On the number of linear uniform hypergraphs with girth constraint

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Abstract

For an integer $r \geqslant 3$, a hypergraph on vertex set $[n]$ is r-uniform if each edge is a set of r vertices, and is said to be linear if every two distinct edges share at most one vertex. Given a family $\mathcal{H}$ of linear r-uniform hypergraphs, let $\mathrm{Forb}_r^L(n, \mathcal{H})$ be the set of linear r-uniform hypergraphs on vertex set $[n]$, which do not contain any member from $\mathcal{H}$ as a subgraph. An r-uniform linear cycle of length ℓ, denoted by C_ℓ^r, is a linear r-uniform hypergraph on $(r-1)\ell$ vertices whose edges can be ordered as $\boldsymbol{e}_1, \dots, \boldsymbol{e}_\ell$ such that $|\boldsymbol{e}_i \cap \boldsymbol{e}_j| = 1$ if $j = i \pm 1$ (indices taken modulo ℓ) and $|\boldsymbol{e}_i \cap \boldsymbol{e}_j| = 0$ otherwise. The *girth* of a linear r-uniform hypergraph is the smallest integer ℓ such that it contains a C_ℓ^r. Let $\mathrm{Forb}_L(n, r, \ell) = \mathrm{Forb}_r^L(n, \mathcal{H})$ when $\mathcal{H} = \{C_i^r : 3 \leqslant i \leqslant \ell\}$, that is, $\mathrm{Forb}_L(n, r, \ell)$ is the set of all linear r-uniform hypergraphs on $[n]$ with girth larger than ℓ. For integers $r \geqslant 3$ and $\ell \geqslant 4$, Balogh and Li [On the number of linear hypergraphs of large girth, J. Graph Theory, **93**(1) (2020), 113-141] showed that $|\mathrm{Forb}_L(n, r, \ell)| = 2^{O(n^{1+1/\lfloor \ell/2 \rfloor})}$ based on the graph container method, while its sharpness remains open. In this paper, we prove that $|\mathrm{Forb}_L(n, r, \ell)| > 2^{n^{1+1/(\ell-1)-O(\log\log n/\log n)}}$ by analyzing the random greedy high girth linear r-uniform hypergraph process. It partially generalizes some known results on linear Turán number of linear cycles in higher uniformities.

Keywords: linear hypergraphs, linear Turán number, linear cycles, girth, random greedy graph process.

Mathematics Subject Classifications: 05C35,05C65,05C80

1 Introduction

For an integer $r \geqslant 3$, a hypergraph $H = (V_H, E_H)$ on vertex set $[n]$ is an *r-uniform hypergraph* (*r-graph* for short) if each edge is an r-element subset (*r-set* for short) of $[n]$, and is said to be *linear* if any two distinct edges share at most one vertex. Write v_H and e_H to denote the number of vertices and edges of H. An r-graph H is called a partial Steiner (n, r, s)-system for an integer s with $2 \leqslant s \leqslant r-1$, if every s-set lies in at most one edge of it. In particular, partial Steiner $(n, r, 2)$-systems are linear r-graphs, partial Steiner $(n, 3, 2)$-systems are partial Steiner triple systems. Given a family $\mathcal{H}$ of linear r-graphs, define the *linear Turán number* of $\mathcal{H}$, denoted by $\mathrm{ex}_r^L(n, \mathcal{H})$, is the maximum number of edges among linear r-graphs on vertex set $[n]$ that contain no r-graphs of $\mathcal{H}$ as subgraphs (*$\mathcal{H}$-free* for short). Recently, the research on the Turán problem in linear hypergraphs has attracted considerable attention. Let $\mathrm{Forb}_r^L(n, \mathcal{H})$ be the set of all linear $\mathcal{H}$-free r-graphs on vertex set $[n]$. When $\mathcal{H}$ consists of a single linear r-graph H, we simply write them as $\mathrm{ex}_r^L(n, H)$ and $\mathrm{Forb}_r^L(n, H)$ instead. The Turán problem and the enumeration problem for a family $\mathcal{H}$ of r-graphs are closely related. Every subgraph of a $\mathcal{H}$-free graph is also $\mathcal{H}$-free, then a trivial bound on the size of $\mathrm{Forb}_r^L(n, \mathcal{H})$ is given in [2] as follows

$$2^{\mathrm{ex}_r^L(n,\mathcal{H})} \leqslant |\mathrm{Forb}_r^L(n, \mathcal{H})| \leqslant \sum_{i \leqslant \mathrm{ex}_r^L(n,\mathcal{H})} \binom{\binom{n}{r}}{i} \leqslant 2n^{r \cdot \mathrm{ex}_r^L(n,\mathcal{H})}.$$

There are several different notions of hypergraph cycles. For an integer $\ell \geqslant 3$, a Berge cycle B_ℓ^r of length ℓ is an alternating sequence of distinct vertices and distinct edges of the form $v_1, \boldsymbol{e}_1, v_2, \boldsymbol{e}_2, \ldots, v_\ell, \boldsymbol{e}_\ell$, where $v_i, v_{i+1} \in \boldsymbol{e}_i$ for each $i \in [\ell-1]$ and $v_\ell, v_1 \in \boldsymbol{e}_\ell$. This definition of a hypergraph cycle is the classical definition due to Berge. Note that forbidding a Berge cycle B_ℓ^r actually forbids a family of hypergraphs. Another notion of hypergraph cycles that has been actively investigated is that of a linear cycle, which is unique up to isomorphism. An r-uniform linear cycle C_ℓ^r of length ℓ is an alternating sequence $v_1, \boldsymbol{e}_1, v_2, \boldsymbol{e}_2, \ldots, v_\ell, \boldsymbol{e}_\ell$ of distinct vertices and distinct edges such that $\boldsymbol{e}_i \cap \boldsymbol{e}_{i+1} = \{v_{i+1}\}$ for each $i \in [\ell-1]$, $\boldsymbol{e}_1 \cap \boldsymbol{e}_\ell = \{v_1\}$ and $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ if $1 < |j-i| < \ell-1$. The *girth* of a linear r-graph is to be the smallest integer ℓ such that it contains a C_ℓ^r. If no such ℓ exists, the girth of the linear r-graph is defined to be infinite.

Determining $\mathrm{ex}_3^L(n, C_3^3)$ is equivalent to the famous $(6,3)$-problem, which aims at determining the maximum number of edges of triple systems not carrying three edges on six vertices. The $(6,3)$-problem is studied by Brown, Erdős and Sós [8], and it is known that for some constant $c > 0$,

$$n^{2-c\sqrt{\log n}} < \mathrm{ex}_3^L(n, C_3^3) = o(n^2),$$

where the lower bound is given by Behrend [3] and the upper bound is given by Rusza and Szemerédi [22]. Erdős, Frankl and Rödl [10] showed that for every $r \geqslant 3$, $\mathrm{ex}_r^L(n, C_3^r) = o(n^2)$ and $\mathrm{ex}_r^L(n, C_3^r) > n^{2-o(1)}$. Using the so-called 2-fold Sidon sets, Lazebnik and Verstraëte [19] constructed linear 3-graphs with girth 5 and showed that

$$\mathrm{ex}_3^L\big(n, \{B_3^3, B_4^3\}\big) = \frac{1}{6}n^{\frac{3}{2}} + O(n).$$

Kostochka, Mubayi and Verstraëte asked if for integers $r \geqslant 3$ and $\ell \geqslant 4$,

$$\mathrm{ex}_r^L\big(n, C_\ell^r\big) = \Theta\big(n^{1+\frac{1}{\lfloor \ell/2 \rfloor}}\big).$$

Later, Collier-Cartaino, Graber and Jiang [9] proved that $\mathrm{ex}_r^L(n, C_\ell^r) = O(n^{1+\frac{1}{\lfloor \ell/2 \rfloor}})$. In detail, they obtained there exist positive constants $c(r,k)$ and $d(r,k)$ such that

$$\mathrm{ex}_r^L(n, C_{2k}^r) \leqslant c(r,k)n^{1+\frac{1}{k}} \text{ and } \mathrm{ex}_r^L(n, C_{2k+1}^r) \leqslant d(r,k)n^{1+\frac{1}{k}},$$

where the integer $k \geqslant 2$, the constants $c(r,k)$ and $d(r,k)$ are exponential in k for fixed r. As an immediate corollary of the main results in [17], Jiang, Ma and Yepremyan further improved the coefficient $c(r,k)$ to be linear in k. Ergemlidze et al. [11] strengthened some results in 3-graphs to be $\mathrm{ex}_3^L(n, B_4^3) = \frac{1}{6}n^{3/2} + O(n)$, $\mathrm{ex}_3^L(n, B_5^3) = \frac{1}{3\sqrt{3}}n^{3/2} + O(n)$, and for $k = 2, 3, 4$ and 6,

$$\mathrm{ex}_3^L\big(n, \{C_3^3, C_5^3, \ldots, C_{2k+1}^3\}\big) = \Theta\big(n^{1+\frac{1}{k}}\big).$$

In general, for $k \geqslant 2$, they also showed that

$$\mathrm{ex}_3^L\big(n, \{C_3^3, C_5^3, \ldots, C_{2k+1}^3\}\big) = \Omega\big(n^{1+\frac{2}{3k-4+\epsilon}}\big),$$

where $\epsilon = 0$ if k is odd and $\epsilon = 1$ if k is even. As a special case of the linear hypergraph extension of Kővári–Sós–Turán's theorem in [14], Gao and Chang showed that $\mathrm{ex}_3^L(n, C_4^3) = \frac{1}{6}n^{3/2} + O(n)$, and Gao et al. in [15] further proved that $\mathrm{ex}_3^L(n, C_5^3) = \frac{1}{3\sqrt{3}}n^{3/2} + O(n)$. For the lower bound of $\mathrm{ex}_r^L(n, C_\ell^r)$, the best known lower bound for $r = 3$ is

$$\mathrm{ex}_3^L\big(n, C_\ell^3\big) = \Omega(n^{1+\frac{1}{\ell-1}})$$

that was shown in [9, 11] due to Verstraëte, by taking a random subgraph of a Steiner triple system. Using generalized Sidon sets [11], it is conceivable that one can obtain a similar (or better) constructive lower bound of $\mathrm{ex}_3^L(n, C_\ell^3)$ (and maybe also for all $r \geqslant 3$), which is an area worth some exploration. The lower bound on the linear Turán number of linear cycles is still far from what is conjectured, especially in higher uniformities $r \geqslant 4$. Sometimes to explore the lower bound of the corresponding Turán problem is more difficult than the upper one.

Following the same logic with the Turán problems of cycles, for integers $r \geqslant 3$ and $\ell \geqslant 4$, Balogh and Li [2] conjectured that $|\mathrm{Forb}_r^L(n, C_\ell^r)| = 2^{\Theta(n^{1+1/\lfloor \ell/2 \rfloor})}$. They confirmed the conjecture in the case of $\ell = 4$. For $\ell > 4$, they provided a result on the girth version, that is, there exists a constant $c = c(r, \ell)$ such that the number of linear r-graphs with girth larger than ℓ is at most $2^{c \cdot n^{1+1/\lfloor \ell/2 \rfloor}}$. Define $\mathrm{ex}_L(n, r, \ell)$ and $\mathrm{Forb}_L(n, r, \ell)$ as $\mathrm{ex}_r^L(n, \mathcal{H})$ and $\mathrm{Forb}_r^L(n, \mathcal{H})$

of $\mathcal{H} = \{C_i^r : 3 \leqslant i \leqslant \ell\}$, respectively. Hence, it implies that $|\text{Forb}_L(n, r, \ell)| = 2^{O(n^{1+1/\lfloor \ell/2 \rfloor})}$, while its lower bound remains open. The idea of the proof in [2] is to reduce the hypergraph enumeration problems to some graph enumeration problems, and then the graph container method is implemented in it. Balogh, Narayanan and Skokan [1] indeed made use of the hypergraph container method and provided a balanced supersaturation theorem for linear cycles to resolve a conjecture on the enumeration of r-uniform hypergraphs with a forbidden C_ℓ^r.

In order to solve a conjecture of Erdős about the existence of sparse Steiner triple systems, Glock, Kühn, Lo and Osthus [16], and Bohman and Warnke [7], studied a constrained triple process along similar lines, which can be seen as a generalization of the *random greedy triangle removal process* [6], to iteratively build a sparse partial Steiner triple system, that is, adding the sparseness constraint does not affect the evolution of the process significantly. To be precise, a Steiner triple system is called k-sparse if it does not contain any $(i+2, i)$-configurations for integers $2 \leqslant i \leqslant k$, where a $(i+2, i)$-configuration is a set of i triples which span at most $i+2$ vertices. Erdős made his conjecture in Steiner triple systems. Some further problems to higher uniformities have been proposed and considered in [13, 16].

The *random greedy r-clique removal process* starts with a complete graph on vertex set $[n]$, denoted as $\mathbb{G}(0)$, and $\mathbb{G}(i+1)$ is the remaining graph from $\mathbb{G}(i)$ by selecting one r-clique uniformly at random out of all r-cliques in $\mathbb{G}(i)$ and deleting all its edges from the edge set $E(i)$ of $\mathbb{G}(i)$. The process terminates once the remaining graph contains no r-cliques, and along the way produces a linear r-graph, where it starts with the empty r-graph on vertex set $[n]$, iteratively adding the vertex set of each chosen r-clique as a new edge in the hypergraph. Let $M = \min\{i : \mathbb{G}(i) \text{ is } r\text{-clique free}\}$, and $E(M)$ be the set of edges left unsaturated by the produced r-cliques. In particular, this problem attracted much attention when $r = 3$ and it is also called the random greedy triangle removal process. Fewer results are known on $r \geqslant 4$. Bennett and Bohman [4] considered the random greedy hypergraph matching process, which can be seen as a generalization of the r-clique removal process. Tian et al. [23] directly discussed the structure of the random greedy r-clique removal process to a natural barrier, using a heuristic assumption to find the trajectories of an ensemble of random variables when the process evolves.

Inspired by the above works, for integers $r \geqslant 3$ and $\ell \geqslant 3$, we consider the constrained random greedy r-clique removal process so that it does not just produce a linear r-graph but with girth larger than ℓ, and the corresponding process is called the *random greedy high girth r-clique removal process* or *high girth linear r-uniform hypergraph process*. Compared with their analysis in [7, 16] discussing sparse Steiner triple systems, the challenge arising in the analysis of random greedy high girth r-clique removal process is that each individual forbidden configuration C_i^r with $3 \leqslant i \leqslant \ell$ should be considered. It is more sensitive and stricter to establish the trajectories and error functions of an ensemble of random variables which we track in this constrained process with a higher uniformity, such that the one-step changes, trend hypotheses and boundedness hypotheses of these tracked variables can support the proof about the concentration of all these variables to the end. Finally, we show the following theorem.

Theorem 1.1. *For every pair of fixed integers $r, \ell \geqslant 3$, consider the random greedy high girth r-clique removal process on vertex set $[n]$. Let M be the number of edges in the generated linear r-graph with girth larger than ℓ when the process terminates. With high probability, there exists some positive constant $\lambda = \lambda(\ell)$ such that*

$$M \geqslant n^{1+\frac{1}{\ell-1}-\lambda\frac{\log\log n}{\log n}}.$$

In the following proof of Theorem 1.1, we will show the term $O(\log\log n/\log n)$ is necessary in the exponent of the lower bound of M. Furthermore, it is impossible to apply this approach to obtain a better lower bound than $\Omega(n^{1+\frac{1}{\ell-1}})$. As a corollary of Theorem 1.1, it is natural to obtain lower bounds of $\mathrm{ex}_L(n, r, \ell)$ and $|\mathrm{Forb}_L(n, r, \ell)|$ when $\ell \geqslant 3$.

Corollary 1.2. *For every pair of fixed integers $r, \ell \geqslant 3$, there exists $\lambda = \lambda(\ell) > 0$ such that as $n \to \infty$,*

$$ex_L(n, r, \ell) > n^{1+\frac{1}{\ell-1}-\frac{\lambda\log\log n}{\log n}} \quad \text{and} \quad |Forb_L(n, r, \ell)| > 2^{n^{1+\frac{1}{\ell-1}-\frac{\lambda\log\log n}{\log n}}}.$$

The remainder of the paper is structured as follows. Notation and auxiliary results used throughout the paper are presented in Section 2. We define the random greedy high girth r-clique removal process in Section 3, introducing some key random variables of the process that we wish to track, estimating the corresponding expected trajectory and choosing error function for each tracked variable. We formally prove the concentration of all these variables in Section 4, where the required one-step changes, trend hypotheses and boundedness hypotheses for these tracked variables are analyzed in higher uniformities. Two important claims to bound the overcounting and boundedness parameters are discussed in detail in the final section.

