

Automaticity of non-positively curved k -fold triangle groups

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Abstract

We show that non-positively curved k -fold triangle groups have finitely many cone types, and hence a regular language of all geodesics. Further, we prove that the language of lexicographically first geodesics is both regular and satisfies the fellow traveller property, giving an automatic structure for this family of groups.

1 Introduction

In this paper, we look at the class of groups called *k -fold triangle groups*. These groups are important because they are the simplest groups that don't fit into the cubical paradigm, which has been well understood by the work of Agol, Wise and others. Despite their convenient presentations, k -fold triangle groups for $k \geq 3$ can behave in unpredictable ways, very differently from triangle groups which they generalise. For this reason, they were explored in [10] and [5] as candidates for non-residually finite hyperbolic groups.

The main result of this paper is the following theorem, which can be seen as an analogue of the Niblo–Reeves theorem in the cubical setting (see [12]) and paves the way for performing explicit computations in k -fold triangle groups:

Theorem 1.1. *Let G be a non-positively curved k -fold triangle group. Then G admits a language which is both regular and satisfies the fellow traveller property. Hence, G is automatic.*

Non-positive curvature plays a crucial part in all of the arguments of the paper. Hyperbolic groups have long been known to be automatic (see [4]), but it is still unknown if all of non-positively curved groups are. Furthermore, by the work of Leary and Minasyan [9], we know that non-positive curvature doesn't guarantee *biautomaticity*. Coxeter groups were proven to be automatic by Davis and Shapiro in 1991 [6], but they were only shown to be biautomatic in 2022 by Osajda and Przytycki [13]. Recently, more biautomatic structures were given in [14]. Another class, which is in spirit close to k -fold triangle groups, are small cancellation groups, which are known to be biautomatic by [8].

The main techniques used in proving Theorem 1.1 are disc diagram techniques and the angle structure setting introduced by Wise in [16]. A crucial step is the following lemma, which relates combinatorial geodesics in a non-positively curved k -fold triangle group with the CAT(0) geodesics in its corresponding complex:

Lemma 1.2. *Let G be a k -fold triangle group and X the corresponding triangle complex. A (combinatorial) geodesic in $\text{Cay}(G, S)$ is of length $n \iff$ any CAT(0)-geodesic between (the interiors of) the corresponding faces in X crosses exactly n edges of X .*

Roadmap: In Section 2, we give some background on the key notions of the paper: k -fold triangle groups, automaticity, and cone types. In Section 3, we prove Lemma 1.2, relating combinatorial and metric structures of k -fold triangle groups. In Section 4, we prove Theorem 4.6 about the finiteness of cone types, upon which all the further results are based. Finally, in Section 5, we complete the proof of Theorem 1.1, by showing that their geodesics satisfy the *fellow traveller property*.

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2 Preliminaries

2.1 k -fold triangle groups

Here we introduce k -fold triangle groups and a structure theorem by Stallings which gives a geometric approach to the non-positively curved groups of this class. For more information on k -fold triangle groups, see [5], and for more information on complexes of groups, see [2].

Definition 2.1. A *generalised triangle group* is a fundamental group of a triangle of groups whose face group is trivial (see Figure 1). A *k -fold triangle group* is a generalised triangle group whose edge groups are all isomorphic to $\mathbb{Z}/k\mathbb{Z}$, for some $k \in \mathbb{N}$.

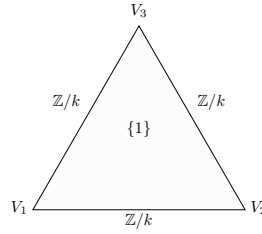


Figure 1: A k -fold triangle group with vertex groups V_1, V_2 and V_3

In the definition above, when $k = 2$ and the vertex groups are dihedral groups, we get a classical triangle group. Furthermore, all the examples where $k = 2$ are iso-

morphic to an amalgamated product of triangle groups and finite groups.

Under mild hypotheses, a k -fold triangle group admits a natural action on a CAT(0) triangle complex with explicit links. To formalise this, we need a notion of a coset graph.

Definition 2.2. A *coset graph* of a group H with two subgroups A and B is the bipartite graph $\Gamma_{H(A,B)}$ with vertex set $H/A \cup H/B$ and edge set $H/(A \cap B)$, where the incidence relation is the relation of inclusion.

Theorem 2.3 (Stallings, [15]). *A k -fold triangle group G with coset graphs $\Gamma_i := \Gamma_{V_i(\mathbb{Z}/k, \mathbb{Z}/k)}$ of respective half-girths r_i that satisfy $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \leq 1$ acts isometrically, by simplicial automorphisms, on a CAT(0) simplicial complex X (see Figure 2) of dimension 2. The action has a 2-simplex as a strict fundamental domain, which is isometric to a Euclidean or hyperbolic triangle with angles $(\pi/r_1, \pi/r_2, \pi/r_3)$. The links of the vertices in X are isomorphic to the coset graphs Γ_i , and the stabilisers of the vertices are isomorphic to the vertex groups V_i .*

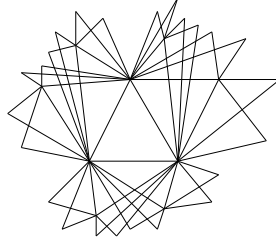


Figure 2: A part of the complex X for a 3-fold triangle group

Remark 2.4. As X is a CAT(0) complex, every torsion subgroup of G is conjugate into a vertex stabiliser.

2.2 Automaticity

In this section, we define automaticity, following [7]. Informally, a group is considered automatic if all of its elements can be iteratively constructed by a simple recipe, encoded through *finite state automata*, which we are also able to use to check if two words in a specific presentation give the same element or differ by right multiplication by a single generator.

Definition 2.5. A *finite state automaton* over S is a finite directed graph $\Gamma = \Gamma(A, E)$, with a labeling of the edges given by $\phi : E \rightarrow \mathcal{P}(S^*)$ and:

- a vertex $a_0 \in A$ called the *start state* of Γ ;
- a subset $A_\infty \subseteq A$ called *the accept states* of Γ .

We say that a set of words $\Lambda \subseteq S^*$ forms a *regular language* for the finite state automaton Γ if it is *accepted* by Γ , i.e. if for every word $w \in \Lambda$ there is an edge path in Γ from the start state a_0 to an accept state a , whose labels make up w .

We are now ready to give a definition of automaticity for which we use a characterisation from [7] (see Theorem 2.3.5).

Remark 2.6. For a word $w \in \Lambda$, $w = s_1 s_2 \dots s_n$ we will use $w(i)$ to denote the word up to the i -th letter, i.e. $w(i) = s_1 s_2 \dots s_i$ and $\bar{w}$ to denote the group element represented by w .

Definition 2.7. Let G be a finitely generated group. An *automatic structure* for G is a pair (S, Λ) , where S is a symmetric finite generating set for G and Λ is a regular language, such that the following conditions hold:

1. Every element of G is represented by some word in Λ .
2. There is a uniform constant δ such that for every $w, w' \in \Lambda$ with $w' = ws$, for some $s \in S$, $d(w(i), w'(i)) \leq \delta$, for every $i \geq 1$.

Remark 2.8. Property 2 is often referred to as the *fellow traveller property*. If the same holds not only for words differing by a right multiplication by a single generator, but also by a left multiplication by a single generator, then the group in question is called *biautomatic*.

2.3 Cone types

Cone types play an important role in the theory of automatic structures. At a group element g , the cone type encodes the geodesics from the identity that pass through g . Here we give a definition and the main connection to automaticity. For more details see [3] or [4].

Definition 2.9. For a group G with a generating set S , we define the *cone type* of an element $g \in G$ as:

$$\text{Cone}(g) = \{h \in G \mid \text{there is a geodesic in } \text{Cay}(G, S) \text{ from id to } gh \text{ which passes through } g\}.$$

Theorem 2.10 (Cannon [4]). *In a finitely generated group G with finitely many cone types, the language of all geodesics is regular. Furthermore, if S is a finite, symmetric generating set for G with a fixed total order $\leq$, then the language of lexicographically first (with respect to the order $\leq$) geodesics is also regular.*

3 Relating combinatorial and CAT(0) geodesics

The main step in understanding cone types and proving Theorem 1.1 is relating the geodesics in the Cayley graph (combinatorial geodesics) to the geodesics in the cell complex (CAT(0) geodesics). This relies on the combinatorial Gauss–Bonnet formula discussed below. The formula was first proven in [1], and later observed in [11]. Here, we use the angle structure setting introduced by Wise in [16].

3.1 Combinatorial Gauss–Bonnet theorem and angle structures

Definition 3.1. Let Y be a 2-complex with angles, i.e. non-negative real numbers $\angle c$ attached to every corner c of a 2-cell in Y . We denote by $|\partial f|$ the length of the attaching map of a 2-cell f in Y . The curvature at a vertex $v \in Y^{(0)}$ is:

$$\kappa(v) = 2\pi - \pi\chi(\text{Link}(v)) - \left(\sum_{c \in \text{Corners}(v)} \angle c \right). \quad (1)$$

The curvature at a face $f \in Y^{(2)}$ is:

$$\kappa(f) = \left(\sum_{c \in \text{Corners}(f)} \angle c \right) - (|\partial f| - 2)\pi. \quad (2)$$

Theorem 3.2 (Combinatorial Gauss–Bonnet). *Let Y be a finite 2-complex with angles. With the notation as above:*

$$\sum_{v \in Y^{(0)}} \kappa(v) + \sum_{f \in Y^{(2)}} \kappa(f) = 2\pi\chi(Y). \quad (3)$$

To be able to use the tools above for a k -fold triangle group G , we need to introduce a suitable 2-dimensional cell complex C .

