STEINER SYSTEMS $S(2,6,226)$ AND $S(2,6,441)$ DO EXIST!

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Abstract. A Steiner system $S(2,k,v)$ is a set X of cardinality v endowed with a family $\mathcal{L}$ of k-element subsets of X such that any two distinct points of X belong to a unique set of the family $\mathcal{L}$. If a Steiner system $S(2,k,v)$ exists, then $k-1$ divides $v-1$ and $k(k-1)$ divides $v(v-1)$. Those divisibility conditions are necessary but not sufficient for the existence of a Steiner system $S(2,k,v)$. For instance, the Bruck–Ryser Theorem implies that no Steiner system $S(2,6,36)$ exists, despite 5 divides 35 and $6\cdot 5$ divides $36\cdot 35$. On the other hand, Wilson showed that for every natural number $k\ge 2$, a Steiner system $S(2,k,v)$ exists for all but finitely many natural numbers v satisfying the above divisibility conditions. The Handbook of Combinatorial Designs lists 29 numbers v for which the existence of a Steiner system $S(2,6,v)$ is not known: 51, 61, 81, 166, 226, 231, 256, 261, 286, 316, 321, 346, 351, 376, 406, 411, 436, 441, 471, 501, 561, 591, 616, 646, 651, 676, 771, 796, 801. In this paper we present seven Steiner systems $S(2,6,226)$ and six Steiner systems $S(2,6,441)$ thus resolving two of those 29 undecided cases. The discovered Steiner systems $S(2,6,226)$ and $S(2,6,441)$ were found by computer search, as 1-rotational difference families and difference families for the groups $(\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3$ and $(\mathbb{Z}_7\rtimes\mathbb{Z}_3)\times(\mathbb{Z}_7\rtimes\mathbb{Z}_3)$, respectively.

1. Introduction

A *Steiner system* $S(2,k,v)$ is a set X of cardinality v endowed with a family $\mathcal{L}$ of k-element subsets of X such that any two distinct points of X belong to a unique set of the family $\mathcal{L}$. It is well known [1, §II.1.2] that for any Steiner system $S(2,k,v)$, the number $k-1$ divides $v-1$ and $k(k-1)$ divides $v(v-1)$. Those divisibility conditions are necessary but not sufficient for the existence of a Steiner system $S(2,k,v)$. For instance, the famous Bruck–Ryser Theorem implies that no Steiner system $S(2,6,36)$ exists, despite 5 divides 35 and $6\cdot 5$ divides $36\cdot 35$. On the other hand, Wilson [3, 4, 5] showed that for every natural number $k\ge 2$, a Steiner system $S(2,k,v)$ exists for all but finitely many natural numbers v satisfying the above divisibility conditions. The Handbook of Combinatorial Designs [1, §IV.3.4] lists 29 numbers v for which the existence of a Steiner system $S(2,6,v)$ is not known: 51, 61, 81, 166, 226, 231, 256, 261, 286, 316, 321, 346, 351, 376, 406, 411, 436, 441, 471, 501, 561, 591, 616, 646, 651, 676, 771, 796, 801. In this paper we present seven Steiner systems $S(2,6,226)$ and six Steiner systems $S(2,6,441)$ thus resolving two of those 29 undecided cases. The discovered Steiner systems $S(2,6,226)$ and $S(2,6,441)$ were found by computer search, as 1-rotational difference families and difference families for the groups $(\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3$ and $(\mathbb{Z}_7\rtimes\mathbb{Z}_3)\times(\mathbb{Z}_7\rtimes\mathbb{Z}_3)$, respectively. For any natural

2020 *Mathematics Subject Classification.* 05B05, 51E05.

number n, we denote by $\mathbb{Z}_n$ the set $\{0, 1, \dots, n-1\}$, which is the group with the addition modulo n.

Difference families are convenient and compact way of producing Steiner systems. Given a group G of order v, we look for a family $\mathcal{B}$ of k-element subsets of G such that the family $\{gB : g \in G, \quad B \in \mathcal{B}\}$ of its left shifts is a Steiner system $S(2, k, v)$. In this case [1, §VI.16.1], the members of the family $\mathcal{B}$ are called *basic blocks*, and the family $\mathcal{B}$ is called a $(v, k, 1)$ *difference family*.

