

UNIFORM DIMENSION THEOREMS FOR PARABOLIC SPDES

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ABSTRACT. Consider the following p -dimensional system of Itô type stochastic PDEs,

$$\begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x) + b(u(t, x)) + \sigma(u(t, x))\xi(t, x) \\ \text{for } (t, x) \in (0, \infty) \times \mathbb{T}, \text{ subject to } u(0) \equiv u_0 \text{ on } \mathbb{T}, \end{cases}$$

where $\mathbb{T}$ denotes a given one-dimensional torus, the initial data $u_0 : \mathbb{T} \rightarrow \mathbb{R}^p$ is assumed to be fixed and non-random and in $C^{1/2}(\mathbb{T}; \mathbb{R}^p)$, and ξ denotes a p -dimensional space-time white noise. Under certain regularity conditions on b and σ , it is proved that, if $p \geq 4$, then

$$\mathbb{P}\{\dim_{\mathbb{H}} u(\{t\} \times F) = 2 \dim_{\mathbb{H}} F \ \forall \text{ compact } F \subset \mathbb{T}, t > 0\} = 1.$$

If in addition the matrix $\sigma(v)$ does not depend on $v \in \mathbb{R}^p$, and is nonsingular, then the above equality holds for all $p \geq 2$.

1. INTRODUCTION

Throughout this paper, we choose and fix a positive integer p , and consider the following p -dimensional system of Itô stochastic PDEs,¹

$$\begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x) + b(u(t, x)) + \sigma(u(t, x))\xi(t, x) \\ \text{for } (t, x) \in (0, \infty) \times \mathbb{T}, \text{ subject to } u(0) \equiv u_0 \text{ on } \mathbb{T}, \end{cases} \quad (1.1)$$

where $\mathbb{T}$ denotes a given one-dimensional torus and the initial data $u_0 : \mathbb{T} \rightarrow \mathbb{R}^p$ is assumed to be fixed and non-random and in $C^{1/2}(\mathbb{T}; \mathbb{R}^p)$.² The random forcing term ξ denotes a p -dimensional space-time white noise, equivalently, 1-D white noise on $\{1, \dots, p\} \times \mathbb{R}_+ \times \mathbb{T}$. In brief terms, this means that ξ is a centered, generalized Gaussian random field such that

$$\text{Cov}[\xi_i(t, x), \xi_j(s, y)] = \delta_0(i - j)\delta_0(t - s)\delta_0(x - y),$$

for all $i, j \in \{1, \dots, p\}$, $t, s \geq 0$, and $x, y \in \mathbb{T}$. Finally, $b : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $\sigma : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^p$ are assumed to be Lipschitz continuous. In this way, standard methods such as those in Walsh [22] can be employed to show that (1.1) is well posed and the

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¹To be concrete, we mention that, in (1.1), the quantity $\sigma(u(t, x))\xi(t, x)$ is understood as an Itô type random-matrix product whose i th coordinate can be written explicitly as $\sum_{j=1}^p \sigma_{i,j}(u(t, x))\xi_j(t, x)$.

²As is customary, for $\alpha \in (0, 1]$, $C^\alpha(\mathbb{T}; \mathbb{R}^p)$ denotes the linear space of all functions $f : \mathbb{T} \rightarrow \mathbb{R}^p$ such that $\sup_{a, b \in \mathbb{T}: a \neq b} \|f(b) - f(a)\|/|b - a|^\alpha < \infty$.

solution $(t, x) \mapsto u(t, x)$ is almost surely in $C^{a,b}(\mathbb{R}_+ \times \mathbb{T}; \mathbb{R}^p)$ for every $a \in (0, \frac{1}{4})$ and $b \in (0, \frac{1}{2})$.

Equations such as (1.1) arise in the analysis of reaction-diffusion systems, or multi-component reaction-diffusion equations; see Fife [9] for a masterly account. In this context, b denotes the sink/source terms, p = the number of components (i.e., the number of interacting reactions), ξ = the external forcing term, and σ encodes the interactions between the components and the forcing. It has been known for a long time that when ξ is a nice and smooth function, the system (1.1) can act phenomenologically differently for higher values of p than it would for instance when $p = 1$. For an early example see Turing [21]; more modern examples abound in the literature on bifurcation theory.

On one hand, the results of the present paper demonstrate that when ξ is white noise, some of the fractal behavior of the solution has the same type of phenomenological dimension dependence as one sees in the deterministic theory. On the other hand, we will see that, in stark contrast with the deterministic theory, dimension dependence can arise solely because the system “suffers from too much noise” when p is large, essentially, regardless of the details of the construction of b and σ . Before we explain our results more precisely, let us discuss a little of the requisite background.

The following is due to Dalang, Khoshnevisan, and Nualart [5, 6], except that they consider equations with Neumann boundary instead of the present setting which can be viewed as an equation with a periodic boundary. The next result can be proved in almost exactly the same manner as the results of Dalang et al. (*ibid.*). As usual, $\dim_{\text{H}}$ denotes the Hausdorff dimension.

Proposition 1.1. *Suppose, in addition, that σ and b are uniformly bounded and C^∞ and σ is uniformly elliptic, then $\mathbb{P}\{\dim_{\text{H}} u(\{t\} \times \mathbb{T}) = 2\} = 1$ for all $t > 0$ when $p \geq 3$. If $\sigma(v)$ does not depend on $v \in \mathbb{R}^p$ (additive noise), then in fact $\mathbb{P}\{\dim_{\text{H}} u(\{t\} \times \mathbb{T}) = 2\} = 1$ for all $t > 0$ when $p \geq 2$.*

One can anticipate Proposition 1.1 using the following heuristic: Recall that the random function $x \mapsto u(t, x)$ is a “smooth perturbation” of a Brownian motion for each fixed $t > 0$. This can be proved by combining the localization results of Foon-dun et al [10] with the structure theory in [13, §3]. Since Hausdorff dimension is a local quantity, one would then expect that $\dim_{\text{H}} u(\{t\} \times \mathbb{T})$ ought to be the same as $\dim_{\text{H}} B(\mathbb{T})$ when $\{B(x)\}_{x \in \mathbb{T}}$ (say) denotes a two-sided, p -dimensional Brownian motion indexed by $\mathbb{T} \cong [-1, 1]$, where $B(0) = 0$. Thus, we can anticipate Proposition 1.1 since a celebrated theorem of Paul Lévy asserts that $\dim_{\text{H}} B(\mathbb{T}) = 2$ almost surely. If we use the same heuristic but replace Lévy’s theorem with the theorem of McKean [17], then we might expect that if, in addition, σ and b are uniformly bounded and C^∞ and σ is uniformly elliptic, then

$$\mathbb{P}\{\dim_{\text{H}} u(\{t\} \times F) = 2 \dim_{\text{H}} F\} = 1 \quad \forall \text{compact } F \subset \mathbb{T}, t > 0, \quad (1.2)$$

when $p \geq 3$. And that, if $\sigma(v)$ does not depend on $v \in \mathbb{R}^p$ (additive noise), then in fact (1.2) holds for all $p \geq 2$. Because this is an assertion about the local behavior of the solution to (1.1), it should be possible to refine the method of proof of Proposition 1.1 in order to prove such a result. Instead, we plan to prove the following Theorem 1.2, though it has a very different proof. It might help to recall that, for a closed non-random set $A \subset \mathbb{R}^p$,

$$A \text{ is polar if } \mathbb{P}\{\exists(t, x) \in (0, \infty) \times \mathbb{T} : u(t, x) \in A\} = 0.$$

Theorem 1.2. *Suppose that the underlying probability space $(\Omega, \mathcal{F}, P)$ is complete. If $\{v \in \mathbb{R}^p : \inf_{x \in \mathbb{R}^d : \|x\|=1} \|\sigma(v)x\| = 0\}$ is polar for the random field u (Assumption 2.3) and $p \geq 4$, then*

$$P\{\dim_{\mathbb{H}} u(\{t\} \times F) = 2 \dim_{\mathbb{H}} F \ \forall \text{ compact } F \subset \mathbb{T}, t > 0\} = 1. \quad (1.3)$$

If in addition the matrix $\sigma(v)$ does not depend on $v \in \mathbb{R}^p$, and is nonsingular, then (1.3) holds for all $p \geq 2$.

Theorem 1.2 presents a nontrivial extension of (1.2): First of all, the polarity condition of Theorem 1.2 is essentially unimprovable and includes but is not limited to strongly elliptic σ . More significantly, the dimension formula (1.3) is valid, off a single null set, simultaneously for all compacts F and times $t > 0$, including random ones. In a sense, Theorem 1.2 can be anticipated (at least for a fixed $t > 0$) from the celebrated uniform dimension theorem of Kaufman [12] which states that

$$P\{\dim_{\mathbb{H}} B(F) = 2 \dim_{\mathbb{H}} F \ \forall \text{ compact } F \subset \mathbb{T}\} = 1 \quad \text{when } p \geq 2,$$

where B once again denotes a two-sided, p -dimensional Brownian motion indexed by $\mathbb{T}$. However, this heuristic comparison to Brownian motion does not appear to be rigorizable since, in contrast for example with (1.2), uniform dimension results cannot be based solely on local arguments.

In Conjecture 2.4 below, we will state a series of uniform dimension conjectures of a similar nature that we have no idea how to prove except in the additive case. One of them can be stated right here, as it pertains directly to the material of the Introduction.

Conjecture 1.3. The uniform dimension theorem (1.3) is valid whenever $p \geq 2$ provided that the set $\{v \in \mathbb{R}^p : \inf_{\|x\|=1} \|\sigma(v)x\| = 0\}$ is polar for u .

Suppose that $p = 1$, and consider the random compact set

$$F_1 = \{x \in \mathbb{T} : u(1, x) = 0\}.$$

If in addition b, σ are bounded and smooth and σ is strongly elliptic, then Corollary 1.7(d) of [6] implies that $P\{F_1 \neq \emptyset\} > 0$, and $\dim_{\mathbb{H}} F_1 = 1/2$ a.s. on $\{F_1 \neq \emptyset\}$, whence $P\{\dim_{\mathbb{H}} u(1, F_1) = 0 \neq 1 = 2 \dim_{\mathbb{H}} F_1\} > 0$. This yields the optimality of the preceding conjecture. And of course, the polarity condition of Conjecture 1.3 is also sharp. For instance, suppose that $\sigma(c) = 0$ for some $c \in \mathbb{R}^p$ and $u_0 \equiv c$. Then, $u(t, x) \equiv c$ for all t and x and so (1.3) fails manifestly.

We include an outline of the proof of Theorem 1.2 in §2, together with statements of more detailed results about the SPDE (1.1) in the case that it is driven by additive noise. The remainder of the paper is devoted to the proof of these results. Let us conclude the Introduction by setting forth a long series of notational conventions that will be used throughout the paper.