Firstly, let X be a CAT(0) cell complex associated to a k -fold triangle group G , $k \geq 2$. Denote the vertex groups of G by $V_1 = \langle a, b \rangle$, $V_2 = \langle b, c \rangle$ and $V_3 = \langle c, a \rangle$ and the edge groups by $E_1 = \langle c \rangle$, $E_2 = \langle a \rangle$, and $E_3 = \langle b \rangle$. For simplicity of the argument, we redefine the metric on X to be such that each triangle is a Euclidean equilateral triangle, which does not change any important properties of X .

Furthermore, let $\text{Cay}(G, S)$ be the Cayley graph of G with respect to the generating set $S = \{a, \dots, (k-1)a, b, \dots, (k-1)b, c, \dots, (k-1)c\}$. Although $\text{Cay}(G, S)$ is not embedded in X (because of torsion), G still acts transitively on the faces of X , so there is a correspondence between the faces of X and the elements of G , i.e. the vertices in $\text{Cay}(G, S)$. Furthermore, there is a map $i : \text{Cay}(G, S) \rightarrow X$, which injectively maps vertices to the centres of triangles in X and maps edges of the Cayley graph to the geodesic segments connecting the relevant centres. This correspondence also allows us to define a *cone type* of $t_g \in X^{(2)}$, the face of X corresponding to an element $g \in G$:

$$\text{Cone}(t_g) = \{t_h \in X^{(2)} \mid h \in \text{Cone}(g)\},$$

where h is the element of G corresponding to t_h .

For $x \in X^{(0)}$, denote by $L^*(x)$ the subgraph of $\text{Cay}(G, S)$ dual to the link $\text{Link}(x)$.

We can now define C as a 2-dimensional cell complex whose 1-skeleton is $\text{Cay}(G, S)$. A 2-cell is attached along every loop in every $L^*(x)$ in $\text{Cay}(G, S)$. Also, for every edge stabiliser $\mathbb{Z}/k\mathbb{Z}$ of G , and every $a, b \in \mathbb{Z}/k\mathbb{Z}$, a triangle is attached with vertices $-a, -b, a + b$.

The metric on X induces the angle structure on C . By construction, the triangular cells will have all angles equal to 0, whereas the other cells will have all angles equal to $\frac{2\pi}{3}$.

3.2 Relating the combinatorial and the metric structure of G

In this section we prove Lemma 1.2. It provides the key geometric insight, by redefining the distance function on G as the number of edges crossed by a CAT(0) geodesic in X :

Lemma 1.2. *Let G be a k -fold triangle group and X the corresponding triangle complex. A (combinatorial) geodesic in $\text{Cay}(G, S)$ is of length $n \iff$ any CAT(0)-geodesic between the interiors of the corresponding faces in X crosses exactly n edges of X .*

Remark 3.3. In the case that a CAT(0) geodesic γ passes through a vertex $x \in X^{(0)}$ we will look at that as γ crossing the smallest sequence of triangles between the one where γ enters x and the one where γ exits x . Similarly, if γ passes through a path $p = (p_0, \dots, p_n) \in X^{(1)}$ we will look at that as γ crossing the smallest sequence of triangles between the one where γ enters p_0 and the one where γ exits p_n . Such a sequence of triangles is not necessarily unique.

Because the triangles of X correspond to vertices in $\text{Cay}(G, S)$, any CAT(0)-geodesic γ in X induces a path g_γ in $\text{Cay}(G, S)$, where crossing an edge between two triangles corresponds to traversing the suitable edge of the Cayley graph. In the case that γ passes through a path in $X^{(1)}$, the induced combinatorial geodesic will follow the sequence of triangles from the remark above.

The proof of the lemma relies on showing that the induced path g_γ must be a combinatorial geodesic. This is done by looking at a minimal disc diagram bounded by g_γ and a combinatorial geodesic.

Before we move on to the proof of the lemma, we state and prove one of its ingredients, which gives more insight into paths induced by geometric geodesics:

Definition 3.4. Let $\mathcal{D}$ be a disc diagram and g a path in $\partial\mathcal{D}$. Then the *vertex curvature* of g in $\mathcal{D}$ is

$$\kappa_{\mathcal{D}}^v(g) = \sum_{v \in \mathcal{D}^{(0)} \cap \text{int}(g)} \kappa(v)$$

and the *face curvature* of g in $\mathcal{D}$ is

$$\kappa_{\mathcal{D}}^f(g) = \sum_{\substack{f \in X^{(2)} \\ f \cap g \neq \emptyset}} \kappa(f).$$

Furthermore, if there is another path g in $\partial\mathcal{D}$,

$$\kappa_{\mathcal{D}}^f(g, g') = \sum_{\substack{f \in X^{(2)} \\ f \cap g \neq \emptyset \\ f \cap g' \neq \emptyset}} \kappa(f).$$

Lemma 3.5. *Let X, C, γ and g_γ be as above and let $\mathcal{D}$ be a minimal disc diagram in C such that $g_\gamma \subset \partial\mathcal{D}$. Then*

$$\kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) \leq \frac{2\pi}{3}.$$

Proof. We will compare the direction of g_γ , to the direction of γ (see Figures 3a, 3b).

Take a sequence of triangles $t_i \in X, i \in \{0, \dots, n\}$ which correspond to vertices $p_i \in X, i \in \{0, \dots, n\}$ of g_γ from Remark 3.3. Moreover, we pick a leftmost such sequence (with respect to the direction of γ). Next, take a subsequence $t_{i_j} \in X, j \in \{0, \dots, m\}$ such that $t_{i_j} \cap \gamma \neq \{pt\}$. We will denote the subset of indices $\{i_j\}$ by I_s . Since γ starts and ends at interiors of triangles, we have that $i_0 = 0$ and $i_m = n$.

We start with the definition of vertex curvature over g_γ :

$$\kappa_{\mathcal{D}}^v(g_\gamma) = \sum_{i=1}^{n-1} \kappa(p_i) = \sum_{i=1}^{n-1} \left(\pi - \sum_{c \in \text{Corners}(p_i)} \angle c \right). \quad (4)$$

We assume that g_γ bounds the disk diagram $\mathcal{D}$ from the right, i.e. that $\mathcal{D}$ is on the left of g_γ . The sum of the corner angles at p_i is bounded below by the angle between the segments $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$. Let $g_\gamma(p_i)$ be the unit vector corresponding to the segment $[p_i, p_{i+1}]$ at p_i . Because $[p_i, p_{i+1}]$ can be seen in X inside two adjacent triangles, so inside a part of a Euclidean plane, the vector corresponding to $[p_{i+1}, p_i]$ at p_{i+1} is $-g_\gamma(p_i)$. Because all the triangles in X are equilateral, $\angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$. Furthermore, angle zero would imply that γ enters and exits some triangle via the same edge, which would contradict the convexity of triangles. So:

$$\angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \in \left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\} \quad (5)$$

From equation 4 we get the following inequality:

$$\kappa_{\mathcal{D}}^v(g_\gamma) \leq \sum_{i=1}^{n-1} \pi - \angle(g_\gamma(p_i), -g_\gamma(p_{i-1})). \quad (6)$$

For each triangle in the subsequence $\{t_{i_j}\}$, $\text{int}(\gamma \cap t_{i_j}) \neq \emptyset$, so for each j , we can pick a point $q_j \in \text{int}(\gamma \cap t_{i_j})$ and define $\gamma(p_i)$ to be the unit vector parallel to the tangent vector of γ at q_i . Because the triangles are Euclidean, this is well-defined.

We want to compare the directions of different vectors. For this purpose we fix an orientation on the triangle containing p_0 and propagate it to all the other triangles in the sequence $\{t_i\}$.

Because γ and g_γ always cross the same edges of the underlying triangles t_{i_j} in X , the angle between them is always $\leq \pi/2$. We separate the indices I_s into two sets, depending on which side of $g_\gamma(p_{i_j})$ the vector $\gamma(p_{i_j})$ is. In the first case, $\gamma(p_i)$ is on the left and $\angle(g_\gamma(p_i), \gamma(p_i)) \in [0, \frac{\pi}{2}]$. We denote the set of such indices by I_l . In the second case, $\gamma(p_i)$ is on the right, and $\angle(g_\gamma(p_i), \gamma(p_i)) \in [\frac{3\pi}{2}, 2\pi)$. This set of indices will be denoted by I_r . Hence we can rewrite 6 as:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g_\gamma) &\leq \sum_{i \in \{1, \dots, n-1\} \setminus I_s} \pi - \angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) + \sum_{i \in I_l} \pi - \angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \\ &\quad + \sum_{i \in I_r} \pi - \angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \end{aligned} \quad (7)$$

In the case of I_l , because (by 5) $\angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \geq \frac{2\pi}{3}$, we have that $\gamma(p_i)$ is between $g_\gamma(p_i)$ and $-g_\gamma(p_{i-1})$, so

$$\angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) = \angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_{i-1})). \quad (8)$$

In the case of I_r , because (by 5) $\angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \leq \frac{4\pi}{3}$, we have that $\gamma(p_i)$ is between $-g_\gamma(p_{i-1})$ and $g_\gamma(p_i)$, so

$$\angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) = \angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_{i-1})) - 2\pi. \quad (9)$$

