $\deg g = n$ and $\deg f = m$, $m < n$, and let $p_1, \dots, p_r$ be the distinct prime factors of $m \cdot n$. Let also p be a prime element of R that divides none of the leading coefficients of f and g, and let k be any positive integer with*

$$\gcd(k, \nu_{p_1}(n) - \nu_{p_1}(m), \dots, \nu_{p_r}(n) - \nu_{p_r}(m)) = 1. \tag{22}$$

If $f(s) + p^k g(s)$ is algebraically primitive and can be written as a product of two nonconstant Dirichlet polynomials with coefficients in R, say $f_1(s)$ and $f_2(s)$, then one of the leading coefficients of f_1 and f_2 must be divisible by p^k, and the other must be coprime to p.

Proof: Let $f(s) = \frac{a_1}{1^s} + \cdots + \frac{a_m}{m^s}$ and $g(s) = \frac{b_1}{1^s} + \cdots + \frac{b_n}{n^s}$, $a_m b_n \neq 0$, and let us write $f(s) + p^k g(s) = \frac{c_1}{1^s} + \cdots + \frac{c_n}{n^s}$, where $c_i = a_i + p^k b_i$ for $i = 1, \dots, m$, while $c_i = p^k b_i$ for $i = m+1, \dots, n$. Our assumption that $p \nmid a_m b_n$ implies that c_m is not divisible by p, $c_{m+1}, \dots, c_n$ are all divisible by p^k, and moreover, $p^{k+1} \nmid c_n$. Thus, in the Newton log-polygon of $f + p^k g$ with respect to the prime element p, the rightmost edge will join the points $(\log m, 0)$ and $(\log n, k)$, that are labelled in Figure 3 below by $Q_{\ell-1}$ and Q_ℓ, respectively.

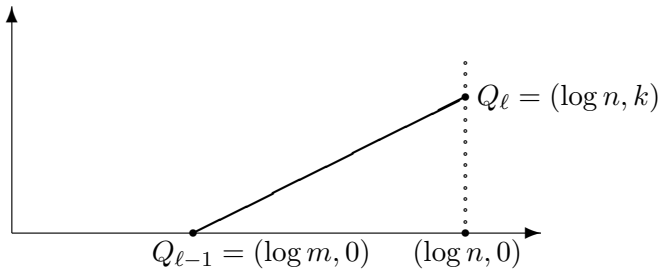

Figure 3. The rightmost segment in the Newton log-polygon of $f(s) + p^k g(s)$ with respect to p.

Moreover, since k satisfies (22), we deduce by Lemma 3.3 that $Q_{\ell-1}Q_\ell$ is in fact a segment of the Newton log-polygon, as it contains no log-integral points other than its endpoints $Q_{\ell-1}$ and Q_ℓ. Since the sequence of the slopes of the edges of the Newton log-polygon, when considered from left to the right, is a strictly increasing one, we deduce that $Q_{\ell-1}Q_\ell$ is the only segment with positive slope. On the other hand, if $f + p^k g$ is the product of two non-constant Dirichlet polynomials f_1 and f_2 with integer coefficients, then each of the factors f_1 and f_2 must have at least one coefficient which is not divisible by p, as $p \nmid c_m$. If we assume now that each of the leading coefficients of f_1 and f_2 is divisible by p, then each of the Newton log-polygons of f_1 and f_2 with respect to p will contain at least one segment with positive slope, which by Theorem 3.1 would produce at least two segments with positive slopes in the Newton log-polygon of $f + p^k g$, a contradiction. Therefore, one of the leading coefficients of f_1 and f_2 must be divisible by p^k, while the other must be coprime to p. This completes the proof of our lemma. □

We mention that Lemma 3.11 is the logarithmic analogue of Lemma 1.4 in [21], which provides information on the leading coefficients of the factors of integer polynomials of the form $f(X) + p^k g(X)$, with $\deg f < \deg g$ and k coprime to $\deg g - \deg f$.

As an application of Lemma 3.11 we have the following irreducibility criterion for linear combinations of Dirichlet polynomials.

Theorem 3.12. *Let f and g be two Dirichlet polynomials with coefficients in a unique factorization domain R, of coprime degrees m and n, respectively, with $m < n$, and let $p_1, \dots, p_r$ be the distinct prime factors of $m \cdot n$. Let also p be a prime element of R that divides none of the leading coefficients of f and g, and let k be any positive integer with*

$$\gcd(k, \nu_{p_1}(n) - \nu_{p_1}(m), \dots, \nu_{p_r}(n) - \nu_{p_r}(m)) = 1.$$

Then $f(s) + p^k g(s)$ is irreducible over $Q(R)$.

Proof: We first observe that the Dirichlet polynomial $f(s) + p^k g(s)$ is algebraically primitive, as $\gcd(m, n) = 1$ and p divides none of the leading coefficients of f and g. Let us assume to the contrary that $f(s) + p^k g(s) = f_1(s) f_2(s)$ with f_1, f_2 nonconstant Dirichlet polynomials with coefficients in R, say $f_1(s) = \frac{a_{c_1}}{c_1^s} + \cdots + \frac{a_{d_1}}{d_1^s}$ and $f_2(s) = \frac{b_{c_2}}{c_2^s} + \cdots + \frac{b_{d_2}}{d_2^s}$ with $d_1 \geq 2$, $d_2 \geq 2$ and $d_1 d_2 = n$. By Lemma 3.11, one of the leading coefficients of f_1 and f_2 must be divisible by p^k, and the other must be coprime to p. Without loss of generality we may assume that $p^k \mid a_{d_1}$ and $p \nmid b_{d_2}$. Let us denote by $\overline{f}, \overline{f_1}$ and $\overline{f_2}$ the Dirichlet polynomials obtained by reducing modulo p the coefficients of f, f_1 and f_2, respectively. By reducing modulo p the equality $f(s) + p^k g(s) = f_1(s) f_2(s)$, we obtain $\overline{f} = \overline{f_1} \cdot \overline{f_2}$. Since the leading coefficient of f is not divisible by p, we have $\deg \overline{f} = \deg f = m$, and since $p \nmid b_{d_2}$, we have $\deg \overline{f_2} = \deg f_2 = d_2$. If we denote $\deg \overline{f_1}$ by d_1', say, we have $m = d_1' d_2$, while $n = d_1 d_2$, so d_2 must be a common divisor of m and n. Since m and n are coprime, d_2 must be equal to 1, which is a contradiction. This completes the proof. □

Cavachi [32] proved the following irreducibility criterion for linear combinations of relatively prime polynomials.

Theorem (Cavachi) *Let $f(X)$ and $g(X)$ be relatively prime polynomials with integer coefficients, with $\deg f < \deg g$. Then $f(X) + pg(X)$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p.*

In particular, for $k = 1$ we obtain from Theorem 3.12 the following analogue for Dirichlet polynomials of Cavachi's Theorem.

Corollary 3.13. *Let $f(s)$ and $g(s)$ be two Dirichlet polynomials with integer coefficients, of coprime degrees m and n, respectively, with $m < n$. Then $f(s) + pg(s)$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p.*

Proof: Since $k = 1$, condition (22) is trivially satisfied, and the irreducibility of $f(s) + pg(s)$ is ensured for every prime p that divides none of the leading coefficients of f and g. □

Another simple instance of Theorem 3.12 is the following irreducibility criterion.

Corollary 3.14. *Let $f(s)$ and $g(s)$ be two Dirichlet polynomials with integer coefficients, of coprime squarefree degrees m and n, respectively, with $m < n$, and let p be a prime number that divides none of the leading coefficients of f and g. Then $f(s) + p^k g(s)$ is irreducible over $\mathbb{Q}$ for all positive integers k.*

Proof: Since m and n are coprime and squarefree, $\nu_{p_i}(n) - \nu_{p_i}(m) = \pm 1$, so condition (22) is trivially satisfied for every k. □

Our following result is similar to Lemma 3.11, and refers to the min-degree coefficients of the factors of $f + p^k g$, instead of their leading coefficients.

Lemma 3.15. *Let f, g be two Dirichlet polynomials with coefficients in a unique factorization domain R, with $\deg_{\min} f = m$, $\deg_{\min} g = n$, $m > n$, and let $p_1, \dots, p_r$ be the distinct prime factors of $m \cdot n$. Let also p be a prime element of R that divides none of the min-degree coefficients of f and g, and let k be any positive integer with*

$$\gcd(k, \nu_{p_1}(n) - \nu_{p_1}(m), \dots, \nu_{p_r}(n) - \nu_{p_r}(m)) = 1. \tag{23}$$

If $f(s) + p^k g(s)$ is algebraically primitive and can be written as a product of two nonconstant Dirichlet polynomials with coefficients in R, say $f_1(s)$ and $f_2(s)$, then one of the min-degree coefficients of f_1 and f_2 must be divisible by p^k.

Proof: Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_M}{M^s}$ and $g(s) = \frac{b_n}{n^s} + \cdots + \frac{b_N}{N^s}$, $a_m a_M b_n b_N \neq 0$, and let us write $f(s) + p^k g(s) = \frac{c_n}{n^s} + \cdots + \frac{c_u}{u^s}$, where $c_i = p^k b_i$ for $i = n, \dots, m-1$, while $c_i = a_k + p^k b_i$ for $i = m, \dots, u$. Our assumption that $p \nmid a_m b_n$ implies that c_m is not divisible by p, $c_n, \dots, c_{m-1}$ are all divisible by p^k, and moreover, $p^{k+1} \nmid c_n$. Thus, in the Newton log-polygon of $f + p^k g$ with respect to the prime element p, the leftmost edge will join the points $(\log n, k)$ and $(\log m, 0)$, that are labelled in Figure 4 below by $Q_{\ell-1}$ and Q_ℓ, respectively.

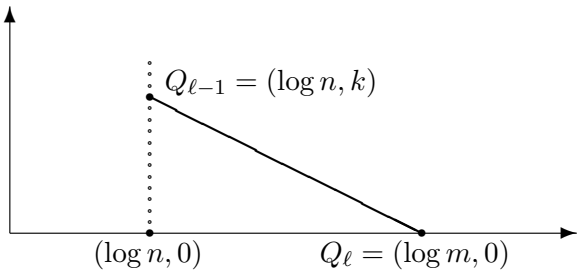

Figure 4. The leftmost segment in the Newton log-polygon of $f(s) + p^k g(s)$ with respect to p.

Since k satisfies (22), we see by Lemma 3.3 that the edge $Q_{\ell-1}Q_\ell$ is in fact a segment of the Newton log-polygon, as it contains no log-integral points other than its endpoints $Q_{\ell-1}$ and Q_ℓ, and moreover, it is the only segment with negative slope. Note that if $f + p^k g = f_1 f_2$ with f_1 and f_2 non-constant Dirichlet polynomials with integer coefficients, then each of the factors f_1 and f_2 must have at least one coefficient not divisible by p, as $p \nmid c_m$. If we assume now that each of the min-degree coefficients of f_1 and f_2 is divisible by p, then each of the Newton log-polygons

of f_1 and f_2 with respect to p will contain at least one segment with negative slope, which by Theorem 3.1 will produce at least two segments with negative slopes in the Newton log-polygon of $f + p^k g$, which is a contradiction. Therefore, one of the min-degree coefficients of f_1 and f_2 must be divisible by p^k, and the other must be coprime to p. This completes the proof. □

As an application of Lemma 3.15 we obtain the following irreducibility criterion for linear combinations of Dirichlet polynomials, in terms of their min-degrees and their min-degree coefficients.

Theorem 3.16. *Let f and g be Dirichlet polynomials with coefficients in a unique factorization domain R, of coprime min-degrees m and n, respectively, with $m > n > 1$, and assume that $\max\{\deg f, \deg g\} \leq 2n$. Let $p_1, \ldots, p_r$ be the distinct prime factors of $m \cdot n$, let p be a prime element of R that divides none of the min-degree coefficients of f and g, and let k be any positive integer with*

$$\gcd(k, \nu_{p_1}(n) - \nu_{p_1}(m), \ldots, \nu_{p_r}(n) - \nu_{p_r}(m)) = 1.$$

Then $f(s) + p^k g(s)$ is irreducible over $Q(R)$.

Proof: We notice that $f(s) + p^k g(s)$ is algebraically primitive, as $\gcd(m, n) = 1$ and p divides none of the min-degree coefficients of f and g. Let us assume to the contrary that $f(s) + p^k g(s) = f_1(s) f_2(s)$ with f_1, f_2 nonconstant Dirichlet polynomials with coefficients in R, say $f_1(s) = \frac{a_{c_1}}{c_1^s} + \cdots + \frac{a_{d_1}}{d_1^s}$ and $f_2(s) = \frac{b_{c_2}}{c_2^s} + \cdots + \frac{b_{d_2}}{d_2^s}$ with $d_1 \geq 2$, $d_2 \geq 2$ and $c_1 c_2 = n$. By Lemma 3.15, one of the min-degree coefficients of f_1 and f_2 must be divisible by p^k, and the other must be coprime to p, say $p^k \mid a_{c_1}$ and $p \nmid b_{c_2}$. As before, let us denote by $\overline{f}, \overline{f_1}$ and $\overline{f_2}$ the Dirichlet polynomials obtained by reducing modulo p the coefficients of f, f_1 and f_2, respectively. By reducing modulo p the equality $f(s) + p^k g(s) = f_1(s) f_2(s)$, we obtain $\overline{f} = \overline{f_1} \cdot \overline{f_2}$. Since the min-degree coefficient of f is not divisible by p, we have $\deg_{\min} \overline{f} = \deg_{\min} f = m$, and since $p \nmid b_{c_2}$, we have $\deg_{\min} \overline{f_2} = \deg_{\min} f_2 = c_2$. If we denote $\deg_{\min} \overline{f_1}$ by c_1', say, we have $m = c_1' c_2$, while $n = c_1 c_2$, so c_2 must be a common divisor of m and n. Since m and n are coprime, c_2 must be equal to 1, so c_1 must be equal to n. This in turn forces d_1 to be greater than n, as $f + p^k g$ cannot have a factor of the form $\frac{a_n}{n^s}$, being algebraically primitive. We thus have $d_2 = \frac{\deg(f + p^k g)}{d_1} \leq \frac{2n}{d_1} < 2$, so d_2 is forced to be equal to 1, a contradiction. □

In particular, we obtain for $k = 1$ the following corollary for Dirichlet polynomials with integer coefficients.

Corollary 3.17. *Let f and g be two Dirichlet polynomials with integer coefficients, of coprime min-degrees m and n, respectively, with $m > n > 1$, and assume that $\max\{\deg f, \deg g\} \leq 2n$. Then $f(s) + pg(s)$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p.*

Proof: Since $k = 1$, condition (23) is obviously satisfied, and the irreducibility of $f(s) + pg(s)$ is now guaranteed for every p that divides none of the min-degree coefficients of f and g. □

Another immediate consequence of Theorem 3.16 is the following irreducibility criterion.

Corollary 3.18. *Let f and g be two Dirichlet polynomials with integer coefficients, of coprime squarefree min-degrees m and n, respectively, with $m > n > 1$ and $\max\{\deg f, \deg g\} \leq 2n$. Let p be a prime number that divides none of the min-degree coefficients of f and g. Then $f(s)+p^k g(s)$ is irreducible over $\mathbb{Q}$ for all positive integers k.*

Proof: Since m and n are coprime and squarefree, $\nu_{p_i}(n) - \nu_{p_i}(m) = \pm 1$, so condition (23) is obviously satisfied for every k. □

In the case when the degrees of f and g are not relatively prime, it is considerably more difficult to draw a conclusion on the irreducibility of $f + p^k g$, but for small values of k, we can at least find some information on the relative degrees of its possible factors, as we will see in the following section in Proposition 4.6 and Proposition 4.12. Other irreducibility criteria for linear combinations of two Dirichlet polynomials, analogous to Schönemann's criterion for polynomials and its variations will be given in Appendix A.

We end this section with the following conjecture, in which we drop the conditions on the degrees of f and g in Corollary 3.13, and instead ask f and g to be relatively prime.

Conjecture 5. *Let f and g be relatively prime Dirichlet polynomials with integer coefficients. Then $f(s) + pg(s)$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p.*

4. The relative degrees of the factors of a Dirichlet polynomial

When we study the canonical decomposition of a Dirichlet polynomial, it is useful to have criteria to exclude some of the possible values of the relative degrees of its hypothetical factors. For polynomials, an important such result was obtained by Filaseta [47], and this result was crucial in proving the irreducibility of all but finitely many Bessel polynomials (see also Filaseta and Trifonov [49], and Filaseta, Finch and Leidy [48], where this result was used to prove the irreducibility of some classes of generalized Laguerre polynomials):

Lemma (Filaseta, [47, Lemma 2]) *Let k and l be integers with $k > l \geq 0$. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$, $p \mid b_j$ for all $j \in \{0, 1, \ldots, n-l-1\}$, and the rightmost edge of the Newton polygon for $g(x)$ with respect to p has slope $< 1/k$. Then for any integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j \in \mathbb{Z}[x]$ cannot have a factor with degree in the interval $[l+1, k]$.*

Here the Newton polygon of $\sum_{j=0}^{n} b_j x^j$ is the lower convex hull of the set of points in the extended plane $\{(0, \nu_p(b_n)), (1, \nu_p(b_{n-1})), \ldots, (n, \nu_p(b_0))\}$.

For Dirichlet polynomials we will first prove the following analogous result.

Lemma 4.1. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p be a prime element of R, c_1, c_2 divisors of m and d_1, d_2 divisors of n with $1 < \frac{d_1}{c_1} \leq \frac{d_2}{c_2} < \frac{n}{m}$. Suppose that $p \nmid a_i$ for some index $i < \frac{md_1}{c_1}$, $p \mid a_j$ for all $j \in \{\frac{md_1}{c_1}, \ldots, n\}$, and the rightmost edge in the Newton log-polygon of f with respect to p has slope $< 1/\log \frac{d_2}{c_2}$. Then for any nonzero elements $b_m, \ldots, b_n \in R$ with*

$p \nmid b_i b_n$, the Dirichlet polynomial $g(s) = \frac{a_m b_m}{m^s} + \cdots + \frac{a_n b_n}{n^s}$ cannot have factors whose relative degrees belong to $[\frac{d_1}{c_1}, \frac{d_2}{c_2}] \cup [\frac{nc_2}{md_2}, \frac{nc_1}{md_1}]$.

Proof: We will first consider the case that $b_j = 1$ for all j, so $g(s) = f(s)$. The conditions of the lemma imply that the Newton log-polygon of f contains at least one edge with positive slope, as $p \nmid a_i$ and $p \mid a_n$, and moreover, at least one edge with slope ≤ 0, if $i > m$. Besides, all the edges in the Newton log-polygon of f with respect to p have slopes less than $1/\log \frac{d_2}{c_2}$. Consider now an edge PQ with positive slope, and assume that $(\log x_1, y_1)$ and $(\log x_2, y_2)$ are two distinct log-integral points on such an edge, with $x_1 < x_2$ and $y_1 < y_2$, say. Then the slope S of this edge satisfies

$$\frac{1}{\log x_2 - \log x_1} \leq S = \frac{y_2 - y_1}{\log x_2 - \log x_1} < \frac{1}{\log \frac{d_2}{c_2}},$$

so

$$\frac{x_2}{x_1} > \frac{d_2}{c_2}. \tag{24}$$

Let us assume to the contrary that f has a factor f_1 with relative degree $\frac{d}{c}$ in the interval $[\frac{d_1}{c_1}, \frac{d_2}{c_2}]$. In particular, in view of (24) we have $\frac{d}{c} \leq \frac{d_2}{c_2} < \frac{x_2}{x_1}$, so the relative degree of f_1 must be smaller than the relative degree of any segment that belongs to an edge with positive slope. Thus, since all the translates of the edges in the Newton log-polygon of f_1 with respect to p become either edges or parts of edges in the Newton log-polygon of f, we conclude that all these translates must be actually found on edges with slope ≤ 0 of the Newton log-polygon of f. If there exist no edges with slope ≤ 0 (for instance if $i = m$ and $p \mid a_j$ for all $j > m$), then such a factor f_1 cannot exist. So let us assume that there exist edges with slope ≤ 0, and among them let PQ be the rightmost one. Thus the sum L of the lengths of the projections on the x-axis of all the edges with slopes ≤ 0 must be at least $\log \frac{d}{c}$. On the other hand, since $p \mid a_j$ for all $j \in \{\frac{md_1}{c_1}, \ldots, n\}$, the x-coordinate of Q cannot exceed $\log(\frac{md_1}{c_1} - 1)$, so L satisfies

$$L \leq \log\left(\frac{md_1}{c_1} - 1\right) - \log m < \log \frac{d_1}{c_1} \leq \log \frac{d}{c},$$

a contradiction. Therefore f cannot have a factor of relative degree $\frac{d}{c}$ in the interval $[\frac{d_1}{c_1}, \frac{d_2}{c_2}]$. Besides, g cannot have a factor h with relative degree in the interval $[\frac{nc_2}{md_2}, \frac{nc_1}{md_1}]$, for otherwise the relative degree of its complementary factor $\frac{g}{h}$ would lie in the interval $[\frac{d_1}{c_1}, \frac{d_2}{c_2}]$.

We will consider now the general case that $b_m, \ldots, b_n$ are arbitrary nonzero elements of R with $p \nmid b_i b_n$. First of all, we note that g is algebraically primitive too. Now, the fact that b_i and b_n are not divisible by p shows that the log-integral point $(\log i, 0)$ and the rightmost endpoint in the Newton log-polygon of f with respect to p will not be affected by passing from f to g, so their positions will remain the same in the Newton log-polygon of g. Since all the edges in the Newton log-polygon of f lie above or on the line that contains its rightmost edge $P_r Q_r$, and $\nu_p(a_k b_k) \geq \nu_p(a_k)$ for all $k \in \{m, \ldots, n\}$, all the edges of the Newton log-polygon of g too will lie above or on the line containing $P_r Q_r$. Moreover, since the two Newton log-polygons have the same rightmost endpoint Q_r, we also deduce that the slope of the rightmost edge in the Newton

log-polygon of g is less than or equal to the slope of P_rQ_r. This can also be seen by observing that

$$\max_{k<n}\frac{\nu_p(a_n)-\nu_p(a_kb_k)}{\log n-\log k}\leq\max_{k<n}\frac{\nu_p(a_n)-\nu_p(a_k)}{\log n-\log k},$$

with the left and the right sides being the slopes of the rightmost edges in the Newton log-polygons of g and f, respectively. Therefore g satisfies the same conditions that are imposed on f in the statement of the lemma, so by the first part of the proof we conclude that g too cannot have a factor of relative degree belonging to $[\frac{d_1}{c_1},\frac{d_2}{c_2}]\cup[\frac{nc_2}{md_2},\frac{nc_1}{md_1}]$. □

Let us recall the definition of the *rational square root* and of the *rational floor* of a pair (m,n):

$$\begin{aligned}\rho(m,n) &:= \max\left\{\frac{d}{c}\ :\ d\mid n,\ c\mid m,\ \text{ and }\ \frac{d}{c}\leq\sqrt{\frac{n}{m}}\right\},\\ \delta(m,n) &:= \min\left\{\frac{d}{c}\ :\ d\mid n,\ c\mid m,\ \text{ and }\ d>c>0\right\},\end{aligned}$$

and the fact that, according to Proposition 2.8, if $\rho(m,n)=1$, then any algebraically primitive Dirichlet polynomial $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ with $a_ma_n\neq 0$ must be irreducible.

As an immediate consequence of Lemma 4.1, one obtains the following irreducibility conditions for the case that $\rho(m,n)>1$ (which implies that $1<\delta(m,n)\leq\rho(m,n)$).

Theorem 4.2. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_ma_n\neq 0$, and let p be a prime element of R. If $p\nmid a_k$ for some index $k<m\delta(m,n)$, $p\mid a_j$ for all $j\geq m\delta(m,n)$, and*

$$\rho(m,n)<\left(\frac{n}{i}\right)^{\frac{1}{\nu_p(a_n)-\nu_p(a_i)}}\ \text{ for all } i<n \text{ with } \nu_p(a_i)<\nu_p(a_n),\tag{25}$$

then for any nonzero elements $b_m,\ldots,b_n\in R$ with $p\nmid b_kb_n$, the Dirichlet polynomial $g(s)=\frac{a_mb_m}{m^s}+\cdots+\frac{a_nb_n}{n^s}$ is irreducible over $Q(R)$.

Proof: If $\rho(m,n)=1$, then f and g are irreducible by Proposition 2.8, hence we may assume that $\rho(m,n)>1$, so $1/\log\rho(m,n)$ makes sense. Note that k is among the indices i with $\nu_p(a_i)<\nu_p(a_n)$, so the set of indices $i<n$ with $\nu_p(a_i)<\nu_p(a_n)$ in (25) is nonempty. We observe that condition (25) forces the slope S of the rightmost edge in the Newton log-polygon of f with respect to p to satisfy the inequality

$$S=\max_{i<n}\frac{\nu_p(a_n)-\nu_p(a_i)}{\log n-\log i}<\frac{1}{\log\rho(m,n)}.$$

According to Lemma 4.1 with $\frac{d_1}{c_1}=\delta(m,n)$, $\frac{d_2}{c_2}=\rho(m,n)$, and $i=k$, our assumptions that $p\nmid a_k$ and $p\mid a_j$ for all $j>m\delta(m,n)$, together with the fact that the rightmost edge in the Newton log-polygon of f has slope smaller than $1/\log\rho(m,n)$, will prevent g to have factors with relative degree in the interval $[\delta(m,n),\rho(m,n)]$. On the other hand, as g is algebraically primitive, if it would be reducible, it should obviously have a nonconstant factor, say $h=\frac{e_c}{c^s}+\cdots+\frac{e_d}{d^s}$ with $d>c$, and whose relative degree $\frac{d}{c}$ belongs to the interval $[\delta(m,n),\rho(m,n)]$. This completes the proof. □

In particular, for $m = 1$ one obtains the following simpler instance of Lemma 4.1.

Lemma 4.3. *Let $f(s) = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \cdots + \frac{a_n}{n^s}$ be a Dirichlet polynomial with coefficients in a unique factorization domain R, $a_1 a_n \neq 0$, p a prime element of R, and d_1, d_2 divisors of n with $1 < d_1 \leq d_2 < n$. Suppose that $p \nmid a_i$ for some index $i < d_1$, $p \mid a_j$ for all $j \geq d_1$, and the rightmost edge in the Newton log-polygon of f with respect to p has slope $< 1/\log d_2$. Then for any nonzero elements $b_1, \ldots, b_n \in R$ with $p \nmid b_i b_n$, the Dirichlet polynomial $g(s) = \frac{a_1 b_1}{1^s} + \frac{a_2 b_2}{2^s} + \cdots + \frac{a_n b_n}{n^s}$ cannot have factors whose degrees belong to $[d_1, d_2] \cup [\frac{n}{d_2}, \frac{n}{d_1}]$.*

A slightly weaker instance of Theorem 4.2 is the following result.

Theorem 4.4. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p be a prime element of R, and assume that $p \nmid a_m$ and $p \mid a_j$ for all $j > m$. If*

$$\rho(m,n) < \left(\frac{n}{i}\right)^{\frac{1}{\nu_p(a_n) - \nu_p(a_i)}} \text{ for all } i < n \text{ with } \nu_p(a_i) < \nu_p(a_n), \tag{26}$$

then for any nonzero elements $b_m, \ldots, b_n \in R$ with $p \nmid b_m b_n$, the Dirichlet polynomial $g(s) = \frac{a_m b_m}{m^s} + \cdots + \frac{a_n b_n}{n^s}$ is irreducible over $Q(R)$.

Remark 4.5. We mention here that Theorem 2.14 ii) is a special case of Theorem 4.4, for if $p \mid a_i$ for each $i = m+1, \ldots, n$, $p \nmid a_m$ and $p^2 \nmid a_n$, then condition (26) reduces to $\rho(m,n) < \frac{n}{m}$, which is obviously satisfied.

In particular, Lemma 4.1 provides information on the factors for some classes of linear combinations of Dirichlet polynomials. We will only present here the simplest such result for linear combinations of the form $f(s) + p^k g(s)$, but one may easily prove more general statements, with less restrictive conditions on the coefficients and on the min-degrees of the Dirichlet polynomial that we are studying.

Proposition 4.6. *Let $f(s) = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \cdots + \frac{a_t}{t^s}$ and $g(s) = \frac{b_1}{1^s} + \frac{b_2}{2^s} + \cdots + \frac{b_n}{n^s}$ be two Dirichlet polynomials with coefficients in a unique factorization domain R, with $a_1 a_t b_1 b_n \neq 0$. Assume that $t < d \leq D$, with d, D divisors of n, $D = \max\{\delta : \delta \mid n,\ \delta \leq \sqrt{n}\}$, and let p be a prime element of R with $p \nmid a_1 a_t b_n$. Then for any positive integer k with*

$$k < \log_D \frac{n}{t}, \tag{27}$$

the Dirichlet polynomial $f(s) + p^k g(s)$ cannot have factors with degrees in $[d, D] \cup [\frac{n}{D}, \frac{n}{d}]$.

