

MODEL-THEORETIC K_1 FOR MODULES OVER SEMISIMPLE RINGS: (WEAK) MORITA INVARIANCE

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ABSTRACT. This paper is a sequel to a paper by the same authors, where they defined K -groups of model-theoretic structures, and computed K_1 of free modules over PIDs. In this paper, we compute K_1 of a right $M_q(R)$ -module M , where R is a division ring, $q \geq 1$, and $|M_q(R)| \neq 2$. As a consequence, we obtain a (weak) Morita invariance $K_1(R_R) \cong K_1((M_q(R))_{M_q(R)})$ for all division rings R and $q \geq 1$. Finally, we compute K_1 of a module over a semisimple ring by showing that the model-theoretic K_1 commutes with finite product of modules. We also show that the algebraic K_1 of a finite product of infinite matrix rings embeds into the model-theoretic K_1 of their right regular modules.

1. INTRODUCTION

Let L be a language and M be a first-order L -structure. Motivated by, and extending Krajiček and Scanlon's definition of the Grothendieck ring $K_0(M)$ [KS00], the authors defined model-theoretic K -groups $K_n(M)$ for $n \geq 0$ in [BK25]. The latter definition employs Quillen's famous $S^{-1}S$ -construction applied to the small symmetric monoidal groupoid $(\mathcal{S}(M), \sqcup, \emptyset)$, whose objects are definable subsets (with parameters) of finite powers of M , whose morphisms are definable bijections, and $\sqcup$ is the disjoint union.

Given a unital ring R , a right module M_R could be thought of as a structure for the language $L_R := \langle +, -, 0, \{\cdot_r \mid r \in R\} \rangle$, where $\cdot_r$ is unary function symbol describing the right action of the scalar r . After meticulous computations of $K_1(M_R)$ for free modules over some PIDs in [BK25], we shift our attention to modules over semisimple rings in this paper.

Our first major contribution is the following.

Theorem A. *Suppose R be a division ring, $q \geq 1$, $|M_q(R)| \neq 2$, and M is a non-zero $M_q(R)$ -module. If $R^\times$ is the group of units in R , then*

$$K_1(M_{M_q(R)}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } M \text{ is finite;} \\ \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{\infty} (GL_{iq}(R)^{ab} \oplus \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{\infty} \left(\frac{R^\times}{[R^\times, R^\times]} \oplus \mathbb{Z}_2 \right) & \text{otherwise.} \end{cases}$$

Recall that a unital ring R is Morita equivalent to the matrix ring $M_q(R)$ for each $q \in \mathbb{N}$. A striking consequence of the above result is the following *weak Morita invariance* for model-theoretic K_1 when R is a division ring.

Corollary 1.1. *Suppose R be a division ring, $q \geq 1$, $|M_q(R)| \neq 2$, and M is a non-zero $M_q(R)$ -module. Then $K_1(M_{M_q(R)}) \cong K_1(M_R)$. In particular, $K_1(R_R) \cong K_1((M_q(R))_{M_q(R)})$ for every division ring R and $q \geq 1$.*

Recall that the algebraic K_1 -group of a unital ring R , denoted $K_1^\oplus(R)$, is isomorphic to $K_1^\oplus(M_q(R))$ [Wei13, Example III.1.1.4] since R and $M_q(R)$ are Morita equivalent. Moreover, this isomorphism is functorially induced by a categorical equivalence $F_q : \text{Proj}(R) \rightarrow \text{Proj}(M_q(R))$ between their respective categories of finitely generated projective modules. On the contrast, the isomorphism in Corollary 1.1 between model-theoretic K_1 -groups is not functorial; therefore, we call it a weak Morita invariance of model-theoretic K_1 .

Our second major contribution is the following theorem which states that model-theoretic K_1 commutes with finite products of modules when the ring is semisimple.

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Theorem B. *Let S be a unital semisimple ring written as $\prod_{i=1}^k S_i$, where for each i , $S_i := Se_i = M_{q_i}(R_i)$ for a division ring R_i , idempotent e_i , and $q_i \geq 1$ (thanks to the Wedderburn-Artin theorem). Write a right S -module M_S as $\prod_{i=1}^k (M_i)_{M_{q_i}(R_i)}$, where $M_i := Me_i$ is a right $M_{q_i}(R_i)$ -module. Assume that each M_i is infinite. Then*

$$K_1(M_S) \cong \prod_{i=1}^k K_1\left((M_i)_{M_{q_i}(R_i)}\right).$$

This result is a partial model-theoretic analogue of a similar result for algebraic K_1 [Wei13, Example III.1.1.3] which states that the latter commutes with finite products.

As a consequence of the two theorems above, using the notations of Theorem B and under its hypotheses, if each S_i is infinite, then we show (Corollary 6.13) that $K_1^\oplus(S)$ naturally embeds into $K_1(S_S)$.

The proofs of Theorems A and B use a recipe similar to that in [BK25], albeit with necessary modifications to deal with non-commutative rings (§ 4, 5), and vector dimensions for definable sets (§ 6).

Theorem A generalizes [BK25, Theorem 5.2], and corrects the following two errors in [BK25, Corollaries 7.10, 7.11].

- The authors mistakenly stated $K_1(V_F) \cong K_1(F_F)$ for an infinite vector space over a finite field F other than F_2 ; however, the former is infinite while the latter is (isomorphic to) $\mathbb{Z}_2$.
- The theory $Th(V_F)$ of an infinite vector space V_F over a finite field F is closed under products as a consequence of [Pre88, Lemma 1.2.3] but in those corollaries the authors erroneously used that this condition fails. In fact, the correct corollaries follow from [BK25, Theorem 7.6] instead of [BK25, Theorem 7.8].

The rest of the paper is organised as follows. We recall the definition of model-theoretic K -groups for modules in § 2 along with Bass' formula for computing K_1 (Theorem 2.5). Since semisimple rings form a subclass of the class of von Neumann regular rings, we recall the fundamental results in the model theory of modules over such rings in § 3, notably Theorem 3.2 and Proposition 3.3 stating elimination of quantifiers and of imaginaries respectively. The proof of Theorem A for the case of modules over division rings and over matrix rings is completed in § 4 and § 5 respectively. Finally, Theorem B is proved in § 6 along with a generalization to certain modules over von Neumann regular rings (Theorem 6.14).

