ARTICLE TEMPLATE

Anti-commuting Solutions of the Yang-Baxter-like Matrix Equation

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ABSTRACT
We solve the Yang-Baxter-like matrix equation $AXA = XAX$ for a general given matrix A to get all anti-commuting solutions, by using the Jordan canonical form of A and applying some new facts on a general homogeneous Sylvester equation. Our main result provides all the anti-commuting solutions of the nonlinear matrix equation.

KEYWORDS

Yang-Baxter-like matrix equation; Jordan canonical form; Jordan block; anti-commuting solution; homogeneous Sylvester equation

1. Introduction

In the past decade, motivated by wide applications [10, 15] of the classic Yang-Baxter equation, which was independently proposed by Yang [14] and Baxter [2] in 1967 and 1972 respectively, in the mathematical and physical sciences such as knot theory, braid groups, and quantum groups, solving the *Yang-Baxter-like matrix equation*

$$AXA = XAX, \tag{1}$$

which has been so named because of its similarity in format to the original Yang-Baxter equation, has attracted much attention in the community of linear algebra and other areas such as numerical analysis. In the quadratic equation (1), A is a given $n \times n$ complex matrix and one desires to find all solutions or some solutions satisfying one or several additional properties.

A solution B of (1) such that $AB = BA$ is called a *commuting solution*. All commuting solutions of the Yang-Baxter-like matrix equation for general A have been obtained in theory via the works of [7, 8, 11] with different approaches. Among them, a particular class of commuting solutions were explicitly constructed. More specifically, associated with each eigenvalue of A, the product of A and the spectral projection matrix from $\mathbb{C}^n$ onto the generalized eigenspace of the eigenvalue is a commuting solution

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[4], called the spectral solution with respect to the given eigenvalue. The collection of spectral solutions has been enlarged to the family of all projection-based commuting solutions in [9]. For the class of diagonalizable matrices A, based on a result [12, 16] on the unique solution of a Sylvester equation, all commuting solutions have been constructed by [6].

As for non-commuting solutions, however, the progress is limited. When A belongs to several special matrix classes, such as A is an elementary matrix, has a small rank, a small number of distinct eigenvalues, or a simple minimal polynomial, the solution manifold of (1) has been investigated in the literature; see, for example, [1, 13, 17, 18]. But so far, finding all solutions of (1) for a general matrix A is still an unsolved problem.

Inspired by the techniques employed to find all commuting solutions of the Yang-Baxter-like matrix equation, in this paper we focus on constructing a class of non-commuting solutions of (1). Our task is to find all solutions B of (1) satisfying $AB = -BA$. Such solutions will be referred to as *anti-commuting solutions*. The main result, Theorem 3.9, of the paper will demonstrate that, the problem of obtaining all anti-commuting solutions for a general matrix A can be reduced to that of finding all anti-commuting solutions of a simplified Yang-Baxter-like matrix equation of smaller size, which is determined by the zero eigenvalue of A. In particular, if A is nonsingular, then the zero matrix is the only anti-commuting solution of (1).

Our approach in the following is: To find anti-commuting solutions of the matrix equation, we solve an equivalent equation to (1) after posing the additional anti-commutativity condition. The original two matrix equations are simplified by replacing the given matrix with its Jordan canonical form.

The paper is organized as follows. We prove several preliminary results in the next section, which are of importance by themselves in addition to their usefulness for Section 3, in which we first give some equivalent and sufficient formations of the original problem, followed by considering the special case that A has only one eigenvalue, and then we apply the results of Section 2 to the general case. Two numerical examples will be presented in Section 4 to illustrate the main solution theorem. We conclude with Section 5.

2. Preliminary Results

In this section we investigate the solution structure of the linear matrix equation $AX = -XA$ as the first step towards solving for anti-commuting solutions of the Yang-Baxter-like matrix equation. We shall use Jordan canonical forms throughout the paper to fulfill our purpose.

The Jordan canonical form J of A is a block diagonal matrix with Jordan blocks as its diagonal square matrices. Each Jordan block has its main diagonal entries one eigenvalue of A and the sup-diagonal entries are straight 1's, with all other entries 0. A $t \times t$ Jordan block with its diagonal entries λ is denoted as $J_t(\lambda)$, which is the following upper triangular matrix:

$$J_t(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

It is well-known that the Jordan block $J_t(0)$ corresponding to eigenvalue 0 satisfies $J_t(0)^t = 0$, since it is a nilpotent matrix of index t.

We need a couple of basic facts about Jordan blocks that are the building bricks of the Jordan canonical form of a matrix. There may be several Jordan blocks corresponding to the same eigenvalue λ of A, the largest size of which is the *index* of λ that is, by definition, the smallest positive integer k such that $\dim N[(A-\lambda I)^k] = \dim N[(A-\lambda I)^{k+1}]$.

When studying the anti-commutability equality $AB = -BA$, we shall consider a simplified equality $JK = -KJ$ with J the Jordan canonical form of A. Since J is a block diagonal matrix, we first study a basic equation

$$J_t(\lambda)Y = -YJ_s(\mu) \tag{2}$$

for the $t \times s$ unknown matrix Y, with λ and μ complex numbers. Note that $J_t(\lambda) = \lambda I_t + J_t(0)$ and $J_s(\mu) = \mu I_s + J_s(0)$, where I_t and I_s are the $t \times t$ and $s \times s$ identity matrix, respectively. Then

$$J_t(\lambda)Y = [\lambda I_t + J_t(0)]Y = \lambda Y + J_t(0)Y$$

and

$$YJ_s(\mu) = Y[\mu I_s + J_s(0)] = \mu Y + YJ_s(0).$$

Consequently, a $t \times s$ matrix K satisfies (2) if and only if

$$J_t(0)K = -K[J_s(0) + (\lambda + \mu)I_s]. \tag{3}$$

The following two lemmas will not only be applied to establishing the main result for a general Yang-Baxter-like matrix equation in the next section, but they also provide a basis for a general assertion of this section for solving a homogeneous Sylvester equation, which extends some classic result (for example, Theorem 5.15 of [3]) from the commuting case to the anti-commuting one. They deal with the different cases of $\lambda \neq -\mu$ and $\lambda = -\mu$ for the given eigenvalues λ and μ, respectively.