Proof: First of all, we observe that $D < \frac{n}{t}$, as $t < D \leq \sqrt{n} \leq \frac{n}{D}$, so the set of positive integers k satisfying (27) is nonempty. We also observe that $f(s) + p^k g(s)$ is algebraically primitive, as its constant term $a_1 + p^k b_1$ is nonzero (since $p \nmid a_1$). Note that, according to the definition of $\rho(m,n)$, in this case we have $D = \rho(1,n)$. Now, if we write $f(s) + p^k g(s) = \sum_{i=1}^{n} \frac{c_i}{i^s}$, with $c_1 c_n \neq 0$, we see that $p \nmid c_t$, $p^k \mid c_i$ for $i \geq t+1$, and $p^{k+1} \nmid c_n$, as $p \nmid a_t b_n$. Therefore the rightmost

edge in the Newton log-polygon of $f(s)+p^k g(s)$ with respect to p has endpoints $(\log t, 0)$ and $(\log n, k)$, so has slope S satisfying

$$S = \frac{k}{\log \frac{n}{t}} < \frac{\log_D \frac{n}{t}}{\log \frac{n}{t}} = \frac{1}{\log D}.$$

By Lemma 4.3 with $d_1 = d$, $d_2 = D$ and $i = t$, we deduce that $f(s)+p^k g(s)$ cannot have a factor with degree in the union of intervals $[d, D] \cup [\frac{n}{D}, \frac{n}{d}]$. □

Lemma 4.1, Theorem 4.2, Theorem 4.4 and Proposition 4.6 have some "mirrored" results that rely on information on the slope of the leftmost edge in the Newton log-polygon.

Lemma 4.7. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p be a prime element of R, c_1, c_2 divisors of m and d_1, d_2 divisors of n with $1 < \frac{d_1}{c_1} \leq \frac{d_2}{c_2} < \frac{n}{m}$. Suppose that $p \mid a_j$ for all $j \in \{m, \ldots, \frac{nc_1}{d_1}\}$, $p \nmid a_i$ for some index $i > \frac{nc_1}{d_1}$, and the leftmost edge in the Newton log-polygon of f with respect to p has slope $> -1/\log \frac{d_2}{c_2}$. Then for any nonzero elements $b_m, \ldots, b_n \in R$ with $p \nmid b_i b_m$, the Dirichlet polynomial $g(s) = \frac{a_m b_m}{m^s} + \cdots + \frac{a_n b_n}{n^s}$ cannot have factors whose relative degrees belong to $[\frac{d_1}{c_1}, \frac{d_2}{c_2}] \cup [\frac{nc_2}{md_2}, \frac{nc_1}{md_1}]$.*

Proof: As before, we will first consider the case that $b_j = 1$ for all j, so $g(s) = f(s)$. Our hypotheses imply that the Newton log-polygon of f contains at least one edge with negative slope, as $p \nmid a_i$ and $p \mid a_m$, and moreover, at least one edge with slope ≥ 0, if $i < n$. Besides, all the edges in the Newton log-polygon of f with respect to p have now slopes greater than $-1/\log \frac{d_2}{c_2}$. Let us consider now an edge PQ with negative slope, and assume that $(\log x_1, y_1)$ and $(\log x_2, y_2)$ are two distinct log-integral points on such an edge, with $x_1 < x_2$ and $y_1 > y_2$, say. Then the slope S of this edge satisfies

$$\frac{-1}{\log x_2 - \log x_1} \geq S = \frac{y_2 - y_1}{\log x_2 - \log x_1} > \frac{-1}{\log \frac{d_2}{c_2}},$$

so as in Lemma 4.1 we must have

$$\frac{x_2}{x_1} > \frac{d_2}{c_2}. \tag{28}$$

We will assume to the contrary that f has a factor f_1 with relative degree $\frac{d}{c}$ in the interval $[\frac{d_1}{c_1}, \frac{d_2}{c_2}]$. In this case (28) forces all the translates of the edges in the Newton log-polygon of f_1 with respect to p to belong to the edges with slope ≥ 0 of the Newton log-polygon of f. If there exist no edges with slope ≥ 0 (for instance if $i = n$ and $p \mid a_j$ for all $j < n$), then such a factor f_1 cannot exist. So let us assume that there exist edges with slope ≥ 0, and among them let now PQ be the leftmost one. Thus the sum L of the lengths of the projections on the x-axis of all the edges with slopes ≥ 0 must be at least $\log \frac{d}{c}$. On the other hand, since $p \mid a_j$ for all $j \in \{m, \ldots, \frac{nc_1}{d_1}\}$, the x-coordinate of P cannot be less than $\log(\frac{nc_1}{d_1} + 1)$, so L must satisfy

$$L \leq \log n - \log\left(\frac{nc_1}{d_1} + 1\right) < \log \frac{d_1}{c_1} \leq \log \frac{d}{c},$$

a contradiction. Therefore f cannot have a factor f_1 of relative degree $\frac{d}{c}$ in the interval $[\frac{d_1}{c_1}, \frac{d_2}{c_2}]$. Besides, f can neither have a factor f_1 of relative degree $\frac{d}{c}$ in the interval $[\frac{nc_2}{md_2}, \frac{nc_1}{md_1}]$, for otherwise $\frac{f}{f_1}$ would have relative degree in the interval $[\frac{d_1}{c_1}, \frac{d_2}{c_2}]$.

For the general case that $b_m, \dots, b_n$ are arbitrary nonzero elements of R with $p \nmid b_i b_m$, we first note that g is algebraically primitive too, and then we observe that since b_i and b_m are not divisible by p, the log-integral point $(\log i, 0)$ and the leftmost endpoint $(\log m, \nu_p(a_m))$ in the Newton log-polygon of f with respect to p will not be affected by passing from f to g, so their positions will remain the same in the Newton log-polygon of g. Since all the edges in the Newton log-polygon of f lie above or on the line that contains its leftmost edge $P_l Q_l$, and $\nu_p(a_k b_k) \geq \nu_p(a_k)$ for all $k \in \{m, \dots, n\}$, the edges of the Newton log-polygon of g too will lie above or on the line containing $P_l Q_l$. Moreover, since the two Newton log-polygons have the same leftmost endpoint P_l, we also deduce that the slope of the leftmost edge in the Newton log-polygon of g is greater than or equal to the slope of $P_l Q_l$. This can also be seen by observing that

$$\min_{k>m} \frac{\nu_p(a_k b_k) - \nu_p(a_m)}{\log k - \log m} \geq \min_{k>m} \frac{\nu_p(a_k) - \nu_p(a_m)}{\log k - \log m},$$

with the left and the right sides being the slopes of the leftmost edges in the Newton log-polygons of g and f, respectively. Thus g satisfies the same conditions that are imposed on f in the statement of the lemma, so by the first part of the proof we conclude again that g cannot have a factor of relative degree $\frac{d}{c}$ belonging to $[\frac{d_1}{c_1}, \frac{d_2}{c_2}] \cup [\frac{nc_2}{md_2}, \frac{nc_1}{md_1}]$. □

The mirrored result of Theorem 4.2 is the following application of Lemma 4.7.

Theorem 4.8. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p be a prime element of R, and assume that $p \nmid a_k$ for some index $k > \frac{n}{\delta(m,n)}$ and $p \mid a_j$ for all $j \leq \frac{n}{\delta(m,n)}$. If*

$$\rho(m,n) < \left(\frac{i}{m}\right)^{\frac{1}{\nu_p(a_m) - \nu_p(a_i)}} \quad \text{for all } i > m \text{ with } \nu_p(a_i) < \nu_p(a_m), \tag{29}$$

then for any nonzero elements $b_m, \dots, b_n \in R$ with $p \nmid b_m b_n$, the Dirichlet polynomial $g(s) = \frac{a_m b_m}{m^s} + \cdots + \frac{a_n b_n}{n^s}$ is irreducible over $Q(R)$.

Proof: As in the case of Theorem 4.2, we may assume that $\rho(m,n) > 1$, so $1/\log \rho(m,n)$ makes sense. We observe now that k is among the indices $i > m$ with $\nu_p(a_i) < \nu_p(a_m)$, so the set of indices $i > m$ with $\nu_p(a_i) < \nu_p(a_m)$ in (29) is nonempty. Next, we observe that by (29), the slope S of the leftmost edge in the Newton log-polygon of f with respect to p satisfies

$$S = \min_{i>m} \frac{\nu_p(a_i) - \nu_p(a_m)}{\log i - \log m} > \frac{-1}{\log \rho(m,n)}.$$

By Lemma 4.7 with $\frac{d_1}{c_1} = \delta(m,n)$, $\frac{d_2}{c_2} = \rho(m,n)$, and $i = k$, our assumptions that $p \nmid a_k$ for some index $k > \frac{n}{\delta(m,n)}$ and $p \mid a_j$ for all $j \leq \frac{n}{\delta(m,n)}$, together with the fact that the leftmost edge in the Newton log-polygon of f has slope $> -1/\log \rho(m,n)$, will prevent g to have factors with relative

degree in the interval $[\delta(m,n),\rho(m,n)]$. On the other hand, as g is algebraically primitive, if it would be reducible, it should obviously have a nonconstant factor whose relative degree lies in this interval. This completes the proof. □

In particular, for $m=1$ one obtains the following simpler instance of Lemma 4.7.

Lemma 4.9. *Let $f(s)=\frac{a_1}{1^s}+\frac{a_2}{2^s}\cdots+\frac{a_n}{n^s}$ be a Dirichlet polynomial with coefficients in a unique factorization domain R, $a_1a_n\neq 0$, let p be a prime element of R, and d_1,d_2 divisors of n with $1<d_1\leq d_2<n$. Suppose that $p\mid a_j$ for all $j\in\{1,\ldots,\frac{n}{d_1}\}$, $p\nmid a_i$ for some index $i>\frac{n}{d_1}$, and the leftmost edge in the Newton log-polygon of f with respect to p has slope $>-1/\log d_2$. Then for any nonzero elements $b_1,\ldots,b_n\in R$ with $p\nmid b_1b_i$, the Dirichlet polynomial $g(s)=\frac{a_1b_1}{1^s}+\frac{a_2b_2}{2^s}\cdots+\frac{a_nb_n}{n^s}$ cannot have factors whose degrees belong to $[d_1,d_2]\cup[\frac{n}{d_2},\frac{n}{d_1}]$.*

A slightly weaker instance of Theorem 4.8 is the following result that mirrors Theorem 4.4.

Theorem 4.10. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_ma_n\neq 0$, let p be a prime element of R, and assume that $p\nmid a_n$ and $p\mid a_j$ for all $j<n$. If*

$$\rho(m,n)<\left(\frac{i}{m}\right)^{\frac{1}{\nu_p(a_m)-\nu_p(a_i)}}\quad \text{for all } i>m \text{ with } \nu_p(a_i)<\nu_p(a_m), \tag{30}$$

then for any nonzero elements $b_m,\ldots,b_n\in R$ with $p\nmid b_mb_n$, the Dirichlet polynomial $g(s)=\frac{a_mb_m}{m^s}+\cdots+\frac{a_nb_n}{n^s}$ is irreducible over $Q(R)$.

Remark 4.11. We note that Theorem 2.14 i) is a special case of Theorem 4.10, for if $p\mid a_i$ for each $i=m,\ldots,n-1$, $p\nmid a_n$ and $p^2\nmid a_m$, then condition (30) reduces to $\rho(m,n)<\frac{n}{m}$, which obviously holds.

The final result in this section is the following one, that mirrors Proposition 4.6:

Proposition 4.12. *Let $f(s)=\frac{a_t}{t^s}+\cdots+\frac{a_n}{n^s}$ and $g(s)=\frac{b_1}{1^s}+\frac{b_2}{2^s}+\cdots+\frac{b_n}{n^s}$ be two Dirichlet polynomials with coefficients in a unique factorization domain R, with $a_ta_nb_1b_n\neq 0$. Assume that $t>\frac{n}{d}\geq\frac{n}{D}$, with d,D divisors of n, $D=\max\{\delta:\delta\mid n,\ \delta\leq\sqrt{n}\}$, and let p be a prime element of R with $p\nmid a_ta_nb_1$. Then for any positive integer k with*

$$k<\log_D t, \tag{31}$$

the Dirichlet polynomial $f(s)+p^kg(s)$ cannot have factors with degrees in $[d,D]\cup[\frac{n}{D},\frac{n}{d}]$.

Proof: Here we first observe that since $D\leq\sqrt{n}$, we have $D<t$, as $t>\frac{n}{d}\geq\frac{n}{D}\geq D$, so the set of positive integers k satisfying (31) is nonempty. We also observe that $f(s)+p^kg(s)$ is algebraically primitive, as its constant term p^kb_1 is nonzero. If we write now $f(s)+p^kg(s)=\sum_{i=1}^n\frac{c_i}{i^s}$, with $c_1c_n\neq 0$ ($c_n=a_n+p^kb_n\neq 0$ as $p\nmid a_n$), we see that $p\nmid c_t$, $p^k\mid c_i$ for $i\leq t-1$, and

$p^{k+1} \nmid c_1$, as $p \nmid a_t b_1$. Therefore the leftmost edge in the Newton log-polygon of $f(s) + p^k g(s)$ with respect to p has endpoints $(0, k)$ and $(\log t, 0)$, so has slope S satisfying

$$S = \frac{-k}{\log t} > \frac{-\log_D t}{\log t} = \frac{-1}{\log D}.$$

The conclusion follows now by Lemma 4.9 with $d_1 = d$, $d_2 = D$ and $i = t$. □

5. Irreducibility criteria that use two or more p-adic valuations

Using a Newton polygon method, we provided in [18] several irreducibility criteria that use simultaneously information on the divisibility of the coefficients of a polynomial with respect to two or more prime numbers. Two such results are the following ones.

Theorem A. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree n, let $k \geq 2$, and let $p_1, \ldots, p_k$ be pairwise distinct prime numbers. For $i = 1, \ldots, k$ let us denote by $w_{i,1}, \ldots, w_{i,n_i}$ the widths of all the segments of the Newton polygon of f with respect to p_i, and by $\mathcal{S}_{p_i}$ the set of all the integers in the interval $(0, \lfloor \frac{n}{2} \rfloor]$ that may be written as a linear combination of $w_{i,1}, \ldots, w_{i,n_i}$ with coefficients 0 or 1. If $\mathcal{S}_{p_1} \cap \cdots \cap \mathcal{S}_{p_k} = \emptyset$, then f is irreducible over $\mathbb{Q}$.*

We mention here that the integers $w_{i,1}, \ldots, w_{i,n_i}$ are not necessarily distinct.

Theorem B. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree n, let $k \geq 2$, let $p_1, \ldots, p_k$ be pairwise distinct prime numbers, and for $i = 1, \ldots, k$ let d_{p_i} denote the greatest common divisor of the widths of the segments of the Newton polygon of f with respect to p_i. Then the degree of any factor of f is divisible by $n/\gcd(\frac{n}{d_{p_1}}, \ldots, \frac{n}{d_{p_k}})$. In particular, if $\gcd(\frac{n}{d_{p_1}}, \ldots, \frac{n}{d_{p_k}}) = 1$, then f is irreducible over $\mathbb{Q}$.*

An immediate application of Theorem 3.4 is the following analogue of Theorem A, that allows one to combine information on the Newton log-polygons of a Dirichlet polynomial f with respect to different prime elements in R in order to study its irreducibility. By contrast to its simple proof, this result will provide us with a lot of unexpected corollaries, as we shall see later on. Before stating it, we will recall the definition of $S^1_{rd}(m, n)$:

$$S^1_{rd}(m, n) := \left\{ \frac{d}{c} \ : \ d \mid n, \ c \mid m, \ \text{ and } \ 1 < \frac{d}{c} \leq \sqrt{\frac{n}{m}} \right\},$$

and the fact that for a segment AB of the Newton log-polygon of f with endpoints $A = (\log i, x_i)$ and $B = (\log j, x_j)$ with $i < j$, the rational number $\frac{j}{i} > 1$ is called the *relative degree* of AB.

Theorem 5.1. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, and $p_1, \ldots, p_k$ prime elements in R. For each $i = 1, \ldots, k$ let $q_{i,1}, \ldots, q_{i,n_i}$ be the relative degrees of all the segments of the Newton log-polygon of f with respect to p_i, repetitions allowed, and define*

$$\mathcal{S}_{p_i} = S^1_{rd}(m, n) \cap \left\{ q_{i,1}^{j_1} \cdots q_{i,n_i}^{j_{n_i}} \ : \ j_1, \ldots, j_{n_i} \in \{0, 1\} \right\}.$$

If $\mathcal{S}_{p_1} \cap \cdots \cap \mathcal{S}_{p_k} = \emptyset$, then f is irreducible over $Q(R)$.

Here the relative degrees $q_{i,1},\dots,q_{i,n_i}$ of the segments are not necessarily distinct, and are considered with "their multiplicities", since an edge may consist of more than a single segment, and all the segments on an edge have the same relative degree. We also recall the fact that $q_{i,1},\dots,q_{i,n_i}$ are rational numbers greater than 1, that satisfy the relation $q_{i,1}\cdots q_{i,n_i}=\frac{n}{m}$ for each $i=1,\dots,k$.

Proof: We first note that $1\notin\mathcal{S}_{p_i}$, since $1\notin S^1_{rd}(m,n)$, so in the definition of $\mathcal{S}_{p_i}$ we may assume that at least one of the exponents $j_1,\dots,j_{n_i}$ is nonzero. Let us assume to the contrary that f is reducible, and let $g(s)=\frac{b_c}{c^s}+\cdots+\frac{b_d}{d^s}$ be a nonconstant factor of f with minimum relative degree $\frac{d}{c}$, so $c\mid m$, $d\mid n$ and $\frac{d}{c}\in S^1_{rd}(m,n)$. Let us fix now an integer $i\in\{1,\dots,k\}$, and let us look at the Newton log-polygon of f with respect to p_i. In view of Theorem 3.1, $\log d-\log c$ must be the sum of some of the widths $\log q_{i,1},\dots,\log q_{i,n_i}$ of the segments in the Newton log-polygon of f with respect to p_i (in such a sum, each $\log q_{i,j}$ may enter at most once, but two such terms entering the sum, say $\log q_{i,j_1}$ and $\log q_{i,j_2}$, may be equal, if they are the widths of two segments lying on the same edge). In other words, $\frac{d}{c}$ must be of the form

$$\frac{d}{c}=q_{i,1}^{j_1}\cdots q_{i,n_i}^{j_{n_i}},$$

for some exponents $j_1,\dots,j_{n_i}\in\{0,1\}$, not all zero, so $\frac{d}{c}$ must belong to $\mathcal{S}_{p_i}$. Since i was chosen arbitrarily, $\frac{d}{c}$ must belong to each $\mathcal{S}_{p_i}$, $i=1,\dots,k$, which obviously cannot hold, as $\mathcal{S}_{p_1}\cap\cdots\cap\mathcal{S}_{p_k}=\emptyset$, according to our hypotheses. Therefore f must be irreducible over $Q(R)$, and this completes the proof of the theorem. □

Another immediate application of Theorem 3.4 that allows one to combine information on the Newton log-polygons of f with respect to different prime elements in R, is the following analogue of Theorem B.

Theorem 5.2. *Let f be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, and let $p_1,\dots,p_k$ be prime elements in R. For each $i=1,\dots,k$ let $q_{i,1},\dots,q_{i,n_i}$ be the relative degrees of all the segments of the Newton log-polygon of f with respect to p_i, and let d_i be the greatest common divisor of the numerators of the rationals $q_{i,1},\dots,q_{i,n_i}$ written in lowest terms. If $\operatorname{lcm}(d_1,\dots,d_k)=\deg f$, then f is irreducible over $Q(R)$.*

Proof: Assume that $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$, and let $g(s)=\frac{b_c}{c^s}+\cdots+\frac{b_d}{d^s}$ be a nonconstant factor of f of minimum degree. Obviously $c\mid m$ and $d\mid n$. If we fix an integer $i\in\{1,\dots,k\}$ and look at the Newton log-polygon of f with respect to p_i, then by Theorem 3.1, $\log d-\log c$ must be the sum of some of the widths $\log q_{i,1},\dots,\log q_{i,n_i}$ of the segments, so we deduce as in the proof of Theorem 5.1 that

$$\frac{d}{c}=q_{i,1}^{j_1}\cdots q_{i,n_i}^{j_{n_i}},$$

for some exponents $j_1,\dots,j_{n_i}\in\{0,1\}$, not all zero. On the other hand, since $q_{i,1},\dots q_{i,n_i}$ are written in lowest terms, d_i must be coprime to all the denominators of $q_{i,1},\dots q_{i,n_i}$, so d_i must divide d. Since this must hold for all $i=1,\dots,k$, d must be a common multiple of $d_1,\dots,d_k$,

and hence a multiple of $\operatorname{lcm}(d_1,\dots,d_k)$. Therefore, if $\operatorname{lcm}(d_1,\dots,d_k)=n$, d must be equal to n, forcing f to be irreducible. This completes the proof. □

In particular, for Dirichlet polynomials $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ with $\gcd(m,n)=1$ we have the following result, which allows us to extend Theorem 3.4 by using divisibility conditions for the coefficients of f with respect to any number of prime elements in R.

Theorem 5.3. *Let $f(s)=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n\neq 0$ and $\gcd(m,n)=1$, and let $p_1,\dots,p_t$ be prime elements of R. Let $q_1,\dots,q_k$ be all the prime factors of $m\cdot n$. If*

i) $\nu_{p_j}(a_m)\neq\nu_{p_j}(a_n)$, for $j=1,\dots,t$,
ii) $\left(\frac{n}{i}\right)^{\nu_{p_j}(a_i)-\nu_{p_j}(a_m)}>\left(\frac{m}{i}\right)^{\nu_{p_j}(a_i)-\nu_{p_j}(a_n)}$ for $m<i<n$, $j=1,\dots,t$,
iii) $\gcd(\nu_{p_1}(a_n)-\nu_{p_1}(a_m),\dots,\nu_{p_t}(a_n)-\nu_{p_t}(a_m),\nu_{q_1}(n)-\nu_{q_1}(m),\dots,\nu_{q_k}(n)-\nu_{q_k}(m))=1$,

then f is irreducible over $Q(R)$.

Proof: Note first that f is algebraically primitive, as m and n are coprime. Conditions i) and ii) show that for each $j=1,\dots,t$ the Newton log-polygon of f with respect to p_j consists of a single edge P_jQ_j, say. Let us fix an index $j\in\{1,\dots,t\}$ and let $g(s)=\frac{b_c}{c^s}+\cdots+\frac{b_d}{d^s}$ be a nonconstant factor of f. Obviously $c\mid m$ and $d\mid n$. By Theorem 3.1, $\log d-\log c$ must be the sum of some of the widths of the segments that the edge P_jQ_j is composed of, so by Lemma 3.3, it must be a multiple of $\frac{1}{\delta_j}(\log n-\log m)$, say

$$\log d-\log c=\frac{M_j}{\delta_j}(\log n-\log m)$$

with $\delta_j=\gcd(\nu_{p_j}(a_n)-\nu_{p_j}(a_m),\nu_{q_1}(n)-\nu_{q_1}(m),\dots,\nu_{q_k}(n)-\nu_{q_k}(m))$, and M_j an integer with $0<M_j\leq\delta_j$. This equality leads us to

$$\frac{d}{c}=\left(\frac{n}{m}\right)^{\frac{M_j}{\delta_j}}\quad\text{for } j=1,\dots,t. \tag{32}$$

We shall prove now that $M_j=\delta_j$ for each j. Let us assume to the contrary that this is false. Note that $\frac{M_j}{\delta_j}$ are all rational numbers, which by (32) must all be equal, so we may write

$$\frac{M_1}{\delta_1}=\cdots=\frac{M_t}{\delta_t}=\frac{a}{b},$$

with $0<a<b$ and $\gcd(a,b)=1$. This yields $a\delta_j=bM_j$ for each j, which, since a and b are coprime, forces b to divide δ_j for each j. On the other hand, by condition iii) we see that $\delta_1,\dots,\delta_t$ must be coprime, so $b=1$, which leaves no possibility for a. Therefore $M_j=\delta_j$ for each j, so by (32) $\frac{d}{c}$ and $\frac{n}{m}$ must be equal too. Since $d\mid n$, $c\mid m$, and m and n are coprime, we finally see that d must be equal to n (and c must be equal to m), which proves that f has no nonconstant factors other than itself, thus being irreducible. □

In what follows, we will give a series of explicit irreducibility criteria that use divisibility conditions with respect to two prime elements $p,q\in R$ for the case that each of the Newton

log-polygons with respect to p and q has at most two edges. Figure 5 below displays all the relevant such combinations of shapes for the two Newton log-polygons, depending on the sign of the slopes of the two edges. Here we had to exclude the four cases obtained by flipping horizontally Figures 5.g, 5.h, 5.i and 5.k, as they would contain a rightmost horizontal edge. Since we will be only interested in edges that contain a single segment, such a horizontal edge would have endpoints $(\log(n-1), 0)$ and $(\log n, 0)$, thus forcing $a_{n-1} \neq 0$ and hence the irreducibility of f by Corollary 2.5. Note that the situation analysed in Theorem 5.3 corresponds to Figure 5.n. For the 13 remaining cases we will provide only the simplest possible irreducibility conditions, where inequalities like the one in condition ii) in Theorem 5.3 are automatically satisfied, thus keeping the statements as close as possible to the one of the Schönemann-Eisenstein criterion for polynomials. By also considering these inequalities, one may immediately state the corresponding 13 results in full generality, but with more involved statements. For the proof of the following 13 results we actually don't need the full strength of Theorem 5.1, in the sense that we will not have to compute $S^1_{rd}(m,n)$. Instead, to exclude some possible values of the relative degrees of the hypothetical factors of f, it will suffice to invoke Theorem 3.1 and Lemma 3.3.

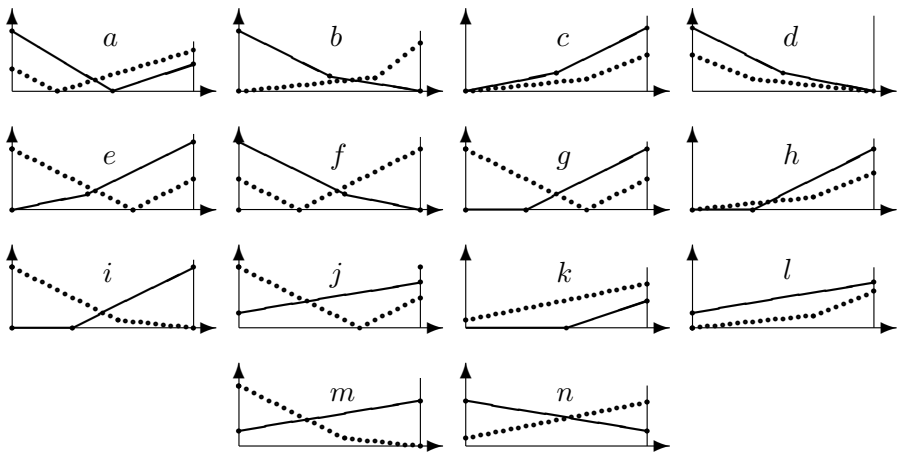

Figure 5. *Relevant combinations of shapes of Newton log-polygons of a Dirichlet polynomial with respect to a pair of prime elements, each one having at most two segments.*

Most of the irreducibility conditions for polynomials in the literature are asymmetric with respect to the usual and reverse ordering of the coefficients. Among the results in [18], we will also mention one providing irreducibility conditions which use two prime numbers and are invariant under reversing the order of the coefficients:

Theorem C. ([18, Corollary 1.14]) *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$, $a_0a_n \neq 0$. If there exist two distinct indices $j, k \in \{1, \ldots, n-1\}$ such that $j + k \neq n$ and two distinct prime numbers p and q such that*

i) $p \mid a_i$ for $i \neq j$, $p \nmid a_j$, $p^2 \nmid a_0$ and $p^2 \nmid a_n$,

ii) $q \mid a_i$ for $i \neq k$, $q \nmid a_k$, $q^2 \nmid a_0$ and $q^2 \nmid a_n$,

then f is irreducible over $\mathbb{Q}$.

The following result, that uses symmetric irreducibility conditions too, may be considered as an analogue for Dirichlet polynomial of Theorem C, and corresponds to Figure 5.a.

Theorem 5.4. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exist two different indices $j_1, j_2 \in \{m+1, \ldots, n-1\}$ with $j_1 j_2 \neq mn$ such that:*

i) $p \mid a_i$ for all $i \neq j_1$, $p \nmid a_{j_1}$, $p^2 \nmid a_m$ and $p^2 \nmid a_n$;

ii) $q \mid a_i$ for all $i \neq j_2$, $q \nmid a_{j_2}$, $q^2 \nmid a_m$ and $q^2 \nmid a_n$;

Then f is irreducible over $Q(R)$.

Proof: Let us represent in Figure 6 the Newton log-polygons with respect to two prime elements p and q, each one consisting of two edges having slopes with different signs, as in Figure 5.a.

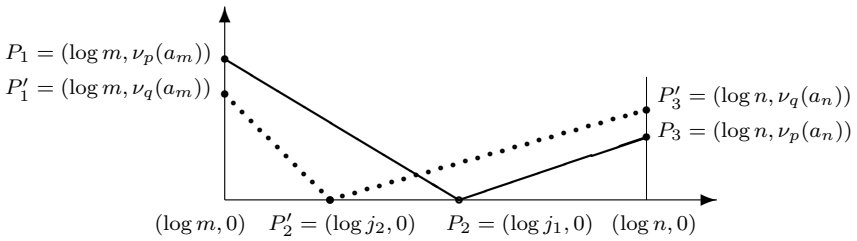

Figure 6.