As is the convention in logic, the set $\mathbb{N}$ of natural numbers includes 0. For group-theoretic preliminaries regarding semi-direct products, wreath products, finitary symmetric groups on a countable set, and their abelianizations, we refer the interested reader to [BK25, § 2].

2. MODEL-THEORETIC K -GROUPS OF MODULES

Quillen's definition of K -groups of a skeletally small symmetric monoidal groupoid $(\mathcal{S}, *, e)$ uses his famous $\mathcal{S}^{-1}\mathcal{S}$ construction (see [Wei13, § IV.4] for more details). He defined the K -theory space $K^*(\mathcal{S})$ of $\mathcal{S}$ to be the geometric realization of $\mathcal{S}^{-1}\mathcal{S}$ and the K -groups of $\mathcal{S}$ as $K_n^*(\mathcal{S}) := \pi_n K^*(\mathcal{S})$. We used this construction to associate K -theory to model-theoretic structures in [BK25], which we recall below.

Let L be a language, M a first-order L -structure and $m \geq 1$. By an abuse of notation, we denote the domain of the structure M again with the same notation. We will always assume that definable means definable with parameters from the universe. For each $m \geq 1$, let $\text{Def}(M^m)$ be the collection of all definable subsets of M^m , and set $\overline{\text{Def}}(M) := \bigcup_{m \geq 1} \text{Def}(M^m)$.

Let $\mathcal{S}(M)$ denote the groupoid whose objects are $\overline{\text{Def}}(M)$ and morphisms are definable bijections between definable sets, i.e., bijections whose graphs are definable sets. Note that $(\mathcal{S}(M), \sqcup, \emptyset)$ is a symmetric monoidal groupoid, where $\sqcup$ is the disjoint union. Moreover, the Cartesian product of definable sets induces a pairing (see [Wei13, § IV.4] for a definition) on this monoidal category.

Definition 2.1. [BK25, Definition 4.5] *Define $K_n(M) := K_n^\sqcup(\mathcal{S}(M))$ for each $n \geq 0$.*

Remark 2.2. The Grothendieck ring $K_0(M)$ defined this way coincides with that of [KS00].

Example 2.3. It follows from Barratt-Priddy-Quillen-Segal theorem [Wei13, Theorem IV.4.9.3] that if M is a finite structure with at least two elements then $K_n(M) \cong \pi_n^s$, where π_n^s is the n^{th} stable homotopy group of spheres. In particular, $K_0(M) \cong \mathbb{Z}$ and $K_1(M) \cong \mathbb{Z}_2$. $\diamond$

Remark 2.4. For a model-theoretic structure M , *translations are faithful* in $\mathcal{S}(M)$, i.e., for all $A, B \in \mathcal{S}(M)$, the translation $\text{Aut}_{\mathcal{S}(M)}(A) \rightarrow \text{Aut}_{\mathcal{S}(M)}(A \sqcup B)$ defined by $f \mapsto f \sqcup \text{id}_B$ is an injective map.

Bass was the first to introduce the group K_1 , and in view of the above remark, the following theorem could be used for the computation of $K_1(M)$.

Theorem 2.5. [Bas68] *Suppose $\mathcal{S}$ is a symmetric monoidal groupoid with faithful translations. Then $K_1(\mathcal{S}) \cong \varinjlim_{s \in \mathcal{S}} (\text{Aut}_{\mathcal{S}}(s))^{ab}$.*

Remark 2.6. Suppose that $(\mathcal{S}, *, e)$ is a symmetric monoidal groupoid whose translations are faithful. Further suppose that $\mathcal{S}$ has a countable sequence of objects $s_1, s_2, \dots$ such that $s_{n+1} \cong s_n * a_n$ for some $a_n \in \mathcal{S}$, and satisfying the cofinality condition that for every $s \in \mathcal{S}$ there is an s' and an n such that $s * s' \cong s_n$. In this case, we can form the colimit $\text{Aut}(\mathcal{S}) := \varinjlim_{n \in \mathbb{N}} \text{Aut}_{\mathcal{S}}(s_n)$, and hence $K_1(\mathcal{S}) = (\text{Aut}(\mathcal{S}))^{ab}$.

Every right R -module M is a first-order structure for the language L_R of right R -modules, where $L_R := \langle +, -, 0, \{\cdot_r : r \in R\} \rangle$ with $\cdot_r$ a unary function symbol for the scalar multiplication by $r \in R$ on the right. The right R -module structure M will be denoted as M_R .

The theory of the L_R -structure M_R admits a partial elimination of quantifiers with respect to positive primitive (*pp*) formulas (see [Pre88, § 2.1] for a definition) as a consequence of the fundamental theorem of the model theory of modules due to Baur and Monk ([Bau76]).

Say that a subset B of M^m is *pp-definable* if it is definable by a *pp*-formula, and a *pp-definable function* is a function between two definable sets whose graph is *pp*-definable. Every *pp*-definable set is either the empty set or a coset of a *pp*-definable subgroup of M^n . Furthermore, the conjunction of two *pp*-formulas is (logically equivalent to) a *pp*-formula.

Let $\mathcal{L}_n(M_R)$ (or just $\mathcal{L}_n$, if the module is clear from the context) denote the meet-semilattice of all *pp*-definable subsets of M^n under intersection. Set $\overline{\mathcal{L}}(M_R) := \bigcup_{n \geq 1} \mathcal{L}_n(M_R)$. Let $\mathcal{L}_n^\circ(M_R)$ be the sub-meet-semilattice of $\mathcal{L}_n(M_R)$ consisting only of the *pp*-definable subgroups. The notation $\overline{\mathcal{X}}(M_R)$ (or just $\overline{\mathcal{X}}$, if the module is clear from the context) will denote the set of *colours*, i.e., *pp*-definable bijection classes of elements of $\overline{\mathcal{L}}(M_R)$. For $A \in \overline{\mathcal{L}}$, the notation $[[A]]$ will denote its *pp*-definable bijection class. The set $\overline{\mathcal{X}}^* := \overline{\mathcal{X}} \setminus [[\emptyset]]$ of non-trivial colours is a monoid under multiplication induced by Cartesian product.