2 Preliminaries

We adopt the standard asymptotic Landau notation and expressions from [7] throughout. All asymptotics in this paper are with respect to $n \to \infty$. For two positive-valued functions $a(n)$, $b(n)$ on the variable n, write $a(n) = O(b(n))$ if there exists a constant $C > 0$ such that for all n, $|a(n)| \leqslant C \cdot |b(n)|$. Write $a(n) = o(b(n))$ or $a(n) \ll b(n)$ if eventually $b(n) > 0$ and $\lim_{n\to\infty} a(n)/b(n)$ exists and equals 0; $a(n) = b(n)(1 + o(1))$ or $a(n) \sim b(n)$ means $\lim_{n\to\infty} a(n)/b(n) = 1$. Denote $a(n) = \Omega(b(n))$ if eventually $a(n) > 0$ and $b(n) = O(a(n))$. Further, $a(n) = \Theta(b(n))$ if both $a(n) = O(b(n))$ and $b(n) = \Omega(a(n))$. Let $a = b \pm c$ be short for $a \in [b - c, b + c]$. Equations containing $\pm$ are always to be interpreted from left to right, i.e. $b_1 \pm c_1 = b_2 \pm c_2$ means that $b_1 - c_1 \geqslant b_2 - c_2$ and $b_1 + c_1 \leqslant b_2 + c_2$. Moreover, given a set S and integers $a, b \geqslant 0$, we write $\binom{S}{b}$ for the collection of all b-sets of S; in particular, $\binom{S}{b} = \emptyset$ and $\binom{a}{b} = 0$ if $b > |S|$ and $b > a$, $a \wedge b$ denotes the minimum of a and b. All logarithms are natural. The floor and ceiling signs are omitted whenever they are not crucial.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space, and $(\mathcal{F}_i)_{i\geqslant 0}$ be the filtration given by the evolution of the process. For an event $\mathcal{A}$ or a random variable Z in $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathbb{P}[\mathcal{A}]$, $\mathbb{1}_{\mathcal{A}}$ and $\bar{\mathcal{A}}$ denote the probability of $\mathcal{A}$, the indicator function of $\mathcal{A}$ and the complement of $\mathcal{A}$; $\mathbb{E}[Z]$ and $\mathbb{V}[Z]$ denote the expectation and variance of Z. Let $X(i)$ be a set which contains all elements with a certain characteristics at the step i in the process. For a sequence of random variables $\{|X(i)|\}_{i\geqslant 0}$, let $\Delta X(i) = |X(i+1)| - |X(i)|$ denote the one-step change of the random variable $|X(i)|$. The pair $\{|X(i)|, \mathcal{F}_i\}_{i\geqslant 0}$ is then called a submartingale or a supermartingale if $|X(i)|$ is $\mathcal{F}_i$-measurable and $\mathbb{E}[\Delta X(i)|\mathcal{F}_i] \geqslant 0$ or $\mathbb{E}[\Delta X(i)|\mathcal{F}_i] \leqslant 0$ for all $i \geqslant 0$, respectively. As usual, an event is said to occur with high probability (*w.h.p.* for short), if the probability that it holds tends to 1 when $n \to \infty$.

Rescale the number of steps i to be $t = t_i = \frac{i}{n^2}$ such that $t_{i+1} = t + \frac{1}{n^2}$ and every function of the discrete variable i can be viewed as a continuous function of t. It is sometimes more convenient to think of our random variables as depending on a continuous variable t rather than the discrete variable i. We pass between these interpretations without comments. The main tool in the proof of our main results is the differential equations method, which applies a pseudo-random heuristic to divine the trajectories of a group of graph parameters that govern the evolution of the process. It plays a central role in the understanding of several other constrained random processes to produce interesting combinatorial objects [4–7, 16, 18, 20, 21, 23]. The key in applying the method is to find a collection of random variables containing $|X(i)|$ whose one step change can be expressed in terms of other variables in the collection. These expressions form a system of difference equations and the solution to the initial value problem for the corresponding system of differential equations gives a natural guess for the expected trajectories of the variables. The one-step change and boundedness hypotheses for each tracked variable are analyzed, then the last step is to apply a martingale concentration inequality and a union bound to prove that all of the variables in the collection are concentrated around their expected trajectories.

Let the event $\mathcal{E}_X$ be of the form $|X(i)| = x(t) \pm \epsilon_x(t)$ for all $i \leqslant M$, where $x(t)$ is the expected trajectory of $|X(i)|$, $\epsilon_x(t)$ is the error function to control the deviation of $|X(i)|$ from $x(t)$. Let the stopping time τ be the smallest index i such that any one of the random variables that we are tracking has strayed far from its expected trajectory at or before the ith step, making it possible to establish the martingale condition. We show that the event $\{\tau = M\}$ holds by means of $\{\tau = M\} = \cap_{X\in\mathcal{I}}\mathcal{E}_X$, where $\mathcal{I}$ is the family of all key variables at every step and $|\mathcal{I}|$ is polynomial in n. We define a pair of sequences of auxiliary random variables for each $|X(i)|$ as

$$X^{\pm}(i) = \pm\Big[|X(i)| - x(t)\Big] - \epsilon_x(t). \tag{2.1}$$

Note that the desired estimate $|X(i)| = x(t) \pm \epsilon_x(t)$ follows if the estimate $X^{\pm}(i) \leqslant 0$ holds. To prove $X^{\pm}(i) \leqslant 0$ holds *w.h.p.*, we first establish that the sequence $X^{\pm}(i)$ is a supermartingale, and then provide bounds on the one-step change $\Delta X^{\pm}(i) = X^{\pm}(i+1) - X^{\pm}(i)$ for each variable.

From (2.1), for $0 \leqslant i \leqslant M$, observe that

$$\begin{aligned}\Delta X^{\pm}(i) &= \pm\Delta X(i) \mp \Big[x\big(t+\frac{1}{n^2}\big) - x(t)\Big] - \Big[\epsilon_x\big(t+\frac{1}{n^2}\big) - \epsilon_x(t)\Big] \\ &= \pm\Big[\Delta X(i) - \frac{x'(t)}{n^2}\Big] - \frac{\epsilon_x'(t)}{n^2} + O\bigg(\sup_{\varsigma\in[t,t+\frac{1}{n^2}]} \frac{|x''(\varsigma)| + |\epsilon_x''(\varsigma)|}{n^4}\bigg),\end{aligned} \tag{2.2}$$

where the second equality is derived by applying Taylor's theorem with remainder in Lagrange form on $x(t+\frac{1}{n^2}) - x(t)$ and $\epsilon_x(t+\frac{1}{n^2}) - \epsilon_x(t)$ until the second order. Based on the estimated expression $x(t)$ and error function $\epsilon_x(t)$ into the equations (2.1) and (2.2), we will show that the terms $\mathbb{E}[\Delta X(i)\,|\mathcal{F}_i]$ and $\frac{x'(t)}{n^2}$ cancel each other out; $\frac{\epsilon_x'(t)}{n^2}$ dominates the remaining terms such that $\mathbb{E}[\Delta X^{\pm}(i)\,|\mathcal{F}_i] \leqslant 0$ and $\mathbb{E}[|\Delta X^{\pm}(i)|\,|\mathcal{F}_i] = O(\frac{|\epsilon_x'(t)|}{n^2})$; then we also have $|\Delta X^{\pm}(i)| \leqslant |\Delta X(i)| + O(\max\{\frac{|x'(t)|}{n^2}, \frac{|\epsilon_x'(t)|}{n^2}\})$ and the initial value $X^{\pm}(0) \leqslant -\frac{1}{2}\epsilon_x(0)$. Applying Freedman's martingale concentration inequality [12] (see also [24]) and a union bound of the events $X^{\pm}(i) \geqslant 0$ for all tracked variables at every step $i \leqslant M$, we prove that the designated variable $X^{\pm}(i) \geqslant 0$ from some $i \leqslant M$ has extremely low probability. For this to work, it is essential to strictly characterize $x(t)$ and $\epsilon_x(t)$ for each tracked variable, where $\epsilon_x(t)$ has a large enough growth rate throughout the process, but $\epsilon_x(t)$ must not grow too fast, otherwise we would lose control of $X(i)$.

3 High girth r-clique removal process

3.1 Definitions and Variables

Definition 3.1. *For any given integer i with $3 \leqslant i \leqslant \ell$, define a Type-$i$ configuration to be a linear r-graph L with $e_L = i$ which spans at most $v_L \leqslant (r-1)\cdot i$ vertices, with the property that L contains no subhypergraph $G \subsetneq L$ with $3 \leqslant e_G < i$ edges on $v_G \leqslant (r-1)\cdot e_G$ vertices. Let $\mathcal{L} = \mathcal{L}_\ell$ denote the collection of all Type-i configurations L with $3 \leqslant i \leqslant \ell$.*

Remark 3.2. *It is observed that the structure of Type-i configuration is a linear cycle C_i^r, because it is linear with girth exactly ℓ, which implies that $v_L = (r-1)\cdot e_L$ for any $L \in \mathcal{L}$, and $v_G \geqslant (r-1)\cdot e_G + 1$ for any $G \subsetneq L$ with $L \in \mathcal{L}$. For any fixed integer ℓ, $|\mathcal{L}| = O(1)$.*

The *random greedy high girth r-clique removal process* starts with a complete graph on vertex set $[n]$, denoted by $\mathbb{G}(0)$, and $\mathbb{G}(i+1)$ is the remaining graph from $\mathbb{G}(i)$ by selecting one r-clique uniformly at random out of all *available* r-cliques in $\mathbb{G}(i)$ and deleting all its edges from the edge set $E(i)$ of $\mathbb{G}(i)$, where an r-clique is called *available* in $\mathbb{G}(i)$ means that adding the vertex set of this r-clique as an edge does not produce an r-graph in $\mathcal{L}$ with the edges generated by the vertices of previously chosen available r-cliques. Let $M = \min\{i : \mathbb{G}(i)$ is *available r-clique* free$\}$. This process is equivalently viewed as creating a random linear r-graph with girth greater than ℓ, that is, the *random greedy high girth linear r-uniform hypergraph process.* Beginning with the empty r-graph $\mathbb{H}(0)$ on the same vertex set $[n]$ we sequentially set $\mathbb{H}(i+1) = \mathbb{H}(i) + \boldsymbol{e}_{i+1}$, where the added r-set $\boldsymbol{e}_{i+1}$ is the vertex set of the chosen available r-clique in $\mathbb{G}(i)$, and $\boldsymbol{e}_{i+1}$ is also called to be an available r-set or available edge for $\mathbb{H}(i)$. The process terminates at a maximal linear r-graph $\mathbb{H}(M)$ with girth larger than ℓ and no available r-sets or available edges. At time i, graphs are defined with respect to edge sets on the vertex set $[n]$, where both $\mathbb{G}(i)$ and $\mathbb{H}(i)$ depend on the outcomes of the process up to this point, and $\mathbb{G}(i)$ is the graph given by the pairs of vertices that do not appear in any edge of $\mathbb{H}(i)$. It will be convenient to study the relation between $\mathbb{G}(i)$ and $\mathbb{H}(i)$ via

$$E(i) = \binom{[n]}{2} \Big\backslash \bigcup_{\boldsymbol{f} \in \mathbb{H}(i)} \left\{ \{x, y\} \in \binom{\boldsymbol{f}}{2} \right\}. \tag{3.1}$$

For $2 \leqslant m \leqslant r$, $u \in [n]$ and $\boldsymbol{f}_m = \{u_1, \cdots, u_m\} \in \binom{[n]}{m}$, let $N_u(i) = \{x \in [n] : \{x, u\} \in E(i)\}$, $N_{\boldsymbol{f}_m}(i) = \cap_{j=1}^{m} N_{u_j}$ and $K_m(i)$ be the set of vertex sets of m-cliques in $\mathbb{G}(i)$. Due to the nature of the random greedy high girth r-clique removal process, it should come as no surprise that the most important variable for us to track is the number of available r-cliques in $\mathbb{G}(i)$. To this end, define $Q(i)$ to be the set of vertex sets of available r-cliques in $\mathbb{G}(i)$. Our goal is to estimate the random variable $|Q(i)|$. Note that $|Q(0)| = \binom{n}{r}$, and at termination of the process $|Q(M)| = 0$. Fix one $\boldsymbol{f}_m \in K_m(i)$, and define the set $Y_{\boldsymbol{f}_m}(i)$ to be

$$Y_{\boldsymbol{f}_m}(i) = \left\{ \boldsymbol{f}'_{r-m} \in \binom{N_{\boldsymbol{f}_m(i)}}{r-m} \, : \, \boldsymbol{f}_m \cup \boldsymbol{f}'_{r-m} \in Q(i) \right\}. \tag{3.2}$$

Thus, $|Y_{\boldsymbol{f}_m}(i)|$ counts the number of available r-cliques in $\mathbb{G}(i)$ extended from the vertex set $\boldsymbol{f}_m \in K_m(i)$, and it is also called to be the *available* $(r-m)$*-codegree* of $\boldsymbol{f}_m$. Particularly, $|Y_{\boldsymbol{f}}(i)| = \mathbb{1}_{\boldsymbol{f}}$, where $\mathbb{1}_{\boldsymbol{f}}$ is the indicator random variable with $\mathbb{1}_{\boldsymbol{f}} = 1$ if $\boldsymbol{f} \in Q(i)$, instead $\mathbb{1}_{\boldsymbol{f}} = 0$ otherwise.

We also track all possible routes to generate any copy of $L \in \mathcal{L}$ in $\mathbb{H}(i)$. For any $L \in \mathcal{L}$, let $\mathcal{F}_L$ be the set of all copies of L in the complete n-vertex r-graph. Given $L \in \mathcal{L}$, $\boldsymbol{f} \in Q(i)$ and $0 \leqslant k \leqslant e_L - 2$, define $W_{\boldsymbol{f},L,k}(i)$ to be the set of $L' \in \mathcal{F}_L$ which is extended from the r-set $\boldsymbol{f} \in Q(i)$ such that $|L' \cap \mathbb{H}(i)| = k$ and $|L' \cap Q(i)| = e_L - k$, that is,

$$W_{\boldsymbol{f},L,k}(i) = \left\{ L' \in \mathcal{F}_L : \, \boldsymbol{f} \in L' \cap Q(i), \ |L' \cap Q(i)| = e_L - k \text{ and } |L' \cap \mathbb{H}(i)| = k \right\}. \tag{3.3}$$

3.2 Estimates on the variables

A pseudo-random heuristic for divining the evolution of variables plays a central role in the understanding of graph processes. The following idea of estimating on the key variables has been applied in the environments of some interesting combinatorial objects [4–7, 16, 18]. We extend the ideas in [7, 16] to show the expected trajectories for each tracked variable of the high girth linear uniform hypergraph process with a higher uniformity.

Suppose that the events that each available r-set, denoted by $\boldsymbol{f}$, is chosen to become an edge in $\mathbb{H}(i)$ are independent with probability $\pi_i = \pi(t)$ without considering the forbidden configurations in $\mathcal{L}$ defined in Definition 3.1, that is, $\mathbb{P}[\boldsymbol{f} \in \mathbb{H}(i)] = \pi_i$, which implies that $\mathbb{H}(i)$ is regarded as the binomial random r-graph $\mathcal{H}_r(n, \pi_i)$. Recalling that $t = t_i = \frac{i}{n^2}$ and in view of $|\mathbb{H}(i)| = i \sim \binom{n}{r} \cdot r!t/n^{r-2}$, we have

$$\pi_i \sim \frac{r!t}{n^{r-2}}. \tag{3.4}$$

From another viewpoint, it means that the graph $\mathbb{G}(i)$ is the random graph whose variables are roughly the same as they are in the binomial random graph $\mathcal{G}(n, p_i)$ with

$$\begin{aligned} p_i = p(t) = \mathbb{P}\big[\{x,y\} \in E(i)\big] &= \mathbb{P}\Big[\bigcap_{\{\boldsymbol{f} \,|\, \{x,y\} \subseteq \boldsymbol{f}\}} \Big\{\boldsymbol{f} \notin \mathbb{H}(i)\Big\}\Big] = 1 - \sum_{\{\boldsymbol{f} \,|\, \{x,y\} \subseteq \boldsymbol{f}\}} \mathbb{P}\big[\boldsymbol{f} \in \mathbb{H}(i)\big] \\ &\sim 1 - \frac{n^{r-2}}{(r-2)!}\pi_i \sim 1 - r(r-1)t, \end{aligned} \tag{3.5}$$

where the last approximate equality is derived from (3.4).

Under the assumption that $\mathbb{H}(i)$ and $\mathbb{G}(i)$ respectively resemble $\mathcal{H}_r(n, \pi_i)$ and $\mathcal{G}(n, p_i)$, for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, we try to estimate the expected trajectory of $|Q(i)|$ as $|Q(i)| = q(t) \pm \epsilon_q(t)$, the likely values of auxiliary random variables $|Y_{\boldsymbol{f}_m}(i)|$ and $|W_{\boldsymbol{f},L,k}(i)|$ as $|Y_{\boldsymbol{f}_m}(i)| = y_m(t) \pm \epsilon_{y_m}(t)$ and $|W_{\boldsymbol{f},L,k}(i)| = w_{L,k}(t) \pm \epsilon_{w_{L,k}}(t)$.