For every $m \in \mathbb{N}$ and $x \in \mathbb{R}^m$, the point x is written coordinatewise as $x = (x_1, \dots, x_m)$, and similarly, the i th coordinate of every function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is written as f_i and the (i, j) th coordinate of a matrix M is written as $M_{i,j}$. For every $n \times m$ matrix M we let $\|M\| = (\sum_{i,j} M_{i,j}^2)^{1/2}$ denote the Hilbert–Schmidt norm of M , so that $\|Mx\| \leq \|M\| \|x\|$ for every $x \in \mathbb{R}^m$. We also write

$$\log_+ x = \log(x \vee e) \quad \forall x \geq 0.$$

Throughout, $\mathbb{T}$ denotes the set $[-1, 1]$ endowed with addition mod 2 so that $\mathbb{T} \cong \mathbb{R}/2\mathbb{Z}$, as an abelian group. We also identify $\mathbb{T}$ with the circle group $\mathbb{S} = \{x \in$

$\mathbb{C} : \|x\| = 1\}$, endowed with multiplication on $\mathbb{C}$, using the group homomorphism $h : x \mapsto \exp(i\pi x)$. The mapping h is a 1-1 isometry when we view $\mathbb{S}$ as a manifold with Riemannian distance and yields $\text{dist}(a, b) = |a - b| \wedge (2 - |b - a|)$ for $a, b \in \mathbb{T}$. Throughout, we follow standard practice and use additive notation for $\mathbb{T}$. Thus, “+”, “−”, and “0” respectively denote the group multiplication, inversion, and identity. Similarly, we write $|a| = \text{dist}(a, 0)$ for all $a \in \mathbb{T}$. We write “ $g_1(x) \lesssim g_2(x)$ ” for all $x \in X$ ” when there exists a positive real number L such that $g_1(x) \leq Lg_2(x)$ for all $x \in X$. Alternatively, we might write “ $g_2(x) \gtrsim g_1(x)$ for all $x \in X$.” By “ $g_1(x) \asymp g_2(x)$ for all $x \in X$ ” we mean that $g_1(x) \lesssim g_2(x)$ and $g_2(x) \lesssim g_1(x)$ for all $x \in X$.

If $k \in [1, \infty)$ is a real number and Y is a random $n \times m$ matrix, then we write $\|Y\|_k = \mathbb{E}(\|Y\|^k)^{1/k}$ regardless of the values of $n, m \in \mathbb{N}$. If $\Phi = \{\Phi(x)\}_{x \in \mathbb{T}}$ is a $\mathbb{T}$ -indexed random field with values in $\mathbb{R}^p$, then for every real number $k \in [2, \infty)$ we write

$$\mathcal{S}_k(\Phi) = \sup_{a \in \mathbb{T}} \|\Phi(a)\|_k \quad \text{and} \quad \mathcal{H}_k(\Phi) = \sup_{\substack{a, b \in \mathbb{T} \\ a \neq b}} \frac{\|\Phi(b) - \Phi(a)\|_k}{|b - a|^{1/2}}. \quad (1.4)$$

For every function $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$, we write

$$\text{Lip}(f) = \sup_{\substack{a, b \in \mathbb{R}^p \\ a \neq b}} \frac{\|f(b) - f(a)\|}{\|b - a\|} \quad \text{and} \quad \mathcal{M}(f) = \sup_{v \in \mathbb{R}^p} \|f(v)\|. \quad (1.5)$$

The Fourier transform on $\mathbb{R}$ is denoted by “ $\hat{\cdot}$ ” and is normalized so that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx \quad \forall \xi \in \mathbb{R}, f \in L^1(\mathbb{R}). \quad (1.6)$$

We will denote the Fourier transform on $\mathbb{T}$ by $\mathcal{F}$, in order to distinguish it from the Fourier transform on $\mathbb{R}$, and normalize it so that

$$\mathcal{F}f(n) = \int_{-1}^1 e^{-\pi inz} f(z) dz \quad \text{and} \quad \mathcal{F}^{-1}g(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{i\pi nz} g(n), \quad (1.7)$$

for every $n \in \mathbb{N}$, $f \in L^1(\mathbb{T})$, $g \in L^1(\mathbb{Z})$, and $z \in \mathbb{T}$.

The open and closed balls in $\mathbb{R}^p$ centered at $x \in \mathbb{R}^p$ with radius $r > 0$ are respectively denoted by

$$\mathbb{B}(x, r) = \{y \in \mathbb{R}^p : \|x - y\| < r\} \quad \text{and} \quad B(x, r) = \overline{\mathbb{B}(x, r)}. \quad (1.8)$$

Throughout, we frequently refer to the set,

$$\mathcal{X} = \mathbb{R}_+ \times \mathbb{T}, \quad (1.9)$$

as *space-time*, and view it as a metric space that is endowed with the following so-called *parabolic metric* ρ :

$$\rho((s, y), (t, x)) = |s - t|^{1/4} + |x - y|^{1/2} \quad \forall (s, y), (t, x) \in \mathcal{X}. \quad (1.10)$$

The corresponding open and closed balls are denoted, respectively, by

$$\mathbb{B}_\rho(a, r) = \{b \in \mathcal{X} : \rho(a, b) < r\}, \quad B_\rho(a, r) = \{b \in \mathcal{X} : \rho(a, b) \leq r\},$$

whenever $a \in \mathcal{X}$ and $r > 0$.

There are associated notions of Hausdorff measure and Hausdorff dimension on space-time $\mathcal{X}$: Whenever $E \subset \mathcal{X}$ and $\beta \geq 0$, the β -dimensional Hausdorff measure of E with respect to the metric ρ is defined by

$$\mathcal{H}_\beta^\rho(E) = \lim_{\delta \rightarrow 0} \inf \sum_{n=1}^{\infty} (2r_n)^\beta,$$

where the inf is taken over all countable open covers $\mathbb{B}_\rho(a_1, r_1), \mathbb{B}_\rho(a_2, r_2), \dots$ of E such that $\sup_{n \geq 1} r_n \leq \delta$. The Hausdorff dimension of E with respect to ρ is defined by

$$\dim_{\mathbb{H}}^\rho E = \inf\{\beta \geq 0 : \mathcal{H}_\beta^\rho(E) = 0\} = \sup\{\beta \geq 0 : \mathcal{H}_\beta^\rho(E) = \infty\}. \quad (1.11)$$

It helps to have a slightly broader definition of Hausdorff dimension for some of our later purposes. Suppose $N \in \mathbb{N}$ is fixed and $\mathbf{d}$ is a metric on $\mathbb{R}^N$. Then, we can define the Hausdorff dimension $\dim_{\mathbb{H}}^{\mathbf{d}}$ and associated Hausdorff measure $\mathcal{H}_\beta^{\mathbf{d}}$ on the metric space $(\mathbb{R}^N, \mathbf{d})$ as above, but replace all references to ρ there by their counterparts that use the metric $\mathbf{d}$ here.

Throughout, $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the filtration generated by the Gaussian noise ξ . That is, $\mathcal{F}_t$ denotes the σ -algebra generated by all Wiener integrals of the form $\int_{[0,t] \times \mathbb{T}} f(y) \cdot \xi(\mathrm{d}s \mathrm{d}y)$ as (t, f) ranges over $(0, \infty) \times L^2(\mathbb{T}; \mathbb{R}^p)$. Without loss of generality, we may and will assume that $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions of martingale theory. In particular, every $\mathcal{F}_t$ is $\mathbb{P}$ -complete. We assume also that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete. These completeness assumptions are critical to ensure the measurability of various sets that are of interest.

2. A PROOF OUTLINE, THE ADDITIVE-NOISE CASE, AND SOME CONJECTURES

The nonlinear part of Theorem 1.2 naturally includes two very different statements. The first is that, off a single $\mathbb{P}$ -null set,

$$\dim_{\mathbb{H}} u(\{t\} \times F) \leq 2 \dim_{\mathbb{H}} F \quad \forall \text{ compact } F \subset \mathbb{T}, t > 0. \quad (2.1)$$

This statement in fact holds for all $p \geq 1$, does not require the polarity assumption of Theorem 1.2, and is an immediate consequence of the definition of Hausdorff dimension and the following version of the well-known modulus of continuity of u : For every $\alpha \in (0, \frac{1}{2})$ and $T > 0$,

$$\sup_{t \in [0, T]} \sup_{x, y \in \mathbb{T} : x \neq y} \frac{\|u(t, x) - u(t, y)\|}{|x - y|^\alpha} < \infty \quad \text{a.s.} \quad (2.2)$$

See Walsh [22, Chapter 3] for very closely related results with essentially the same proofs. Moreover, the preceding argument implies also that the event defined in (2.1) includes the event defined in (2.2). Since the probability space is complete and the event of (2.2) is measurable, then so is the event defined by (2.1). Therefore, measurability issues do not arise in this first step.

One can readily fill in the few remaining gaps to see that the preceding outline yields a complete proof of (2.1), modulo a simple and well-known covering argument, such as Proposition 2.3 of Falconer [8], and holds without need for the additional technical assumptions of Theorem 1.2. We will concentrate our efforts on proving the second implication of Theorem 1.2. Namely that, if the set $\{v \in \mathbb{R}^p : \inf_{\|x\|=1} \|\sigma(v)x\| = 0\}$ is polar for u and $p \geq 4$, then

$$\dim_{\mathbb{H}} u(\{t\} \times F) \geq 2 \dim_{\mathbb{H}} F \quad \forall \text{ compact } F \subset \mathbb{T}, t > 0, \quad (2.3)$$

off a single P-null set. Because $\dim_{\mathbb{H}}$ is countably stable [8], it suffices to prove that (2.3) holds simultaneously for all $t \in [a, b]$ and $F \subseteq B(0, c)$ where $0 < a < b$ and $c > 0$ are non-random and fixed. Then, one can try to emulate the method of Kaufman [12], introduced first in the context of Brownian motion. Namely, choose and fix some $\alpha \in (0, 2)$, and for every $n \in \mathbb{N}$ cover $B(0, c)$ as optimally as possible with finitely many balls $B(a_1, 2^{-\alpha n}), B(a_2, 2^{-\alpha n}), \dots$, and then prove that simultaneously for every subset $a_{i_1}, \dots, a_{i_N}$ of the a_j s, and for all $t \in [a, b]$, the number of inverse images of the form $\{x \in \mathbb{T} : u(t, x) \in B(a_j, 2^{-\alpha n})\}$ [$1 \leq j \leq N$] that intersect any dyadic arc in $\mathbb{T}$ grows at most polylogarithmically in 2^n . Since the underlying probability space is complete, this takes care of all measurability issues as well.

In the context of Kaufman's theorem [12], the latter intersection estimate can be carried out readily thanks to the fact that Brownian motion has stationary and independent Gaussian increments. In the present setting, the required estimates are very difficult to develop in part because of the inherent nonlinear dependence of u on the underlying noise. Thus, as a first step, we study the special case of (1.1) where it is driven by a non-degenerate additive noise. In that case, a standard stopping argument reduces the problem to the one where b is uniformly bounded. Then, an appeal to the Girsanov theorem allows us to reduce the problem to the case that $b \equiv 0$ when the solution is now a Gaussian random field. Moreover, barring the relatively easy-to-manage effects of the initial data, that Gaussian random field has all but one requisite property of Brownian motion: It only does not have independent increments! While this is a significant loss of information, it was shown by Wu and Xiao [23] and Xiao [24] that one can appeal to the theory of strong local non-determinism (SLND) in order to at least partially overcome the lack of independence; see also Berman [3], Lee and Xiao [16], and Monrad and Pitt [18]. Thus, the main impediment to a proof of the following is to develop a suitable notion of SLND. As it turns out, there are a number of variations of the notion of SLND that are available here. Once we identify the senses in which the process u has SLND, we are led to the three parts of the next theorem, each part corresponding to a different notion of SLND. Note that part (i) of Theorem 2.1 below shows the definitive form of Theorem 1.2, with optimal conditions, valid in the case of additive noise. Now recall (1.9) and (1.10).

Theorem 2.1 (Additive noise). *If σ is a constant nonsingular matrix, then:*

- (i) $\mathbb{P}\{\dim_{\mathbb{H}} u(\{t\} \times A) = 2 \dim_{\mathbb{H}} A \ \forall \text{ compact } A \subset \mathbb{T}, t > 0\} = 1 \ \forall p \geq 2;$
- (ii) $\mathbb{P}\{\dim_{\mathbb{H}} u(B \times \{x\}) = 4 \dim_{\mathbb{H}} B \ \forall \text{ compact } B \subset (0, \infty), x \in \mathbb{T}\} = 1 \ \forall p \geq 4;$
- (iii) $\mathbb{P}\{\dim_{\mathbb{H}} u(C) = \dim_{\mathbb{H}}^{\rho} C \ \forall \text{ compact } C \subset \mathcal{X}\} = 1 \ \forall p \geq 6.$

Next, we consider (1.1) in the multiplicative case that $\sigma : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^p$ is not necessarily a constant matrix, and discuss how we prove a weaker form of (2.3) in which $t > 0$ is fixed (so that the null set off which (2.3) can depend on t). The proof of the uniform-in- t result is more complicated, and discussed in detail in the forthcoming arguments. Barring the uniformity issue in t , this is the general setting under which Theorem 1.2 is posed.