In our case $\nu_p(a_m) = \nu_p(a_n) = \nu_q(a_m) = \nu_q(a_n) = 1$, so by Lemma 3.3, each one of the edges P_1P_2, P_2P_3 and $P_1'P_2', P_2'P_3'$ contains a single segment. Let $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ be a nonconstant factor of f, and let us assume that $d < n$. By using Theorem 3.1 for the prime elements p and q, we deduce that the relative degree $\frac{d}{c}$ of g must belong to the intersection

$$\left\{\frac{j_1}{m}, \frac{n}{j_1}\right\} \bigcap \left\{\frac{j_2}{m}, \frac{n}{j_2}\right\}.$$

On the other hand this intersection is empty, as $j_1 \neq j_2$ and $j_1 j_2 \neq mn$, so d must be equal to n, and c to m, implying $g = f$. Therefore f must be irreducible. □

Theorem 5.5. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exist indices j_1, j_2 with $m < j_1 < \sqrt{mn} < j_2 < n$ and $j_1 j_2 \neq mn$ such that:*

i) $p^2 \mid a_i$ for $m \leq i < j_1$, $p \mid a_i$ for $j_1 \leq i < n$, $p \nmid a_n$, $p^2 \nmid a_{j_1}$ and $p^3 \nmid a_m$;

ii) $q^2 \mid a_i$ for $j_2 < i \leq n$, $q \mid a_i$ for $m < i \leq j_2$, $q \nmid a_m$, $q^2 \nmid a_{j_2}$ and $q^3 \nmid a_n$;

Then f is irreducible over $Q(R)$.

Proof: Let us represent now in Figure 7 the Newton log-polygons with respect to two prime elements p and q, one consisting of two edges having negative slopes, and the other consisting of two edges having positive slopes, as in Figure 5.b.

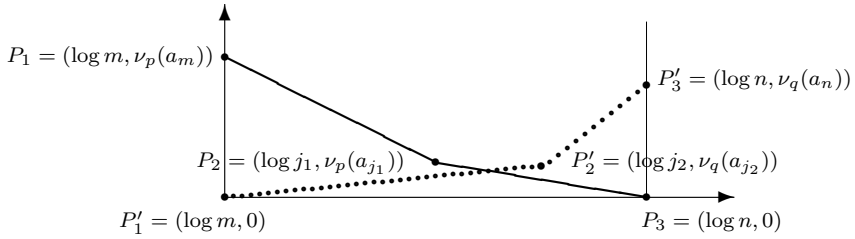

Figure 7.

In our particular case we have $\nu_p(a_m) = \nu_q(a_n) = 2$ and $\nu_p(a_{j_1}) = \nu_q(a_{j_2}) = 1$, so conditions i) and ii) together with Lemma 3.3 show that each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. The inequalities $m < j_1 < \sqrt{mn} < j_2 < n$ show that the slopes of P_1P_2 and $P_1'P_2'$ are smaller than the slopes of P_2P_3 and $P_2'P_3'$, respectively. In particular, P_1, P_2 and P_3 are not collinear, and P_1', P_2' and P_3' too are not collinear. As before, f cannot have a nonconstant factor $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ with $d < n$, since by Theorem 3.1, $\frac{d}{c}$ would be forced to belong to the intersection

$$\left\{\frac{j_1}{m}, \frac{n}{j_1}\right\} \bigcap \left\{\frac{j_2}{m}, \frac{n}{j_2}\right\},$$

which is again empty, as $j_1 \neq j_2$ and $j_1 j_2 \neq mn$. So again f must be irreducible. □

Theorem 5.6. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exist indices $j_1 \neq j_2$ with $j_1, j_2 \in (\sqrt{mn}, n)$ such that:*

i) $p \nmid a_m$, $p \mid a_i$ for $m < i \leq j_1$, $p^2 \nmid a_{j_1}$, $p^2 \mid a_i$ for $j_1 < i \leq n$ and $p^3 \nmid a_n$;

ii) $q \nmid a_m$, $q \mid a_i$ for $m < i \leq j_2$, $q^2 \nmid a_{j_2}$, $q^2 \mid a_i$ for $j_2 < i \leq n$ and $q^3 \nmid a_n$.

Then f is irreducible over $Q(R)$.

Proof: We represent now in Figure 8 the Newton log-polygons with respect to two prime elements p and q, both consisting of two edges having positive slopes, as in Figure 5.c.

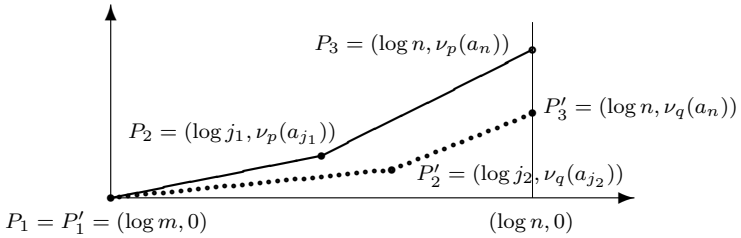

Figure 8.

Here $\nu_p(a_n) = \nu_q(a_n) = 2$ and $\nu_p(a_{j_1}) = \nu_q(a_{j_2}) = 1$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. Using the fact that both indices j_1 and j_2 belong to the interval $(\sqrt{mn}, n)$, one may check that the slopes of P_2P_3 and $P_2'P_3'$ are greater than the slopes of P_1P_2 and $P_1'P_2'$, respectively. In particular, P_1, P_2 and P_3 are not collinear, and P_1', P_2' and P_3' too are not collinear. Now, since $j_1 \neq j_2$ and $j_1 j_2 > mn$, we conclude that the intersection

$$\left\{\frac{j_1}{m}, \frac{n}{j_1}\right\} \bigcap \left\{\frac{j_2}{m}, \frac{n}{j_2}\right\}$$

is empty, thus forcing f to be irreducible. □

Theorem 5.7. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exist indices $j_1 \neq j_2$ with $j_1, j_2 \in (m, \sqrt{mn})$ such that:*

i) $p^2 \mid a_i$ for $m \leq i < j_1$, $p \mid a_i$ for $j_1 \leq i < n$, $p \nmid a_n$, $p^2 \nmid a_{j_1}$ and $p^3 \nmid a_m$;

ii) $q^2 \mid a_i$ for $m \leq i < j_2$, $q \mid a_i$ for $j_2 \leq i < n$, $q \nmid a_n$, $q^2 \nmid a_{j_2}$ and $q^3 \nmid a_m$;

Then f is irreducible over $Q(R)$.

Proof: We represent now in Figure 9 the Newton log-polygons with respect to two prime elements p and q, both consisting of two edges having negative slopes, as in Figure 5.d.

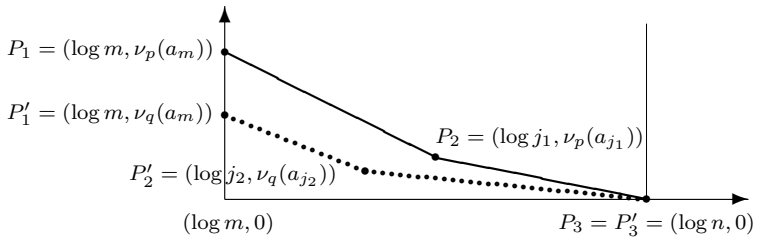

Figure 9.

Here $\nu_p(a_m) = \nu_q(a_m) = 2$ and $\nu_p(a_{j_1}) = \nu_q(a_{j_2}) = 1$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. The fact that $j_1, j_2 \in (m, \sqrt{mn})$ forces the slopes of P_2P_3 and $P_2'P_3'$ to be greater than the slopes of P_1P_2 and $P_1'P_2'$, respectively. In particular, P_1, P_2 and P_3 are not collinear, and P_1', P_2' and P_3' too are not collinear. Since $j_1 \neq j_2$ and $j_1 j_2 < mn$, the intersection

$$\left\{\frac{j_1}{m}, \frac{n}{j_1}\right\} \bigcap \left\{\frac{j_2}{m}, \frac{n}{j_2}\right\}$$

is empty, so f must be irreducible. □

Theorem 5.8. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exist indices $j_1 \neq j_2$ with $\sqrt{mn} < j_1 < n$, $m < j_2 < n$, $j_1 j_2 \neq mn$ such that:*

i) $p \nmid a_m$, $p \mid a_i$ for $m < i \leq j_1$, $p^2 \nmid a_{j_1}$, $p^2 \mid a_i$ for $j_1 < i \leq n$ and $p^3 \nmid a_n$;

ii) $q \mid a_i$ for $i \neq j_2$, $q \nmid a_{j_2}$, $q^2 \nmid a_m$ and $q^2 \nmid a_n$.

Then f is irreducible over $Q(R)$.

Proof: We represent now in Figure 10 the Newton log-polygons with respect to two prime elements p and q, one consisting of two edges having positive slopes, and the other consisting of two edges having slopes of different signs, as in Figure 5.e.

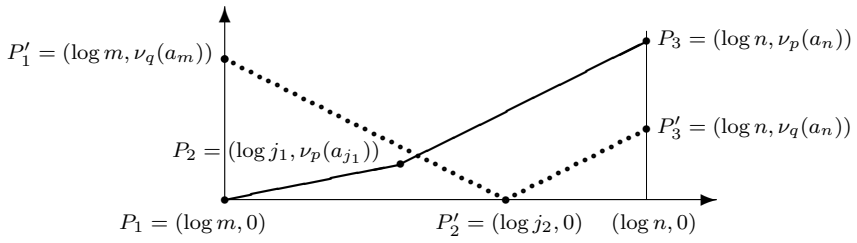

Figure 10.

In our case $\nu_q(a_n) = \nu_p(a_{j_1}) = \nu_q(a_m) = 1$ and $\nu_p(a_n) = 2$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. Moreover, the slope of P_1P_2 is smaller than the slope of P_2P_3, as $\sqrt{mn} < j_1 < n$, so in particular the log-integral points P_1, P_2 and P_3 are not collinear. The intersection

$$\left\{\frac{j_1}{m}, \frac{n}{j_1}\right\} \bigcap \left\{\frac{j_2}{m}, \frac{n}{j_2}\right\},$$

is again empty, showing that f must be irreducible. □

Theorem 5.9. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exist indices $j_1 \neq j_2$ with $m < j_1 < \sqrt{mn}$, $m < j_2 < n$ and $j_1 j_2 \neq mn$ such that:*

i) $p^2 \mid a_i$ for $m \leq i < j_1$, $p \mid a_i$ for $j_1 \leq i < n$, $p \nmid a_n$, $p^2 \nmid a_{j_1}$ and $p^3 \nmid a_m$;

ii) $q \mid a_i$ for $i \neq j_2$, $q \nmid a_{j_2}$, $q^2 \nmid a_m$ and $q^2 \nmid a_n$.

Then f is irreducible over $Q(R)$.

Proof: Figure 11 shows the Newton log-polygons with respect to two prime elements p and q, one consisting of two edges having negative slopes, and the other consisting of two edges having slopes of different sign, as in Figure 5.f.

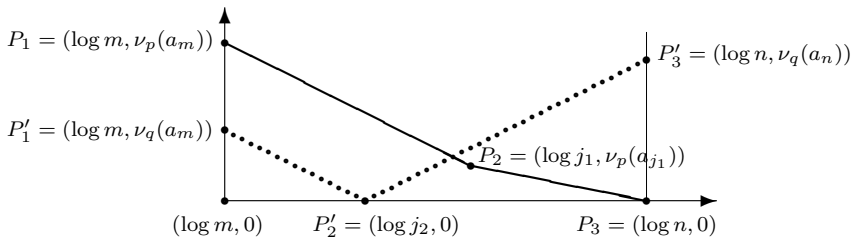

Figure 11.

In our case $\nu_q(a_n) = \nu_p(a_{j_1}) = \nu_q(a_m) = 1$ and $\nu_p(a_m) = 2$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. The slope of P_1P_2 is smaller than the slope of P_2P_3, this time because $m < j_1 < \sqrt{mn}$, so in particular the log-integral points P_1, P_2 and P_3 are not collinear. The intersection

$$\left\{\frac{j_1}{m}, \frac{n}{j_1}\right\} \bigcap \left\{\frac{j_2}{m}, \frac{n}{j_2}\right\},$$

is again empty, which shows that f is irreducible. □

Theorem 5.10. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exists an index $j \in \{m+2, \ldots, n-1\}$ with $j \neq mn/(m+1)$ such that:*

i) $p \mid a_i$ for all $i \geq m+2$, $p \nmid a_m a_{m+1}$, $p^2 \nmid a_n$;

ii) $q \mid a_i$ for all $i \neq j$, $q \nmid a_j$, $q^2 \nmid a_m$ and $q^2 \nmid a_n$;

Then f is irreducible over $Q(R)$.

Proof: Let us represent in Figure 12 the Newton log-polygons with respect to two prime elements p and q, the first one consisting of an horizontal edge followed by an edge with positive slope, and the second one consisting of two edges having slopes with different signs, as in Figure 5.g.

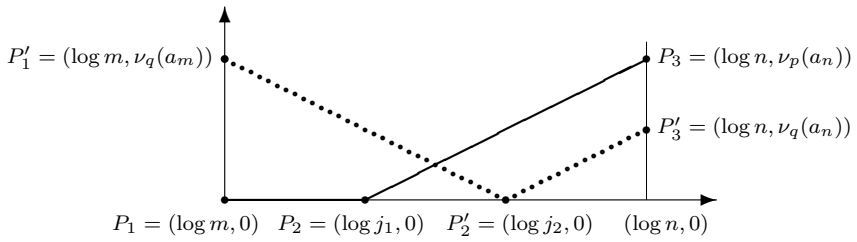

Figure 12.

In our case $j_1 = m+1$, j_2 is denoted by j, $\nu_p(a_n) = \nu_q(a_m) = \nu_q(a_n) = 1$, so by i), ii) and Lemma 3.3, each of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. Again, let $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ be a nonconstant factor of f. If d would be less than n, then by Theorem 3.1, $\frac{d}{c}$ would belong to

$$\left\{\frac{m+1}{m}, \frac{n}{m+1}\right\} \bigcap \left\{\frac{j}{m}, \frac{n}{j}\right\}.$$

Since this intersection is empty (as $j \neq m+1$ and $j \neq mn/(m+1)$), d must be equal to n (and c to m), proving that f is irreducible. □

Theorem 5.11. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$, and let p and q be two prime elements of R. Assume that there exists an index $j \in \{m+2, \ldots, n-1\}$ with $j \neq mn/(m+1)$ and $j > \sqrt{mn}$ such that:*

i) $p \mid a_i$ for all $i \geq m+2$, $p \nmid a_m a_{m+1}$, $p^2 \nmid a_n$;

ii) $q \nmid a_m$, $q \mid a_i$ for $m < i \leq j$, $q^2 \nmid a_j$, $q^2 \mid a_i$ for $j < i \leq n$ and $q^3 \nmid a_n$.

Then f is irreducible over $Q(R)$.

Proof: Let us represent in Figure 13 the Newton log-polygons with respect to two prime elements p, q, the first one consisting of an horizontal edge followed by an edge with positive slope, and the second one consisting of two edges having positive slopes, as in Figure 5.h.

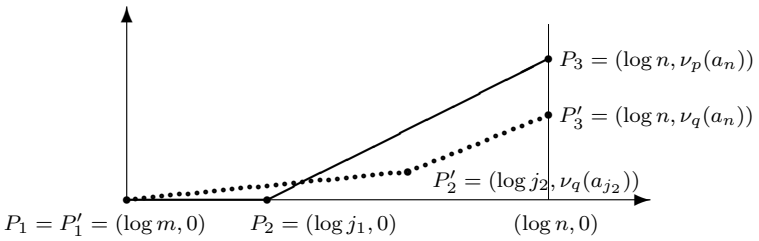

Figure 13.

In our particular case $j_1 = m+1$, j_2 is denoted by j, $\nu_p(a_n) = \nu_q(a_j) = 1$ and $\nu_q(a_n) = 2$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. Also, the slope of $P_2'P_3'$ is greater than the slope of $P_1'P_2'$, as $j > \sqrt{mn}$, so in particular

the log-integral points P_1', P_2' and P_3' are not collinear. Again, let $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ be a nonconstant factor of f. If d would be less than n, by Theorem 3.1, $\frac{d}{c}$ would belong to

$$\left\{\frac{m+1}{m}, \frac{n}{m+1}\right\} \bigcap \left\{\frac{j}{m}, \frac{n}{j}\right\}.$$

Since this intersection is empty (as $j \neq m+1$ and $j \neq mn/(m+1)$), d must be equal to n (and c to m), so f must be irreducible. □

Theorem 5.12. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, with $a_m a_n \neq 0$ and let p, q be two prime elements of R. Assume there exists an index j with $m+1 < j < \sqrt{mn}$ such that:*

i) $p \mid a_i$ for all $i \geq m+2$, $p \nmid a_m a_{m+1}$, $p^2 \nmid a_n$;

ii) $q^2 \mid a_i$ for $m \leq i < j$, $q \mid a_i$ for $j \leq i < n$, $q \nmid a_n$, $q^2 \nmid a_j$ and $q^3 \nmid a_m$;

Then f is irreducible over $Q(R)$.

Proof: Figure 14 displays the Newton log-polygons with respect to two prime elements p and q, the first one consisting of a horizontal edge followed by an edge with positive slope, and the second one consisting of two edges having negative slopes, as in Figure 5.i.

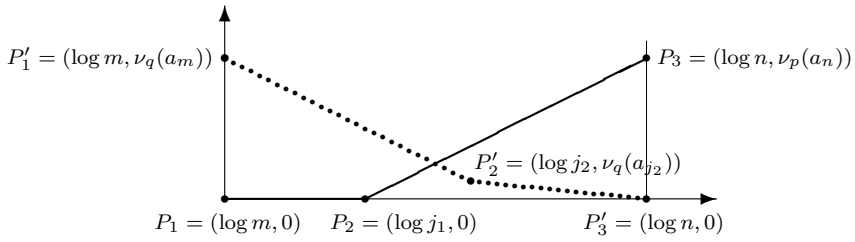

Figure 14.

In our particular case $j_1 = m+1$, j_2 is denoted by j, $\nu_p(a_n) = \nu_q(a_j) = 1$ and $\nu_q(a_m) = 2$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2, P_2P_3, $P_1'P_2'$ and $P_2'P_3'$ contains a single segment. Moreover, the slope of $P_2'P_3'$ is greater than the slope of $P_1'P_2'$, as $j < \sqrt{mn}$, so in particular the log-integral points P_1', P_2' and P_3' are not collinear. Now, if $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ is a nonconstant factor of f with $d < n$, then by Theorem 3.1, $\frac{d}{c}$ must belong to

$$\left\{\frac{m+1}{m}, \frac{n}{m+1}\right\} \bigcap \left\{\frac{j}{m}, \frac{n}{j}\right\}.$$

The existence of an index j with $m+1 < j < \sqrt{mn}$ requires that $\sqrt{mn} > m+2$, and also implies that $\frac{mn}{m+1} > \sqrt{mn}$, so our j cannot be equal to $\frac{mn}{m+1}$. Thus, the above intersection is empty, so d must be equal to n, and c to m, which proves the irreducibility of f. □

Theorem 5.13. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p, q be prime elements of R, and assume that $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$. Let j be an index with $m < j < n$ and let $\delta := \gcd(\nu_p(a_n) - \nu_p(a_m), \nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m))$. If*

i) $\nu_p(a_m) \neq \nu_p(a_n)$ and $\left(\frac{n}{i}\right)^{\nu_p(a_i)-\nu_p(a_m)} > \left(\frac{m}{i}\right)^{\nu_p(a_i)-\nu_p(a_n)}$ for $m < i < n$,

ii) $q \mid a_i$ *for all* $i \neq j$, $q \nmid a_j$, $q^2 \nmid a_m$ *and* $q^2 \nmid a_n$;
iii) $j \notin \{m^{1-\frac{i}{\delta}} n^{\frac{i}{\delta}} \ : \ 0 < i < \delta\}$,

then f *is irreducible over* $Q(R)$.

Proof: Figure 15 displays the Newton log-polygons with respect to two prime elements p and q, the first one consisting of a single edge, and the second one consisting of two edges with slopes of different signs, as in Figure 5.j.

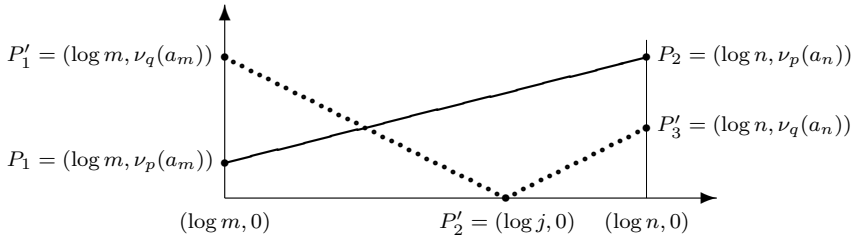

Figure 15.

In our particular case $\nu_q(a_m) = \nu_q(a_n) = 1$, so by i), ii) and Lemma 3.3, each one of the edges $P_1'P_2'$ and $P_2'P_3'$ contains a single segment, while P_1P_2 contains precisely δ segments. If $\delta = 1$, then f is irreducible by Theorem 3.4, so we may assume that $\delta \geq 2$. If $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ is a nonconstant factor of f with $d < n$, then by Theorem 3.1, $\frac{d}{c}$ must belong to

$$\left\{\frac{j}{m}, \frac{n}{j}\right\} \bigcap \left\{\left(\frac{n}{m}\right)^{\frac{i}{\delta}} \ : \ 0 < i < \delta\right\}.$$

In view of iii), this intersection is empty, so d must be equal to n, and c to m, which proves the irreducibility of f. □

Theorem 5.14. *Let* $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ *be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain* R, $a_m a_n \neq 0$, *and let* p, q *be prime elements of* R. *If*

i) $\nu_p(a_m) \neq \nu_p(a_n)$ *and* $\left(\frac{n}{i}\right)^{\nu_p(a_i)-\nu_p(a_m)} > \left(\frac{m}{i}\right)^{\nu_p(a_i)-\nu_p(a_n)}$ *for* $m < i < n$,
ii) $q \nmid a_m a_{m+1}$, $q \mid a_i$ *for all* $i > m+1$, *and* $q^2 \nmid a_n$,

then f *is irreducible over* $Q(R)$.

Proof: In Figure 16 we present the Newton log-polygons with respect to two prime elements p and q, the first one consisting of a single edge, and the second one consisting of a horizontal edge, followed by an edge with positive slope, as in Figure 5.k.

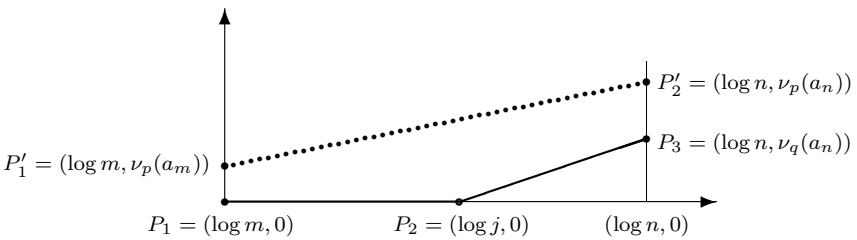

Figure 16.

In our particular case $j = m+1$ and $\nu_q(a_n) = 1$, so by i), ii) and Lemma 3.3, each one of the edges P_1P_2 and P_2P_3 contains a single segment, while P_1P_2 contains precisely δ segments, with

$$\delta := \gcd(\nu_p(a_n) - \nu_p(a_m), \nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m)),$$

where $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$. If $\delta = 1$, the irreducibility of f follows by Theorem 3.4, so we may assume that $\delta \geq 2$. Now, if $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ is a nonconstant factor of f with $d < n$, then by Theorem 3.1, $\frac{d}{c}$ must belong to

$$\left\{\frac{m+1}{m}, \frac{n}{m+1}\right\} \bigcap \left\{\left(\frac{n}{m}\right)^{\frac{i}{\delta}} \ : \ 0 < i < \delta\right\}.$$

To prove that this intersection is empty, we observe that an equality of the form $\frac{m+1}{m} = (\frac{n}{m})^{\frac{i}{\delta}}$ with $i \in \{1, \ldots, \delta - 1\}$ leads to $(m+1)^\delta = n^i m^{\delta - i}$, and an equality of the form $\frac{n}{m+1} = (\frac{n}{m})^{\frac{i}{\delta}}$ with $i \in \{1, \ldots, \delta - 1\}$ leads to $(m+1)^\delta = n^{\delta - i} m^i$. As m and $m+1$ are coprime, and i is at least 1, none of these equalities can hold, so d must be equal to n, and c to m, proving that f is irreducible. □

Theorem 5.15. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p, q be prime elements of R, and assume that $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$. Let j be an index, $\sqrt{mn} < j < n$ and let $\delta := \gcd(\nu_p(a_n) - \nu_p(a_m), \nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m))$. If*

i) $\nu_p(a_m) \neq \nu_p(a_n)$ and $\left(\frac{n}{i}\right)^{\nu_p(a_i) - \nu_p(a_m)} > \left(\frac{m}{i}\right)^{\nu_p(a_i) - \nu_p(a_n)}$ for $m < i < n$,

ii) $q \nmid a_m$, $q \mid a_i$ for $m < i \leq j$, $q^2 \nmid a_j$, $q^2 \mid a_i$ for $j < i \leq n$ and $q^3 \nmid a_n$,

iii) $j \notin \{m^{1 - \frac{i}{\delta}} n^{\frac{i}{\delta}} \ : \ 0 < i < \delta\}$,

then f is irreducible over $Q(R)$.

Proof: Figure 17 displays the Newton log-polygons with respect to two prime elements p and q, the first one consisting of a single edge, and the second one consisting of two edges with positive slopes, as in Figure 5.l.

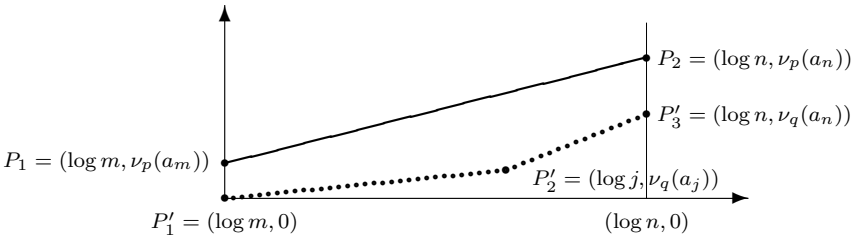

Figure 17.

In our particular case $\nu_q(a_j) = 1$ and $\nu_q(a_n) = 2$, so by i), ii) and Lemma 3.3, each one of the edges $P_1'P_2'$ and $P_2'P_3'$ contains a single segment, while P_1P_2 contains precisely δ segments. Also, the slope of $P_2'P_3'$ is greater than the slope of $P_1'P_2'$, as $\sqrt{mn} < j < n$. Again, if $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ is a nonconstant factor of f with $d < n$, then by Theorem 3.1, $\frac{d}{c}$ must belong to

$$\left\{\frac{j}{m}, \frac{n}{j}\right\} \bigcap \left\{\left(\frac{n}{m}\right)^{\frac{i}{\delta}} \ : \ 0 < i < \delta\right\}.$$

On the other hand, according to iii), this intersection must be empty, so d must be equal to n, and c to m, which proves the irreducibility of f. □

Our last result in this section is the following one.

Theorem 5.16. *Let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be an algebraically primitive Dirichlet polynomial with coefficients in a unique factorization domain R, $a_m a_n \neq 0$, let p, q be prime elements of R, and assume that $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$. Let j be an index, $m < j < \sqrt{mn}$ and let $\delta := \gcd(\nu_p(a_n) - \nu_p(a_m), \nu_{q_1}(n) - \nu_{q_1}(m), \ldots, \nu_{q_k}(n) - \nu_{q_k}(m))$. If*

i) $\nu_p(a_m) \neq \nu_p(a_n)$ and $\left(\frac{n}{i}\right)^{\nu_p(a_i)-\nu_p(a_m)} > \left(\frac{m}{i}\right)^{\nu_p(a_i)-\nu_p(a_n)}$ for $m < i < n$,

ii) $q^2 \mid a_i$ for $m \leq i < j$, $q \mid a_i$ for $j \leq i < n$, $q \nmid a_n$, $q^2 \nmid a_j$ and $q^3 \nmid a_m$,

iii) $j \notin \{m^{1-\frac{i}{\delta}} n^{\frac{i}{\delta}} \ : \ 0 < i < \delta\}$,

then f is irreducible over $Q(R)$.

Proof: In Figure 18 we plot the Newton log-polygons with respect to two prime elements p and q, the first one consisting of a single edge, and the second one consisting of two edges with negative slopes, as in Figure 5.m.

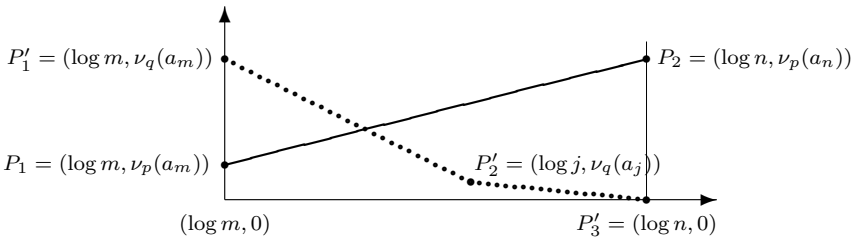

Figure 18.