Lemma 2.1. *Let t and s be positive integers and let λ and μ be complex numbers such that $\lambda \neq -\mu$. Then a $t \times s$ matrix K satisfies the equality $J_t(\lambda)K = -KJ_s(\mu)$ if and only if $K = 0$.*

Proof Only the necessity part needs a proof. Suppose that $J_t(\lambda)K = -KJ_s(\mu)$. Then

(3) is satisfied. Since $J_t(0)^t = 0$, using (3) repeatedly t times, we see that

$$\begin{aligned}
0 &= J_t(0)^t K = J_t(0)^{t-1} \cdot J_t(0)K \\
&= -J_t(0)^{t-1} K[J_s(0) + (\lambda+\mu)I_s] \\
&= -J_t(0)^{t-2} \cdot J_t(0)K \cdot [J_s(0) + (\lambda+\mu)I_s] \\
&= J_t(0)^{t-2} K[J_s(0) + (\lambda+\mu)I_s]^2 \\
&= \cdots = (-1)^{t-1} \cdot J_t(0)K \cdot [J_s(0) + (\lambda+\mu)I_s]^{t-1} \\
&= (-1)^t K[J_s(0) + (\lambda+\mu)I_s]^t.
\end{aligned}$$

The assumption $\lambda \neq -\mu$ guarantees that $J_s(0) + (\lambda+\mu)I_s$ is a nonsingular matrix, so the equality $0 = (-1)^t K[J_s(0) + (\lambda+\mu)I_s]^t$ implies $K = 0$. □

Corollary 2.2. *Let $\lambda \neq 0$. A $t \times s$ matrix K solves the equation $J_t(\lambda)Y = -YJ_s(\lambda)$ if and only if $K = 0$.*

Lemma 2.3. *Let t and s be positive integers and let λ and μ be complex numbers such that $\lambda = -\mu$. Then a $t \times s$ matrix K satisfies the equality $J_t(\lambda)K = -KJ_s(\mu)$ if and only if $K = [0 \;\; \hat{K}]$ when $t \leq s$ or*

$$K = \begin{bmatrix} \hat{K} \\ 0 \end{bmatrix}$$

when $t \geq s$, where $\hat{K}$ is an upper triangular matrix of the form

$$\hat{K} = \begin{bmatrix}
\hat{k}_1 & \hat{k}_2 & \hat{k}_3 & \cdots & \hat{k}_{r-1} & \hat{k}_r \\
0 & -\hat{k}_1 & -\hat{k}_2 & \cdots & -\hat{k}_{r-2} & -\hat{k}_{r-1} \\
0 & 0 & \hat{k}_1 & \cdots & \hat{k}_{r-3} & \hat{k}_{r-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (-1)^{r-2}\hat{k}_1 & (-1)^{r-2}\hat{k}_2 \\
0 & 0 & 0 & \cdots & 0 & (-1)^{r-1}\hat{k}_1
\end{bmatrix}, \tag{4}$$

with $r = \min\{t, s\}$ and $\hat{k}_1, \ldots, \hat{k}_r$ are arbitrary complex numbers.

Proof We only prove the lemma for the case of $t \leq s$; the other case can be done by the same idea. Since $\lambda + \mu = 0$, it is clear that $J_t(\lambda)K = -KJ_s(\mu)$ if and only if

$$J_t(0)K = -KJ_s(0). \tag{5}$$

First we verify that, if $K = [0 \;\; \hat{K}]$ with $\hat{K}$ given by (4), then (5) is true. This follows from the observation that

$$J_t(0)K = J_t(0)[0 \;\; \hat{K}] = [0 \;\; J_t(0)\hat{K}]$$

and

$$-KJ_s(0) = -[0 \;\; \hat{K}]J_s(0) = [0 \;\; -\hat{K}J_t(0)],$$

and the direct computation:

$$
\begin{aligned}
&J_t(0)\hat{K}\\
= &\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\begin{bmatrix} \hat{k}_1 & \hat{k}_2 & \hat{k}_3 & \cdots & \hat{k}_{r-1} & \hat{k}_r \\ 0 & -\hat{k}_1 & -\hat{k}_2 & \cdots & -\hat{k}_{r-2} & -\hat{k}_{r-1} \\ 0 & 0 & \hat{k}_1 & \cdots & \hat{k}_{r-3} & \hat{k}_{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{r-2}\hat{k}_1 & (-1)^{r-2}\hat{k}_2 \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{r-1}\hat{k}_1 \end{bmatrix}\\
= &\begin{bmatrix} 0 & -\hat{k}_1 & -\hat{k}_2 & \cdots & -\hat{k}_{t-2} & -\hat{k}_{t-1} \\ 0 & 0 & \hat{k}_1 & \cdots & \hat{k}_{t-3} & \hat{k}_{t-2} \\ 0 & 0 & 0 & \cdots & -\hat{k}_{t-4} & -\hat{k}_{t-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & (-1)^{t-1}\hat{k}_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&-\hat{K}J_t(0)\\
= &-\begin{bmatrix} \hat{k}_1 & \hat{k}_2 & \hat{k}_3 & \cdots & \hat{k}_{r-1} & \hat{k}_r \\ 0 & -\hat{k}_1 & -\hat{k}_2 & \cdots & -\hat{k}_{r-2} & -\hat{k}_{r-1} \\ 0 & 0 & \hat{k}_1 & \cdots & \hat{k}_{r-3} & \hat{k}_{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{r-2}\hat{k}_1 & (-1)^{r-2}\hat{k}_2 \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{r-1}\hat{k}_1 \end{bmatrix}
\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\\
= &\begin{bmatrix} 0 & -\hat{k}_1 & -\hat{k}_2 & \cdots & -\hat{k}_{t-2} & -\hat{k}_{t-1} \\ 0 & 0 & \hat{k}_1 & \cdots & \hat{k}_{t-3} & \hat{k}_{t-2} \\ 0 & 0 & 0 & \cdots & -\hat{k}_{t-4} & -\hat{k}_{t-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & (-1)^{t-1}\hat{k}_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = J_t(0)\hat{K}.
\end{aligned}
$$