Given $\mathbb{H}(i)$, an r-set $\boldsymbol{f} \in Q(i)$ means that $\{\{x,y\} \in \binom{\boldsymbol{f}}{2}\} \subseteq E(i)$ and $\mathbb{H}(i) + \boldsymbol{f}$ contains no $L' \in \cup_{L \in \mathcal{L}} \mathcal{F}_L$ satisfying $\boldsymbol{f} \in L'$, then it follows that

$$\mathbb{E}[|Q(i)|] = \sum_{\boldsymbol{f} \in \binom{[n]}{r}} p_i^{\binom{r}{2}} \mathbb{P}\Big[\bigcap_{L \in \mathcal{L}} \big\{\mathbb{H}(i) + \boldsymbol{f} \text{ contains no } L' \in F_L \text{ with } \boldsymbol{f} \in L'\big\}\Big]. \tag{3.6}$$

Let

$$\mathcal{N}_{\boldsymbol{f},L} = \big\{L' \in \mathcal{F}_L : \boldsymbol{f} \in L'\big\}$$

denote the set of $L' \in \mathcal{F}_L$ satisfying $\boldsymbol{f} \in L'$ for any fixed r-set $\boldsymbol{f} \in \binom{[n]}{r}$ and $L \in \mathcal{L}$. We have

$$|\mathcal{N}_{\boldsymbol{f},L}| = \frac{r!e_L}{|\mathtt{Aut}(L)|} n^{v_L - r} + O(n^{v_L - r - 1}) \tag{3.7}$$

because $|\mathcal{N}_{\boldsymbol{f},L}| \cdot \binom{n}{r} = |\mathcal{F}_L| \cdot e_L$ and $|\mathcal{F}_L| = n(n-1)\cdots(n-v_L+1)/|\mathtt{Aut}(L)| = n^{v_L}/|\mathtt{Aut}(L)| + O(n^{v_L-1})$, where $\mathtt{Aut}(L)$ is the automorphism group of L. The main term $\frac{r!e_L}{|\mathtt{Aut}(L)|} n^{v_L-r}$ in (3.7) counts the number of $L' \in \mathcal{F}_L$ in $\mathcal{N}_{\boldsymbol{f},L}$ in which the intersection of all of them is exactly $\boldsymbol{f}$. Recall that $\mathbb{H}(i)$ is regarded as $\mathcal{H}_r(n, \pi_i)$ and $v_L = (r-1) \cdot e_L$ for any $L \in \mathcal{L}$ by Remark 3.2, then we estimate

$$\begin{aligned} &\mathbb{P}\Big[\bigcap_{L \in \mathcal{L}} \big\{\mathbb{H}(i) + \boldsymbol{f} \text{ contains no } L' \in \mathcal{F}_L \text{ with } \boldsymbol{f} \in L'\big\}\Big] \\ &\sim \prod_{L \in \mathcal{L}} \big(1 - \pi_i^{e_L - 1}\big)^{\frac{r!e_L}{|\mathtt{Aut}(L)|} n^{(r-1)e_L - r}} \\ &\sim \xi_i, \end{aligned} \tag{3.8}$$

where

$$\xi_i = \xi(t) = \exp\left[-\sum_{L\in\mathcal{L}} \frac{r!e_L}{|\mathtt{Aut}(L)|}(r!t)^{e_L-1}n^{e_L-2}\right] \tag{3.9}$$

is the probability of availability for each vertex set of an r-clique in $\mathbb{G}(i)$. As the equations shown in (3.4), (3.8) and (3.9), we anticipate the expected value of $\mathbb{E}[|Q(i)|]$ in (3.6) as

$$\mathbb{E}[|Q(i)|] \sim \frac{n^r}{r!}p_i^{\binom{r}{2}}\xi_i. \tag{3.10}$$

Remark 3.3. *As the equation of ξ_i shown in (3.9), we also have*

$$\xi_i' = -\xi_i \cdot \tilde{\xi}_i \tag{3.11}$$

with

$$\tilde{\xi}_i = \tilde{\xi}(t) = \sum_{L\in\mathcal{L}} \frac{e_L(e_L-1)}{|\boldsymbol{Aut}(L)|}(r!)^{e_L}(nt)^{e_L-2}. \tag{3.12}$$

Similarly, for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, as the definitions of $Y_{\boldsymbol{f}_m}(i)$ in (3.2), we estimate the expected value of $|Y_{\boldsymbol{f}_m}(i)|$ to be

$$\mathbb{E}[|Y_{\boldsymbol{f}_m}(i)|] \sim \frac{n^{r-m}}{(r-m)!}p_i^{\binom{r}{2}-\binom{m}{2}}\xi_i, \tag{3.13}$$

in which $\binom{n-m}{r-m}p_i^{\binom{r}{2}-\binom{m}{2}} \sim \frac{n^{r-m}}{(r-m)!}p_i^{\binom{r}{2}-\binom{m}{2}}$ counts the expected number of r-cliques extended from $\boldsymbol{f}_m$ in $\mathbb{G}(i)$; ξ_i is the probability of availability for each vertex set of an r-clique in $\mathbb{G}(i)$ shown in (3.9). For any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, as the definition of $W_{\boldsymbol{f},L,k}(i)$ in (3.3), we also approximate the expected value of $|W_{\boldsymbol{f},L,k}(i)|$ as

$$\mathbb{E}[|W_{\boldsymbol{f},L,k}(i)|] \sim |\mathcal{N}_{\boldsymbol{f},L}| \cdot \binom{e_L-1}{k} \cdot \pi_i^k \cdot (p_i^{\binom{r}{2}}\xi_i)^{e_L-k-1}, \tag{3.14}$$

where $|\mathcal{N}_{\boldsymbol{f},L}| \cdot \binom{e_L-1}{k}$ counts the ways to choose a $L' \in \mathcal{F}_L$ satisfying $\boldsymbol{f} \in L'$, and then choose k edges in $L' - \boldsymbol{f}$; $\pi_i^k(p_i^{\binom{r}{2}}\xi_i)^{e_L-k-1}$ represents the probability that these chosen k edges are in $\mathbb{H}(i)$ and the remaining $e_L - k - 1$ edges are available to $\mathbb{H}(i)$.

Our main result below shows that these random variables are concentrated around the trajectories we have heuristically divined in (3.10), (3.13) and (3.14) to verify our pseudo-random intuitions.

Theorem 3.4. *Given any $\mu > 0$ and fixed integers $r, \ell \geqslant 3$, there exist $\lambda > 0$ and $\alpha \in (0,1)$ such that, for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, with probability at least $1 - n^{-\mu}$,*

$$|Q(i)| = q(t) \pm \epsilon_q(t), \tag{3.15}$$

$$|Y_{\boldsymbol{f}_m}(i)| = y_m(t) \pm \epsilon_{y_m}(t), \tag{3.16}$$

$$|W_{\boldsymbol{f},L,k}(i)| = w_{L,k}(t) \pm \epsilon_{w_{L,k}}(t), \tag{3.17}$$

holding when i satisfying $0 \leqslant i \leqslant M$ with M defined as

$$M = n^{1+\frac{1}{\ell-1}-\lambda\frac{\log\log n}{\log n}}. \tag{3.18}$$

Here, for $t \in [0, t_M]$,

$$q(t) = \frac{n^r}{r!} p_i^{\binom{r}{2}} \xi_i, \tag{3.19}$$

$$y_m(t) = \frac{n^{r-m}}{(r-m)!} p_i^{\binom{r}{2}-\binom{m}{2}} \xi_i, \tag{3.20}$$

$$w_{L,k}(t) = \frac{r! e_L}{|\boldsymbol{Aut}(L)|} \binom{e_L - 1}{k} (r!t)^k (p_i^{\binom{r}{2}} \xi_i)^{e_L-1-k} n^{(r-1)(e_L-k)+k-r}, \tag{3.21}$$

and

$$\epsilon_q(t) = \sigma_i n^{\alpha+r-1}, \tag{3.22}$$

$$\epsilon_{y_m}(t) = \sigma_i n^{\alpha+r-m-1}, \tag{3.23}$$

$$\epsilon_{w_{L,k}}(t) = \beta_k \sigma_i^{k+2} n^{\alpha+(r-1)(e_L-k)+k-r-1} t_M^k, \tag{3.24}$$

in which

$$\sigma_i = \sigma(t) = \log(n^\alpha + n^2 t), \tag{3.25}$$

β_k is a constant depending on k and

$$t_M = \frac{M}{n^2} = n^{-1+\frac{1}{\ell-1}-\lambda\frac{\log\log n}{\log n}}. \tag{3.26}$$

Theorem 3.4 is proved in Section 4 and we always tacitly assume $0 \leqslant i \leqslant M$ with M defined in (3.18). It implies that the process produces a linear r-graph of size at least M with its girth greater than ℓ. Theorem 3.4 verifies Theorem 1.2 with room to spare in the power of the logarithmic factor. We make no attempt to optimize the constants λ, α and the coefficients β_k for any $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$. There are many choices of them that can be balanced to satisfy certain inequalities. For example, for given positive integers r, ℓ and k, we choose these constants to satisfy the equations

$$\lambda > \frac{\ell}{\ell-1}, \quad \alpha_0 < \alpha < 1 \quad \text{and} \quad \beta_k = \Big(\frac{3\ell r!}{p_M^{\binom{r}{2}} \xi_M}\Big)^k, \tag{3.27}$$

where $\alpha_0 = \frac{\ell-2}{\ell-1}$, p_M and ξ_M are constants when $i = M$ defined in (3.5) and (3.9). We will see these choices are sufficient for our proof of Theorem 3.4 in next section. We do not replace them with their actual values, which is for the interest of understanding the role of these constants played in the calculations.

The equation of $|Q(i)|$ shown in (3.15) follows from the equation of $|Y_{\boldsymbol{f}_2}(i)|$ in (3.16) for any $\boldsymbol{f}_2 \in E(i)$. Indeed, any r-set $\boldsymbol{f} \in Q(i)$ is counted $\binom{r}{2}$ times in $\sum_{\boldsymbol{f}_2 \in E(i)} |Y_{\boldsymbol{f}_2}(i)|$ for any $\boldsymbol{f}_2 \in E(i)$. Since $|E(i)| = \binom{n}{2} - \binom{r}{2} i$ and $p_i = 1 - r(r-1)t$ in (3.5), it follows that

$$|E(i)| = \frac{n^2 p_i}{2} \cdot (1 - \frac{1}{np_i})$$

and

$$
\begin{aligned}
|Q(i)| &= \frac{1}{\binom{r}{2}} \sum_{\boldsymbol{f}_2 \in E(i)} |Y_{\boldsymbol{f}_2}(i)| \\
&= \frac{n^2 p_i}{[r]_2}\Big(1 - \frac{1}{np_i}\Big)\big(y_2(t) \pm \epsilon_{y_2}(t)\big) \\
&= q(t) \pm \epsilon_q(t),
\end{aligned} \tag{3.28}
$$

where $[r]_i = \prod_{j=0}^{i-1}(r-j)$ denotes the falling factorial, $\frac{n^2 p_i}{[r]_2} \cdot y_2(t) = q(t)$ by (3.19) and (3.20), $\frac{n^2 p_i}{[r]_2} \cdot \epsilon_{y_2}(t) < \epsilon_q(t)$ and $\frac{n}{[r]_2} \cdot (y_2(t) \pm \epsilon_{y_2}(t)) = o(\epsilon_q(t))$ by (3.22) and (3.23). It thus suffices to establish the bounds in Theorem 3.4 of $|Y_{\boldsymbol{f}_m}(i)|$ for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$ and of $|W_{\boldsymbol{f},L,k}(i)|$ for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$ in next section.

Remark 3.5. *According to (3.5) and (3.9), note that $p_i, \xi_i = \Theta(1)$ when $t \in [0, t_M]$ with t_M defined in (3.26). Unfortunately, it is not allowed to choose $t_M = n^{-1+\frac{1}{\ell-1}}$ in the proof of our main results in Section 4 because we can not remove σ_i or change the power of σ_i in (3.22)-(3.24). Based on the analysis and calculations in Section 4, the factor σ_i is necessary to show the sequences of auxiliary random variables for each tracked variable are supermartingales. However, $\xi_i \to 0$ as $t = n^{-1+\frac{1}{\ell-1}+\varepsilon}$ for any arbitrarily small $\varepsilon > 0$, which implies it is impossible to apply this approach to show a better lower bound of $ex_L(n, r, \ell)$ than $\Omega(n^{1+\frac{1}{\ell-1}})$.*

Remark 3.6. *Set $\hat{\epsilon}_q(t) = \epsilon_q(t)/q(t)$ and $\hat{\epsilon}_{y_m}(t) = \epsilon_{y_m}(t)/y_m(t)$ for any $2 \leqslant m \leqslant r-1$. We have*

$$
\hat{\epsilon}_q(t) = \hat{\epsilon}_{y_m}(t) = O(\sigma_i n^{\alpha-1}) \tag{3.29}
$$

as the equations shown in Theorem 3.4.

Remark 3.7. *In the following sections, in order to express the formulas concise and clear, unless otherwise specified, we often suppress the dependence on t in notation $q(t)$, $y_m(t)$, $w_{L,k}(t)$, $\epsilon_q(t)$, $\epsilon_{y_m}(t)$ and $\epsilon_{w_{L,k}}(t)$ in Section 4.*

4 Analysis of the high-girth process

For any integer $0 \leqslant i < M$, let $\mathcal{G}(i)$ denote the event that the estimates in Theorem 3.4 hold for any $\boldsymbol{f}_m \in K_m(j)$ with $2 \leqslant m \leqslant r-1$, $\boldsymbol{f} \in Q(j)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$ when $0 \leqslant j \leqslant i$. The initial conditions in $\mathbb{G}(0)$ satisfy the equations in (3.15)-(3.17), which implies that the event $\mathcal{G}(0)$ always holds. Assume that the event $\mathcal{G}(i)$ holds with probability at least $1 - n^{-\mu}$ for some $\mu > 0$. Following the approach in [7, 16] to consider sparse Steiner triple systems, we analyze the random greedy high girth linear r-uniform hypergraph process in this section and finally show that the event $\mathcal{G}(M)$ holds with high probability to complete the proof of Theorem 3.4.

Compared with the analysis of triple systems, conditioned on the event $\mathcal{G}(i)$ holds, it is challenging to establish $\mathbb{E}[\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$ and $\mathbb{E}[\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$, $|\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)|$ and $|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)|$ in higher uniformities such that they comply with martingale concentration inequalities. We bound the probability of the event that the tracked variable $Y^{\pm}_{\boldsymbol{f}_m}(i) \geqslant 0$ for some $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, or $W^{\pm}_{\boldsymbol{f},L,k}(i) \geqslant 0$ for some $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$ such that an application of the union bound for all variables over all steps $i \leqslant M$ shows that the probability of the occurrence of the event $\mathcal{G}(i+1)$ is extremely high.

For technical reasons we regulate $\Delta Y_{\boldsymbol{f}_m}(i) = 0$ as soon as $\boldsymbol{f}_m \notin K_m(i{+}1)$ and $\Delta W_{\boldsymbol{f},L,k}(i) = 0$ as soon as $\boldsymbol{f} \notin Q(i{+}1)$ since the relevant structural constraints are violated. In the following, in order to analyze $\Delta Y_{\boldsymbol{f}_m}(i)$ (resp. $\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)$) and $\Delta W_{\boldsymbol{f},L,k}(i)$ (resp. $\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)$), we always assume $\boldsymbol{f}_m \in K_m(i+1)$ and $\boldsymbol{f} \in Q(i+1)$.

4.1 The one-step (expected) changes of $|Y_{\boldsymbol{f}_m}(i)|$ and $|W_{\boldsymbol{f},L,k}(i)|$

4.1.1 $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$

Given one $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, consider $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$. Choosing one element from $Y_{\boldsymbol{f}_m}(i)$, denoted as $\boldsymbol{f}^c_m$, let $\boldsymbol{f} = \boldsymbol{f}_m \cup \boldsymbol{f}^c_m$. We enumerate all possibilities to remove an r-set from $Q(i)$ such that $\boldsymbol{f}^c_m$ is removed from $Y_{\boldsymbol{f}_m}(i)$. The first way is that the chosen r-set makes $\boldsymbol{f}$ not the vertex set of an r-clique. Let $Q_{\boldsymbol{f}^c_m}(i)$ denote the collection of such available r-sets in $Q(i)$. The second way is that the chosen r-set makes $\boldsymbol{f}$ still the vertex set of an r-clique, while $\boldsymbol{f}$ is not available anymore. Note that a bijection is established between such r-sets and the elements in $\bigcup_{L\in\mathcal{L}} W_{\boldsymbol{f},L,e_L-2}(i)$, where no r-sets are associated with these two ways because $L \in \mathcal{L}$ is linear. Combining these two ways together, it follows that

$$\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i] = -\sum_{\boldsymbol{f}^c_m \in Y_{\boldsymbol{f}_m}(i)} \frac{|Q_{\boldsymbol{f}^c_m}(i)| + |\bigcup_{L\in\mathcal{L}} W_{\boldsymbol{f},L,e_L-2}(i)|}{|Q(i)|}. \tag{4.1}$$

For $\boldsymbol{f}$, $\boldsymbol{g} \in Q(i)$ with $\boldsymbol{f} \neq \boldsymbol{g}$, since $\mathbb{H}(i) + \{\boldsymbol{f},\boldsymbol{g}\}$ may contain multiple linear r-graphs $L \in \mathcal{L}$, it might not be true to estimate $|\bigcup_{L\in\mathcal{L}} W_{\boldsymbol{f},L,e_L-2}(i)|$ as $\sum_{L\in\mathcal{L}} |W_{\boldsymbol{f},L,e_L-2}(i)|$. To overcome this kind of possibility for overcounting, for any $\boldsymbol{f} \in Q(i)$, define $\Upsilon_{\boldsymbol{f}}(i)$ as

$$\Upsilon_{\boldsymbol{f}}(i) = \Big\{\boldsymbol{g} \in Q(i)\backslash\{\boldsymbol{f}\} : \mathbb{H}(i) + \{\boldsymbol{f},\boldsymbol{g}\} \text{ contains multiple } L \in \mathcal{L} \text{ containing } \{\boldsymbol{f},\boldsymbol{g}\}\Big\}, \tag{4.2}$$

which means that the selection of $\boldsymbol{g} \in Q(i)\backslash\{\boldsymbol{f}\}$ into $\mathbb{H}(i)$ makes $\boldsymbol{f}$ unavailable more than one time. Rearrange the equation in (4.1) to be

$$\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i] = -\sum_{\boldsymbol{f}^c_m \in Y_{\boldsymbol{f}_m}(i)} \frac{|Q_{\boldsymbol{f}^c_m}(i)| + \sum_{L\in\mathcal{L}} |W_{\boldsymbol{f},L,e_L-2}(i)| + O(|\Upsilon_{\boldsymbol{f}}(i)|)}{|Q(i)|}. \tag{4.3}$$

We analyze the enumerative formula of $|Q_{\boldsymbol{f}^c_m}(i)|$ in (4.3). Recall that $Q_{\boldsymbol{f}^c_m}(i)$ is the collection of r-sets in $Q(i)$ such that its choice into $\mathbb{H}(i)$ as an edge makes $\boldsymbol{f} = \boldsymbol{f}_m \cup \boldsymbol{f}^c_m$ not the vertex set of an r-clique in $\mathbb{G}(i+1)$.