In our particular case $\nu_q(a_j) = 1$ and $\nu_q(a_m) = 2$, so by i), ii) and Lemma 3.3, each one of the edges $P_1'P_2'$ and $P_2'P_3'$ contains a single segment, while P_1P_2 contains precisely δ segments. Also, the slope of $P_2'P_3'$ is greater than the slope of $P_1'P_2'$, as $m < j < \sqrt{mn}$. Again, if $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$ is a nonconstant factor of f with $d < n$, then by Theorem 3.1, $\frac{d}{c}$ must belong to

$$\left\{\frac{j}{m}, \frac{n}{j}\right\} \bigcap \left\{\left(\frac{n}{m}\right)^{\frac{i}{\delta}} \ : \ 0 < i < \delta\right\}.$$

By iii), this intersection must be empty, so d must be equal to n, and c to m, proving the irreducibility of f. □

6. The irreducibility of Dirichlet polynomials that assume a prime value

Some of the classical or more recent irreducibility criteria for integer polynomials rely on information on the prime factorizations of the values that they assume at some integral arguments. Among the first such results that appeared over the time, we mention here the following irreducibility criteria of Stäckel [101], Ore [82], and Weisner [105].

Theorem (Stäckel, 1918) *Let $f(X) \in \mathbb{Z}[X]$ and let A denote the maximum of the absolute values of the coefficients of $f(X)$. If there is an integer t with $|t| > 1 + A$ for which $f(t)$ is a prime number, then $f(X)$ is irreducible in $\mathbb{Z}[X]$.*

Theorem (Ore, 1934) *Let $f \in \mathbb{Z}[X]$ be an arbitrary polynomial of degree $n \geq 2$. Assume that for an integer t we have $|f(t)| = pq$ for some positive integers p, q with p prime. Assume also that all the roots of f lie outside the disk $\{z : |z - t| \leq \rho\}$ for some real number $\rho \geq q$. Then f is irreducible in $\mathbb{Q}[X]$.*

Theorem (Weisner, 1934) *Let $f \in \mathbb{Z}[X]$ be an arbitrary polynomial of degree $n \geq 2$ and leading coefficient a_n. Assume that all the roots of f are in the disk $\{z : |z| < \rho\}$ for some real number $\rho \geq 1$. Assume also that for an integer t we have $|f(t)| = p^\ell q$ and $p \nmid f'(t)^{\ell-1}$ for some positive integers p, q, ℓ with p prime. If $|t| \geq q + \rho$ or $p^\ell \geq |a_n|(|t| + \rho)^{n-1}$, then f is irreducible in $\mathbb{Q}[X]$.*

Similar related results have been proved by G. Pólya and G. Szegö [91], Girstmair [54] and Guersenzvaig [55], to name just a few authors. Other famous such results are, for instance, Cohn's irreducibility criterion and its generalization to an arbitrary base obtained by Brillhart, Filaseta and Odlyzko [24], and to function fields over finite fields by Murty [76].

The aim of this section is to find irreducibility criteria for Dirichlet polynomials that assume a value divisible by a prime, or by a prime power. Some of the results that we will obtain are analogous to the above criteria of Stäckel, Ore and Weisner. In the proof of our results we will need an inequality of Gelfond on the heights of the factors of a multivariate polynomial (see Mahler [72] for a proof). We will first need to fix some notations. For a polynomial

$$f(z_1, \dots, z_n) = \sum_{h_1=0}^{m_1} \cdots \sum_{h_n=0}^{m_n} a_{h_1 \dots h_n} z_1^{h_1} \cdots z_n^{h_n}$$

in n variables $z_1, \dots, z_n$ with arbitrary real or complex coefficients, we let

$$H(f) = \max_{h_i=0,\dots,m_i,\ i=1,\dots,n} |a_{h_1 \dots h_n}|,$$

the height of f, and denote by $\nu(f)$ the number of variables $z_1, \dots, z_n$ that occur in f at least to the first degree. With this notation we have the following result.

Theorem (Gelfond). *Let $f(z_1, \dots, z_n) = \prod_{i=1}^{\ell} f_i(z_1, \dots, z_n)$ with $f_1, \dots, f_\ell$ polynomials with arbitrary real or complex coefficients. Then we have the inequality*

$$\prod_{i=1}^{\ell} H(f_i) \leq 2^{m_1 + \cdots + m_n - \nu(f)} \{(m_1 + 1) \cdots (m_n + 1)\}^{\frac{1}{2}} H(f). \tag{33}$$

This inequality will be particularly useful in the case of polynomials with integer coefficients, for which the right side provides an upper bound for the heights of the factors $f_1, \dots, f_\ell$.

Definition 6.1. For a Dirichlet polynomial $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ with integer coefficients and $a_m a_n \neq 0$ we define $H(f) = \max\{|a_m|, \dots, |a_n|\}$ and call it the "height" of f.

Before stating our first result in this section, we will mention that despite its somewhat unaestethic presence in inequality (34) below, the constant $1 + \frac{\log_3(\log_2 12)}{2}$ plays a crucial role in

obtaining conditions in closed form for t in Corollaries 6.4, 6.5 and 6.6, and also in Corollaries 6.9, 6.10 and 6.11.

Theorem 6.2. *Let f be a Dirichlet polynomial of composite degree n with integer coefficients, and assume that p is the least prime factor of n, and that f has $k \geq 1$ relevant primes. If $|f(-t)| = Pq$ with P a prime number, q a positive integer, and t an integer satisfying*

$$t > \frac{n}{p} \log\left(\frac{nqH(f)}{p} \cdot \left(\frac{n}{2}\right)^{k\left(1+\frac{\log_3(\log_2 12)}{2}\right)} \right), \tag{34}$$

then f is irreducible.

Proof: Let us assume to the contrary that $f(s) = g(s)h(s)$ with g, h nonconstant Dirichlet polynomials with integer coefficients. Since $|f(-t)| = Pq$ and P is prime, we deduce from the equality $|f(-t)| = |g(-t)| \cdot |h(-t)|$ that one of the integers $|g(-t)|$ and $|h(-t)|$ must be divisible by P, so the other must divide q, say $|g(-t)| \mid q$. In particular we have

$$|g(-t)| \leq q. \tag{35}$$

Let now $F(X_1, \ldots, X_k) \in \mathbb{Z}[X_1, \ldots, X_k]$ be the polynomial obtained from $f(s)$ by letting $X_i = p_i^{-s}$ for each of the k relevant primes $p_1, \ldots, p_k$ of f. We obviously have $H(f) = H(F)$. Suppose now that $g(s) = \frac{b_c}{c^s} + \cdots + \frac{b_d}{d^s}$, with d a proper divisor of n, and $c \leq d$. Since g is a factor of f, we can use the same identification above and obtain a polynomial $G(X_1, \ldots, X_k) \in \mathbb{Z}[X_1, \ldots, X_k]$, which is a factor of F and satisfies $H(g) = H(G)$. Note that some of the indeterminates $X_1, \ldots, X_k$ might not effectively appear in the expression of G. Let $\nu_{p_i}(\cdot)$ denote the usual p_i-adic valuation, for $i = 1, \ldots, k$. By (33) we must have

$$H(g) \leq 2^{m_1 + \cdots + m_k - k} \{(m_1 + 1) \cdots (m_k + 1)\}^{\frac{1}{2}} H(f),$$

where for each $i = 1, \ldots, k$, $m_i = \max\limits_{N \in Supp(f)} \nu_{p_i}(N)$, the greatest multiplicity of p_i in the prime factorizations of the integers in $Supp(f)$. Since a uniform upper bound for $m_1, \ldots, m_k$ is obviously $\log_2 n$, we deduce that

$$H(g) \leq 2^{k \log_2 \frac{n}{2}} (\log_2 2n)^{\frac{k}{2}} H(f) = \left(\frac{n}{2}\right)^k (\log_2 2n)^{\frac{k}{2}} H(f). \tag{36}$$

We will first assume that $c = d$, so $g(-t) = b_d d^t$. According to (35), we have $|g(-t)| \leq q$, so in this case we reach a contradiction if $|d^t| > q$, that is for $t > \log_d q$, as $|b_d| \geq 1$. In particular, we obtain a contradiction for $t > \log_2 q$, as $d \geq 2$. Since the right side in (34) exceeds $\log_2 q$, we conclude that g cannot consist of a single term, so f must be algebraically primitive. We will therefore assume that $c < d$. By the triangle inequality and (36) we successively deduce in this

case that

$$\begin{aligned} q &\geq |b_c c^t + \cdots + b_d d^t| \geq |b_d d^t| - H(g)(c^t + \cdots + (d-1)^t) \\ &\geq d^t - H(g)(1 + \cdots + (d-1)^t) \\ &\geq d^t - \left(\frac{n}{2}\right)^k (\log_2 2n)^{\frac{k}{2}} H(f)(1 + \cdots + (d-1)^t) \\ &\geq d^t - \left(\frac{n}{2}\right)^k (\log_2 2n)^{\frac{k}{2}} H(f)(d-1)^{t+1}. \end{aligned}$$

To reach a contradiction, it suffices to show that $d^t > (\frac{n}{2})^k (\log_2 2n)^{\frac{k}{2}} H(f)(d-1)^{t+1} + q$. We will actually prove that $(\frac{d}{d-1})^t > (\frac{n}{2})^k (\log_2 2n)^{\frac{k}{2}} H(f)(d-1) + q$, that is

$$t > \log_{\frac{d}{d-1}} \left(\left(\frac{n}{2}\right)^k (\log_2 2n)^{\frac{k}{2}} H(f)(d-1) + q \right).$$

Observing that the right side in last inequality is an increasing function of d, and $d \leq \frac{n}{p}$, it will be sufficient to prove that

$$t > \log_{\frac{\frac{n}{p}}{\frac{n}{p}-1}} \left(\left(\frac{n}{2}\right)^k (\log_2 2n)^{\frac{k}{2}} H(f) \left(\frac{n}{p} - 1\right) + q \right), \tag{37}$$

or even more, to simplify computations, that

$$t > \log_{\frac{\frac{n}{p}}{\frac{n}{p}-1}} \left(\left(\frac{n}{2}\right)^k (\log_2 2n)^{\frac{k}{2}} H(f) \cdot \frac{nq}{p} \right). \tag{38}$$

Let us denote by A the term under the logarithm in (38). At this point, we can use the fact that $\left(\frac{\frac{n}{p}}{\frac{n}{p}-1}\right)^{\frac{n}{p}} > e$, the base of natural logarithm, to deduce that

$$\frac{n}{p} \log A > \log_{\frac{\frac{n}{p}}{\frac{n}{p}-1}} A,$$

which shows that we can actually ask t to satisfy the inequality

$$t > \frac{n}{p} \left(k \log \frac{n}{2} + \frac{k}{2} \log(\log_2 2n) + \log\left(\frac{nq}{p} H(f)\right) \right).$$

Since n is composite, we have $n \geq 4$. We will study the case $n = 4$ separately, so we will first assume that $n \geq 6$. We search for a real positive δ as small as possible such that $(\frac{n}{2})^\delta \geq \log_2 2n$ for $n \geq 6$. It is easy to check that $\delta = \log_3(\log_2 12)$. Thus, it suffices to ask t to satisfy the inequality

$$t > \frac{n}{p} \left(k \log \frac{n}{2} + k \frac{\log_3(\log_2 12)}{2} \log \frac{n}{2} + \log\left(\frac{nq}{p} H(f)\right) \right),$$

or, equivalently,

$$t > \frac{n}{p} \log\left(\left(\frac{n}{2}\right)^{k\left(1 + \frac{\log_3(\log_2 12)}{2}\right)} \cdot \frac{nqH(f)}{p} \right),$$

which is precisely condition (34). This completes the proof for $n \geq 6$.

We consider now the case $n = 4$. Here we have $p = d = 2$, and if f would have a nonzero term $\frac{a_3}{3^s}$, its irreducibility would follow by Proposition 2.3, so we can assume that the only

relevant prime of f is 2, meaning that $k = 1$. Thus, we need to prove that the conclusion on the irreducibility of f holds for t satisfying

$$t > 2 \log \left(2^{2+\frac{\log_3(\log_2 12)}{2}} \cdot qH(f) \right). \tag{39}$$

On the other hand, condition (37) reduces in this case to

$$t > \log_2(2\sqrt{3}H(f) + q). \tag{40}$$

We will first consider the case that $q \geq 2$. In this case (40) obviously holds if t satisfies (39), since $\log_2(2\sqrt{3}qH(f)) > \log_2(2\sqrt{3}H(f) + q)$, $2\big(2 + \frac{\log_3(\log_2 12)}{2}\big) \log 2 > 3 > \log_2(2\sqrt{3})$, and $2 \log(qH(f)) = 2(\log 2)(\log_2(qH(f))) > \log_2(qH(f))$.

For $q = 1$ we need to show that

$$2 \log \left(2^{2+\frac{\log_3(\log_2 12)}{2}} \cdot H(f) \right) > \log_2(2\sqrt{3}H(f) + 1),$$

or, equivalently, that

$$\log_2 \left(2^{2+\frac{\log_3(\log_2 12)}{2}} \cdot H(f) \right)^{2 \log 2} > \log_2(2\sqrt{3}H(f) + 1), \tag{41}$$

which obviously holds for $H(f) \geq 2$, since the term under the logarithm in the left side is greater than $3.72H(f)^{1.38}$, which in turn is greater than $2\sqrt{3}H(f) + 1$ for $H(f) \geq 2$.

However, inequality (41) fails for $H(f) = 1$, so in this case we need to analyse separately the Dirichlet polynomials of the form $f(s) = \frac{a}{1^s} + \frac{b}{2^s} + \frac{c}{4^s}$ with integers a, b, c satisfying $|a| \leq 1$, $|b| \leq 1$ and $|c| = 1$. If both a and b are zero, $|f(-t)|$ cannot be a prime number for positive integers t, so such an f cannot satisfy the hypotheses of the theorem. If $a = 0$ and $b \neq 0$, then f must be of the form $\pm(\frac{1}{2^s} + \frac{1}{4^s})$, or of the form $\pm(\frac{-1}{2^s} + \frac{1}{4^s})$, which are not algebraically primitive. Note that even if we ignore the fact that f must be algebraically primitive, both $\pm(\frac{1}{2^s} + \frac{1}{4^s})$ and $\pm(\frac{-1}{2^s} + \frac{1}{4^s})$ fail to satisfy the conditions in the statement. In the first case $|f(-t)|$ cannot be a prime number for positive integers t, while in the second case, the only prime value of the form $4^t - 2^t$ is obtained for $t = 1$, but $t = 1$ does not satisfy condition (34), since the right side in (34) is in this case greater than 3.

Assume now that $b = 0$ and $a \neq 0$, so f is now of the form $\pm(\frac{1}{1^s} + \frac{1}{4^s})$, or of the form $\pm(\frac{-1}{1^s} + \frac{1}{4^s})$. In the first case, the irreducibility of f follows by direct computation, without looking at its possible prime values. However, in this case for $|f(-t)| = 4^t + 1$ to be a prime, it must in fact be a Fermat prime, and $t = 4$ and $t = 8$ satisfy condition (34), with corresponding prime values the Fermat primes $F_3 = 257$ and $F_4 = 65537$. In our second case, the Dirichlet polynomials $f(s) = \pm(\frac{-1}{1^s} + \frac{1}{4^s})$ are obviously reducible, and the only prime value of the form $|f(-t)| = 4^t - 1$ is 3, and is obtained for $t = 1$, which fails to satisfy condition (34).

We are thus left with the case that none of a, b and c is zero, so $|a| = |b| = |c| = 1$. It is easy to see that all the corresponding eight such Dirichlet polynomials are irreducible, since their hypothetical nonconstant factors would be forced to be of the form $\pm(\frac{1}{1^s} + \frac{1}{2^s})$, or of the form $\pm(\frac{-1}{1^s} + \frac{1}{2^s})$, and any product of two such factors would force b to be zero or ± 2. This completes the proof. □

Remark 6.3. It should be noted that to improve (34), instead of bounding $1+\cdots+(d-1)^t$ from above by $(d-1)^{t+1}$, one might use Faulhaber's formula

$$\sum_{i=1}^{n}(d-1)^t=\frac{1}{t+1}\sum_{i=0}^{t}\binom{t+1}{i}B_i(d-1)^{t-i+1},$$

combined with majorants for the absolute values of the Bernoulli numbers B_i. Another way to improve (34) would be to search for a sharper bound on $H(g)$ than the right side in (33), but such attempts might produce much more involved formulae.

In particular, we obtain from Theorem 6.2 the following result for the case that no information on the least prime factor of f and on the number of relevant primes of f is known.

Corollary 6.4. *Let f be a Dirichlet polynomial of degree $n\geq 2$ with integer coefficients. If $|f(-t)|=Pq$ with P a prime number, q a positive integer, and t an integer satisfying*

$$t>n^2+\frac{n}{2}\log(qH(f)),$$

then f is irreducible.

Proof: If n is prime, the irreducibility of f follows by Proposition 2.2, without any additional condition. For a composite n we may ask t to satisfy (34) with 2 instead of p, and with $\pi(n)$, the prime counting function, instead of k, since the number of relevant primes of f is at most $\pi(n)$. In this case condition (34) becomes

$$t>\left(1+\frac{\log_3(\log_2 12)}{2}\right)\frac{n\pi(n)}{2}\log\left(\frac{n}{2}\right)+\frac{n}{2}\log\left(\frac{nqH(f)}{2}\right).$$

By [3, (3.6)] we have $\pi(n)<1.25506\frac{n}{\log n}$ for $n>1$, so we can ask t to satisfy

$$t>\frac{1.25506}{2}\cdot\left(1+\frac{\log_3(\log_2 12)}{2}\right)\cdot\frac{n^2}{\log n}\log\left(\frac{n}{2}\right)+\frac{n}{2}\log\left(\frac{nqH(f)}{2}\right).$$

Rather surprisingly, the constant $\frac{1.25506}{2}\cdot\left(1+\frac{\log_3(\log_2 12)}{2}\right)$ is quite close to 1, more precisely, it is equal to 0.99217077..., so we can ask t to satisfy

$$t>\frac{n^2}{\log n}(\log n-\log 2)+\frac{n}{2}\log\left(\frac{n}{2}\right)+\frac{n}{2}\log(qH(f)),$$

or, equivalently,

$$t>n^2-\frac{n^2\log 2}{\log n}+\frac{n}{2}\log\left(\frac{n}{2}\right)+\frac{n}{2}\log(qH(f)).$$

The proof finishes by observing that $\frac{n^2\log 2}{\log n}>\frac{n}{2}\log\left(\frac{n}{2}\right)$ for $n\geq 2$. □

In particular, we obtain for $q=1$ the following result for Dirichlet polynomials that assume a prime value, that may be regarded as an analogue of Stäckel's irreducibility criterion.

Corollary 6.5. *Let f be a Dirichlet polynomial of degree $n\geq 2$ with integer coefficients. If $|f(-t)|$ is a prime number for some integer $t>n^2+\frac{n}{2}\log H(f)$, then f is irreducible.*

The lower bound for t simplifies even more if all the coefficients of f have absolute values at most 1, as in the following result.

Corollary 6.6. *Let f be a Dirichlet polynomial of degree $n \geq 2$ with integer coefficients $a_i \in \{-1, 0, 1\}$. If $|f(-t)|$ is a prime number for some integer $t > n^2$, then f is irreducible.*

We mention here that for $\ell > 1$, Weisner's technical condition that $p \nmid f'(t)^{\ell-1}$ reduces to $p \nmid f'(t)$, and in case f can be written as a product of two nonconstant factors g and h, prevents $g(t)$ and $h(t)$ to be both divisible by p. As we shall see in the following result, in the case of a Dirichlet polynomial $f(s)$ that factors nontrivially as $g(s) \cdot h(s)$, to prevent $g(-t)$ and $h(-t)$ to be both divisible by a certain prime P, we will need a surprisingly different kind of condition on the derivative of f, namely an irrationality condition for the principal Pth root of a certain expression depending on f.

Theorem 6.7. *Let f be a Dirichlet polynomial of composite degree $n \geq 2$ with integer coefficients a_i, and assume that p is the smallest prime factor of n, and that f has $k \geq 1$ relevant primes. Assume that $|f(-t)| = P^\ell q$ with P a prime number, ℓ and q positive integers, $P \nmid q$, $\ell \geq 2$, and t an integer satisfying*

$$t > \frac{n}{p} \log \left(\frac{nqH(f)}{p} \cdot \left(\frac{n}{2}\right)^{k\left(1+\frac{\log_3(\log_2 12)}{2}\right)} \right).$$

If the principal Pth root of $\prod_i i^{a_i \cdot i^t}$ is an irrational number, then f is irreducible.

Proof: Assume that $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$, and that $f = g \cdot h$, with $g(s) = \frac{b_{m_1}}{m_1^s} + \cdots + \frac{b_{n_1}}{n_1^s}$ and $h(s) = \frac{c_{m_2}}{m_2^s} + \cdots + \frac{c_{n_2}}{n_2^s}$ nonconstant Dirichlet polynomials with integer coefficients. Since $|f(-t)| = P^\ell q$ and $f(-t) = g(-t)h(-t)$, we have $|g(-t)| = P^a q_1$ and $|h(-t)| = P^b q_2$ for two nonnegative integers a, b with $a + b = \ell$ and two positive integers q_1, q_2 with $q_1 q_2 = q$ (hence none of q_1 and q_2 is divisible by P). We will prove now that one of the integers a and b must be zero. Let us assume to the contrary that both a and b are positive. Observe that

$$\begin{aligned} f'(-t) &= -\sum_{m \leq i \leq n} a_i i^t \log i \\ &= -h(-t) \cdot \left(\sum_{m_1 \leq i \leq n_1} b_i i^t \log i \right) - g(-t) \cdot \left(\sum_{m_2 \leq i \leq n_2} c_i i^t \log i \right) \\ &= -P^b q_2 \left(\sum_{m_1 \leq i \leq n_1} b_i i^t \log i \right) - P^a q_1 \left(\sum_{m_2 \leq i \leq n_2} c_i i^t \log i \right), \end{aligned}$$

from which we deduce that

$$\prod_i i^{a_i \cdot i^t} = \left(\prod_i i^{b_i \cdot i^t} \right)^{P^b q_2} \cdot \left(\prod_i i^{c_i \cdot i^t} \right)^{P^a q_1}. \tag{42}$$

Observe now that the right side in (42) is the Pth power of a rational number, implying that the principal Pth root of $\prod_i i^{a_i \cdot i^t}$ is a rational number, a contradiction. Therefore one of a and b

must be zero, say $a = 0$, implying that $|g(-t)| = q_1$. In particular, this shows that $|g(-t)| \leq q$. The proof continues now using precisely the same lines as in the proof of Theorem 6.2. □

Remark 6.8. We notice that for a given Dirichlet polynomial with integer coefficients a_i, if $\prod_i i^{a_i \cdot i^t} \neq 1$ for some positive integer t, then the principal Pth root of $\prod_i i^{a_i \cdot i^t}$ is an irrational number for all but finitely many prime numbers P.

In particular, we obtain from Theorem 6.7 the following result for the case that no information on the least prime factor of f and on the number of relevant primes of f is known.

Corollary 6.9. *Let f be a Dirichlet polynomial of degree $n \geq 2$ with integer coefficients a_i. Assume that $|f(-t)| = P^\ell q$ with P a prime number, q a positive integer, $P \nmid q$, $\ell \geq 2$ an integer, and t an integer satisfying*

$$t > n^2 + \frac{n}{2} \log(qH(f)).$$

If the principal Pth root of $\prod_i i^{a_i \cdot i^t}$ is an irrational number, then f is irreducible.

We note that Corollary 6.4 and Corollary 6.9 can be viewed as analogous results for Dirichlet polynomials of the irreducibility criteria of Ore and Weissner.

As a particular case, we obtain for $q = 1$ the following irreducibility conditions for Dirichlet polynomials that assume a prime power value.

Corollary 6.10. *Let f be a Dirichlet polynomial of degree $n \geq 2$ with integer coefficients a_i. Assume that $|f(-t)| = P^\ell$ with $t > n^2 + \frac{n}{2} \log H(f)$ an integer, P a prime, and $\ell \geq 2$ an integer. If the principal Pth root of $\prod_i i^{a_i \cdot i^t}$ is an irrational number, then f is irreducible.*

Here too, the simplest conditions on t appear when all the coefficients a_i are zero, or ± 1.

Corollary 6.11. *Let f be a Dirichlet polynomial of degree $n \geq 2$ with integer coefficients $a_i \in \{-1, 0, 1\}$. Assume that $|f(-t)| = P^\ell$ with t, P and ℓ integers with $t > n^2$, P prime, and $\ell \geq 2$. If the principal Pth root of $\prod_i i^{a_i i^t}$ is an irrational number, then f is irreducible.*

7. The multiplicities of the irreducible factors of Dirichlet polynomials and square-free criteria

A direct consequence of the definition is the following square-free criterion.

Proposition 7.1. *A primitive Dirichlet polynomial of square-free degree is square-free.*

In the forthcoming results we will use the following definition.

Definition 7.2. For any integer $n \geq 2$ we let $M(n) = \max_{p|n} \nu_p(n)$, and for any positive integers n and k we let $\Delta_k(n) = \max\{d : d \mid n \text{ and } d^k \mid n\}$, and also

$$n_{<k} = \prod_{p \text{ prime}, \nu_p(n) < k} p^{\nu_p(n)} \quad \text{and} \quad n_{\geq k} = \prod_{p \text{ prime}, \nu_p(n) \geq k} p^{\nu_p(n)},$$

with an empty product equal to 1. Thus $n = n_{<k} \cdot n_{\geq k}$, $\Delta_1(n) = n$, and for $k \geq 2$ we have $\Delta_k(n) = 1$ if and only if n is k-power-free, that is if and only if $n = n_{<k}$.

For a primitive Dirichlet polynomial $f(s) \in DP(R)[s]$ having the canonical decomposition $f(s) = c \cdot f_1(s)^{n_1} \cdots f_k(s)^{n_k}$, with $c \in R \setminus \{0\}$ and $f_i \in DP(R)[s]$ irreducible and pairwise nonassociated in divisibility, we let $M(f) = \max\{n_1, \ldots, n_k\}$.

We will first prove two k-power-freeness criteria for Dirichlet polynomials $f(s) \in DP(R)[s]$ whose degrees have at least one prime factor with multiplicity $\geq k$, which complement Proposition 2.3 and Proposition 2.4.

Theorem 7.3. *Let $k \geq 2$ and n be integers with $M(n) \geq k$, let q_k be the smallest prime factor of n with multiplicity $\geq k$, and f a Dirichlet polynomial of degree n. If f has a nonzero coefficient a_i with $n - \min\{n_{<k}, q_k\}q_k^{k-1} < i < n$, then f is k-power-free.*

Proof: Let us assume that $n_{\geq k} = p_1^{n_1} \cdots p_r^{n_r}$, say, with $p_1, \ldots, p_r$ distinct prime numbers and $n_i \geq k$ for $i = 1, \ldots, r$. Then we have $\Delta_k(n) = p_1^{\lfloor \frac{n_1}{k} \rfloor} \cdots p_r^{\lfloor \frac{n_r}{k} \rfloor}$ and $q_k = \min\{p_1, \ldots, p_r\}$. If one of $n_1, \ldots, n_r$ is at least $k+1$, say $n_1 \geq k+1$, then $\frac{n}{\Delta_k(n)} \geq n_{<k} \cdot p_1^{n_1 - \lfloor \frac{n_1}{k} \rfloor} \geq n_{<k} \cdot p_1^k \geq n_{<k} \cdot q_k^k$. If $r \geq 2$ and all the n_i's are equal to k, then $\frac{n}{\Delta_k(n)} = n_{<k} \cdot p_1^{k-1} \cdots p_r^{k-1} > n_{<k} \cdot q_k^{2k-2} \geq n_{<k} \cdot q_k^k$. We are left with the case that $r = 1$ and $n_1 = k$, so $q_k = p_1$ and $n_{\geq k} = q_k^k$. In this case we have $\Delta_k(n) = q_k$ and $\frac{n}{\Delta_k(n)} = n_{<k} \cdot q_k^{k-1}$. Thus, we conclude that

$$\frac{n}{\Delta_k(n)} \geq n_{<k} \cdot q_k^{k-1}. \tag{43}$$

Let $f(s) = \frac{a_1}{1^s} + \cdots + \frac{a_n}{n^s}$ and assume towards a contradiction that $f(s) = g(s)^k h(s)$ for some Dirichlet polynomials g and h, say $g(s) = \frac{b_1}{1^s} + \cdots + \frac{b_t}{t^s}$ and $h(s) = \frac{c_1}{1^s} + \cdots + \frac{c_\ell}{\ell^s}$, with $t > 1$. We may then write $g^k(s) = \frac{d_1}{1^s} + \cdots + \frac{d_{t^k}}{t^{ks}}$ with coefficients $d_i = \sum_{j_1 \cdots j_k = i} b_{j_1} \cdots b_{j_k}$ for $i = 1, \ldots, t^k$. Let now $i_{max} = \max\{i : i < n \text{ and } a_i \neq 0\}$. By the multiplication rule, the coefficient $a_{i_{max}}$ is given by $a_{i_{max}} = \sum_{j \cdot j' = i_{max}} d_j c_{j'}$, and in this sum we must have $d_j c_{j'} \neq 0$ for at least one pair (j, j') with $j \cdot j' = i_{max}$, as $a_{i_{max}} \neq 0$. Consider such a pair (j, j'). Since $t^k \cdot \ell = n$ and $i_{max} < n$, at least one of the inequalities $j \leq t^k$ and $j' \leq \ell$ must be a strict one (note that if $n = p^k$ for some prime p, then we must have $t = p = q_k$ and $\ell = 1$, so for such integers n the inequality $j' < \ell$ cannot hold).