Definition 2.7. *The theory of the module M_R is said to be closed under products if for each $n \geq 1$ and for any subgroups $A, B \in \mathcal{L}_n$, the index $[A : A \cap B]$ is either 1 or ∞ .*

The next result follows immediately from [Pre09, Lemma 1.2.3].

Proposition 2.8. *The following hold for an infinite free module M_R over a ring R :*

- (1) $\mathcal{L}_n^\circ(M_R) \cong \mathcal{L}_n^\circ(R_R)$;
- (2) if R is finite, then the theory of M_R is closed under products.

The second author computed K_0 for all modules; we mention only a special case below.

Theorem 2.9. [Kub15, Theorem 4.1.2] *Suppose M_R is a module whose theory is closed under products. Then $K_0(M_R)$ is isomorphic to the monoid ring $\mathbb{Z}[\overline{\mathcal{X}}^*(M_R)]$.*

3. MODEL THEORY OF MODULES OVER INFINITE SEMISIMPLE RINGS

Recall that a unital ring R is *von Neumann regular* if for every element $a \in R$ there exists an element $r \in R$ such that $ara = a$. A few notable examples of von Neumann regular rings include fields, division rings (ring where any non-zero element is a unit), semisimple rings, and the endomorphism ring of F -linear morphisms, $\text{End}_F(V)$, for any vector space V over a field F .

Remark 3.1. The class of von Neumann rings is closed under finite direct products and opposite rings.

The class of von Neumann regular rings can be completely characterized model-theoretically as described in the next result—several cases of this result were proven by multiple authors over a period of few decades but we only cite a book.

Theorem 3.2. [Hod93, A.2.1] *A ring R is von Neumann regular iff every pp -formula in the language L_R is equivalent to one without quantifiers.*

Moreover, when R is von Neumann regular, we also get a complete elimination of pp -imaginaries—this result is stated as [Pre09, Proposition 10.2.38] in functor-category-theoretic language. The next statement is an algebraic consequence of this result that provides a complete description of all pp -definable subsets of R_R^n as well as of pp -definable bijections between them.

Proposition 3.3. *If R is von Neumann regular, and $f : D_1 \rightarrow D_2$ is a pp -definable bijection between pp -definable subsets of R_R^n for some $n \geq 1$, then both D_1 and D_2 are cosets of right R -submodules of R_R^n , and there are $\bar{d}_i \in D_i$ and $A \in GL_n(R)$ such that for each $\bar{x} \in D_1$, we have $f(\bar{x}) = (\bar{x} - \bar{d}_1)A + \bar{d}_2$.*

Our main object of study is the class of *semisimple rings*, i.e., the class of von Neumann regular rings satisfying the (equivalent) conditions of the next theorem.

Theorem 3.4. [Rot09, § 4.1,4.2] *The following are equivalent for a ring R .*

- (1) *The ring R is Noetherian and von Neumann regular.*
- (2) *All right R -modules are projective.*
- (3) *All right R -modules are injective.*
- (4) *(Wedderburn-Artin) There are division rings $R_1, \dots, R_k$ such that $R \cong \prod_{i=1}^k M_{q_i}(R_i)$.*

From the perspective of the computation of model-theoretic K_1 , the above characterization of semisimple rings demands that we first need to compute $K_1(M_R)$, where R is a division ring or a matrix ring over a division ring. We already addressed the computation of $K_1(M_R)$, where R is a finite division ring, or equivalently, a finite field, in [BK25, § 7]. Therefore, in this paper, we focus our attention to infinite division rings.

We require some standard properties of modules over a division ring R .

Proposition 3.5. (1) *Finitely generated modules over a division ring R satisfy the invariant basis property, i.e., isomorphic finitely generated modules have equal rank.*
 (2) *There is an obvious isomorphism $(M_q(R))^{op} \cong M_q(R^{op})$ of matrix rings which restricts to a group isomorphism $(GL_q(R))^{op} \cong GL_q(R^{op})$.*

Since a division ring R is Noetherian and every R -module is free, thanks to Proposition 2.8(1), the proof of [BK25, Proposition 6.1], that does not use commutativity, could be adapted to obtain the following.

Theorem 3.6. *If R is a division ring and M_R is an infinite module then $\overline{\mathcal{X}}^*(M_R) \cong \mathbb{N}$.*

Recall that a unital ring R is Morita equivalent to the matrix ring $M_q(R)$ for any $q \geq 1$, i.e., there is an equivalence $F_q : \text{Mod-}R \rightarrow \text{Mod-}M_q(R)$ between the module categories.

Remark 3.7. Every right ideal of $M_q(R)$ is isomorphic to $F_q(N)$ for a submodules N of R_R^q . In particular, the $M_q(R)$ -module $M_{1 \times q}(R)$ is isomorphic to $F_q(R_R)$. Moreover, if R is a division ring, then $F_q(R_R^k)$ is a free module over R with rank kq thanks to 3.5(1). In particular, if $M_{M_q(R)}$ is finite then so is R .

This remark together with Propositions 2.8(1) and 3.3 ensure that the proof of Theorem 3.6 could be readily adapted to yield the following.

Theorem 3.8. *Suppose R is a division ring and $M_{M_q(R)}$ is infinite. Then $\overline{\mathcal{X}}^*(M_{M_q(R)}) \cong q\mathbb{N} \cong \mathbb{N}$.*

The above theorem together with Theorem 2.9 and Example 2.3 yields the following.