We had mentioned that locally $x \mapsto u(t, x)$ is approximately a Brownian motion. Instead of that approximation, our proof hinges on the fact that locally $t \mapsto u(t, x)$ is approximately a fractional Brownian motion (fBm) of index $\frac{1}{4}$; see the unpublished manuscript by Khoshnevisan, Swanson, Xiao, and Zhang [15]. We use this approximation theorem by conditioning on everything up to time $t - \eta$ for a suitable

choice of $\eta = \eta(t) \in (0, t)$. If η is sufficiently close to t , then one can approximate $[t - \eta, t] \ni s \mapsto u(s)$ by a Gaussian process that resembles a scaled version of an fBm with index $\frac{1}{4}$; the scaled version arises since the variance process of the Gaussian approximation to $[t - \eta, t] \ni s \mapsto u(s)$ is small when $\eta \approx t$. Pitt [19] showed that fBm has the SLND property. A similar argument shows that our Gaussian approximation to $[t - \eta, t] \ni s \mapsto u(s)$ enjoys a similar property. This, and the Gaussian approximation itself, together yield the probability estimates that are needed in order to carry out a combinatorial analysis of the intersection numbers of inverse images of the form $\{x \in \mathbb{T} : u(t, x) \in B(a_j, 2^{-\alpha n})\}$ [$1 \leq j \leq N$], brought up earlier in the context of equations with additive noise. And the condition $p \geq 4$, which we believe is too strong, comes up in this part for technical reasons that roughly say that the scaling imposed by replacing $[0, t]$ by $[t - \eta, t]$ – and then conditioning on everything by time $t - \eta$ – does not do a great deal of harm, so that the approximating Gaussian random field behaves essentially as does fBm with index $\frac{1}{4}$. In this way, the analysis of the inverse images resembles that of the inverse images of fBm. Suffices it to say only that the proof of (2.3) that includes the uniformity in t rests on a similar method but uses a much more delicate form of SLND that is valid on many fine scales of t simultaneously.

Let us conclude this section by introducing notation for the singular values of the matrix $\sigma(v)$. Recall that the *singular values* of a $p \times p$ matrix M are the eigenvalues of $M'M$. As such, singular values are nonnegative.

Definition 2.2. Throughout, let $\lambda(v) = \inf_{x \in \mathbb{R}^p \setminus \{0\}} \|\sigma(v)x\|^2 / \|x\|^2$ denote the smallest singular value of $\sigma(v)$ for every $v \in \mathbb{R}^p$.

Note that the polarity condition of Theorem 1.2 can be restated, a little more succinctly, as follows.

Assumption 2.3 (Polarity). $\lambda^{-1}\{0\}$ is polar for u .

We have no idea how to prove the following, but in light of Theorem 2.1, and despite the nonlocal nature of uniform dimension results, we feel that the following is likely to be true.

Conjecture 2.4. Theorem 2.1(ii) and (iii) are valid in the presence of multiplicative noise provided that $\lambda^{-1}\{0\}$ is polar for u .

Let us conclude this section by presenting the next observation which readily implies that λ is locally Lipschitz on the open set $\mathbb{R}^p \setminus \lambda^{-1}\{0\}$.

Lemma 2.5. $\lambda^{1/2}$ is Lipschitz continuous on $\mathbb{R}^p$.

This must be well known. But the proof is so short that it might be easier to simply present the proof.

Proof. For every $v \in \mathbb{R}^p$, by compactness there exists $x(v) \in \mathbb{R}^p$ such that $\|x(v)\| = 1$ and $\|\sigma(v)x(v)\|^2 = \lambda(v)$. Thus,

$$\sqrt{\lambda(v)} = \|\sigma(v)x(v)\| \geq \|\sigma(w)x(v)\| - \text{Lip}(\sigma)\|v - w\| \geq \sqrt{\lambda(w)} - \text{Lip}(\sigma)\|v - w\|,$$

valid for every $v, w \in \mathbb{R}^p$. Reverse the roles of v and w to finish the proof. $\square$

3. THE GAUSSIAN CASE

In this section, we study the specialization of the SPDE (1.1) to the case that $\sigma(v)$ is equal to the $p \times p$ identity matrix $\mathbf{I}$ for all $v \in \mathbb{R}^p$, and that $u_0 \equiv 0$. Because of the important role of this specialization, we reserve the symbol H – for “heat” – for the solution to (1.1) in that case. In other words,

$$\begin{cases} \partial_t H = \partial_x^2 H + \xi & \text{on } (0, \infty) \times \mathbb{T}, \\ \text{subject to } H(0) = 0, \end{cases} \quad (3.1)$$

where $\mathbb{0} : \mathbb{T} \rightarrow \mathbb{R}^p$ takes the value $0 \in \mathbb{R}^p$ everywhere on $\mathbb{T}$. The unique (mild) solution to (3.1) is the following Wiener-integral process,

$$H(0, x) = 0 \quad \text{and} \quad H(t, x) = \int_{(0, t) \times \mathbb{T}} G_{t-s}(x, y) \xi(ds dy), \quad (3.2)$$

for all $t > 0$ and $x \in \mathbb{T}$, where

$$G_r(a, b) = \frac{1}{\sqrt{4\pi r}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(a-b+2n)^2}{4r}\right) \quad [r > 0, a, b \in \mathbb{T}] \quad (3.3)$$

denotes the heat kernel for $\partial_t - \partial_x^2$ on $(0, \infty) \times \mathbb{T}$. See da Prato and Zabczyk [4] and Walsh [22]. We pause to observe that the random function $H = (H_1, \dots, H_p)$ is made of p i.i.d. coordinate processes $H_1, \dots, H_p$.

Our analysis of the random field H will rely on various heat-kernel estimates for the heat kernel G that are closely related to some of the technical estimates in Khoshnevisan, Kim, Mueller, and Shiu [14]. For instance, Lemma B.1 and Remark B.2 of [14] together assert that

$$\frac{1}{4} \max(t^{-1/2}, 1) \leq \sup_{x, y \in \mathbb{T}} G_t(x, y) \leq 2 \max(t^{-1/2}, 1) \quad \forall t > 0. \quad (3.4)$$

Next we present three lemmas that tighten up some of the other heat-kernel estimates of [14], improving them to what we believe are their respective essentially-optimal forms. We have resisted the temptation of deriving matching lower bounds as we will not need them in the sequel.

Lemma 3.1. *For every fixed $T > 0$,*

$$\int_0^t ds \int_{\mathbb{T}} dy [G_s(x, y) - G_s(z, y)]^2 \lesssim \sqrt{t} \wedge |x - z|,$$

uniformly for all $x, z \in \mathbb{T}$ and $t \in (0, T]$.

Proof. We can reiterate (3.3) as follows: For all $a, b \in \mathbb{T}$, $v \in \mathbb{R}$, and $s > 0$,

$$G_s(a, b) = \sum_{n=-\infty}^{\infty} \phi_s(a - b + 2n),$$

where

$$\phi_s(v) = \frac{\exp(-v^2/(4s))}{(4\pi s)^{1/2}} \quad (3.5)$$

denotes the heat kernel for $\partial_t - \partial_x^2$ on $\mathbb{R}$ (vs. $\mathbb{T}$). We can use the semigroup property of the heat kernel in order to see that

$$\begin{aligned} \int_{\mathbb{T}} [G_s(x, y) - G_s(z, y)]^2 dy &= 2G_{2s}(0, 0) - 2G_{2s}(x, z) \\ &= 2 \sum_{n=-\infty}^{\infty} [\phi_{2s}(2n) - \phi_{2s}(x - z + 2n)] = \sum_{n=-\infty}^{\infty} \left(e^{-2\pi^2 s n^2} - e^{\pi i(x-z)n - 2\pi^2 s n^2} \right) \\ &= 2 \sum_{n=1}^{\infty} e^{-2\pi^2 s n^2} (1 - \cos(\pi(x-z)n)), \end{aligned}$$

where we appealed to the Poisson summation formula (Lemma A.3) in order to deduce the third identity. Thus,

$$\begin{aligned} \int_0^t ds \int_{\mathbb{T}} dy [G_s(x, y) - G_s(z, y)]^2 &\leq 2 \sum_{n=1}^{\infty} \frac{1 - e^{-2\pi^2 t n^2}}{2\pi^2 n^2} (2 \wedge (\pi^2(x-z)^2 n^2)) \\ &\lesssim \left(\sum_{n=1}^{\infty} \frac{1 \wedge (t n^2)}{n^2} \right) \wedge \left(\sum_{n=1}^{\infty} \frac{1 \wedge ((x-z)^2 n^2)}{n^2} \right) \lesssim \sqrt{t} \wedge |x-z| \quad (\text{Lemma A.2}), \end{aligned}$$

uniformly for all $t > 0$ and $x, z \in \mathbb{T}$. This does the job. $\square$

Lemma 3.2. *For every $T > 0$ fixed,*

$$\int_0^r ds \int_{\mathbb{T}} dy [G_{t-s}(x, y) - G_{r-s}(x, y)]^2 \lesssim \sqrt{t-r},$$

uniformly for all $x \in \mathbb{T}$ and $0 < r < t \leq T$.

Proof. Thanks to the semigroup property of the heat kernel,

$$\begin{aligned} \int_{\mathbb{T}} [G_{t-r+s}(x, y) - G_s(x, y)]^2 dy &= G_{2(t-r+s)}(0, 0) + G_{2s}(0, 0) - 2G_{t-r+2s}(0, 0) \\ &= \sum_{n=-\infty}^{\infty} \phi_{2(t-r+s)}(2n) + \sum_{n=-\infty}^{\infty} \phi_{2s}(2n) - 2 \sum_{n=-\infty}^{\infty} \phi_{t-r+2s}(2n). \end{aligned}$$

Then, by the Poisson summation formula (Lemma A.3), we have

$$\begin{aligned} \int_{\mathbb{T}} [G_{t-r+s}(x, y) - G_s(x, y)]^2 dy &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(e^{-2\pi^2(t-r+s)n^2} + e^{-2\pi^2 s n^2} - 2e^{-2\pi^2(t-r+2s)n^2} \right) \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(e^{-\pi^2(t-r+s)n^2} - e^{-\pi^2 s n^2} \right)^2 = \sum_{n=1}^{\infty} e^{-2\pi^2 s n^2} \left(1 - e^{-\pi^2(t-r)n^2} \right)^2, \end{aligned}$$

for all $t > r > 0$, $s \geq 0$ and $x \in \mathbb{T}$. Hence, a change of variable yields that

$$\begin{aligned} \int_0^r ds \int_{\mathbb{T}} dy [G_{t-s}(x, y) - G_{r-s}(x, y)]^2 &= \int_0^r ds \int_{\mathbb{T}} [G_{t-r+s}(x, y) - G_s(x, y)]^2 dy \\ &= \sum_{n=1}^{\infty} \frac{1 - e^{-2\pi^2 r n^2}}{2\pi^2 n^2} \left(1 - e^{-\pi^2(t-r)n^2} \right)^2 \lesssim \sum_{n=1}^{\infty} \frac{1 \wedge ((t-r)^2 n^4)}{n^2} \lesssim \sqrt{|t-r|}, \end{aligned}$$

where the last inequality holds by Lemma A.2. The proof is complete. $\square$

Next, we present a result about the size of certain heat integrals of powers of the “parabolic distance” $|t - s|^{1/2} + |x - y|$ between space-time points (t, x) and (s, y) .