Assume first that $j < t^k$ and $j' \leq \ell$. Since $t^k \mid n$, we have $t \leq \Delta_k(n)$, and since j is an index smaller than t^k such that $d_j \neq 0$, we observe that $j \leq t^{k-1}(t-1)$, with equality if $b_{t-1} \neq 0$. Using also (43), we successively deduce that

$$i_{max} = j \cdot j' \leq t^{k-1}(t-1)j' \leq (t^k - t^{k-1})\ell = n - \frac{n}{t} \leq n - \frac{n}{\Delta_k(n)} \leq n - n_{<k} \cdot q_k^{k-1}.$$

Let us assume now that $j \leq t^k$ and $j' < \ell$. In this case we have

$$i_{max} = j \cdot j' \leq t^k(\ell - 1) = n - t^k \leq n - q_k^k,$$

as $t > 1$ is a divisor of n with $t^k \mid n$, and hence $t \geq q_k$. By these inequalities we deduce that i_{max} is at most $n - \min\{n_{<k}, q_k\}q_k^{k-1}$. We conclude that if f has a nonzero coefficient a_i with $n - \min\{n_{<k}, q_k\}q_k^{k-1} < i < n$, then f must be k-power-free. □

Let us denote by p_n the smallest prime factor of n. Observe that $p_n \leq q_k$, and that if $n_{<k} \neq 1$, then we also have $p_n \leq n_{<k}$, so we obtain as a corollary the following weaker, but simpler result.

Corollary 7.4. *Let $k \geq 2$ and n be integers with $M(n) \geq k$, p_n the smallest prime factor of n, and f a primitive Dirichlet polynomial of degree n. Then f is k-power-free in each of the following two cases:*

i) $n_{<k} \neq 1$ and f has a nonzero coefficient a_i with $n - p_n^k < i < n$;

ii) $n_{<k} = 1$ and f has a nonzero coefficient a_i with $n - p_n^{k-1} < i < n$.

Theorem 7.5. *Let $k \geq 2$ be an integer, $f = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ a primitive Dirichlet polynomial with $a_m a_n \neq 0$, $M(m) \geq k$ and $M(n) \geq k$. Let q_k and r_k be the smallest prime factors of m and n that have multiplicities $\geq k$, respectively. If $m > \frac{n}{r_k^k}$ and f has a nonzero coefficient a_i with $m < i < m + \min\{m_{<k}, q_k\} \cdot q_k^{k-1}$, then f is k-power-free.*

Proof: We deduce as in the proof of Theorem 7.3 that

$$\frac{m}{\Delta_k(m)} \geq m_{<k} \cdot q_k^{k-1}. \tag{44}$$

Let us assume again towards a contradiction that $f(s) = g(s)^k h(s)$ for some Dirichlet polynomials g and h, say $g(s) = \frac{b_u}{u^s} + \cdots + \frac{b_t}{t^s}$ and $h(s) = \frac{c_v}{v^s} + \cdots + \frac{c_\ell}{\ell^s}$, with $t > 1$. We may then write $g^k(s) = \frac{d_{u^k}}{u^{ks}} + \cdots + \frac{d_{tk}}{t^{ks}}$ with coefficients $d_i = \sum_{j_1 \cdots j_k = i} b_{j_1} \cdots b_{j_k}$ for $i = u^k, \ldots, t^k$. Let now $i_{min} = \min\{i : i > m \text{ and } a_i \neq 0\}$. By the multiplication rule, the coefficient $a_{i_{min}}$ is given by $a_{i_{min}} = \sum_{j \cdot j' = i_{min}} d_j c_{j'}$, and in this sum we must have $d_j c_{j'} \neq 0$ for at least one pair (j, j') with $j \cdot j' = i_{min}$, as $a_{i_{min}} \neq 0$. Let (j, j') be such a pair. Since $u^k \cdot v = m$ and $i_{min} > m$, at least one of the inequalities $j \geq u^k$ and $j' \geq v$ must be a strict one (note that if $m = p^k$ for some prime p, then we must have $u = p = q_k$ and $v = 1$, so for such m the inequality $j' < v$ cannot hold).

Assume first that $j > u^k$ and $j' \geq v$. Since $u^k \mid m$, we have $u \leq \Delta_k(m)$, and since j is an index greater than u^k such that $d_j \neq 0$, we observe that $j \geq u^{k-1}(u+1)$, with equality if $b_{u+1} \neq 0$. In view of (44), we deduce that

$$i_{min} = j \cdot j' \geq u^{k-1}(u+1)j' \geq (u^k + u^{k-1})v = m + \frac{m}{u} \geq m + \frac{m}{\Delta_k(m)} \geq m + m_{<k} \cdot q_k^{k-1}.$$

Assume now that $j \geq u^k$ and $j' > v$. In this case we have

$$i_{min} = j \cdot j' \geq u^k(v+1) = m + u^k \geq m + q_k^k,$$

provided we show that u is a divisor of m greater than 1, and hence $u \geq q_k$, as $u^k \mid m$. To see this, assume to the contrary that $u = 1$. This will force v to be equal to m, implying that $\ell \geq m$, which leads to $n = t^k \ell \geq t^k m \geq r_k^k m$ (the inequality $t \geq r_k$ holds due to the fact that $t^k \mid n$), thus contradicting our assumption that $m > \frac{n}{r_k^k}$, as needed.

Since in both cases i_{min} is at least $m+\min\{m_{<k}, q_k\}\cdot q_k^{k-1}$, we conclude that if f has a nonzero coefficient a_i with $m < i < m + \min\{m_{<k}, q_k\} \cdot q_k^{k-1}$, then f must be k-power-free. □

If we denote by p_m and p_n the smallest prime factors of m and n, respectively, we have $p_m \leq q_k$, and that if $m_{<k} \neq 1$, then we also have $p_m \leq m_{<k}$. Besides, $p_n \leq r_k$, so we have as a corollary the following simpler k-power-freeness criterion.

Corollary 7.6. *Let $k \geq 2$ be an integer and $f = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ a primitive Dirichlet polynomial with $a_m a_n \neq 0$, $M(m) \geq k$ and $M(n) \geq k$. Let p_m and p_n be the smallest prime factors of m and n, respectively. If $m > \frac{n}{p_n^k}$, then f is k-power-free in each of the following two cases:*

i) $m_{<k} \neq 1$ and f has a nonzero coefficient a_i with $m < i < m + p_m^k$;

ii) $m_{<k} = 1$ and f has a nonzero coefficient a_i with $m < i < m + p_m^{k-1}$.

In particular, from Corollary 7.4 and Corollary 7.6 we obtain for $k = 2$ the following square-free criterion.

Corollary 7.7. *Let $m < n$ be non square-free integers, each one having at least one prime factor with multiplicity 1, and let $f(s) = \frac{a_m}{m^s} + \cdots + \frac{a_n}{n^s}$ be a Dirichlet polynomial with $a_m a_n \neq 0$. If one of the coefficients a_{n-1}, a_{n-2} or a_{n-3} is nonzero, or, if $m > n/4$ and one of the coefficients a_{m+1}, a_{m+2} or a_{m+3} is nonzero, then f is square-free.*

From Theorem 7.3 and Theorem 7.5 one may obtain upper bounds for the multiplicities of the irreducible factors of f that depend only on information on the prime factorizations of m and n, and on $Supp(f)$, regardless of the values of the nonzero coefficients of f. These bounds are given in full generality in Corollary 7.8 and Corollary 7.10, and in weaker, but more tractable form in Corollary 7.9 and Corollary 7.11.

Corollary 7.8. *Let n be a positive integer with $M(n) = M \geq 2$. For any integer k with $2 \leq k \leq M$ denote by q_k the smallest prime factor of n with multiplicity $\geq k$. Let $f(s) = \frac{a_1}{1^s} + \cdots + \frac{a_n}{n^s}$ be a primitive Dirichlet polynomial of degree n, and let $i_{max} = \max\{i : i < n \text{ and } a_i \neq 0\}$. If $i_{max} > n - \min\{n_{<M}, q_M\} \cdot q_M^{M-1}$, then*

$$M(f) < \min\{k : \ 2 \leq k \leq M \text{ and } i_{max} > n - \min\{n_{<k}, q_k\} \cdot q_k^{k-1}\}. \tag{45}$$

Proof: For $k_1 < k_2$ we have $q_{k_1} \leq q_{k_2}$ and $n_{<k_1} \leq n_{<k_2}$, so

$$\min\{n_{<k_1}, q_{k_1}\} \cdot q_{k_1}^{k_1-1} < \min\{n_{<k_2}, q_{k_2}\} \cdot q_{k_2}^{k_2-1},$$

which shows that the sequence $n - \min\{n_{<k}, q_k\} \cdot q_k^{k-1}$ is a strictly decreasing one. Thus, if $i_{max} > n - \min\{n_{<M}, q_M\} \cdot q_M^{M-1}$, there exists a smallest integer $k \in \{2, 3, \ldots, M\}$ such that $n - \min\{n_{<k}, q_k\} \cdot q_k^{k-1} < i_{max}$, say k_{min}. By Theorem 7.3 we deduce that f is k_{min}-power-free, so $M(f)$ satisfies (45). □

If we replace in (45) the term $\min\{n_{<k}, q_k\} \cdot q_k^{k-1}$ by p_n^{k-1}, with p_n the smallest prime factor of n, we obtain a weaker, but more tractable bound for $M(f)$, as in the following result.

Corollary 7.9. *Let n be a positive integer with $M(n)=M\geq 2$, and let p_n be the smallest prime factor of n. Let $f(s)=\frac{a_1}{1^s}+\cdots+\frac{a_n}{n^s}$ be a primitive Dirichlet polynomial of degree n, and let $i_{max}=\max\{i: i<n \text{ and } a_i\neq 0\}$. If $i_{max}>n-p_n^{M-1}$, then*

$$M(f)<\begin{cases}1+\lceil\log_{p_n}(n-i_{max})\rceil, & \text{if } i_{max}\notin\{n-p_n, n-p_n^2,\ldots,n-p_n^{M-1}\},\\ 2+\log_{p_n}(n-i_{max}), & \text{otherwise.}\end{cases} \tag{46}$$

Proof: If $i_{max}>n-p_n^{M-1}$, there exists a smallest $k\in\{2,3,\ldots,M\}$ such that $n-p_n^{k-1}<i_{max}$, say k_{min}, displayed in curly brace in (46). By Corollary 7.8 we conclude that $M(f)<k_{min}$. □

Corollary 7.10. *Let m and n be positive integers with $m<n$ and such that $M(n)\geq M(m)=M\geq 2$, and let $f=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be a primitive Dirichlet polynomial with $a_m a_n\neq 0$. For any integer $k\in\{2,3,\ldots,M\}$ let q_k and r_k be the smallest prime factors of m and n that have multiplicities $\geq k$, respectively. Let $i_{min}=\min\{i: i>m \text{ and } a_i\neq 0\}$. If $m\geq\frac{n}{r_M^M}$ and $i_{min}<m+\min\{m_{<M},q_M\}\cdot q_M^{M-1}$, then*

$$M(f)<\min\{k:\ 2\leq k\leq M,\ m>\frac{n}{r_k^k} \text{ and } i_{min}<m+\min\{m_{<k},q_k\}\cdot q_k^{k-1}\}. \tag{47}$$

Proof: For $k_1<k_2$ we have $r_{k_1}\leq r_{k_2}$, $q_{k_1}\leq q_{k_2}$ and $m_{<k_1}\leq m_{<k_2}$, so $\frac{n}{r_{k_1}^{k_1}}\geq\frac{n}{r_{k_2}^{k_2}}$ and

$$\min\{m_{<k_1},q_{k_1}\}\cdot q_{k_1}^{k_1-1}<\min\{m_{<k_2},q_{k_2}\}\cdot q_{k_2}^{k_2-1},$$

which shows that the sequence $\frac{n}{r_k^k}$ is a decreasing one, while the sequence $m+\min\{m_{<k},q_k\}\cdot q_k^{k-1}$ is a strictly increasing one. Thus, if $m>\frac{n}{r_M^M}$ and $i_{min}<m+\min\{m_{<M},q_M\}\cdot q_M^{M-1}$, there exists a smallest integer $k\in\{2,3,\ldots,M\}$ such that $m>\frac{n}{r_k^k}$ and $i_{min}<m+\min\{m_{<k},q_k\}\cdot q_k^{k-1}$, say k_{min}. By Theorem 7.5 we deduce that f is k_{min}-power-free, so $M(f)$ satisfies (47). □

If we replace in (47) the term $\min\{m_{<k},q_k\}\cdot q_k^{k-1}$ by p_m^{k-1}, and r_k by p_n, with p_m and p_n the smallest prime factors of m and n, respectively, we obtain the following weaker, but more tractable bound for $M(f)$.

Corollary 7.11. *Let m and n be integers with $2\leq m<n$ and $M(n)\geq M(m)=M\geq 2$, and let p_m and p_n be the smallest prime factors of m and n, respectively. Let $f=\frac{a_m}{m^s}+\cdots+\frac{a_n}{n^s}$ be a primitive Dirichlet polynomial with $a_m a_n\neq 0$, and let $i_{min}=\min\{i: i>m \text{ and } a_i\neq 0\}$. If $m\geq\frac{n}{p_n^M}$ and $i_{min}<m+p_m^{M-1}$, then $M(f)<k_{min}$ with*

$$k_{min}=\min\left\{k:\ 2\leq k\leq M,\ k>\max\{\log_{p_n}(\frac{n}{m}),1+\log_{p_m}(i_{min}-m)\}\right\}. \tag{48}$$

Proof: If $m\geq\frac{n}{p_m^M}$ and $i_{min}<m+p_m^{M-1}$, there exists a smallest $k\in\{2,3,\ldots,M\}$ such that $m\geq\frac{n}{p_m^k}$ and $i_{min}<m+p_m^{k-1}$, say k_{min}, given by (48). By Corollary 7.10 we conclude that $M(f)<k_{min}$. □

In [2] we proved the following square-free criterion for polynomials over unique factorization domains, that uses no derivatives.

Proposition *Let R be a unique factorization domain and $f \in R[X]$ a primitive polynomial with $\deg(f) = n \geq 2$. Fix an arbitrarily chosen integer $m \geq 2$. Then the following are equivalent:*

i) The polynomial f is square-free;

ii) For every $g \in R[X]$ with $\deg(g) = n-1$, g^m is not divisible by f.

For unique factorization domains R that include the real numbers (in particular logarithms of positive rationals), one may define the derivative and the higher derivatives with respect to s of a Dirichlet polynomial $f \in DP(R)[s]$, and use them to study its repeated irreducible factors. For Dirichlet polynomials over arbitrary unique factorization domains that do not necessarily include the real numbers, we will devise analogous square-free conditions that require no derivatives.

Theorem 7.12. *Let R be a unique factorization domain and $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree $n \geq 2$, with n not square-free. Fix an arbitrarily chosen integer $m \geq 2$. Then the following statements are equivalent:*

i) The Dirichlet polynomial f is square-free;

ii) For every prime factor p of n with multiplicity at least 2, and every Dirichlet polynomial $g \in DP(R)[s]$ of degree at most $\frac{n}{p}$, g^m is not divisible by f;

iii) For every prime factor p of n with multiplicity at least 2, and every Dirichlet polynomial $g \in DP(R)[s]$ of degree $\frac{n}{p}$, g^m is not divisible by f.

Proof: *i)* $\Rightarrow$ *ii)* Let $f = f_1 \cdots f_k$, with $f_1, \ldots, f_k \in DP(R)[s]$ irreducible and pairwise nonassociated in divisibility. Let p be a prime factor of n and assume there exists a Dirichlet polynomial $g \in DP(R)[s]$ with $\deg g \leq \frac{n}{p}$, such that f divides g^m. Then each one of the f_i's will divide g, and hence f will divide g, which is impossible since $\deg g < n$.

ii) $\Rightarrow$ *iii)* The argument is trivial.

iii) $\Rightarrow$ *i)* This implication is a particular case of the following more general result on the multiplicities of the irreducible factors of Dirichlet polynomials. $\square$

Theorem 7.13. *Let $k \geq 2$ be an integer, R a unique factorization domain and $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree n. Let us fix an arbitrarily chosen integer $m \geq k$.*

i) If for every prime factor p of n with $p^k \mid n$, and for every Dirichlet polynomial $g \in DP(R)[s]$ of degree $\frac{n}{p^{k-1}}$, g^m is not divisible by f, then f is k-power-free;

ii) If f is k-power-free, then for every prime factor p of n with $p^k \mid n$, and for every Dirichlet polynomial $g \in DP(R)[s]$ of degree $\frac{n}{p^{\nu_p(n)-1}}$, g^m is not divisible by f.

Proof: i) The conclusion is trivial if all the prime factors of n have multiplicities at most $k-1$, so we may assume that n has at least one prime factor with multiplicity $\geq k$. Now let us assume to the contrary that $f = c \cdot f_1^{n_1} \cdots f_r^{n_r}$ with $c \in R \setminus \{0\}$ and $f_1, \ldots, f_r \in DP(R)[s]$ irreducible and pairwise nonassociated in divisibility, with at least one of the multiplicities $\geq k$, say $n_1 \geq k$. Thus, the multiplicity of f_1 in the canonical decomposition of $(\frac{f}{f_1^{k-1}})^m$ is $m(n_1 - k + 1)$, which is greater than or equal to n_1, as $m \geq k$ and $k \geq \frac{n_1}{n_1 - k + 1}$. In particular, this shows that f will divide $(\frac{f}{f_1^{k-1}})^m$. Let p be a prime factor of $\deg f_1$, and observe that p has multiplicity at least k

in the prime factorization of n, since $n_1 \geq k$ and $(\deg f_1)^{n_1} \mid n$. Let now $q = (\frac{\deg f_1}{p})^{k-1}$. In case $q = 1$ we take $g = \frac{f}{f_1^{k-1}}$, while if $q > 1$ we multiply $\frac{f}{f_1^{k-1}}$ by a Dirichlet polynomial $h \in DP(R)[s]$ with $\deg h = q$, and take instead $g = \frac{fh}{f_1^{k-1}}$. In both situations we conclude that g^m is divisible by f and $\deg g = \frac{n}{p^{k-1}}$ with p a prime factor of n with $p^k \mid n$, a contradiction.

ii) Conversely, assume that $f = c \cdot f_1^{n_1} \cdots f_r^{n_r}$ with $c \in R \setminus \{0\}$ and $f_1, \ldots, f_r \in DP(R)[s]$ irreducible and pairwise nonassociated in divisibility, and with all the multiplicities n_i at most $k-1$, and that there is a prime p with $p^k \mid n$ and $g \in DP(R)[s]$ of degree $\frac{n}{p^{\nu_p(n)-1}}$ such that $f \mid g^m$. Then $f_1 \cdots f_r$ must divide g, and since $n_i \leq k-1$ for each i, f must in fact divide g^{k-1}, which in turn forces n to divide $(\frac{n}{p^{\nu_p(n)-1}})^{k-1}$. In particular, this divisibility forces $\nu_p(n)$ to satisfy $\nu_p(n) \leq (k-1)(\nu_p(n) - (\nu_p(n)-1)) = k-1$, which cannot hold, since $p^k \mid n$, and this completes the proof. □

In particular, by taking $m = k = M(n)$ in Theorem 7.13, we obtain necessary and sufficient conditions for $M(f)$ to be smaller than $M(n)$.

Corollary 7.14. *Let R be a unique factorization domain, $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree n, with $M(n) \geq 2$, and let us fix an arbitrarily chosen integer $m \geq M(n)$. Then $M(f) < M(n)$ if and only if for every prime factor p of n with multiplicity $M(n)$, and for every Dirichlet polynomial $g \in DP(R)[s]$ of degree $\frac{n}{p^{M(n)-1}}$, g^m is not divisible by f.*

The square-free conditions in Theorem 7.12 simplify when the order of f is a prime power, as follows.

Corollary 7.15. *Let R be a unique factorization domain, and $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree p^d, with p a prime number and $d \geq 2$. Fix an arbitrarily chosen integer $m \geq 2$. Then the following statements are equivalent:*

i) The Dirichlet polynomial f is square-free;

ii) For every $g \in DP(R)[s]$ of degree at most p^{d-1}, g^m is not divisible by f;

iii) For every $g \in DP(R)[s]$ of degree p^{d-1}, g^m is not divisible by f.

Remark 7.16. When R is a unique factorization domain with positive characteristic $\mathfrak{p} \geq k$, we may take the integer m in Theorem 7.13 (and also in Theorem 7.12 if $k = 2$) to be equal to $\mathfrak{p}$, so that a condition like f divides g^m may be regarded as a homogeneous system of linear equations. Indeed, if $f(s) = \sum_{i=1}^{n} \frac{a_i}{i^s}$ and we assume that for a prime divisor p of n with $p^k \mid n$ and a Dirichlet polynomial g of degree $\frac{n}{p^{k-1}}$, say $g(s) = \sum_{i=1}^{n/p^{k-1}} \frac{c_i}{i^s}$ we have

$$f(s)h(s) = g(s)^{\mathfrak{p}} \tag{49}$$

for some Dirichlet polynomial h of degree t, say $h(s) = \sum_{i=1}^{t} \frac{b_i}{i^s}$, we see that $t = \frac{n^{\mathfrak{p}-1}}{p^{(k-1)\mathfrak{p}}}$ and equality (49) reads

$$\sum_{i=1}^{n} \frac{a_i}{i^s} \sum_{i=1}^{t} \frac{b_i}{i^s} = \sum_{i=1}^{\frac{n}{p^{k-1}}} \frac{c_i^{\mathfrak{p}}}{i^{\mathfrak{p}s}}.$$

By equating the coefficients in this equality one obtains a homogeneous system of $(\frac{n}{p^{k-1}})^{\mathfrak{p}}$ linear equations in the unknowns $b_1,\dots,b_t,c_1^{\mathfrak{p}},\dots,c_{n/p^{k-1}}^{\mathfrak{p}}$, of matrix form

$$\mathcal{A}_{\mathfrak{p},n,p,k}\cdot X=\mathbf{0}, \tag{50}$$

with $\mathcal{A}_{\mathfrak{p},n,p,k}$ the coefficient matrix depending on $\mathfrak{p},n,p$ and k, having $(\frac{n}{p^{k-1}})^{\mathfrak{p}}$ rows and $t+\frac{n}{p^{k-1}}$ columns, X the column vector $(b_1,\dots,b_t,c_1^{\mathfrak{p}},\dots,c_{n/p^{k-1}}^{\mathfrak{p}})^T$, and $\mathbf{0}$ a zero column matrix.

One may easily check that the entries $\mathfrak{a}_{i,j}$ of $\mathcal{A}_{\mathfrak{p},n,p,k}$ are given by

$$\mathfrak{a}_{i,j}=\begin{cases} a_{\frac{i}{j}}, & \text{if } j\mid i \text{ and } j\in\{1,\dots,t\},\\ 0, & \text{if } j\nmid i \text{ and } j\in\{1,\dots,t\},\\ -1, & \text{if } i=(j-t)^{\mathfrak{p}} \text{ and } j\in\{t+1,\dots,t+\frac{n}{p^{k-1}}\},\\ 0, & \text{if } i\neq(j-t)^{\mathfrak{p}} \text{ and } j\in\{t+1,\dots,t+\frac{n}{p^{k-1}}\},\end{cases} \tag{51}$$

with the convention that $a_{\mathfrak{i}}=0$ for $\mathfrak{i}>n$. We notice that the system (50) has more equations than unknowns, that is $(\frac{n}{p^{k-1}})^{\mathfrak{p}}>t+\frac{n}{p^{k-1}}$, which is equivalent to $n^{\mathfrak{p}-2}(n-1)>p^{(\mathfrak{p}-1)(k-1)}$. Indeed, since we may write $n=p^k\ell$ for some positive integer ℓ, we need to check that $p^{\mathfrak{p}-k-1}\ell^{\mathfrak{p}-2}(p^k\ell-1)>1$. Recall now that $\mathfrak{p}\geq k$. Last inequality obviously holds for $\mathfrak{p}\geq k+1$, while for $\mathfrak{p}=k$, it reduces to $\ell^{k-2}(p^k\ell-1)>p$, which also holds, since $\ell^{k-2}(p^k\ell-1)\geq p^2-1>p$, as $k\geq 2$, $\ell\geq 1$ and $p\geq 2$. However, $\mathcal{A}_{\mathfrak{p},n,p,k}$ may have rows r_i consisting only of zero entries, if for instance i is not a $\mathfrak{p}$th power and $a_{\frac{i}{j}}=0$ for each divisor j of i with $1\leq j\leq t$. On the other hand, if p^k is a proper divisor of n, that is $\ell>1$ (so that $t>1$ as well), some of the zero rows may simply occur due to some arithmetic constraints, regardless of the values of the coefficients of f. This is the case of the rows with index $i>n$ (implying that $a_i=0$ by our convention), with i not a $\mathfrak{p}$th power, and not a multiple of any of the integers $2,3,\dots,t$, and the removal of these rows does not affect the rank of $\mathcal{A}_{\mathfrak{p},n,p,k}$. Letting $\pi_t=\prod_{p \text{ prime},\ p\leq t}p$, we see that these are the rows whose indices i belong to the interval $(\max\{t,n\},(\frac{n}{p^{k-1}})^{\mathfrak{p}})$, are not $\mathfrak{p}$th powers, and are coprime to π_t. Let us denote by $\mathcal{S}$ the set of these indices, and by β its cardinality. To count these indices, one may use the fact that given two positive integers A and B, the number $\mathfrak{n}(A,B)$ of positive integers not greater than A and which are coprime to B is $\mathfrak{n}(A,B)=\sum_{d\mid B}\mu(d)\lfloor\frac{A}{d}\rfloor$, with μ the Möbius function. Let $\mathcal{S}'$ be the set of integers in the interval $(\max\{t,n\},(\frac{n}{p^{k-1}})^{\mathfrak{p}})$ that are coprime to π_t, and γ its cardinality. Then we have

$$\gamma=\mathfrak{n}((n/p^{k-1})^{\mathfrak{p}},\pi_t)-\mathfrak{n}(\max\{t,n\},\pi_t), \tag{52}$$

as $t\mid(\frac{n}{p^{k-1}})^{\mathfrak{p}}$. We will prove now that $\mathfrak{p}\leq t$, hence $\mathfrak{p}\mid\pi_t$, implying that all the integers in $\mathcal{S}'$ are in fact coprime to $\mathfrak{p}$, so in particular none of them can be a $\mathfrak{p}$th power. To do this, we first notice that $t=p^{\mathfrak{p}-k}\ell^{\mathfrak{p}-1}$, and $t<n$ if and only if $p^{\mathfrak{p}-2k}<\ell^{2-\mathfrak{p}}$, which, since $\mathfrak{p}\geq 2$, is equivalent to $\mathfrak{p}<2k$. We distinguish the following three cases:

Case 1: $\mathfrak{p}=k$; in this case we have $\mathfrak{p}=k\leq 2^{k-1}\leq\ell^{\mathfrak{p}-1}=t$;

Case 2: $k+1\leq\mathfrak{p}<2k$; here we have $\mathfrak{p}<2k\leq 2\cdot 2^k\leq 2\cdot 2^{\mathfrak{p}-1}\leq 2\cdot\ell^{\mathfrak{p}-1}\leq p^{\mathfrak{p}-k}\ell^{\mathfrak{p}-1}=t$;

Case 3: $\mathfrak{p}>2k$; in this case we have $\mathfrak{p}<4\cdot 2^{\mathfrak{p}-1}\leq p^k\cdot 2^{\mathfrak{p}-1}\leq p^k\cdot\ell^{\mathfrak{p}-1}\leq p^{\mathfrak{p}-k}\ell^{\mathfrak{p}-1}=t$.

This shows us that $\mathcal{S} = \mathcal{S}'$, and hence $\beta = \gamma$. We thus conclude that

$$\operatorname{rank}(\mathcal{A}_{\mathfrak{p},n,p,k}) \leq \min\left\{\frac{n^{\mathfrak{p}-1}}{p^{(k-1)\mathfrak{p}}} + \frac{n}{p^{k-1}}, \frac{n^{\mathfrak{p}}}{p^{(k-1)\mathfrak{p}}} - \gamma\right\}, \tag{53}$$

with γ given by (52).

Note that if

$$\operatorname{rank}(\mathcal{A}_{\mathfrak{p},n,p,k}) = \frac{n^{\mathfrak{p}-1}}{p^{(k-1)\mathfrak{p}}} + \frac{n}{p^{k-1}}, \tag{54}$$

then the system (50) will have no nontrivial solutions. Let us also note that the rightmost $\frac{n}{p^{k-1}}$ columns of $\mathcal{A}_{\mathfrak{p},n,p,k}$ contain a square diagonal submatrix of order $\frac{n}{p^{k-1}}$ whose diagonal entries are all equal to -1, corresponding to the rows of $\mathcal{A}_{\mathfrak{p},n,p,k}$ with indices $1, 2^{\mathfrak{p}}, \ldots, (\frac{n}{p^{k-1}})^{\mathfrak{p}}$.