Corollary 3.9 (Weak Morita invariance of model-theoretic K_0). *If R is a division ring, $q \geq 1$, and M is an infinite $M_q(R)$ -module, then $K_0(M_{M_q(R)}) \cong K_0(M_R)$. As a consequence, for all division rings R and $q \geq 1$, we have $K_0((M_q(R))_{M_q(R)}) \cong K_0(R_R)$.*

Remark 3.10. The above corollary is a weak Morita invariance of model-theoretic K_0 , and not the usual Morita invariance as seen in its algebraic counterpart [Wei13, Corollary II.2.7.1] since the isomorphism $K_0(R_R) \cong K_0(M_q(R)_{M_q(R)})$ is not induced by the functor F_q between module categories when R is infinite. Indeed, the functor F_q naturally yields a bijective map from $\mathcal{L}_k^\circ(R_R)$ to $\mathcal{L}_k^\circ(M_q(R)_{M_q(R)})$ thanks to Proposition 3.3, but the latter bijection fails to extend to a bijective map from $\mathcal{L}_k(R_R)$ to $\mathcal{L}_k(M_q(R)_{M_q(R)})$ since there are far too many parameters on the right side compared to the left side.

4. K_1 OF MODULES OVER A DIVISION RING

The goal of this section is to prove Theorem A when $q = 1$, i.e., the computation of $K_1(M_R)$ for a non-zero right R -module M_R for a division ring R . Note that M is a free R -module. We may assume that M is infinite for otherwise the conclusion follows from Example 2.3.

Recall from Proposition 2.8(2) that the theory of M_R is closed under products. Moreover, Theorem 3.6 yields that $\overline{\mathcal{X}}^*(M_R) \cong \mathbb{N}$. The proof for this case of the theorem is along lines similar to the computation of $K_1(V_F)$ [BK25, § 5], where V_F is an infinite vector space over an infinite field F . We follow all three steps of the proof of the latter while highlighting the changes for the division ring case. For brevity, we denote $\mathcal{S}(M_R)$ by $\mathcal{S}$ and $\overline{\mathcal{X}}^*(M_R)$ by $\overline{\mathcal{X}}^*$.

Step I: In this step, we associate a “dimension” $\dim(f)$ to each automorphism f of a definable set through its “support”, and show that the groups of bounded-dimension automorphisms of sufficiently large definable sets are isomorphic.

Recall the definition of dimension of a definable set from [BK25, Definition 6.3]: for $E \in \mathcal{S}$, set $\dim(E) := \begin{cases} -\infty & \text{if } E = \emptyset; \\ \max\{\mathfrak{A} \in \overline{\mathcal{X}}^* \mid \Lambda_{\mathfrak{A}}(E) \neq 0\} & \text{otherwise,} \end{cases}$ where $\Lambda_{\mathfrak{A}}$ is a definable-bijection-invariant integer-valued function defined in [Kub15, § 5.2] that takes value 0 at all but finitely many inputs. In other words, if $E \neq \emptyset$, then $\dim(E)$ is the degree of the polynomial $[E] \in K_0(M_R) \cong \mathbb{Z}[\mathbb{N}] \cong \mathbb{Z}[X]$.

Definition 4.1. Let $E \in \mathcal{S}$ and $f \in \text{Aut}_{\mathcal{S}}(E)$. The support of f is the (definable) set $\text{Supp}(f) := \{a \in E : f(a) \neq a\}$. Set $\dim(f) := \dim(\text{Supp}(f))$.

The main result in this step is the following.

Proposition 4.2. [BK25, Proposition 5.4] For $E \in \mathcal{S}$, let $\Omega_m(E) := \{f \in \text{Aut}_{\mathcal{S}}(E) : \dim(f) \leq m\}$ be the subgroup of $\text{Aut}_{\mathcal{S}}(E)$ of automorphisms fixing all elements of E outside a subset of dimension at most m . If $E_1, E_2 \in \mathcal{S}$ have dimension strictly greater than m , then $\Omega_m(E_1) \cong \Omega_m(E_2)$.

Step II: We first recall the basic notations for the reader’s ease. For all $n \in \mathbb{N}$, set $\Omega_n^n := \text{Aut}_{\mathcal{S}}(M^n)$. For each $0 \leq m < n$, let $\Omega_m^n := \Omega_m(M^n)$ and Σ_m^n denote the finitary permutation group on a countable set of cosets of an m -rank submodule of M^n . For each $n \geq 1$, let $\Upsilon^n := \Upsilon^n(M_R)$ denote the subgroup of $\text{Aut}_{\mathcal{S}}(M^n)$ consisting only of pp -definable bijections. Since the group of R -linear automorphisms of R_R^n is $GL_n(R)^{op} \cong GL_n(R^{op})$ (see Proposition 3.5(2) for the last isomorphism), it follows from Proposition 3.3 that a pp -definable bijection is in fact a definable linear bijection and that Υ^n is the group $GL_n(R^{op}) \ltimes M^n$, where $GL_n(R^{op})$ acts on M^n on the right by matrix multiplication. Furthermore, subgroups $\Upsilon_m^n := \Upsilon_m^n \wr \Sigma_m^n$ of Ω_m^n for $1 \leq m < n$ satisfy $\Omega_m^n \cong \Upsilon_m^n \ltimes \Omega_{m-1}^n$ as shown in Step II of [BK25, § 5] using Proposition 4.2. The rest of the proof of Step II there follows verbatim to conclude

$$\Omega_n^n \cong \Upsilon^n \ltimes (\Upsilon_{n-1}^n \ltimes (\Upsilon_{n-2}^n \ltimes (\cdots (\Upsilon_1^n \ltimes \Omega_0^n) \cdots))).$$

Step III: In this final step, we first compute $(\Omega_n^n)^{ab}$ exactly as in Step III of [BK25, § 5] except for a very subtle change where we replace $GL_n(R)$ with $(GL_n(R))^{op}$ as explained in the step above.