Lemma 3.3. *For all $q \in (0, 1]$ and $T > 0$ fixed,*

$$\int_0^t ds \int_{\mathbb{T}} dy [G_s(x, y)]^2 [(t - s)^{q/2} + |y - x|^q] \lesssim t^{(1+q)/2},$$

uniformly for every $t \in (0, T]$ and $x \in \mathbb{T}$.

Proof. An inspection of (3.3) tells us that the integral of the lemma is independent of $x \in \mathbb{T}$. Therefore, we consider the case that $x = 0$. Since $\int_{\mathbb{T}} [G_{t-s}(0, y)]^2 dy = G_{2(t-s)}(0, 0) \lesssim (t-s)^{-1/2}$ uniformly for all $t \in (0, T]$ [see (3.4)], we change variables in order to deduce the inequality

$$\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(0, y)]^2 s^{q/2} \lesssim t^{(1+q)/2},$$

uniformly for $t \in (0, T]$. It remains to establish the same upper bound when the integrand $s^{q/2}$ is replaced by $|y|^q$. With this aim in mind, note that $|w + 2n| \geq |n|$ uniformly for every $n \in \mathbb{Z} \setminus \{0\}$ and $-1 \leq w \leq 1$. Therefore, we may appeal to (3.5) to deduce the heat-kernel estimate,

$$|G_r(a, b) - \phi_r(b - a)| \leq \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{-n^2/(4r)}}{\sqrt{4\pi r}} \leq \int_0^\infty \frac{\exp\{-w^2/(4r)\}}{\sqrt{\pi r}} dw \leq 2,$$

valid for all $a, b \in \mathbb{T}$ and $r > 0$. Consequently, we may write

$$\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(0, y)]^2 |y|^q \lesssim \int_0^t \frac{ds}{s} \int_{-\infty}^\infty dy [\phi_1(y/\sqrt{s})]^2 |y|^q + t,$$

and change variables in the integral in order to conclude the proof. $\square$

It is well known that the variance of every coordinate of $H(t, x)$ is of order $\sqrt{t}$ when $t \ll 1$. The next result shows that the said variance is in fact nearly $\sqrt{t/\pi}$ when $t \ll 1$.

Lemma 3.4. *$\sqrt{t/\pi} \leq \text{Var } H_1(t, x) \leq \sqrt{t/\pi} + (t/2)$ for all $t \geq 0$ and $x \in \mathbb{T}$.*

Proof. Thanks to stationarity and symmetry, we may and will assume that $x = 0$. Now, the semigroup property and the symmetry of the heat kernel together yield $\text{Var } H_1(t, 0) = \int_0^t ds \int_{\mathbb{T}} dy [G_s(0, y)]^2 = \int_0^t G_{2s}(0, 0) ds$, which can be simplified as

$$\int_0^t \frac{ds}{\sqrt{4\pi s}} + 2 \sum_{n=1}^\infty \int_0^t e^{-n^2/s} \frac{ds}{\sqrt{4\pi s}} = \sqrt{\frac{t}{\pi}} + 2 \sum_{n=1}^\infty \int_0^t e^{-n^2/s} \frac{ds}{\sqrt{4\pi s}}.$$

This implies the result since the preceding infinite sum is on one hand ≥ 0 and on the other hand bounded above by $\int_0^\infty dw \int_0^t \exp(-w^2/s) ds / \sqrt{4\pi s} = t/4$. This proves the lemma. $\square$

Our next result presents a way to quantify the notion that $H(t, x)$ and $H(t, z)$ are close to being independent when $t \ll 1$ and $|x - z| \gg \sqrt{t}$.

Lemma 3.5. *Uniformly for all $t > 0$ and $x, z \in \mathbb{T}$,*

$$\text{Cov}[H_1(t, x), H_1(t, z)] \lesssim \max(\sqrt{t}, t) \exp\left(-\frac{|x - z|^2}{8t}\right).$$

Consequently, there exists $c > 1$ such that

$$\begin{aligned} 0 &< \inf_{t \in (0,1]} \inf_{|x-z| \geq c\sqrt{t}} t^{-1/2} \text{Var}[H_1(t, x) - H_1(t, z)] \\ &\leq \sup_{t \in (0,1]} \sup_{x, z \in \mathbb{T}} t^{-1/2} \text{Var}[H_1(t, x) - H_1(t, z)] < \infty. \end{aligned}$$

Proof. Thanks to stationarity, we may (and will) assume without loss of generality, that $z = 0$. Now, for every $t > 0$ and $x \in \mathbb{T}$, we invoke the semigroup property of G to find that $\mathcal{C} = \text{Cov}[H_1(t, x), H_1(t, 0)]$ satisfies

$$\mathcal{C} = \int_0^t G_{2s}(0, x) ds \leq \sum_{n=0}^{\infty} \int_0^t \exp\left(-\frac{x^2 - 4n|x| + 4n^2}{8s}\right) \frac{ds}{\sqrt{2\pi s}};$$

see (3.3). Therefore,

$$\begin{aligned} \mathcal{C} &\leq \int_0^t \exp\left(-\frac{x^2}{8s}\right) \frac{ds}{\sqrt{2\pi s}} + \sum_{n=1}^{\infty} \int_0^t \exp\left(-\frac{x^2 + 4n(n-1)}{8s}\right) \frac{ds}{\sqrt{2\pi s}} \\ &\leq \exp\left(-\frac{x^2}{8t}\right) \left[\sqrt{2t} + \sum_{n=1}^{\infty} \int_0^t \exp\left(-\frac{n(n-1)}{2s}\right) \frac{ds}{\sqrt{s}} \right]. \end{aligned}$$

We may consider the sum before the integral. Since

$$\sum_{n=2}^{\infty} \exp\left(-\frac{n(n-1)}{2s}\right) \leq \sum_{n=2}^{\infty} \exp\left(-\frac{n^2}{4s}\right) \leq \int_0^{\infty} \exp\left(-\frac{y^2}{4s}\right) dy \propto \sqrt{s},$$

uniformly for all $s \in (0, 1]$, the lemma follows. $\square$

Armed with the preceding technical lemmas, we are ready to present one of the main results of this section. The following proposition asserts that the Gaussian random field $H = \{H(t, x)\}_{(t,x) \in (0,\infty) \times \mathbb{T}}$ is strongly locally nondeterministic [3, 18, 24] and provides sharp bounds on conditional variances for H . See [16, Proposition 5.1] and [11, Section 3.2] for similar results.

Proposition 3.6. *For every $T > 0$,*

$$\begin{aligned} \text{Var}(H_1(t, x) \mid H_1(t_1, x_1), \dots, H_1(t_m, x_m)) \\ \asymp \min \left\{ \sqrt{t}, \min_{1 \leq j \leq m} \left(|t - t_j|^{1/2} + |x - x_j| \right) \right\}, \end{aligned} \quad (3.6)$$

uniformly for all $m \in \mathbb{N}$ and $(t, x), (t_1, x_1), \dots, (t_m, x_m) \in (0, T] \times \mathbb{T}$.

Proof. Recall that for any centered Gaussian vector $(Z, Z_1, \dots, Z_m)$,

$$\text{Var}(Z \mid Z_1, \dots, Z_m) = \inf_{a_1, \dots, a_m \in \mathbb{R}} \left\| Z - \sum_{j=1}^m a_j Z_j \right\|_2^2 \leq \min_{0 \leq j \leq m} \|Z - Z_j\|_2^2, \quad (3.7)$$

where $Z_0 = 0$. This implies that $\text{Var}(Z \mid Z_1, \dots, Z_m)$ is bounded above by

$$\min \left\{ \text{Var} H_1(t, x), \min_{1 \leq j \leq m} \mathbb{E} \left(|H_1(t, x) - H_1(t_j, x_j)|^2 \right) \right\}.$$

Thus, Lemmas 3.1, 3.2, and 3.4 together yield the upper bound in (3.6).

In order to prove the lower bound in (3.6), let us choose and fix $m \in \mathbb{N}$, $(t, x), (t_1, x_1), \dots, (t_m, x_m) \in (0, T] \times \mathbb{T}$ as well as numbers $a_1, \dots, a_m \in \mathbb{R}$, and define

$$r = (\sqrt{T} \vee 1)^{-1} \min \left\{ \sqrt{t}, \min_{1 \leq j \leq m} \left(|t - t_j|^{1/2} \vee |x - x_j| \right) \right\}. \quad (3.8)$$

Note that $r \leq \sqrt{t/T} \leq 1$. We may and will assume that $r > 0$, for, otherwise there is nothing to prove. Thanks to (3.2),

$$\begin{aligned} \mathfrak{E} &:= \mathbb{E} \left(\left| H_1(t, x) - \sum_{j=1}^m a_j H_1(t_j, x_j) \right|^2 \right) \\ &= \int_{-\infty}^{\infty} ds \int_{\mathbb{T}} dy \left| G_{t-s}(x, y) \mathbb{1}_{[0, t]}(s) - \sum_{j=1}^m a_j G_{t_j-s}(x_j, y) \mathbb{1}_{[0, t_j]}(s) \right|^2 \\ &= \sum_{j, k=0}^m \alpha_j \alpha_k \int_0^{t_j \wedge t_k} ds \int_{\mathbb{T}} dy G_{t_j-s}(x_j, y) G_{t_k-s}(x_k, y), \end{aligned}$$

where $t_0 = t$, $x_0 = x$, $\alpha_0 = 1$, and $\alpha_j = -a_j$ for $j = 1, \dots, m$. Thanks to the semigroup property of the heat kernel,

$$\begin{aligned} \int_{\mathbb{T}} G_{t_j-s}(x_j, y) G_{t_k-s}(x_k, y) dy &= G_{t_j+t_k-2s}(x_j, x_k) \\ &= \sum_{n=-\infty}^{\infty} \phi_{t_j+t_k-2s}(x_j - x_k - 2n). \end{aligned}$$

According to the Poisson summation formula (Lemma A.3),

$$\sum_{n=-\infty}^{\infty} \phi_{t_j+t_k-2s}(x_j - x_k - 2n) = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{\pi i n(x_j - x_k) - \pi^2 n^2(t_j + t_k - 2s)}.$$

Consequently,

$$\begin{aligned} \mathfrak{E} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{j, k=0}^m \alpha_j \alpha_k e^{\pi i n(x_j - x_k)} \int_0^{t_j \wedge t_k} e^{-\pi^2 n^2(t_j + t_k - 2s)} ds \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{j=0}^m \alpha_j e^{\pi i n x_j} e^{-\pi^2 n^2(t_j - s)} \mathbb{1}_{[0, t_j]}(s) \right|^2 ds \\ &= (4\pi)^{-1} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{j=0}^m \alpha_j e^{\pi i n x_j} \frac{e^{-i\tau t_j} - e^{-\pi^2 n^2 t_j}}{\pi^2 n^2 - i\tau} \right|^2 d\tau \quad (\text{Plancherel}) \\ &= (4\pi)^{-1} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \left(e^{-i\tau t} - e^{-\pi^2 n^2 t} \right) - \right. \\ &\quad \left. - \sum_{j=1}^m a_j e^{\pi i n(x_j - x)} \left(e^{-i\tau t_j} - e^{-\pi^2 n^2 t_j} \right) \right|^2 \frac{d\tau}{\pi^4 n^4 + \tau^2}. \end{aligned} \quad (3.9)$$

In order to derive a lower bound for (3.9), we employ Fourier transforms for functions on $\mathbb{T}$. Recall that we denote the Fourier transform on $\mathbb{T}$ by $\mathcal{F}$ in order to distinguish it from the Fourier transform on $\mathbb{R}$; see (1.6) and (1.7).