This allows us to rewrite the bound 7 for the curvature of g_γ as:

$$\begin{aligned} \kappa_v^{\mathcal{D}}(g_\gamma) &\leq \sum_{i \in \{1, \dots, n-1\} \setminus I_s} \pi - \angle(g_\gamma(p_i), -g_\gamma(p_{i-1})) \\ &\quad + \sum_{i \in I_l} \pi - (\angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_{i-1}))) \\ &\quad + \sum_{i \in I_r} \pi - (\angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_{i-1})) - 2\pi). \end{aligned} \quad (10)$$

We will now use the fact that γ is a geodesic. If two consecutive triangles with indices in I_s , $t_{i_{j-1}}$ and t_{i_j} share an edge, then $\gamma(p_{i_{j-1}})$ is parallel to $\gamma(p_{i_j})$ in a locally Euclidean space, so we can regard them as the same vector. However if $t_{i_{j-1}}$ and t_{i_j} only share a vertex v_j , then the sequence of triangles $t_{i_{j-1}}, t_{i_{j-1}+1}, \dots, t_{i_j}$ forms a non-positively curved space, with negative curvature concentrated at v_j , through which γ passes. Hence $\gamma(p_{i_{j-1}})$ is not necessarily parallel to $\gamma(p_{i_j})$ along g_γ . However, we still have all the needed information about the change of angles.

Assume that v_j has curvature zero. Then $\gamma(p_{i_j})$ and $\gamma(p_{i_{j-1}})$ are parallel even though γ passes through v_j . Furthermore, since g_γ and γ cross the same edges of triangles $t_{i_{j-1}}$ and t_{i_j} , we know the following:

$$\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_{j-1}})) = \begin{cases} \frac{2\pi}{3}, & \text{if } v_j \notin \mathcal{D} \\ \frac{4\pi}{3}, & \text{if } v_j \in \mathcal{D} \end{cases}, \text{ and} \quad (11)$$

$$\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) \in \begin{cases} \frac{4\pi}{3}, & \text{if } v_j \notin \mathcal{D} \\ \frac{2\pi}{3}, & \text{if } v_j \in \mathcal{D} \end{cases}. \quad (12)$$

We can now rewrite one of these angles in terms of the other:

$$\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_j-1})) = \begin{cases} \angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) - \frac{2\pi}{3}, & \text{if } v_j \notin \mathcal{D} \\ \angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) + \frac{2\pi}{3}, & \text{if } v_j \in \mathcal{D} \end{cases} \quad (13)$$

To conclude the analysis of this case, we observe that $\pm \frac{2\pi}{3}$ is exactly the curvature of the two vertices between $p_{i_{j-1}}$ and p_{i_j} on the path of length 3 which connects them, so:

$$\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_j-1})) = \angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) + \sum_{i_{j-1} < i < i_j} \kappa(p_i) \quad (14)$$

The second case is when v_j is negatively curved. Using that γ and g_γ always intersect the same sides of the triangle they pass through, we get the following two estimates:

$$\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_j-1})) \in \begin{cases} [\frac{\pi}{2}, \frac{5\pi}{6}], & \text{if } v_j \notin \mathcal{D} \\ [\frac{7\pi}{6}, \frac{3\pi}{2}], & \text{if } v_j \in \mathcal{D} \end{cases}, \text{ and} \quad (15)$$

$$\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) \in \begin{cases} [\frac{7\pi}{6}, \frac{3\pi}{2}], & \text{if } v_j \notin \mathcal{D} \\ [\frac{\pi}{2}, \frac{5\pi}{6}], & \text{if } v_j \in \mathcal{D} \end{cases}. \quad (16)$$

Consequently:

$$-\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_j-1})) \leq \begin{cases} -\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) + \pi, & \text{if } v_j \notin \mathcal{D} \\ -\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) - \frac{\pi}{3}, & \text{if } v_j \in \mathcal{D}. \end{cases} \quad (17)$$

Hence, in formula 10, we can replace $-\angle(\gamma(p_i), -g_\gamma(p_{i-1}))$ by $-\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}}))$ which adds at most π curvature when $v_j \notin \mathcal{D}$ and $-\frac{\pi}{3}$ curvature when $v_j \in \mathcal{D}$.

When $v_j \notin \mathcal{D}$ the vertices of the path around v_j are negatively curved and there are $l_j - 1 \geq 3$ of them, so

$$-\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_j-1})) + \sum_{i_{j-1} < i < i_j} \kappa(p_i) \leq -\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})). \quad (18)$$

On the other hand, when $v_j \in \mathcal{D}$, the $l_j - 1$ vertices of the path around v_j may all be positively curved, but then there is a cell $c_j \in X^{(2)}$ which contains them. Since the path of triangles chosen in the beginning was a leftmost one, the girth of the cell has to be larger than $2l$, otherwise g_γ would contain exactly the right half of the boundary

of the cell c_j , which could be replaced by the left half of the boundary. So, the girth of c_j is at least $2(l_j + 1)$ and the curvature of c_j is

$$\kappa(c) \leq -(2l_j + 2 - 6)\frac{\pi}{3},$$

This further means that:

$$\frac{1}{2}\kappa(c) \leq -(l_j - 2)\frac{\pi}{3},$$

which is enough to cover all but one positively curved vertex, which can be covered by $-\pi/3$ curvature we get from the angle replacement above. Hence:

$$-\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_{j-1}})) + \sum_{i_{j-1} < i < i_j} \kappa(p_i) + \kappa(c_j) \leq -\angle(\gamma(p_{i_{j-1}}), -g_\gamma(p_{i_{j-1}})) \quad (19)$$

Finally, we incorporate estimates 14, 18 and 19 into 10 and add the face curvature (where all the cells we don't need to offset the vertex curvature can be ignored, as all the cells of $\mathcal{D}$ are non-positively curved). This gives us:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) &\leq -\angle(\gamma(p_0), -g_\gamma(p_0)) + \angle(\gamma(p_{i_{m-1}}), -g_\gamma(p_{i_{m-1}})) \\ &\quad + \sum_{i \in I_l} \pi - (\angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_i))) \\ &\quad + \sum_{i \in I_r} \pi - (\angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_i)) - 2\pi). \end{aligned} \quad (20)$$

Now, in the first case, we have that $\gamma(p_i)$ is between $g_\gamma(p_i)$ and $-g_\gamma(p_i)$, so

$$\angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_i)) = \angle(g_\gamma(p_i), -g_\gamma(p_i)) = \pi. \quad (21)$$

In the second case, $\gamma(p_i)$ is between $-g_\gamma(p_i)$ and $g_\gamma(p_i)$, so

$$\angle(g_\gamma(p_i), \gamma(p_i)) + \angle(\gamma(p_i), -g_\gamma(p_i)) = \angle(g_\gamma(p_i), -g_\gamma(p_i)) + 2\pi = \pi + 2\pi. \quad (22)$$

Hence, the only non-zero contribution to the curvature bound comes from the 0-th and the $(n-1)$ -st term:

$$\kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) \leq -\angle(\gamma(p_0), -g_\gamma(p_0)) + \angle(\gamma(p_{i_{m-1}}), -g_\gamma(p_{i_{m-1}})). \quad (23)$$

In general $\angle(\gamma(p_{i_j}), -g_\gamma(p_{i_j})) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ for $0 \leq j \leq m-1$, but since we have that γ starts and ends at interiors of triangles, we have sharper bounds for the end cases: $\angle(\gamma(p_0), -g_\gamma(p_0)), \angle(\gamma(p_{i_{m-1}}), -g_\gamma(p_{i_{m-1}})) \in (\frac{\pi}{2}, \frac{3\pi}{2})$. So:

$$\kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) < \pi. \quad (24)$$

As the direction of g_γ can only change for a multiple of $\frac{\pi}{3}$ at a time, we can sharpen the estimate above to get the wanted one:

$$\kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) \leq \frac{2\pi}{3}. \quad (25)$$

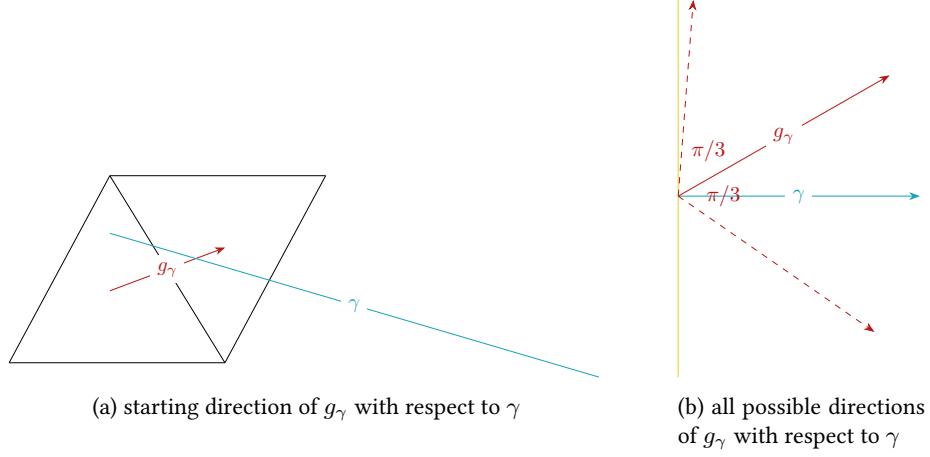


Figure 3: Comparing the directions of γ and g_γ

□

We now introduce two notions that will help us analyse disk diagrams appearing in the rest of the paper.