Conversely, let a $t \times s$ matrix $K = [k_{ij}]$ satisfy (5). Because of the special structure of $J_t(0)$, we see that rows 1 through $t-1$ of $J_t(0)K$ are rows 2 through t of K respectively and row t of $J_t(0)K$ is zero. Similarly, column 1 of $KJ_s(0)$ is zero and columns 2 through s of $KJ_s(0)$ are columns 1 through $s-1$ of K respectively. Thus, $J_t(0)K = -KJ_s(0)$ implies

$$
\begin{bmatrix} k_{21} & k_{22} & \cdots & k_{2,s-1} & k_{2s} \\ k_{31} & k_{32} & \cdots & k_{3,s-1} & k_{3s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{t1} & k_{t2} & \cdots & k_{t,s-1} & k_{ts} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & k_{11} & \cdots & k_{1,s-2} & k_{1,s-1} \\ 0 & k_{21} & \cdots & k_{2,s-2} & k_{2,s-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & k_{t-1,1} & \ddots & k_{t-1,s-2} & k_{t-1,s-1} \\ 0 & k_{t1} & \cdots & k_{t,s-2} & k_{t,s-1} \end{bmatrix}.
$$

It follows that $k_{i1} = 0$ for $i = 2, \ldots, t$ and $k_{tj} = 0$ for $j = 1, \ldots, s-1$. Comparing the corresponding entries of the both side matrices, we see that

$$k_{i+1,j} = -k_{i,j-1}, \quad i = 1, \dots, t-1, \; j = 2, \dots, s,$$

The above recursive relation means that when the row and column indices increase by 1 each, the corresponding entry changes the sign. Hence, since all the entries of row t of K are 0 except for the last one, starting from $k_{t-1,2} = -k_{t3} = 0$, we obtain in succession that $k_{ij} = 0$ for all (i, j) with $1 \le i \le t$ and $1 \le j \le s-t$, so $K = [0 \;\; \hat{K}]$ with 0 the $t \times (s-t)$ zero matrix and $\hat{K} = [\hat{k}_{ij}]$ is $t \times t$. The same argument shows that $\hat{k}_{ij} = 0$ for all $i, j = 1, \dots, t$ such that $i > j$ and $\hat{k}_{ii} = (-1)^{i-1}\hat{k}_{11}$ for $i = 2, \dots, t$, $\hat{k}_{i,i+1} = (-1)^{i-1}\hat{k}_{12}$ for $i = 2, \dots, t-1$, $\dots$, $\hat{k}_{2t} = -\hat{k}_{1,t-1}$. Hence $\hat{K}$ is given by (4) with $\hat{k}_i \equiv \hat{k}_{1i}$ being arbitrary numbers for $i = 1, \dots t$. □

Corollary 2.4. *In particular, Lemma 2.3 gives all solutions of the equation* $J_t(0)Y = -YJ_s(0)$.

Denote by $J(\lambda)$ a block diagonal matrix with its diagonal consisting of several Jordan blocks associated to the same eigenvalue λ. Then Lemmas 2.1 and 2.3 can be extended to the following proposition.

Proposition 2.5. *Let K be a solution of the matrix equation*

$$J(\lambda)Y = -YJ(\mu),$$

where $J(\lambda) = \mathrm{diag}[J_{r_1}(\lambda), \dots, J_{r_m}(\lambda)]$ *and* $J(\mu) = \mathrm{diag}[J_{s_1}(\mu), \dots, J_{s_l}(\mu)]$.

(i) *If $\lambda \neq -\mu$, then $K = 0$.*

(ii) *If $\lambda = -\mu$, then K is an $m \times l$ block matrix with its (i, j) block an $r_i \times s_j$ matrix K_{ij} that has the same pattern as given in Lemma 2.3.*

Proof Partition K according to the diagonal blocks size of $J(\lambda)$ and $J(\mu)$, which is consistent with the equality $J(\lambda)K = -KJ(\mu)$. Using the block matrix multiplication, we have

$$J_{r_i}(\lambda)K_{ij} = -K_{ij}J_{s_j}(\mu), \quad i = 1, \dots, m, \; j = 1, \dots, l.$$

Then immediately, Lemma 2.1 implies (i) and Lemma 2.3 gives (ii). □

We generalize the above proposition to the case of multi-eigenvalues. For this purpose, consider a general linear matrix equation

$$UX = -XV, \tag{6}$$

where U and V are respectively $u \times u$ and $v \times v$ known matrices, so the unknown matrix X is $u \times v$.

Let J_U and J_V be the Jordan canonical forms of U and V, respectively. There are nonsingular matrices P and Q such that $U = PJ_UP^{-1}$ and $V = QJ_VQ^{-1}$. Then the equation (6) for X is equivalent to the equation

$$J_UY = -YJ_V \tag{7}$$

for Y with $Y = P^{-1}XQ$. Write

$$J_U = \text{diag}[J(\lambda_1), \ldots, J(\lambda_p)] \quad \text{and} \quad J_V = \text{diag}[J(\mu_1), \ldots, J(\mu_q)],$$

where $\lambda_1, \ldots, \lambda_p$ are all distinct eigenvalues of U and $\mu_1, \ldots, \mu_q$ are all distinct eigenvalues of V. Partitioning $Y = [Y_{ij}]$ accordingly as a $p \times q$ block matrix, we see that solving (7) is equivalent to solving the pq sub-matrix equations

$$J(\lambda_i)Y_{ij} = -Y_{ij}J(\mu_j), \quad i = 1, \ldots, p; \ j = 1, \ldots, q.$$

Proposition 2.5 then gives rise to the main result of this section.