A slight modification of this construction is a 1-rotational difference family, defined as follows. Given a group G of order $v-1$, denote the group G by ∞ and extend the group operation from G to $G \cup \{\infty\} = G \cup \{G\}$ putting $\infty \cdot \infty = \infty \cdot g = \infty = g \cdot \infty$ for all $g \in G$. So, ∞ is the two-sided zero of the semigroup $G \cup \{\infty\}$, and ∞ is the fixed point of the multiplicative action of the group G on $G \cup \{\infty\}$. Then we look for a family $\mathcal{B}$ of k-element subsets of the semigroup $G \cup \{\infty\}$ such that the family $\{gB : g \in G, \quad B \in \mathcal{B}\}$ of its left shifts is a Steiner system $S(2, k, v)$. In this case [1, §VI.16.6], the members of the family $\mathcal{B}$ are called *basic blocks*, and the family $\mathcal{B}$ is called a $(v, k, 1)$ 1*-rotational difference family*.

2. Steiner systems $S(2, 6, 226)$

The group whose one-point extension admits 1-rotational difference families that determine seven Steiner systems $S(2, 6, 226)$ is the semidirect product $(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$, which is the Cartesian product $(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3) \times \mathbb{Z}_3$ endowed with the operation

$$(\vec{x}, a) \cdot (\vec{y}, b) = (\vec{x} + M^a(\vec{y}), a + b),$$

where $\vec{x}, \vec{y} \in \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3$, $a, b \in \mathbb{Z}_3$, and M is the automorphism of the group $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3$, defined by the matrix multiplication

$$M(\vec{x}) = \begin{pmatrix} 4 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \cdot x_1 + x_2 \\ 4 \cdot x_1 \\ x_3 \end{pmatrix}$$

for any vector $\vec{x} = (x_1, x_2, x_3) \in \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3$.

By computer search we have found seven $(226, 6, 1)$ 1-rotational difference families on the semigroup $\big((\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3\big) \cup \{\infty\}$, determining seven Steiner system $S(2, 6, 226)$. Each of those seven 1-rotational difference families consists of 15 blocks (preceded by the fingerprints):

(1) $[1 = 59400, 2 = 1336500, 3 = 11590200, 4 = 31761900]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 1402, 2312, 3212, 4102\},$
$\{0000, 0011, 1110, 2021, 3021, 4410\}, \{0000, 0012, 0120, 0420, 2002, 3002\},$
$\{0000, 0101, 1102, 1421, 2320, 3022\}, \{0000, 0201, 1012, 2202, 2311, 4110\},$
$\{0000, 0401, 2022, 3220, 4121, 4402\}, \{0000, 0301, 1410, 3211, 3302, 4012\},$
$\{0000, 0102, 2222, 3121, 4001, 4320\}, \{0000, 0202, 1211, 3001, 3110, 4412\},$
$\{0000, 0121, 2401, 3120, 3422, 4202\}, \{0000, 0311, 1201, 2102, 4212, 4310\},$
$\{0000, 0111, 1122, 3222, 4211, 4300\}, \{0000, 0221, 1412, 2212, 3100, 3421\},$
$\{0000, 1400, 2300, 3200, 4100, \infty\};$

(2) $[1 = 91800, 2 = 1250100, 3 = 11557800, 4 = 31848300]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 1301, 2111, 3411, 4201\},$
$\{0000, 0011, 1001, 2210, 3310, 4001\}, \{0000, 0012, 0210, 0310, 1002, 4002\},$
$\{0000, 0101, 1102, 1320, 2021, 3422\}, \{0000, 0301, 1011, 3302, 3410, 4212\},$
$\{0000, 0122, 2401, 3121, 3420, 4202\}, \{0000, 0212, 1211, 1310, 3402, 4301\},$
$\{0000, 0111, 1122, 3222, 4211, 4300\}, \{0000, 0221, 1412, 2212, 3100, 3421\},$
$\{0000, 0121, 2410, 3022, 3311, 4302\}, \{0000, 0211, 1012, 1121, 3102, 4320\},$
$\{0000, 0421, 1202, 2022, 2211, 3110\}, \{0000, 0311, 1220, 2402, 4012, 4421\},$
$\{0000, 1300, 2100, 3400, 4200, \infty\};$