Definition 3.6. Let $\mathcal{D}$ be a disk diagram and $g \subset \partial\mathcal{D}$ a part of its boundary. If a cell c in $\mathcal{D}$ intersects g in at least one edge, we call it a *g-cell*. If two *g-cells* c_1 and c_2 have a non-empty intersection on g we call them *g-adjacent*.

Proof of Lemma 1.2. Let g be a geodesic in $\text{Cay}(G, S)$ from a vertex u to a vertex v and let γ be the unique geodesic in X from a point p_u in triangle u to a point p_v in triangle v . Further, let g_γ be the path in $\text{Cay}(G, S)$ induced by γ .

We want to show that g_γ is also a geodesic in $\text{Cay}(G, S)$, i.e. that we can geodesically transform g into g_γ . This is done by observing the two combinatorial paths inside the cell complex C and doing induction on the area between them.

Take the first segment $[t_1, t_2]$ such that g and g_γ only meet at $g(t_1) = g_\gamma(t_1) = u'$ and $g(t_2) = g_\gamma(t_2) = v'$. On this segment, the paths bound a disc diagram $\mathcal{D}$ over the cell complex C . Moreover, we can take the disc diagram to be of minimal area. Then the area between the two paths will be the sum of the areas of minimal disc diagrams over all segments where the paths are disjoint.

We now look at the curvature of the faces and vertices of $\mathcal{D}$.

By construction of C , $\mathcal{D}$ contains two types of cells. The first type are n -gons of size ≥ 6 , which by the formula for curvature of faces have non-positive curvature $-(n-6)\frac{\pi}{3}$. The second type are triangles, coming from the order k -elements. Since their corners have angle zero, the triangles will be negatively curved with curvature $-\pi$.

By minimal area, the interior vertices can also be shown to be of non-positive curvature. Namely, if a vertex v is in the interior of $\mathcal{D}$, then $\text{Link}(v)$ is either a loop, which needs to be of length at least 2π , or it is a tree, which means that $\mathcal{D}$ could be further reduced, contradicting the minimal area assumption.

Consider a vertex v on the boundary. It has

$$\sum_{c \in \text{Corners}(v)} c = s \frac{2\pi}{3}, s \geq 0,$$

because all the interior angles of the complex C are either $\frac{2\pi}{3}$ or 0. We will show that $s \geq 1$. The case $s = 0$ is excluded by disjointness at ends, geodesicity of g or underlying geodesic of g_γ . Namely, if $v \in \{u', v'\}$ is such that the sum of the corners at v is zero, it would mean that g and g_γ intersect at a whole segment containing u' or v' , which was excluded by a restriction above. Further, $v \in g$ with corner angle zero would mean backtracking, contradicting the combinatorial geodesicity of g . Finally, we saw in the proof of Lemma 3.5 that γ being a CAT(0) geodesic excludes the case of corner angle zero on g_γ .

Hence, for a vertex v on the boundary,

$$\sum_{c \in \text{Corners}(v)} c = s \frac{2\pi}{3}, s \geq 1,$$

which corresponds to $\kappa(v) = \pi - s \frac{2\pi}{3}, s \geq 1$. Particularly, the curvature at vertices u', v' is $\kappa(u'), \kappa(v') \leq \frac{\pi}{3}$.

Further, by Lemma 3.5,

$$\kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2} \kappa_{\mathcal{D}}^f(g_\gamma) \leq \frac{2\pi}{3}. \quad (26)$$

By the combinatorial Gauss–Bonnet theorem for $\mathcal{D}$, we have the following:

$$2\pi\chi(\mathcal{D}) = \sum_{v \in \partial\mathcal{D}^{(0)}} \kappa(v) + \sum_{v \in \text{int } \mathcal{D}^{(0)}} \kappa(v) - \sum_{f \in \mathcal{D}^{(2)}} \kappa(f). \quad (27)$$

This can be rewritten as:

$$\begin{aligned}
2\pi\chi(\mathcal{D}) &= \kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^v(g_\gamma) + \kappa(u') + \kappa(v') + \sum_{v \in \text{int } \mathcal{D}^{(0)}} \kappa(v) \\
&\quad + \kappa_{\mathcal{D}}^f(g) + \kappa_{\mathcal{D}}^f(g_\gamma) - \kappa_{\mathcal{D}}^f(g, g_\gamma) + \sum_{\substack{f \in \mathcal{D}^{(2)} \\ f \cap (g \cup g_\gamma) = \emptyset}} \kappa(f). \quad (28)
\end{aligned}$$

We know that $\kappa(u') \leq \frac{\pi}{3}, \kappa(v') \leq \frac{\pi}{3}, \chi(\mathcal{D}) = 1$, and vertices in the interior as well as all the faces have non-positive curvature. Incorporating these estimates into the equation 28 gives the following:

$$\kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^v(g_\gamma) + \kappa_{\mathcal{D}}^f(g) + \kappa_{\mathcal{D}}^f(g_\gamma) - \kappa_{\mathcal{D}}^f(g, g_\gamma) \geq \frac{2\pi}{3}. \quad (29)$$

As $\kappa_{\mathcal{D}}^f(g_\gamma) \leq \kappa_{\mathcal{D}}^f(g, g_\gamma)$, we know that $\frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) - \kappa_{\mathcal{D}}^f(g, g_\gamma) \leq -\frac{1}{2}\kappa_{\mathcal{D}}^f(g, g_\gamma)$. Moreover, by lemma 3.5, $\kappa_{\mathcal{D}}^v(g_\gamma) + \frac{1}{2}\kappa_{\mathcal{D}}^f(g_\gamma) \leq \frac{2\pi}{3}$, so we get:

$$\kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^f(g) - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g_\gamma) \geq \frac{2\pi}{3}. \quad (30)$$

Although all faces have non-positive curvature, we kept the ones intersecting g . Inspecting them closer will lead to a sharper bound.

Assume that there are m_i n_i -gons, $n_i \geq 6$, that are $g|_{[u', v']}$ -cells. Then they must intersect $g|_{[u', v']}$ in a connected sequence of edges, because $g|_{[u', v]}$ is a combinatorial geodesic and the disc diagram is reduced. The edges of $g|_{[u', v]}$ preceding and following this sequence can either belong to another n_i' -gon or to a triangle. We differentiate between n_i -gons which have zero, one or two g -adjacent triangles.

Denote those respectively by $n_i^{(0)}$ -gons, $n_i^{(1)}$ -gons and $n_i^{(2)}$ -gons. Further, denote the number of each of them by $m_i^{(0)}$, $m_i^{(1)}$ and $m_i^{(2)}$, where $m_i^{(0)} + m_i^{(1)} + m_i^{(2)} = m_i$. Finally, denote the number triangular $g|_{[u', v]}$ -cells by s . We can see that $s \geq \sum_i \frac{m_i^{(1)} + 2m_i^{(2)}}{2}$. As previously mentioned, the curvature of each triangle is $\kappa(t) = -\pi$. and the curvature of each $n_i^{(j)}$ -gon $-(n_i^{(j)} - 6)\frac{\pi}{3}$. Additionally, as $\mathcal{D}$ has no cut points, none of the triangles can be shared cells with g_γ , so we get their full curvature. On the other hand, the n_i -gons can be shared, so we might not get their full curvature, but we get at least half of it. Hence, 30 transforms into the following:

$$\kappa_{\mathcal{D}}^v(g) \geq \frac{2\pi}{3} + \frac{1}{2} \sum_i \sum_{0 \leq j \leq 2} m_i^{(j)} (n_i^{(j)} - 6) \frac{\pi}{3} + s\pi. \quad (31)$$

We now analyse the curvature of the vertices on $g|_{(u', v')}$, to give a contradicting upper bound.

By previous analysis, the vertices on the boundary can take curvature $\frac{\pi}{3} - s\frac{2\pi}{3}$, $s \geq 0$.

If there are $\frac{n_i^{(0)}}{2} - 1$ consecutive vertices of positive curvature on an $n_i^{(0)}$ -gon, we can retract over it (see Figure 4), getting a disk of smaller area which satisfies the same conditions and hence allows us to argue by induction. If there are more than

$\frac{n_i^{(0)}}{2} - 1$ consecutive vertices of positive curvature, a similar move can reduce g to a shorter path, contradicting the fact that it is a combinatorial geodesic. Thus, we can assume that there are at most $\frac{n_i^{(0)}}{2} - 2$ consecutive vertices of positive curvature on any $n_i^{(0)}$ -gon.

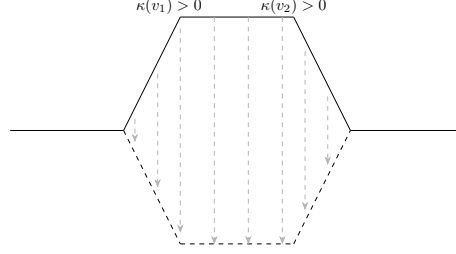


Figure 4: Retracting over a hexagon with two positively curved vertices and no g -adjacent triangles.