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth, symmetric test function supported in $[-\frac{1}{2}, \frac{1}{2}]$ with $\psi(0) = 1$. Let us write ψ_r for its rescaled version

$$\psi_r(y) = \psi(r^{-1}y) \quad \text{for all } y \in \mathbb{R}. \quad (3.10)$$

Display (3.10) defines a function ψ_r on $\mathbb{R}$. We now define a function $\Psi_r : \mathbb{T} \rightarrow \mathbb{R}$ on $\mathbb{T}$ by restricting it as follows:

$$\Psi_r(x) = \psi_r(x) \quad \text{for all } x \in [-1, 1]. \quad (3.11)$$

Recall that $\mathcal{F}$ denotes the Fourier transform on the torus, and consider the following expression:

$$\begin{aligned} I &= \sum_{n=-\infty}^{\infty} \mathcal{F}\Psi_r(n) \int_{-\infty}^{\infty} d\tau \times \\ &\quad \times \left[\left(e^{-i\tau t} - e^{-\pi^2 n^2 t} \right) - \sum_{j=1}^m a_j e^{\pi i n(x_j - x)} \left(e^{-i\tau t_j} - e^{-\pi^2 n^2 t_j} \right) \right] e^{i\tau t} \hat{\psi}_{r,2}(\tau) \\ &= \sum_{n=-\infty}^{\infty} \hat{\psi}_r(\pi n) \int_{-\infty}^{\infty} d\tau \times \\ &\quad \times \left[\left(e^{-i\tau t} - e^{-\pi^2 n^2 t} \right) - \sum_{j=1}^m a_j e^{\pi i n(x_j - x)} \left(e^{-i\tau t_j} - e^{-\pi^2 n^2 t_j} \right) \right] e^{i\tau t} \hat{\psi}_{r,2}(\tau); \end{aligned} \quad (3.12)$$

see (1.7) and (3.11) for the second equality which holds since $0 < r \leq 1$. Because $\hat{\psi}_r(y) = r\hat{\psi}(ry)$ for all $y \in \mathbb{R}$ and $r > 0$, and since $\hat{\psi}$ is a function of rapid decrease, the sum in (3.12) converges absolutely. In particular, we may appeal to the inverse Fourier transform on $\mathbb{R}$ to rewrite I as follows:

$$\begin{aligned} I &= 2\pi \sum_{n=-\infty}^{\infty} \mathcal{F}\Psi_r(n) \left[\left(\psi_{r,2}(0) - \psi_{r,2}(t) e^{-\pi^2 n^2 t} \right) \right. \\ &\quad \left. - \sum_{j=1}^m a_j e^{\pi i n(x_j - x)} \left(\psi_{r,2}(t - t_j) - \psi_{r,2}(t) e^{-\pi^2 n^2 t_j} \right) \right]. \end{aligned}$$

Next, we make two more observations about the quantity I .

Firstly, observe that $t/r^2 \geq 1$ by the definition (3.8) of r . This together with the fact that ψ is supported in $[-\frac{1}{2}, \frac{1}{2}]$ implies that $\psi_{r,2}(t) = 0$, and hence

$$\begin{aligned} I &= 2\pi \sum_{n=-\infty}^{\infty} \mathcal{F}\Psi_r(n) \left[1 - \sum_{j=1}^m a_j e^{\pi i n(x_j - x)} \psi_{r,2}(t - t_j) \right] \\ &= 4\pi \left[\Psi_r(0) - \sum_{j=1}^m a_j \Psi_r(x_j - x) \psi_{r,2}(t - t_j) \right], \end{aligned}$$

where we have used (1.7) to obtain the last equality. Furthermore, according to the definition of r , $|t - t_j| \geq r^2$ or $|x - x_j| \geq r$ for every $j \in \{1, \dots, m\}$. This implies

that $\Psi_r(x_j - x)\psi_{r,2}(t - t_j) = 0$, and hence,

$$I = 4\pi\Psi_r(0) = 4\pi. \quad (3.13)$$

Our second observation about the quantity I is the following, which is a consequence of the preceding and the Cauchy–Schwarz inequality:

$$\begin{aligned} |I|^2 &\leq 4\pi\mathbb{E} \left(\left| H_1(t, x) - \sum_{j=1}^m a_j H_1(t_j, x_j) \right|^2 \right) \times \\ &\quad \times \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{\psi}_r(\pi n)|^2 |\hat{\psi}_{r,2}(\tau)|^2 (\pi^4 n^4 + \tau^2) d\tau. \end{aligned}$$

Since $\hat{\psi}_s(a) = s\hat{\psi}(sa)$ for all $s > 0$ and $a \in \mathbb{R}$, and $\hat{\psi}$ is a function of rapid decrease, we have

$$\begin{aligned} &\sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{-\infty}^{\infty} |\hat{\psi}_r(\pi n)|^2 |\hat{\psi}_{r,2}(\tau)|^2 \pi^4 n^4 d\tau \\ &\lesssim \sum_{n=1}^{\infty} \frac{r^2 n^4}{1 + r^6 n^6} \int_{-\infty}^{\infty} \frac{r^4 d\tau}{1 + r^4 \tau^2} \lesssim r \int_{-\infty}^{\infty} \frac{d\tau}{1 + r^4 \tau^2} = r^{-1} \int_{-\infty}^{\infty} \frac{d\tau}{1 + \tau^2}, \end{aligned}$$

where in the second inequality we have used the following estimate

$$\sum_{n=1}^{\infty} \frac{n^4}{1 + r^6 n^6} = \sum_{n \leq 1/r} \frac{n^4}{1 + r^6 n^6} + \sum_{n > 1/r} \frac{n^4}{1 + r^6 n^6} \leq r^{-5} + \int_{1/r}^{\infty} \frac{y^4}{1 + r^6 y^6} dy \asymp r^{-5}.$$

Similar arguments yield

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} |\hat{\psi}_r(\pi n)|^2 \int_{-\infty}^{\infty} |\hat{\psi}_{r,2}(\tau)|^2 \tau^2 d\tau = 2 \int_{\mathbb{T}} dz |\Psi_r(z)|^2 \int_{-\infty}^{\infty} |\hat{\psi}_{r,2}(\tau)|^2 \tau^2 d\tau \\ &\lesssim \int_{-\infty}^{\infty} dy |\psi_r(y)|^2 \int_{-\infty}^{\infty} \frac{r^4 \tau^2 d\tau}{1 + r^8 \tau^4} = r^{-1} \int_{-\infty}^{\infty} dy |\psi(y)|^2 \int_{-\infty}^{\infty} \frac{\tau^2 d\tau}{1 + \tau^4}, \end{aligned}$$

where the implied constants in the preceding two display depend only on ψ and do not depend on $m, t, t_1, \dots, t_m, x, x_1, \dots, x_m, a_1, \dots, a_m, r$. It follows from the preceding calculations that

$$|I|^2 \lesssim \mathbb{E} \left(\left| H_1(t, x) - \sum_{j=1}^m a_j H_1(t_j, x_j) \right|^2 \right) \times r^{-1}, \quad (3.14)$$

where the implied constant depends only on ψ . Finally, we combine (3.13) and (3.14), and use (3.7) to find that

$$\begin{aligned} &\text{Var}(H_1(t, x) \mid H_1(t_1, x_1), \dots, H_1(t_m, x_m)) \\ &= \inf_{a_1, \dots, a_m \in \mathbb{R}} \mathbb{E} \left(\left| H_1(t, x) - \sum_{j=1}^m a_j H_1(t_j, x_j) \right|^2 \right) \gtrsim r, \end{aligned}$$

with an implied constant that depends only on the function ψ – which was simply an artifact of the proof and does not have any bearing on the process H . In particular, the implied constant above does not depend on any of the parameters

$m, t, t_1, \dots, t_m, x, x_1, \dots, x_m, r$. Recall the definition of r in (3.8) and use the elementary inequality $a \vee b \geq (a + b)/2$ for all $a, b \geq 0$ to finish the proof. $\square$

4. LINEARIZATION

We will need to consider more general initial data than constants: Let us say that the initial data $h : \mathbb{T} \rightarrow \mathbb{R}^p$ is random but independent of ξ .³ We will tacitly assume that h has regularity as well; see for example the upper bound of Lemma 4.1 where we will need $\mathcal{S}_k(h) < \infty$ unless the lemma is vacuous. The same happens in Lemmas 4.2, 4.3, 4.4 and 4.5. In any case, there is no problem with existence, regularity, etc., and the solution can be written as a system of mild integral equations,

$$\begin{aligned} u_i(t, x) &= (\mathcal{G}_t h_i)(x) + \int_{(0, t) \times \mathbb{T}} G_{t-s}(x, y) b_i(u(s, y)) \, ds \, dy \\ &\quad + \sum_{j=1}^p \int_{(0, t) \times \mathbb{T}} G_{t-s}(x, y) \sigma_{i,j}(u(s, y)) \, \xi_j(ds \, dy), \end{aligned} \quad (4.1)$$

for $i = 1, \dots, p$, where $\{\mathcal{G}_t\}_{t \geq 0}$ denotes the semigroup associated to G [see (3.3)]. That is,

$$(\mathcal{G}_0 \psi)(x) = \psi(x) \quad \text{and} \quad (\mathcal{G}_t \psi)(x) = \int_{\mathbb{T}} G_t(x, y) \psi(y) \, dy,$$

for all $t > 0$ and $x \in \mathbb{T}$ for all measurable scalar functions $\psi : \mathbb{T} \rightarrow \mathbb{R}$ for which the above integral converges absolutely.

The primary goal of this section is to prove that when t is very small, the conditional law of $\{u(t, x)\}_{x \in \mathbb{T}}$ is approximately that of a certain p -dimensional Gaussian random field, given the initial data h ; see Lemma 4.5 and its seemingly stronger, but in fact essentially equivalent, consequence (7.2) for a precise formulation. When $p = 1$, such a result appears in various forms in da Prato and Zabczyk [4] and Walsh [22], and many subsequent papers. Here, we prove improvements of these results which are valid for general $p \geq 1$ and will appear in essentially-optimal form, have essentially-optimal error-rate estimates, and possess optimal assumptions on the diffusion coefficient σ .

For the following two lemmas, it might help to recall that $\mathcal{S}_k(\cdot)$, $\mathcal{H}_k(\cdot)$, and $\mathcal{M}(\cdot)$ are defined in (1.4) and (1.5), respectively.

Lemma 4.1. *Choose and fix some $T > 0$. Then, the following is valid*

$$\sup_{x \in \mathbb{T}} \|u(t, x)\|_k \lesssim \mathcal{S}_k(h) + \mathcal{M}(b) \sqrt{t} \left(t^{1/4} \vee t^{1/2} \right) + \mathcal{M}(\sigma) \sqrt{k} \left(t^{1/4} \vee t^{1/2} \right),$$

where the implied constant does not depend on σ , h , $t \in [0, T]$, or $k \in [2, \infty)$.

Proof. The lemma has content iff $\mathcal{S}_k(h) \vee \mathcal{M}(b) \vee \mathcal{M}(\sigma) < \infty$, a condition which we assume. Because $\|(\mathcal{G}_t h)(x)\|_k \leq \mathcal{S}_k(h)$, the Minkowski inequality, Cauchy–Schwarz inequality, and a suitable application of the Burkholder–Davis–Gundy inequality

³We will use the symbol h consistently instead of u_0 , in this section, in order to remind that the initial data can be random though independent of ξ .

