CLASSIFICATION OF FRACTIONAL, SINGULAR YAMABE METRICS ON A TWICE-PUNCTURED SPHERE I

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Abstract. The Delaunay metrics form a family of conformally flat, constant fractional (order $s \in (0,1)$) Q-curvature metrics on a twice-punctured sphere. They are all (after a Möbius transformation) rotationally symmetric and periodic, and admit several elegant variational descriptions. We prove that there exists $\delta > 0$ such that when $s \in (1-\delta, 1)$ any complete, conformally flat constant Q-curvature metric on a twice-punctured sphere is a Delaunay metric. Along the way, we prove a sharp *a priori* bound for the conformal factor of these metrics, which may be of independent interest.

1. Introduction

Much of geometric analysis in the last 40 years has concentrated on conformally invariant equations. The first step in this program for a given equation is to demonstrate the existence of a solution in a given conformal class. As a next step, one can try to classify the solutions, or at least prove compactness of the solution set in the form of *a priori* bounds. In the case of the conformal class of the round sphere, the conformal invariance of the equation combines with the noncompactness of the group of conformal transformations to complicate both the *a priori* bounds and the uniqueness of solutions.

In the present paper, we consider the following conformally invariant equation

$$\mathbb{P}_s u = c_{n,s} u^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \Omega, \tag{1}$$

where $s \in (0, \frac{n}{2})$, $\mathbb{P}_s$ is the family of fractional Paneitz operators constructed by Graham and Zworski in [20] (see also [8]) and $c_{n,s} > 0$ is the normalization constant defined as

$$c_{n,s} = 2^{2s} \left(\frac{\Gamma\left(\frac{1}{2}\left(\frac{n}{2}+s\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{n}{2}-s\right)\right)} \right)^2 .$$

The function Γ is the classical Gamma function defined as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$. Here we assume $\Omega := M \setminus \Lambda$ to be the so-called regular set, where (M, g) is an n-dimensional smooth, compact Riemannian manifold without boundary, with $n \geq 3$ and $\Lambda \subset M$ is a closed subset called the singular set. We adopt the same terminology as in the study of constant scalar curvature metrics and denote a solution $u \in \mathcal{C}^\infty(\Omega)$ of (1) as a fractional, singular Yamabe solution, and the associated metric $\widetilde{g} = u^{\frac{4}{n-2s}} g$ as a fractional, singular Yamabe metric.

We are mostly interested in this equation on subdomains of the sphere, *i.e.* $\Omega := \mathbf{S}^n \setminus \Lambda$ with $\Lambda \subset \mathbf{S}^n$ a closed set. We denote the usual round metric as $g_\circ$. In this locally conformally flat

2020 *Mathematics Subject Classification.* 35J60, 35R11, 53C21, 53A30, 35B33, 35B40.

Key words and phrases. fractional Yamabe problem, Delaunay metrics, fractional Paneitz operator, singular solutions, conformal geometry, uniqueness and classification.

setting, $\mathbb{P}_s$ has the explicit form (originally given by Branson in [5]) given by

$$\mathbb{P}_s = \frac{\Gamma\left(\frac{n}{2}-s\right)\Gamma\left(B+s+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(B-s+\frac{1}{2}\right)} \quad \text{with} \quad B=\sqrt{-\Delta_{g_\circ}+\left(\frac{n-1}{2}\right)^2}. \tag{2}$$

Later, we will see a similar expansion for $\mathbb{P}_s$ in the cylindrical setting, which was first derived in [17].

Now, let $\Lambda = \{N,S\}$ with $N,S \in \mathbf{S}^n$ denote its north and south poles, respectively. The conformally flat metric $g = U^{\frac{4}{n-2s}} g_\circ$ has constant Q_s-curvature if the (positive) conformal factor $U : \mathbf{S}^n \backslash \{N,S\} \to (0,\infty)$ satisfies the PDE

$$\mathbb{P}_s U = c_{n,s} U^{\frac{n+2s}{n-2s}} \quad \text{on} \quad \mathbf{S}^n \setminus \{N,S\}. \tag{$\mathcal{Q}^\circ_{n,s}$}$$

In this setting, completeness of the metric $g \in [g_\circ]$ is equivalent to the boundary condition

$$\liminf_{p \to \{N,S\}} U(p) = \infty.$$

Equivalently, we say that $U \in \mathcal{C}^\infty(\mathbf{S}^n \setminus \{N,S\})$ is a singular solution to ($\mathcal{Q}^\circ_{n,s}$).

Let g_{flat} be the standard Euclidean metric on $\mathbf{R}^n$. We can use conformal invariance to transfer this problem to (a subdomain of) Euclidean space. Let $\Pi : \mathbf{S}^n \backslash \{N\} \to \mathbf{R}^n$ be the standard stereographic projection and let $u_\circ : \mathbf{R}^n \to (0,\infty)$ be the standard spherical solution (*i.e.* Aubin–Talenti-type bubble or Chen–Li–Ou bubble [11]) given by

$$u_\circ(x) = \hat{c}_{n,s}\left(\frac{1+|x|^2}{2}\right)^{\frac{2s-n}{2}} \tag{3}$$

with

$$\hat{c}_{n,s} = 2^{\frac{2s-n}{2}}\left(\frac{\Gamma\left(\frac{1}{2}\left(\frac{n}{2}+s\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{n}{2}-s\right)\right)}\right)^{\frac{2s-n}{2s}}\left(\frac{\Gamma\left(\frac{n}{2}-s\right)}{\Gamma\left(\frac{n}{2}+s\right)}\right)^{\frac{2s-n}{4s}}.$$

Here we remark that $\hat{c}_{n,s} > 1$ (cf. [17, Proposition 6.1]).

We can now write a conformally flat metric in both the spherical and Euclidean gauges, where the two are related by

$$g = U^{\frac{4}{n-2s}} g_\circ = u^{\frac{4}{n-2s}} g_{\text{flat}} \quad \text{with} \quad u = U \cdot u_\circ. \tag{4}$$

We observe that the spherical solution described above does not correspond to the unit sphere, but rather to a sphere $\mathbf{S}^n(\rho)$ of radius $\rho = \rho(n,s) > 0$ given by

$$\rho_{n,s} = 2^{-2}\left(\frac{\Gamma\left(\frac{1}{2}\left(\frac{n}{2}+s\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{n}{2}-s\right)\right)}\right)^{-2/s}\left(\frac{\Gamma\left(\frac{n}{2}-s\right)}{\Gamma\left(\frac{n}{2}+s\right)}\right)^{-1/s}.$$

We choose this scaling so that the sphere fits with the form of the Delaunay solutions we derive below in § 2.2.

It will be useful to transfer without further comment between these two gauges, as well as the cylindrical gauge mentioned earlier. In the Euclidean gauge, the positive conformal factor $u : \mathbf{R}^n \backslash \{0\} \to (0,\infty)$ satisfies

$$(-\Delta)^s u = c_{n,s} u^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \mathbf{R}^n \setminus \{0\}, \tag{$\overline{\mathcal{Q}}_{n,s}$}$$

where Δ is the standard Euclidean Laplacian and $(-\Delta)^s$ is the fractional Laplacian, which can be understood as the principal part of a singular integral. Here, the boundary condition translates to $\liminf_{x\to 0} u(x) = \infty$.

In previous work, DelaTorre, del Pino, González and Wei [16] constructed a family of solutions on the twice-punctured sphere. These solutions are, after an appropriate conformal dilation, all rotationally symmetric. They generalize the well-known Delaunay metrics in the scalar curvature setting, which were found by Schoen [32], and we call them Delaunay metrics as well. A Delaunay metric is uniquely determined by its fundamental period, or equivalently by its necksize. More precisely, the fractional Delaunay family can be equivalently characterized as the set of even, positive, periodic solutions $v_\varepsilon : \mathbf{R} \to (0,\infty)$ of a nonlocal initial value problem (see ($\widehat{\mathcal{Q}}_{n,s}$)) that fixes the necksize parameter $\varepsilon = \min_{t\in\mathbf{R}} v_\varepsilon(t)$. For each necksize $\varepsilon \in (0,1]$, the Euclidean conformal factor

$$u_\varepsilon(x) = |x|^{\frac{2s-n}{2}} v_\varepsilon(-\ln|x|), \tag{5}$$

solves ($\overline{\mathcal{Q}}_{n,s}$) and defines $g_\varepsilon = u_\varepsilon^{\frac{4}{n-2s}} g_{\text{flat}}$ a complete constant Q_s–curvature metric on $\mathbf{R}^n\backslash\{0\}$. We can also write out the Delaunay metric in the spherical gauge as

$$g_\varepsilon = U_\varepsilon^{\frac{4}{n-2s}} g_\circ \quad \text{and} \quad u_\varepsilon = U_\varepsilon \cdot u_\circ.$$

In what follows, we call $u \in \mathcal{C}^\infty(\mathbf{R}^n \setminus \{0\})$ a singular solution to ($\overline{\mathcal{Q}}_{n,s}$) if it does not extend continuously to the origin, *i.e.* $\lim_{x\to 0} u(x) = \infty$; otherwise, we call it a nonsingular solution.

Our first main result is a partial classification of singular solutions to the fractional Yamabe equation, which can be stated as

Theorem 1. *Let $n \in \mathbf{N}$ with $n \geq 3$. There exists $\delta > 0$ such that if $s \in (1-\delta, 1)$ and $u \in \mathcal{C}^\infty(\mathbf{R}^n \setminus \{0\})$ is a singular solution to ($\overline{\mathcal{Q}}_{n,s}$), then there exists $\varepsilon \in (0,1]$ and $R > 0$ such that*

$$u = \bar{u}_{\varepsilon,R} \quad \text{with} \quad \bar{u}_{\varepsilon,R}(x) = R^{\frac{2s-n}{2}} u_\varepsilon\left(\frac{x}{R}\right),$$

where $\bar{u}_\varepsilon \in \mathcal{C}^\infty(\mathbf{R}^n \setminus \{0\})$ is the Euclidean Delaunay solution of necksize ε (see (12)).

For a given doubleton set $\{p,q\} \in \mathbf{S}^n$ with $p \neq q$ we define the marked moduli space of complete metrics with constant fractional Q-curvature as

$$\mathcal{M}_{\{p,q\}} = \left\{ g = U^{\frac{4}{n-2s}} g_\circ \in [g_\circ] : U \in \mathcal{C}^\infty(\mathbf{S}^n \setminus \{p,q\}) \text{ is a positive solution to } (\mathcal{Q}^\circ_{n,s}) \right\} / \sim,$$

where the quotient is taken under diffeomorphism preserving the singular set $\{N,S\}$ and is furnished with the Gromov-Hausdorff topology.