Theorem 2.6. *The solutions of $UX = -XV$ are $X = PYQ^{-1}$ in which the block matrix $Y = [Y_{ij}]$ has the property: For $i = 1, \ldots, p$ and $j = 1, \ldots, q$,*
(i) *if $\lambda_i \neq -\mu_j$, then $Y_{ij} = 0$;*
(ii) *if $\lambda_i = -\mu_j$, then Y_{ij} can be partitioned as a block matrix with each block as given by* (ii) *in Proposition 2.5.*

Corollary 2.7. *If the eigenvalues $\lambda_1, \ldots, \lambda_p$ of U and $\mu_1, \ldots, \mu_q$ of V satisfy the condition that $\lambda_i \neq -\mu_j$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$, then $X = 0$ is the only solution of the equation $UX = -XV$.*

Corollary 2.8. *If the eigenvalues $\lambda_1, \ldots, \lambda_p$ of U are nonzero and $\lambda_i \neq -\lambda_j$ for all $i \neq j$, then $X = 0$ is the only solution of the equation $UX = -XU$.*

3. Anti-commuting Solutions of the Yang-Baxter-like Matrix Equation

Suppose that A is a given $n \times n$ complex matrix. We first give some useful results that will be applied to proving the main theorems of the paper. The first lemma below provides an equivalent condition for a matrix to be an anti-commuting solution of the Yang-Baxter-like matrix equation.

Lemma 3.1. *Let an $n \times n$ complex matrix B satisfy the equality $AB = -BA$. Then B solves (1) if and only if*

$$B(B - A)A = 0. \tag{8}$$

Proof Since $AB = -BA$,

$$ABA - BAB = -BAA + BBA = B(B - A)A.$$

Thus, $ABA = BAB$ if and only if $B(B - A)A = 0$. □

Remark 1. Another equivalent condition for B to be an anti-commuting solution of (1) is $A(B-A)B = 0$, which follows from $ABA-BAB = -AAB+ABB = A(B-A)B$.

Remark 2. Suppose that $AB = -BA$. Then a sufficient condition for (8) to be valid is $BA = A^2$ or $BA = B^2$, since the former implies $(B - A)A = 0$ and the latter implies $B(B - A) = 0$, from which $B(B - A)A = 0$.

It is well-known (see, for example, Lemma 3.1 of [5]) that, if J is the Jordan canonical form of A, then solving (1) for X is equivalent to solving the simplified Yang-Baxter-like matrix equation

$$JYJ = YJY \tag{9}$$

for Y in the sense that B is a solution of (1) if and only if K is a solution of (9) with their relation $B = WKW^{-1}$, where W is the similarity matrix connecting the given matrix A to its Jordan canonical form J, that is, $A = WJW^{-1}$. Furthermore, B and K share some common properties, such as B commutes with A if and only if K commutes with J.

The same idea implies that with the similarity relation $B = WKW^{-1}$, the matrix B is an anti-commuting solution of (1) if and only if the matrix K is an anti-commuting solution of (9). Thus in our analysis below, it is sufficient to find all the anti-commuting solutions of (9). As a consequence, all anti-commuting solutions in this section will be stated with respect to the simplified Yang-Baxter-like matrix equation (9), from which all anti-commuting solutions of the original Yang-Baxter-like matrix equation (1) are available immediately via the similarity matrix W.

Our strategy is, first we find all $n \times n$ matrices K satisfying the anti-commutability equation $JY = -YJ$, for which we can apply the general results of the previous section; then, among all such matrices K we search for those satisfying the equation $JYJ = YJY$. By Lemma 3.1, this latter task is reduced to solving the homogeneous equation $Y(Y - J)J = 0$.

We first consider the simplest case that $J = J_n(\lambda)$, which can be analyzed by applying Corollaries 2.2 and 2.4.

Theorem 3.2. *If $\lambda \neq 0$, then $K = 0$ is the only anti-commuting solution of $J_n(\lambda)YJ_n(\lambda) = YJ_n(\lambda)Y$. If $\lambda = 0$, then all anti-commuting solutions of $J_n(0)YJ_n(0) = YJ_n(0)Y$ are*

$$K = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \quad \textit{for } n = 2,$$

$$K = \begin{bmatrix} 0 & y & x \\ 0 & 0 & -y \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{for } n = 3, \quad \textit{and}$$

$$K = \begin{bmatrix} 0 & 0 & \cdots & & 0 & y & x \\ 0 & 0 & \cdots & & 0 & 0 & -y \\ 0 & 0 & \cdots & & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{10}$$

where x and y are arbitrary complex numbers.

Proof The result for $\lambda \neq 0$ is immediate from Corollary 2.2 applied to $J_n(\lambda)Y = -YJ_n(\lambda)$. Suppose that $\lambda = 0$. The case $n \leq 3$ is easy to verify, so we assume that $n \geq 4$. By Corollary 2.4 with $t = s = n$, the condition $J_n(0)K = -KJ_n(0)$ implies that K has the structure given by (4). In other words, all such K are

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & \cdots & k_{n-1} & k_n \\ 0 & -k_1 & -k_2 & \cdots & -k_{n-2} & -k_{n-1} \\ 0 & 0 & k_1 & \cdots & k_{n-3} & k_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{n-2}k_1 & (-1)^{n-2}k_2 \\ 0 & 0 & 0 & \cdots & 0 & (-1)^{n-1}k_1 \end{bmatrix}, \tag{11}$$

in which $k_1, \ldots, k_n$ are arbitrary complex numbers.