(3) $[1 = 95400, 2 = 1328400, 3 = 11586600, 4 = 31737600]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 1212, 2402, 3102, 4312\},$
$\{0000, 0011, 1110, 2021, 3021, 4410\}, \{0000, 0012, 0120, 0420, 2002, 3002\},$
$\{0000, 0101, 1102, 1321, 2022, 3420\}, \{0000, 0201, 1310, 2111, 2202, 4012\},$
$\{0000, 0401, 2120, 3022, 4221, 4402\}, \{0000, 0301, 1012, 3302, 3411, 4210\},$
$\{0000, 0102, 2222, 3121, 4001, 4320\}, \{0000, 0202, 1211, 3001, 3110, 4412\},$
$\{0000, 0121, 2401, 3120, 3422, 4202\}, \{0000, 0311, 1201, 2102, 4212, 4310\},$
$\{0000, 0111, 1122, 3222, 4211, 4300\}, \{0000, 0221, 1412, 2212, 3100, 3421\},$
$\{0000, 1300, 2100, 3400, 4200, \infty\};$

(4) $[1 = 106200, 2 = 1169100, 3 = 11201400, 4 = 32271300]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 1311, 2101, 3401, 4211\},$
$\{0000, 0011, 1122, 2102, 3402, 4422\}, \{0000, 0012, 2021, 2311, 3021, 3211\},$
$\{0000, 0101, 0320, 1422, 2111, 3112\}, \{0000, 0110, 0201, 1222, 2312, 4221\},$
$\{0000, 0120, 0421, 1411, 3302, 4412\}, \{0000, 0211, 0310, 2222, 3221, 4402\},$
$\{0000, 0102, 2421, 3022, 3311, 4310\}, \{0000, 0202, 1012, 1121, 3120, 4311\},$
$\{0000, 0402, 1210, 2022, 2211, 3121\}, \{0000, 0302, 1211, 2420, 4012, 4421\},$
$\{0000, 1002, 1101, 2301, 2402, 3400\}, \{0000, 1202, 1401, 3002, 3301, 4200\},$
$\{0000, 1100, 2200, 3300, 4400, \infty\};$

(5) $[1 = 113400, 2 = 1290600, 3 = 11563200, 4 = 31780800]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 1202, 2412, 3112, 4302\},$
$\{0000, 0011, 1110, 2401, 3101, 4410\}, \{0000, 0012, 2002, 2110, 3002, 3410\},$
$\{0000, 0100, 0200, 0300, 0400, \infty\}, \{0000, 0101, 0420, 1102, 2222, 4121\},$
$\{0000, 0201, 0310, 2202, 3211, 4412\}, \{0000, 0120, 0401, 1421, 3322, 4402\},$
$\{0000, 0121, 0320, 1402, 2101, 3122\}, \{0000, 0102, 2421, 3022, 3311, 4310\},$
$\{0000, 0202, 1012, 1121, 3120, 4311\}, \{0000, 0402, 1210, 2022, 2211, 3121\},$
$\{0000, 0302, 1211, 2420, 4012, 4421\}, \{0000, 0111, 1122, 3222, 4211, 4300\},$
$\{0000, 0221, 1412, 2212, 3100, 3421\};$

(6) $[0 = 4500, 1 = 70200, 2 = 1228500, 3 = 11100600, 4 = 32344200]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 0101, 0211, 0311, 0401\},$
$\{0000, 0011, 1302, 2422, 3122, 4202\}, \{0000, 0012, 1411, 2121, 3421, 4111\},$
$\{0000, 0100, 2002, 2301, 3102, 3301\}, \{0000, 0200, 1101, 1202, 4002, 4101\},$
$\{0000, 0102, 2421, 3022, 3311, 4310\}, \{0000, 0202, 1012, 1121, 3120, 4311\},$
$\{0000, 0402, 1210, 2022, 2211, 3121\}, \{0000, 0302, 1211, 2420, 4012, 4421\},$
$\{0000, 0111, 1122, 1410, 2302, 3021\}, \{0000, 0221, 1011, 2212, 2320, 4102\},$

$\{0000, 0122, 1321, 2101, 2310, 3212\}, \{0000, 0312, 1301, 1420, 3411, 4122\}$, $\{0000, 1300, 2100, 3400, 4200, \infty\}$;