In this fashion, we may reformulate our main theorem as

Corollary 2. *Let $n \in \mathbf{N}$ with $n \geq 3$ and let $p,q \in \mathbf{S}^n$ with $p \neq q$. There exists $\delta > 0$ such that if $s \in (1-\delta, 1)$ then each element of $\mathcal{M}_{\{p,q\}}$ is a Delaunay metric.*

Alternatively, we could say that $\mathcal{M}_{\{p,q\}}$ is a half-open interval in the real line, where the parameter represents the minimal period of the corresponding Delaunay solution or (equivalently) the minimum value of the Delaunay solution, *i.e.* the necksize.

We use the same technique as in the proof of Theorem 1 to prove that all Delaunay solutions are nondegenerate. This condition is easiest to define in the cylindrical gauge. For any $v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0,\infty)$ solution to ($\widehat{\mathcal{Q}}_{n,s}$), we set the associated linearized operator as

$$\mathbb{L}_v := P_s + c_{n,s} - \tfrac{n+2s}{n-2s}\, c_{n,s}\, v^{\frac{4s}{n-2s}}$$

We say that v is nondegenerate if $w \in L^2(\mathbf{R} \times \mathbf{S}^{n-1})$ and $\mathbb{L}_v(w) = 0$ implies $w \equiv 0$.

In this setting, our second main result is

Theorem 3. *Let $n \in \mathbf{N}$ with $n \geq 3$. There exists $\delta > 0$ such that if $s \in (1-\delta, 1)$ and $v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0,\infty)$ satisfies ($\widehat{\mathcal{Q}}_{n,s}$), then v is nondegenerate.*

In the scalar curvature setting, the conformal factor satisfies a second order ODE, and so the uniqueness follows very quickly from phase-plane analysis. However, as demonstrated in [17], one cannot apply these techniques directly in the nonlocal setting. Nonetheless, we can prove uniqueness if s is close to, but less than one.

Our proof of uniqueness relies on two new ideas. In § 3.2, we use the fact that the spherical solution has Morse index one to derive a sharp *a priori* upper bound for a solution to the Q-curvature equation of order s on a twice-punctured sphere, with the spherical solution giving us a universal upper bound. This allows us to associate a Delaunay solution with the same L^∞-norm to any solution on the twice-punctured sphere. We prove Theorem 1 by contradiction, assuming there exists a sequence $s_k \nearrow 1$ and solutions v_k that are not Delaunay solutions. Letting $\overline{v}_k$ be the matched Delaunay solution, in § 4.1 we prove that a rescaling of the difference $v_k - \overline{v}_k$ converges locally to a solution of a homogeneous second order ODE. In § 4.2 we extend this local convergence to global convergence. This limit must be zero by the conditions of the ODE, which contradicts the normalization of our rescaling. The passage from local convergence to global convergence relies on a sharp estimate for the integral kernel and the solution operator of the nonlocal equation, derived in [16] and described below in § 2.2. We expect this strategy to be robust and to have applications in other fractional geometric problems, and indeed we adapt our technique to prove Theorem 3 in § 5. We conclude this paper with a brief outlook in § 6. First, in § 6.1, we discuss work in progress [1] in which we extend the classification to $s \in (1, 1+\delta)$, where once again $\delta > 0$ is sufficiently small. Next, in § 6.2, we describe some possible applications of our classification result, including refined asymptotics of solutions with isolated singularities and a general compactness result.

2. Qualitative properties of Delaunay solutions

In this section, we first describe the reduction to a one-dimensional problem. Next, we describe the Delaunay solutions. Finally, we recall that Dávila, Del Pino and Sire [15] (see also [4]) proved that the standard bubble has Morse index one.

2.1. Rotational symmetry.

Here, we describe why all the complete, constant Q_s-curvature metrics on a twice-punctured sphere depend only on one variable. We begin with a complete, conformally flat, constant Q_s-curvature metric $g = U^{\frac{4}{n-2s}} g_\circ$ on a twice-punctured sphere. After a conformal dilation, we can assume the two punctures are antipodal, and after a global rotation, we can take the punctures to be the north and south poles, which we denote respectively as N and S. We now use stereographic projection to transfer the problem to $\mathbf{R}^n \backslash \{0\}$. Following this change of coordinates, we arrive at ($\overline{\mathcal{Q}}_{n,s}$).

One can now apply [6] to see that u is rotationally invariant. Alternatively, one can argue as follows. Ngô and Ye [28] prove that, because the singular set $\{0\}$ has capacity 0, the function u is s'-superharmonic for each $0 < s' < s$. One can now use the method of moving planes to show that u is radial. One can also use the method of moving planes in integral form (see [13] and [12]) to show the radial symmetry directly.

2.2. Existence of Delaunay solutions.

The Delaunay solutions form a particularly nice family of rotationally symmetric solutions of ($\overline{\mathcal{Q}}_{n,s}$) on $\mathbf{R}^n \backslash \{0\}$, or, equivalently, of ($\mathcal{Q}^\circ_{n,s}$) on $\mathbf{S}^n \backslash \{N, S\}$. In the classical setting (*i.e.* $s = 1$) the existence of the Delaunay solutions goes back to the 1930s, due to the work of Fowler [18]. In the fractional setting, DelaTorre, Del Pino, González and Wei [16] first established their existence provided $s \in (0, 1)$. Later, Jin and Xiong [23] extended this construction to the range of exponents $s \in (1, \frac{n}{2})$.

The Emden-Fowler change of coordinates lies at the heart of this construction. Given $u : \mathbf{R}^n\backslash\{0\} \to \mathbf{R}$, we let

$$v(t,\theta) = \mathfrak{F}(u)(t,\theta) = e^{\left(\frac{2s-n}{2}\right)t}u(e^{-t}\theta) \quad \text{with} \quad t = -\ln|x| \quad \text{and} \quad \theta = \frac{x}{|x|}, \tag{6}$$

obtaining a new function $v : \mathbf{R} \times \mathbf{S}^{n-1} \to \mathbf{R}$. One can, of course, invert this change of coordinates, giving us

$$u(x) = \mathfrak{F}^{-1}(v)(x) = |x|^{\frac{2s-n}{2}} v\left(-\ln|x|, \frac{x}{|x|}\right).$$

The construction in [16] starts with the Ansatz that the solution u has the form

$$u(x) = |x|^{\frac{2s-n}{2}}\widetilde{v}(-\ln|x|),$$

where $\widetilde{v} : (0,\infty) \to (0,\infty)$ is a positive smooth function. In fact, one can use the *a priori* bounds of [6] to show that $\widetilde{v}$ is bounded, and so the integral operators written below are convergent. This change of variables transforms ($\overline{\mathcal{Q}}_{n,s}$) into

$$\kappa_{n,s}\int_0^\infty \int_{\mathbf{S}^{n-1}} \frac{(r^{\frac{2s-n}{2}}\widetilde{v}(r) - \rho^{\frac{2s-n}{2}}\widetilde{v}(\rho))\rho^{n-1}}{|r^2+\rho^2-2r\rho\langle\theta,\varphi\rangle|^{\frac{n+2s}{2}}} d\varphi d\rho = c_{n,s}u(x)^{\frac{n+2s}{n-2s}} \tag{7}$$

with

$$\kappa_{n,s} = \pi^{-n/2}2^{2s}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}s.$$

They then make the change of variables $\overline{\rho} = \rho/r$. Using the identity

$$\widetilde{v}(r) = \left(1-\overline{\rho}^{\frac{2s-n}{2}}\right)\widetilde{v}(r) + \overline{\rho}^{\frac{2s-n}{2}}\widetilde{v}(r),$$

the equation (7) becomes

$$\kappa_{n,s}\int_0^\infty \int_{\mathbf{S}^{n-1}} \frac{\overline{\rho}^{n-1-\frac{n-2s}{2}}(\widetilde{v}(r) - \widetilde{v}(\overline{\rho}r))}{|1+\overline{\rho}^2-2\overline{\rho}\langle\theta,\varphi\rangle|^{\frac{n+2s}{2}}} d\overline{\rho} d\varphi + A\widetilde{v}(r) = c_{n,s}\widetilde{v}(r)^{\frac{n+2s}{n-2w}}, \tag{8}$$

where

$$A = \int_0^\infty \int_{\mathbf{S}^{n-1}} \frac{(1-\overline{\rho}^{\frac{2s-n}{2}})\overline{\rho}^{n-1}}{(1+\overline{\rho}^2-2\overline{\rho}\langle\theta,\varphi\rangle)^{\frac{n+2s}{2}}} d\varphi d\overline{\rho}$$

is a positive constant; in fact, our normalizations imply $A = c_{n,s}$ (see [17, Proposition 2.7]). Finally, we complete the Emden-Fowler change of variables by letting $t = -\ln r$, $\tau = -\ln\rho$ and $v(t) = \widetilde{v}(-\ln r)$, which transforms (8) into

$$P_s(v) + c_{n,s}v = c_{n,s}v^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \mathbf{R}, \tag{$\widehat{\mathcal{Q}}_{n,s}$}$$

where

$$P_s(v)(t) = \int_{-\infty}^\infty (v(t)-v(\tau))K_s(t-\tau)d\tau \tag{9}$$

and

$$K_s(\xi) = \kappa_{n,s}\int_{\mathbf{S}^{n-1}} \frac{e^{\left(\frac{2s-n}{2}\right)\xi}}{(1+e^{-2\xi}-2e^{-2\xi}\langle\theta,\varphi\rangle)^{\frac{n+2s}{2}}} d\varphi.$$

From this formulation, one can search for periodic solutions with minimal period L by setting

$$K_{s,L}(t-\tau) = \sum_{j\in\mathbb{Z}} K_s(t-\tau-jL) \tag{10}$$

and minimizing

$$\mathcal{E}_{s,L}(v)=\frac{\frac{1}{2}\int_0^L\int_0^L(v(t)-v(\tau))^2K_{s,L}(t-\tau)dtd\tau+c_{n,s}\int_0^L v(t)^2dt}{\left(\int_0^L v(t)^{\frac{2n}{n-2s}}dt\right)^{\frac{n-2s}{n}}},\tag{11}$$

where one minimizes over all the functions such that the numerator is finite. The main theorem of [16, Theorem 1.1] states that for each $L>0$ the functional $\mathcal{E}_{s,L}$ in (11) has a nonzero minimizer. Furthermore, when $0<L\ll 1$ is sufficiently small, this minimizer is a positive constant; however, it is not constant when $L\gg 1$ is sufficiently large.