The remaining thing is to select K among the above matrices (11) to satisfy the second requirement $K[K - J_n(0)]J_n(0) = 0$. A direct computation gives that $[K - J_n(0)]J_n(0) =$

$$\begin{bmatrix} 0 & k_1 & k_2 - 1 & k_3 & \cdots & k_{n-2} & k_{n-1} \\ 0 & 0 & -k_1 & -k_2 - 1 & \cdots & -k_{n-3} & -k_{n-2} \\ 0 & 0 & 0 & k_1 & \cdots & k_{n-4} & k_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{n-3}k_1 & (-1)^{n-3}k_2 - 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & (-1)^{n-2}k_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

so the entries of the first row of $K[K - J_n(0)]J_n(0)$ is $0, k_1^2, -k_1, 2k_1k_3 - k_2(k_2 + 1), \ldots, k_1k_{n-1} - k_2k_{n-2} + \cdots + k_{n-2}[(-1)^{n-3}k_2 - 1] + (-1)^{n-2}k_{n-1}k_1$. Since the $(1,2)-$entry and $(1,3)-$entry of $K[K - J_n(0)]J_n(0) = 0$ are zero, $k_1 = 0$ and $k_2 = 0$ or -1. But since the $(2,5)-$entry of $K[K - J_n(0)]J_n(0)$ is zero, the possibility of $k_2 = -1$ is excluded, so $k_2 = 0$. Then we obtain in succession that $k_3 = k_4 = \cdots = k_{n-2} = 0$, with k_{n-1} and k_n being arbitrary numbers. Hence all the solutions K of (9) are given by (10). $\square$

Now we let A be a general $n \times n$ complex matrix. For the simplicity of presentation and analysis, we can write the Jordan form J of A as

$$J = \begin{bmatrix} J(\lambda_1) & & & \\ & J(\lambda_2) & & \\ & & \ddots & \\ & & & J(\lambda_d) \end{bmatrix}, \tag{12}$$

in which $\lambda_1, \ldots, \lambda_d$ are all distinct eigenvalues of A and

$$J(\lambda_i) = \begin{bmatrix} J_{t_{i1}}(\lambda_i) & & \\ & \ddots & \\ & & J_{t_{im_i}}(\lambda_i) \end{bmatrix}, \quad i = 1, \ldots, d.$$

In the following, we first consider the case that J is a nonsingular matrix, so its eigenvalues are all nonzero.

Theorem 3.3. *If J is nonsingular, then $K = 0$ is the only anti-commuting solution of $JYJ = YJY$.*

Proof Let the d distinct eigenvalues of J, which are all nonzero by assumption, be written as

$$\lambda_1, \dots, \lambda_{d-2k}, \mu_1, -\mu_1, \dots, \mu_k, -\mu_k,$$

in which $-\lambda_i$ is not an eigenvalue of J for $i = 1, \dots, d-2k$ with $0 \le k \le d/2$. Without loss of generality, we assume that

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{13}$$

where J_1 and J_2 correspond to the eigenvalues $\lambda_1, \dots, \lambda_{d-2k}, \mu_1, \dots, \mu_k$ and $-\mu_1, \dots, -\mu_k$, respectively.

Suppose that K is an anti-commuting solution of (9). Partition K as a 2×2 block matrix

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

according to the sizes of the diagonal blocks of J as in (13). Then the equality $JK = -KJ$ becomes

$$\begin{bmatrix} J_1K_1 & J_1K_2 \\ J_2K_3 & J_2K_4 \end{bmatrix} = -\begin{bmatrix} K_1J_1 & K_2J_2 \\ K_3J_1 & K_4J_2 \end{bmatrix}.$$

By Corollary 2.8, $K_1 = 0$ from $J_1K_1 = -K_1J_1$ and $K_4 = 0$ from $J_2K_4 = -K_4J_2$. On the other hand, Lemma 3.1 ensures $K(K - J)J = 0$, from which $KJ = K^2$ since J is nonsingular. It follows that

$$\begin{bmatrix} 0 & K_2J_2 \\ K_3J_1 & 0 \end{bmatrix} = \begin{bmatrix} K_2K_3 & 0 \\ 0 & K_3K_2 \end{bmatrix}.$$

Since J_1 and J_2 are nonsingular, $K_2J_2 = 0$ implies $K_2 = 0$ and $K_3J_1 = 0$ implies $K_3 = 0$. Therefore, $K = 0$. $\square$

Now we consider the remaining case that the given matrix A is singular, in other words, 0 is an eigenvalue of A. Due to the fact that J is a block diagonal matrix, we need the following lemma that concerns the structure of K in the equality $HK = -KH$ when H is a block diagonal matrix.

Lemma 3.4. *Let $H = \text{diag}(H_1, \dots, H_d)$ with H_i being $h_i \times h_i$ for $i = 1, \dots, d$. Suppose that $HK = -KH$ for a $d \times d$ block matrix $K = [K_{ij}]$ with K_{ij} being $h_i \times h_j$ for $i, j = 1, \dots, d$. Then for each pair (i, j), if no eigenvalue of H_i is opposite to any eigenvalue of H_j in sign, then $K_{ij} = 0$; otherwise, $K_{ij} \neq 0$.*

Proof Multiplying

$$\text{diag}(H_1, \dots, H_d)[K_{ij}] = -[K_{ij}]\text{diag}(H_1, \dots, H_d)$$

out gives

$$H_iK_{ij} = -K_{ij}H_j, \ \ \forall\, i, j = 1, \dots, d.$$

For the given i and j, since H_i and H_j have no eigenvalues with opposite signs, Theorem 2.6 (i) guarantees that $K_{ij} = 0$. Otherwise, there are at least one eigenvalue λ of H_i and one eigenvalue μ of H_j such that $\lambda = -\mu$, Theorem 2.6 (ii) implies that, as a block matrix, all blocks of K_{ij} are nonzero matrices, which are structured as in (ii) of Proposition 2.5. □