Proposition 4.3. If R is a division ring, $|R| \neq 2$, and M is infinite, then for each $n \geq 1$, we have

$$(\Omega_n^n)^{ab} \cong (GL_n(R))^{ab} \oplus \bigoplus_{i=0}^{n-1} \left((GL_i(R))^{ab} \oplus \mathbb{Z}_2 \right) \cong \frac{R^\times}{[R^\times, R^\times]} \oplus \bigoplus_{i=1}^{n-1} \left(\frac{R^\times}{[R^\times, R^\times]} \oplus \mathbb{Z}_2 \right) \oplus \mathbb{Z}_2.$$

Proof. The proof of the first isomorphism follows that of [BK25, Proposition 5.5] verbatim since $GL_n(R^{op}) \cong GL_n(R)^{op}$. For the second isomorphism, we use a result of Dieudonné from 1943 where

he explicitly proved that $(GL_n(R))^{ab} \cong \frac{R^\times}{[R^\times, R^\times]}$ for all $n \geq 1$ (except for $n = 2$, when $|R| = 2$) [Wei13, III.1.2.4]. $\square$

Recall from Theorem 2.5 and Remark 2.6 that $K_1(M_R) \cong \varinjlim_{n \in \mathbb{N}} (\Omega_n^n)^{ab}$, and hence the proof of this case of Theorem A is complete thanks to the above proposition.

Remark 4.4. If R is a von Neumann regular ring and M_R is an infinite right R -module such that the theory of M_R is closed under products and $\bar{\mathcal{X}}^*(M_R) \cong \mathbb{N}$, then Steps I and II of the proof of Theorem A follow verbatim.

Let us recall the computation of $K_1^\oplus(R)$ from algebraic K-theory.

Theorem 4.5. [Wei13, III.1.2.4] *If R is a division ring, then $K_1^\oplus(R) \cong \frac{R^\times}{[R^\times, R^\times]}$.*

There is a beautiful connection between algebraic and model-theoretic K_1 -groups whose proof follows that of [BK25, Theorem 9.1] verbatim.

Theorem 4.6. *For an infinite division ring R , there is a natural embedding of $K_1^\oplus(R)$ into $K_1(R_R)$ induced by the inclusion functor $\text{Free}(R) \rightarrow \mathcal{S}(R_R)$ (thanks to Proposition 3.3), where $\text{Free}(R)$ is the full subcategory of $\text{Mod-}R$ consisting of finitely generated free modules.*

Remark 4.7. Suppose $GL(R) := \varinjlim_{n \in \mathbb{N}} GL_n(R)$ for an infinite division ring R . Then the composition $GL(R) \rightarrow K_1^\oplus(R) \hookrightarrow K_1(R_R)$ can be described as follows: if $A \in GL_n(R)$ is not in the image of the natural embedding of $GL_{n-1}(R)$ into $GL_n(R)$, then it maps to $\det(A) \in (GL_n(R))^{ab}$, where $(GL_n(R))^{ab}$ is the leading term of $(\Omega_n^n)^{ab}$.

5. K_1 OF $M_q(R)$ -MODULES FOR A DIVISION RING R

The main goal of this short section is to prove Theorem A in its full generality by computing $K_1(M_{M_q(R)})$ when $M_q(R)$ is the matrix ring over a division ring R , $q \geq 1$, and $|M_q(R)| \neq 2$.

The proof is divided into three steps similar to the proof in the section above. Thanks to the last sentence of Remark 3.7 and Example 2.3, we assume that M is infinite. We argued in Theorem 3.8 that $\bar{\mathcal{X}}^*(M_{M_q(R)}) \cong q\mathbb{N} \cong \mathbb{N}$, and noted in Proposition 2.8(2) that the theory of $M_{M_q(R)}$ is closed under products. Therefore, Remark 4.4 yields that we only need to deal with Step III.

The next result computes $(\Upsilon^n(M_{M_q(R)}))^{ab}$ for most values of q and n , and its proof is essentially that of [BK25, Lemma 7.1]—the latter result is stated for commutative rings but its proof only uses elementary matrices and does not depend on the commutativity of the ring.

Lemma 5.1. *Suppose R is a unital ring and M_R is a right R -module. Then for each $q \geq 2$ we have $(GL_q(R)^{op} \ltimes M^q)^{ab} \cong (GL_q(R))^{ab}$. Moreover, if the multiplicative identity 1 in R can be written as a sum of two units, then the conclusion also holds true for $q = 1$.*

The hypothesis of the above lemma fails when $|R| = 2$ and $q = 1$. For all other cases, combining Dieudonné's result and Lemma 5.1 with the fact that $GL_k(M_q(R)) \cong GL_{kq}(R)$ for all $k, q \in \mathbb{N}$, we can follow the computations in Step III of § 5 to conclude the proof of Theorem A.

Remark 5.2. The isomorphism $K_1(R_R) \cong K_1((M_q(R))_{(M_q(R))})$ for each division ring R stated in Corollary 1.1 could be interpreted as weak Morita invariance of model-theoretic K_1 , and not the usual Morita invariance as seen in its algebraic counterpart [Wei13, Example III.1.1.4]—this property is shared with model-theoretic K_0 as explained in Remark 3.10. When R is infinite, we overcome the difference between the sets of parameters for different rings thanks to Lemma 5.1 as well as the use wreath products with finitary permutation groups in Step II of the proof via Proposition 4.2.

6. K_1 OF MODULES OVER SEMISIMPLE RINGS

Throughout this section, S will denote a unital semisimple ring. Thanks to the Wedderburn-Artin theorem (Theorem 3.4(4)), we have $S \cong \prod_{i=1}^k S_i$, where $S_i := M_{q_i}(R_i)$ for some division ring R_i and $q_i \geq 1$. Let e_i be the idempotent such that $Se_i = M_{q_i}(R_i)$.

Remark 6.1. Theorem 3.4(2) yields that a right S -module M_S can be written as $M \cong \prod_{i=1}^k M_i$, where $M_i := Me_i$ is a right S_i -module. As a consequence, the category of $\text{Mod-}S = \text{Mod}(\prod_{i=1}^k S_i)$ is equivalent to the category of $\prod_{i=1}^k (\text{Mod-}S_i)$.