We complete this description by remarking that the preceding paragraphs classify the Delaunay solutions in the cylindrical gauge. We can subsequently change coordinates to write the Delaunay solution in the Euclidean or the spherical gauges. In the Euclidean gauge, we have

$$\bar{u}_\varepsilon(x)=\mathfrak{F}^{-1}(v_\varepsilon)(x)=|x|^{\frac{2s-n}{2}}\bar{v}_\varepsilon(-\ln|x|).\tag{12}$$

One can then transfer this solution to the sphere using stereographic projection, obtaining

$$U_\varepsilon=\bar{u}_\varepsilon\circ\Pi=\mathfrak{F}^{-1}\circ\bar{v}_\varepsilon\circ\Pi$$

2.3. **Morse index and stability.** We discuss the Morse index of the standard Aubin–Talenti bubbles classified by Chen, Li and Ou [11, Theorem 1.1]. Let us recall that this family of bubbles of order $s\in(0,n/2)$ solving ($\overline{\mathcal{Q}}_{n,s}$) such that $\lim_{x\to 0}u(x)<\infty$ is given by

$$u_{x_0,\varepsilon}(x)=u_\circ\left(\frac{x-x_0}{\varepsilon}\right)=\hat{c}_{n,s}\left(\frac{\varepsilon+|x-x_0|^2}{2\varepsilon}\right)^{\frac{2s-n}{2}},$$

for some $x_0\in\mathbf{R}^n$ and $\varepsilon>0$, where $u_{0,1}=u_\circ$ is the spherical solution defined as (3). These solutions transform, after the Emden-Fowler change of variables, into

$$v_{t_0,\varepsilon}(t)=v_\circ\left(\frac{t-t_0}{\varepsilon}\right)=\hat{c}_{n,s}\cosh\left(\frac{t-t_0}{\varepsilon}\right)^{\frac{2s-n}{2}}$$

for some $t_0\in\mathbf{R}$ and $\varepsilon>0$, where $v_{0,1}(t)=v_\circ(t)=\hat{c}_{n,s}\cosh(t)^{\frac{2s-n}{2}}$.

The main result of Dávila, del Pino and Sire [15, Theorem 1.1] characterizes the kernel of

$$\overset{\circ}{\mathcal{L}}_s=(-\Delta)^s-\tfrac{n+2s}{n-2s}c_{n,s}u_\circ^{\frac{4s}{n-2s}}.\tag{13}$$

In fact, they are generated by the Jacobi fields arising from the action of two conformal transformations, namely translations and dilations. Later, Andrade, Wei and Ye [4, Lemma 4.6] (see also [10, Lemma 5.1]) extended this nondegeneracy result to the full range of exponents, namely $s\in(0,\frac{n}{2})$. Changing from the Euclidean gauge to the cylindrical gauge transforms the spherical linearized operator into

$$\begin{aligned}\overset{\circ}{\mathcal{L}}_s(v)(t)&=P_s(v)(t)+c_{n,s}v(t)-\tfrac{n+2s}{n-2s}c_{n,s}v_\circ(t)^{\frac{4s}{n-2s}}v(t)\\&=\int_{-\infty}^{\infty}K_s(t-\tau)(v(t)-v(\tau))d\tau+c_{n,s}v(t)-\tfrac{n+2s}{n-2s}c_{n,s}v_\circ(t)^{\frac{4s}{n-2s}}v(t).\end{aligned}$$

Theorem A ([15]). *Let $n\in\mathbf{N}$ and $s\in(0,1)$ with $n\geq 3$. There holds*

$$\ker(\overset{\circ}{\mathcal{L}}_s)\cap L^\infty(\mathbb{R})=\operatorname{span}\left\{\frac{\partial u_\circ}{\partial x^i}:i=0,1,\ldots,n\right\},$$

where

$$\frac{\partial u_\circ}{\partial x^0}(x):=\tfrac{n-2s}{2}u_\circ(x)+\langle x,\nabla u_\circ\rangle.$$

In particular, the Morse index of the spherical solutions

$$u_\circ(x) = \hat{c}_{n,s}\left(\frac{1+|x|^2}{2}\right)^{\frac{2s-n}{2}} \quad \text{and} \quad v_\circ(t) = \hat{c}_{n,s}\cosh(t)^{\frac{2s-n}{2}}$$

are both equal to one.

The last result states that the standard bubble solution above is nondegenerate in a sense, the set of bounded solutions in the kernel of the spherical linearized operator is spanned by the functions

$$\frac{\partial u_\circ}{\partial x^0}(x) = \tfrac{n-2s}{2}u_\circ(x) + \langle x, \nabla u_\circ\rangle \quad \text{and} \quad \frac{\partial u_\circ}{\partial x^i}(x) \quad \text{for} \quad i \in \{1,\dots,n\}.$$

On a related note, we observe that the results of [14, Theorem 1.2] prove that all the Delaunay solutions (and its deformations) $\bar{u}_\varepsilon \in \mathcal{C}^\infty(\mathbf{R}^n\setminus\{0\})$ and $\bar{v}_\varepsilon \in \mathcal{C}^\infty(\mathbf{R})$ described above have infinite Morse index.

3. A priori and sharp supremum estimates

In this section, we establish two types of upper bounds for positive solutions to the critical fractional equation on the punctured sphere and its cylindrical counterpart. We begin with a general upper bound in the setting that the singular set is a closed subset of $\mathbf{S}^n$ with zero capacity. This estimate, though quite general, is not sharp enough for our purposes, so after establishing this general upper bound we specialize to the case of a twice-punctured sphere. Using the spectral properties of the linearized operator around the spherical profile $v_\circ$, we show that this profile realizes the global supremum among all positive solutions to the logarithmic cylindrical equation. This rigidity result exploits the fact that $v_\circ$ has Morse index one and illustrates the optimality of the bubble in this conformal class.

3.1. **Upper bound estimate.** In this section, we derive *a priori* upper bounds in a more general setting.

Lemma 4. *Let $n \in \mathbf{N}$ and $s \in (0,1)$ with $n \geq 3$, and let $\Lambda \subset \mathbf{S}^n$ be a closed set with zero capacity. If $U : \mathbf{S}^n\setminus\Lambda \to (0,\infty)$ satisfies ($\mathcal{Q}^\circ_{n,s}$), then there exists $C > 0$ depending only on n and s such that*

$$U(x) \leq C\,\mathrm{dist}_{g_\circ}(x,\Lambda)^{\frac{2s-n}{2}}. \tag{14}$$

Remark 5. *In the case that Λ is finite, this result follows from [6, Proposition 3.1]. This more general proof when Λ is a closed set with zero capacity follows the model first used by Pollack in [29, Theorem 4.3] and has since been adapted in a number of other places, such as [26], [30], [9], and [3]. Since this same argument has appeared now a number of times in the literature, we only provide its sketch.*

Proof. We argue by contradiction, adapting the blow-up argument of Pollack [29, Theorem 4.3], combined with the convexity argument of Schoen [33] and its conformal extension by Qing and Raske [30, Theorem 4.1].

Fix a regular point $x_\infty \in \mathbf{S}^n\setminus\Lambda$ and consider the stereographic projection with pole at x_∞, identifying $\mathbf{S}^n\setminus\{x_\infty\}$ with $\mathbf{R}^n$. Let $u : \mathbf{R}^n\setminus\widehat{\Lambda} \to (0,\infty)$ be the stereographic projection of U, where $\widehat{\Lambda}$ is the image of Λ under the projection. Then u satisfies the Euclidean conformally invariant equation

$$(-\Delta)^s u = c_{n,s}u^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \mathbf{R}^n\setminus\widehat{\Lambda},$$

where $\widehat{\Lambda}$ is a finite set. Since the singular set $\widehat{\Lambda}$ has capacity zero, the results of [28] imply that u is s'-superharmonic for every $s' \in (0,s)$.

Next, we define the conformal metric $g_u = u^{\frac{4}{n-2s}} g_{\mathrm{flat}}$. By applying the moving planes method of Schoen [33] adapted to the fractional setting by Qing and Raske [30, Theorem 4.1], we deduce that every Euclidean ball in the domain $\mathbf{R}^n \setminus \widehat{\Lambda}$ is strictly mean-convex with respect to g_u. This key convexity property will be used to obtain a contradiction.

Now, we suppose by contradiction that the desired estimate (14) fails. Then, there exists a sequence of data Λ_k, solutions u_k, points $x_{0,k}$, and radii $\rho_k > 0$ such that $B_{\rho_k}(x_{0,k}) \subset \mathbf{R}^n \setminus \widehat{\Lambda}_k$ and the function

$$f_k(x) := (\rho_k - |x - x_{0,k}|)^{\frac{2s-n}{2}} u_k(x)$$

attains arbitrarily large supremum in $B_{\rho_k}(x_{0,k})$. That is, there exists $x_{1,k} \in B_{\rho_k}(x_{0,k})$ such that

$$M_k := f_k(x_{1,k}) = \sup_{x \in B_{\rho_k}(x_{0,k})} f_k(x) \to \infty \quad \text{as} \quad k \to \infty.$$

We define the scaling factor $S_k := \frac{1}{2} M_k^{-\frac{2}{n-2s}}$, and consider the blow-up sequence

$$w_k(y) := S_k^{\frac{n-2s}{2}} u_k(x_{1,k} + S_k y),$$

which, by construction, satisfies $w_k(0) = 2^{\frac{2s-n}{2}}$ and

$$(-\Delta)^s w_k = c_{n,s} w_k^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \Omega_k := \frac{1}{S_k} \left(\mathbf{R}^n \setminus \widehat{\Lambda}_k - x_{1,k} \right).$$

Because the singular sets $\widehat{\Lambda}_k \subset \mathbf{R}^n$ diverge to infinity in the rescaled variables, and thanks to the uniform a priori Harnack inequality from [35, Theorem 1.2], we obtain local $\mathcal{C}^{2,\alpha}$ estimates for w_k on compact subsets. Passing to a subsequence, we conclude that $w_k \to w_\infty$ in $\mathcal{C}^2_{\mathrm{loc}}(\mathbf{R}^n)$, where $w_\infty \in \mathcal{C}^2(\mathbf{R})$ satisfies $w_\infty(0) = 2^{\frac{2s-n}{2}}$ and solves

$$(-\Delta)^s w_\infty = c_{n,s} w_\infty^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \mathbf{R}^n.$$

By the classification result of Chen, Li and Ou [12, Theorem 1.1] w_∞ must be the standard spherical solution

$$w_\infty(y) = \left(\frac{\varepsilon}{1 + \varepsilon^2 |y - y_0|^2} \right)^{\frac{n-2s}{2}}.$$

for some $\varepsilon > 0$ and $y_0 \in \mathbf{R}^n$.

Since $w_k \to w_\infty$ locally uniformly, we find that for large $k \gg 1$, the metrics g_{u_k} become nearly spherical on large balls centered at $x_{1,k}$. In particular, there exists $R > 0$ such that the ball $B_R(x_{1,k})$ has a boundary that is strictly mean-concave with respect to the conformal metric g_{u_k}. But this contradicts the earlier geometric result that all Euclidean balls must have strictly mean-convex boundaries for g_{u_k}, as deduced from the fractional moving planes method.