Claim 4.1.1

$$|Q_{\boldsymbol{f}_m^c}(i)| = \left[\binom{r}{2} - \binom{m}{2}\right] \cdot \big(y_2 \pm \epsilon_{y_2}\big) + O\big(n^{r-3}\big).$$

Proof of Claim 4.1.1. Let $\bar{\boldsymbol{f}}_s \subseteq \boldsymbol{f}$ with $|\bar{\boldsymbol{f}}_s| = s$. Denote $Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i) \subseteq Q_{\boldsymbol{f}_m^c}(i)$ as the collection of available r-sets such that every r-set in $Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i)$ exactly contains the vertices $\bar{\boldsymbol{f}}_s$ in $\boldsymbol{f}$. Since $\boldsymbol{f}_m \in K_m(i+1)$, to ensure that the removal of the edges of any one of these corresponding r-cliques in $Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i)$ results in $\boldsymbol{f}$ not the vertex set of an r-clique, we have $2 \leqslant s \leqslant r$ and $|\bar{\boldsymbol{f}}_s \cap \boldsymbol{f}_m| \leqslant 1$.

Let $\binom{\boldsymbol{f}_m}{j} \oplus \binom{\boldsymbol{f}_m^c}{s-j}$ denote the collection of s-sets consisting of the union of j vertices in $\boldsymbol{f}_m$ and $s-j$ vertices in $\boldsymbol{f}_m^c$. We have $\bar{\boldsymbol{f}}_s \in \cup_{j=0}^1 \binom{\boldsymbol{f}_m}{j} \oplus \binom{\boldsymbol{f}_m^c}{s-j}$ and $|Q_{\boldsymbol{f}_m^c}(i)|$ is decomposed into

$$|Q_{\boldsymbol{f}_m^c}(i)| = \sum_{s=2}^{r-m} \sum_{\bar{\boldsymbol{f}}_s \in \binom{\boldsymbol{f}_m^c}{s}} |Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i)| + \sum_{s=2}^{r-m+1} \sum_{\bar{\boldsymbol{f}}_s \in \binom{\boldsymbol{f}_m}{1} \oplus \binom{\boldsymbol{f}_m^c}{s-1}} |Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i)|. \tag{4.4}$$

Apply inclusion-exclusion to obtain $|Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i)|$ as

$$\begin{aligned}|Q_{\boldsymbol{f}_m^c}^{\bar{\boldsymbol{f}}_s}(i)| &= |Y_{\bar{\boldsymbol{f}}_s}(i)| - \sum_{\boldsymbol{t}_1 \in \binom{\boldsymbol{f} \setminus \bar{\boldsymbol{f}}_s}{1}} |Y_{\bar{\boldsymbol{f}}_s \cup \boldsymbol{t}_1}(i)| + \cdots + \sum_{\boldsymbol{t}_{r-s} \in \binom{\boldsymbol{f} \setminus \bar{\boldsymbol{f}}_s}{r-s}} (-1)^{r-s} |Y_{\bar{\boldsymbol{f}}_s \cup \boldsymbol{t}_{r-s}}(i)| \\ &= \sum_{j=0}^{r-s} \sum_{\boldsymbol{t}_j \in \binom{\boldsymbol{f} \setminus \bar{\boldsymbol{f}}_s}{j}} (-1)^j |Y_{\bar{\boldsymbol{f}}_s \cup \boldsymbol{t}_j}(i)|.\end{aligned}$$

Combining with the equation in (4.4), it follows that

$$\begin{aligned}|Q_{\boldsymbol{f}_m^c}(i)| = &\sum_{s=2}^{r-m} \sum_{j=0}^{r-s} \sum_{\bar{\boldsymbol{f}}_s \in \binom{\boldsymbol{f}_m^c}{s}} \sum_{\boldsymbol{t}_j \in \binom{\boldsymbol{f} \setminus \bar{\boldsymbol{f}}_s}{j}} (-1)^j |Y_{\bar{\boldsymbol{f}}_s \cup \boldsymbol{t}_j}(i)| \\ &+ \sum_{s=2}^{r-m+1} \sum_{j=0}^{r-s} \sum_{\bar{\boldsymbol{f}}_s \in \binom{\boldsymbol{f}_m}{1} \oplus \binom{\boldsymbol{f}_m^c}{s-1}} \sum_{\boldsymbol{t}_j \in \binom{\boldsymbol{f} \setminus \bar{\boldsymbol{f}}_s}{j}} (-1)^j |Y_{\bar{\boldsymbol{f}}_s \cup \boldsymbol{t}_j}(i)|.\end{aligned} \tag{4.5}$$

As to the equation of $|Y_{\boldsymbol{f}_m}(i)|$ for any $2 \leqslant m \leqslant r-1$ in (3.16), $|Y_{\bar{\boldsymbol{f}}_s \cup \boldsymbol{t}_j}(i)|$ with $(s,j) = (2,0)$ dominates the sum on the right side of (4.5), then $|Q_{\boldsymbol{f}_m^c}(i)|$ satisfies

$$|Q_{\boldsymbol{f}_m^c}(i)| = \left[\binom{r-m}{2} + m(r-m)\right] \cdot \big(y_2 \pm \epsilon_{y_2}\big) + O\big(n^{r-3}\big),$$

where $\binom{r-m}{2}$ enumerates $|Y_{\bar{\boldsymbol{f}}_s}(i)|$ when $\bar{\boldsymbol{f}}_s \in \binom{\boldsymbol{f}_m^c}{2}$, $m(r-m)$ enumerates $|Y_{\bar{\boldsymbol{f}}_s}(i)|$ when $\bar{\boldsymbol{f}}_s \in \binom{\boldsymbol{f}_m}{1} \oplus \binom{\boldsymbol{f}_m^c}{1}$, and $O(n^{r-3})$ is derived from the terms when $(s,j) \neq (2,0)$ in (4.5). Note that $\binom{r-m}{2} + m(r-m) = \binom{r}{2} - \binom{m}{2}$, we complete the proof of Claim 4.1.1. $\square$

4.1.2 $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$

Given $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, we now consider $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$. We consider all possibilities to make $|W_{\boldsymbol{f},L,k}(i)|$ changed by adding an available r-set into $\mathbb{H}(i)$.

Choose one $L' \in W_{\boldsymbol{f},L,k}(i)$, we firstly enumerate ways to remove L' from $W_{\boldsymbol{f},L,k}(i)$ by deleting an r-set in $Q(i)\backslash\{\boldsymbol{f}\}$, which implies that some $\boldsymbol{g} \in (L' \cap Q(i))\backslash\{\boldsymbol{f}\}$ becomes unavailable. Given one $\boldsymbol{g} \in (L' \cap Q(i))\backslash\{\boldsymbol{f}\}$, similarly, the ways to make $\boldsymbol{g}$ unavailable are in two cases. One case is that $\boldsymbol{g}$ is not the vertex set of an r-clique. Let $Q_{\boldsymbol{g}}(i)$ denote the collection of such available r-sets in $Q(i)$. The other case is that the chosen r-set makes $\boldsymbol{g}$ still the vertex set of an r-clique, while $\boldsymbol{g}$ is unavailable. The number of such r-sets is counted as $\sum_{K\in\mathcal{L}} |W_{\boldsymbol{g},K,e_K-2}(i)| + O(|\Upsilon_{\boldsymbol{g}}(i)|)$. Define $\Psi_{L'}(i)$ as another overcounting collection of r-sets to be

$$\Psi_{L'}(i) = \Big\{\boldsymbol{h} \in Q(i)\backslash\{\boldsymbol{f}\} : \text{two } r\text{-sets in } (L' \cap Q(i))\backslash\{\boldsymbol{h}\} \text{ are unavailable in } \mathbb{H}(i) + \{\boldsymbol{h}\}\Big\}, \tag{4.6}$$

which implies that the selection of $\boldsymbol{h} \in Q(i)\backslash\{\boldsymbol{f}\}$ into $\mathbb{H}(i)$ makes two different r-sets in $(L' \cap Q(i))\backslash\{\boldsymbol{h}\}$ unavailable. Hence, for any given $L' \in W_{\boldsymbol{f},L,k}(i)$, the number of ways to remove L' from $W_{\boldsymbol{f},L,k}(i)$ is counted by

$$\sum_{\boldsymbol{g}\in(L'\cap Q(i))\backslash\{\boldsymbol{f}\}} \Big(|Q_{\boldsymbol{g}}(i)| + \sum_{K\in\mathcal{L}} |W_{\boldsymbol{g},K,e_K-2}(i)| + O(|\Upsilon_{\boldsymbol{g}}(i)|)\Big) + O\big(|\Psi_{L'}(i)|\big), \tag{4.7}$$

where $\Upsilon_{\boldsymbol{g}}(i)$ is defined in (4.2).

Secondly, for $k \geqslant 1$, the ways to make one $L' \in W_{\boldsymbol{f},L,k-1}(i)$ move into $W_{\boldsymbol{f},L,k}(i+1)$ are counted by

$$\sum_{L'\in W_{\boldsymbol{f},L,k-1}(i)} (e_L - k) + O\big(|\Lambda_{\boldsymbol{f},L,k-1}(i)|\big), \tag{4.8}$$

where $\Lambda_{\boldsymbol{f},L,k-1}(i)$ is the last overcounting collection of r-sets to be

$$\Lambda_{\boldsymbol{f},L,k-1}(i) = \Big\{\boldsymbol{h} \in L' \cap Q(i)\backslash\{\boldsymbol{f}\} : \text{an } r\text{-set in } (L' \cap Q(i))\backslash\{\boldsymbol{h}\} \text{ is unavailable in } \mathbb{H}(i) + \{\boldsymbol{h}\}, \text{ where } L' \in W_{\boldsymbol{f},L,k-1}(i)\Big\}, \tag{4.9}$$

that is, the selection of $\boldsymbol{h} \in L' \cap Q(i)\backslash\{\boldsymbol{f}\}$ for some $L' \in W_{\boldsymbol{f},L,k-1}(i)$ into $\mathbb{H}(i)$ as an edge makes another r-set of $(L' \cap Q(i))\backslash\{\boldsymbol{h}\}$ unavailable.

Considering all these effects shown in (4.7) and (4.8) together, it follows that

$$\begin{aligned}
&\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i] \\
&= -\frac{1}{|Q(i)|} \sum_{L'\in W_{\boldsymbol{f},L,k}(i)} \Bigg[\sum_{\boldsymbol{g}\in(L'\cap Q(i))\backslash\{\boldsymbol{f}\}} \Big(|Q_{\boldsymbol{g}}(i)| + \sum_{K\in\mathcal{L}} |W_{\boldsymbol{g},K,e_K-2}(i)| + O(|\Upsilon_{\boldsymbol{g}}(i)|)\Big) + O\big(|\Psi_{L'}(i)|\big)\Bigg] \\
&\quad + \frac{\mathbb{1}_{\{k\geqslant 1\}}}{|Q(i)|} \Bigg[\sum_{L'\in W_{\boldsymbol{f},L,k-1}(i)} (e_L - k) + O\big(|\Lambda_{\boldsymbol{f},L,k-1}(i)|\big)\Bigg].
\end{aligned} \tag{4.10}$$

We analyze the enumerative formula of $|Q_{\boldsymbol{g}}(i)|$ in (4.10). Recall that $Q_{\boldsymbol{g}}(i)$ is the collection of r-sets in $Q(i)$ such that its selection into $\mathbb{H}(i)$ as an edge makes $\boldsymbol{g}$ not the vertex set of an r-clique in $\mathbb{G}(i+1)$, where $\boldsymbol{g} \in (L' \cap Q(i)) \backslash \{\boldsymbol{f}\}$ for some $L' \in W_{\boldsymbol{f},L,k}(i)$.

Claim 4.1.2

$$|Q_{\boldsymbol{g}}(i)| = \binom{r}{2} \cdot \big(y_2 \pm \epsilon_{y_2}\big) + O\big(n^{r-3}\big).$$

Proof of Claim 4.1.2. Let $\bar{\boldsymbol{g}}_s \subseteq \boldsymbol{g}$ with $|\bar{\boldsymbol{g}}_s| = s$ for an integer s satisfying $2 \leqslant s \leqslant r$. Denote $Q_{\boldsymbol{g}}^{\bar{\boldsymbol{g}}_s}(i) \subseteq Q_{\boldsymbol{g}}(i)$ to be the collection of available r-sets such that every r-set in $Q_{\boldsymbol{g}}^{\bar{\boldsymbol{g}}_s}(i)$ exactly contains the vertices $\bar{\boldsymbol{g}}_s$ in $\boldsymbol{g}$. Hence, we rewrite $|Q_{\boldsymbol{g}}(i)|$ into

$$|Q_{\boldsymbol{g}}(i)| = \sum_{s=2}^{r} \sum_{\bar{\boldsymbol{g}}_s \in \binom{\boldsymbol{g}}{s}} |Q_{\boldsymbol{g}}^{\bar{\boldsymbol{g}}_s}(i)|. \tag{4.11}$$

We also apply the inclusion-exclusion to obtain $|Q_{\boldsymbol{g}}^{\bar{\boldsymbol{g}}_s}(i)|$ as

$$\begin{aligned}|Q_{\boldsymbol{g}}^{\bar{\boldsymbol{g}}_s}(i)| &= |Y_{\bar{\boldsymbol{g}}_s}(i)| - \sum_{\boldsymbol{t}_1 \in \binom{\boldsymbol{g}\backslash\bar{\boldsymbol{g}}_s}{1}} |Y_{\bar{\boldsymbol{g}}_s \cup \boldsymbol{t}_1}(i)| + \cdots + \sum_{\boldsymbol{t}_{r-s} \in \binom{\boldsymbol{g}\backslash\bar{\boldsymbol{g}}_s}{r-s}} (-1)^{r-s} |Y_{\bar{\boldsymbol{g}}_s \cup \boldsymbol{t}_{r-s}}(i)| \\ &= \sum_{j=0}^{r-s} \sum_{\boldsymbol{t}_j \in \binom{\boldsymbol{g}\backslash\bar{\boldsymbol{g}}_s}{j}} (-1)^j |Y_{\bar{\boldsymbol{g}}_s \cup \boldsymbol{t}_j}(i)|.\end{aligned}$$

Combining with the equation in (4.11), it follows that

$$|Q_{\boldsymbol{g}}(i)| = \sum_{s=2}^{r} \sum_{j=0}^{r-s} \sum_{\bar{\boldsymbol{g}}_s \in \binom{\boldsymbol{g}}{s}} \sum_{\boldsymbol{t}_j \in \binom{\boldsymbol{g}\backslash\bar{\boldsymbol{g}}_s}{j}} (-1)^j |Y_{\bar{\boldsymbol{g}}_s \cup \boldsymbol{t}_j}(i)|,$$

in which

$$\sum_{\bar{\boldsymbol{g}}_s \in \binom{\boldsymbol{g}}{s}} \sum_{\boldsymbol{t}_j \in \binom{\boldsymbol{g}\backslash\bar{\boldsymbol{g}}_s}{j}} |Y_{\bar{\boldsymbol{g}}_s \cup \boldsymbol{t}_j}(i)| = \sum_{\bar{\boldsymbol{g}}_{s+j} \in \binom{\boldsymbol{g}}{s+j}} \binom{s+j}{s} |Y_{\bar{\boldsymbol{g}}_{s+j}}(i)|$$

because each $\bar{\boldsymbol{g}}_{s+j} \in \binom{\boldsymbol{g}}{s+j}$ on the right side is counted $\binom{s+j}{s}$ times to be $\bar{\boldsymbol{g}}_s \cup \boldsymbol{t}_j$ on the left side. We further have

$$|Q_{\boldsymbol{g}}(i)| = \sum_{s=2}^{r} \sum_{j=0}^{r-s} \sum_{\bar{\boldsymbol{g}}_{s+j} \in \binom{\boldsymbol{g}}{s+j}} (-1)^j \binom{s+j}{s} |Y_{\bar{\boldsymbol{g}}_{s+j}}(i)|. \tag{4.12}$$