Corollary 3.5. *Under the conditions of Lemma 3.2, if in addition no eigenvalue of H_i is opposite in sign to any eigenvalue of H_j for all $i \neq j$, then K is a block diagonal matrix.*

Proof By Corollary 2.7, $K = \mathrm{diag}(K_1, \ldots, K_d)$ with $K_i = K_{ii}$ for $i = 1, \ldots, d$. □

Now we examine the equation $JY = -YJ$. According to the block structure of J as in (12), partition K as

$$K = \begin{bmatrix} K_{11} & \cdots & K_{1d} \\ \vdots & \vdots & \vdots \\ K_{d1} & \cdots & K_{dd} \end{bmatrix}. \tag{14}$$

Then $JK = -KJ$ if and only if

$$J(\lambda_i)K_{ij} = -K_{ij}J(\lambda_j), \; i, j = 1, \ldots, d. \tag{15}$$

Proposition 2.5 directly gives rise to the following result.

Proposition 3.6. *Let $K = [K_{ij}]$ be a solution of the equation $JY = -YJ$. Then for each pair (i, j), if $\lambda_i \neq -\lambda_j$, then $K_{ij} = 0$; if $\lambda_i = -\lambda_j$, then K_{ij} is an $m_i \times m_j$ block matrix with its (k, l) block a $t_{ik} \times t_{il}$ matrix whose pattern is given in Lemma 2.3.*

Based on the above results, the next theorem gives the structure of all anti-commuting solutions of (9) for a general Jordan canonical form J of A.

Theorem 3.7. *Let the distinct eigenvalues $\lambda_1, \ldots, \lambda_d$ of J be such that $\lambda_i \neq -\lambda_j$ for all $i \neq j$. Then all anti-commuting solutions of $JYJ = YJY$ are $K = \mathrm{diag}(K_1, \ldots, K_d)$ in which for $i = 1, \ldots, d$,*

(i) *if $\lambda_i \neq 0$, then $K_i = 0$;*

(ii) *if $\lambda_i = 0$, then K_i is an $m_i \times m_i$ block matrix with each block structured in Lemma 2.3 as stated by Corollary 2.4 and satisfies $K_i[K_i - J(0)]J(0) = 0$.*

Proof The given assumption on the eigenvalues of A, by Corollary 3.5, ensures that $K_{ij} = 0$ is the only solution of (15) whenever $i \neq j$. Hence all anti-commuting solutions K of $JY = -YJ$ are $d \times d$ block diagonal matrices. Conclusions (i) and (ii) are immediate from Proposition 3.1. □

In Theorem 3.7 (ii), the structure of K_i corresponding to $\lambda_i = 0$ is determined by the equality $K_i[K_i - J(0)]J(0) = 0$, which can be decomposed into a system of m_i^2 matrix equations according to the diagonal block structure of $J(0)$. As an application of Theorem 3.7, we consider a special case that J is singular and the eigenvalue 0 possesses only one Jordan block. The truth of the following result is then obvious by means of Theorems 3.2 and 3.7.

Corollary 3.8. *Let $n \geq 4$. Suppose that all the distinct eigenvalues of J are $\lambda_1 = 0, \lambda_2, \ldots, \lambda_d$ with $J(0) = J_m(0)$ and $\lambda_i \neq -\lambda_j$ for $i \neq j$ from 2 to d. Then all anti-*

commuting solutions of the equation $JYJ = YJY$ *are* $K = \mathrm{diag}(K_1, 0, \dots, 0)$*, in which* K_1 *is an* $m \times m$ *matrix whose expression is given by (10).*

Using the same method for the proof of Theorem 3.2, we can get rid of the additional assumption that $\lambda_i \neq -\lambda_j$ for $i \neq j$ from 2 to d in Theorem 3.7 and Corollary 3.8. For this purpose, we write $J(0)$ as J_0 and rewrite all the nonzero eigenvalues of J as

$$\mu_1, \dots, \mu_{d-2k-1}, \nu_1, -\nu_1, \dots, \nu_k, -\nu_k,$$

in which $-\mu_i$ is not an eigenvalue of J for $i = 1, \dots, d-2k-1$ with $0 \leq k < d/2$. Then

$$J = \begin{bmatrix} J_0 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}, \tag{16}$$

where J_1 and J_2 correspond to the eigenvalues $\mu_1, \dots, \mu_{d-2k-1}, \nu_1, \dots, \nu_k$ and $-\nu_1, \dots, -\nu_k$, respectively.

In $JK = -KJ$, partitioning K as a 3×3 block matrix

$$K = \begin{bmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \\ K_7 & K_8 & K_9 \end{bmatrix}$$

according to the block sizes of J by (16), we obtain that

$$\begin{bmatrix} J_0K_1 & J_0K_2 & J_0K_3 \\ J_1K_4 & J_1K_5 & J_1K_6 \\ J_2K_7 & J_2K_8 & J_2K_9 \end{bmatrix} = -\begin{bmatrix} K_1J_0 & K_2J_1 & K_3J_2 \\ K_4J_0 & K_5J_1 & K_6J_2 \\ K_7J_0 & K_8J_1 & K_9J_2 \end{bmatrix}.$$

From Corollary 2.3, $J_0K_2 = -K_2J_1$ implies $K_2 = 0, J_0K_3 = -K_3J_2$ implies $K_3 = 0, J_1K_4 = -K_4J_0$ implies $K_4 = 0, J_1K_5 = -K_5J_1$ implies $K_5 = 0, J_2K_7 = -K_7J_0$ implies $K_7 = 0$, and $J_2K_9 = -K_9J_2$ implies $K_9 = 0$. Thus

$$K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & 0 & K_6 \\ 0 & K_8 & 0 \end{bmatrix}.$$