Finally, this contradiction proves the desired a priori upper bound (14). The constant $C > 0$ depends only on n and s, completing the proof. □

3.2. **Sharp supremum estimate.** In the special case where $\Lambda = \{N, S\}$ is a pair of points, we can transform this problem to the cylinder using (6), and so (14) reads $v(t) \leq C$, where C is a positive constant depending on n and s. However, because the proof of (14) uses an argument by contradiction, we do not obtain a sharp constant.

In preparation, we need the following result.

Lemma 6. *Let $n \in \mathbf{N}$ and $s \in (0,1)$ with $n \geq 3$. If $v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0,\infty)$ solves $(\widehat{\mathcal{Q}}_{n,s})$ we cannot have $v > v_\circ$ on all of $\mathbf{R} \times \mathbf{S}^{n-1}$.*

Proof. This is an adaptation of the proof of [14, Proposition 2.5]. Following their notation, for any $L > 0$ we define

$$\lambda_1(L) = \inf\left\{ \frac{\frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}} K_s(t-\tau)(v(t)-v(\tau))^2 dt d\tau}{\int_{\mathbf{R}} v^2(t)dt} : v \in H^s(\mathbf{R})\backslash\{0\} \text{ and } \operatorname{supp}(v) \subset [-L, L] \right\}$$

and let $\phi_1 \in H^s(\mathbf{R})\backslash\{0\}$ be the associated eigenfunction. They show in [14, Lemma 2.4] that the eigenvalue is positive for each $L > 0$ and that $\lim_{L\to\infty} \lambda_1(L) = 0$, as well as the fact that the eigenfunction is a positive solution to

$$P_s(\phi_1) = \lambda_1(L)\phi_1 \quad \text{in} \quad (-L, L). \tag{15}$$

Now we analyze $P_s(v - v_\circ - \alpha\phi_1)$ on the interval $[-L, L]$ where $\alpha > 0$ and $L > 0$ are parameters to be chosen later. First we estimate $P_s(v - v_\circ)$. The function $\eta_1(\xi) = \xi^{\frac{n+2s}{n-2s}} - \xi$ is convex, so its graph lies above each of its tangent lines. In other words,

$$\eta_1(v) \geq \eta_1'(v_\circ)(v - v_\circ) + \eta_1(v_\circ),$$

and so

$$\begin{aligned} P_s(v - v_\circ)(t) &= c_{n,s}((v(t))^{\frac{n+2s}{n-2s}} - v(t) - (v_\circ(t))^{\frac{n+2s}{n-2s}} + v_\circ(t)) \\ &= c_{n,s}(\eta_1(v)(t) - \eta_1(v_\circ)(t)) \\ &\geq c_{n,s}\eta_1'(v_\circ)(t)(v(t) - v_\circ(t)) \\ &= c_{n,s}\left(\left(\frac{n+2s}{n-2s}\right)\hat{c}_{n,s}^{\frac{4s}{n-2s}}(\cosh t)^{-2s} - 1\right)(v(t) - v_\circ(t)). \end{aligned} \tag{16}$$

If we choose $L_0 > 0$ such that

$$\left(\frac{n+2s}{n-2s}\right)\hat{c}_{n,s}^{\frac{4s}{n-2s}}(\cosh L_0)^{-2s} = \frac{3}{2}$$

then (16) implies

$$P_s(v - v_\circ)(t) \geq \frac{1}{2}c_{n,s}(v(t) - v_\circ(t)) \quad \text{for} \quad t \in [-L_0, L_0]. \tag{17}$$

Now, we choose $\alpha > 0$ such that

$$\frac{1}{\alpha} = \sup_{t\in[-L_0,L_0]} \frac{\phi_1(t)}{v(t) - v_\circ(t)}.$$

We then have

$$w(t) = v(t) - v_\circ(t) - \alpha\phi_1(t) \geq 0, \tag{18}$$

and, by continuity, there exists $t_0 \in [-L_0, L_0]$ such that $w(t_0) = 0$. Combining (17) and (18), we obtain

$$P_s(w)(t) \geq \left(\frac{1}{2}c_{n,s} - \lambda_1(L_0)\right)(v(t) - v_\circ(t)),$$

so we can conclude that $P_s(w) \geq 0$ and use the maximum principle (see [14, Proposition 2.2]) as soon as we know $\lambda_1(L_0) \leq \frac{c_{n,s}}{2}$. However, this inequality follows from [14, Eq. (2.4)] and the subsequent scaling law. $\square$

Proposition 7. *Let $n \in \mathbf{N}$ and $s \in (0, 1)$ with $n \geq 3$. If $v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0, \infty)$ solves ($\widehat{\mathcal{Q}}_{n,s}$), then*

$$\sup_{t\in\mathbf{R}} v(t) \leq \sup_{t\in\mathbf{R}} v_\circ(t) = \hat{c}_{n,s}.$$

Remark 8. *In some sense, this estimate is similar to the height estimate of a constant mean curvature graph in [25, Lemma 1.7].*

Proof. Suppose by contradiction that there exists a solution $v : \mathbf{R} \to (0, \infty)$ to (9) such that

$$\sup_{t\in\mathbf{R}} v(t) > \sup_{t\in\mathbf{R}} v_\circ(t) = \hat{c}_{n,s}.$$

After a translation, we may assume

$$v(0) > v_\circ(0) = \hat{c}_{n,s} = \sup_{t\in\mathbf{R}} v_\circ(t).$$

By the lemma above, we cannot have $v \geq v_\circ$, so there is a maximal open interval $(t_{0,-}, t_{0,+})$ containing 0 on which $v(t) > v_\circ(t)$.

Let us define

$$t_{0,+} := \sup \left\{ t > 0 : v(\tau) > v_\circ(\tau) \text{ for all } \tau \in (0,t) \right\},$$

and

$$t_{0,-} := \inf \left\{ t < 0 : v(\tau) > v_\circ(\tau) \text{ for all } \tau \in (t,0) \right\}.$$

Then, by the classification result in [12, Theorem 1.1], we know that any solution $v \in \mathcal{C}^2(\mathbf{R})$ with fast-decay $\lim_{|t|\to\infty} v(t) = 0$ must be a translation of the spherical solution $v_\circ$. Since $v \not\equiv v_\circ$, there holds

$$\limsup_{t\to\pm\infty} v(t) > 0.$$

This ensures that the function $v - v_\circ$ oscillates and changes sign. In particular, there exist maximal intervals $(t_{0,+}, t_{1,+})$ and $(t_{1,-}, t_{0,-})$ such that

$$v - v_\circ < 0 \quad \text{in} \quad (t_{0,+}, t_{1,+}) \cup (t_{1,-}, t_{0,-}).$$

We sketch the intervals $(t_{0,+}, t_{1,+})$ and $(t_{1,-}, t_{0,-})$ in Figure 1 below.

Now, let us define the test functions

$$w_+(t) := \begin{cases} v(t) - v_\circ(t), & \text{if } t \in (t_{0,+}, t_{1,+}), \\ 0, & \text{if } t \in (-\infty, t_{0,+}) \cup (t_{1,+}, \infty) \end{cases}$$

and

$$w_-(t) := \begin{cases} v(t) - v_\circ(t), & \text{if } t \in (t_{1,-}, t_{0,-}), \\ 0, & \text{if } t \in (-\infty, t_{1,-}) \cup (t_{0,-}, \infty). \end{cases}$$

Notice that both $w_\pm \leq 0$ on $\mathrm{supp}(w_\pm)$, and are compactly supported, piecewise smooth functions in the energy space.

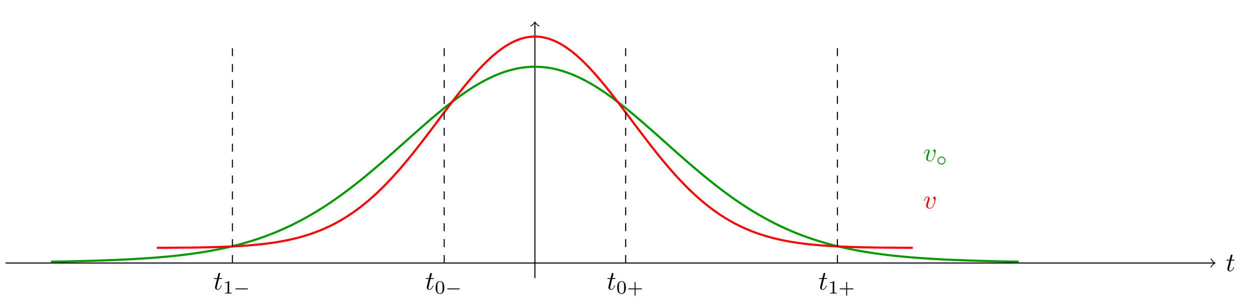

FIGURE 1. This figure shows the spherical solution in green and the supposed solution v whose maximum is greater than the spherical solution in red.

We compute the quadratic form associated with the linearized operator $\overset{\circ}{\mathcal{L}}_s$ at $v_\circ$ on w_+ to get

$$\begin{aligned}\int_{-\infty}^{\infty} w_+\overset{\circ}{\mathcal{L}}_s(w_+)\,dt &= \int_{t_{0,+}}^{t_{1,+}} (v-v_\circ)\left(\overset{\circ}{\mathcal{L}}_s(v-v_\circ)\right)dt\\ &= \int_{t_{0,+}}^{t_{1,+}} (v-v_\circ)\left(P_s(v-v_\circ)+c_{n,s}(v-v_\circ)-\left(\frac{n+2s}{n-2s}\right)c_{n,s}v_\circ^{\frac{4s}{n-2s}}(v-v_\circ)\right)dt\\ &= c_{n,s}\int_{t_{0,+}}^{t_{1,+}} (v-v_\circ)\left(v^{\frac{n+2s}{n-2s}}-v_\circ^{\frac{n+2s}{n-2s}}-\left(\frac{n+2s}{n-2s}\right)v_\circ^{\frac{4s}{n-2s}}(v-v_\circ)\right)dt. \end{aligned} \tag{19}$$

Observe that $\eta_2(\xi)=\xi^{\frac{n+4}{n-4}}$ is a strictly convex function, and that the integrand has the form

$$(v-v_\circ)(\eta_2(v)-\eta_2(v_\circ)-\eta_2'(v_\circ)(v-v_\circ)).$$

Since η_2 is convex, the second term

$$\eta_2(v)-\eta_2(v_\circ)-\eta_2'(v_\circ)(v-v_\circ)>0$$

is strictly positive. By our construction $v-v_\circ<0$ on the interval $(t_{0,+},t_{1,})$, and so the total integral is negative *i.e.*

$$\int_{-\infty}^{\infty} w_+\overset{\circ}{\mathcal{L}}_s(w_+)\,dt<0 \quad\text{and}\quad \int_{-\infty}^{\infty} w_-\overset{\circ}{\mathcal{L}}_s(w_-)\,dt<0.$$

Since $\operatorname{supp}(w_+)\cap\operatorname{supp}(w_-)=\varnothing$, these test functions are orthogonal in L^2, and $\operatorname{span}\{w_+,w_-\}$ is a two-dimensional subspace on which the linearized operator around spherical solutions is negative definite. Therefore, the quadratic form has at least two negative directions, contradicting the fact that $v_\circ$ has Morse index one. This contradiction proves that no such function v with $\sup_{t\in\mathbf{R}} v(t)>\hat{c}_{n,s}$ can exist. Hence, one has

$$\sup_{t\in\mathbf{R}} v(t)\leq \sup_{t\in\mathbf{R}} v_\circ(t)=\hat{c}_{n,s}.$$

and the proof of the proposition is finished. □

4. The proof of classification

We now prove Theorem 1. The argument has two parts. First, a local compactness step shows that the normalized difference between a solution and the corresponding Delaunay profile vanishes on compact intervals. Second, a global step enhances the local convergence to the entire line by combining the Green representation, the decay of the kernel, the weighted norm, and an ODE barrier that excludes nontrivial limit profiles in the noncompact recentering regime.