Hence, $|Q_{\boldsymbol{g}}(i)|$ can be regarded as the sum of all elements in the $(r-1) \times (r-1)$ upper triangular matrix below

$$\begin{pmatrix} \sum\limits_{\bar{\boldsymbol{g}}_2 \in \binom{\boldsymbol{g}}{2}} (-1)^0 \binom{2}{2} |Y_{\bar{\boldsymbol{g}}_2}(i)| & \cdots & \sum\limits_{\bar{\boldsymbol{g}}_r \in \binom{\boldsymbol{g}}{r}} (-1)^{r-2} \binom{r}{2} |Y_{\bar{\boldsymbol{g}}_r}(i)| \\ \vdots & \ddots & \vdots \\ \sum\limits_{\bar{\boldsymbol{g}}_r \in \binom{\boldsymbol{g}}{r}} (-1)^0 \binom{r}{r} |Y_{\bar{\boldsymbol{g}}_r}(i)| & \cdots & 0 \end{pmatrix},$$

where the row corresponds to the index s and the column corresponds to the index j in (4.12), respectively. Summing these elements again according to all back diagonal lines,

$$|Q_{\boldsymbol{g}}(i)| = \sum_{s=2}^{r} \sum_{\bar{\boldsymbol{g}}_s \in \binom{\boldsymbol{g}}{s}} \sum_{j=2}^{s} (-1)^{s-j} \binom{s}{j} |Y_{\bar{\boldsymbol{g}}_s}(i)| = \sum_{s=2}^{r} \sum_{\bar{\boldsymbol{g}}_s \in \binom{\boldsymbol{g}}{s}} (-1)^s (s-1) |Y_{\bar{\boldsymbol{g}}_s}(i)|,$$

where $\sum_{j=2}^{s}(-1)^{s-j}\binom{s}{j} = (-1)^s(s-1)$. According to the expressions of $|Y_{\bar{\boldsymbol{g}}_s}(i)|$ in (3.16) for $2 \leqslant s \leqslant r$, the term $|Y_{\bar{\boldsymbol{g}}_2}(i)|$ dominates the sum on the right side of (4.12). As $s = 2$, as the equations shown in (3.20) and (3.23), we complete the proof of Claim 4.1.2, where the last term $O(n^{r-3})$ comes from those terms when $s \neq 2$. □

4.1.3 $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$ and $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$ again

On the basis of the equations in (4.3) and (4.10), the enumerative formulas in Claim 4.1.1 and Claim 4.1.2, in order to obtain $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$ for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, and $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$ for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, we also require the following estimates hold on $|\Upsilon_{\boldsymbol{f}}(i)|$ in (4.2), $|\Psi_{L'}(i)|$ in (4.6) and $|\Lambda_{\boldsymbol{f},L,k-1}(i)|$ in (4.9).

Claim 4.1.3 Let $\mathcal{A}(i)$ denote the event that

$$|\Upsilon_{\boldsymbol{f}}(i)| = O\big(n^{\alpha'+r-3}(nt_M)^{\ell-2}\big), \tag{4.13}$$

$$|\Psi_{L'}(i)| = O\big(n^{\alpha'+r-3}(nt_M)^{\ell-2}\big), \tag{4.14}$$

$$\mathbb{1}_{\{k\geqslant 1\}} \cdot |\Lambda_{\boldsymbol{f},L,k-1}(i)| = O\big(n^{\alpha'+(r-1)(e_L-k)-2}(nt_M)^{k-1}\big) \tag{4.15}$$

hold for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $L' \in W_{\boldsymbol{f},L,k}(i)$ with $0 \leqslant k \leqslant e_L - 2$, where $\alpha' \in (\alpha_0, \alpha)$. On the condition of the event $\mathcal{G}(i)$, the event $\mathcal{A}(i)$ holds with probability $1 - o(n^{-\mu})$, that is,

$$\mathbb{P}\big[\bar{\mathcal{A}}(i)\,|\mathcal{G}(i)\big] = o(n^{-\mu}).$$

The proof of each equation in (4.13)-(4.15) is left to be analyzed in Section 5.2.

Assuming the event $\mathcal{G}(i) \cap \mathcal{A}(i)$ holds, we now have $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$ and $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$ as follows.

Claim 4.1.4 On the condition of the event $\mathcal{G}(i)\cap\mathcal{A}(i)$, for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$,

$$\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i] = \frac{y_m'}{n^2} + O\Big(\sigma_i^{\ell} n^{\alpha+r-m-3}(nt_M)^{\ell-2}\Big).$$

Proof of Claim 4.1.4. Note that $\epsilon_{y_2} = O(\sigma_i n^{\alpha+r-3})$ in (3.23),

$$\sum_{L\in\mathcal{L}} \epsilon_{w_{L,e_L-2}} = O\Big(\sigma_i^{\ell} n^{\alpha+r-3}(nt_M)^{\ell-2}\Big)$$

in (3.24), and the equation of $|\Upsilon_{\boldsymbol{f}}(i)|$ for any $\boldsymbol{f} \in Q(i)$ shown in (4.13),

$$\sum_{L\in\mathcal{L}} \epsilon_{w_{L,e_L-2}} \gg \epsilon_{y_2} \text{ and } \sum_{L\in\mathcal{L}} \epsilon_{w_{L,e_L-2}} \gg |\Upsilon_{\boldsymbol{f}}(i)|.$$

As the equation of $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$ in (4.3), $|Q_{\boldsymbol{f}_m^c}(i)|$ in Claim 4.1.1, and $\hat{\epsilon}_q = \hat{\epsilon}_{y_m}$ for any $2 \leqslant m \leqslant r-1$ in (3.28), $\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]$ is rearranged as

$$\begin{aligned}
&\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i]\\
&= -\frac{(1+O(\hat{\epsilon}_{y_m}))y_m}{q}\left[\left(\binom{r}{2}-\binom{m}{2}\right)y_2 + \sum_{L\in\mathcal{L}} w_{L,e_L-2} + O\Big(\sigma_i^{\ell} n^{\alpha+r-3}(nt_M)^{\ell-2}\Big)\right].
\end{aligned}$$

Since $\tilde{\xi}_i = O\big((nt)^{\ell-2}\big)$ in (3.12), q in (3.19), y_m in (3.20), $w_{L,k}$ in (3.21) and $\hat{\epsilon}_{y_m}$ in (3.28), we have

$$\begin{aligned}
\sum_{L\in\mathcal{L}} \frac{w_{L,e_L-2}}{q} &= \frac{\tilde{\xi}_i}{n^2},\\
\frac{\hat{\epsilon}_{y_m} y_m}{q}\Big(y_2 + \sum_{L\in\mathcal{L}} w_{L,e_L-2}\Big) &= O\Big(\sigma_i n^{\alpha+r-m-3}(nt_M)^{\ell-2}\Big),
\end{aligned}$$

and then

$$\mathbb{E}[\Delta Y_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i] = -\left[\left(\binom{r}{2}-\binom{m}{2}\right)\frac{y_m y_2}{q} + \frac{y_m\tilde{\xi}_i}{n^2}\right] + O\Big(\sigma_i^{\ell} n^{\alpha+r-m-3}(nt_M)^{\ell-2}\Big).$$

Here, it is checked that

$$\frac{y_m'}{n^2} = -\left[\left(\binom{r}{2}-\binom{m}{2}\right)\frac{y_m y_2}{q} + \frac{y_m\tilde{\xi}_i}{n^2}\right],$$

by differentiating y_m in (3.20) with respect to t, the equations of p_i, ξ_i and $\tilde{\xi}_i$ in (3.5), (3.9) and (3.12), to complete the proof of Claim 4.1.4. $\square$

Claim 4.1.5 On the condition of the event $\mathcal{G}(i)\cap\mathcal{A}(i)$,

$$\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i] = \begin{cases} \frac{w_{L,0}'}{n^2} + O\Big(\sigma_i^{\ell} n^{\alpha+(r-1)e_L-r-3}(nt_M)^{\ell-2}\Big), & \text{if } k=0;\\ \frac{w_{L,k}'}{n^2} \pm \beta_k\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r-2}(nt_M)^{k-1}, & \text{if } k\geqslant 1. \end{cases}$$

Proof of Claim 4.1.5. In the case of $k \geqslant 1$, we consider $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)|\mathcal{F}_i]$ by (4.10). By the equations shown in (3.23) and (3.24),

$$\begin{aligned}
\epsilon_{y_2} &= \sigma_i n^{\alpha+r-3},\\
\epsilon_{w_{L,k-1}} &= O\big(\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-2}(nt_M)^{k-1}\big),\\
\sum_{L\in\mathcal{L}}\epsilon_{w_{L,e_L-2}} &= O\big(\sigma_i^{\ell} n^{\alpha+r-3}(nt_M)^{\ell-2}\big).
\end{aligned}$$

By the equations of $|\Upsilon_{\boldsymbol{g}}(i)|$, $|\Psi_{L'}(i)|$ and $|\Lambda_{\boldsymbol{f},L,k-1}(i)|$ in (4.13)-(4.15), we further have

$$\begin{aligned}
\sum_{L\in\mathcal{L}}\epsilon_{w_{L,e_L-2}} &\gg \epsilon_{y_2}, &\qquad \sum_{L\in\mathcal{L}}\epsilon_{w_{L,e_L-2}} &\gg |\Upsilon_{\boldsymbol{g}}(i)|,\\
\sum_{L\in\mathcal{L}}\epsilon_{w_{L,e_L-2}} &\gg |\Psi_{L'}(i)|, &\qquad \epsilon_{w_{L,k-1}} &\gg |\Lambda_{\boldsymbol{f},L,k-1}(i)|.
\end{aligned}$$

Hence, as the equation of $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i]$ in (4.10), $|Q_{\boldsymbol{g}}(i)|$ in Claim 4.1.2, $\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)|\mathcal{F}_i]$ is rearranged as

$$\begin{aligned}
&\mathbb{E}[\Delta W_{\boldsymbol{f},L,k}(i)\,|\mathcal{F}_i] \\
&= -\frac{(w_{L,k}+O(\epsilon_{w_{L,k}}))}{(1+O(\hat{\epsilon}_q))q}\left[\big(e_L-k-1\big)\left(\binom{r}{2}y_2+\sum_{L\in\mathcal{L}} w_{L,e_L-2}\right)+O\Big(\sigma_i^{\ell}n^{\alpha+r-3}(nt_M)^{\ell-2}\Big)\right] \\
&\quad+\frac{1}{(1+O(\hat{\epsilon}_q))q}\left[\big(e_L-k\big)w_{L,k-1}\pm 2(e_L-k)\epsilon_{w_{L,k-1}}\right].
\end{aligned} \tag{4.16}$$

By the equations of $\tilde{\xi}_i$, q, y_2 and $w_{L,k}$ shown in (3.12), (3.19)-(3.21), we have

$$\frac{\binom{r}{2}y_2+\sum_{L\in\mathcal{L}} w_{L,e_L-2}}{q}=\frac{r^2(r-1)^2}{2n^2p_i}+\frac{\tilde{\xi}_i}{n^2}.$$

The first term on the right side of the equation in (4.16) is reduced to

$$\begin{aligned}
&-\big(w_{L,k}+O(\epsilon_{w_{L,k}})\big)\left[\big(e_L-k-1\big)\left(\frac{r^2(r-1)^2}{2n^2p_i}+\frac{\tilde{\xi}_i}{n^2}\right)+O\Big(\sigma_i^{\ell}n^{\alpha-3}(nt_M)^{\ell-2}\Big)\right] \\
&=-\,w_{L,k}\big(e_L-k-1\big)\left(\frac{r^2(r-1)^2}{2n^2p_i}+\frac{\tilde{\xi}_i}{n^2}\right)+O\Big(\sigma_i^{\ell}n^{\alpha+(r-1)(e_L-k)-r-3}(nt_M)^{k+\ell-2}\Big).
\end{aligned} \tag{4.17}$$

By the equations of $w_{L,k-1}$, $\epsilon_{w_{L,k-1}}$ and β_k in (3.21), (3.24) and (3.27), we have

$$w_{L,k-1}\cdot\hat{\epsilon}_q \ll \epsilon_{w_{L,k-1}} \text{ and } \frac{2r!(e_L-k)\beta_{k-1}}{p_i^{\binom{r}{2}}\xi_i}<\beta_k.$$

The second term on the right side of the equation in (4.16) is reduced to

$$\begin{aligned}
&\left[\big(e_L-k\big)\frac{w_{L,k-1}}{q}\pm\frac{2r!(e_L-k)}{p_i^{\binom{r}{2}}\xi_i}\beta_{k-1}\sigma_i^{k+1}n^{\alpha+(r-1)(e_L-k)-r-2}(nt_{\mathrm{M}})^{k-1}\right] \\
=&\left[\big(e_L-k\big)\frac{w_{L,k-1}}{q}\pm\beta_k\sigma_i^{k+1}n^{\alpha+(r-1)(e_L-k)-r-2}(nt_{\mathrm{M}})^{k-1}\right].
\end{aligned} \tag{4.18}$$

Here, it is also checked that

$$\frac{w'_{L,k}}{n^2}=-w_{L,k}\big(e_L-k-1\big)\left(\frac{r^2(r-1)^2}{2n^2p_i}+\frac{\tilde{\xi}_i}{n^2}\right)+\big(e_L-k\big)\frac{w_{L,k-1}}{q}$$

by differentiating $w_{L,k}$ in (3.21) with respect to t. Combining the equations shown in (4.16)-(4.18), we complete the proof of Claim 4.2.4 in the case of $k\geqslant 1$, where the error term in (4.17) is negligible compared to the one in (4.18) because

$$\beta_k\sigma_i^{k+1}n^{\alpha+(r-1)(e_L-k)-r-2}(nt_M)^{k-1}\gg\sigma_i^{\ell}n^{\alpha+(r-1)(e_L-k)-r-3}(nt_M)^{k+\ell-2}$$

with t_M in (3.26) and $\lambda>\frac{\ell}{\ell-1}$ in (3.27). For $k=0$, we only have the error term in (4.17), which is $O(\sigma_i^{\ell}n^{\alpha+(r-1)e_L-r-3}(nt_M)^{\ell-2})$. $\square$

4.1.4 $|\Delta Y_{\boldsymbol{f}_m}(i)|$ and $|\Delta W_{\boldsymbol{f},L,k}(i)|$

Next consider $|\Delta Y_{\boldsymbol{f}_m}(i)|$ for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, and $|\Delta W_{\boldsymbol{f},L,k}(i)|$ for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$. Let $\boldsymbol{e}_{i+1} \in Q(i)$ denote the available r-set that added to $\mathbb{H}(i)$. In fact, based on the analysis in Section 4.1.1 and Section 4.1.2, we have the ways to bound the one-step changes of $\Delta Y_{\boldsymbol{f}_m}(i)$ and $\Delta W_{\boldsymbol{f},L,k}(i)$. We formally characterize these ways into three boundedness parameters in higher uniformities below.

Define $\Pi_{\boldsymbol{f}_m,\boldsymbol{e}_{i+1}}(i)$ as the collection of available r-sets in $Y_{\boldsymbol{f}_m}(i)$ that become unavailable due to the addition of the available edge $\boldsymbol{e}_{i+1}$ into $\mathbb{H}(i)$, where $|\boldsymbol{e}_{i+1} \cap \boldsymbol{f}_m| \leqslant 1$ because $\boldsymbol{f}_m \in K_m(i+1)$. Thus, we have

$$|\Delta Y_{\boldsymbol{f}_m}(i)| = O\big(|\Pi_{\boldsymbol{f}_m,\boldsymbol{e}_{i+1}}(i)|\big). \tag{4.19}$$

To bound $\Delta W_{\boldsymbol{f},L,k}(i)$ for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, we firstly define $\Pi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)$ as the collection of $L' \in W_{\boldsymbol{f},L,k}(i)$ with $\boldsymbol{e}_{i+1} \in (L' \cap Q(i)) \backslash \{\boldsymbol{f}\}$, that is, $|\Pi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)|$ is the number of L' that is removed from $W_{\boldsymbol{f},L,k}(i)$ due to $\boldsymbol{e}_{i+1}$ equals one available r-set in $L' \backslash \{\boldsymbol{f}\}$. Specially, in the case of $k \geqslant 1$, $|\Pi_{\boldsymbol{f},L,k-1,\boldsymbol{e}_{i+1}}(i)|$ refers to the growth in $\Delta W_{\boldsymbol{f},L,k}(i)$. We also define $\Phi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)$ as the collection of $L' \in W_{\boldsymbol{f},L,k}(i)$ that is removed from $W_{\boldsymbol{f},L,k}(i)$ due to the addition of $\boldsymbol{e}_{i+1} \notin L'$ to $\mathbb{H}(i)$. Thus, we have

$$|\Delta W_{\boldsymbol{f},L,k}(i)| = O\Big(|\Pi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)| + |\Phi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)| + \mathbb{1}_{\{k \geqslant 1\}} \cdot |\Pi_{\boldsymbol{f},L,k-1,\boldsymbol{e}_{i+1}}(i)|\Big). \tag{4.20}$$

On the basis of the equations in (4.19) and (4.20), in order to bound $|\Delta Y_{\boldsymbol{f}_m}(i)|$ and $|\Delta W_{\boldsymbol{f},L,k}(i)|$, we also require the following estimates to hold on $|\Pi_{\boldsymbol{f}_m,\boldsymbol{e}_{i+1}}(i)|$, $|\Phi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)|$ and $|\Pi_{\boldsymbol{f},L,k,\boldsymbol{e}_{i+1}}(i)|$.