In the case of $n_i^{(1)}$ -gons, having $\frac{n_i^{(1)}}{2} - 1$ adjacent vertices of curvature $\frac{\pi}{3}$ might not be enough to do a retraction move, because one of the positive vertices might come from a place where the adjacent triangle connects to the cell. Nevertheless, having $\frac{n_i^{(1)}}{2}$ consecutive vertices of positive curvature is always enough to retract and always produces a shorter path (because the move can be followed by a retraction over a triangle; see the paragraph below), leading to a contradiction. So, we can allow at most $\frac{n_i^{(1)}}{2} - 1$ consecutive vertices of positive curvature on any $n_i^{(1)}$ -gon.

Whenever we are in the situation of having an n -gon with a g -adjacent triangle such that the vertex where they connect is positively curved, we are able to modify the disk diagram so that the g -adjacent triangle not only shares an edge with g , but also with the n -gon. (See Figure 5.) Namely, the only other cells incident to that vertex can be triangles, because the one n -gon already uses up all the corner angle amount we are allowed for the curvature to stay positive. The sequence of triangles from the n -gon to the g -adjacent triangle forms an l -gon (for some l) which is triangulated and can be retriangulated so that the g -adjacent triangle also shares an edge with the n -gon and none of the properties of the disk diagram are changed (it is still of minimal area and bounded by g).

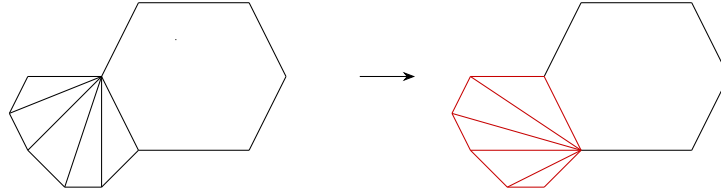


Figure 5: Retriangulating to get a g -adjacent triangle which shares an edge with the polygon.

Finally, for $n_i^{(2)}$ -gons, if we account for the g -adjacent triangles, we are looking to retract over an $(n_i^{(2)} + 2)$ -gon. (See Figure 6.) For this we also need at least $\frac{n_i^{(2)}}{2}$ consecutive vertices of positive curvature. Hence, $n_i^{(2)}$ -gons can also admit at most $\frac{n_i^{(2)}}{2} - 1$ consecutive vertices of positive curvature.

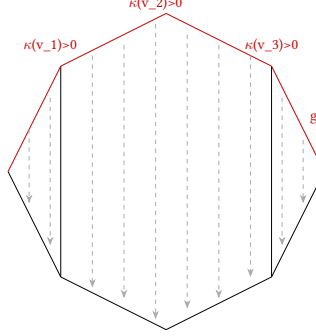


Figure 6: Retracting over a polygon with two g -adjacent triangles.

So far, we produced an upper limit on the number of positively curved vertices on $g|_{(u',v')}$, now we look into what we can say about the negatively curved ones. Each sequence of consecutive vertices of positive curvature on $g|_{(u',v')}$ coming from an $n_i^{(0)}$ -gon is between two vertices of negative curvature (unless it is at one of the ends of the segment we're considering). Similarly, each sequence of positive vertices on $g|_{(u',v')}$ coming from an $n_i^{(1)}$ -gon, has a negative vertex on one of the sides (unless at an end). Hence, we have at least $\sum_i \frac{2m_i^{(0)} + m_i^{(1)} - 2}{2} \frac{\pi}{3}$ negative curvature on $g|_{(u',v')}$.

The observations above imply the following upper bound on the curvature:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g) &\leq \sum_i \left(m_i^{(0)} \left(\frac{n_i^{(0)}}{2} - 2 \right) \frac{\pi}{3} + m_i^{(1)} \left(\frac{n_i^{(1)}}{2} - 1 \right) \frac{\pi}{3} + m_i^{(2)} \left(\frac{n_i^{(2)}}{2} - 1 \right) \frac{\pi}{3} \right) \\ &\quad - \sum_i \frac{2m_i^{(0)} + m_i^{(1)} - 2}{2} \frac{\pi}{3}. \end{aligned} \quad (32)$$

This can further be rewritten as:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g) &\leq \sum_i \left(m_i^{(0)} \left(\frac{n_i^{(0)}}{2} - 3 \right) \frac{\pi}{3} + m_i^{(1)} \left(\frac{n_i^{(1)}}{2} - \frac{3}{2} \right) \frac{\pi}{3} + m_i^{(2)} \left(\frac{n_i^{(2)}}{2} - 1 \right) \frac{\pi}{3} \right) \\ &\quad + \frac{\pi}{3} - s\pi + s\pi. \end{aligned} \quad (33)$$

Finally, we get:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g) &\leq \sum_i \left(m_i^{(0)} \left(\frac{n_i^{(0)}}{2} - 3 \right) \frac{\pi}{3} + m_i^{(1)} \left(\frac{n_i^{(1)}}{2} - 3 \right) \frac{\pi}{3} + m_i^{(2)} \left(\frac{n_i^{(2)}}{2} - 4 \right) \frac{\pi}{3} \right) \\ &\quad + \frac{\pi}{3} + s\pi. \end{aligned} \quad (34)$$

Which can be rewritten as:

$$\kappa_{\mathcal{D}}^v(g) \leq \frac{\pi}{3} + \frac{1}{2} \sum_i \sum_{0 \leq j \leq 2} m_i^{(j)} \left(n_i^{(j)} - 6 \right) \frac{\pi}{3} + s\pi, \quad (35)$$

contradicting the lower bound 31 from before. $\square$

4 Cone types of k -fold triangle groups

In this section, we use Lemma 1.2 to show the finiteness of cone types for non-positively curved k -fold triangle groups. For this purpose, we introduce some additional notions, used to better understand and work with the local and global structures of X .

Take $d_e : X^{(2)} \rightarrow \mathbb{Z}_{\geq 0}$ to be the function on the faces of X induced by the length function on G , i.e. distance from the vertex e in $\text{Cay}(G, S)$ (in X this is the distance from a base face). Similarly to $L^*(x)$ defined in Section 3.1, we can define $\mathcal{L}^*(x)$ to be the directed graph on the same set of vertices as $L^*(x)$, where $e = (t_1, t_2)$ is an edge from $t_1 \in X^{(2)}$ to $t_2 \in X^{(2)}$ if $d_e(t_2) = d_e(t_1) + 1$. The map i sends $L^*(x) \subset \text{Cay}(G, S)$ to $i(L^*(x)) \subset i(\text{Cay}(G, S)) \subset X$.

Similarly, for $A \subseteq X^{(0)}$, denote by $\mathcal{L}^*(A) = \bigcup_{x \in A} \mathcal{L}^*(x)$.

Definition 4.1. We say that a triangle $t \in X^{(2)}$ at $v \in X^{(0)}$ is *minimal* if $d_e(t) \leq d_e(t')$ for every other triangle $t' \in X^{(2)}$ containing v .

Remark 4.2. As triangles $t \in X^{(2)}$ at $v \in X^{(0)}$ can be identified with vertices of $L^*(v)$ (or equivalently vertices of $\mathcal{L}^*(v)$) we can also talk about *minimal vertices* of $L^*(v)$, respectively $\mathcal{L}^*(v)$.

We are now ready to state some corollaries of Lemma 1.2 which elaborate on the interplay between the metric and combinatorial structure of G :

Lemma 4.3. A CAT(0) geodesic in X between a point p_e in the base face e and a vertex $v \in X^{(0)}$ approaches v from a minimal triangle at v . (In other words, the geodesic intersects a minimal triangle at more than just a vertex v .) Moreover, if there are more minimal triangles at v , they are all adjacent.

Proof. Let γ be the CAT(0) geodesic in X between a point p_e in the base face e and a vertex $v \in X^{(0)}$. Assume γ approaches v from a triangle t in X .

Suppose that γ goes through the interior of t . Look at the ϵ -ball around v in X , $B(v, \epsilon)$. Take a point $p_{t'} \in B(v, \epsilon) \cap t'$, for some triangle $t' \neq t$ at v . Consider the CAT(0) geodesic $\gamma_{t'}$ in X between p_e and $p_{t'}$.

In a CAT(0) space, geodesics whose ends are close stay uniformly close to each other. Hence, by making ϵ small enough, we can ensure that $\gamma_{t'}$ crosses the same set of edges as γ up to the neighbourhood of v , which by Lemma 1.2 means that $d_e(t) < d_e(t')$. If γ passes through some points or edges of $X^{(1)}$ on the way from p_e to v , the induced combinatorial path will still have the same number of edges as the combinatorial path induced by $\gamma_{t'}$, because the set of triangles whose interiors they cross is the same conclusive with t .

Suppose now that γ approaches v through an edge $[v, w]$ of t . There are k triangles at $[v, w]$, $t_1, \dots, t_k$ and their pairwise distances differ by at most one. Now for an arbitrary triangle $t' \in X^{(2)}$, the geodesic $\gamma_{t'}$ (constructed as before) need not necessarily go through the triangle t , but needs to either go through the interior of one of the triangles $t_1, \dots, t_k$, denoted by t_i , in which case $d(t') > d(t_i)$ or through the edge $[v, w]$, in which case the distance is defined using the combinatorially shortest path. However, this shortest path also needs to go through one of $t_1, \dots, t_k$, as they are the only ones at v connected to w which γ also passes through. Either way, we get $d(t') > d(t_i)$, for some i . Hence, at least one of the triangles $t_1, \dots, t_k$ needs to be minimal, and if there are more minimal triangles they are all among $t_1, \dots, t_k$. This finishes the proof. $\square$

From the lemma above, we see that the distance function is in some sense determined locally. We formalise this by introducing the notion of local distance functions:

Definition 4.4. The distance function $d_v : L^*(v)^{(0)} \rightarrow \mathbb{Z}_{\geq 0}$ is defined as the shortest distance between a vertex $t \in L^*(v)^{(0)}$ and a minimal vertex $t_v \in L^*(v)^{(0)}$.