Assume that each M_i is infinite. The main goal of this section is to prove Theorem B. As in § 4, the proof of this theorem is along lines similar to the computation of K_1 of infinite vector spaces [BK25, § 5], and we only focus on Steps I and II.

Remark 6.2. Since each M_i is infinite, Proposition 2.8 and Remark 3.7 together yield that the theory of the module M_S is closed under products.

Combining Remark 6.1 and Theorem 3.8, we obtain the following.

Theorem 6.3. *Using the notations and hypotheses of Theorem B, $\overline{\mathcal{X}}^*(M_S) \cong \prod_{i=1}^k q_i \mathbb{N} \cong \mathbb{N}^k$.*

Corollary 6.4 (Model-theoretic K_0 commutes with products). *Using the notations and hypotheses of Theorem B, we have $K_0(M_S) \cong K_0(M_{S_1}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K_0(M_{S_k}) \cong \mathbb{Z}[X_1, \dots, X_k]$.*

The monoid $\mathbb{N}^k$ is naturally equipped with a partial order where $\overline{m} \leq \overline{m}'$ if $m_i \leq m'_i$ for each $1 \leq i \leq k$. Let $\overline{1} := (1, 1, \dots, 1) \in \mathbb{N}^k$ and $m \cdot \overline{1}$ for the constant tuple $(m, \dots, m) \in \mathbb{N}^k$.

Set $\mathcal{S} := \mathcal{S}(M_S)$ and $\mathcal{S}_i := \mathcal{S}((M_i)_{S_i})$ for every $1 \leq i \leq k$. Let $\pi_i : \mathcal{S} \rightarrow \mathcal{S}_i$ be the natural projection functors. For $E \in \mathcal{S}_i$, let $\dim_i(E)$ denote $\dim_{\mathcal{S}_i}(E)$ as defined in the previous sections. The above theorem forces us to assign a “dimension vector” to definable sets in $\mathcal{S}$.

Definition 6.5. *For $E \in \mathcal{S}$, define $\dim_{\mathcal{S}}(E) := \begin{cases} -\infty & \text{if } E = \emptyset; \\ (\dim_i(\pi_i(E)))_{i=1}^k & \text{otherwise.} \end{cases}$*

For simplicity, we denote $\dim_{\mathcal{S}}(E)$ by $\dim(E)$ and write its i^{th} component as $(\dim(E))_i$. The definition of dimension of automorphisms remains the same as in Definition 4.1. For a nonempty $E \in \mathcal{S}$ and $\overline{m} \in \mathbb{N}^k$, let $\Omega_{\overline{m}}(E) := \{f \in \text{Aut}_{\mathcal{S}}(E) : \dim(f) \leq \overline{m}\}$ be the subgroup of $\text{Aut}_{\mathcal{S}}(E)$ of automorphisms fixing all elements of E outside a subset of dimension at most $\overline{m}$.

We wish to prove the following vector analogue of Proposition 4.2 and [BK25, Proposition 5.4].

Proposition 6.6. *If $\overline{m} \in \mathbb{N}^k$, $E_1, E_2 \in \mathcal{S}$ and $\dim(E_l) \geq \overline{m} + \overline{1}$ for $l = 1, 2$, then $\Omega_{\overline{m}}(E_1) \cong \Omega_{\overline{m}}(E_2)$.*

The proof of [BK25, Proposition 5.4] can be readily adapted to obtain a proof of the above except for its first line, which is the content of Lemma 6.8. (Recall that for the case of PIDs, [BK25, Lemma 6.4] plays the same role as Lemma 6.8.)

Before stating and proving the lemma, we need some technical details about special definable sets called ‘blocks’. Recall from [Kub15, Definition and Lemma 3.1.6] that $B \in \mathcal{S}$ is a *block* if $B = A \setminus \bigcup_{j=1}^t A_j$ for some $A, A_j \in \overline{\mathcal{L}}(M_S)$ with $A_j \subsetneq A$ for each $1 \leq j \leq t$. In view of Remark 6.2, [Kub15, Remark 3.1.7] yields that each block is nonempty.

Remark 6.7. It follows from [Kub15, Lemma 2.5.7] that every set in $\mathcal{S}$ can be written as a finite disjoint union of blocks. Suppose $B = A \setminus \bigcup_{j=1}^t A_j, B' = A \setminus \bigcup_{j=1}^{t'} A'_j \in \mathcal{S}$ are blocks. Then $\pi_i(B)$ is a block in $\mathcal{S}_i$ for each $1 \leq i \leq k$. Moreover, $B \cap B'$ is non-empty, and $\dim(B) = \dim(A) = \dim(B')$.

Now we are ready to state and prove the anticipated lemma.

Lemma 6.8. *Let $D_1, D_2 \in \mathcal{S}$ and $\overline{m} = (m_1, \dots, m_k) \in \mathbb{N}^k$. If $\dim(D_l) \geq \overline{m} + \overline{1}$ for $l = 1, 2$, then there exists $D \in \mathcal{S}$ with a definable bijection $g : D \rightarrow D_2$ such that $\dim(D_1 \cap D) \geq \overline{m} + \overline{1}$.*

Proof. Suppose $D_1^1 := D_1 = \bigsqcup_{p=1}^{t_1} B_p^1$ is a decomposition of D_1 into blocks guaranteed by Remark 6.7. Then there exists some $p \in [t_1]$ such that $\dim(\pi_1(B_p^1)) = (\dim(D_1))_1$. Thus, there exist $a_i \in \pi_i(D_1^1)$ for $i \neq 1$ such that $C_1^1 := \{(x, a_2, \dots, a_n) \mid x \in \pi_1(B_p^1)\} \subseteq D_1^1$. Note that $\dim(C_1^1) = ((\dim(D_1))_1, 0, \dots, 0)$.

Successively incrementing i from 1 to $k - 1$ in steps of size 1, we set $D_1^{i+1} := D_1^i \setminus C_1^i$ and repeat the process to obtain C_1^{i+1} for each $i \in \{1, \dots, k - 1\}$. The construction ensures that $C_1^{i_1} \cap C_1^{i_2} = \emptyset$ for $1 \leq i_1 < i_2 \leq k$, $C_1 := \bigsqcup_{i=1}^k C_1^i \subseteq D_1$ and $\dim(C_1) = \dim(D_1)$. We can also obtain $C_2 := \bigsqcup_{i=1}^k C_2^i \subseteq D_2$ in a similar manner.