Claim 4.1.6 Let $\mathcal{B}(i)$ denote the event that

$$\mathbb{1}_{\{|\boldsymbol{f}_m \cap \boldsymbol{g}| \leqslant 1\}} \cdot |\Pi_{\boldsymbol{f}_m,\boldsymbol{g}}(i)| = O\big(n^{\alpha'+r-m-1}\big), \tag{4.21}$$

$$\mathbb{1}_{\{\boldsymbol{f} \neq \boldsymbol{g}\}} \cdot |\Pi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)| = O\big(n^{\alpha'+(r-1)(e_L-k-2-\mathbb{1}_{\{k<e_L-2\}})}\big), \tag{4.22}$$

$$\mathbb{1}_{\{\boldsymbol{f} \neq \boldsymbol{g}\}} \cdot |\Phi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)| = O\big(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k\big), \tag{4.23}$$

hold for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, $\boldsymbol{f}$, $\boldsymbol{g} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, where $\alpha' \in (\alpha_0, \alpha)$. On the condition of the event $\mathcal{G}(i)$, the event $\mathcal{B}(i)$ holds with probability $1 - o(n^{-\mu})$, that is,

$$\mathbb{P}[\bar{\mathcal{B}}_i \,|\, \mathcal{G}_i] = o(n^{-\mu}).$$

The proof of each equation in (4.21)-(4.23) is left to be analyzed in Section 5.2.

Assuming the event $\mathcal{G}(i) \cap \mathcal{B}(i)$ holds, we now have $|\Delta Y_{\boldsymbol{f}_m}(i)|$ and $|\Delta W_{\boldsymbol{f},L,k}(i)|$ as follows.

Claim 4.1.7 On the condition of the event $\mathcal{G}(i) \cap \mathcal{B}(i)$,

$$|\Delta Y_{\boldsymbol{f}_m}(i)| = O\big(n^{\alpha'+r-m-1}\big),$$
$$|\Delta W_{\boldsymbol{f},L,k}(i)| = O\big(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k\big).$$

4.2 The one-step (expected) changes of $Y^{\pm}_{\boldsymbol{f}_m}(i)$ and $W^{\pm}_{\boldsymbol{f},L,k}(i)$

4.2.1 $\mathbb{E}\big[|\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)||\mathcal{F}_i\big]$ and $|\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)|$

By (2.1) and (2.2), we have $Y^{\pm}_{\boldsymbol{f}_m}(i) = \pm[|Y_{\boldsymbol{f}_m}(i)| - y_m(t)] - \epsilon_{y_m}(t)$ and

$$\Delta Y^{\pm}_{\boldsymbol{f}_m}(i) = \pm\left[\Delta Y_{\boldsymbol{f}_m}(i) - \frac{y'_m(t)}{n^2}\right] - \frac{\epsilon'_{y_m}(t)}{n^2} + O\left(\sup_{\varsigma\in[t,t+\frac{1}{n^2}]} \frac{|y_m{}''(\varsigma)| + |\epsilon''_{y_m}(\varsigma)|}{n^4}\right). \tag{4.24}$$

Claim 4.2.1 For any $t \in [0, t_M]$ and $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, we have $\{Y^{\pm}_{\boldsymbol{f}_m}(i)\}_{i\geqslant 0}$ is a supermartingale and

$$\mathbb{E}\big[|\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)|\,|\mathcal{F}_i\big] = O\Big(n^{\alpha+r-m-1}\big(n^{\alpha} + n^2 t\big)^{-1}\Big).$$

Proof of Claim 4.2.1. In order to establish the desired supermartingale condition

$$\mathbb{E}[\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)\,|\mathcal{F}_i] \leqslant 0$$

by Claim 4.1.4 and the equation shown in (4.24), it is sufficient to show that the error function ϵ_{y_m} in (3.23) satisfies the following inequality

$$\epsilon'_{y_m} \gg \max\left\{\sigma_i^{\ell} n^{\alpha+r-m-1}(nt_M)^{\ell-2},\ \sup_{\zeta\in[t,t+\frac{1}{n^2}]} \frac{|y''_m(\zeta)| + |\epsilon''_{y_m}(\zeta)|}{n^2}\right\}. \tag{4.25}$$

We have obtained that $\epsilon'_{y_m} = \frac{n^{\alpha+r-m+1}}{n^{\alpha}+n^2 t}$ by (3.23), and it is checked that the equation in (4.25) is true, where t_M in (3.26) and $\lambda > \frac{\ell}{\ell-1}$ in (3.27).

Finally, it is natural to obtain $\mathbb{E}[|\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)|\,|\mathcal{F}_i] = O\big(\frac{|\epsilon'_{y_m}|}{n^2}\big)$. □

Claim 4.2.2 Assuming the event $\mathcal{G}(i) \cap \mathcal{B}(i)$ holds,

$$|\Delta Y^{\pm}_{\boldsymbol{f}_m}| = O\big(n^{\alpha'+r-m-1}\big)$$

for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$.

Proof of Claim 4.2.2. Following the equation in (4.24), we indeed have shown

$$|\Delta Y^{\pm}_{\boldsymbol{f}_m}(i)| \leqslant |\Delta Y_{\boldsymbol{f}_m}(i)| + O\left(\max\left\{\frac{|y'_m|}{n^2}, \frac{|\epsilon'_{y_m}|}{n^2}\right\}\right).$$

As the equations of y_m in (3.20) and ϵ_{y_m} in (3.23), we have obtained

$$\frac{|y'_m|}{n^2} = O\big(n^{r-m-2}(1 + (nt)^{\ell-2})\big) \quad \text{and} \quad \frac{|\epsilon'_{y_m}|}{n^2} = O\big(n^{\alpha+r-m-1}(n^{\alpha} + n^2 t)^{-1}\big),$$

which are negligible compared with $|\Delta Y_{\boldsymbol{f}_m}| = O(n^{\alpha'+r-m-1})$ shown in Claim 4.1.7. □

4.2.2 $\mathbb{E}\big[|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)||\mathcal{F}_i\big]$ and $|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)|$

By (2.1) and (2.2), we have $W^{\pm}_{\boldsymbol{f},L,k}(i) = \pm[|W_{\boldsymbol{f},L,k}(i)| - w_{L,k}(t)] - \epsilon_{w_{L,k}}(t)$ and

$$\Delta W^{\pm}_{\boldsymbol{f},L,k}(i) = \pm\Big[\Delta W_{\boldsymbol{f},L,k}(i) - \frac{w'_{L,k}(t)}{n^2}\Big] - \frac{\epsilon'_{w_{L,k}}(t)}{n^2} + O\Big(\sup_{\varsigma\in[t,t+\frac{1}{n^2}]} \frac{|w''_{L,k}(\varsigma)| + |\epsilon''_{w_{L,k}}(\varsigma)|}{n^4}\Big). \tag{4.26}$$

Claim 4.2.3 For any $t \in [0, t_M]$, $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, we have $\{W^{\pm}_{\boldsymbol{f},L,k}(i)\}_{i\geqslant 0}$ is a supermartingale and

$$\mathbb{E}\big[|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)|\,|\mathcal{F}_i\big] = O\Big(\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r-1}(n^\alpha + n^2 t)^{-1}(nt_M)^k\Big).$$

Proof of Claim 4.2.3. In order to establish the desired supermartingale condition

$$\mathbb{E}\big[\Delta W^{\pm}_{\boldsymbol{f},L,k}\,|\mathcal{F}_i\big] \leqslant 0$$

by Claim 4.1.5 and the equation shown in (4.26), we will show that the error function $\epsilon_{w_{L,k}}$ in (3.24) satisfies the following inequalities

$$\epsilon'_{w_{L,k}} \gg \sup_{\zeta\in[t,t+\frac{1}{n^2}]} \frac{|w''_{L,k}(\zeta)| + |\epsilon''_{w_{L,k}}(\zeta)|}{n^2}, \tag{4.27}$$

and

$$\epsilon'_{w_{L,k}} \begin{cases} \gg \sigma_i^{\ell} n^{\alpha+(r-1)e_L-r-1}(nt_M)^{\ell-2}, & \text{if } k = 0; \\ > \beta_k \sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r}(nt_M)^{k-1}, & \text{if } k \geqslant 1. \end{cases} \tag{4.28}$$

Differentiating $\epsilon_{w_{L,k}}$ in (3.24) with respect to t, we have

$$\epsilon'_{w_{L,k}} = (k+2)\beta_k \sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r+1}(n^\alpha + n^2 t)^{-1}(nt_M)^k,$$

and it is checked that the equations shown in (4.27) and (4.28) are true.

Finally, it is natural to obtain $\mathbb{E}[|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)|\,|\mathcal{F}_i] = O\big(\frac{|\epsilon'_{w_{L,k}}|}{n^2}\big)$. □

Claim 4.2.4 Assuming the event $\mathcal{G}(i) \cap \mathcal{B}(i)$ holds,

$$|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)| = O\big(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k\big)$$

for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$.

Proof of Claim 4.2.4. Following the equation shown in (4.26), we indeed have shown

$$|\Delta W^{\pm}_{\boldsymbol{f},L,k}| \leqslant |\Delta W_{\boldsymbol{f},L,k}(i)| + O\Big(\max\Big\{\frac{|w'_{L,k}|}{n^2}, \frac{|\epsilon'_{w_{L,k}}|}{n^2}\Big\}\Big). \tag{4.29}$$

As the equations of $w_{L,k}$ shown in (3.21) and $\epsilon_{w_{L,k}}$ shown in (3.24), we have obtained that

$$\frac{|w'_{L,k}|}{n^2} = \mathbb{1}_{\{k\geqslant 1\}} \cdot O\Big(n^{(r-1)(e_L-k)+k-r-2}t^{k-1}\Big) + O\Big(n^{(r-1)(e_L-k)+k-r-2}(1+(nt)^{\ell-2})t^k\Big)$$

and

$$\frac{|\epsilon'_{w_{L,k}}|}{n^2} = O\Big(\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r-1}(n^\alpha + n^2 t)^{-1}(nt_M)^k\Big),$$

which are all negligible compared with $|\Delta W_{\boldsymbol{f},L,k}(i)| = O(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k)$ shown in Claim 4.1.7. $\square$

4.3 Proof of Theorem 3.4

For $i \geqslant 0$, define the event $\mathcal{G}^+(i) = \mathcal{G}(i) \cap \mathcal{A}(i) \cap \mathcal{B}(i)$. Recall that $|Y_{\boldsymbol{f}_m}(0)| = \binom{n-m}{r-m}$ for any $\boldsymbol{f}_m \in K_m(0)$ with $2 \leqslant m \leqslant r-1$, and $|W_{\boldsymbol{f},L,k}(0)| = \mathbb{1}_{\{k=0\}} \cdot |N_{\boldsymbol{f},L}|$, where $|\mathcal{N}_{\boldsymbol{f},L}|$ is shown in (3.7) for any $\boldsymbol{f} \in Q(0)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$. Firstly, we show the event $\mathcal{G}(0)$ always holds. By the expressions of y_m and $w_{L,k}$ in (3.20) and (3.21), and the expressions of ϵ_{y_m} and $\epsilon_{w_{L,k}}$ in (3.23) and (3.24) when $t = 0$, we have

$$\begin{aligned} Y^{\pm}_{\boldsymbol{f}_m}(0) &= \pm[|Y_{\boldsymbol{f}_m}(0)| - y_m(0)] - \epsilon_{y_m}(0) \\ &< -\frac{1}{2}\sigma_0 n^{\alpha+r-m-1}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} W^{\pm}_{\boldsymbol{f},L,k}(0) &= \pm[|W_{\boldsymbol{f},L,k}(0)| - w_{L,k}(0)] - \epsilon_{w_{L,k}}(0) \\ &= \mathbb{1}_{\{k=0\}} \cdot O\big(n^{(r-1)e_L-r-1}\big) - \epsilon_{w_{L,k}}(0) \\ &< -\frac{1}{2}\beta_k \sigma_0^{k+2} n^{\alpha+(r-1)(e_L-k)-r-1}(nt_M)^k, \end{aligned} \tag{4.31}$$

which implies that $|Y_{\boldsymbol{f}_m}(0)| = y_m(0) \pm \epsilon_{y_m}(0)$ and $|W_{\boldsymbol{f},L,k}(0)| = w_{L,k}(0) \pm \epsilon_{w_{L,k}}(0)$ hold. The equation of $|Q(0)| = q(0) \pm \epsilon_q(0)$ shown in (3.15) follows from the analysis of the equation shown in (3.28).

Assume the event $\mathcal{G}(i-1)$ holds for $i \geqslant 1$. By Claim 4.1.3, Claim 4.1.6 and the law of total probability, $\mathcal{G}^+(i-1)$ holds with probability $1 - o(n^{-\mu})$. Hence,

$$\mathbb{P}[\bar{\mathcal{G}}(M)] \leqslant \mathbb{P}\big[\bar{\mathcal{G}}(i) \cap \mathcal{G}^+(i-1) \text{ for some } 1 \leqslant i \leqslant M\big] + o\big(n^{-\mu}\big), \tag{4.32}$$

where the second term $o(n^{-\mu})$ refers to the probability that the event $\mathcal{G}^+(i-1)$ does not hold.

Let the stopping time $\tau_{\boldsymbol{f}_m}$ be the minimum of M, the last step $i \geqslant 1$ such that $\boldsymbol{f}_m \in K_m(i)$, and the minimum $i \geqslant 1$ that the event $\bar{\mathcal{G}}^+(i)$ holds. Similarly, let the stopping time $\tau_{\boldsymbol{f}}$ be the minimum of M, the last step $i \geqslant 1$ such that $\boldsymbol{f} \in Q(i)$, and the minimum $i \geqslant 1$ that the event $\bar{\mathcal{G}}^+(i)$ holds. It follows that

$$\begin{aligned}
&\mathbb{P}\Big[\bar{\mathcal{G}}(i)\cap\mathcal{G}^+(i-1)\text{ for some }1\leqslant i\leqslant M\Big]\\
&\leqslant\sum_{2\leqslant m\leqslant r-1}\sum_{\boldsymbol{f}_m\in K_m(i)}\sum_{\flat\in\{+,-\}}\mathbb{P}\Big[Y^\flat_{\boldsymbol{f}_m}(i\wedge\tau_{\boldsymbol{f}_m})\geqslant 0\text{ for some }i\geqslant 1\Big]\\
&\quad+\sum_{\boldsymbol{f}\in Q(i)}\sum_{L\in\mathcal{L}}\sum_{0\leqslant k\leqslant e_L-2}\sum_{\flat\in\{+,-\}}\mathbb{P}\Big[W^\flat_{\boldsymbol{f},L,k}(i\wedge\tau_{\boldsymbol{f}})\geqslant 0\text{ for some }i\geqslant 1\Big].
\end{aligned}\tag{4.33}$$

In order to bound the above probability, the following tail probability formula in [7, 12, 16, 24] is applied to each inequality in (4.33). The following tail probability is due to Freedman [12], which is called to be Freedman's martingale concentration inequality and was originally stated for martingales, while the proof for supermartingales is the same. It is also applied to study the existence of sparse Steiner triple systems to approximately answer a conjecture of Erdős in the previous works of Glock, Kühn, Lo and Osthus [16] and of Bohman and Warnke [7]. The simplified version of the Freedman's martingale concentration inequality here [7, 16, 24] improves the Azuma-Hoeffding inequality when the expected one-step changes are much smaller than the worst case ones, that is, $\mathbb{E}[|\Delta S(i)|\,|\mathcal{F}_i]\ll\max_{i\geqslant 0}|\Delta S(i)|$.