4.1. **Local classification.** The local step relies on a compactness argument for the normalized difference between two solutions. After normalizing and rescaling on $[-L,L]$, we extract a limit profile that solves a homogeneous limiting equation with zero Cauchy data; hence, it must be trivial.

Proposition 9. *Let $n\in\mathbf{N}$ and $s\in(0,1)$ with $n\geq 3$. For each fixed $L>0$, there exists $\delta>0$ such that if $1-\delta<s<1$ and $v:\mathbf{R}\to(0,\infty)$ satisfies ($\widehat{\mathcal{Q}}_{n,s}$), then the restriction of v to $[-L,L]$ is a Delaunay solution.*

Proof. We prove this proposition by contradiction. Assume there exist a sequence $\{s_k\}_{k\in\mathbf{N}}\subset(0,1)$ of fractional powers such that $s_k\nearrow 1$ as $k\to\infty$ and a sequence of positive smooth solutions $\{v_k\}_{k\in\mathbf{N}}\subset\mathcal{C}^2(\mathbf{R})$ to ($\widehat{\mathcal{Q}}_{n,s}$), that is

$$P_{s_k}(v_k)+c_{n,s_k}v_k=c_{n,s_k}v_k^{\frac{n+2s_k}{n-2s_k}} \quad\text{in}\quad \mathbf{R}\quad\text{for}\quad k\in\mathbf{N}. \tag{20}$$

Furthermore, we assume v_k is not a Delaunay solution for each k.

Now, for each $k \in \mathbf{N}$, we choose a Delaunay solution $\overline{v}_k$ so that

$$\|v_k\|_{L^\infty([-L,L])} = \|\overline{v}_k\|_{L^\infty([-L,L])} \quad \text{and} \quad v_k(0) = \|v_k\|_{L^\infty([-L,L])} = \|\overline{v}_k\|_{L^\infty([-L,L])} = \overline{v}_k(0).$$

Let us set

$$\phi_k := \overline{v}_k - v_k, \tag{21}$$

which implies

$$\phi_k(0) = \phi_k'(0) = 0. \tag{22}$$

Also, setting

$$\mathcal{N}_k(v) := P_{s_k}(v) + c_{n,s_k} v - c_{n,s_k}\, v^{\frac{n+2s_k}{n-2s_k}},$$

it holds

$$\begin{aligned} 0 &= \mathcal{N}_k(\overline{v}_k) - \mathcal{N}_k(v_k) \\ &= P_{s_k}(\phi_k) + c_{n,s_k}\phi_k - c_{n,s_k}\left(\overline{v}_k^{\frac{n+2s_k}{n-2s_k}} - (\overline{v}_k - \phi_k)^{\frac{n+2s_k}{n-2s_k}}\right) \\ &=: \mathcal{L}_k(\phi_k) - \mathcal{R}_k(\phi_k), \end{aligned} \tag{23}$$

where

$$\mathcal{L}_k(\phi_k) = P_{s_k}(\phi_k) + c_{n,s_k}\phi_k - \left(\tfrac{n+2s_k}{n-2s_k}\right) c_{n,s_k}\, \overline{v}_k^{\frac{4s_k}{n-2s_k}}\phi_k \quad \text{and} \quad \mathcal{R}_k(\phi_k) = \mathcal{O}(\|\phi_k\|^2) \quad \text{as} \quad k \to \infty.$$

Define a normalized sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{C}^2(\mathbf{R})$ such that

$$\psi_k(t) := \frac{\phi_k(t)}{A_k} \quad \text{with} \quad A_k := \max_{t\in[-L,L]} |\phi_k(t)| \quad \text{for} \quad k \in \mathbf{N}.$$

Then, it holds $\|\psi_k\|_{L^\infty([-L,L])} = 1$ and $\psi_k(0) = \psi_k'(0) = 0$. Moreover, one has

$$\mathcal{L}_k(\psi_k) = \frac{1}{A_k}\mathcal{R}_k(\phi_k) = \mathcal{O}\big(\|\phi_k\|_{L^\infty([-L,L])}\big) \to 0 \quad \text{as} \quad k \to \infty.$$

Recall that each $\overline{v}_k$ is a Delaunay-type solution to ($\widehat{\mathcal{Q}}_{n,s}$) corresponding to the same necksize normalization

$$\overline{v}_k(0) = v_k(0) = \|v_k\|_{L^\infty(\mathbf{R})}.$$

Since the parameters $s_k \nearrow 1$, the coefficients of the equation are uniformly bounded, and by Proposition 7, we can extract a subsequence such that $\overline{v}_k \longrightarrow \overline{v}_*$ in $\mathcal{C}^{2,\alpha}_{\rm loc}(\mathbf{R})$ for some $\alpha \in (0,1)$, where $\overline{v}_* \in \mathcal{C}^2(\mathbf{R})$ is the classical (local) Delaunay profile solving

$$-\overline{v}_*'' + c_{n,1}\,\overline{v}_* = c_{n,1}\,\overline{v}_*^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{R}.$$

The corresponding linearized operator around the local Delaunay profile is therefore

$$\mathcal{L}_*(\psi) := -\psi'' + c_{n,1}\,\psi - \left(\tfrac{n+2}{n-2}\right) c_{n,1}\,\overline{v}_*^{\frac{4}{n-2}}\psi \quad \text{for} \quad \psi \in \mathcal{C}^2(\mathbf{R}). \tag{24}$$

This operator is precisely the local ($s = 1$) limit of the family $\mathcal{L}_k$, since $P_{s_k} \to -\partial_t^2$ in the strong resolvent sense as $s_k \to 1^-$ (see the estimates (35), (36) and (37) below) and $\overline{v}_k \to \overline{v}_*$ in $\mathcal{C}^2_{\rm loc}$.

Next, recall from (23) that

$$\mathcal{L}_k(\psi_k) = \frac{1}{A_k}\mathcal{R}_k(\phi_k) = \mathrm{o}(1) \quad \text{as} \quad k \to \infty, \tag{25}$$

and $\|\psi_k\|_{L^\infty([-L,L])} = 1$ with $\psi_k(0) = \psi_k'(0) = 0$. By Lemma 4 and standard regularity estimates, the sequence $\{\psi_k\}_{k\in\mathbf{N}} \subset \mathcal{C}^{2,\alpha}([-L,L])$ is equicontinuous and uniformly bounded.

From this, we can apply the Arzelà–Ascoli theorem to find a function $\psi_\infty \in \mathcal{C}^{2,\alpha}([-L,L])$ such that, up to subsequence, $\psi_k \to \psi_\infty$ in $\mathcal{C}^{2,\alpha}([-L,L])$ satisfying

$$\begin{cases} -\psi_\infty'' + \left(\frac{n-2}{2}\right)^2 \psi_\infty - \left(\frac{n(n+2)}{4}\right) v_*^{\frac{4}{n-2}} \psi_\infty = 0 \quad \text{in} \quad (-L,L) \\ \psi_\infty(0) = 0, \quad \psi_\infty'(0) = 0. \end{cases} \tag{26}$$

Hence, passing to the limit in the equation (25) gives us

$$\mathcal{L}_*(\psi_\infty) = 0 \quad \text{in} \quad [-L,L].$$

In view of (24) and the initial conditions $\psi_\infty(0) = \psi_\infty'(0) = 0$, the uniqueness theorem for ODEs implies $\psi_\infty \equiv 0$ on $[-L,L]$, which in turn contradicts the normalization $\|\psi_k\|_{L^\infty([-L,L])} = 1$ for each k. □

4.2. **Global classification.** In this section, we complete the proof of Theorem 1 by showing the convergence of $\{\psi_k\}$ to ψ_∞ is global, in $L^\infty(\mathbf{R})$. In other words, we show that one can choose the same δ for each L in Proposition 9.

Before proving global convergence, we record the ODE barrier that excludes fast left decay for global limit profiles.

Lemma 10. *Let $\sigma_* := \frac{n-2}{2} > 0$, $V \in L^\infty((-\infty,0])$ be a bounded nonnegative potential satisfying $0 \le V \le c_2$, and $\psi \in \mathcal{C}^2(\mathbf{R})$ be a solution to*

$$\begin{cases} -\psi'' + \sigma_*^2 \psi - V(\zeta)\psi = 0 \quad \text{in} \quad \mathbf{R}, \\ \psi(0) = 1. \end{cases} \tag{27}$$

If for some $\sigma > \sigma_*$ one has

$$|\psi(\zeta)| \le e^{\sigma\zeta} \quad \text{for all} \quad \zeta \le 0, \tag{28}$$

then $\psi \equiv 0$.