Lemma 3.1 gives rise to $K(K-J)J = 0$, which is

$$\begin{aligned} &\begin{bmatrix} K_1 & 0 & 0 \\ 0 & 0 & K_6 \\ 0 & K_8 & 0 \end{bmatrix} \begin{bmatrix} K_1 - J_0 & 0 & 0 \\ 0 & -J_1 & K_6 \\ 0 & K_8 & -J_2 \end{bmatrix} \begin{bmatrix} J_0 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \\ = &\begin{bmatrix} K_1 & 0 & 0 \\ 0 & 0 & K_6 \\ 0 & K_8 & 0 \end{bmatrix} \begin{bmatrix} (K_1 - J_0)J_0 & 0 & 0 \\ 0 & -J_1^2 & K_6J_2 \\ 0 & K_8J_1 & -J_2^2 \end{bmatrix} \\ = &\begin{bmatrix} K_1(K_1 - J_0)J_0 & 0 & 0 \\ 0 & K_6K_8J_1 & -K_6J_2^2 \\ 0 & -K_8J_1^2 & K_8K_6J_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence $K_1(K_1 - J_0)J_0 = 0, K_6K_8J_1 = 0, -K_6J_2^2 = 0, -K_8J_1^2 = 0$, and $K_8K_6J_1 = 0$. Since J_1 and J_2 are nonsingular, $K_6 = 0$ from $-K_6J_2^2 = 0$ and $K_8 = 0$ from $-K_8J_1^2 = 0$. Therefore we have the following theorem.

Theorem 3.9. *Let the Jordan canonical form J of A be given by (16). Then all anti-commuting solutions of $JYJ = YJY$ are*

$$K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where K_1 satisfy $J_0K_1 = -K_1J_0$ and $K_1(K_1 - J_0)J_0 = 0$; in other words, K_1 are all the anti-commuting solutions of $J_0YJ_0 = YJ_0Y$.

4. Numerical Examples

To illustrate the main result of this paper, we present two concrete Yang-Baxter-like matrix equations with a given singular constant matrix A.

Example 4.1. Let

$$A = \frac{1}{33}\begin{bmatrix} 12 & 9 & -66 & 21 & 36 & 36 & 0 & 9 \\ -12 & 57 & 0 & 12 & -36 & 30 & 33 & 57 \\ 64 & 4 & 88 & -64 & -72 & -72 & 0 & 4 \\ 2 & -4 & -55 & 31 & 39 & 6 & 0 & -4 \\ 90 & 18 & 99 & -90 & -93 & -93 & 0 & 18 \\ -8 & 16 & 22 & 8 & -24 & 9 & 0 & 16 \\ 11 & -22 & -55 & -11 & 33 & 33 & -66 & 11 \\ 52 & -5 & 55 & -52 & -42 & -42 & -33 & -5 \end{bmatrix}.$$

Then, $A = WJW^{-1}$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 2 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

By Theorem 3.9, all anti-commuting solutions of the simplified Yang-Baxter-like matrix equation $JYJ = YJY$ are $K = \mathrm{diag}(K_1, 0, 0)$, where

$$K_1 = \begin{bmatrix} 0 & y & x \\ 0 & 0 & -y \\ 0 & 0 & 0 \end{bmatrix}, \forall\, x, y \in \mathbb{C}.$$

Hence, all anti-commuting solutions of the original Yang-Baxter-like matrix equation

$AXA = XAX$ are $B = WKW^{-1} = (1/33)\times$

$$\begin{bmatrix} 8y+4x & 17y-8x & 44y-11x & -8y-4x & 12x-42y & 12x-42y & 0 & 17y-8x \\ 12y+4x & 9y-8x & 33y-11x & -12y-4x & 12x-30y & 12x-30y & 0 & 9y-8x \\ 64y+16x & 4y-32x & 88y-44x & -64y-16x & 48x-72y & 48x-72y & 0 & 4y-32x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 96y+24x & 6y-48x & 132y-66x & -96y-24x & 72x-108y & 72x-108y & 0 & 6y-48x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 48y+12x & 3y-24x & 66y-33x & -48y-12x & 36x-54y & 36x-54y & 0 & 3y-24x \end{bmatrix}$$

with x and y being arbitrary complex numbers.

Example 4.2. Let

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{diag}[J_3(0), J_4(0)].$$

Partition a 7×7 matrix K accordingly as

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}.$$

Then by Lemma 2.3, $JK = -KJ$ if and only if

$$K_1 = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & -k_{11} & -k_{12} \\ 0 & 0 & k_{11} \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & k_{22} & k_{23} & k_{24} \\ 0 & 0 & -k_{22} & -k_{23} \\ 0 & 0 & 0 & k_{22} \end{bmatrix},$$

$$K_3 = \begin{bmatrix} k_{31} & k_{32} & k_{33} \\ 0 & -k_{31} & -k_{32} \\ 0 & 0 & k_{31} \\ 0 & 0 & 0 \end{bmatrix}, \quad K_4 = \begin{bmatrix} k_{41} & k_{42} & k_{43} & k_{44} \\ 0 & -k_{41} & -k_{42} & -k_{43} \\ 0 & 0 & k_{41} & k_{42} \\ 0 & 0 & 0 & -k_{41} \end{bmatrix}.$$