(7) $[0 = 4500, 1 = 113400, 2 = 1225800, 3 = 11541600, 4 = 31862700]$
$\{0000, 0001, 0112, 0222, 0322, 0412\}, \{0000, 0010, 1102, 2212, 3312, 4402\}$,
$\{0000, 0011, 1422, 2112, 3412, 4122\}, \{0000, 0012, 2301, 2421, 3121, 3201\}$,
$\{0000, 0102, 1302, 1400, 2401, 4001\}, \{0000, 0202, 2102, 2300, 3001, 4301\}$,
$\{0000, 0121, 1112, 1301, 2022, 3410\}, \{0000, 0211, 1320, 2101, 2222, 4012\}$,
$\{0000, 0421, 2110, 3022, 4201, 4412\}, \{0000, 0311, 1012, 3322, 3401, 4220\}$,
$\{0000, 0122, 2201, 3120, 4011, 4312\}, \{0000, 0212, 1210, 3021, 3122, 4401\}$,
$\{0000, 0422, 1011, 1212, 2420, 3301\}, \{0000, 0312, 1101, 2021, 2422, 4310\}$,
$\{0000, 1100, 2200, 3300, 4400, \infty\}$.

The generating blocks consist of elements of the group $G = (\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ denoted by quadruples of numbers (without commas and parentheses) or the symbol ∞ (which is the fixed point of the action of the group G on $G \cup \{\infty\}$).

The fingerprint of a Steiner system is defined in the paper [2]. It is a simple isomorphic invariant that can be quickly calculated and can distinguish non-isomorphic Steiner systems, in the sense that two Steiner systems with distinct fingerprints are surely non-isomorphic, but the converse is not always true and to check isomorphism of Steiner systems with coinciding fingerprints one should apply more powerful (but also more time-consuming) tools like `nauty`.

As we can see, the fingerprints of all seven discovered Steiner systems $S(2, 6, 226)$ are distinct, implying that these seven Steiner systems are pairwise non-isomorphic.

3. Steiner systems $S(2, 6, 441)$

The group, used for finding difference families that generate the Steiner systems $S(2, 6, 441)$, is the Cartesian square $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ of the group $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, endowed with the operation $(x, a) \cdot (y, b) = (x + 2^a y, a + b)$, where $x, y \in \mathbb{Z}_7$ and $a, b \in \mathbb{Z}_3$.

By computer search we have found six Steiner systems $S(2, 6, 441)$, generated by six $(441, 6, 1)$ difference families on the group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$. Each of those six difference families consists of 20 blocks (preceded by the fingerprints):

(1) $[2 = 2079756, 3 = 45666432, 4 = 289883412]$
$\{0000, 0001, 1012, 1102, 6062, 6102\}, \{0000, 0010, 0142, 0242, 2002, 5012\}$,
$\{0000, 0011, 0121, 1132, 1250, 4252\}, \{0000, 0042, 0130, 3251, 6161, 6252\}$,
$\{0000, 0110, 1261, 3161, 4012, 6262\}, \{0000, 0160, 1212, 3062, 4111, 6211\}$,
$\{0000, 0012, 0051, 3032, 3040, 3061\}, \{0000, 1130, 2100, 3030, 4230, 6200\}$,
$\{0000, 0020, 1031, 3150, 4140, 6061\}, \{0000, 1040, 2162, 3022, 5022, 6152\}$,
$\{0000, 0022, 0031, 4010, 4051, 4062\}, \{0000, 1030, 2200, 3100, 5130, 6230\}$,
$\{0000, 0030, 2102, 3220, 4210, 5132\}, \{0000, 0112, 2161, 4131, 6020, 6112\}$,
$\{0000, 0101, 1152, 2211, 5261, 6122\}, \{0000, 0102, 1210, 2131, 5141, 6260\}$,
$\{0000, 0122, 2041, 2232, 3111, 3250\}, \{0000, 1141, 2201, 3252, 4150, 5042\}$,
$\{0000, 0152, 4161, 4220, 5031, 5242\}, \{0000, 1151, 2250, 3212, 4102, 5011\}$;