Lemma 4.5. For any triangle $t \in X^{(2)}$ and an adjacent vertex $v \in X^{(0)}$:

$$d_e(t) = d_e(t_v) + d_v(t),$$

for some minimal triangle t_v at v .

This leads to the first theorem of the paper:

Theorem 4.6. For every $\tau \in X^{(2)}$, $\mathcal{L}^*(\tau)$ determines the cone type of τ .

By Cannon's theorem (Theorem 2.10) we instantly get the following corollary:

Corollary 4.7. The language of all geodesics in G is regular.

Proof of Theorem 4.6. From the definition, we can see that the cone of τ can inductively be constructed by adding the neighbouring faces of greater distance. For a triangle $t \in \text{Cone}(\tau)$, $\mathcal{L}^*(t)$ encodes all the local triangles of greater distance. We will show that whenever we add a neighbouring triangle t' of greater distance to t ,

we can determine $\mathcal{L}^*(t')$ from $\mathcal{L}^*(t)$, allowing us to propagate this process and retrieve all the triangles of the cone.

Namely, assume $t = \{x, y, z\}$ is of distance n . If $t' = \{y, z, w\}$ is a neighbouring triangle of distance $n + 1$ we want to say that we also know $\mathcal{L}^*(t')$, i.e. how $\mathcal{L}^*(w)$ fits together with $\mathcal{L}^*(\{y, z\})$. We claim that $\mathcal{L}^*(w)$ has to have its minimal vertices in $\mathcal{L}^*(\{y, z\})$. By Lemma 4.5, we can then get the rest of $\mathcal{L}^*(w)$ by constructing paths to each vertex from the closest minimal one in $L^*(w)$.

Assume the contrary, i.e. that a minimal triangle t_w at w does not have either y or z as its vertices. Then by Lemma 4.5, there is a sequence of neighbouring triangles at w from some minimal triangle at w to t' increasing in distance. Furthermore, Lemma 4.3 tells us that all minimal triangles at a vertex are adjacent, so we can see this sequence as starting from t_w , where the first step is not necessarily distance increasing, but all the others are. Without loss of generality, we can assume that the second-to-last member of that sequence is a triangle t'' sharing vertices w and z with t' (see Figure 7). The distance function at t'' is n .

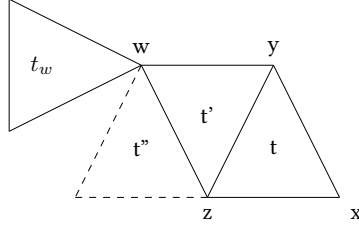


Figure 7: Triangles t, t', t'' and t_w

Denote the third vertex of t'' by v . Now look at the CAT(0) geodesics γ_v and γ_z from p_e to the vertices v and z respectively. Lemma 4.3 tells us these geodesics must approach their respective end vertices through minimal triangles. We further know that smaller distance triangles surround the triangle t' from both sides. Take the shortest path p in $L^*(z)$ from a minimal vertex of $L^*(z)$ to t and the shortest path p'' in $L^*(z)$ from a minimal vertex of $L^*(z)$ to t'' . Because t and t'' are of the same distance, paths p and p'' are of the same length. Since minimal points of $L^*(z)$ are at most distance one apart, we can use these paths to complete a cycle c in $L^*(z)$ containing the points t'', t', t . As c is a cycle in $L^*(z)$, its girth is at least 6, so the length of each of the paths is at least 2. Hence, there is at least one triangle between t'' and any minimal triangle at z . This implies that the angle between the edge $\{v, z\}$ and γ_z is larger than $\frac{\pi}{3}$. Similarly, triangle t'' has a smaller distance triangle on one side and a smaller or equal distance triangle on the other side. If both are smaller, analogous to the argument above, we have a path of distance at least 3 from the minimal triangle at v to t'' , so the angle between the edge γ_v and $\{v, z\}$ is larger than $\frac{2\pi}{3}$. If there is a triangle of equal distance on one side of t'' at v , then that triangle and t'' have to be the minimal triangles at w , so the other triangles containing v, w cannot be of smaller distance. Hence, we can use the same approach of looking at the paths to the two

triangles and the cycle c' induced by them, which will not be trivial because the paths cannot enter the triangles from the same edge. In this case, the path from a minimal triangle at v to t'' is at least 2, but it can only be two if $L^*(v)$ contains two minimal vertices. This would further mean that γ_v approaches v at the intersection of the two minimal triangles, so the angle between the edge γ_v and $\{v, z\}$ would be equal to $\frac{2\pi}{3}$. To sum up, the angle between the edge γ_v and $\{v, z\}$ is larger or equal to $\frac{2\pi}{3}$ and the angle between the edge $\{v, z\}$ and γ_z is larger than $\frac{\pi}{3}$, so the sum of the angles of the triangle $\{z, v, p_e\}$ (see Figure 8) is greater than π , which is a contradiction.

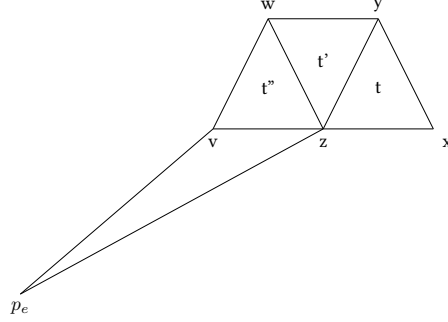


Figure 8: The triangle $\{p_e, v, z\}$

To sum up, we proved that $\mathcal{L}^*(t)$ determines $\mathcal{L}^*(t')$, which inductively gives us the cone type of τ . $\square$

5 The language of lexicographically first geodesics

In this section, we show that the language of lexicographically first geodesics in k -fold triangle groups satisfies the fellow traveller property and hence, provides an automatic structure for G .

Definition 5.1. Let $\mathcal{D}$ be a disk diagram over C , and g a combinatorial geodesic such that $g \subset \partial\mathcal{D}$. We say that an n -gonal g -cell $c \in \mathcal{D}$, $n \geq 6$, is *g -maximally positively curved* if it contains the maximal possible number of vertices of positive curvature. In the case of n -gonal g -cells with at most one g -adjacent triangle this number is $\frac{n}{2} - 1$, and in the case of n -gonal g -cells with two g -adjacent triangles it is $\frac{n}{2}$. We say that c is *almost g -maximally positively curved* if it contains one fewer positively curved vertices than possible.

In Lemma 1.2, the g -maximally positively curved polygons with no g -adjacent triangles were the ones over which we were able to retract to reduce the area of the disc diagram. In this section, we will not be able to do that. Still, we will prove that if g is a combinatorial geodesic, then there has to be additional negative curvature between such polygons.

Lemma 5.2. *Let $\mathcal{D}$ be a disk diagram over C , and g a combinatorial geodesic such that $g \subset \partial\mathcal{D}$ and g contains no cut points of $\mathcal{D}$. Further, let all the non-triangular g -cells be g -maximally positively curved with the exception of the ones with no g -adjacent triangles, which are also allowed to be almost g -maximally positively curved. Then for every two g -maximally positively curved cells with no g -adjacent triangles, there is a cell $c \in \mathcal{D}$ between them, which intersects g but is not a g -cell.*

Proof. Let c_1 and c_2 be two g -maximally positively curved cells with no g -adjacent triangles with no other such cells between them. We will look at a disc diagram $\mathcal{D}'$ consisting of all cells intersecting g between c_1 and c_2 . Assume that $\mathcal{D}'$ consists only of g -cells.

Because all the non-triangular g -cells are (almost) g -maximally positively curved, all of them, apart from c_1 and c_2 , have at least as many vertices on g as they do on $\partial\mathcal{D}' - g$. (See Figure 9) But c_1 and c_2 have strictly more vertices on g than on $\partial\mathcal{D}' - g$. This contradicts g being a combinatorial geodesic. $\square$

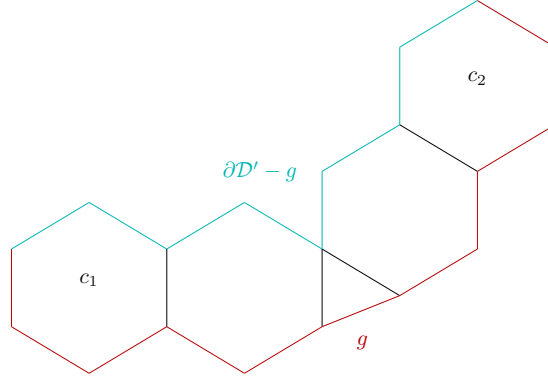


Figure 9: The disk diagram $\mathcal{D}'$ consists only of g -maximally positively curved g -cells and their g -adjacent triangles.