Let now $\mathcal{B}_{\mathfrak{p},n,p,k}$ be the matrix obtained from $\mathcal{A}_{\mathfrak{p},n,p,k}$ by removing its rightmost $\frac{n}{p^{k-1}}$ columns and its rows with indices $1, 2^{\mathfrak{p}}, \ldots, (\frac{n}{p^{k-1}})^{\mathfrak{p}}$. The matrix $\mathcal{B}_{\mathfrak{p},n,p,k}$ has $(\frac{n}{p^{k-1}})^{\mathfrak{p}} - \frac{n}{p^{k-1}}$ rows and $\frac{n^{\mathfrak{p}-1}}{p^{(k-1)\mathfrak{p}}}$ columns, and its entries $x_{i,j}$ can be described as follows. The set $S = \{1, 2, \ldots, (\frac{n}{p^{k-1}})^{\mathfrak{p}} - \frac{n}{p^{k-1}}\}$ of indices of the rows of $\mathcal{B}_{\mathfrak{p},n,p,k}$ can be partitioned into $\frac{n}{p^{k-1}} - 1$ disjoint subsets of consecutive integers

$$S = \bigcup_{\delta=1}^{\frac{n}{p^{k-1}}-1} S_\delta$$

with $S_\delta = \{\delta^{\mathfrak{p}} - \delta + 1, \delta^{\mathfrak{p}} - \delta + 2, \ldots, (\delta+1)^{\mathfrak{p}} - (\delta+1)\}$ for $\delta = 1, \ldots, \frac{n}{p^{k-1}} - 1$. Thus, to any integer $i \in S$ we can associate a unique index $\delta_i \in \{1, \ldots, \frac{n}{p^{k-1}} - 1\}$ such that $i \in S_{\delta_i}$, and an entry $x_{i,j}$ of the matrix $\mathcal{B}_{\mathfrak{p},n,p,k}$ is now easily seen to be

$$x_{i,j} = \begin{cases} a_{\frac{i+\delta_i}{j}}, & \text{if } j \mid (i+\delta_i), \\ 0, & \text{otherwise,} \end{cases}$$

with the shift of i with the corresponding δ_i appearing as a consequence of the removal of the rows of $\mathcal{A}_{\mathfrak{p},n,p,k}$ with indices $1, 2^{\mathfrak{p}}, \ldots, (\frac{n}{p^{k-1}})^{\mathfrak{p}}$. Here we use again the convention that $a_{\mathfrak{i}} = 0$ for $\mathfrak{i} > n$. Next, we observe that if instead of (54) we ask $\mathcal{B}_{\mathfrak{p},n,p,k}$ to have full rank, that is to satisfy

$$\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,k}) = \frac{n^{\mathfrak{p}-1}}{p^{(k-1)\mathfrak{p}}}, \tag{55}$$

then the system (50) will have no nontrivial solutions. We thus conclude that if (55) holds for each prime factor p of n that has multiplicity at least k, then f must be k-power-free.

Note that if f is as in Corollary 7.15, we need to test (55) only for the prime p in the statement, so we only need to check the equality $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) = p^{(d-1)\mathfrak{p}-d}$.

By Theorem 7.13 and Remark 7.16 one obtains the following k-power-free criterion.

Theorem 7.17. *Let $k \geq 2$ be an integer, R a unique factorization domain with prime characteristic $\mathfrak{p} \geq k$ and $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree n. If for every prime factor p of n with $p^k \mid n$ we have $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,k}) = \frac{n^{\mathfrak{p}-1}}{p^{(k-1)\mathfrak{p}}}$, then f is k-power-free.*

We may prove now the following square-free criterion, that relies on Theorem 7.12 and on the study of the rank of the matrix $\mathcal{B}_{\mathfrak{p},n,p,2}$.

Theorem 7.18. *Let R be a unique factorization domain with prime characteristic $\mathfrak{p}$, and $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree $n \geq 2$.*

i) If $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) = \frac{n^{\mathfrak{p}-1}}{p^{\mathfrak{p}}}$ for every prime factor p of n with $p^2 \mid n$, then f is square-free.

ii) If the Frobenius map of R is surjective, then f is square-free if and only if the equality $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) = \frac{n^{\mathfrak{p}-1}}{p^{\mathfrak{p}}}$ holds for every prime factor p of n with $p^2 \mid n$.

Proof: i) We may obviously assume that n is not square-free. If for every prime factor p of n with $p^2 \mid n$ we have $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) = \frac{n^{\mathfrak{p}-1}}{p^{\mathfrak{p}}}$, then all the corresponding systems of equations (50) will have only trivial solutions, so no equality of the form (49) with g of degree $\frac{n}{p}$ will hold. By the equivalence i) $\Leftrightarrow$ iii) in Theorem 7.12 we conclude that f must be square-free.

ii) If we assume that the Frobenius map of R is surjective, then we may replace $c_1^{\mathfrak{p}}, \ldots, c_{n/p}^{\mathfrak{p}}$ by some arbitrary unknowns $d_1, \ldots, d_{n/p}$ and regard the system (50) as a linear one. Therefore, if $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) < \frac{n^{\mathfrak{p}-1}}{p^{\mathfrak{p}}}$, the system (50) will have a nontrivial solution in $Q(R)$, the quotient field of R. Multiplying by a common denominator if necessary, which is itself the $\mathfrak{p}$th power of an element in R, will provide a solution in R having at least one nonzero b_i or one nonzero d_j. We notice that in fact, such a nonzero solution must have at least one nonzero b_i and at least one nonzero d_j (and hence at least one nonzero c_j, as $c_j^{\mathfrak{p}} = d_j$). For such a nonzero solution we let $i_{max} = \max\{i : b_i \neq 0\}$ and $j_{max} = \max\{j : c_j \neq 0\}$ with $i_{max} \leq t = \frac{n^{\mathfrak{p}-1}}{p^{\mathfrak{p}}}$, $j_{max} \leq \frac{n}{p}$, and $n \cdot i_{max} = j_{max}^{\mathfrak{p}}$, so equality (49) holds for some $h(s)$ and $g(s)$ with $\deg h = i_{max}$ and $\deg g = j_{max}$. In particular, we found a Dirichlet polynomial $g \in DP(R)[s]$ of degree at most $\frac{n}{p}$ such that $g^{\mathfrak{p}}$ is divisible by f. By the equivalence i) $\Leftrightarrow$ ii) in Theorem 7.12, we conclude that f cannot be square-free. □

Corollary 7.19. *Let R be a unique factorization domain with prime characteristic $\mathfrak{p}$, and $f \in DP(R)[s]$ a primitive Dirichlet polynomial of degree p^d, with p a prime number and $d \geq 2$.*

i) If $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) = p^{(d-1)\mathfrak{p}-d}$, then f is square-free.

ii) If the Frobenius map of R is surjective, then f is square-free if and only if we have $\operatorname{rank}(\mathcal{B}_{\mathfrak{p},n,p,2}) = p^{(d-1)\mathfrak{p}-d}$.

As in the case of polynomials, we can search the repeated irreducible factors of a Dirichlet polynomial with complex coefficients by studying its derivative. We will first study the common factors of two Dirichlet polynomials f and g by adapting some well-known techniques used for polynomials. In this respect, we have the following result.

Proposition 7.20. *Let $f(s) = \frac{a_1}{1^s} + \cdots + \frac{a_m}{m^s}$ and $g(s) = \frac{b_1}{1^s} + \cdots + \frac{b_n}{n^s}$ be two Dirichlet polynomials with coefficients in a unique factorization domain R, with $a_m b_n \neq 0$.*

i) If f and g have a common factor of degree d, then there exist two Dirichlet polynomials $u, v \in DP(R)[s]$ with $\deg u = \frac{n}{d}$, $\deg v = \frac{m}{d}$, and such that $f(s)u(s) + g(s)v(s) = 0$.

ii) Conversely, if $f(s)u(s)+g(s)v(s)=0$ *for two Dirichlet polynomials* $u,v\in DP(R)[s]$ *with* $\deg u=\frac{n}{d}$ *and* $\deg v=\frac{m}{d}$ *for some divisor* d *of* $\gcd(m,n)$*, then* $\gcd(f,g)$ *has degree a multiple of* d*, with* $\deg\gcd(f,g)=d$ *if* u *and* v *are relatively prime.*

Proof: i) If f and g share a factor $h(s)$ of degree d, with $d\mid\gcd(m,n)$, then one may take $u(s)=\frac{g(s)}{h(s)}$ and $v(s)=-\frac{f(s)}{h(s)}$, and u and v have the desired properties.

ii) Assume now that $f(s)u(s)+g(s)v(s)=0$ for two Dirichlet polynomials $u,v\in DP(R)[s]$ with $\deg u=\frac{n}{d}$ and $\deg v=\frac{m}{d}$. Let $h(s)=\gcd(f(s),g(s))$, and let $\deg h=k$, say, with $1\leq k\leq\gcd(m,n)$. Writing $f=hf_1$ and $g=hg_1$ with f_1 and g_1 relatively prime yields $f_1(s)u(s)=-g_1(s)v(s)$, which further implies the divisibilities $f_1(s)\mid v(s)$ and $g_1(s)\mid u(s)$. In particular, we must have $\frac{m}{k}\mid\frac{m}{d}$ and $\frac{n}{k}\mid\frac{n}{d}$, so k must be a multiple of d.

Moreover, if u and v are relatively prime, the equality $f_1(s)u(s)=-g_1(s)v(s)$ forces f_1 and v to be associated in divisibility, and also f_2 and u to be associated in divisibility, so we must have the equalities $\deg f_1=\deg v$ and $\deg g_1=\deg u$. Thus k must be equal to d, and in this case f and g have h as a common factor of degree d. □

Let now $u(s)=\frac{u_1}{1^s}+\cdots+\frac{u_{n/d}}{(n/d)^s}$ and $v(s)=\frac{v_1}{1^s}+\cdots+\frac{v_{m/d}}{(m/d)^s}$ in $DP(R)[s]$ be such that

$$f(s)u(s)+g(s)v(s)=0, \tag{56}$$

with f,g as in Proposition 7.20, and d a divisor of $\gcd(m,n)$. By equating the coefficients, we may regard equality (56) as a homogeneous system of $\frac{mn}{d}$ linear equations in the $\frac{n}{d}+\frac{m}{d}$ unknowns $u_1,\ldots,u_{n/d},v_1,\ldots,v_{m/d}$, written in matrix form as

$$\mathcal{R}_d\cdot X=\mathbf{0}, \tag{57}$$

with $\mathcal{R}_d$ the coefficient matrix having $\frac{mn}{d}$ rows and $\frac{m+n}{d}$ columns, $X=(u_1,\ldots,u_{\frac{n}{d}},v_1,\ldots,v_{\frac{m}{d}})^T$, and $\mathbf{0}$ a zero column matrix. We notice that except for the case that $m=n=2$, the matrix $\mathcal{R}_d$ has more rows than columns. One may easily check that the entries $r_{i,j}$ of $\mathcal{R}_d$ are given by

$$r_{i,j}=\begin{cases}a_{\frac{i}{j}}, & \text{if } j\mid i \text{ and } j\in\{1,\ldots,\frac{n}{d}\},\\ 0, & \text{if } j\nmid i \text{ and } j\in\{1,\ldots,\frac{n}{d}\},\\ b_{\frac{i}{j-\frac{n}{d}}}, & \text{if } j-\frac{n}{d}\mid i \text{ and } j\in\{\frac{n}{d}+1,\ldots,\frac{n}{d}+\frac{m}{d}\},\\ 0, & \text{if } j-\frac{n}{d}\nmid i \text{ and } j\in\{\frac{n}{d}+1,\ldots,\frac{n}{d}+\frac{m}{d}\},\end{cases} \tag{58}$$

where by convention, we put $a_{\mathfrak{i}}=0$ for $\mathfrak{i}>m$ and $b_{\mathfrak{i}}=0$ for $\mathfrak{i}>n$. Here too, as in the case of the matrices $\mathcal{A}_{\mathfrak{p},n,p,k}$, there are rows of $\mathcal{R}_d$ that have only zero entries, no matter what coefficients we choose for f and g. These are the rows whose indices i exceed $\max\{m,n\}$ (so that $a_i=b_i=0$, by convention) and are not a multiple of any of the integers in $\{2,3,\ldots,\frac{\max\{m,n\}}{d}\}$, that is the rows with indices i in the closed interval $[\max\{m,n\}+1,\frac{mn}{d}]$, with i not divisible by any prime number less than or equal to $\frac{\max\{m,n\}}{d}$. An account of these indices may be done by the method described in Remark 7.16.

We may now state the following criterion for two Dirichlet polynomials to be relatively prime.

Theorem 7.21. *Let f and g be two Dirichlet polynomials with coefficients in a unique factorization domain R, of degrees m and n, respectively, with* $\gcd(m,n) > 1$, *and let d be a divisor of* $\gcd(m,n)$. *Then f and g have no common factors of degree $\geq d$ if and only if* $\operatorname{rank}(\mathcal{R}_d) = \frac{m+n}{d}$. *In particular, f and g are relatively prime if and only if* $\operatorname{rank}(\mathcal{R}_1) = m+n$.

Proof: If $\operatorname{rank}(\mathcal{R}_d) = \frac{m+n}{d}$ for some divisor d of $\gcd(m,n)$, then the system (57) has only the trivial solution, so any equality of the form (56) with $1 \leq \deg u \leq \frac{n}{d}$ and $1 \leq \deg v \leq \frac{m}{d}$ must fail. In particular, will fail any equality of the form (56) with $\deg u = \frac{n}{D}$ and $\deg v = \frac{m}{D}$ with D a divisor of n greater than or equal to d. By Proposition 7.20 i) we deduce that f and g can have no common factors of degree $\geq d$.

Assume now that $\operatorname{rank}(\mathcal{R}_d) < \frac{m+n}{d}$. Then the system (57) will have a nonzero solution in the quotient field of R, which after multiplication by a common denominator will give a nonzero solution in R, that is a pair of nonzero u and v with $1 \leq \deg u \leq \frac{n}{d}$ and $1 \leq \deg v \leq \frac{m}{d}$ such that $f(s)u(s) = -g(s)v(s)$. Let again $h(s) = \gcd(f(s), g(s))$, and assume that $\deg h = k$, say, with $1 \leq k \leq \gcd(m,n)$. Writing again $f = hf_1$ and $g = hg_1$ with f_1 and g_1 relatively prime yields $f_1(s)u(s) = -g_1(s)v(s)$, so f_1 must divide v. In particular, we must have $\frac{m}{k} \mid \deg v$, and since $\deg v \leq \frac{m}{d}$, we conclude that $k \geq d$, so f and g have h as a common factor of degree $\geq d$.

In particular, f and g are relatively prime if and only if $\operatorname{rank}(\mathcal{R}_1) = m+n$. This also shows that if $\mathcal{R}_1$ has full rank, then for each divisor d of $\gcd(m,n)$, $\mathcal{R}_d$ must also have full rank. □

The matrix $\mathcal{R}_1$ may be regarded as an analogue of the Sylvester matrix associated to two univariate polynomials, and has the following shape, with n columns with a_i's, followed by m columns with b_j's, the mn rows being filled out with zeros. Its entries are given by (58) for $d = 1$. Here we will only display $\mathcal{R}_1$ for the case that $m < n$.

$$\mathcal{R}_1 = \begin{bmatrix}
a_1 & & & & & & & & & & & b_1 & & & & & & & & \\
a_2 & a_1 & & & & & & & & & & b_2 & b_1 & & & & & & & \\
a_3 & 0 & a_1 & & & & & & & & & b_3 & 0 & b_1 & & & & & & \\
a_4 & a_2 & 0 & a_1 & & & & & & & & b_4 & b_2 & 0 & b_1 & & & & & \\
a_5 & 0 & 0 & 0 & a_1 & & & & & & & b_5 & 0 & 0 & 0 & b_1 & & & & \\
a_6 & a_3 & a_2 & 0 & 0 & a_1 & & & & & & b_6 & b_3 & b_2 & 0 & 0 & b_1 & & & \\
a_7 & 0 & 0 & 0 & 0 & 0 & a_1 & & & & & b_7 & 0 & 0 & 0 & 0 & 0 & b_1 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
a_m & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & & & b_m & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & a_m & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & b_n & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & a_m & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & & & \vdots & & & & b_n & & & & & \vdots \\
 & & & & & & & & & & a_m & & & & & & & & & b_n
\end{bmatrix},$$

The lowest nonzero entry in each one of the first n columns is a_m, and the lowest nonzero entry in each one of the following m columns is b_n. We also note that in a column c_j with $j \in \{1, \ldots, n\}$ the number of zeros between a_i and a_{i+1} is equal to $j-1$, while in a column c_j with $j \in \{n+1, \ldots, n+m\}$ the number of zeros between b_i and b_{i+1} is equal to $j-n-1$.

In comparison to the case of univariate polynomials, where we have to compute the resultant of $f(X)$ and $g(X)$ to see if they are relatively prime, in the case of Dirichlet polynomials we have to compute the rank of $\mathcal{R}_1$, which seems to be a more involved task.

For a Dirichlet polynomial f of order m with complex coefficients a_i, we may let g to be $f^{(k)}$, one of the higher derivatives of f, and denote the corresponding matrices $\mathcal{R}_d$ (with $d \mid m$) by $\mathcal{D}_d^{(k)}$, with entries $d_{i,j}^{(k)}$ given by

$$d_{i,j}^{(k)} = \begin{cases} a_{\frac{i}{j}}, & \text{if } j \mid i \text{ and } j \in \{1, \ldots, \frac{m}{d}\}, \\ 0, & \text{if } j \nmid i \text{ and } j \in \{1, \ldots, \frac{m}{d}\}, \\ (-1)^k a_{\frac{i}{j-\frac{m}{d}}} \log^k(\frac{i}{j-\frac{m}{d}}), & \text{if } j - \frac{m}{d} \mid i \text{ and } j \in \{\frac{m}{d}+1, \ldots, \frac{2m}{d}\}, \\ 0, & \text{if } j - \frac{m}{d} \nmid i \text{ and } j \in \{\frac{m}{d}+1, \ldots, \frac{2m}{d}\}, \end{cases} \tag{59}$$

with the convention that $a_{\mathfrak{i}} = 0$ for $\mathfrak{i} > m$. In comparison to the case of a univariate polynomial, where we need to compute its discriminant to see if it has repeated factors, for a Dirichlet polynomial we need to compute the rank of its associated matrix $\mathcal{D}_1^{(1)}$. In this respect, as an immediate consequence of Theorem 7.21 we have the following square-free criterion.

Corollary 7.22. *Let $f(s) = \frac{a_1}{1^s} + \cdots + \frac{a_m}{m^s}$ be an algebraically primitive Dirichlet polynomial of degree m with complex coefficients, with m not square-free, and let d be a divisor of m. Then f and f' have no common factors of degree $\geq d$ if and only if $\operatorname{rank}(\mathcal{D}_d^{(1)}) = \frac{2m}{d}$. In particular, f is square-free if and only if $\operatorname{rank}(\mathcal{D}_1^{(1)}) = 2m$.*

Proof: It remains to prove that f is square-free if and only if f and f' are relatively prime. Indeed, if $f = g^2 h$ for some Dirichlet polynomials g and h with g nonconstant, the g will be also a factor of f'. Conversely, assume that f and f' share a nonconstant irreducible factor g, say $f = gh$ and $f' = g\ell$. This yields $g'h + gh' = g\ell$, so $g'h = g(\ell - h')$. Since g is irreducible, to conclude that $g \mid h$, we must prove that g cannot divide g'. Let $g(s) = \frac{b_1}{1^s} + \cdots + \frac{b_k}{k^s}$, say, with $k > 1$ a divisor of m and $b_k \neq 0$. If we assume to the contrary that $g'(s) = ag(s)$ for some complex number a, after equating the coefficients we deduce that $-b_i \log i = ab_i$ for each $i = 1, \ldots, k$. These equalities can hold only if $a = -\log k$ and $b_1 = \cdots = b_{k-1} = 0$, so only if $g(s) = \frac{b_k}{k^s}$, and actually with k prime, as g was assumed to be irreducible. But this contradicts our assumption that f is algebraically primitive, so g must divide h, and hence $g^2 \mid f$. □

More generally, the ranks of the matrices $\mathcal{D}_1^{(k)}$ provide information on the maximal multiplicity of the irreducible factors of a Dirichlet polynomial, as follows.

Corollary 7.23. *Let $f(s) = \frac{a_1}{1^s} + \cdots + \frac{a_m}{m^s}$ be a Dirichlet polynomial of degree m with complex coefficients, and assume that $M := \max_{p|m} \nu_p(m) > 1$. If $\operatorname{rank}(\mathcal{D}_1^{(k)}) = 2m$ for some positive integer $k < M$, then the maximal multiplicity of the irreducible factors of f is at most k.*

Computing the ranks of the matrices $\mathcal{D}_d^{(k)}$, with entries $d_{i,j}^{(k)}$ given by (59), requires testing the nonvanishing of some nonlinear forms in logarithms of natural numbers. This ultimately reduces to test the nonvanishing of some nonlinear forms in logarithms of prime numbers, which is by no means an easy task, not even in the simplest case when the coefficients a_i are rational numbers.

8. Logarithmic Newton polytopes and irreducibility criteria for multivariate Dirichlet polynomials

Many foundational results on multiple Dirichlet series appeared in the last decades, with applications in the study of moments of zeta and L-functions, and also in areas related to combinatorics and representation theory. For some of these results we refer the reader to Brubaker, Bump and Friedberg [27], [28], [29], Diaconu, Goldfeld and Hoffstein [42], and Chinta and Gunnels [35], for instance. For some recent results on the residues of quadratic Weyl group multiple Dirichlet series we refer the reader to Diaconu, Ion, Paşol and Popa [41], and for recent developments on the study of moments of quadratic L-functions to Bergström, Diaconu, Petersen and Westerland [4].

To study the factorization of multivariate Dirichlet polynomials, an intuitive approach is to search for connections with the intensively studied geometry of polytopes (see Ewald [46], Grünbaum [53], Webster [103], Ziegler [107] and Schneider [96] for some standard texts). In this section we will use a special type of polytopes that are most suitable to study the factorization of multivariate Dirichlet polynomials, namely logarithmic convex polytopes, that we will briefly call *log-polytopes*. Logarithmic convex hulls are used for instance in the study of logarithmically convex Reinhardt domains (Abhyankar [1]). For applications of logarithmic convexity the reader is referred to Boyd and Vandenberghe [23].

Some notations and definitions are in order.

Definition 8.1. For a field K, we will denote by $DP(K)[s_1, \ldots, s_n]$ the ring of Dirichlet multivariate polynomials with coefficients in K and indeterminates $s_1, \ldots, s_n$. A nonzero such Dirichlet multivariate polynomial f is a finite sum of the form

$$f(s_1, \ldots, s_n) = \sum_{(i_1,\ldots,i_n)\in S_f} \frac{a_{i_1,\ldots,i_n}}{i_1^{s_1} \cdots i_n^{s_n}} \tag{60}$$

with $a_{i_1,\ldots,i_n} \in K \setminus \{0\}$, and with the finite set $S_f \in \mathbb{N}^{*n}$ called the *support* of f.

The *(total) degree* of f, and the *degree of f with respect to s_j* are defined as

$$\deg f = \max_{(i_1,\ldots,i_n)\in S_f} i_1 \cdots i_n \quad \text{and} \quad \deg_j f = \max_{(i_1,\ldots,i_n)\in S_f} i_j,$$

respectively, so we have the inequality $\deg f \leq \prod_{j=1}^n \deg_j f$.

We say that a multivariate Dirichlet polynomial f is *algebraically primitive* if it has no non-constant factor with only one term, that is no factor of the form $\frac{c}{i_1^{s_1}\cdots i_n^{s_n}}$ with $c\neq 0$ and at least one i_j greater than 1.

Unless otherwise specified, we will work over an algebraically closed field K. To simplify notations, we will use natural logarithms, but as in the univariate case, we might as well consider an arbitrary base $\mathfrak{b}>1$ for our logarithms, and this would not affect the study of the factorization properties of f. Since there is no risk of confusion with the definition of log-integral points in Section 3, we will also use the term log-integral in a different setting.

Definition 8.2. A point $v=(v_1,\ldots,v_n)\in\mathbb{R}^n$ is called *log-integral* if its coordinates v_i are of the form $\log k_i$ for some positive integers $k_1,\ldots,k_n$, respectively. A convex polytope in $\mathbb{R}^n$ is called *log-integral* if all of its vertices are log-integral. Thus a *log-integral* polytope will be the convex hull of a finite set S of log-integral points in $\mathbb{R}^n$, so if $S=\{x_1,\ldots,x_k\}\subseteq\mathbb{R}^n$ with $x_1,\ldots,x_k$ log-integral, then the log-integral polytope of S is

$$P^{log}(S)=\left\{\sum_{i=1}^{k}\lambda_i x_i:\lambda_i\in[0,1],\ \lambda_1+\cdots+\lambda_k=1\right\}.$$

Let again $f(s_1,\ldots,s_n)$ be as in (60). To each vector $(i_1,\ldots,i_n)\in S_f$ we will associate its *log-integral companion* $(\log i_1,\ldots,\log i_n)\in\mathbb{R}^n$, and the set of all these log-integral points will be denoted by S_f^{log}, and referred to as the *logarithmic support* of f.

The *log-integral Newton polytope* of f (briefly *the Newton log-polytope* of f) is the convex hull of S_f^{log} and will be denoted by $NP^{log}(f)$. Thus, if $S_f^{log}=\{v_1,\ldots,v_m\}$, say, then

$$NP^{log}(f)=\left\{\sum_{i=1}^{m}\lambda_i v_i:\lambda_i\in[0,1],\ \lambda_1+\cdots+\lambda_m=1\right\}.$$

The definition of the Minkowski sum of two polytopes applies to log-integral polytopes as well, so for two multivariate Dirichlet polynomials g,h having the same indeterminates, one may consider the set $NP^{log}(g)+NP^{log}(h)=\{x+y:x\in NP^{log}(g),\ y\in NP^{log}(h)\}$. With these notations we have the following analogue of Ostrowski's Theorem on Newton polytopes for multivariate polynomials [83] (see also [84]).

Theorem 8.3. *Let f,g,h be multivariate Dirichlet polynomials such that $f=g\cdot h$. Then $NP^{log}(f)=NP^{log}(g)+NP^{log}(h)$.*

Proof: Our proof adapts the ideas in the proof of Ostrowski' s Theorem given in [50]. Let

$$\begin{aligned}
f(s_1,\ldots,s_n) &= \sum_{(i_1,\ldots,i_n)\in S_f}\frac{a_{i_1,\ldots,i_n}}{i_1^{s_1}\cdots i_n^{s_n}},\\
g(s_1,\ldots,s_n) &= \sum_{(j_1,\ldots,j_n)\in S_g}\frac{b_{j_1,\ldots,j_n}}{j_1^{s_1}\cdots j_n^{s_n}},\ \text{and}\\
h(s_1,\ldots,s_n) &= \sum_{(k_1,\ldots,k_n)\in S_h}\frac{c_{k_1,\ldots,k_n}}{k_1^{s_1}\cdots k_n^{s_n}}
\end{aligned}$$

be our multivariate Dirichlet polynomials with nonzero coefficients $a_{i_1,\dots,i_n}$, $b_{j_1,\dots,j_n}$, $c_{k_1,\dots,k_n}$, indeterminates $s_1,\dots,s_n$ and support sets S_f, S_g and S_h, respectively. Let us assume that $S_f^{log} = \{u_1,\dots,u_{d_1}\}$, $S_g^{log} = \{v_1,\dots,v_{d_2}\}$ and $S_h^{log} = \{w_1,\dots,w_{d_3}\}$. A vector $x \in NP^{log}(f)$ may be written as

$$x = \sum_{i=1}^{d_1} \lambda_i u_i \quad \text{with nonnegative } \lambda_i \text{ and } \sum_{i=1}^{d_1} \lambda_i = 1,$$

where each u_i is uniquely written as $u_i = (\log i_1,\dots,\log i_n)$ for a certain $(i_1,\dots,i_n) \in S_f$. By the multiplication rule for Dirichlet polynomials, our assumption that $f = g \cdot h$ shows that each term $\frac{1}{i_1^{s_1}\cdots i_n^{s_n}}$ appearing (multiplied by $a_{i_1,\dots,i_n}$) in the sum defining f must be written as a product $(\frac{1}{j_1^{s_1}\cdots j_n^{s_n}}) \cdot (\frac{1}{k_1^{s_1}\cdots k_n^{s_n}})$ for at least one pair of terms $(\frac{1}{j_1^{s_1}\cdots j_n^{s_n}}, \frac{1}{k_1^{s_1}\cdots k_n^{s_n}})$ with $(j_1,\dots,j_n) \in S_g$ and $(k_1,\dots,k_n) \in S_h$, so $(\log i_1,\dots,\log i_n) = (\log j_1,\dots,\log j_n) + (\log k_1,\dots,\log k_n)$. This shows that every convex combination of the u_i's may be written as the sum of a convex combination of the v_j's and a convex combination of the w_k's, proving the inclusion $NP^{log}(f) \subseteq NP^{log}(g) + NP^{log}(h)$.

To prove the other inclusion, let us consider a vertex x of $NP^{log}(g) + NP^{log}(h)$, so $x = y + z$ for some $y \in NP^{log}(g)$ and $z \in NP^{log}(h)$. We will first prove that the pair (y, z) is uniquely determined. Assume to the contrary that $x = y + z = y' + z'$ for some $y' \in NP^{log}(g)$ and $z' \in NP^{log}(h)$. Then we also have $x = \frac{1}{2}(y + z') + \frac{1}{2}(y' + z)$. Since this is a convex combination of the points $y + z'$ and $y' + z$, both belonging to $NP^{log}(g) + NP^{log}(h)$, and x was assumed to be a vertex, we must actually have $y + z' = y' + z$, which forces $y - y' = y' - y$ and $z - z' = z' - z$, or equivalently $y = y'$ and $z = z'$. As x is a vertex of $NP^{log}(g) + NP^{log}(h)$, we see now that the uniquely determined points y and z must be vertices of $NP^{log}(g)$ and $NP^{log}(h)$, respectively. Thus $y = (\log j_1,\dots,\log j_n)$ and $z = (\log k_1,\dots,\log k_n)$ for some $(j_1,\dots,j_n) \in S_g$ and $(k_1,\dots,k_n) \in S_h$. This further shows that the product $(\frac{1}{j_1^{s_1}\cdots j_n^{s_n}}) \cdot (\frac{1}{k_1^{s_1}\cdots k_n^{s_n}})$ must appear (multiplied by a nonzero constant) in the sum defining f, so it must be of the form $\frac{1}{i_1^{s_1}\cdots i_n^{s_n}}$ for some $(i_1,\dots,i_n) \in S_f$, which finally shows that $x \in S_f^{log} \subseteq NP^{log}(f)$. As any convex combination of the vertices of $NP^{log}(g) + NP^{log}(h)$ must also belong to $NP^{log}(f)$, we conclude that $NP^{log}(g) + NP^{log}(h) \subseteq NP^{log}(f)$, which completes the proof. □

Theorem 8.3 suggests the use of the following definition.