(Lemma A.4 below) together show that $\|u(t, x)\|_k$ is bounded from above by

$$\begin{aligned}
& \mathcal{S}_k(h) + \int_{(0,t) \times \mathbb{T}} \|G_{t-s}(x, y)b(u(s, y))\|_k \, ds \, dy \\
& \quad + \left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y)\sigma(u(s, y)) \, \xi(ds \, dy) \right\|_k \\
& \leq \mathcal{S}_k(h) + m(b)\sqrt{2t} \left[\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \right]^{1/2} \\
& \quad + m(\sigma)\sqrt{4kp} \left[\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \right]^{1/2} \\
& = \mathcal{S}_k(h) + m(b)\sqrt{2t} \left[\int_0^t G_{2s}(0, 0) \, ds \right]^{1/2} + m(\sigma)\sqrt{4kp} \left[\int_0^t G_{2s}(0, 0) \, ds \right]^{1/2},
\end{aligned}$$

thanks to the semigroup property of the heat kernel. In particular, (3.4) yields the lemma. $\square$

Lemma 4.2. *For every $T > 0$,*

$$\|u(t, x) - u(t, z)\|_k \lesssim \mathcal{H}_k(h)\sqrt{|x - z|} + \left(m(b) + m(\sigma)\sqrt{k}\right) \left[t^{1/4} \wedge \sqrt{|x - z|}\right],$$

uniformly in $\sigma, h, t \in [0, T]$, $x, z \in \mathbb{T}$, and $k \in [2, \infty)$.

Proof. Recall (1.4). Without loss of generality, we assume that $\mathcal{H}_k(h) \vee m(b) \vee m(\sigma) < \infty$. Minkowski's inequality shows that, for every $t > 0$, $x, z \in \mathbb{T}$, and $k \in [2, \infty)$,

$$\|(\mathcal{G}_t h)(x) - (\mathcal{G}_t h)(z)\|_k \leq \int_{\mathbb{T}} G_t(0, y) \|h(y + x) - h(y + z)\|_k \, dy \leq \mathcal{H}_k(h)\sqrt{|x - z|}.$$

Therefore, a suitable appeal to the BDG inequality [Lemma A.4] shows that

$$\begin{aligned}
& \|u(t, x) - u(t, z)\|_k \\
& \leq \mathcal{H}_k(h)\sqrt{|x - z|} + \int_0^t ds \int_{\mathbb{T}} dy |G_{t-s}(x, y) - G_{t-s}(z, y)| \|b(u(s, y))\|_k \\
& \quad + \sqrt{4kp} \left[\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y) - G_{t-s}(z, y)]^2 \|\sigma(u(s, y))\|_k^2 \right]^{1/2} \\
& \leq \mathcal{H}_k(h)\sqrt{|x - z|} + \left[m(b)\sqrt{2t} + m(\sigma)\sqrt{4kp}\right] \\
& \quad \times \left[\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y) - G_{t-s}(z, y)]^2 \right]^{1/2}.
\end{aligned}$$

Lemma 3.1 completes the proof. $\square$

Lemma 4.3. *For every $T > 0$,*

$$\sup_{x \in \mathbb{T}} \|u(t, x) - u(r, x)\|_k \lesssim \left(\mathcal{H}_k(h) + m(b) + m(\sigma)\sqrt{k}\right) |t - r|^{1/4},$$

uniformly in $t, r \in [0, T]$, σ, h , and $k \in [2, \infty)$.

Proof. Recall (1.4) and assume without loss of generality that $\mathcal{H}_k(h) \vee m(b) \vee m(\sigma) < \infty$. Choose and fix $T > 0$. Let $t > r$ in $[0, T]$, and write for every $x \in \mathbb{T}$,

$$\|u(t, x) - u(r, x)\|_k = I_1 + I_2 + I_3, \quad (4.2)$$

where

$$\begin{aligned} I_1 &= (\mathcal{G}_t h)(x) - (\mathcal{G}_r h)(x), \\ I_2 &= \int_{(0, r) \times \mathbb{T}} [G_{t-s}(x, y) - G_{r-s}(x, y)] b(u(s, y)) \, ds \, dy \\ &\quad + \int_{(0, r) \times \mathbb{T}} [G_{t-s}(x, y) - G_{r-s}(x, y)] \sigma(u(s, y)) \, \xi(ds \, dy), \\ I_3 &= \int_{(r, t) \times \mathbb{T}} G_{t-s}(x, y) b(u(s, y)) \, ds \, dy + \int_{(r, t) \times \mathbb{T}} G_{t-s}(x, y) \sigma(u(s, y)) \, \xi(ds \, dy). \end{aligned}$$

Recall that $\mathcal{G}$ can be identified with the semigroup for a Brownian motion $\{b(s)\}_{s \geq 0}$ on the torus. Because time increments of b have the same laws as the same time increments but for a Brownian motion on $\mathbb{R}$, (1.4) implies that for $k \in [2, \infty)$,

$$\begin{aligned} \|I_1\|_k &\leq \mathbb{E} \|h(x + b(t)) - h(x + b(r))\|_k \\ &\leq \mathcal{H}_k(h) \mathbb{E} \sqrt{|b(t) - b(r)|} \lesssim \mathcal{H}_k(h) |t - r|^{1/4}, \end{aligned}$$

where the implied constant is universal. This is the desired estimate for I_1 .

In order to bound I_2 , we appeal to the BDG inequality [Lemma A.4] and Lemma 3.2 to obtain the following:

$$\begin{aligned} \|I_2\|_k &\leq \int_0^r ds \int_{\mathbb{T}} dy |G_{t-s}(x, y) - G_{r-s}(x, y)| \|b(u(s, y))\|_k \\ &\quad + \left[4kp \int_0^r ds \int_{\mathbb{T}} dy |G_{t-s}(x, y) - G_{r-s}(x, y)|^2 \|\sigma(u(s, y))\|_k^2 \right]^{1/2} \\ &\leq \left[m(b)\sqrt{2t} + m(\sigma)\sqrt{4kp} \right] \left[\int_0^r ds \int_{\mathbb{T}} dy |G_{t-s}(x, y) - G_{r-s}(x, y)|^2 \right]^{1/2} \\ &\lesssim \left[m(b) + m(\sigma)\sqrt{k} \right] (t - r)^{1/4}, \end{aligned}$$

where the implied constant does not depend on (t, r, x, k, σ, h) .

We proceed in a similar fashion to bound I_3 as follows:

$$\begin{aligned} \|I_3\|_k &\leq \int_r^t ds \int_{\mathbb{T}} dy \|G_{t-s}(x, y) b(u(s, y))\|_k \\ &\quad + \left[4kp \int_r^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \|\sigma(u(s, y))\|_k^2 \right]^{1/2} \\ &\leq \left[m(b)\sqrt{2t} + m(\sigma)\sqrt{4kp} \right] \left[\int_r^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \right]^{1/2} \\ &\lesssim \left[m(b) + m(\sigma)\sqrt{k} \right] \left[\int_0^{t-r} G_{2s}(0, 0) \, ds \right]^{1/2} \\ &\lesssim \left[m(b) + m(\sigma)\sqrt{k} \right] (t - r)^{1/4}, \end{aligned}$$

where the last two inequalities follow from the semigroup property of the heat kernel and (3.4), and the implied constants do not depend on the parameters

(σ, h, t, r, x, k) . Combine the preceding norm bounds for I_1, I_2, I_3 and use (4.2) to complete the proof. $\square$

Lemmas 4.2 and 4.3, and an appeal to Dudley's metric-entropy theorem [7] together yield the following. The proof is omitted as it is nowadays standard.

Lemma 4.4. *Choose and fix some $T > 0$. If there exists $L > 0$ such that $\mathcal{H}_k(h) \leq L\sqrt{k}$ for all $k \in [2, \infty)$, then there exists $c = c(L, \mathcal{M}(b), \mathcal{M}(\sigma), T) > 0$ such that*

$$\mathbb{E} \left[\exp \left(c \sup_{\substack{(t,x), (s,z) \in [0,T] \times \mathbb{T} \\ (t,x) \neq (s,z)}} \left| \frac{\|u(t,x) - u(s,z)\|}{\rho((t,x), (s,z)) \sqrt{\log_+(1/\rho((t,x), (s,z)))}} \right|^2 \right) \right] < \infty,$$

where ρ was defined in (1.10).

For every $t > 0$ and $x \in \mathbb{T}$, define

$$\begin{aligned} I(t, x) &= \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) b(u(s, y)) \, ds \, dy \\ &\quad + \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) \sigma(u(s, y)) \, \xi(ds \, dy). \end{aligned} \tag{4.3}$$

Lemma 4.5. *For every $T > 0$,*

$$\|I(t, x) - \sigma(h(x))H(t, x)\|_k \lesssim [\mathcal{H}_k(h) \vee \mathcal{M}(b) \vee \mathcal{M}(\sigma)] k \sqrt{t},$$

where the implied constant is independent of $t \in (0, T]$, $x \in \mathbb{T}$, $k \in [2, \infty)$ and h , and depends on σ only through its Lipschitz constant $\text{Lip}(\sigma)$.

Proof. Recall (1.4) and assume without loss of generality that $\mathcal{H}_k(h) \vee \mathcal{M}(b) \vee \mathcal{M}(\sigma) < \infty$. For every $t > 0$ and $x \in \mathbb{T}$, write

$$\begin{aligned} I(t, x) - \sigma(h(x))H(t, x) &= \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) b(u(s, y)) \, ds \, dy \\ &\quad + \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) [\sigma(u(s, y)) - \sigma(h(x))] \, \xi(ds \, dy). \end{aligned}$$

We first apply the Minkowski inequality and (3.4) in order to see that

$$\begin{aligned} \left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) b(u(s, y)) \, ds \, dy \right\|_k &\leq \int_0^t ds \int_{\mathbb{T}} dy \|G_{t-s}(x, y) b(u(s, y))\|_k \\ &\leq \mathcal{M}(b) \int_0^t ds \int_{\mathbb{T}} dy G_{t-s}(x, y) \lesssim \mathcal{M}(b) \int_0^t \max(s^{-1/2}, 1) \, ds \lesssim \mathcal{M}(b) \sqrt{t}, \end{aligned}$$

where the implied constant depends only on T .

Next, observe that if A and B are two random variables with values in $\mathbb{R}^p$, then

$$\|\sigma(A) - \sigma(B)\|_k \leq p^{1/k} \text{Lip}(\sigma) \|A - B\|_k.$$

We apply this with $A = u(s, y)$ and $B = h(x)$, together with the Burkholder–Davis–Gundy inequality in the form of Lemma A.4 below, in order to see that

$$\begin{aligned} & \left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) [\sigma(u(s, y)) - \sigma(h(x))] \xi(ds dy) \right\|_k \\ & \leq \left[4kp \int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \|\sigma(u(s, y)) - \sigma(h(x))\|_k^2 \right]^{1/2} \\ & \leq \sqrt{4kp} \text{Lip}(\sigma) \left[\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \|u(s, y) - h(x)\|_k^2 \right]^{1/2}. \end{aligned}$$

As regards the $L^k(\mathbb{P})$ -norm inside the integral, we have, thanks to Lemma 4.3 and (1.4),

$$\begin{aligned} \|u(s, y) - h(x)\|_k & \leq \|u(s, y) - h(y)\|_k + \|h(y) - h(x)\|_k \\ & \lesssim (\mathcal{H}_k(h) + \mathcal{M}(b) + \mathcal{M}(\sigma)\sqrt{k}) s^{1/4} + \mathcal{H}_k(h) \sqrt{|y - x|}, \end{aligned}$$

where the implied constant is independent of $s \in (0, T]$, $x, y \in \mathbb{T}$, $k \in [2, \infty)$ and h , and depends on σ only through $\text{Lip}(\sigma)$. We can combine the preceding two displays to find that

$$\begin{aligned} & \left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) [\sigma(u(s, y)) - \sigma(h(x))] \xi(ds dy) \right\|_k^2 \\ & \lesssim k[\mathcal{H}_k(h) \vee \mathcal{M}(b) \vee \mathcal{M}(\sigma)]^2 \left\{ \int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 [k\sqrt{s} + |y - x|] \right\} \\ & \lesssim [\mathcal{H}_k(h) \vee \mathcal{M}(b) \vee \mathcal{M}(\sigma)]^2 k^2 t, \end{aligned}$$

thanks to Lemma 3.3, where the parameter dependencies are as in the statement of the lemma. This concludes the proof of Lemma 4.5. $\square$

5. SUFFICIENT CONDITIONS FOR UNIFORM DIMENSION

The primary goal of this section is to present results that streamline the process of establishing uniform dimension theorems for space-time random fields.