Proof. Let us define $w(\zeta) := e^{-\sigma_*\zeta}\psi(\zeta)$ for $\zeta \le 0$. A direct computation gives us

$$w'' + 2\sigma_* w' + V(\zeta)w = 0 \quad \text{on} \quad (-\infty,0],$$

which, by multiplying by an integrating factor, yields the exact derivative identity

$$\left(e^{2\sigma_*\zeta} w'(\zeta)\right)' = -e^{2\sigma_*\zeta} V(\zeta) w(\zeta).$$

Next, from the assumed bound (28), it follows that

$$|w(\zeta)| = e^{-\sigma_*\zeta}|\psi(\zeta)| \le e^{(\sigma_*-\sigma)|\zeta|} \to 0 \quad \text{as} \quad \zeta \to -\infty.$$

Hence, $w \in L^1((-\infty,0])$ and $e^{2\sigma_*\zeta}V(\zeta)w(\zeta) \in L^1((-\infty,0])$. In addition, by differentiating (27), we find $e^{2\sigma_* y}w'(\zeta) \to 0$ as $\zeta \to -\infty$. From this, by integrating from $-\infty$ to ζ, we obtain

$$w'(\zeta) = -e^{-2\sigma_*\zeta} \int_{-\infty}^{\zeta} e^{2\sigma_*\tau} V(\tau) w(\tau) d\tau.$$

Integrating by parts and using the Fundamental Theorem of Calculus along with the boundary condition $w(\zeta) \to 0$ as $\zeta \to -\infty$, we get the Volterra-type identity

$$w(\zeta) = -\frac{1}{2\sigma_*} \int_{-\infty}^{\zeta} \left(1 - e^{-2\sigma_*(\zeta-\tau)}\right) V(\tau) w(\tau) d\tau.$$

Thus, taking absolute values and recalling $0 \le V \le c_2$, we deduce

$$|w(\zeta)| \le \frac{c_2}{2\sigma_*} \int_{-\infty}^{\zeta} |w(\tau)| d\tau \quad \text{for} \quad \zeta \le 0.$$

Finally, by setting $\mathcal{G}(\zeta) := \int_{-\infty}^{\zeta} |w(\tau)| d\tau$, one has that

$$\frac{d}{d\zeta}\mathcal{G}(\zeta) \leq 0 \quad \text{and} \quad \lim_{\zeta \to -\infty} \mathcal{G}(\zeta) = 0.$$

and satisfies the differential inequality

$$\frac{d}{d\zeta}\mathcal{G}(\zeta) = |w(\zeta)| \leq \tfrac{c_2}{2\sigma_*}\mathcal{G}(\zeta).$$

By Grönwall's inequality, it follows $\mathcal{G} \equiv 0$ and hence $w \equiv 0$. Therefore $\psi \equiv 0$, contradicting $\psi(0) = 1$. The proof is then finished. □

Next, we also need a uniform interior $\mathcal{C}^{3,\alpha}$ estimate for solutions to the cylindrical linearized operator. For this, we recall the kernel

Lemma 11. *Let $n \in \mathbf{N}$ and $s \in (0,1)$ with $n \geq 3$. Let $I \subset \mathbf{R}$ be a bounded open interval and $V_s \in \mathcal{C}^{2,\alpha}(I)$ be a potential satisfying*

$$0 < c_1 \leq V_s(t) \leq c_2 \quad \text{and} \quad \limsup_{s \to 1^-} \|V_s\|_{\mathcal{C}^{2,\alpha}(I)} < \infty$$

for some $c_1, c_2 > 0$ independent of s. Assume $\phi \in L^\infty(\mathbf{R}) \cap \mathcal{C}^2(I)$ solves

$$P_s(\phi)(t) + c_{n,s}\phi(t) - \ V_s(t)\phi(t) = \varphi_s(t) \quad \text{in} \quad I \tag{29}$$

with $\varphi_s \in \mathcal{C}^{1,\alpha}(I)$ and $\|\varphi_s\|_{\mathcal{C}^{1,\alpha}(I)} < \infty$ uniformly in s. Then, for every $J \subset\subset I$ there exist $\alpha \in (0,1)$, $s_0 \in (\frac{1}{2}, 1)$, and $C > 0$ not depending on $s \in [s_0, 1)$, such that

$$\|\phi\|_{\mathcal{C}^{3,\alpha}(J)} \leq C \left(\|\phi\|_{L^\infty(\mathbf{R})} + \|\varphi_s\|_{\mathcal{C}^{1,\alpha}(I)}\right). \tag{30}$$

In particular, the $\mathcal{C}^{3,\alpha}$ constant is uniform as $s \to 1^-$, and it does not depend on the point $t \in J$ up to third order.

Proof. This is a direct consequence of the regularity results by Silvestre [34, Theorem 5.1] and Ros-Oton and Serra [31, Theorem 1.1] under the verification that the log-cylindrical operator $\mathbb{P}_s = P_s + c_{n,s}$ on the right-hand side of (29) satisfies the ellipticity conditions [34, Eq. (2.2), pg. 1158] and [31, Eq. (1.2), pg. 8676]. Here, we notice that by [22, Theorem 1.1] there exists $s_0 \in (\frac{1}{2}, 1)$ such that the regularity constant $C > 0$ does not depend on $s \in (s_0, 1]$. □

Let us introduce a weighted space suitable for studying our linearized equation.

Definition 12. *Let $n \in \mathbf{N}$ with $n \geq 3$. For any $\frac{n-2}{2} < \sigma \leq \frac{n}{2}$, let us define the weighted norm*

$$\|\phi\|_* := \sup_{t \in \mathbf{R}} |e^{-\sigma|t|}\phi(t)|.$$

Remark 13. *In the last definition, we can assume $\sigma = \frac{n}{2}$. Indeed, it is sufficient to choose any exponent that allows us to annihilate the kernel of the operator P_s, thereby enabling us to apply Lemma 10.*

We are now ready to prove the global compactness.

Proof of Theorem 1 (global part). Initially, from (23), for each $k \in \mathbf{N}$ the difference function $\phi_k \in \mathcal{C}^2(\mathbf{R})$ given by (21) satisfies the linearized equation below

$$\mathcal{L}_{s_k}(\phi_k) = \mathrm{o}_k(1) \quad \text{in} \quad \mathbf{R} \quad \text{as} \quad s_k \to 1^- \quad \text{or} \quad k \to \infty, \tag{31}$$

where

$$\mathcal{L}_{s_k}(\phi_k) := \int_{-\infty}^{\infty}\int_{\mathbf{S}^{n-1}} (\phi_k(t) - \phi_k(\tau))K_{s_k}(t-\tau)d\tau + c_{n,s}v(t) - \left(\tfrac{n+2s_k}{n-2s_k}\right)c_{n,s_k}\,\overline{v}_k^{\frac{4s_k}{n-2s_k}}\,\phi.$$

Thus, by setting

$$P_{s_k}(\phi_k) := \int_{-\infty}^{\infty}\int_{\mathbf{S}^{n-1}} (\phi(t)-\phi(\tau))K_{s_k}(t-\tau)d\tau \quad \text{and} \quad V_{s_k}(t) = \left(\tfrac{n+2s_k}{n-2s_k}\right)c_{n,s_k}\,\overline{v}_k^{\frac{4s_k}{n-2s_k}},$$

we can reformulate (31) as

$$\begin{cases} P_{s_k}(\phi_k) + c_{n,s_k}\phi_k - V_{s_k}(t)\phi_k = \mathrm{o}_k(1) \quad \text{in} \quad \mathbf{R} \\ \phi_k(0) = \phi_k'(0) = 0 \end{cases} \tag{32}$$

with $0 < c_1 < V_{s_k}(t) < c_2$ for $s_k \in (0,1)$, where we used (22).

We are now in a position to prove that the limit $\phi_\infty = \lim_{k\to\infty}\phi_k$ must vanish identically, and so (20) has a unique solution, which is the so-called Delaunay solutions $\bar{v}_k \in \mathcal{C}(\mathbf{R}^2)$.

We argue by contradiction. Suppose that $\phi_k \not\equiv 0$ for each $k \in \mathbf{N}$, then by linearity, we may assume that $\|\phi_k\|_* = 1$. Thus, one can find a sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbf{R}$ such that

$$\|\phi_k\|_* = 1 = e^{-\sigma|t_k|}\phi(t_k),$$

which implies

$$\phi_k(t_k) = e^{\sigma|t_k|} \quad \text{and} \quad |\phi_k(t)| \leq C_0 e^{\sigma|t|} \quad \text{for all} \quad t \in \mathbf{R} \tag{33}$$

for some $C_0 = C_0(n, s_k) > 0$. This will allow us to take the limit as $s_k \to 1^-$.

Setting $\rho = t-\tau$ and using [16, Lemmas 2.5 and 2.6], one knows that there exist $\varepsilon, M \in \mathbf{R}$ with $0 \ll \varepsilon \ll 1$ small and $M \gg 1$ large such that the kernel of P_s acting on radial functions satisfies

$$K_s(\rho) \sim \begin{cases} \varsigma_{n,s}|\rho|^{-(1+2s)}, & \text{if} \quad 0 < |\rho| < \varepsilon \\ \varsigma_{n,s}, & \text{if} \quad 0 \leq \varepsilon < |\rho| \leq M < \infty \\ \varsigma_{n,s}e^{-\frac{n+2s}{s}|\rho|}, & \text{if} \quad |\rho| > M, \end{cases}$$

where

$$\varsigma_{n,s} = \pi^{-\frac{n}{2}}2^{2s}\frac{\Gamma(\frac{n}{2}+s)}{\Gamma(1-s)}s \sim 1-s \quad \text{as} \quad s \to 1^-.$$

Therefore, as $s_k \to 1^-$, we have

$$P_{s_k}(\phi_k) = \int_{-\infty}^{\infty}\int_{\mathbf{S}^{n-1}} (\phi_k(t)-\phi_k(\tau))K_{s_k}(t-\tau)d\tau \sim \varsigma_{n,s}(I_1+I_2+I_3) \sim (1-s_k)(I_1+I_2+I_3), \tag{34}$$

where

$$I_1 := \int_{|t-\tau|<\varepsilon} (\phi_k(t)-\phi_k(\tau))|t-\tau|^{-(1+2s_k)}\,d\tau,$$

$$I_2 := \int_{\varepsilon\leq|t-\tau|\leq M} (\phi_k(t)-\phi_k(\tau))\,d\tau,$$

and

$$I_3 := \int_{|t-\tau|>M} (\phi_k(t)-\phi_k(\tau))e^{-\frac{n+2s_k}{2}|t-\tau|}d\tau.$$

It remains to estimate the integral terms I_1, I_2, and I_3. First, by using the regularity in Lemma 11 and the parity of the kernel, we can Taylor expand $\phi_k(\tau)$ to get

$$\begin{aligned}
I_1 &= \int_{|t-\tau|<\varepsilon} (\phi_k(t)-\phi_k(\tau))|t-\tau|^{-(1+2s_k)}\,d\tau \\
&= \int_{|t-\tau|<\varepsilon} \left[-\phi_k'(t)(\tau-t)-\phi_k''(t)\frac{(\tau-t)^2}{2}+\mathcal{O}(|\tau-t|^3)\right]|t-\tau|^{-(1+2s_k)}\,d\tau \\
&= \int_{-\varepsilon}^{\varepsilon} -|\rho|^{-(1+2s)}\left[\phi_k'(t)\rho+\phi_k''(t)\frac{\rho^2}{2}+\mathcal{O}(|\rho|^3)\right]d\rho \\
&= -\frac{\varepsilon^{2-2s_k}}{2-2s_k}\phi_k''(t)+\mathcal{O}(\varepsilon^2) = -\frac{1}{2(1-s_k)}\phi_k''(t)+\mathcal{O}(\varepsilon^2) \quad \text{as} \quad s_k\to 1^-
\end{aligned} \tag{35}$$