Assume without loss of generality that $m_1^1 := (\dim(C_1^1))_1 \leq (\dim(C_2^1))_1 =: m_2^1$. Since $\pi_1(C_1^1)$ is a block in $\mathcal{S}_1$ for $l = 1, 2$ by Remark 6.7, there is a definable embedding of $\pi_1(C_1^1)$ into $M_1^{m_1^1}$, say with image $C_l'^1$. Clearly $\overline{x} \mapsto (\overline{x}, \overline{0})$ defines an embedding $i_1 : M_1^{m_1^1} \hookrightarrow M_1^{m_2^1}$ with a splitting, say p_1 . Then $E_1 := C_1'^1 \cap p_1(C_2'^1)$ satisfies $\dim(E_1) = m_1^1 > m_1$ thanks to the final statement of

Remark 6.7. Note that $\{(x, a_2, \dots, a_n) \mid x \in E_1\} \subseteq C_1^1$. Analogously, there is an inclusion of E_1 into C_2^1 .

Repeating the above process for each $1 \leq i \leq k$, we obtain E_i that embeds into both C_1^i and C_2^i , and satisfies $\dim(E_i) > m_i$. Therefore, the images of $E := \bigsqcup_{i=1}^k \{(0, \dots, 0, x_i, 0, \dots, 0) \mid x_i \in E_i\}$ in D_1 and D_2 are isomorphic and $\dim(E) \geq \bar{m} + \bar{1}$. By replacing the image of E in D_2 by the image of E in D_1 , we get the required arrow g . $\square$

Given $\bar{m} := (m_1, \dots, m_k) \in \mathbb{N}^k$, define $M^{\bar{m}} := \prod_{i=1}^k M_i^{m_i}$.

Remark 6.9. If $D \in \mathcal{S}$, then there are $l_i \geq 1$ such that $\pi_i(D) \subseteq M_i^{l_i}$, and hence there are natural embeddings

$$\text{Aut}_{\mathcal{S}}(D) \hookrightarrow \text{Aut}_{\mathcal{S}}\left(\prod_{i=1}^k \pi_i(D)\right) \hookrightarrow \text{Aut}_{\mathcal{S}}(M^{\bar{l}}) \hookrightarrow \text{Aut}_{\mathcal{S}}(M^{l \cdot \bar{1}}),$$

where $l := \max\{l_1, l_2, \dots, l_k\}$. This observation yields that $(M^{n \cdot \bar{1}})_{n \geq 1}$ is a cofinal sequence in $\mathcal{S}$, and hence thanks to Remark 2.6, we have

$$K_1(M_{\mathcal{S}}) \cong \varinjlim_{n \in \mathbb{N}} \text{Aut}_{\mathcal{S}}(M^{n \cdot \bar{1}}).$$

For all $n \in \mathbb{N}$, set $\Omega_n^n(\mathcal{S}) := \text{Aut}_{\mathcal{S}}(M^{n \cdot \bar{1}})$. For each $0 \leq m < n$, let $\Omega_m^n(\mathcal{S}) := \Omega_{m \cdot \bar{1}}(M^{n \cdot \bar{1}})$ (Proposition 6.6). For each $n \geq 1$, let $\Upsilon^n(\mathcal{S})$ denote the subgroup of $\text{Aut}_{\mathcal{S}}(M^{n \cdot \bar{1}})$ consisting only of pp -definable bijections. The notations in this paragraph are also used when we replace M with M_i , and $\mathcal{S}$ with $\mathcal{S}_i$.

Remark 6.10. Since S is a von Neumann regular ring, Remark 3.1 yields that the theory of $M_{\mathcal{S}}$ eliminates quantifiers. As a result, $\Upsilon^n(\mathcal{S})$ is the group of definable S -linear automorphisms of $M^{n \cdot \bar{1}}$. In other words, $\Upsilon^n(\mathcal{S}) \cong GL_n(S^{op}) \ltimes M^{n \cdot \bar{1}}$, where $GL_n(S^{op})$ acts on $M^{n \cdot \bar{1}}$ on the right by matrix multiplication.

Lemma 6.11. *Using the notations and hypotheses of Theorem B, we have*

$$GL_n(S^{op}) \ltimes M^{n \cdot \bar{1}} \cong \prod_{i=1}^k (GL_n(S_i^{op}) \ltimes M_i^n).$$

Proof. The proof readily follows from the observation that $GL_n(S^{op}) \cong \prod_{i=1}^k GL_n(S_i^{op})$ acts on $M^{n \cdot \bar{1}} = (\prod_{i=1}^k M_i)^n$ componentwise. $\square$

Combining Remark 6.10 with the lemma above, we get the following conclusion.

Corollary 6.12. *For each $n \in \mathbb{N}$, $\Upsilon^n(\mathcal{S}) \cong \prod_{i=1}^k \Upsilon^n(\mathcal{S}_i)$.*

We have a chain of normal subgroups of $\Omega_n^n(\mathcal{S})$:

$$(6.1) \quad \Omega_0^n(\mathcal{S}) \triangleleft \Omega_1^n(\mathcal{S}) \triangleleft \dots \triangleleft \Omega_{n-1}^n(\mathcal{S}) \triangleleft \Omega_n^n(\mathcal{S}).$$

The group $\Upsilon^n(\mathcal{S})$ acts on $\Omega_{n-1}^n(\mathcal{S})$ by conjugation and $\Omega_n^n(\mathcal{S}) = \Upsilon^n(\mathcal{S}) \ltimes \Omega_{n-1}^n(\mathcal{S})$. Fix some $0 < m < n$. Let $\mathcal{S}_{m \cdot \bar{1}}(M^{n \cdot \bar{1}})$ denote the full subcategory of $\mathcal{S}$ consisting of definable subsets of $M^{n \cdot \bar{1}}$ of dimension at most $m \cdot \bar{1}$. The restriction of $\sqcup$ equips $\mathcal{S}_{m \cdot \bar{1}}(M^{n \cdot \bar{1}})$ with a symmetric monoidal structure and $\Omega_m^n(\mathcal{S}) \cong \text{Aut}(\mathcal{S}_{m \cdot \bar{1}}(M^{n \cdot \bar{1}}))$. We want to find a subgroup $\Upsilon_m^n(\mathcal{S})$ of $\Omega_m^n(\mathcal{S})$ such that $\Omega_m^n(\mathcal{S}) = \Upsilon_m^n(\mathcal{S}) \ltimes \Omega_{m-1}^n(\mathcal{S})$.