Lemma 5.1. *Consider an $\mathbb{R}^p$ -valued random field $\{Z(s, y)\}_{(s,y) \in [S,T] \times \mathbb{T}}$, where $T > S \geq 0$. Choose and fix some number $\alpha \in (0, 1]$. For every $n \in \mathbb{N}$, $\delta \in (0, 1)$, $y \in \mathbb{T}$, $\nu \in \mathbb{R}^p$, and $r > 0$, define*

$$\begin{aligned} \mathcal{N}_n^\delta(s, B(\nu, r)) &= \#\{y \in F_n^\delta : Z(s, y) \in B(\nu, r)\}, \\ \text{where } F_n^\delta &= \mathbb{T} \cap \{j2^{-n(1+\delta)/\alpha} : j \in \mathbb{Z}\}. \end{aligned} \tag{5.1}$$

Recall that $B(\nu, r)$ is the closed ball defined in (1.8). Suppose the following two conditions hold:

- (i) For every $\varepsilon \in (0, 1)$, $\mathbb{P}\{Z(s) \in C^{\alpha(1-\varepsilon)}(\mathbb{T}) \forall s \in [S, T]\} = 1$;
- (ii) For every $R > 0$,

$$\lim_{\delta \rightarrow 0+} \limsup_{n \rightarrow \infty} n^{-1} \sup_{s \in [S, T]} \sup_{\nu \in B(0, R)} \log \mathcal{N}_n^\delta(s, B(\nu, 2^{-n})) = 0 \quad a.s.$$

Then, $\mathbb{P}\{\dim_{\mathbb{H}} Z(\{s\} \times F) = \alpha^{-1} \dim_{\mathbb{H}} F \ \forall \text{ compact } F \subset \mathbb{T}, s \in [S, T]\} = 1$.

Proof. By Lemma A.6, (i) implies that $\dim_{\mathbb{H}} Z(\{s\} \times F) \leq \alpha^{-1} \dim_{\mathbb{H}} F$ for all compact F and $s \in [S, T]$, all valid off a single P-null set. In the remainder of the proof, we aim to show that, off a single P-null set,

$$\dim_{\mathbb{H}} Z(\{s\} \times F) \geq \alpha^{-1} \dim_{\mathbb{H}} F \quad \forall \text{compact } F \subset \mathbb{T}, s \in [S, T]. \quad (5.2)$$

Choose and fix an arbitrary $R > 0$ and observe that taking $F = Z(s)^{-1}(A) = \{y \in \mathbb{T} : Z(s, y) \in A\}$ in (5.2) yields the following equivalent formulation: Off a single P-null set,

$$\dim_{\mathbb{H}} Z(s)^{-1}(A) \leq \alpha \dim_{\mathbb{H}} A \quad \forall \text{compact } A \subset B(0, R), s \in [S, T]. \quad (5.3)$$

In order to establish (5.3), we fix $\delta \in (0, 1)$ and use the condition of part (ii) in order to choose a P-null set Ω_0 off which

$$\sup_{s \in [S, T]} \max_{\nu \in B(0, R)} \mathcal{N}_n^\delta(s, B(\nu, 2^{-n})) = \mathcal{O}(2^{c_\delta n}) \quad \text{as } n \rightarrow \infty, \quad (5.4)$$

where $c_\delta \rightarrow 0$ as $\delta \rightarrow 0$. Also, we can find a P-null set Ω_1 off which the condition of part (i) holds for

$$\varepsilon = \varepsilon(\delta) = 1 - \frac{1 + (\delta/2)}{1 + \delta}. \quad (5.5)$$

Now that we have identified the null set $\Omega_2 := \Omega_0 \cup \Omega_1$ (which depends on δ), we work pathwise, nonprobabilistically, from here on in order to conclude the proof of (5.3), and hence the lemma.

To this end, we deduce from the definition of Hausdorff dimension that for every Borel set $A \subset B(0, R)$ and for all $\kappa > \dim_{\mathbb{H}} A$ and $n \in \mathbb{N}$, we can find $\nu_{1,n}, \nu_{2,n}, \dots \in B(0, R)$ and $r_{1,n}, r_{2,n}, \dots \in (0, 2^{-n})$ such that $A \subset \cup_{i=1}^\infty \mathbb{B}(\nu_{i,n}, r_{i,n})$ – see (1.8) – and

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^\infty r_{i,n}^\kappa < \infty. \quad (5.6)$$

Since $\{\mathbb{B}(\nu_{i,n}, r_{i,n})\}_{i=1}^\infty$ is an open cover of A , $\{Z(s)^{-1}(\mathbb{B}(\nu_{i,n}, r_{i,n}))\}_{i=1}^\infty$ is an open cover of $Z(s)^{-1}(A)$ for every $s \in [S, T]$.

We observe that, off Ω_2 , there exists a random number $C > 0$ and a non-random number $C_0 > 0$ such that for all sufficiently large $n \in \mathbb{N}$, every inverse image of the form $Z(s)^{-1}(\mathbb{B}(\nu_{i,n}, r_{i,n}))$ lies in not more than $C r_{i,n}^{-c_\delta}$ -many intervals of length $C_0 r_{i,n}^{(1+\delta)/\alpha}$. In order to see this, consider a large $n \in \mathbb{N}$, fix $i \in \mathbb{N}$, and let $y \in Z(s)^{-1}(\mathbb{B}(\nu_{i,n}, r_{i,n}))$, where $2^{-m-1} \leq r_{i,n} \leq 2^{-m}$ for some $m \geq n$. Then, the point y is contained in an interval of side length $2^{-(m-1)(1+\delta)/\alpha} [\leq C_0 r_{i,n}^{(1+\delta)/\alpha}]$ centered at a point $y^* \in F_{m-1}^\delta$ that is nearest to y . We see that the center y^* of this interval must belong to $Z(s)^{-1}(\mathbb{B}(\nu_{i,n}, 2^{-m+1})) \cap F_{m-1}^\delta$ uniformly for all large m . This is because the choice of ε in (5.5) together with the triangle inequality implies that

$$\begin{aligned} \|Z(s, y^*) - \nu_{i,n}\| &\leq \|Z(s, y) - \nu_{i,n}\| + \|Z(s, y^*) - Z(s, y)\| \\ &\lesssim 2^{-m} + (2^{-(m-1)(1+\delta)/\alpha})^{\alpha(1-\varepsilon)} \leq 2^{-m} + 2^{-(1+\delta/2)(m-1)} \leq 2^{-m+1}, \end{aligned}$$

uniformly for all sufficiently large $m \in \mathbb{N}$, and where the implied constant does not depend on any of the parameters that arise. According to (5.4),

$$\#[Z(s)^{-1}(\mathbb{B}(\nu_{i,n}, 2^{-m+1})) \cap F_{m-1}^\delta] \leq C r_{i,n}^{-c_\delta} \quad \forall n \text{ large enough.}$$

This shows that, when n is sufficiently large, $Z(s)^{-1}(\mathbb{B}(\nu_{i,n}, r_{i,n}))$ lies in not more than $Cr_{i,n}^{-c_\delta}$ -many intervals of side length $C_0 r_{i,n}^{(1+\delta)/\alpha}$, valid uniformly for all $s \in [S, T]$.

Next, choose and fix $\theta > 0$ to see that, off Ω_2 , the θ -dimensional Hausdorff measure of $Z(s)^{-1}(A)$ is at most

$$C \sum_{i=1}^{\infty} r_{i,n}^{\theta(1+\delta)/\alpha - c_\delta},$$

regardless of the value of n , the choice of the Borel set $A \subset B(0, R)$, or the value of $s \in [S, T]$. Thanks to (5.6), the preceding sum converges uniformly in n provided that $\theta(1+\delta)/\alpha - c_\delta > \kappa$. Since $\kappa > \dim_{\mathbb{H}} A$ is arbitrary, this proves that, off Ω_2 ,

$$\dim_{\mathbb{H}} Z(s)^{-1}(A) \leq \frac{\alpha(\dim_{\mathbb{H}} A + c_\delta)}{1 + \delta},$$

simultaneously for all compact $A \subset B(0, R)$ and $s \in [S, T]$. Let $\delta \downarrow 0$ along a rational sequence in order to deduce that (5.3) holds off a single P-null set. Since $R > 0$ is arbitrary, this completes the proof of Lemma 5.1. $\square$

The next lemma provides a sufficient condition for part (ii) of the previous lemma to be applicable.

Lemma 5.2. *Choose and fix some $T > S \geq 0$, and let $\{Z(s, y)\}_{(s,y) \in [S,T] \times \mathbb{T}}$ be an $\mathbb{R}^p$ -valued random field. Recall (1.10) and (5.1) with $\alpha = \frac{1}{2}$, and suppose that:*

(i) *The following is finite for some $c, \gamma > 0$:*

$$\mathfrak{E}_0 := \mathbb{E} \left[\exp \left(c \sup \left| \frac{\|Z(s, y) - Z(s', y')\|}{\rho((s, y), (s', y')) \sqrt{\log_+(1/\rho((s, y), (s', y'))))}} \right|^\gamma \right) \right],$$

where the sup is over all distinct points $(s, y), (s', y') \in [S, T] \times \mathbb{T}$;

(ii) *For every $\delta \in (0, 1)$, there exist numbers $a, \kappa > 0$ such that*

$$\mathbb{P} \{ \mathcal{N}_n^\delta(s, B(\nu, 2^{-n})) \geq 2^{an} \} \leq 2^{-\kappa n^2},$$

uniformly for all $n \in \mathbb{N}$, $s \in [S, T]$ and $\nu \in \mathbb{R}^p$.

Then, for every $\delta \in (0, 1)$ and $R > 0$, there exist $K, L > 0$ such that

$$\mathbb{P} \left\{ \sup_{s \in [S, T]} \sup_{\nu \in B(0, R)} \mathcal{N}_n^\delta(s, B(\nu, 2^{-n})) \geq K 2^{an} \right\} \leq L^n e^{-n^2/L},$$

uniformly for all $n \in \mathbb{N}$.