Second, using the bound (33), it is straightforward to see that

$$I_2 = \int_{\varepsilon\le|t-\tau|\le M}(\phi_k(t)-\phi_k(\tau))d\tau = \int_{\varepsilon\le|\rho|\le M} C_0 e^{\sigma\rho}d\rho \le C_2 \tag{36}$$

for some $C_2 = C_2(n,\varepsilon,M)>0$. Third, again thanks to the bound (33), one has

$$|I_3| = \left|\int_{|t-\tau|>M}(\phi_k(t)-\phi_k(\tau))e^{-\frac{n+2s_k}{2}|t-\tau|}d\tau\right| \le \int_{|\rho|>M} C_0 e^{\sigma|\rho|}e^{-\frac{n+2s_k}{2}|\rho|}d\rho \le C_3 \tag{37}$$

for some $C_3 = C_3(n,M)>0$. Therefore, we have the convergence below

$$P_{s_k}(\phi_k) \sim (1-s_k)\left[\frac{1}{2(1-s_k)}\phi_k''+C_2+C_3\right] \sim -\frac{1}{2}\phi_k'' \sim -\phi_k'' \quad \text{as} \quad s_k\to 1^-.$$

Hence, when passing to the limit as $s_k\to 1^-$ or $k\to\infty$ in (32), we get

$$\begin{cases} \phi_\infty''+c_{n,1}\phi_\infty - V_\infty(t)\phi_\infty = 0 \quad \text{in} \quad \mathbf{R} \\ \phi_\infty(0)=\phi_\infty'(0)=0, \end{cases} \tag{38}$$

where $V_\infty(t)=\frac{n(n+2)}{4}(v_*(t))^{\frac{4}{n-2}}$ and $v_* = \lim_{k\to\infty}\overline{v}_k$ is the limit Delaunay solution.

We now consider two cases, depending on whether the maximum of $t_\infty = \sup_{t\in\mathbf{R}}\phi_\infty(t)$ is attained on a bounded domain or not:

Case 1: $\limsup_{k\to\infty}|t_k|<\infty$.

In this case, we can take a limit and arrive at (38), which directly gives us $\phi_\infty\equiv 0$, contradicting the normalization $\phi_\infty(0)=0$ and proving the desired classification for (20).

Case 2: $\limsup_{k\to\infty}|t_k|=\infty$.

Without loss of generality, we may assume that $t_k>0$. In this case, we consider the rescaled function

$$\psi_k(\zeta) := \frac{\phi_k(t_k+\zeta)}{\phi_k(t_k)},$$

which satisfies $\psi_k(0)=1$ and

$$|\psi_k(\zeta)| \le e^{\sigma|t_k\zeta|}\frac{1}{\phi(t_k)} \le e^{\sigma|t_k+\zeta|}\frac{1}{e^{\sigma|t_k|}} \le e^{\sigma\zeta} \quad \text{for} \quad \zeta\in(-t_k,\infty),$$

Thus, since $\sigma=\frac{n}{2}<\frac{n+2s_k}{2}$, we can still pass to the limit as $s_k\to 1^-$. Therefore, the limit $\psi_\infty(\zeta)=\lim_{k\to\infty}\psi_k(\zeta)$ satisfies

$$|\psi_\infty(\zeta)| \le e^{\sigma|\zeta|} \quad \text{for} \quad \zeta\in(-t_k,\infty) \tag{39}$$

and solves

$$\begin{cases} -\psi_\infty'' + \left(\frac{n-2}{2}\right)^2 \psi_\infty - V_\infty(\zeta)\psi_\infty = 0 \quad \text{in} \quad \mathbf{R} \\ \psi_\infty(0) = 1 \end{cases} \tag{40}$$

and there exists a constant $c_2 > 0$ such that $0 \le V_\infty(t) \le c_2$. Finally, using (39) we can apply the Lemma 10 and the following remark to (40) and show that the decaying rate as $\zeta \to -\infty$ is too large. Thus, $\psi_\infty \equiv 0$, which is a contradiction to $\psi_\infty(0) = 1$, finishing the proof. □

5. The proof of nondegeneracy

In this section we prove Theorem 3. Since the proof is very similar to that of Theorem 1, we only indicate which changes are necessary.

We let $s_k \nearrow 1$, let $\varepsilon_k \in (0,1]$ and let v_k satisfy $(\widehat{\mathcal{Q}}_{n,s})$ on $\mathbf{R} \times \mathbf{S}^{n-1}$. By Theorem 1, we know that v_k is a Delaunay solution of order s_k, *i.e.* $v_k = v_{\varepsilon_k}$ for some $\varepsilon_k \in (0,1]$. Additionally, we let

$$\mathbb{L}_k := P_{s_k} + c_{n,s_k} - \tfrac{n+2s_k}{n-2s_k} c_{n,s_k} \bar{v}_{\varepsilon_k}^{\frac{4s_k}{n-2s_k}}$$

be the linearization of (9) about v_{ε_k} and let $w_k \in L^2(\mathbf{R} \times \mathbf{S}^{n-1})$ satisfy $\mathbb{L}_k(w_k) = 0$. Here we use two specific functions $\widetilde{w}_k^+$ of $\widetilde{w_k}^-$ in the kernel of the operator $\mathbb{L}_k$, namely

$$\widetilde{w_k}^+ = \dot{v}_{\varepsilon_k} \quad \text{and} \quad \widetilde{w_k}^- = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon_k} v_\varepsilon,$$

where the dot denotes the derivative with respect to t. If we let T_ε be the period of the Delaunay solution with necksize ε and differentiate the relation

$$v_\varepsilon(t + T_\varepsilon) = v_\varepsilon(t) \tag{41}$$

with respect to t, we see that $\widetilde{w}_k^+$ is bounded and periodic. Differentiating (41) with respect to ε we see that $\widetilde{w}_k^-$ grows at most linearly. Furthermore, direct computation tells us

$$\widetilde{w}_k^+(0) > 0, \quad \dot{\widetilde{w}}_k^+(0) = 0, \quad \widetilde{w}_k^-(0) = 0, \quad \text{and} \quad \dot{\widetilde{w}}_k^-(0) > 0.$$

Now we use these two tempered, geometric Jacobi fields to adjust w_k. After adding appropriate multiples of $\widetilde{w}_k^+$ and $\widetilde{w}_k^-$ to w_k we produce a new function $\widehat{w}_k$ that once again satisfies $\mathbb{L}_k(\widehat{w}_k) = 0$. Now fix $L > 0$ and let

$$\overline{w}_k(t) = \frac{1}{A_k} \widehat{w}_k(t)|_{[-L,L]} \quad \text{with} \quad A_k = \sup_{-L \le t \le L} \widehat{w}_k(t).$$

This new function satisfies

$$\mathbb{L}_k(\overline{w}_k) = 0, \quad \|\overline{w}_k\|_{L^\infty([-L,L])} = 1, \quad \text{and} \quad \overline{w}_k(0) = 0 = \dot{\overline{w}}_k(0).$$

Using the same argument as in § 4.1 we show that $\{\overline{w}_k\}$ converges locally to the zero function, and using the techniques in § 4.2 we pass from local convergence to global convergence. Combining these two convergence results, we contradict the normalization that $\|\overline{w}_k\|_{L^\infty([-L,L])} = 1$, which yields that $\widehat{w}_k \equiv 0$ and that w_k is a linear combination of $\widetilde{w}_k^+$ and $\widetilde{w}_k^-$. Clearly neither of them belongs to $L^2(\mathbf{R} \times \mathbf{S}^{n-1})$. This finishes the proof. □

6. Extensions and applications

We conclude this paper by first discussing the possibility of extending our classification to the case that $s > 1$ and then discuss some possible applications.

6.1. **Extending the range of the fractional parameter.** It is quite natural to ask whether our classification result remains valid when the order of the conformal operator exceeds one. In this subsection we discuss our work in progress [1] to classify conformally flat, constant Q_s-curvature metrics on a twice-punctured sphere in the case that $s \in (1, 1+\delta)$, where δ is sufficiently small.

The first complication in this classification lies in the description of the Delaunay solutions of order s when $s > 1$. Indeed, the integral operator P_s as given in (9) is not well-defined in the case $s > 1$, so one needs an alternative formulation of the problem. Jin and Xiong [23] give a dual formulation of the problem, starting with the equation

$$u = c_{n,s}(-\Delta)^{-s}\left(u^{\frac{n+2s}{n-2s}}\right). \tag{42}$$

After the Emden-Fowler change of variables they arrive at an integral operator with better convergence properties.

The larger complication is that the passage from local convergence to global convergence, as described in our Section 4.2 above, relies on a sharp estimate of the growth rate of solutions to the linearized equation. In the regime $s > 1$ these estimates are not currently available, and in particular we do not have an analog of Lemma 10 in this setting. Deriving such estimates associated to the linearization of the dual operator will occupy a large portion of [1].

6.2. **Informed speculation.** As a direct consequence of our classification, one can rapidly prove several related results. First of all, one can combine the nondegenerate with the slide-back technique in the paper by Korevaar, Mazzeo, Pacard and Schoen [24] to obtain refined asymptotics of solutions with isolated singularities. Second, we expect that one can adapt the proof of compactness in the moduli space setting, as in the paper by Pollack [29]. We only sketch the ideas of both of these proofs and leave the details for future projects.

6.2.1. *Local behavior near isolated singularities.* Using a blow-up analysis, Caffarelli, Jin, Sire and Xiong [6] show that any isolated singularity of a singular Yamabe metric on the sphere must be asymptotically radial. Combining this with our classification of the radial solutions, we see that any isolated singularity is asymptotic to a Delaunay solution. In other words, if $s < 1$ is sufficiently close to 1 and if $\Lambda \subset \mathbf{S}^n$ is a finite set and $U : \mathbf{S}^n \backslash \Lambda \to (0, \infty)$ satisfies the boundary condition and solves the PDE

$$\mathbb{P}_s U = c_{n,s} U^{\frac{n+2s}{n-2s}} \quad \text{on} \quad \mathbf{S}^n \setminus \Lambda. \tag{$\mathcal{Q}^{\circ}_{n,s,\Lambda}$}$$

then for each $p_* \in \Lambda$ and sufficiently small radius $0 < r \ll 1$ there exists a (rescaled) Delaunay solution $\bar{U}_\varepsilon$ such that

$$U(p) = \bar{U}_\varepsilon(p)(1 + o(1)) \quad \text{for all} \quad x \in \mathbf{B}_r(p_*)\backslash\{p_*\} \quad \text{as} \quad p \to p_*.$$

Changing to the cylindrical gauge in the punctured ball $\mathbf{B}_r(p)\backslash\{p\}$, we can write this estimate as

$$|v(t,\theta) - v_\varepsilon(t+T)| = \mathrm{o}(1) \quad \text{as} \quad t \to \infty \tag{43}$$

for some $T \in \mathbf{R}$.