Lemma 4.3.1 Let $\{S(i)\}_{i\geqslant 0}$ be a supermartingale with respect to the filtration $\mathcal{F}=(\mathcal{F}_i)_{i\geqslant 0}$. Let $\Delta S(i)=S(i+1)-S(i)$. Suppose that $\sum_{i\geqslant 0}\mathbb{E}[|\Delta S(i)|\,|\mathcal{F}_i]\leqslant C$ and $\max_{i\geqslant 0}|\Delta S(i)|\leqslant V$. Then, for any $z>0$,

$$\mathbb{P}\Big[S(i)\geqslant S(0)+z\text{ for some }i\geqslant 1\Big]\leqslant\exp\Big[-\frac{z^2}{2V(C+z)}\Big].$$

As the equation shown in (4.30), we have

$$Y^\flat_{\boldsymbol{f}_m}(0)<-\frac{1}{2}\sigma_0 n^{\alpha+r-m-1}$$

for $\flat\in\{+,-\}$. By Claim 4.2.1 and Claim 4.2.2, it follows that the sequence $\{Y^\flat_{\boldsymbol{f}_m}(i\wedge\tau_{\boldsymbol{f}_m})\}_{i\geqslant 0}$ is a supermartingale,

$$\sum_{i\geqslant 0}\mathbb{E}[|\Delta Y^\flat_{\boldsymbol{f}_m}(i)|\,|\mathcal{F}_i]=n^2t\cdot O\big(n^{\alpha+r-m-1}(n^\alpha+n^2t)^{-1}\big)=O\big(n^{\alpha+r-m-1}\big)$$

and $|\Delta Y^\flat_{\boldsymbol{f}_m}(i)|=O(n^{\alpha'+r-m-1})$ with $\alpha'\in(\alpha_0,\alpha)$. Applying Lemma 4.3.1 with

$$z=\frac{1}{2}\sigma_0 n^{\alpha+r-m-1},\ C=O(n^{\alpha+r-m-1})\text{ and }V=O(n^{\alpha'+r-m-1})$$

to $Y^\flat_{\boldsymbol{f}_m}(i)$, we have

$$\mathbb{P}\big[Y^\flat_{\boldsymbol{f}_m}(i\wedge\tau_{\boldsymbol{f}_m})\geqslant 0\text{ for some }i\geqslant 1\big]=n^{-\omega(1)}.\tag{4.34}$$

Similarly, we have

$$W^{\flat}_{\boldsymbol{f},L,k}(0) < -\frac{1}{2}\beta_k \sigma_0^{k+2} n^{\alpha+(r-1)(e_L-k)-r-1}(nt_M)^k$$

by (4.31) for $\flat \in \{+,-\}$. By Claim 4.2.3 and Claim 4.2.4, the sequence $\{W^{\flat}_{\boldsymbol{f},L,k}(i \wedge \tau_{\boldsymbol{f}})\}_{i \geqslant 0}$ is also a supermartingale,

$$\begin{aligned}\sum_{i\geqslant 0} \mathbb{E}[|\Delta W^{\flat}_{\boldsymbol{f},L,k}(i)|\,|\mathcal{F}_i] &= n^2 t \cdot O\big(\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r-1}(n^{\alpha}+n^2 t)^{-1}(nt_M)^k\big) \\ &= O\big(\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r-1}(nt_M)^k\big)\end{aligned}$$

and $|\Delta W^{\pm}_{\boldsymbol{f},L,k}(i)| = O(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k)$ with $\alpha' \in (\alpha_0, \alpha)$. Applying Lemma 4.3.1 with

$$\begin{aligned} z &= \frac{1}{2}\beta_k \sigma_0^{k+2} n^{\alpha+(r-1)(e_L-k)-r-1}(nt_M)^k, \\ C &= O\big(\sigma_i^{k+1} n^{\alpha+(r-1)(e_L-k)-r-1}(nt_M)^k\big), \\ V &= O\big(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k\big)\end{aligned}$$

to $W^{\flat}_{\boldsymbol{f},L,k}(i)$, it follows that

$$\mathbb{P}\Big[W^{\flat}_{\boldsymbol{f},L,k}(i \wedge \tau_{\mathbf{f}}) \geqslant 0 \text{ for some } i \geqslant 0\Big] = n^{-\omega(1)}. \tag{4.35}$$

Taking a union bound of the probabilities shown in (4.34) for any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$, and the ones in (4.35) for any $\boldsymbol{f} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, combining the equation shown in (4.33), $|\mathcal{L}| = O(1)$, $|K_m(i)| < n^m$ and $|Q(i)| < n^r$, we have $\mathbb{P}\big[\bar{\mathcal{G}}(i) \cap \mathcal{G}^+(i-1) \text{ for some } 1 \leqslant i \leqslant M\big] = o(n^{-\mu})$. Finally, as the equations shown in (4.32), we complete the proof of Theorem 3.4 by $\mathbb{P}[\bar{\mathcal{G}}(M)] = o(n^{-\mu})$.

5 Claim 4.1.3 and Claim 4.1.6

In this section, for an integer i with $0 \leqslant i < M$, we still assume that the event $\mathcal{G}(i)$ holds, and prove Claim 4.1.3 and Claim 4.1.6.

5.1 A special injective function

Given an r-graph H with $\max\{v_H, e_H\} \leqslant 2r\ell$, let $\hat{H} \subsetneq H$ be a subhypergraph, and θ be a fixed injection from $V(\hat{H})$ to the vertex-set $[n]$ of $\mathbb{H}(i)$.

An injection ψ maps an r-set $\boldsymbol{f} = \{x_1, \cdots, x_r\}$ as $\psi(\boldsymbol{f}) = \psi(x_1)\dots\psi(x_r)$ and maps a collection U of r-sets as $\psi(U) = \cup_{\boldsymbol{f}\in U}\psi(\boldsymbol{f})$. Define a set of injections ψ extended from θ, denoted as $\Psi_{\theta,\hat{H},H}(i)$, to be

$$\Psi_{\theta,\hat{H},H}(i) = \Big\{\psi : V(H) \to [n] \text{ with } \psi|_{V(\hat{H})} \equiv \theta \text{ and } \psi(H \backslash \hat{H}) \subseteq \mathbb{H}(i)\Big\}, \tag{5.1}$$

which implies that the value $|\Psi_{\theta,\hat{H},H}(i)|$ counts copies of H in $\mathbb{H}(i)\cup\theta(\hat{H})$ and these copies all have the fixed image $\theta(\hat{H})$ of $\hat{H}$ under the fixed injection θ as a subhypergraph.

Lemma 5.1. *With the above notation, let $\mathcal{C}(i)$ denote the event that, for any $\epsilon>0$, an r-graph pair $(\hat{H},H)$ and an injection $\theta: V(\hat{H})\to[n]$, the set $\Psi_{\theta,\hat{H},H}(i)$ defined in (5.1) satisfies*

$$\left|\Psi_{\theta,\hat{H},H}(i)\right| \leqslant n^{\epsilon}\cdot \max_{\hat{H}\subseteq\widetilde{H}\subseteq H} n^{[v_H-(r-1)e_H]+[(r-1)e_{\widetilde{H}}-v_{\widetilde{H}}]}\cdot (nt)^{e_H-e_{\widetilde{H}}}.$$

On the condition of the event $\mathcal{G}(i)$, the event $\mathcal{C}(i)$ holds with probability $1-o(n^{-\mu})$, that is,

$$\mathbb{P}\big[\bar{\mathcal{C}}(i)\,|\mathcal{G}(i)\big]=o(n^{-\mu}).$$

Proof of Lemma 5.1. For any i with $0\leqslant i<M$, suppose that the event $\mathcal{G}(i)$ holds. For any collection $U\subseteq\binom{[n]}{r}$ of r-sets, we have

$$\mathbb{P}\big[U\subseteq\mathbb{H}(i)|\mathcal{G}(i)\big]=O\Big(i^{|U|}\cdot\Big(\frac{1}{|Q(i)|}\Big)^{|U|}\Big)$$

because there are at most $i^{|U|}$ ways that all r-sets in U appear in $\mathbb{H}(i)$, and each r-set is added to $\mathbb{H}(i)$ at some step $j\leqslant i$ with probability at most $1/|Q(i)|$. Since $p_i,\xi_i=\Theta(1)$ when $t\in[0,t_M]$ as shown in (3.5) and (3.9), we have $|Q(i)|=\Theta(n^r)$ as the equation shown in (3.19), and it follows that

$$\mathbb{P}\big[U\subseteq\mathbb{H}(i)\,|\mathcal{G}(i)\big]=O\Big(\big(n^{2-r}t\big)^{|U|}\Big). \tag{5.2}$$

Hence, on the definition of $\Psi_{\theta,\hat{H},H}(i)$ in (5.1), for any fixed r-graph pair $(\hat{H},H)$, an injection $\theta: V(\hat{H})\to[n]$, and $s\geqslant 1$ an integer that will be defined later, we have

$$\mathbb{E}\Big[\big|\big(\Psi_{\theta,\hat{H},H}(i)\big)^s\big|\Big|\mathcal{G}(i)\Big]=\sum_{(\psi_1,\cdots,\psi_s)\in(\Psi_{\theta,\hat{H},H}(i))^s}\mathbb{E}\Big[\prod_{j\in[s]}\Big(\mathbb{1}_{\{\psi_j(H\backslash\hat{H})\subseteq\mathbb{H}(i)\}}\Big)\,\Big|\mathcal{G}(i)\Big], \tag{5.3}$$

where $(\Psi_{\theta,\hat{H},H}(i))^s$ is the s-th power set of $\Psi_{\theta,\hat{H},H}(i)$, the sum is over all s-tuples $(\psi_1,\cdots,\psi_s)$ with the injection $\psi_j: V(H)\to[n]$ satisfying $\psi_j|_{V(\hat{H})}\equiv\theta$ for $j\in[s]$.

Let $E_j=\psi_j(H\backslash\hat{H})$ with $j\in[s]$. As the equation shown in (5.2),

$$\mathbb{E}\Big[\mathbb{1}_{\{\psi_j(H\backslash\hat{H})\subseteq\mathbb{H}(i)\}}\,\Big|\mathcal{G}(i)\Big]=O\Big(\big(n^{2-r}t\big)^{|E_j|}\Big).$$

We rearrange the equation in (5.3) to

$$\mathbb{E}\Big[\big|\big(\Psi_{\theta,\hat{H},H}(i)\big)^s\big|\Big|\mathcal{G}(i)\Big]=O\Big(\sum_{(\psi_1,\cdots,\psi_s)\in(\Psi_{\theta,\hat{H},H}(i))^s}\big(n^{2-r}t\big)^{\left|\cup_{j\in[s]}|E_j|\right|}\Big). \tag{5.4}$$

Conversely, for any s-tuple of injections $(\psi_1,\cdots,\psi_s)\in(\Psi_{\theta,\hat{H},H}(i))^s$, there must exist $\widetilde{H}_j$ with $\hat{H}\subseteq\widetilde{H}_j\subseteq H$ satisfying

$$E_j\cap\big(\cup_{k\in[j-1]}E_k\big)\cong\widetilde{H}_j\backslash\hat{H}\quad\text{and}\quad E_j\backslash\cup_{k\in[j-1]}E_k\cong H\backslash\widetilde{H}_j, \tag{5.5}$$

where $j \in [s]$. By double-counting arguments, for any s-tuples of graphs $(\widetilde{H}_1, \dots, \widetilde{H}_s)$ with $\widetilde{H}_j$ satisfying $\hat{H} \subseteq \widetilde{H}_j \subseteq H$, we have at most $\prod_{j\in[s]} n^{v_H - v_{\widetilde{H}_j}}$ ways to generate s-tuples $(\psi_1, \cdots, \psi_s) \in (\Psi_{\theta,\hat{H},H}(i))^s$.

Therefore, for any s-tuples of graphs $(\widetilde{H}_1, \dots, \widetilde{H}_s)$ with $\widetilde{H}_j$ satisfying $\hat{H} \subseteq \widetilde{H}_j \subseteq H$, we rewrite the equation in (5.4) to

$$\begin{aligned}\mathbb{E}\Big[\Big|\big(\Psi_{\theta,\hat{H},H}(i)\big)^s\Big|\Big|\mathcal{G}(i)\Big] &= O\Big(\sum_{(\widetilde{H}_1,\dots,\widetilde{H}_s)}\prod_{j\in[s]} n^{v_H - v_{\widetilde{H}_j}}\big(n^{2-r}t\big)^{e_H - e_{\widetilde{H}_j}}\Big)\\ &= O\bigg(\Big(\max_{\hat{H}\subseteq\widetilde{H}\subseteq H} n^{v_H - v_{\widetilde{H}}}\big(n^{2-r}t\big)^{e_H - e_{\widetilde{H}}}\Big)^s\bigg)\\ &= O\bigg(\Big(\max_{\hat{H}\subseteq\widetilde{H}\subseteq H} n^{[v_H-(r-1)e_H]+[(r-1)e_{\widetilde{H}}-v_{\widetilde{H}}]}\big(nt\big)^{e_H - e_{\widetilde{H}}}\Big)^s\bigg).\end{aligned}$$

For any $\epsilon > 0$, choosing $s = \lceil (2+2r\ell+\mu)/\epsilon \rceil$, by Markov's inequality of conditional probability, we have

$$\begin{aligned}&\mathbb{P}\Big[\Big|\Psi_{\theta,\hat{H},H}(i)\Big| > n^{\epsilon}\cdot \max_{\hat{H}\subseteq\widetilde{H}\subseteq H} n^{[v_H-(r-1)e_H]+[(r-1)e_{\widetilde{H}}-v_{\widetilde{H}}]}\cdot\big(nt\big)^{e_H - e_{\widetilde{H}}}\,\Big|\mathcal{G}(i)\Big]\\ &= O\big(n^{-\epsilon s}\big) = O\big(n^{-(2+2r\ell+\mu)}\big).\end{aligned}$$

Finally, by taking a union bound for all possible $(i, \hat{H}, H, \theta)$, which has at most $o(n^{2+2r\ell})$ ways because there are $M = o(n^2)$ ways to choose i, and $O(n^{2r\ell})$ ways to choose $(\hat{H}, H, \theta)$ as $\max\{v_H, e_H\} \leqslant 2r\ell$, we have $\mathbb{P}[\bar{\mathcal{C}}(i)\,|\mathcal{G}(i)] = o(n^{-\mu})$ to complete the proof of Claim 5.1. □

Remark 5.2. *We make no efforts to characterize the optimal upper bound of* $\max\{v_H, e_H\}$ *in Section 5.1. Indeed, following the same discussions,* $\max\{v_H, e_H\}$ *could be any constant, while the value* $2r\ell$ *is enough to satisfy the number of vertices in the structures counted by the overcounting and boundedness parameters in Claim 4.1.3 and Claim 4.1.6.*

5.2 View the parameters in a different way

By Claim 5.1, for $1 \leqslant i < M$, assuming the event $\mathcal{G}(i) \cap \mathcal{C}(i)$ holds, we introduce some notation to recharacterize the overcounting and boundedness parameters in Claim 4.1.3 and Claim 4.1.6.

Definition 5.3. *For any given* $L \in \mathcal{L}$, $0 \leqslant k \leqslant e_L - 2$, *a collection of nonempty available* r*-sets* $U \subseteq Q(i)$ *and* $\boldsymbol{f}_m = \{x_1, \cdots, x_m\} \subseteq \boldsymbol{f} \in Q(i)\backslash U$ *such that*

$$|U| + \mathbb{1}_{\{m\geqslant 1\}} \leqslant e_L - k.$$

In particular, $\boldsymbol{f}_m = \boldsymbol{f} = \emptyset$ *when* $m = 0$. *Let* $\Gamma_{U,\boldsymbol{f}_m,L,k}(i)$ *be*

$$\Gamma_{U,\boldsymbol{f}_m,L,k}(i) = \Big\{L' \in \mathcal{F}_L : U \subseteq L' \cap Q(i),\ \boldsymbol{f}_m \subseteq V(L') \text{ and } \big|L' \cap \mathbb{H}(i)\big| \geqslant k\Big\}. \tag{5.6}$$

Lemma 5.4. *Assume that the event $\mathcal{G}(i) \cap \mathcal{C}(i)$ holds. With the above notation, it follows that*

$$\left|\Gamma_{U,\boldsymbol{f}_m,L,k}(i)\right| = O\left(n^{\alpha'+(r-1)\cdot\left(e_L-k-|U|-\mathbb{1}_{\{|U|+\mathbb{1}_{\{m\geqslant 1\}}<e_L-k\}}\right)-\mathbb{1}_{\{m\geqslant 1\}}\cdot m}\right),$$

where $\alpha' \in (\alpha_0, \alpha)$.

The proof of Lemma 5.4 is left to Section 5.3. Firstly, we use it to prove the equations shown in (4.21) and (4.22) in Claim 4.1.6.

Proof of (4.21) in Claim 4.1.6. For any $\boldsymbol{f}_m \in K_m(i)$ with $2 \leqslant m \leqslant r-1$ and $\boldsymbol{g} \in Q(i)$, if $|\boldsymbol{f}_m \cap \boldsymbol{g}| \leqslant 1$, by the set $\Pi_{\boldsymbol{f}_m,\boldsymbol{g}}(i)$ defined in Section 4.1.4 and $\Gamma_{U,\boldsymbol{f}_m,L,k}(i)$ in (5.6), we have

$$|\Pi_{\boldsymbol{f}_m,\boldsymbol{g}}(i)| = \sum_{L\in\mathcal{L}} |\Gamma_{\{\boldsymbol{g}\},\boldsymbol{f}_m,L,e_L-2}(i)|.$$

For any $L \in \mathcal{L}$, substituting $U = \{\boldsymbol{g}\}$ and $k = e_L - 2$ in Lemma 5.4, we have

$$\mathbb{1}_{\{|U|+\mathbb{1}_{\{m\geqslant 1\}}<e_L-k\}} = 0,$$

and then $|\Gamma_{\{\boldsymbol{g}\},\boldsymbol{f}_m,L,e_L-2}(i)| = O(n^{\alpha'+r-m-1})$. Since $|\mathcal{L}| = O(1)$, we have the equation in (4.21) is true. $\square$

Proof of (4.22) in Claim 4.1.6. For any $\boldsymbol{f}$, $\boldsymbol{g} \in Q(i)$, $L \in \mathcal{L}$ and $0 \leqslant k \leqslant e_L - 2$, if $\boldsymbol{f} \neq \boldsymbol{g}$, by the set $\Pi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)$ defined in Subsection 4.1.4 and $\Gamma_{U,\boldsymbol{f}_m,L,k}(i)$ in (5.6), we have

$$|\Pi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)| = |\Gamma_{\{\boldsymbol{f},\boldsymbol{g}\},\emptyset,L,k}(i)|.$$

Substituting $U = \{\boldsymbol{f}, \boldsymbol{g}\}$ and $m = 0$ in Lemma 5.4, it follows that

$$|\Gamma_{\{\boldsymbol{f},\boldsymbol{g}\},\emptyset,L,k}(i)| = O\left(n^{\alpha'+(r-1)\cdot(e_L-k-2-\mathbb{1}_{\{k<e_L-2\}})}\right)$$

to show the equation (4.22) is true. $\square$

With the help of $\Gamma_{U,\boldsymbol{f}_m,L,k}(i)$ in (5.6), we further define some sets to collect overlapping copies of the pair L and K for any $(L,K) \in \mathcal{L}^2$, which are extended from certain r-sets in $Q(i)$.