Remark 5.3. By the combinatorial Gauss–Bonnet theorem (Theorem 3.2), $\mathcal{D}$ satisfies the following formula:

$$\sum_{v \in \mathcal{D}^{(0)}} \kappa(v) + \sum_{f \in \mathcal{D}^{(2)}} \kappa(f) = 2\pi\chi(\mathcal{D}). \quad (36)$$

Furthermore, we know that all the interior vertices and faces are non-positively curved, so:

$$LHS \leq \sum_{v \in \partial\mathcal{D}^{(0)}} \kappa(v) + \sum_{f_g \text{ is a } g\text{-cell}} \kappa(f_g) + \sum_{\substack{f \in \mathcal{D}^{(2)} \\ f \text{ is not a } g\text{-cell}}} \kappa(f). \quad (37)$$

The claim of Lemma 5.2 is that if all the g -cells are g -maximally positively curved and there are at least two without g -adjacent triangles, then we can lower this upper bound by detecting more negative curvature, which comes from the cell c (from Lemma 5.2).

Namely, if c is a triangle, then by accounting for its curvature, we can make the upper bound in 37 smaller for π .

Otherwise, if c is not a triangle, its face curvature might not change the upper bound in 37, but the vertex in which c intersects g will have at least $\frac{2\pi}{3}$ smaller curvature, than if c did not exist.

To conclude, in the case of the lemma, the inequality 37 can be improved to:

$$LHS \leq \sum_{v \in \partial \mathcal{D}^{(0)}} \kappa(v) + \sum_{f_g \text{ is a } g\text{-cell}} \kappa(f_g) + \sum_{\substack{f \in \mathcal{D}^{(2)} \\ f \text{ is not a } g\text{-cell}}} \kappa(f) - \frac{2\pi}{3}. \quad (38)$$

Lemma 5.4. *A combinatorial geodesic in $\text{Cay}(G, S)$ from e cannot enter $L^*(x)$, for any x , at a (combinatorial) distance greater than one from any minimal vertex of $L^*(x)$.*

Proof. Let g be a combinatorial geodesic from e to t . Assume that $t \in L^*(x)$, for some x , is of distance at least 2 from a minimal triangle t_x of $L^*(x)$, and no other vertices of g are in $L^*(x)$. Then the vertex t' , preceding t on g is of smaller distance than t . So, t is surrounded by smaller distance vertices from two sides. Hence, by the same argument as in the proof of Theorem 4.6, there is a vertex y of X such that $t, t' \in L^*(y)$ and a CAT(0) geodesic γ_y from e to y such that the angle between γ_y and $[x, y]$ is greater than $\frac{2\pi}{3}$. Furthermore, the fact that $d_x(t_x, t) \geq 2$ means that for the geodesic γ_x from e to x , the angle between γ_x and $[x, y]$ is greater or equal to $\frac{\pi}{3}$. But that means that the triangle $\{e, x, y\}$ has a sum of angles greater than π , which is impossible. $\square$

Theorem 5.5. *The lexicographically first geodesics in $\text{Cay}(G, S)$ satisfy the fellow traveller property.*

Proof. Let g be the lexicographically first geodesic from $u \in \text{Cay}(G, S)^{(0)}$ to $w \in \text{Cay}(G, S)^{(0)}$ and g' the lexicographically first geodesic from u to $v = ws \in \text{Cay}(G, S)$, for some $s \in S$. We will show that such two geodesics are always at most distance $\delta = \max_{x \in X^{(0)}} \text{diam } L^*(x)$, which is finite because $L^*(x)$ can take at most three different forms (one for each vertex group of G).

This uniform bound (which gives what is usually referred to as the *asynchronous fellow traveller property*) in the case of a geodesic language implies the fellow traveller property.

Denote the vertices on the geodesic g by $u_0 = u, u_1, \dots, u_n = w$ and the edges by $s_1, \dots, s_n$. Similarly, denote the vertices of g' by $u'_0 = u, \dots, u'_m = v$ and the edges by $s'_1, \dots, s'_m$. (See Figure 10.)

Assume now, without the loss of generality, that $s_1 < s'_1$. Observe that this implies that $v \notin \text{Cone}(u_1)$, as otherwise g' would not be a lexicographically first geodesic.

Furthermore, no vertices of g' can belong to $\text{Cone}(u_1)$, as otherwise we could get a combinatorial geodesic with the same endpoints and of the same length starting with a smaller letter, contradicting that g' is lexicographically first. On the other hand, all the vertices of g need to be in $\text{Cone}(u_1)$, so $u_i \neq u'_j$ for every i and j .

We will use a disk diagram argument to show that g and g' stay close together.

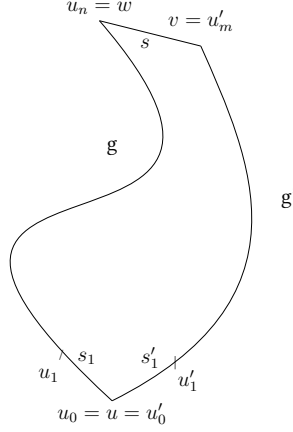


Figure 10: lexicographically first geodesics g and g'

Let $x \in X^{(0)}$ be the farthest point from u such that g and g' enter it at a distance greater than 1. Note that by Lemma 5.4, this distance needs to be exactly 2. If such x doesn't exist, then g and g' enter and consequently exit every $L^*(x)$ at distance one, so they are never at distance larger than δ from each other and the proof is done.

Now, let w' and v' be the vertices of g and g' , respectively, where g and g' enter $L^*(x)$. Consider a disk diagram $\mathcal{D}$ in the cell complex C of minimal area bounded by g , g' , and the geodesic path l in $L^*(x)$ connecting w' and v' .

We now estimate the curvatures. By the combinatorial Gauss–Bonnet theorem for $\mathcal{D}$:

$$\begin{aligned} 2\pi\chi(\mathcal{D}) &= \sum_{v \in \mathcal{D}^{(0)}} \kappa(v) + \sum_{f \in \mathcal{D}^{(2)}} \kappa(f) \\ &= \kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^v(l) + \kappa_{\mathcal{D}}^v(g') + \kappa(u) + \kappa(v') + \kappa(w') + \sum_{f \in \mathcal{D}^{(2)}} \kappa(f), \end{aligned} \quad (39)$$

where we can split the vertices of $\mathcal{D}$ in the way above because g and g' don't share any vertices.

The interior angle at u can either be $\frac{2\pi}{3}$, in which case $\kappa(u) \leq \frac{\pi}{3}$, or it can be zero if both geodesics pass through the same first edge in X , in which case $\kappa(u) = \pi$.

We want to avoid the case of angle zero. This is done in the following way: If we are in the situation of angle zero, instead of looking at the disc diagram $\mathcal{D}$, we look at a modified disc diagram $\tilde{\mathcal{D}}$. The modified disc is bounded by l, g and $\tilde{g}'$, where $\tilde{g}'$ starts from u_1 instead of u . The new path $\tilde{g}'$ is still a combinatorial geodesic, because having a shorter path to some point p of g' from u_1 would imply that that point is in $\text{Cone}(u_1)$. But now we must have an interior angle of at least $\frac{2\pi}{3}$ at the starting vertex. Hence, without loss of generality, we can assume that $\kappa(u) \leq \frac{\pi}{3}$. Furthermore, the interior angles of $\mathcal{D}$ at v' and w' are $\frac{2\pi}{3}$, so we know that $\kappa(v'), \kappa(w') \leq \frac{\pi}{3}$, but we won't use these estimates yet because of potential overlap with later estimates. We also know that $\chi(\mathcal{D}) = 1$, so we can rewrite 39 as:

$$2\pi \leq \kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^v(l) + \kappa_{\mathcal{D}}^v(g') + \sum_{f \in \mathcal{D}^{(2)}} \kappa(f) + \kappa(w') + \kappa(v') + \frac{\pi}{3}. \quad (40)$$

The faces of $\mathcal{D}$ can intersect g, g' , both or neither, so

$$\sum_{f \in \mathcal{D}^{(2)}} \kappa(f) = \kappa_{\mathcal{D}}^f(g) + \kappa_{\mathcal{D}}^f(g') - \kappa_{\mathcal{D}}^f(g, g') + \sum_{\substack{f \in \mathcal{D}^{(2)} \\ f \cap (g \cup g') = \emptyset}} \kappa(f). \quad (41)$$

As all the faces are non-positively curved, the estimate doesn't have to include all of them, so we will only keep the ones intersecting g or g' . The estimate 40 becomes the following:

$$\begin{aligned} \pi \leq & \kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^f(g) - \frac{1}{2} \kappa_{\mathcal{D}}^f(g, g') + \kappa_{\mathcal{D}}^v(g') + \kappa_{\mathcal{D}}^f(g) - \frac{1}{2} \kappa_{\mathcal{D}}^f(g, g') + \\ & \kappa_{\mathcal{D}}^v(l) + \kappa(w') + \kappa(v'). \end{aligned} \quad (42)$$

For estimating the curvature of the geodesic segments g and g' , we use the same vertex curvature analysis as in Lemma 1.2. We start with g' . As before, denote by $m_i^{(j)}$ -gons the number of $n_i^{(j)}$ -gons, $n_i^{(j)} \geq 6$, that are g' -cells with $j \in \{0, 1, 2\}$ g' -adjacent triangles. Further, denote by s' the number of g' -adjacent triangles. Unlike in Lemma 1.2 we are not looking to retract the disk diagram, so the bounds on the maximal positive curvature of g' -cells will be different.

As g' is a combinatorial geodesic, the maximal number of positive vertices on an $n_i^{(0)}$ -gon is $\frac{n_i^{(0)}}{2} - 1$. For $n_i^{(1)}$ -gons it is $\frac{n_i^{(1)}}{2} - 1$, and for $n_i^{(2)}$ -gons it is $\frac{n_i^{(2)}}{2}$.