In the case that $1 - \delta < s < 1$ with $\delta > 0$ sufficiently small, we can use the slide-back techniques of [24] to strengthen this estimate, showing there exist $T \in \mathbf{R}$, $a \in \mathbf{R}^n$, and $\beta > 1$ such that

$$\left| v(t,\theta) - v_\varepsilon(t+T) - e^{-t}\langle\theta, a\rangle\left(-\dot{v}_\varepsilon(t+T) + \frac{n-2s}{2} v_\varepsilon(t+T)\right)\right| = \mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t \to \infty. \tag{44}$$

Here, once again, the dot denotes a derivative with respect to t. We expect that this proof should be very similar to the one in [24], so we provide the strategy for the proof below and leave its details for subsequent work.

We start with $v \in \mathcal{C}^\infty([0,\infty) \times \mathbf{S}^{n-1})$ a positive smooth solution to

$$P_s v + c_{n,s} v = c_{n,s} v^{\frac{n+2s}{n-2s}} \quad \text{in} \quad [0,\infty) \times \mathbf{S}^{n-1}$$

and aim to show that there exists a Delaunay asymptote v_ε and parameters $T \in \mathbf{R}$, $a \in \mathbf{R}^n$, and $\beta > 1$ such that the estimate (44) holds. Let $\{\tau_k\}_{k\in\mathbf{N}} \subset \mathbf{R}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and define the so-called slide-back sequence $\{v_k\}_{k\in\mathbf{N}} \subset \mathcal{C}^2([-\tau_k,\infty) \times \mathbf{S}^{n-1})$ as

$$v_k(t,\theta) = v(t+\tau_k,\theta) \quad \text{for} \quad k \in \mathbf{N}.$$

By the *a priori* estimates proven in Section 3.2 , we can use the Arzela-Ascoli theorem to extract a convergent subsequence $v_k \to v_*$, which (by our classification result) must have the form

$$v_*(t,\theta) = v_\varepsilon(t+T).$$

To obtain simple asymptotics, we must show that both $\varepsilon, T > 0$ do not depend on the choice of the sequence. We prove the independence of ε using the radial Pohozaev invariant (Hamiltonian) in (45) below; here it remains for us to prove that different Delaunay solutions in fact have different Pohozaev invariants. Next we prove the independence of T follows from a delicate rescaling argument similar to the proof of [24, Proposition 5].

Once we have simple asymptotics, we use the nondegeneracy sketched in the previous subsection together with a version of the linear decomposition lemma (see [27, Lemma 4.18]) to get the refined asymptotics.

Remark 14. *The completeness condition implies the lower bound $u(x) \gtrsim |x|^{(2s-n)/2}$, ruling out weakly singular behavior. The asymptotic expansion* (44) *therefore provides a sharp description of all isolated singularities for s close to one.*

6.2.2. *Compactness of the moduli space of complete metrics.* As a final application, we describe a compactness result modeled on Pollack's compactness theorem [29] in the scalar curvature setting. To state the result properly, we define both the marked moduli space and the unmarked moduli space of singular Yamabe metrics. If $\Lambda \subset \mathbf{S}^n$ is finite, we let

$$\mathcal{M}_{s,\Lambda} = \left\{ g = U^{\frac{4}{n-2s}} g_\circ \in [g_\circ] : U \in \mathcal{C}^\infty(\mathbf{S}^n \setminus \Lambda) \text{ is a positive singular solution to } (\mathcal{Q}^\circ_{n,s,\Lambda}) \right\} / \sim,$$

Similarly, for $k \geq 3$, we define

$$\mathcal{M}_{s,k} = \{ g \in \mathcal{M}_{s,\Lambda} : \#\Lambda = k \} / \sim .$$

Here, the quotient is taken over diffeomorphisms preserving the singular set $\Lambda \subset \mathbf{S}^n$ and the Gromov-Hausdorff topology is placed on both moduli spaces.

The compactness result we prove is the following: if $\Omega \subset \mathcal{M}_k$ is such that the radial Pohozaev invariants (Hamiltonian) of all Delaunay asymptotes are bounded away from zero and the distance between singular points remains bounded away from zero, then any sequence in Ω has a convergent subsequence. Again, we expect the proof to be similar in spirit to that of [29], [2], and [3], so we provide only the central ideas of the proof.

Since it plays such a central role, we first take some time to define the (radial) Pohozaev invariant as (see [17, Theorem 1.2])[1]

$$\mathcal{H}^*_s(v) = \frac{c_{n,s}}{\widetilde{d}_s} \left(\frac{n-2s}{2n} v^{\frac{2n}{n-2s}} - \frac{1}{2} v^2 \right) + \frac{1}{2} \int_0^{\rho_0^*} (\rho^*)^{1-2s} \left(e_1^*(\rho)(\partial_t V)^2 - e^*(\rho)(\partial_{\rho^*} V)^2 \right) d\rho^*, \quad (45)$$

[1] Please notice that there is a typo in the coefficients e_1^* and e^* as written in [17], but the asymptotic behavior is correct.

where $V \in \mathcal{C}^\infty(\mathbf{R}^2_+)$ is the Caffarelli-Silvestre extension of $v \in \mathcal{C}^\infty(\mathbf{R})$ and $e_1^*, e^* \in \mathcal{C}^\infty(0,\rho_0^*)$ are positive coefficients depending only on ρ and the dimension such that $e_1^*(\rho), e_*(\rho) \to 1$ as $\rho \to 0$, where $\rho^* \in (0,\rho_0^*)$ with $\rho_0^* > 0$ being the critical extension parameter given by [17, Corollary 4.2]. Here we recall that in Emden–Fowler coordinates, the bulk (Poincaré–Einstein) hyperbolic metric takes the form

$$g_+ = d\rho^* + \left(1 + \tfrac{1-s}{2+2s}(\rho^*)^{n-1}\right)dt^2 + \mathcal{O}(\rho^*)dtd\rho^*.$$

We emphasize that the formula above holds only for the conformal factor expressed in log-cylindrical coordinates. For the spherical and Euclidean gauge, we also set

$$\mathcal{H}_s^*(u) = (\mathcal{P}_s \circ \mathfrak{F}^{-1})(u) \quad \text{and} \quad \mathcal{H}_s(U) = (\mathcal{H}_s^* \circ \mathfrak{F}^{-1})(u \circ \Pi),$$

where $v \in \mathcal{C}^\infty(\mathbf{R})$, $u \in \mathcal{C}^\infty(\mathbf{R}^n \setminus \{0\})$, and $U \in \mathcal{C}^\infty(\mathbf{S}^n \setminus \{N,S\})$ are solutions to ($\widehat{\mathcal{Q}}_{n,s}$), ($\overline{\mathcal{Q}}_{n,s}$), and ($\mathcal{Q}^\circ_{n,s}$), respectively. Here we recall that $\Pi : \mathbf{S} \setminus \{N,S\} \to \mathbf{R}^n \setminus \{0\}$ is the stereographic projection and $\mathfrak{F} : \mathcal{C}^\infty(\mathbf{R}^n \setminus \{0\}) \to \mathcal{C}^\infty(\mathbf{R} \times \mathbf{S}^{n-1})$ is the log-cylindrical transform.

Let $\{g_k = U_k^{\frac{4}{n-2s}} g_\circ\}$ be a sequence of singular Yamabe metrics such that all the asymptotic radial Pohozaev invariants are bounded away from zero, and the distances between singular points are also bounded away from zero. Using (for instance) [2, Lemma 11], it suffices to assume the singular set is a fixed finite set Λ for the entire sequence. Now, let $\{D_\ell\}_{\ell \in \mathbf{N}}$ be a compact exhaustion of $\mathbf{S}^n \backslash \Lambda$ and for any large $\ell \gg 1$, we extract a convergent subsequence $U_k \to U_*(\ell)$. Letting $\ell \to \infty$, we obtain a limit $U_* \in \mathcal{C}^{2,\alpha}(\mathbf{S}^n \backslash \Lambda)$ for some $\alpha \in (0,1)$. Finally, we use the fact that the Pohozaev invariants are bounded away from zero to show that $U_* > 0$ is strictly positive and defines a complete metric on $\mathbf{S}^n \backslash \Lambda$, *i.e.* $\lim_{p \to \Lambda} U(p) = \infty$. If $U_* \equiv 0$, we could rescale the sequence and contradict the positive lower bound for all Pohozaev invariants.

In the regime in which we've established our classification, namely $1 - \delta < s < 1$, we expect the proof described above to carry over from the local setting to the nonlocal setting without any complications. It is worth remarking that in the local setting one can write the radial Pohozaev invariant very explicitly in terms of the necksize, while such a relation in the nonlocal setting is more elusive, mostly because the expression (45) is much more complicated. Thus, it would be very interesting to relate the necksize more explicitly to the radial Pohozaev invariant in the nonlocal setting.

Declarations

Funding. J.H.A. acknowledges financial support from São Paulo Research Foundation (FAPESP) #2020/07566-3, #2021/15139-0, and #2023/15567-8 and National Council for Scientific and Technological Development (CNPq) #409764/2023-0, #443594/2023-6, #441922/2023-6, and #306014/2025-4. A. DlT. acknowledges financial support from the Spanish Ministry of Science and Innovation (MICINN), through the IMAG-Maria de Maeztu Excellence Grant #CEX2020-001105-M/AEI/ 10.13039/501100011033 and FEDER-MINECO Grants #PID2021- 122122NB-I00, #PID2020-113596GB-I00 , and #PID2024-155314NB-I00; RED2022-134784-T, funded by MCIN/AEI/10.13039/501100011033 and by J. Andalucia (FQM-116), Fondi Ateneo – Sapienza Universit'a di Roma, PRIN (Prot. 20227HX33Z) and INdAM-GNAMPA Projects 2023, 2024 and 2025 – CUP #E53C2200193000, CUP #E53C23001670001 and CUP #E532400195000. J. M. do Ó acknowledges financial support from National Council for Scientific and Technological Development (CNPq) #312340/2021-4, #409764/2023-0, #443594/2023-6, Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) MATH AMSUD #88887.878894/2023-00, and Paraíba State Research Foundation (FAPESQ), # 3034/2021. J.R. acknowledges financial support from the Deutsche Forschungsgemeinschaftgrant (DFG) #561401741. JW acknowledges financial support from Hong Kong General Research Fund #14309824 entitled "New frontiers in singular limits of nonlinear partial differential equations".

Conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

Data availability. Data sharing is not applicable as no datasets were generated or analyzed during the study.

Ethics approval. Not applicable.

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