Definition 5.5. *Given r-sets $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h} \in Q(i)$ such that $\boldsymbol{f} \neq \boldsymbol{h}$ and $\boldsymbol{g} \neq \boldsymbol{h}$, $(L,K) \in \mathcal{L}^2$, define the set $R_{\{\boldsymbol{f},\boldsymbol{h}\},L,k,\{\boldsymbol{g},\boldsymbol{h}\},K}(i)$ to be*

$$\begin{aligned} &R_{\{\boldsymbol{f},\boldsymbol{h}\},L,k,\{\boldsymbol{g},\boldsymbol{h}\},K}(i) \\ &\quad= \Big\{(L',K') : \ (L',K') \in \Gamma_{\{\boldsymbol{f},\boldsymbol{h}\},\emptyset,L,k} \times \Gamma_{\{\boldsymbol{g},\boldsymbol{h}\},\emptyset,K,e_K-2} \ \textit{and } L' \neq K'\Big\}. \end{aligned} \tag{5.7}$$

By adding constraint on choosing $\boldsymbol{h}$, define $R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)$ as

$$R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i) = \Big\{(L',K'):\ (L',K') \in \bigcup_{\{\boldsymbol{h}\}=L'\cap K'\cap Q(i)\setminus\{\boldsymbol{f},\boldsymbol{g}\}} R_{\{\boldsymbol{f},\boldsymbol{h}\},L,k,\{\boldsymbol{g},\boldsymbol{h}\},K}(i)\Big\}. \tag{5.8}$$

By further adding constraint on choosing $\boldsymbol{g}$, define $R^{++}_{\boldsymbol{f},L,k,K}(i)$ as

$$R^{++}_{\boldsymbol{f},L,k,K} = \Big\{(L',K'):\ (L',K') \in \bigcup_{\boldsymbol{g}\in L'\cap Q(i)} R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)\Big\}. \tag{5.9}$$

Remark 5.6. *As the equations shown in (5.7)-(5.9), note that $e_L, e_K \leqslant \ell$ for any $(L,K)\in\mathcal{L}^2$ and $|\mathcal{L}| = O(1)$, then it follows that*

$$\big|R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)\big| \leqslant \ell^2 \cdot \big|R_{\{\boldsymbol{f},\boldsymbol{h}\},L,k,\{\boldsymbol{g},\boldsymbol{h}\},K}(i)\big|$$

and

$$\big|R^{++}_{\boldsymbol{f},L,k,K}(i)\big| \leqslant \ell \cdot \big|R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)\big|.$$

Lemma 5.7. *Assume that the event $\mathcal{G}(i)\cap\mathcal{C}(i)$ holds. With the above notation, it follows that*

$$|R_{\{\boldsymbol{f},\boldsymbol{h}\},L,k,\{\boldsymbol{g},\boldsymbol{h}\},K}(i)| = O\Big(n^{\alpha'+(r-1)(e_L-k)-r-1}(nt_M)^k\Big),$$

where $\alpha'\in(\alpha_0,\alpha)$.

The proof of Lemma 5.7 is left to Section 5.3. We firstly use it to prove the equations (4.13)-(4.15) in Claim 4.1.3 and the equation (4.23) in Claim 4.1.6.

Proof of Claim 4.1.3. For any $\boldsymbol{f}\in Q(i)$, $L\in\mathcal{L}$ and $L'\in W_{\boldsymbol{f},L,k}(i)$ with $0\leqslant k\leqslant e_L-2$, by the definitions of $\Upsilon_{\boldsymbol{f}}(i)$ in (4.2), $\Psi_{L'}(i)$ in (4.6), $\Lambda_{\boldsymbol{f},L,k-1}(i)$ in (4.9), $R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)$ and $R^{++}_{\boldsymbol{f},L,k,K}(i)$ in (5.8) and (5.9), we have the parameters $|\Upsilon_{\boldsymbol{f}}(i)|$, $|\Psi_{L'}(i)|$ and $\mathbb{1}_{\{k\geqslant1\}}\cdot|\Lambda_{\boldsymbol{f},L,k-1}(i)|$ in Claim 4.1.3 satisfy

$$\begin{aligned}
|\Upsilon_{\boldsymbol{f}}(i)| &= \sum_{(L,K)\in\mathcal{L}^2} |R^+_{\boldsymbol{f},L,e_L-2,\boldsymbol{f},K}(i)|,\\
|\Psi_{L'}(i)| &= \sum_{\substack{\boldsymbol{f},\boldsymbol{g}\in L'\cap Q(i)\\ \boldsymbol{f}\neq\boldsymbol{g}}}\ \sum_{(L,K)\in\mathcal{L}^2} |R^+_{\boldsymbol{f},L,e_L-2,\boldsymbol{g},K}(i)|,\\
\mathbb{1}_{\{k\geqslant1\}}\cdot|\Lambda_{\boldsymbol{f},L,k-1}(i)| &= \mathbb{1}_{\{k\geqslant1\}}\cdot\sum_{K\in\mathcal{L}} |R^{++}_{\boldsymbol{f},L,k-1,K}(i)|.
\end{aligned}$$

By Remark 5.6 and Lemma 5.7, note that $e_L, e_K\leqslant\ell$ and $|\mathcal{L}|=O(1)$, we complete the proof of the equations (4.11)-(4.13) in Claim 4.1.3. □

Proof of (4.23) in Claim 4.1.6. For any $\boldsymbol{f},\boldsymbol{g}\in Q(i)$, $L\in\mathcal{L}$ and $0\leqslant k\leqslant e_L-2$, by the definitions of $\Phi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)$ in Section 4.1.4 and $R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)$ in (5.8), we have $|\Phi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)|$ satisfies

$$|\Phi_{\boldsymbol{f},L,k,\boldsymbol{g}}(i)| = \sum_{K\in\mathcal{L}} |R^+_{\boldsymbol{f},L,k,\boldsymbol{g},K}(i)|.$$

By Remark 5.6 and Lemma 5.7, note that $e_K\leqslant\ell$ and $|\mathcal{L}|=O(1)$, we complete the proof of the equation (4.23) in Claim 4.1.6. □

5.3 Lemma 5.3 and Lemma 5.7

We will apply Lemma 5.1 to prove Lemma 5.3 and Lemma 5.7 below.

Proof of Lemma 5.3. For any $L \in \mathcal{L}$, let $H \subset L$ on vertex-set $V(H) = V(L)$ and $|H| = k$ with $0 \leqslant k \leqslant e_L - 2$. Choose $U' \subseteq L \backslash H$ with $|U'| = |U|$ and a m-set $\boldsymbol{f}'_m$ with $\boldsymbol{f}'_m \subseteq \boldsymbol{f}'$ for some $\boldsymbol{f}' \in L \backslash (H \cup U')$. Note that $\boldsymbol{f}' = \emptyset$ when $m = 0$.

Define $\hat{H}$ on the vertex-set $V(\hat{H}) = V(U') \cup \boldsymbol{f}'_m$ and $|\hat{H}| = 0$. Let θ be an injection that maps the r-sets in U' onto the r-sets in U, and $\boldsymbol{f}'_m$ onto $\boldsymbol{f}_m$. Since the number of ways to choose H, U', $\boldsymbol{f}'_m$ and θ is finite, we have $|\Gamma_{U,\boldsymbol{f}_m,L,k}(i)| = O(|\Psi_{\theta,\hat{H},H}(i)|)$. We are interested in applying Lemma 5.1 to prove an upper bound of $|\Psi_{\theta,\hat{H},H}(i)|$.

Consider any $\widetilde{H}$ with $\hat{H} \subseteq \widetilde{H} \subseteq H$. Note that for $t \in [0, t_M]$ with t_M in (3.26),

$$n^{\epsilon}(nt)^{e_H - e_{\widetilde{H}}} \leqslant n^{\epsilon}(nt_M)^{\ell-2} = o(n^{\alpha'})$$

in Lemma 5.1 because $\alpha' \in (\alpha_0, \alpha)$ with $\alpha_0 = \frac{\ell-2}{\ell-1}$ and $\alpha \in (\alpha_0, 1)$ in (3.27). By Lemma 5.1, it suffices to show

$$\begin{aligned}
&[v_H - (r-1)e_H] - [v_{\widetilde{H}} - (r-1)e_{\widetilde{H}}] \\
&\leqslant (r-1)\cdot(e_L - k - |U| - \mathbb{1}_{\{|U| + \mathbb{1}_{\{m\geqslant 1\}} < e_L - k\}}) - \mathbb{1}_{\{m \geqslant 1\}} \cdot m \\
&= \begin{cases} (r-1)\cdot(e_L - k - |U| - \mathbb{1}_{\{|U| < e_L - k\}}), \text{ if } m = 0, \\ (r-1)\cdot(e_L - k - |U| - \mathbb{1}_{\{|U| < e_L - k - 1\}}) - m, \text{ otherwise,} \end{cases}
\end{aligned} \tag{5.10}$$

to complete our proof.

If $V(\widetilde{H}) = V(H)$, then the trivial estimate

$$[v_H - (r-1)e_H] - [v_{\widetilde{H}} - (r-1)e_{\widetilde{H}}] = (r-1)(e_{\widetilde{H}} - e_H) \leqslant 0$$

implies the above inequality in (5.10).

Now suppose that $V(\widetilde{H}) \subsetneq V(H)$. Since L is a linear cycle of length at most ℓ and $v_L = (r-1)e_L$ by Remark 3.2, $H \subset L$ with $|H| = k$ for $0 \leqslant k \leqslant e_L - 2$, we have the number of edges of L induced on $V(\widetilde{H}) \cup \boldsymbol{f}'$ is at most $\frac{v_{\widetilde{H}} + \mathbb{1}_{\{m\geqslant 1\}}(r-m-1)}{r-1}$. Note that $U' \cup \{\boldsymbol{f}'\} \subseteq L \backslash H$ and we further delete one from $e_{\widetilde{H}}$ when $|U| + \mathbb{1}_{\{m\geqslant 1\}} < e_L - k$. Finally, we have

$$e_{\widetilde{H}} \leqslant \frac{v_{\widetilde{H}} + \mathbb{1}_{\{m\geqslant 1\}}(r-m-1)}{r-1} - |U| - \mathbb{1}_{\{m\geqslant 1\}} - \mathbb{1}_{\{|U| + \mathbb{1}_{\{m\geqslant 1\}} < e_L - k\}}.$$

Now $[v_H - (r-1)e_H] - [v_{\widetilde{H}} - (r-1)e_{\widetilde{H}}]$ satisfies the inequality shown in (5.12) by replacing $v_H = (r-1)e_L$ and $e_H = k$. $\square$

Proof of Lemma 5.6. For any given $(L, K) \in \mathcal{L}^2$, let $\bar{L} \subset L$ on vertex set $V(\bar{L}) = V(L)$ and $|\bar{L}| = k$ with $0 \leqslant k \leqslant e_L - 2$, and $\bar{K} \subset K$ with $V(\bar{K}) = V(K)$ and $|\bar{K}| = e_K - 2$, such that there are r-sets $\boldsymbol{f}'$, $\boldsymbol{g}'$ and $\boldsymbol{h}'$ satisfying $\bar{L} \cup \{\boldsymbol{f}', \boldsymbol{h}'\} \subseteq L$ and $\bar{K} \cup \{\boldsymbol{g}', \boldsymbol{h}'\} = K$, where we allow $\boldsymbol{f}' = \boldsymbol{g}'$, while $\boldsymbol{f}' \neq \boldsymbol{h}'$ and $\boldsymbol{g}' \neq \boldsymbol{h}'$.

Consider $H = \bar{L} \cup \bar{K}$ with $V(H) = V(L) \cup V(K)$. Define $\hat{H}$ on vertex-set $V(\hat{H}) = \boldsymbol{f}' \cup \boldsymbol{g}' \cup \boldsymbol{h}'$ and $|\hat{H}| = 0$. Let θ be one injection with the property that $\boldsymbol{f}'$, $\boldsymbol{g}'$ and $\boldsymbol{h}'$ are mapped onto $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$. Since the number of ways to choose $\bar{L}$, $\bar{K}$, $\boldsymbol{f}'$, $\boldsymbol{g}'$, $\boldsymbol{h}'$ and θ is finite, we have $|R_{\{\boldsymbol{f},\boldsymbol{h}\},L,k,\{\boldsymbol{g},\boldsymbol{h}\},K}(i)| = O(|\Psi_{\theta,\hat{H},H}(i)|)$. We are interested in applying Lemma 5.1 to show an upper bound of $|\Psi_{\theta,\hat{H},H}(i)|$.

For any $\widetilde{H}$ with $\hat{H} \subseteq \widetilde{H} \subseteq H$, in Lemma 5.1, note that for any $t \in [0, t_M]$ with t_M in (3.26)

$$n^{\epsilon}(nt)^{e_H - e_{\widetilde{H}}} \leqslant n^{\epsilon}(nt_M)^{k+e_K-2} = o\big(n^{\alpha'}(nt_M)^k\big),$$

where $\alpha' \in (\alpha_0, \alpha)$ with $\alpha_0 = \frac{\ell-2}{\ell-1}$ and $\alpha \in (\alpha_0, 1)$ in (3.27). It suffices to show

$$[v_H - (r-1)e_H] - [v_{\widetilde{H}} - (r-1)e_{\widetilde{H}}] \leqslant (r-1)(e_L - k) - r - 1 \tag{5.11}$$

to complete our proof.

Let

$$A = \widetilde{H} \cap \bar{L} \qquad \text{and} \qquad B = \bar{K} \cap (\widetilde{H} \cup \bar{L}) \tag{5.12}$$

with $V(A) = V(\widetilde{H}) \cap V(\bar{L})$ and $V(B) = V(\bar{K}) \cap (V(\widetilde{H}) \cup V(\bar{L}))$. In fact, based on $H = \bar{L} \cup \bar{K}$ and $\widetilde{H} \subseteq H$, it follows that

$$B = (\widetilde{H} \backslash \bar{L}) \cup (\bar{K} \cap \bar{L}). \tag{5.13}$$

From the definitions of the set A and B in (5.12) and (5.13), we have

$$\begin{aligned} e_A + e_B &= e_{\bar{L} \cap \bar{K}} + e_{\widetilde{H}}, \\ v_A + v_B &= v_{\bar{L} \cap \bar{K}} + v_{\widetilde{H}}, \end{aligned}$$

and then

$$\begin{aligned} &(v_H - v_{\widetilde{H}}) - (r-1)(e_H - e_{\widetilde{H}}) \\ &= \big(v_{\bar{L}} + v_{\bar{K}}\big) - \big(v_A + v_B\big) - (r-1)\big(e_{\bar{L}} + e_{\bar{K}}\big) + (r-1)\big(e_A + e_B\big). \end{aligned} \tag{5.14}$$

Note that $V(A) \subseteq V(L)$, $\boldsymbol{f}', \boldsymbol{h}' \subseteq V(A)$ while $\boldsymbol{f}', \boldsymbol{h}' \notin \bar{L}$, thus we have $e_A \leqslant \frac{v_A - \mathbb{1}_{\{v_A \neq v_L\}}}{r-1} - 2$, which implies

$$v_A - (r-1)e_A \geqslant 2r - 2 + \mathbb{1}_{\{v_A \neq v_L\}}. \tag{5.15}$$

Similarly, since $V(B) \subseteq V(K)$, $\boldsymbol{g}'$, $\boldsymbol{h}' \subseteq V(B)$, $\boldsymbol{g}'$, $\boldsymbol{h}' \notin \bar{K}$, we also have

$$v_B - (r-1)e_B \geqslant 2r - 2 + \mathbb{1}_{\{v_B \neq v_K\}}. \tag{5.16}$$

Combined the equations shown in (5.14)-(5.16) with

$$v_{\bar{L}} - (r-1)e_{\bar{L}} = (r-1)(e_L - k) \text{ and } v_{\bar{K}} = (r-1)(e_{\bar{K}} + 2)$$

because L, $K \in \mathcal{L}$, we have the equation shown in (5.11) holds when $r \geqslant 3$. $\square$

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