Definition 8.4. A log-integral polytope is *log-integrally decomposable* if it can be written as the sum of two log-integral polytopes each one containing at least two points, and *log-integrally indecomposable* otherwise. We also mention here that a multivariate Dirichlet polynomial is called absolutely irreducible, if it is irreducible over an algebraically closed field K.

This definition allows one to transform Theorem 8.3 into an absolute irreducibility criterion for multivariate Dirichlet polynomials, that may be regarded as an analogue of an absolute irreducibility criterion of Gao [50, p. 507] for multivariate polynomials:

Theorem 8.5. *If f is algebraically primitive and $NP^{log}(f)$ is log-integrally indecomposable, then f is absolutely irreducible.*

Proof: We first note that f has no nonconstant factors with only one term, as it was assumed agebraically primitive. Thus, if we suppose now that f factors as a product of two nonconstant multivariate Dirichlet polynomials g and h, then each one of g and h should have at least two terms, which by Theorem 8.3 would force $NP^{log}(f)$ to be log-integrally decomposable, a contradiction. □

We will prove now a result that may be regarded as a logarithmic analogue of Lemma 4.1 in [50], which will describe all the log-integral points belonging to the convex hull of a set of two different log-integral points in $\mathbb{R}^n$. We will first need some additional terminology. Let $\mathbf{vw}$ be any segment with log-integral endpoints $\mathbf{v} = (\log a_1, \dots, \log a_n)$ and $\mathbf{w} = (\log b_1, \dots, \log b_n)$ in $\mathbb{R}^n$, and for each $i = 1, \dots, n$ let $d_i = \gcd\{v_p(b_i) - v_p(a_i) : p \text{ prime}, \ p \mid a_i \cdot b_i\}$. Let us define

$$\overline{\gcd}(\mathbf{vw}) = \overline{\gcd}(\mathbf{w} - \mathbf{v}) = \gcd(d_1, \dots, d_n). \tag{61}$$

Note that $\mathbf{w} - \mathbf{v}$ might no longer be a log-integral point in $\mathbb{R}^n$. Moreover, for any k segments $\mathbf{v_1w_1}, \dots, \mathbf{v_kw_k}$ with log-integral endpoints, we define

$$\overline{\gcd}(\mathbf{v_1w_1}, \dots, \mathbf{v_kw_k}) = \overline{\gcd}(\mathbf{w_1} - \mathbf{v_1}, \dots, \mathbf{w_k} - \mathbf{v_k}) = \gcd(\overline{\gcd}(\mathbf{v_1w_1}), \dots, \overline{\gcd}(\mathbf{v_kw_k})).$$

Lemma 8.6. *Let $\mathbf{v} = (\log a_1, \dots, \log a_n)$ and $\mathbf{w} = (\log b_1, \dots, \log b_n)$ be two different log-integral points in $\mathbb{R}^n$. Then the log-integral points belonging to the line segment $\mathbf{vw}$ are*

$$\mathbf{u}_i = \left(\log\left(a_1^{1-\frac{i}{d}} b_1^{\frac{i}{d}}\right), \dots, \log\left(a_n^{1-\frac{i}{d}} b_n^{\frac{i}{d}}\right)\right), \quad i = 0, \dots, d,$$

with $d = \overline{\gcd}(\mathbf{vw})$. Moreover, if $\mathbf{u_i}$ is such a log-integral point, then $\frac{|\mathbf{u_i} - \mathbf{v}|}{|\mathbf{w} - \mathbf{v}|} = \frac{\overline{\gcd}(\mathbf{vu_i})}{\overline{\gcd}(\mathbf{vw})} = \frac{i}{d}$, where $|\mathbf{x}|$ denotes the Euclidean length of the vector $\mathbf{x}$.

Proof: A point $\mathbf{u}$ belonging to the line segment $\mathbf{vw}$ has the form $(1-t)\mathbf{v} + t\mathbf{w}$ for some real number $t \in [0, 1]$. For $\mathbf{u}$ to be log-integral we need $a_i^{1-t} b_i^t$ to be integer for each $i = 1, \dots, n$. Let us fix an index $i \in \{1, \dots, n\}$ for which $a_i \neq b_i$, and assume that $p_1, \dots, p_k$ are all the prime factors of $a_i \cdot b_i$, so we may write

$$a_i^{1-t} b_i^t = p_1^{(1-t)v_{p_1}(a_i) + tv_{p_1}(b_i)} \cdots p_k^{(1-t)v_{p_k}(a_i) + tv_{p_k}(b_i)},$$

where for each index j at most one of the multiplicities $v_{p_j}(a_i)$ and $v_{p_j}(b_i)$ might be zero. Thus, for $a_i^{1-t} b_i^t$ to be an integer, we need $t \cdot (v_{p_j}(b_i) - v_{p_j}(a_i))$ to be an integer for each $j = 1, \dots, k$ for which $v_{p_j}(a_i) \neq v_{p_j}(b_i)$, which shows in particular that t must be rational, say $t = \frac{r}{s} \in [0, 1]$, with r and s coprime if $t \neq 0$ and $t \neq 1$. Thus, we need s to divide $v_{p_j}(b_i) - v_{p_j}(a_i)$ for each $j = 1, \dots, k$, as r and s are coprime, or equivalently, we need s to divide d_i. Since this must hold for each i, s must be a divisor of d. The conclusion follows by observing that the set of all quotients $\frac{r}{s} \in [0, 1]$ with s a divisor of d and $\gcd(r, s) = 1$ if $r \neq 0$ and $r \neq s$, is in fact the set of all the quotients $\frac{i}{d}$ with $i \in \{0, \dots, d\}$.

For the second statement, observe that $\mathbf{u_i} = (1 - \frac{i}{d})\mathbf{v} + \frac{i}{d}\mathbf{w}$, so $\mathbf{u_i} - \mathbf{v} = \frac{i}{d}(\mathbf{w} - \mathbf{v})$. Thus, it remains to prove that $\overline{\gcd}(\mathbf{vu_i}) = i$. To do this, we will apply the definition of $\overline{\gcd}$ with $\mathbf{u_i}$ instead of $\mathbf{w}$. For each $j = 1, \ldots, n$ let

$$\begin{aligned} d'_j &= \gcd\Big\{v_p(a_j^{1-\frac{i}{d}} b_j^{\frac{i}{d}}) - v_p(a_j) : p \text{ prime},\ p \mid a_j \cdot b_j\Big\} \\ &= \gcd\Big\{\frac{i}{d}(\nu_p(b_j) - \nu_p(a_j)) : p \text{ prime},\ p \mid a_j \cdot b_j\Big\}, \end{aligned}$$

with $d = \overline{\gcd}(\mathbf{vw})$ given by (61). According to the definition, $\overline{\gcd}(\mathbf{vu_i}) = \gcd(d'_1, \ldots, d'_n)$. Recall now that d is the greatest common divisor of all the differences of the form $\nu_p(b_j) - \nu_p(a_j)$ obtained when j runs in $\{1, \ldots, n\}$ and for a given j, p runs in the set of all the primes dividing $a_j \cdot b_j$. Therefore, dividing all these differences by d results in a set of coprime integers, which in view of the expressions of d'_j above, shows that $\gcd(d'_1, \ldots, d'_n) = i$, as claimed. □

In particular, from Lemma 8.6 we obtain the following corollary.

Corollary 8.7. *Let* $\mathbf{v} = (\log a_1, \ldots, \log a_n)$ *and* $\mathbf{w} = (\log b_1, \ldots, \log b_n)$ *be two different log-integral points in* $\mathbb{R}^n$*, and let* $d_i = \gcd\{v_p(a_i) - v_p(b_i) : p \text{ prime},\ p \mid a_i \cdot b_i\}$ *for* $i = 1, \ldots, n$*. If* $\gcd(d_1, \ldots, d_n) = 1$*, then the line segment* $\mathbf{vw}$ *is log-integrally indecomposable.*

Proof: By Lemma 8.6 we see that the only log-integral points belonging to the line segment $\mathbf{vw}$ are $\mathbf{v}$ and $\mathbf{w}$. Let us assume to the contrary that $\mathbf{vw} = P^{log}(S_1) + P^{log}(S_2)$ for two finite sets of log-integral points $S_1, S_2 \subset \mathbb{R}^n$, each one containing at least two points, say $X_1, X_2 \in S_1$ and $Y_1, Y_2 \in S_2$, with $X_1 \neq X_2$ and $Y_1 \neq Y_2$. As $X_1 + Y_1$ and $X_1 + Y_2$ are different log-integral points and both belong to $\{\mathbf{v}, \mathbf{w}\}$, we may assume without loss of generality that

$$X_1 + Y_1 = \mathbf{v} \quad \text{and} \quad X_1 + Y_2 = \mathbf{w}. \tag{62}$$

Now, since $X_2 + Y_1$ and $X_2 + Y_2$ are also different log-integral points belonging to $\{\mathbf{v}, \mathbf{w}\}$, we can neither have $X_2 + Y_1 = \mathbf{v}$ and $X_2 + Y_2 = \mathbf{w}$, as $X_1 \neq X_2$, nor $X_2 + Y_1 = \mathbf{w}$ and $X_2 + Y_2 = \mathbf{v}$, for this will lead us after subtracting these two relations from the relations in (62) to $X_1 - X_2 = \mathbf{v} - \mathbf{w}$ and $X_1 - X_2 = \mathbf{w} - \mathbf{v}$, which cannot hold simultaneously, as $\mathbf{v} \neq \mathbf{w}$. Thus the line segment $\mathbf{vw}$ must be log-integrally indecomposable. □

Ostrowski [84] obtained the following irreducibility criterion for multivariate polynomials:

Theorem (Ostrowski) *A two-term polynomial*

$$aX_1^{i_1} \cdots X_k^{i_k} + bX_{k+1}^{i_{k+1}} \cdots X_n^{i_n} \in K[X_1, \ldots, X_n], \quad a, b \in K \setminus \{0\},$$

is absolutely irreducible iff $\gcd(i_1, \ldots, i_n) = 1$.

Combining now Theorem 8.5 and Corollary 8.7, we easily obtain a criterion of absolute irreducibility for multivariate Dirichlet polynomials consisting of two terms, that is similar to Ostrowski's irreducibility criterion above.

Theorem 8.8. *Let K be an algebraically closed field and $f(s_1,\dots,s_n)=\frac{a}{a_1^{s_1}\cdots a_n^{s_n}}+\frac{b}{b_1^{s_1}\cdots b_n^{s_n}}$, with $a,b\in K$, $ab\neq 0$, and a_i coprime to b_i for each $i\in\{1,\dots,n\}$. Then f is absolutely irreducible if and only if the multiplicities of all the primes that appear in the prime factorizations of $a_1,\dots,a_n,b_1,\dots,b_n$ are relatively prime.*

Proof: As a_i is coprime to b_i for each i, f is algebraically primitive, and any term of the form $v_p(a_i)-v_p(b_i)$ with p a prime dividing $a_i\cdot b_i$ will either be equal to $v_p(a_i)$, or to $-v_p(b_i)$. Our assumption that all these multiplicities are coprime leads via Corollary 8.7 to the conclusion that the convex hull of $\{(\log a_1,\dots,\log a_n),(\log b_1,\dots,\log b_n)\}$, which is $NP^{log}(f)$, must be log-integrally indecomposable. This shows by Theorem 8.5 that f must be absolutely irreducible. Conversely, let us assume that the greatest common divisor of the multiplicities of all the primes that appear in the prime factorizations of $a_1,\dots,a_n,b_1,\dots,b_n$, say d, is greater than 1. Then each of the integers $a_1,\dots,a_n,b_1,\dots,b_n$ is a d-power of some positive integer, say $a_i=\alpha_i^d$ and $b_i=\beta_i^d$ for $i=1,\dots,n$, and moreover, α_i and β_i are coprime for each i. On the other hand, as K is algebraically closed, there exist nonzero elements $\alpha,\beta\in K$ such that $a=\alpha^d$ and $-b=\beta^d$. Then f must be reducible, being divisible by $\alpha\cdot\alpha_1^{s_1}\cdots\alpha_n^{s_n}-\beta\cdot\beta_1^{s_1}\cdots\beta_n^{s_n}$ (which is a nonzero multivariate Dirichlet polynomial, as α_i and β_i are coprime for each i). □

Gao proved the following elegant criterion for integrally indecomposability of polytopes.

Theorem [50, Theorem 4.2]. *Let Q be any integral polytope in $\mathbb{R}^n$ contained in a hyperplane H and let $\mathbf{v}\in\mathbb{R}^n$ be an integral point lying outside of H. Suppose that $\mathbf{v_1},\dots,\mathbf{v_k}$ are all the vertices of Q. Then the polytope* $\mathrm{conv}(v,Q)$ *is integrally indecomposable if and only if* $\gcd(\mathbf{v}-\mathbf{v_1},\dots,\mathbf{v}-\mathbf{v_k})=1$.

A key ingredient in the proof of this result is the following lemma that counts integral points on a segment with integral endpoints.

Lemma [50, Lemma 4.1]. *Let $\mathbf{v_0}$ and $\mathbf{v_1}$ be two integral points in $\mathbb{R}^n$. Then the number of integral points on the line segment $\mathbf{v_0v_1}$, including $\mathbf{v_0}$ and $\mathbf{v_1}$, is equal to* $\gcd(\mathbf{v_0}-\mathbf{v_1})+1$. *Further, if $\mathbf{v_2}$ is any integral point on $\mathbf{v_0v_1}$, then $\frac{|\mathbf{v_2}-\mathbf{v_0}|}{|\mathbf{v_1}-\mathbf{v_0}|}=\frac{\gcd(\mathbf{v_2}-\mathbf{v_0})}{\gcd(\mathbf{v_1}-\mathbf{v_0})}$, where $|\mathbf{v}|$ denotes the Euclidean length of the vector $\mathbf{v}$.*

A logarithmic analogue of Gao's Theorem is the following result.

Theorem 8.9. *Let Q be any log-integral polytope in $\mathbb{R}^n$ contained in a hyperplane H, and let $\mathbf{v}\in\mathbb{R}^n$ be a log-integral point lying outside of H. Suppose that $\mathbf{v_1},\dots,\mathbf{v_k}$ are all the vertices of the log-integral polytope Q. Then the polytope* $\mathrm{conv}(v,Q)$ *is log-integrally indecomposable if and only if* $\overline{\gcd}(\mathbf{v}-\mathbf{v_1},\dots,\mathbf{v}-\mathbf{v_k})=1$.

Proof: The proof follows the same lines used in the proof of Gao's Theorem, with "integral" replaced by "log-integral", gcd replaced by $\overline{\gcd}$, and the above lemma replaced by Lemma 8.6. The details will be omitted. □

The reader may combine Theorem 8.5 and Theorem 8.9 to obtain criteria for absolute irreducibility for multivariate Dirichlet polynomials whose Newton log-integral polytopes have more than two vertices. Other criteria for log-integrally indecomposability may be obtained by adapting the remaining results of Gao in [50] for multivariate polynomials, for instance. For an algorithm for testing indecomposability we refer the reader to Gao and Lauder [51], and for absolute irreducibility modulo large prime numbers to Gao and Rodrigues [52].

9. Logarithmic upper Newton polygons and Stepanov-Schmidt type results

In the case of multivariate polynomials, apart from Newton polytopes, another geometrical object useful in their study is the *upper Newton polygon*, used by Stepanov and Schmidt [95] (see also Gao [50]) to prove the following absolute irreducibility criterion for bivariate polynomials.

The Stepanov-Schmidt criterion. *Let K be a field and let $f \in K[X,Y]$ with degree n in X. If the upper Newton polygon of f with respect to Y has only one line segment from $(0,m)$ to $(n,0)$ and $\gcd(m,n)=1$, then f is absolutely irreducible over K.*

Here the upper Newton polygon of $f(X,Y)=a_0(Y)+a_1(Y)X+\cdots+a_n(X)X^n$ is the upper convex hull of the set of points $(0,\deg a_0),(1,\deg a_1),\dots,(n,\deg a_n)$.

In this section we will construct a logarithmic analogue for algebraically primitive multivariate Dirichlet polynomials of the upper Newton polygon for multivariate polynomials, which we will call the *logarithmic upper Newton polygon*, or briefly the *upper Newton log-polygon*. We will fix again an arbitrarily chosen real number $\mathfrak{b}$ greater than 1, which will be the base our logarithms will be considered to, and for a given base $\mathfrak{b}$, we will call *log-integral* any point having coordinates $(\log_{\mathfrak{b}} x,\ \log_{\mathfrak{b}} y)$ with x,y positive integers. As we shall see later, in this case too the base $\mathfrak{b}$ will be in some sense irrelevant, so to simplify notations we will choose again to work with natural logarithms, keeping in mind that the construction of the upper Newton log-polygon below can be done using any base $\mathfrak{b}>1$.

Now let K be a field, and $f(s_1,\dots,s_t)$ be an algebraically primitive multivariate Dirichlet polynomial in $t\geq 2$ indeterminates $s_1,\dots,s_t$ and coefficients in K. Fix one of the indeterminates, say s_t, and denote it by s, then write f as $f(s)=\sum_{i=n'}^{n}\frac{a_i}{i^s}$ with a_i multivariate Dirichlet polynomials with indeterminates $s_1,\dots,s_{t-1}$ and coefficients in K, $a_{n'}a_n\neq 0$. As f is algebraically primitive, we have $1\leq n'<n$. Let us fix another indeterminate, say s_r with $r\in\{1,\dots,t-1\}$, and then let us consider the degrees of the coefficients a_i regarded as Dirichlet polynomials in indeterminate s_r and coefficients Dirichlet polynomials in the remaining indeterminates. Denote these degrees by $\deg_r a_i$, and to each term $\frac{a_i}{i^s}$ of f with $a_i\neq 0$ let us associate a point in plane with coordinates $(\log i,\log\deg_r a_i)$. The *upper* convex hull of all of these points will be called the *upper Newton log-polygon* of f with respect to indeterminate s_r.

At this point we have to mention that in the case of our multivariate Dirichlet polynomials, when applied to the coefficients a_i, $-\log\deg_r(\cdot)$ has the properties of a valuation, as in the case of $\nu_p(\cdot)$, that was used in the construction of the Newton log-polygon in Section 3. In this

respect one might use the *lower* convex hull of the set of points $(\log i, -\log \deg_r a_i)$, to derive results similar to those in Section 3. However, to avoid carrying the minus sign, we will prefer to use the upper convex hull of the set of points $(\log i, \log \deg_r a_i)$. More precisely, this upper convex hull is obtained as follows: Let $i_1 = n'$, $P_1 = (\log i_1, \log \deg_r a_{i_1}) = (\log n', \log \deg_r a_{n'})$, and let $P_2 = (\log i_2, \log \deg_r a_{i_2})$ with i_2 the largest integer with the property that there are no points $(\log i, \log \deg_r a_i)$ lying above the line passing through P_1 and P_2. Then let $P_3 = (\log i_3, \log \deg_r a_{i_3})$, with i_3 the largest integer such that there are no points $(\log i, \log \deg_r a_i)$ lying above the line passing through P_2 and P_3, and so on. The last line segment built in this way will be $P_{v-1}P_v$, with $P_v = (\log n, \log \deg_r a_n)$. The broken line $P_1 \dots P_v$ thus constructed is the upper Newton log-polygon of f with respect to the indeterminate s_r. We will refer to the log-integral points $P_1, \dots, P_v$ as the *vertices* of the upper Newton log-polygon of f. We note that when we construct this upper convex hull, the missing terms in the sum defining f, that is the terms $\frac{a_i}{i^s}$ with $a_i = 0$, are irrelevant, as we can formally consider them as having $-\infty$ for their y-coordinate (in view of the convention that the zero Dirichlet polynomial has degree 0).

The line segments P_lP_{l+1} will be called *edges*. An edge P_lP_{l+1} might pass through some log-integral points, other than its endpoints P_l and P_{l+1}. These points might have coordinates of the form $(\log i, \log y_i)$ with $i, y_i \in \mathbb{Z}_+$ and y_i not necessarily equal to $\deg_r a_i$. Let us denote all these intermediate points by $Q_1, \dots, Q_k$, say. Then P_lQ_1, Q_1Q_2, $\dots$, $Q_{k-1}Q_k$ and Q_kP_{l+1} will be called *segments* of the upper Newton log-polygon. Thus, as in the case of Newton log-polygons, a segment will contain no log-integral points other than its endpoints, and an edge consists of a number of segments, possibly just one, in which case the edge itself will be called a segment. Note that any two edges must have different slopes, while two segments that belong to the same edge will obviously have the same slope.

The vectors $\overline{P_iP_{i+1}}$ will be called *vectors* of the edges of the upper Newton log-polygon, and *the vector system* of the upper Newton log-polygon will be the union of all the vectors of its edges, repetitions allowed.

At the cost of a translation on the x-axis, this construction too may be also used for multivariate Dirichlet polynomials that are not algebraically primitive. The use of logarithms brings here too more geometric insight, while arithmetical conditions lead to effective results on the factorization of these multivariate Dirichlet polynomials. The construction of the vertices $P_1, \dots, P_r$ and the identification of all the intermediate log-integral points lying on the edges $P_1P_2, \dots, P_{v-1}P_v$ depends now on some more sophysticated "log-diophantine" inequalities and equations. Thus, if we choose two indices i and j with $i < j$, we see that a log-integral point $(\log k, \log \deg_r a_k)$ lies below or on the line passing through the log-integral points $(\log i, \log \deg_r a_i)$ and $(\log j, \log \deg_r a_j)$ if and only if

$$\deg_r a_k \leq (\deg_r a_i)^{\log_{\frac{j}{i}} \frac{j}{k}} (\deg_r a_j)^{1-\log_{\frac{j}{i}} \frac{j}{k}}, \tag{63}$$

while a log-integral point $(\log k, \log y)$ with y and k integers with $i < k < j$, lies on this line if and only if

$$y = (\deg_r a_i)^{\log_{\frac{j}{i}} \frac{j}{k}} (\deg_r a_j)^{1-\log_{\frac{j}{i}} \frac{j}{k}}. \tag{64}$$

We recall that the log-integral points lying on a line segment with endpoints that are log-integral too are described in Lemma 8.6, which will allow us to find explicit irreducibility conditions. We mention here that conditions (63) and (64) also show that the construction of the upper Newton log-polygon doesn't essentially depend on the base of the logarithms that we are using. So changing this base will not affect the intrinsic arithmetic properties of the upper Newton log-polygon that we will rely on when studying the factorization of a multivariate Dirichlet polynomial.

We are now ready to state the following result, that is in some analogous to Theorem 3.1.

Theorem 9.1. *Let f, g, h be nonconstant algebraically primitive multivariate Dirichlet polynomials with $f = g \cdot h$. Then the upper Newton log-polygon of f with respect to an indeterminate s_r is obtained by using translates of the edges of the upper Newton log-polygons of g and h with respect to s_r, using precisely one translate for each edge, so as to form a polygonal line with decreasing slopes, when considered from left to the right.*

In other words, this theorem says that the vector system of the upper Newton log-polygon of f is obtained by considering the union of the vector systems of the upper Newton log-polygons of g and h, repetitions allowed, and then replacing any two vectors having the same orientation by their sum.

Proof: The proof is similar to that of Theorem 3.1, with almost the same phrasing, and only some minor modifications, so some of the details will be omitted.

Let $f(s) = \sum\limits_{i=n'}^{n} \frac{a_i}{i^s}$, $g(s) = \sum\limits_{j=d'}^{d} \frac{b_j}{j^s}$ and $h(s) = \sum\limits_{k=n'/d'}^{n/d} \frac{c_k}{k^s}$ with a_i, b_j and c_k multivariate Dirichlet polynomials in indeterminates $s_1, \ldots, s_r$. Let us consider an edge of the upper Newton log-polygon of f, say $P_l P_{l+1}$ (that may consist of more than a single segment), and let us assume that the points P_l and P_{l+1} have coordinates $(\log i_m, \log \deg_r a_{i_m})$, and $(\log i_M, \log \deg_r a_{i_M})$, respectively, with $i_m < i_M$. The slope of $P_l P_{l+1}$ is

$$S = \frac{\log \deg_r a_{i_M} - \log \deg_r a_{i_m}}{\log i_M - \log i_m}.$$

Let now $\log \deg_r a_{i_M} - \log \deg_r a_{i_m} = A$ and $\log i_M - \log i_m = I$, so $S = \frac{A}{I}$. The edge $P_l P_{l+1}$ of the Schmidt-Stepanov log-polygon of f belongs to the line $Iy - Ax = F$, where

$$F = I \log \deg_r a_{i_M} - A \log i_M = I \log \deg_r a_{i_m} - A \log i_m.$$

According to the definition, all the points $(\log i, \log \deg_r a_i)$, $i = n', \ldots, n$, either lie on this line, or below it, so $I \log \deg_r a_i - A \log i \leq F$ for all i, with a strict inequality for $i < i_m$ and $i > i_M$, and we have equality in the endpoints corresponding to i_m and i_M (and for the points $(\log i, \log \deg_r a_i)$ with $i_m < i < i_M$, if any, any of the cases "<" and "=" being possible).

To each term $\frac{a_i}{i^s}$ we will associate a real number

$$w\left(\frac{a_i}{i^s}\right) := I \log \deg_r a_i - A \log i,$$

called its *weight* with respect to the edge P_lP_{l+1}. We may then regard i_m and i_M as the least and the largest i such that the corresponding term $\frac{a_i}{i^s}$ of f has maximum weight (which is F) with respect to P_lP_{l+1}, so i_m and i_M are uniquely determined. For g we define

$$G := \max_{d'\le j\le d}\{I\log\deg_r b_j - A\log j\}$$

and we will also define j_m and j_M as the least and the largest index, respectively, such that

$$G = I\log\deg_r b_{j_m} - A\log j_m = I\log\deg_r b_{j_M} - A\log j_M. \tag{65}$$

We notice that $j_m \le j_M$. Similarly, for h we define

$$H := \max_{n'/d'\le k\le n/d}\{I\log\deg_r c_k - A\log k\}$$

and we denote by k_m and k_M the least and the largest index, respectively, such that

$$H = I\log\deg_r c_{k_m} - A\log k_m = I\log\deg_r c_{k_M} - A\log k_M. \tag{66}$$

Here too, we have $k_m \le k_M$. By the multiplication rule for Dirichlet series, we have

$$\frac{a_{j_mk_m}}{(j_mk_m)^s} = \sum_{j\cdot k=j_m\cdot k_m}\frac{b_j}{j^s}\cdot\frac{c_k}{k^s}. \tag{67}$$

For two terms that are multiplied in (67) we have $w(\frac{b_j}{j^s}\cdot\frac{c_k}{k^s}) = w(\frac{b_j}{j^s}) + w(\frac{c_k}{k^s})$, so the weight of the summand with $j = j_m$ and $k = k_m$ in (67) is $G + H$, while the weights of the rest of the summands are all less than $G + H$, since for these ones we either have $j < j_m$, or $k < k_m$. Besides, when $j\cdot k$ is constant, the weight of a product $\frac{b_j}{j^s}\cdot\frac{c_k}{k^s}$ regarded as a function on $\log\deg_r b_j + \log\deg_r c_k$ is a strictly increasing function, as $I > 0$. Therefore, since in (67) we have $j\cdot k = j_m\cdot k_m$, hence $j\cdot k$ is constant, the sum $\log\deg_r b_j + \log\deg_r c_k$ (and hence $\deg_r b_jc_k$) has maximum value only for $j = j_m$ and $k = k_m$, for any other pair (j, k) being strictly smaller. Thus $\deg_r a_{j_mk_m} = \deg_r b_{j_m}c_{k_m}$, so

$$w\Big(\frac{a_{j_mk_m}}{(j_mk_m)^s}\Big) = G + H. \tag{68}$$

Reasoning similarly with (67) replaced by $\frac{a_i}{i^s} = \sum_{j\cdot k=i}\frac{b_j}{j^s}\cdot\frac{c_k}{k^s}$, one may prove that

$$w\Big(\frac{a_i}{i^s}\Big) < G + H \qquad \text{for } i < j_m\cdot k_m, \tag{69}$$

$$w\Big(\frac{a_i}{i^s}\Big) \le G + H \qquad \text{for } i \ge j_m\cdot k_m. \tag{70}$$

By (68), (69) and (70), we must have $G + H = F$ and $j_m\cdot k_m = i_m$. Similarly, we also obtain $j_M\cdot k_M = i_M$, leading to $\log i_M - \log i_m = (\log j_M - \log j_m) + (\log k_M - \log k_m)$, with at least one of $\log j_M - \log j_m$ and $\log k_M - \log k_m$ positive. If both $\log j_M - \log j_m$ and $\log k_M - \log k_m$ are nonzero, then the geometric segment with endpoints $(\log j_m, \log\deg_r b_{j_m})$ and $(\log j_M, \log\deg_r b_{j_M})$ must be an edge of the upper Newton log-polygon of g, and the geometric segment with endpoints

$(\log k_m, \log \deg_r c_{k_m})$ and $(\log k_M, \log \deg_r c_{k_M})$ must be an edge of the upper Newton log-polygon of h, the slopes of these two edges being $S = \frac{A}{I}$, since from (65) and (66) one obtains

$$\frac{\log \deg_r b_{j_M} - \log \deg_r b_{j_m}}{\log j_M - \log j_m} = \frac{A}{I} = \frac{\log \deg_r c_{k_M} - \log \deg_r c_{k_m}}{\log k_M - \log k_m}.$$

Thus the sum of the lengths of the edges with slope S in the upper Newton log-polygons of g and h is precisely the length of the edge with slope S in the upper Newton log-polygon of f, and the vector of the edge with slope S in the upper Newton log-polygon of f is the sum of the vectors of the edges with slope S in the upper Newton polygons of g and h. Finally, one may prove that if one of g and h has an edge with slope S in its upper Newton log-polygon, then S must appear among the slopes of the edges in the upper Newton log-polygon of f. Also, the upper Newton log-polygons of g and h cannot have edges with slopes other than the ones appearing in the upper Newton log-polygon of f. This completes the proof. □

In particular, Theorem 9.1 shows that each nonconstant factor of a multivariate Dirichlet polynomial f must contribute at least one edge (that may consist of a single segment) to the upper Newton log-polygon of f. Therefore, we have the following analogue for multivariate Dirichlet polynomials of the Stepanov-Schmidt criterion for bivariate polynomials:

Corollary 9.2. *If with respect to an indeterminate s_r, the upper Newton log-polygon of an algebraically primitive multivariate Dirichlet polynomial f consists of a single edge, and this edge contains no log-integral points other than its endpoints, then f must be irreducible.*

Here too it is crucial to make distinction between edges and segments. It is not sufficient to ask the upper Newton log-polygon of f to have a single edge, for an edge might in principle consist of multiple segments coming from the upper Newton log-polygons of two or more of the nonconstant factors of f. That's why in Corollary 9.2 we had to ask the edge to have no log-integral points other than its endpoints. In order to obtain an effective instance of this corollary, we will make use of Lemma 8.6, as follows.