(2) $[2 = 2233224, 3 = 47296368, 4 = 288100008]$
$\{0000, 0001, 1032, 1102, 6042, 6102\}, \{0000, 0010, 0142, 0242, 1012, 6002\},$
$\{0000, 0011, 3032, 5142, 5262, 6200\}, \{0000, 0012, 1100, 3240, 4221, 6222\},$
$\{0000, 0021, 1250, 3252, 4211, 6121\}, \{0000, 0042, 1242, 2121, 2211, 4061\},$
$\{0000, 0020, 3140, 3262, 4150, 4232\}, \{0000, 1030, 2201, 3052, 5052, 6231\},$
$\{0000, 0022, 0031, 3010, 3051, 3062\}, \{0000, 1040, 2200, 3100, 5140, 6240\},$
$\{0000, 0030, 0151, 0250, 3152, 4152\}, \{0000, 0101, 2042, 3220, 4250, 5032\},$
$\{0000, 0112, 1160, 3150, 4040, 4132\}, \{0000, 1010, 2052, 3250, 5230, 6032\},$
$\{0000, 0131, 0222, 1162, 2210, 4051\}, \{0000, 1052, 2202, 3101, 5130, 6261\},$
$\{0000, 0141, 0252, 3021, 5260, 6112\}, \{0000, 1031, 2252, 3141, 5132, 6230\},$
$\{0000, 1020, 2242, 3161, 5131, 6252\}, \{0000, 1131, 2262, 3222, 4151, 5010\};$

(3) $[1 = 24696, 2 = 2085048, 3 = 46008648, 4 = 289511208]$
$\{0000, 0001, 1022, 1102, 6052, 6102\}, \{0000, 0010, 0142, 0242, 3012, 4002\},$
$\{0000, 0011, 2111, 3101, 5001, 5010\}, \{0000, 0012, 0222, 3022, 3030, 3212\},$
$\{0000, 0020, 0110, 0212, 3211, 4211\}, \{0000, 0022, 2161, 3230, 5201, 6252\},$
$\{0000, 0122, 2200, 3042, 5212, 6151\}, \{0000, 0152, 1121, 2262, 4032, 5200\},$
$\{0000, 0041, 1210, 2242, 4251, 5162\}, \{0000, 0030, 1201, 2041, 5061, 6231\},$
$\{0000, 0112, 1150, 3120, 4000, 4162\}, \{0000, 0101, 1021, 2160, 5110, 6051\},$
$\{0000, 1010, 2222, 3131, 5151, 6262\}, \{0000, 1060, 2252, 3141, 5121, 6212\},$
$\{0000, 1030, 2221, 3162, 5142, 6211\}, \{0000, 1112, 2152, 3060, 4221, 6241\},$
$\{0000, 1111, 2220, 3232, 4142, 5051\}, \{0000, 1161, 2052, 3242, 4231, 5120\},$
$\{0000, 1152, 2031, 4161, 5242, 6220\}, \{0000, 1141, 2211, 3252, 4120, 5062\};$

(4) $[1 = 31752, 2 = 2259684, 3 = 45179568, 4 = 290158596]$
$\{0000, 0001, 1012, 1100, 6062, 6100\}, \{0000, 0010, 0042, 0142, 1202, 6212\},$
$\{0000, 0012, 1001, 1140, 4162, 4252\}, \{0000, 0160, 1200, 4061, 5232, 6231\},$
$\{0000, 0110, 1241, 2242, 3011, 6200\}, \{0000, 0021, 3140, 3260, 6022, 6111\},$
$\{0000, 0020, 0131, 0161, 0242, 0252\}, \{0000, 1101, 2202, 3202, 4101, 5000\},$
$\{0000, 0022, 0031, 3010, 3051, 3062\}, \{0000, 1040, 2200, 3100, 5140, 6240\},$
$\{0000, 0030, 1120, 3021, 4011, 6110\}, \{0000, 0112, 1060, 1152, 2032, 6032\},$
$\{0000, 0101, 2041, 3120, 4150, 5031\}, \{0000, 0102, 1210, 2131, 5141, 6260\},$
$\{0000, 0130, 4151, 4241, 5062, 5212\}, \{0000, 0140, 2012, 2262, 3121, 3231\},$
$\{0000, 1250, 2141, 3142, 4001, 6252\}, \{0000, 1151, 2150, 3212, 4002, 5211\},$
$\{0000, 1020, 2161, 3152, 5142, 6131\}, \{0000, 1112, 2240, 3220, 4152, 5060\};$