Also, as before, for every $n_i^{(0)}$ -gon, we get two negatively curved vertices on g' and for every $n_i^{(1)}$ -gon one, so, by excluding the end vertices, we get that there are at least $\sum_i \frac{2m_i^{(0)} + m_i^{(1)} - 2}{2} \frac{\pi}{3}$ negative curvature on g' .

Finally, any $n_i^{(0)}$ -gon is either not g' -maximally positively curved, which means that the total curvature over g' is $\pi/3$ smaller than it could be, or it is g' -maximally positively curved and, by Lemma 5.2, produces $-\frac{\pi}{3}$ extra curvature with another g' -maximally positively curved $n_i^{(0)}$ -gon, if such exists. This extra negative curvature

either comes from a non- g -cell c between the $n'^{(0)}$ -gons or from one of the g -cells between them not being maximally curved in this context. There are no cut points on g' because of how $\mathcal{D}$ was defined. In total, there is at least $(m'_i{}^{(0)} - 1)\frac{\pi}{3}$ of extra negative curvature for every i .

As for the curvature of g' -adjacent faces that was not accounted for above (as a cell c from Lemma 5.2), we have g' -cells contributing $-\sum_i \sum_{0 \leq j \leq 2} m'_i{}^{(j)} (n'_i{}^{(j)} - 6) \frac{\pi}{3}$ to the curvature and their g' -adjacent triangles contributing $-s'\pi$.

Lastly, $\kappa_{\mathcal{D}}^f(g, g')$ can be at most $\sum_i \sum_{0 \leq j \leq 2} m'_i{}^{(j)} (n'_i{}^{(j)} - 6) \frac{\pi}{3}$ if all the g' -cells are also g -cells.

Adding up all those estimates, we get the following upper bound:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g') + \kappa_{\mathcal{D}}^f(g') - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') \leq \\ \sum_i \left(m'_i{}^{(0)} \left(\frac{n'_i{}^{(0)}}{2} - 1 \right) \frac{\pi}{3} + m'_i{}^{(1)} \left(\frac{n'_i{}^{(1)}}{2} - 1 \right) \frac{\pi}{3} + m'_i{}^{(2)} \left(\frac{n'_i{}^{(2)}}{2} \right) \frac{\pi}{3} \right) \\ - \sum_i \frac{2m'_i{}^{(0)} + m'_i{}^{(1)} - 2}{2} \frac{\pi}{3} - \left(\sum_i m'_i{}^{(0)} - 1 \right) \frac{\pi}{3} \\ - \sum_i \sum_{0 \leq j \leq 2} m'_i{}^{(j)} (n'_i{}^{(j)} - 6) \frac{\pi}{3} - s'\pi + \frac{1}{2} \sum_i \sum_{0 \leq j \leq 2} m'_i{}^{(j)} (n'_i{}^{(j)} - 6) \frac{\pi}{3}. \end{aligned} \quad (43)$$

This further transforms into:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g') + \kappa_{\mathcal{D}}^f(g') - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') \leq \\ \sum_i \left(m'_i{}^{(0)} \left(\frac{n'_i{}^{(0)}}{2} - 3 \right) \frac{\pi}{3} + m'_i{}^{(1)} \left(\frac{n'_i{}^{(1)}}{2} - \frac{3}{2} \right) \frac{\pi}{3} + m'_i{}^{(2)} \left(\frac{n'_i{}^{(2)}}{2} \right) \frac{\pi}{3} \right) \\ + \frac{2\pi}{3} - \frac{1}{2} \sum_i \sum_{0 \leq j \leq 2} m'_i{}^{(j)} (n'_i{}^{(j)} - 6) \frac{\pi}{3} - s'\pi. \end{aligned} \quad (44)$$

Moreover, using that $s' \geq \sum_i \frac{m'_i{}^{(1)} + 2m'_i{}^{(2)}}{2}$, we get:

$$\begin{aligned} \kappa_{\mathcal{D}}^v(g') + \kappa_{\mathcal{D}}^f(g') - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') \leq \\ \sum_i \left(m'_i{}^{(0)} \left(\frac{n'_i{}^{(0)}}{2} - 3 \right) \frac{\pi}{3} + m'_i{}^{(1)} \left(\frac{n'_i{}^{(1)}}{2} - 3 \right) \frac{\pi}{3} + m'_i{}^{(2)} \left(\frac{n'_i{}^{(2)}}{2} - 3 \right) \frac{\pi}{3} \right) \\ + \frac{2\pi}{3} - \sum_i \sum_{0 \leq j \leq 2} m'_i{}^{(j)} \left(\frac{n'_i{}^{(j)}}{2} - 3 \right) \frac{\pi}{3}. \end{aligned} \quad (45)$$

This finally means that:

$$\kappa_{\mathcal{D}}^v(g') + \kappa_{\mathcal{D}}^f(g') - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') \leq \frac{2\pi}{3}. \quad (46)$$

We could get the same estimate for g , but in this case we can do better. Again, denote by $m_i^{(j)}$ -gons the number of $n_i^{(j)}$ -gons, $n_i^{(j)} \geq 6$, that are g -cells with $j \in \{0, 1, 2\}$ g -adjacent triangles. Further, denote by s the number of g -adjacent triangles.

Let us first assume that g has no g -maximally curved $n^{(0)}$ -gons. The only thing that changes compared to the estimate for g' is that at the first step instead of having $\sum_i m_i'^{(0)} \left(\frac{n_i'^{(0)}}{2} - 1 \right) \frac{\pi}{3} - (\sum_i m_i'^{(0)} - 1) \frac{\pi}{3}$ we have $\sum_i m_i'^{(0)} \left(\frac{n_i'^{(0)}}{2} - 2 \right) \frac{\pi}{3}$.

This means that entire estimate only changes for $\frac{\pi}{3}$, so the inequality bounding the curvature of $\mathcal{D}$ that comes from g is:

$$\kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^f(g) - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') \leq \frac{\pi}{3}. \quad (47)$$

Now assume that there are g -maximally positively curved $n^{(0)}$ -gons. Using that g is the lexicographically first geodesic to w , we will show that the bound above still needs to hold.

From Lemma 5.2 we get extra negative curvature between every two g -maximally positively curved $n^{(0)}$ -gons. We will use the additional information we have about g to show that there must also be extra negative curvature after the last g -maximally positively curved $n^{(0)}$ -gon, which we will denote by f_0 .

Assume that there is no extra negative curvature after f_0 along g . Since g is a combinatorial geodesic from e and f_0 is g -maximally positively curved, the last vertex of f_0 on g , v_0 , has to be the unique maximal vertex of f_0 . The unique minimal vertex of f_0 also needs to be contained in g , which implies that all vertices of f_0 belong to $\text{Cone}(u_1)$.

Because g is a geodesic, the next vertex on it after v_0 , v_1 , has to be of strictly larger distance. In addition, the neighbouring vertex to v_0 that is not on g , v_{-1} , has to be of strictly smaller distance. These three vertices v_{-1} , v_0 and v_1 have to be on the next g -cell f_1 , as otherwise v_0 would be contained in more than two cells, which would give extra negative curvature.

Now, f_1 needs to be almost g -maximally positively curved, but as g enters f_1 from a non-minimal vertex, this implies that f_1 also has a unique maximal vertex which is on g . As v_{-1} needs to be the unique minimal vertex of f_1 , all vertices of f_1 also belong to $\text{Cone}(u_1)$.

We can continue this analysis along g until we reach w' . Denote the g -almost maximally curved cell containing w' by $f_{w'}$. All the vertices of $f_{w'}$ are in $\text{Cone}(u_1)$. We already observed that w' and v' are of distance two, but Lemma 5.4 also tells us that the vertex between them, $v_{\min}$ has to be the unique minimal vertex of $L^*(x)$ for some $x \in X^{(0)}$ containing w' and v' .

If we assume that $\kappa(w') < \frac{\pi}{3}$, then $\kappa(w') \leq 0$ and though we might have $\kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^f(g) - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') = \frac{2\pi}{3}$, we still get:

$$\kappa_{\mathcal{D}}^v(g) + \kappa_{\mathcal{D}}^f(g) - \frac{1}{2}\kappa_{\mathcal{D}}^f(g, g') + \kappa(w') \leq \frac{2\pi}{3}. \quad (48)$$

Otherwise, if $\kappa(w') = \frac{\pi}{3}$, since there is no extra negative curvature along g , the vertex v_{min} is also a part of $f_{w'}$, so it is in $\text{Cone}(u_1)$. But this would mean that v' is also in $\text{Cone}(u_1)$, which is a contradiction.

To conclude, the estimate 48 always holds, either because g has smaller curvature than a regular combinatorial geodesic, or because w' is negatively curved.

Incorporating the curvature estimates 48 and 46 for g and g' into the inequality 42 (along with $\kappa(v') \leq \frac{\pi}{3}$) we get:

$$0 \leq \kappa_{\mathcal{D}}^v(l) \quad (49)$$

However, the disc diagram $\mathcal{D}$ at v_{min} has an interior angle $\frac{4\pi}{3}$, so we get a contradictory upper bound:

$$\kappa_{\mathcal{D}}^v(l) = \kappa(v_{min}) \leq -\frac{\pi}{3}. \quad (50)$$

□

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