Theorem 9.3. *Let K be a field, $m < n$ two positive integers and $a_m(t), \ldots, a_n(t)$ Dirichlet polynomials in the indeterminate t over K, with $a_m a_n \neq 0$ and $\deg a_m \neq \deg a_n$. Let also $f(s,t) = \sum_{i=m}^{n} \frac{a_i(t)}{i^s}$, assume that f is algebraically primitive, and let*

$$\begin{aligned} d_1 &= \gcd\{\nu_p(m) - \nu_p(n) : p \text{ prime},\ p \mid m \cdot n\} \text{ and} \\ d_2 &= \gcd\{\nu_p(\deg a_m) - \nu_p(\deg a_n) : p \text{ prime},\ p \mid \deg a_m \cdot \deg a_n\}. \end{aligned}$$

If $\deg a_i < (\deg a_m)^{\log \frac{n}{m} \frac{n}{i}} (\deg a_n)^{1 - \log \frac{n}{m} \frac{n}{i}}$ for $m < i < n$ and $\gcd(d_1, d_2) = 1$, then f is irreducible.

Proof: In view of (63), our hypothesis that $\deg a_i < (\deg a_m)^{\log \frac{n}{m} \frac{n}{i}} (\deg a_n)^{1 - \log \frac{n}{m} \frac{n}{i}}$ for $m < i < n$ shows that all the log-integral points $(\log i, \log \deg a_i)$ with $m < i < n$ lie below the line passing through the log-integral points $A = (\log m, \log \deg a_m)$ and $B = (\log n, \log \deg a_n)$, so the upper Newton polygon of f consists of a single edge joining A and B. Conditions $\deg a_m \neq \deg a_n$

and $\gcd(d_1, d_2) = 1$ show that the only log-integral points on the edge AB are its endpoints A and B, so the edge AB consists of a single segment. By Corollary 9.2, f must be irreducible. $\square$

We mention here that if the coefficients a_i are multivariate Dirichlet polynomials, one may simultaneously use information on the upper Newton log-polygons with respect to different indeterminates, and obtain results similar to those in Section 5, but with considerably more involved statements.

10. Examples

1) For every integer $n > 1$, the Dirichlet polynomials $\zeta_n(s) = 1 + \frac{1}{2^s} + \cdots + \frac{1}{n^s}$ that approximate the Riemann zeta-function are irreducible. The claim follows by Proposition 2.2 for prime n, and by Proposition 2.3 for composite n.

2) For every nonzero integers a, b, c, d, the Dirichlet polynomial $f(s) = \frac{a}{10^s} + \frac{b}{11^s} + \frac{c}{14^s} + \frac{d}{16^s}$ is irreducible. Here one may apply Proposition 2.4 with $m = 10$, $n = 16$ and $i = 11$.

3) Let $f(s)$ be a Dirichlet polynomial with integer coefficients and support $\{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$, and let a be a nonzero integer and $n > i_k$ an integer with $\gcd(i_1, \ldots, i_k, n) = 1$. Then the Dirichlet polynomial $p \cdot f(s) + \frac{a}{n^s}$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p.

Indeed, assume that $f(s) = \frac{a_1}{i_1^s} + \cdots + \frac{a_k}{i_k^s}$ with $a_1, \ldots, a_k \in \mathbb{Z}$, $a_1 \cdots a_k \neq 0$. To apply Theorem 2.14 i) to the Dirichlet polynomial $\frac{p \cdot a_1}{i_1^s} + \cdots + \frac{p \cdot a_k}{i_k^s} + \frac{a}{n^s}$, which is algebraically primitive (as $\gcd(i_1, \ldots, i_k, n) = 1$), it suffices to choose any prime p such that $p \nmid a_1 a$.

4) Let $f(s)$ be a Dirichlet polynomial with integer coefficients and support $\{i_1, i_2, \ldots, i_k\}$ with $1 < i_1 < i_2 < \cdots < i_k$, and let a be a nonzero integer and $m < i_1$ a positive integer with $\gcd(m, i_1, \ldots, i_k) = 1$. Then the Dirichlet polynomial $\frac{a}{m^s} + p \cdot f(s)$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p.

To see this, assume again that $f(s) = \frac{a_1}{i_1^s} + \cdots + \frac{a_k}{i_k^s}$ with $a_1, \ldots, a_k \in \mathbb{Z}$, $a_1 \cdots a_k \neq 0$. Here one may choose $m = 1$, for instance. To apply Theorem 2.14 ii) to the Dirichlet polynomial $\frac{a}{m^s} + \frac{p \cdot a_1}{i_1^s} + \cdots + \frac{p \cdot a_k}{i_k^s}$, which is also algebraically primitive, it suffices to choose any prime p such that $p \nmid a_k a$.

5) Let $m < i < n$ be integers with m coprime to n, and let $f(s) = \frac{a_m}{m^s} + \frac{a_i}{i^s} + \frac{a_n}{n^s}$ be a Dirichlet polynomial with integer coefficients, $a_m a_i a_n \neq 0$. Assume that $q_1, \ldots, q_k$ are all the prime factors of $m \cdot n$, and that their multiplicities in the prime decomposition of mn are all odd. If a_n is odd, $4 \mid a_m$, $8 \mid a_i$ but $8 \nmid a_m$, then f is irreducible over $\mathbb{Q}$.

Note first that f is algebraically primitive, as m an n are coprime. The conclusion follows from Theorem 3.4 with $p = 2$. Indeed, we have $\nu_2(a_i) > 2 = \nu_2(a_m) > \nu_2(a_n) = 0$. Condition ii) is obviously satisfied, since $\frac{n}{i} > 1$ while $\frac{m}{i} < 1$ and the exponents $\nu_p(a_i) - \nu_p(a_m)$ and $\nu_p(a_i) - \nu_p(a_n)$ are both positive. Condition iii) also holds, as $\nu_2(a_n) - \nu_2(a_m) = -2$, and for each $i = 1, \ldots, k$, $\nu_{q_i}(n) - \nu_{q_i}(m)$ is odd, being either equal to $\nu_{q_i}(n)$, or to $-\nu_{q_i}(m)$, as m and n are coprime.

6) Let $m < n$ be coprime positive integers, at least one of them being not a square. Let also i be an integer with $m < i < \sqrt{mn}$. Then for every prime number p, the Dirichlet polynomial

$f(s) = \frac{1}{m^s} + \frac{p}{i^s} + \frac{p^2}{n^s}$ is irreducible. Indeed, f is algebraically primitive, and since at least one of m and n is not a square, we have $\rho(m,n) < \sqrt{\frac{n}{m}}$. We may now apply Theorem 4.4, since condition (26) reduces to the inequalities $\rho(m,n) < \sqrt{\frac{n}{m}}$ and $\rho(m,n) < \frac{n}{i}$, the second one being satisfied, as $i < \sqrt{mn}$. Moreover, for any nonzero integers a, b, c with $p \nmid ac$, the Dirichlet polynomial $\frac{a}{m^s} + \frac{pb}{i^s} + \frac{p^2c}{n^s}$ is also irreducible over $\mathbb{Q}$.

7) Let $m < n$ be coprime positive integers, with at least one of them not a square, and let i be an integer with $\sqrt{mn} < i < n$. Then for every prime number p, the Dirichlet polynomial $f(s) = \frac{p^2}{m^s} + \frac{p}{i^s} + \frac{1}{n^s}$ is irreducible. Again, f is algebraically primitive, and since at least one of m and n is not a square, we have $\rho(m,n) < \sqrt{\frac{n}{m}}$. Here we will apply Theorem 4.10, since condition (29) reduces to the inequalities $\rho(m,n) < \sqrt{\frac{n}{m}}$ and $\rho(m,n) < \frac{i}{m}$, the second one being satisfied, as $\sqrt{mn} < i$. Besides, for any nonzero integers a, b, c with $p \nmid ac$, the Dirichlet polynomial $\frac{ap^2}{m^s} + \frac{pb}{i^s} + \frac{c}{n^s}$ is irreducible over $\mathbb{Q}$.

8) For every distinct prime numbers p and q, the Dirichlet polynomial $f(s) = \frac{pq}{10^s} + \frac{q}{15^s} + \frac{p}{18^s} + \frac{pq}{38^s}$ is irreducible over $\mathbb{Q}$. Note that f is algebraically primitive. Its irreducibility follows by Theorem 5.4 with $j_1 = 15$ and $j_2 = 18$.

9) For any prime numbers p, q, r with $p \neq q$, and any positive integer n, the Dirichlet polynomial $p(1 + \frac{1}{r^s})^n + q(2 + \frac{1}{r^s})^n$ is irreducible over $\mathbb{Q}$. To prove this, one may apply Corollary A.4 to the pair (F, G) with $F(s) = 1 + \frac{1}{r^s}$ and $G(s) = 2 + \frac{1}{r^s}$.

10) For every integers m, n with $3 \nmid m$ and $n \geq 2$, and every integers b, c with $b \not\equiv c \pmod 3$, the Dirichlet polynomial $m(-1 + \frac{1}{2^s})^n + 3(-b + \frac{c}{2^s})$ is irreducible over $\mathbb{Q}$. Here one may apply Corollary A.6 with $k = 1$, $F_1(s) = -1 + \frac{1}{2^s}$, $p = 3$, $G(s) = -b + \frac{c}{2^s}$ and $a = -2$.

11) Let $\{p_1, p_2, \ldots, p_{2n}\}$ and $\{q_1, q_2, \ldots, q_{2n}\}$ be two sets of prime numbers, not necessarily disjoint. Then for every $a, b \in \mathbb{C} \setminus \{0\}$, the multivariate Dirichlet polynomial $f(s_1, \ldots, s_n) = \frac{a}{(p_1^{q_1})^{s_1} \cdots (p_n^{q_n})^{s_n}} + \frac{b}{(p_{n+1}^{q_{n+1}})^{s_1} \cdots (p_{2n}^{q_{2n}})^{s_n}}$ is absolutely irreducible. Here we may apply Theorem 8.8, since the multiplicities $q_1, q_2, \ldots, q_{2n}$ are obviously relatively prime.

12) Let $f(s,t) = \frac{a}{(p_1^{q_1})^s (p_2^{q_2})^t} + \frac{b}{(p_3^{q_3})^s (p_4^{q_4})^t} + \frac{c}{(p_5^{q_5})^s (p_6^{q_6})^t}$ with $p_1, \ldots, p_6, q_1, \ldots, q_6$ distinct prime numbers, and $a, b, c \in \mathbb{C} \setminus \{0\}$. Let us denote $p_i^{q_i}$ by r_i for $i = 1, \ldots, 6$, and assume without loss of generality that $r_1 < r_3 < r_5$. If

$$\log_{\frac{r_4}{r_2}} \left(\frac{r_3}{r_1} \right) \neq \log_{\frac{r_6}{r_4}} \left(\frac{r_5}{r_3} \right),$$

then f is absolutely irreducible. Observe first that f is algebraically primitive. We then see that $S_f^{log} = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ with $\mathbf{v_1} = (\log r_1, \log r_2)$, $\mathbf{v_2} = (\log r_3, \log r_4)$ and $\mathbf{v_3} = (\log r_5, \log r_6)$, so condition $\overline{\gcd}(\mathbf{v_3} - \mathbf{v_1}, \mathbf{v_3} - \mathbf{v_2}) = 1$ in Theorem 8.9 is obviously satisfied. Since the condition above involving logarithms prevents $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ to be collinear, the conclusion follows by Theorem 8.9 and Theorem 8.5.

13) Let $f(s,t) = 1 + (1 + \frac{1}{2^t})\frac{1}{8^s} + (1 + \frac{1}{32^t})\frac{1}{16^s}$. We will conclude that f is irreducible by applying Theorem 9.3. We may write $f(s,t) = \frac{a_1(t)}{1^s} + \frac{a_2(t)}{8^s} + \frac{a_3(t)}{16^s}$ with $a_1 = 1$, $a_8 = 1 + \frac{1}{2^t}$ and $a_{16} = 1 + \frac{1}{32^t}$. In this case we have $d_1 = \gcd(\nu_2(16) - \nu_2(1)) = 4$ and $d_2 = \gcd(\nu_2(32) - \nu_2(1)) = 5$, so $\gcd(d_1, d_2) = 1$, and condition $\deg a_i < (\deg a_m)^{\log_{\frac{n}{m}} \frac{n}{i}} (\deg a_n)^{1 - \log_{\frac{n}{m}} \frac{n}{i}}$ for $m = 1$, $i = 8$ and

$n = 16$ reduces to $2 < 1 \cdot 32^{\frac{3}{4}} = 13.4543...$, which obviously holds. Note that the same conclusion holds if we replace $a_8(t)$ by any Dirichlet polynomial in t of degree at most 13.

Appendix A. Variations on a theme by Schönemann

Schönemann [97] proved the following irreducibility criterion for polynomials with integer coefficients, that admits a straightforward generalization to unique factorization domains.

Irreducibility criterion of Schönemann *Suppose that a polynomial $f(X) \in \mathbb{Z}[X]$ has the form $f(X) = \phi(X)^e + pM(X)$, where p is a prime number, $\phi(X)$ is an irreducible polynomial modulo p, and $M(X)$ is a polynomial relatively prime to $\phi(X)$ modulo p, with $\deg M < \deg f$. Then f is irreducible over $\mathbb{Q}$.*

For some generalizations of this result we refer the reader to Panaitopol and Ştefănescu [85], [86], and Jakhar and Sangwan [65], for instance.

In this section we will provide several irreducibility criteria for some classes of linear combinations of Dirichlet polynomials, that may be regarded as analogous results of Schönemann's criterion and its variations. Our first result that relies on Schönemann's ideea is the following.

Theorem A.1. *Let $f(s) = qF(s)^n + pG(s)$ where n is a positive integer, F, G are Dirichlet polynomials with coefficients in a unique factorization domain R, p is a prime element of R, and q a nonzero element of R. If the leading coefficient of f is not divisible by p, F is irreducible modulo p, and F and G are relatively prime modulo p, then f is irreducible over $Q(R)$, the quotient field of R.*

Proof: The proof is similar to that of Schönemann's irreducibility criterion. For an arbitrary Dirichlet polynomial $h(s)$ we will denote by $\overline{h}(s)$ the Dirichlet polynomial obtained from h by reducing its coefficients modulo p. Let us first note that our assumption that the leading coefficient of f is not divisible by p forces the equality $\deg G \leq (\deg F)^n$ and also the fact that q is not divisible by p. We also note that the leading coefficient of F is allowed to be divisible by p, but as F is irreducible modulo p, f cannot have all the coefficients divisible by p. By Proposition 2.2 we may assume that the degree of f is composite. Now let us assume to the contrary that $f(s) = F_1(s)F_2(s)$, with F_1, F_2 Dirichlet polynomials of degrees $r \geq 2$ and $t \geq 2$, respectively. By reducing modulo p the relation $f(s) = qF(s)^n + pG(s) = F_1(s)F_2(s)$, one obtains

$$\overline{f}(s) = \overline{q} \cdot \overline{F}(s)^n = \overline{F_1}(s)\overline{F_2}(s). \tag{71}$$

Since the leading coefficient of f is not divisible by p, the same will hold for the leading coefficients of the hypothetical factors F_1 and F_2, hence $\deg \overline{F}_1 = \deg F_1 \geq 2$ and $\deg \overline{F}_2 = \deg F_2 \geq 2$. Since $\overline{F}$ is irreducible modulo p, we deduce by (71) that

$$\overline{F_1}(s) = \overline{c} \cdot \overline{F}(s)^a \quad \text{and} \quad \overline{F_2}(s) = \overline{d} \cdot \overline{F}(s)^b \tag{72}$$

for two integers $a, b \geq 0$ with $a + b = n$, and two elements $c, d \in R$, with $\overline{c}\overline{d} = \overline{q}$. We note here that in view of (72), none of a and b can be zero, as $\deg \overline{F}_1 \geq 2$ and $\deg \overline{F}_2 \geq 2$, so for $n = 1$ we

already have a contradiction. Therefore, in what follows we may assume that $a \geq 1$, $b \geq 1$ and $n \geq 2$. As we shall see, the fact that both a and b are positive will be of crucial importance for the remaining part of the proof. Next, we see that (72) implies the existence of two Dirichlet polynomials h_1, h_2 such that

$$F_1(s) = cF(s)^a + ph_1(s) \quad \text{and} \quad F_2(s) = dF(s)^b + ph_2(s),$$

so $qF^n + pG = (cF^a + ph_1)(dF^b + ph_2) = cdF^n + pch_2F^a + pdh_1F^b + p^2h_1h_2$, that is

$$(q - cd)F^n + pG = pch_2F^a + pdh_1F^b + p^2h_1h_2. \tag{73}$$

Recalling now that $\overline{c}\overline{d} = \overline{q}$, there exists an element $e \in R$ such that

$$cd + pe = q. \tag{74}$$

Combining (73) and (74), we deduce after division by p that

$$G(s) = ch_2(s)F(s)^a + dh_1(s)F(s)^b - eF(s)^n + ph_1(s)h_2(s). \tag{75}$$

Reducing now (75) modulo p, one obtains

$$\overline{G} = \overline{c} \cdot \overline{h}_2 \cdot \overline{F}^a + \overline{d} \cdot \overline{h}_1 \cdot \overline{F}^b - \overline{e} \cdot \overline{F}^n,$$

with the right term being divisible by $\overline{F}$, as both a and b are positive. This is obviously a contradiction, since F and G are supposed to be relatively prime modulo p, and this completes the proof of the theorem. □

In particular, as a corollary of the proof of Theorem A.1, we have the following result.

Corollary A.2. *Let $f(s) = qF(s)^n + pG(s)$ where n is a positive integer, F, G are Dirichlet polynomials with integer coefficients, p a prime number, and q a nonzero integer. If the leading coefficient of f is not divisible by p, F is irreducible modulo p, and there exists an integer m with $p \mid F(m)$ and $p \nmid G(m)$, then f is irreducible over $\mathbb{Q}$.*

Proof: The proof follows exactly the same lines as in the case of Theorem A.1, until we reach (75). The contradiction follows now by taking $s = m$ in equation (75). □

We will also prove here a similar result that provides symmetric irreducibility conditions with respect to two non-associated prime elements of R.

Theorem A.3. *Let $f(s) = pF(s)^n + qG(s)^n$ with n a positive integer, F, G monic nonconstant Dirichlet polynomials with coefficients in a unique factorization domain R with $\deg F = \deg G$, and p, q non-associated prime elements of R. If F is irreducible modulo q, G is irreducible modulo p, and $F \not\equiv G \pmod{pq}$, then f is irreducible over $Q(R)$.*

Proof: For a Dirichlet polynomial h we will denote by $\tilde{h}$ and $\overline{h}$ the Dirichlet polynomials obtained by reducing the coefficients of h modulo p and modulo q, respectively. Here the leading coefficient of f is $p + q$, which is coprime to both p and q, and $\deg f = n \deg F$, which is a prime number only if $n = 1$ and $\deg F$ is prime. Assume again that $\deg f$ is composite and that

$f(s) = F_1(s)F_2(s)$, with F_1, F_2 Dirichlet polynomials of degrees $r \geq 2$ and $t \geq 2$, respectively. By reducing modulo q the relation $f(s) = pF(s)^n + qG(s)^n = F_1(s)F_2(s)$, one obtains

$$\overline{f}(s) = \overline{p} \cdot \overline{F}(s)^n = \overline{F_1}(s)\overline{F_2}(s). \tag{76}$$

Since the leading coefficient of f is not divisible by p, the same will hold for the leading coefficients of the hypothetical factors F_1 and F_2, hence $\deg \overline{F}_1 = \deg F_1 \geq 2$ and $\deg \overline{F}_2 = \deg F_2 \geq 2$. As before, since $\overline{F}$ is irreducible modulo q, we deduce by (76) that

$$\overline{F_1}(s) = \overline{c} \cdot \overline{F}(s)^a \quad \text{and} \quad \overline{F_2}(s) = \overline{d} \cdot \overline{F}(s)^b \tag{77}$$

for two integers $a, b \geq 0$ with $a + b = n$, and two elements $c, d \in R$, with $\overline{c}\overline{d} = \overline{p}$. By (77) we see that in this case too, none of a and b can be zero, for $\deg \overline{F}_1 \geq 2$ and $\deg \overline{F}_2 \geq 2$, so for $n = 1$ we already have a contradiction. Therefore, in what follows we will assume that $a \geq 1$, $b \geq 1$ and $n \geq 2$. By (77) there exist two Dirichlet polynomials h_1, h_2 such that

$$F_1(s) = cF(s)^a + qh_1(s) \quad \text{and} \quad F_2(s) = dF(s)^b + qh_2(s),$$

so $pF^n + qG^n = (cF^a + qh_1)(dF^b + qh_2) = cdF^n + qch_2F^a + qdh_1F^b + q^2h_1h_2$, which yields

$$(p - cd)F^n + qG^n = qch_2F^a + qdh_1F^b + q^2h_1h_2. \tag{78}$$

As before, since $\overline{c}\overline{d} = \overline{p}$, there exists an element $e \in R$ such that $p - cd = qe$, so by (78) we deduce after division by q that

$$G^n = ch_2F^a + dh_1F^b - eF^n + qh_1h_2. \tag{79}$$

Reducing now (79) modulo q, one obtains

$$\overline{G}^n = \overline{c} \cdot \overline{h}_2 \cdot \overline{F}^a + \overline{d} \cdot \overline{h}_1 \cdot \overline{F}^b - \overline{e} \cdot \overline{F}^n,$$

with the right side being divisible by $\overline{F}$, as both a and b are positive. This shows that $\overline{G}^n$ must be divisible by $\overline{F}$. Recalling that F is irreducible modulo q, we see that $\overline{G}$ must be divisible by $\overline{F}$. Now, since $\deg \overline{F} = \deg \overline{G}$, we actually have $\overline{G} = \overline{r} \cdot \overline{F}$ for some element $r \in R$ not divisible by q. As both F and G are monic, we conclude that $\overline{r} = \overline{1}$, so $\overline{F} = \overline{G}$. Absolutely similar, by reducing the coefficients modulo p, one proves that $\tilde{F} = \tilde{G}$. This obviously cannot hold, as $F \not\equiv G \pmod{pq}$, and this completes the proof. □

In particular, we have the following irreducibility criterion for Dirichlet polynomials of prime power degree.

Corollary A.4. *Let $f(s) = pF(s)^n + qG(s)^n$ with p, q different positive primes, n a positive integer, and F, G monic Dirichlet polynomials with integer coefficients of equal degree r, with r a prime number. If $F \not\equiv G \pmod{pq}$, then f is irreducible over $\mathbb{Q}$.*

Proof: By Proposition 2.2, F is irreducible modulo q and G is irreducible modulo p, as both F and G have prime degree. □

One may find similar symmetric irreducibility conditions that do not require F and G to be monic, with slightly more involved statements. We will end this section with a result for Dirichlet polynomials with integer coefficients, that uses information on the values that such series take at a certain negative integral argument.

Theorem A.5. *Let $f(s) = mF_1(s)^{n_1} \cdots F_k(s)^{n_k} + pG(s)$ with $F_1, \ldots, F_k, G$ Dirichlet polynomials with integer coefficients, m a nonzero integer, p a prime number, $n_1, \ldots, n_k$ positive integers, and assume that $F_1, \ldots, F_k$ are irreducible modulo p. If the leading coefficient of f is not divisible by p, and there exists a negative integer a such that $F_1(a), \ldots, F_k(a)$ are all divisible by p, but $G(a)$ is not divisible by p, then $f(s)$ is irreducible over $\mathbb{Q}$.*

Proof: Our assumption that the leading coefficient of f is not divisible by p forces $\deg G$ to be less than or equal to $\deg F_1^{n_1} \cdots F_k^{n_k}$, and prevents m to be divisible by p. We note that the leading coefficients of $F_1, \ldots, F_k$ are allowed to be divisible by p, but $f \not\equiv 0 \pmod{p}$, as $F_1, \ldots, F_k$ are irreducible modulo p. As in previous results, we may assume that f has composite degree. Assume to the contrary that f is reducible, say $f(s) = f_1(s)f_2(s)$ with $\deg f_1 \geq 2$ and $\deg f_2 \geq 2$. Reducing modulo p, we obtain $\overline{m}\overline{F_1}^{n_1} \cdots \overline{F_k}^{n_k} = \overline{f_1} \cdot \overline{f_2}$. We note that the leading coefficients of f_1 and f_2 are not divisible by p, so $\deg \overline{f_1} = \deg f_1$ and $\deg \overline{f_2} = \deg f_2$. Therefore there exist integers m_1, m_2 with $\overline{m} = \overline{m_1} \cdot \overline{m_2}$ and nonnegative integers $a_1, \ldots a_k$, $b_1, \ldots, b_k$ with $a_i + b_i = n_i$ for $i = 1, \ldots, k$ such that

$$\overline{f_1} = \overline{m_1}\overline{F_1}^{a_1} \cdots \overline{F_k}^{a_k} \quad \text{and} \quad \overline{f_2} = \overline{m_2}\overline{F_1}^{b_1} \cdots \overline{F_k}^{b_k}.$$

Observe now that we can neither have $a_1 = \cdots = a_k = 0$, nor $b_1 = \cdots = b_k = 0$, as $\deg \overline{f_1} \geq 2$ and $\deg \overline{f_2} \geq 2$. The two equalities above imply the existence of two Dirichlet polynomials g_1, g_2 such that $f_1 = m_1 F_1^{a_1} \cdots F_k^{a_k} + pg_1$ and $f_2 = m_2 F_1^{b_1} \cdots F_k^{b_k} + pg_2$, so we have

$$mF_1^{n_1} \cdots F_k^{n_k} + pG = (m_1 F_1^{a_1} \cdots F_k^{a_k} + pg_1)(m_2 F_1^{b_1} \cdots F_k^{b_k} + pg_2). \tag{80}$$

Since $m - m_1 m_2 = pe$ for a certain integer e, we obtain by (80) after divison by p that

$$\begin{aligned} G(s) &= m_1 g_2(s) F_1(s)^{a_1} \cdots F_k(s)^{a_k} + m_2 g_1(s) F_1(s)^{b_1} \cdots F_k(s)^{b_k} \\ &\quad - eF_1(s)^{n_1} \cdots F_k(s)^{n_k} + pg_1(s)g_2(s). \end{aligned}$$

Recall now our assumption that for some integer a, all of $F_1(a), \ldots, F_k(a)$ are divisible by p, while $G(a)$ is not a multiple of p. The desired contradiction is obtained by letting $s = a$ in our previous equality. Therefore f must be irreducible over $\mathbb{Q}$. □

Corollary A.6. *Let $f(s) = mF_1(s)^{n_1} \cdots F_k(s)^{n_k} + pG(s)$ with $F_1, \ldots, F_k, G$ Dirichlet polynomials with integer coefficients, $F_1, \ldots, F_k$ monic and of prime degree, m a nonzero integer, p a prime number, and $n_1, \ldots, n_k$ positive integers. If the leading coefficient of f is not divisible by p, and there exists a negative integer a such that $F_1(a), \ldots, F_k(a)$ are all divisible by p, but $G(a)$ is not divisible by p, then $f(s)$ is irreducible over $\mathbb{Q}$.*

Proof: By Proposition 2.2, $F_1, \ldots, F_k$ are all irreducible modulo p. □

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