From Theorem 3.7 (ii), to satisfy the further equality $K(K-J)J = 0$, the above sub-matrices K_1, K_2, K_3, K_4 of K must satisfy

$$\begin{aligned} & \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} [K_1 - J_3(0)]J_3(0) & K_2J_4(0) \\ K_3J_3(0) & [K_4 - J_4(0)]J_4(0) \end{bmatrix} \\ = & \begin{bmatrix} K_1[K_1 - J_3(0)]J_3(0) + K_2K_3J_3(0) & K_1K_2J_4(0) + K_2[K_4 - J_4(0)]J_4(0) \\ K_3[K_1 - J_3(0)]J_3(0) + K_4K_3J_3(0) & K_3K_2J_4(0) + K_4[K_4 - J_4(0)]J_4(0) \end{bmatrix} \\ = & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

from which

$$\begin{cases} K_1[K_1 - J_3(0)]J_3(0) + K_2K_3J_3(0) = 0, \\ K_1K_2J_4(0) + K_2[K_4 - J_4(0)]J_4(0) = 0, \\ K_3[K_1 - J_3(0)]J_3(0) + K_4K_3J_3(0) = 0, \\ K_3K_2J_4(0) + K_4[K_4 - J_4(0)]J_4(0) = 0. \end{cases} \tag{17}$$

The above expressions of K_1, K_2, K_3, and K_4 lead to

$$\begin{aligned} & K_1[K_1 - J_3(0)]J_3(0) + K_2K_3J_3(0) \\ = & \begin{bmatrix} 0 & k_{11}^2 & -k_{11} - k_{22}k_{31} \\ 0 & 0 & k_{11}^2 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & K_1K_2J_4(0) + K_2[K_4 - J_4(0)]J_4(0) \\ = & \begin{bmatrix} 0 & 0 & -k_{22}k_{41} & -k_{12}k_{22} - k_{22}k_{42} - k_{22} + k_{23}k_{41} \\ 0 & 0 & 0 & -k_{22}k_{41} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & K_3[K_1 - J_3(0)]J_3(0) + K_4K_3J_3(0) \\ = & \begin{bmatrix} 0 & k_{41}k_{31} & k_{31}k_{12} - k_{31} + k_{41}k_{32} - k_{42}k_{31} \\ 0 & 0 & -k_{41}k_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & K_3K_2J_4(0) + K_4[K_4 - J_4(0)]J_4(0) \\ = & \begin{bmatrix} 0 & k_{41}^2 & k_{31}k_{22} - k_{41} & k_{31}k_{23} - k_{32}k_{22} + 2k_{41}k_{43} - k_{42}^2 - k_{42} \\ 0 & 0 & k_{41}^2 & k_{31}k_{22} + k_{41} \\ 0 & 0 & 0 & k_{41}^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, (17) is equivalent to the system

$$\begin{cases} k_{11} = 0, \\ k_{41} = 0, \\ k_{22}k_{31} = 0, \\ k_{22}(1 + k_{12} + k_{42}) = 0, \\ k_{31}(k_{12} - 1 - k_{42}) = 0, \\ k_{31}k_{23} - k_{32}k_{22} - k_{42}^2 - k_{42} = 0. \end{cases}$$

Solving the above, we obtain all anti-commuting solutions of (9):

$$K = \begin{bmatrix} 0 & k_{12} & k_{13} & 0 & 0 & k_{23} & k_{24} \\ 0 & 0 & -k_{12} & 0 & 0 & 0 & -k_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{32} & k_{33} & 0 & 0 & k_{43} & k_{44} \\ 0 & 0 & -k_{32} & 0 & 0 & 0 & -k_{43} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0 & k_{12} & k_{13} & 0 & 0 & k_{23} & k_{24} \\ 0 & 0 & -k_{12} & 0 & 0 & 0 & -k_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{32} & k_{33} & 0 & -1 & k_{43} & k_{44} \\ 0 & 0 & -k_{32} & 0 & 0 & 1 & -k_{43} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0 & -(k_{42}+1) & k_{13} & 0 & k_{22} & k_{23} & k_{24} \\ 0 & 0 & k_{42}+1 & 0 & 0 & -k_{22} & -k_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{22} \\ 0 & -\frac{k_{42}(k_{42}+1)}{k_{22}} & k_{33} & 0 & k_{42} & k_{43} & k_{44} \\ 0 & 0 & \frac{k_{42}(k_{42}+1)}{k_{22}} & 0 & 0 & -k_{42} & -k_{43} \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad k_{22} \neq 0;$$

$$K = \begin{bmatrix} 0 & k_{42}+1 & k_{13} & 0 & 0 & \frac{k_{42}(k_{42}+1)}{k_{31}} & k_{24} \\ 0 & 0 & -(k_{42}+1) & 0 & 0 & 0 & -\frac{k_{42}(k_{42}+1)}{k_{31}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31} & k_{32} & k_{33} & 0 & k_{42} & k_{43} & k_{44} \\ 0 & -k_{31} & -k_{32} & 0 & 0 & -k_{42} & -k_{43} \\ 0 & 0 & k_{31} & 0 & 0 & 0 & k_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad k_{31} \neq 0.$$

5. Conclusions

We have found all anti-commuting solutions of the quadratic matrix equation (1) with an arbitrary given matrix A. Our main result indicates that, it is the Yang-Baxter-like matrix equation associated to the eigenvalue 0 of A that leads to all such solutions.

The study of this problem was motivated in the continued effort to find all solutions of (1), and our approach was based on a general study of some special type Sylvester equation and an equivalent formation of (1) via Lemma 3.1. The resulting solution process is divided into two steps. The first step was to solve the anti-commutability equation $JY = -YJ$ and the second one focused on the equation $Y(Y - J)J = 0$ in which J is the Jordan canonical form of A. It turned out that eventually, solving $JY = -YJ$ and $Y(Y - J)J = 0$ is essentially the same as solving $J(0)Y = -YJ(0)$ and $Y[Y - J(0)]J(0) = 0$, as Theorem 3.9 has demonstrated.

We hope to generalize our result in the future to find other types of non-commuting solutions of (1), and our goal is to find all solutions of the Yang-Baxter-like matrix equation.

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