(5) $[1 = 42336, 2 = 1815156, 3 = 46213272, 4 = 289558836]$
$\{0000, 0001, 1032, 1101, 6042, 6101\}, \{0000, 0010, 0042, 0242, 2112, 5102\},$
$\{0000, 0012, 1202, 3162, 3201, 4121\}, \{0000, 0021, 3152, 4140, 4222, 6220\},$
$\{0000, 0121, 0230, 2021, 4212, 5032\}, \{0000, 0130, 0211, 3111, 4102, 6022\},$
$\{0000, 0020, 2061, 3230, 4260, 5031\}, \{0000, 0131, 3050, 3121, 4061, 6061\},$
$\{0000, 0022, 0031, 3010, 3051, 3062\}, \{0000, 1040, 2200, 3100, 5140, 6240\},$
$\{0000, 0030, 1111, 3061, 4041, 6121\}, \{0000, 1010, 2031, 3240, 5240, 6051\},$
$\{0000, 0101, 1022, 2260, 5210, 6052\}, \{0000, 0102, 1220, 3141, 4131, 6250\},$
$\{0000, 0110, 4111, 4231, 5002, 5232\}, \{0000, 0160, 2002, 2242, 3161, 3241\},$

{0000, 0120, 1002, 1262, 5121, 5261}, {0000, 0150, 2151, 2211, 6002, 6212},
{0000, 1020, 2261, 3142, 5152, 6231}, {0000, 1122, 2251, 3231, 4162, 5010};

(6) [1 = 42336, 2 = 1973916, 3 = 46283832, 4 = 289329516]
{0000, 0001, 1032, 1102, 6042, 6102}, {0000, 0010, 0142, 0242, 1012, 6002},
{0000, 0011, 3110, 3231, 4042, 6141}, {0000, 0042, 1120, 3021, 4102, 4260},
{0000, 0122, 1022, 4150, 4242, 5101}, {0000, 0112, 2032, 2250, 3051, 5232},
{0000, 0012, 2000, 2062, 3141, 6131}, {0000, 0020, 1220, 3131, 4161, 6200},
{0000, 0022, 0031, 5010, 5051, 5062}, {0000, 1020, 2200, 3100, 5120, 6220},
{0000, 0030, 2240, 3002, 4032, 5260}, {0000, 0130, 1112, 5152, 6060, 6130},
{0000, 0101, 2231, 3152, 4122, 5241}, {0000, 0102, 1212, 2031, 5041, 6262},
{0000, 0120, 4022, 4212, 6151, 6221}, {0000, 0150, 1121, 1251, 3052, 3262},
{0000, 0131, 0222, 1221, 2002, 4130}, {0000, 0141, 0252, 3140, 5002, 6251},
{0000, 1040, 2212, 3151, 5161, 6232}, {0000, 1151, 2131, 3010, 4262, 6222}.

Writing the generating blocks, we denoted elements of the group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ by quadruples of numbers (without commas and parentheses).

As we can see, the fingerprints of all six discovered Steiner systems $S(2,6,441)$ are distinct, implying that these six systems are pairwise non-isomorphic.

4. An Open Problem

Problem 1. Calculate (the order of) the automorphism groups of the Steiner systems $S(2,6,226)$ and $S(2,6,441)$ from Sections 2 and 3.

Remark 1. To find the Steiner systems $S(2,6,226)$ we have used a semidirect product $\big((\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3\big)\rtimes\mathbb{Z}_4$ acting on the semigroup $((\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)\cup\{\infty\}$, which implies that the orders of the automorphism groups of the discovered Steiner systems $S(2,6,226)$ are divisible by the order 900 of their subgroup $\big((\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3\big)\rtimes\mathbb{Z}_4$.

Remark 2. To find the Steiner systems $S(2,6,441)$ we have used a semidirect product $\big((\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)\big) \rtimes \mathbb{Z}_4$ acting on the group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, which implies that the orders of the automorphism groups of the discovered Steiner systems $S(2,6,441)$ are divisible by the order 1764 of their subgroup $\big((\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)\big) \rtimes \mathbb{Z}_4$.

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