Let Σ_j denote the permutation group of a finite set of size j , and $\bar{\Sigma}$ denote the finitary permutation group of a countably infinite set.

For $\bar{j} := (j_1, \dots, j_k) \in \mathbb{N}^k$, let $S_{m, \bar{j}} \in \mathcal{S}_{m \cdot \bar{1}}(M^{n \cdot \bar{1}})$ denote a copy of $\prod_{i=1}^k (\bigsqcup_{l=1}^{j_i} M_i^m)$ in such a way that $S_{m, \bar{j}} \subseteq S_{m, \bar{j}'}$ whenever $\bar{j} \leq \bar{j}'$ in $\mathbb{N}^k$. Note that

$$\text{Aut}_{\mathcal{S}}(S_{m, \bar{1}}) \cong \Omega_m^m(\mathcal{S}) \cong \Upsilon^m(\mathcal{S}) \ltimes \Omega_{m-1}^m(\mathcal{S}) \cong \Upsilon^m(\mathcal{S}) \ltimes \Omega_{m-1}^n(\mathcal{S}),$$

where the action of $\Upsilon^m(\mathcal{S})$ on $\Omega_{m-1}^n(\mathcal{S})$ is induced by the isomorphism $\Omega_{m-1}^m(\mathcal{S}) \cong \Omega_{m-1}^n(\mathcal{S})$ given by Proposition 6.6. For similar reasons, we also have

$$\text{Aut}_{\mathcal{S}}(S_{m,\bar{j}}) \cong (\Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \Sigma_{j_i}) \ltimes \Omega_{m-1}^m(\mathcal{S}) \cong (\Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \Sigma_{j_i}) \ltimes \Omega_{m-1}^n(\mathcal{S}),$$

where the group $(\Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \Sigma_{j_i})$ acts on $\Omega_{m-1}^m(\mathcal{S})$ by conjugation and permutes lower dimensional subsets of $S_{m,\bar{j}} \subset M^{n,\bar{1}}$. Since $(S_{m,j,\bar{1}})_{j \in \mathbb{N}}$ is a cofinal sequence in $\mathcal{S}_{m,\bar{1}}(M^{n,\bar{1}})$, Remark 2.6 yields

$$\begin{aligned} \Omega_m^n(\mathcal{S}) &\cong \varinjlim_{j \in \mathbb{N}} \text{Aut}_{\mathcal{S}}(S_{m,j,\bar{1}}) \\ &\cong \varinjlim_{j \in \mathbb{N}} \left((\Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \Sigma_j) \ltimes \Omega_{m-1}^n(\mathcal{S}) \right) \\ &\cong \left(\varinjlim_{j \in \mathbb{N}} (\Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \Sigma_j) \right) \ltimes \Omega_{m-1}^n(\mathcal{S}) \\ &\cong \left(\Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \bar{\Sigma} \right) \ltimes \Omega_{m-1}^n(\mathcal{S}). \end{aligned}$$

Let $\Upsilon_m^n(\mathcal{S}) := \Upsilon^m(\mathcal{S}) \wr \prod_{i=1}^k \bar{\Sigma}$. Note that $\Upsilon_m^n(\mathcal{S})$ acts on $\Omega_{m-1}^n(\mathcal{S})$ by conjugation. Thus each $\Omega_m^n(\mathcal{S})$ is an iterated semi-direct product of certain wreath products as in the following expression.

$$\Omega_n^n(\mathcal{S}) \cong \Upsilon^n(\mathcal{S}) \ltimes (\Upsilon_{n-1}^n(\mathcal{S}) \ltimes (\Upsilon_{n-2}^n(\mathcal{S}) \ltimes (\cdots (\Upsilon_1^n(\mathcal{S}) \ltimes \Omega_0^n(\mathcal{S})) \cdots))).$$

Thanks to Corollary 6.12 and the fact that all conjugation actions in the above expression are componentwise, we conclude $\Omega_n^n(\mathcal{S}) \cong \prod_{i=1}^k \Omega_n^n(\mathcal{S}_i)$. This completes the proof of Theorem B.

Since $\text{Free}(S)$ is cofinal in $\prod_{i=1}^k \text{Free}(S_i)$ (Remark 6.9) and algebraic K_1 commutes with finite products, we obtain the following consequence of Theorems 4.6, A and B.

Corollary 6.13. *If $S = \prod_{i=1}^k S_i$ is a semisimple ring with each S_i infinite, then there is a natural embedding of $K_1^\oplus(S)$ into $K_1(S_S)$ given by a k -tuple of maps as in Remark 4.7.*

The proof of Theorem B follows verbatim to yield the following stronger result.

Theorem 6.14. *For $1 \leq i \leq k$, let S_i be a von Neumann regular ring and M_i be a right S_i -module satisfying the hypotheses of Remark 4.4. Suppose $\prod_{i=1}^k S_i$ acts on $\prod_{i=1}^k M_i$ componentwise. Then*

$$K_1 \left(\left(\prod_{i=1}^k M_i \right)_{(\prod_{i=1}^k S_i)} \right) \cong \prod_{i=1}^k K_1((M_i)_{S_i}).$$

If the hypotheses of Remark 4.4 fail for a von Neumann regular ring R , then our recipe fails to compute the group $K_1(M_R)$.

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