Proof. Fix $\delta \in (0, 1)$ and $R > 0$. Define

$$A(n) = \left\{ \frac{j}{2^{4n(1+\delta)}} : j \in \mathbb{Z} \right\} \quad \text{and} \quad D(n) = \left\{ \frac{\ell}{2^{2n}\sqrt{p}} : \ell \in \mathbb{Z}^p \right\}.$$

First, note that there is a number $K > 0$, depending only on δ , such that for every $n \in \mathbb{N}$ and every $y^* \in F_{n-1}^\delta$,

$$\# \left(B(y^*, 2^{-2(n-1)(1+\delta)}) \cap F_n^\delta \right) \leq K. \quad (5.7)$$

We can observe that the following holds for all $n \in \mathbb{N}$:

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{\substack{s \in [S, T], \\ \nu \in B(0, R)}} N_n^\delta(s, B(\nu, 2^{-n})) \geq K2^{an}, \sup_{s, s', y, y'} \|Z(s, y) - Z(s', y')\| \leq 2^{-n-1} \right\} \\ & \leq \mathbb{P} \left\{ \max_{s \in A(n) \cap [S, T]} \max_{\nu \in D(n) \cap B(0, R)} N_{n-1}^\delta(s, B(\nu, 2^{-n+1})) \geq 2^{an} \right\}, \end{aligned} \quad (5.8)$$

where $\sup_{s, s', y, y'}$ denotes, here and throughout the proof, the sup operator over all $(s, y), (s', y') \in [S, T] \times \mathbb{T}$ that satisfy $|s - s'| \leq 2^{-4n(1+\delta)}$ and $|y - y'| \leq 16 \times 2^{-2n(1+\delta)}$. In order to see why (5.8) is true, let us suppose that there exist $s \in [0, T]$, $\nu \in B(0, R)$ and $y_1, \dots, y_m \in F_n^\delta$ with $\#\{y_1, \dots, y_m\} = m \geq K2^{an}$ such that $\|Z(s, y_i) - \nu\| \leq 2^{-n}$ for all $1 \leq i \leq m$, and that $\sup_{s, s', y, y'} \|Z(s, y) - Z(s', y')\| \leq 2^{-n-1}$. Then, we can choose nearest neighbors $s^* \in A(n) \cap [S, T]$, $\nu^* \in D(n) \cap B(0, R)$ and $y_1^*, \dots, y_m^* \in F_{n-1}^\delta$ that respectively satisfy $|s^* - s| \leq 2^{-4n(1+\delta)}$, $\|\nu^* - \nu\| \leq 2^{-2n}$ and $|y_i^* - y_i| \leq 2^{-2(n-1)(1+\delta)} [\leq 16 \times 2^{-2n(1+\delta)}]$, in order to see that

$$\begin{aligned} \|Z(s^*, y_i^*) - \nu^*\| & \leq \|Z(s^*, y_i^*) - Z(s, y_i)\| + \|Z(s, y_i) - \nu\| + \|\nu - \nu^*\| \\ & \leq 2^{-n-1} + 2^{-n} + 2^{-2n} \leq 2^{-n+1}. \end{aligned}$$

By (5.7), $\#\{y_1^*, \dots, y_m^*\} \geq m/K$, whence it follows that $N_{n-1}^\delta(s^*, B(\nu^*, 2^{-n+1}))$ is at least $\#\{y_1^*, \dots, y_m^*\} \geq m/K \geq 2^{an}$. This proves (5.8).

Next, condition (ii) together with a simple union bound yields the following, valid uniformly for all $n \in \mathbb{N}$:

$$\mathbb{P} \left\{ \max_{\substack{s \in A(n) \cap [S, T], \\ \nu \in D(n) \cap B(0, R)}} N_{n-1}^\delta(s, B(\nu, 2^{-n+1})) \geq 2^{an} \right\} \leq L_1^n e^{-n^2/L_1}, \quad (5.9)$$

for a constant $L_1 > 0$ that depends only on $(p, S, T, \delta, R, \kappa)$.

Now, we can put together (5.9) and (5.8) in order to conclude that

$$\mathbb{P} \left\{ \sup_{s \in [S, T]} \sup_{\nu \in B(0, R)} N_n^\delta(s, B(\nu, 2^{-n})) \geq K2^{an} \right\} \leq L_1^n e^{-n^2/L_1} + \mathcal{P}_n, \quad (5.10)$$

where $\mathcal{P}_n = \mathbb{P}\{\sup_{s, s', y, y'} \|Z(s, y) - Z(s', y')\| > 2^{-n-1}\}$. Recall the metric ρ from (1.10) and observe that $\sup_{s, s', y, y'} \rho((s, y), (s', y')) \leq 5 \times 2^{-n(1+\delta)}$. Therefore, condition (i) and Chebyshev's inequality together imply that there exist $L_2 = L_2(\delta) > 0$ and $L_3 > 0$ that depend only on $(c, \gamma, \epsilon_0, \delta, L_2)$ and satisfy

$$\begin{aligned} \mathcal{P}_n & \leq \mathbb{P} \left\{ \sup_{\substack{(s, y), (s', y') \in [S, T] \times \mathbb{T} \\ (s, y) \neq (s', y')}} \frac{\|Z(s, y) - Z(s', y')\|}{\rho((s, y), (s', y')) \sqrt{\log_+(1/\rho((s, y), (s', y'))))}} \geq \frac{2^{\delta n}}{\sqrt{L_2 n}} \right\} \\ & \leq \epsilon_0 \exp \left(-c \left[\frac{2^{\delta n}}{\sqrt{L_2 n}} \right]^\gamma \right) \leq L_3^n e^{-n^2/L_3}, \end{aligned}$$

uniformly for all $n \in \mathbb{N}$. Combine this with (5.10) to conclude the proof. $\square$

The preceding lemma is key to deriving lower bounds for the Hausdorff dimension of random sets of the form $Z(\{s\} \times F)$, valid uniformly in $s \in [0, T]$ and compact

sets $F \subset \mathbb{T}$. We now turn to sufficient conditions for deriving lower bounds for the anisotropic Hausdorff dimension $\dim_{\mathbb{H}}^\rho$ of random sets of the form $Z(G)$, valid uniformly for all compact $G \subset [0, T] \times \mathbb{T}$; see (1.11) and the subsequent paragraph. In fact, we state and prove a more general result since we anticipate future applications in other contexts.

Lemma 5.3. *Consider an $\mathbb{R}^p$ -valued random field $\{Z(s)\}_{s \in I}$ where $I \subset \mathbb{R}^N$ is a non-random, compact, upright rectangle. Choose and fix $\alpha_1, \dots, \alpha_N \in (0, 1]$, and metrize I using $\mathbf{d}(s, s') = \sum_{j=1}^N |s_j - s'_j|^{\alpha_j}$ for all $s, s' \in I$. For every $n \in \mathbb{N}$, $\delta \in (0, 1)$, $\nu \in \mathbb{R}^p$, and $r > 0$, define*

$$\begin{aligned} \mathcal{N}_n^\delta(\nu, r) &= \#\{s \in F_n^\delta : Z(s) \in B(\nu, r)\}, \text{ where} \\ F_n^\delta &= I \cap \bigcup_{j_1, \dots, j_N \in \mathbb{Z}} \left\{ (j_1 2^{-(1+\delta)n/\alpha_1}, \dots, j_N 2^{-(1+\delta)n/\alpha_N}) \right\}, \end{aligned} \quad (5.11)$$

and suppose that the following two conditions hold:

- (i) For every $\varepsilon \in (0, 1)$, $\sup_{s, s' \in I: s \neq s'} \|Z(s) - Z(s')\| / \mathbf{d}(s, s')^{1-\varepsilon} < \infty$;
- (ii) For every $R > 0$, $\lim_{\delta \rightarrow 0+} \limsup_{n \rightarrow \infty} n^{-1} \sup_{\nu \in B(0, R)} \log \mathcal{N}_n^\delta(\nu, 2^{-n}) = 0$ a.s.

Then, there exists a $\mathbb{P}$ -null set off which $\dim_{\mathbb{H}} Z(G) = \dim_{\mathbb{H}}^\rho G$, uniformly for all compact sets $G \subset I$.

The proof of the above lemma requires making only minor adaptations to that of Lemma 5.2 and therefore is left to the interested reader. It might help to emphasize that the following lemma provides a sufficient condition for part (ii) of Lemma 5.3 to be applicable.

Lemma 5.4. *Let $\{Z(s)\}_{s \in I}$ be an $\mathbb{R}^p$ -valued random field, where $I \subset \mathbb{R}^N$ is a non-random, compact, upright rectangle. Let $\mathbf{d}$, $\mathcal{N}_n^\delta(\nu, r)$, and F_n^δ be as defined in Lemma 5.3, and suppose that the following two conditions hold:*

- (i) The following is finite for some $c > 0$ and $\gamma > 0$:

$$\varepsilon_1 := \mathbb{E} \left[\exp \left(c \sup_{\substack{s, s' \in I \\ s \neq s'}} \left[\frac{\|Z(s) - Z(s')\|}{\mathbf{d}(s, s') |\log_+(1/\mathbf{d}(s, s'))|^{1/2}} \right]^\gamma \right) \right]; \quad (5.12)$$

- (ii) For every $\delta \in (0, 1)$, there exist $a, b > 0$ and $\kappa > 0$ such that

$$\mathbb{P} \left\{ \mathcal{N}_n^\delta(\nu, 2^{-n}) \geq 2^{an} \right\} \leq b 2^{-\kappa n^2} \quad \forall n \in \mathbb{N}, s \in I, \nu \in \mathbb{R}^p. \quad (5.13)$$

Then, for every $\delta \in (0, 1)$ and $R > 0$, there exists $L > 0$ such that

$$\mathbb{P} \left\{ \sup_{\nu \in B(0, R)} \mathcal{N}_n^\delta(\nu, 2^{-n}) \geq 2^{an} \right\} \leq L^n e^{-n^2/L} \quad \text{uniformly for all } n \in \mathbb{N}.$$

Proof. Fix $\delta \in (0, 1)$ and $R > 0$. Similarly to the proof of Lemma 5.2, we can use interpolation and (5.13) to write

$$\begin{aligned} &\mathbb{P} \left\{ \sup_{\nu \in B(0, R)} \mathcal{N}_n^\delta(\nu, 2^{-n}) \geq 2^{an} \right\} \\ &\leq L_1^n e^{-n^2/L_1} + \mathbb{P} \left\{ \sup_{s, s' \in I: \mathbf{d}(s, s') \leq 2^{-n(1+\delta)}} \|Z(s) - Z(s')\| > 2^{-n-1} \right\}, \end{aligned}$$

for all $n \in \mathbb{N}$, where $L_1 > 0$ is a number depending only on (p, δ, R, κ) . Then, we further use Chebyshev's inequality and (5.12) to see that there exist numbers $L_2 = L_2(\delta) > 0$ and $L_3 = L_3(c, \gamma, \mathfrak{E}_1, \delta, L_2) > 0$ such that

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{s, s' \in I: d(s, s') \leq 2^{-n(1+\delta)}} \|Z(s) - Z(s')\| > 2^{-n-1} \right\} \\ & \leq \mathbb{P} \left\{ \sup_{\substack{s, s' \in I: \\ s \neq s'}} \frac{\|Z(s) - Z(s')\|}{d(s, s') \sqrt{\log_+(1/d(s, s'))}} \geq \frac{2^{\delta n}}{\sqrt{L_2 n}} \right\} \leq \mathfrak{E}_1 \exp \left(-c \left[\frac{2^{\delta n}}{\sqrt{L_2 n}} \right]^\gamma \right) \\ & \leq L_3^n \exp(-n^2/L_3) \quad \text{uniformly for all } n \in \mathbb{N}. \end{aligned}$$

This completes the proof. $\square$

6. PROOF OF THEOREM 2.1

We are now ready to prove Theorem 2.1; this is our broadest uniform dimension result, available when the noise term in (1.1) is additive.

The proof is divided in two parts. In the first part, we prove Theorem 2.1 in the special case that $b \equiv 0$. In the second part, we extend the result to general non-random Lipschitz continuous functions $b : \mathbb{R}^p \rightarrow \mathbb{R}^p$.

Part 1. Suppose $b \equiv 0$, and σ is a constant nonsingular $p \times p$ matrix. We may write

$$u(t, x) = (\mathcal{G}_t u_0)(x) + \sigma H(t, x), \quad (6.1)$$

where H is the solution to (3.1) which is a centered Gaussian random field with i.i.d. coordinate processes.

Proof of (i). Suppose $p \geq 2$. In order to prove (i) let us fix $0 < S < T < \infty$, and set $Z(t, x) = u(t, x)$ for $(t, x) \in [S, T] \times \mathbb{T}$ and $\alpha = \frac{1}{2}$ in Lemma 5.1. Let $\mathcal{N}_n^\delta(t, B(\nu, r))$ and F_n^δ be defined by (5.1) with $\alpha = \frac{1}{2}$, and define $\tilde{\mathcal{N}}_n^\delta(t, B(\nu, r))$ to be the total number of all n -tuples of all $x_1 < \dots < x_n$ in F_n^δ such that $u(t, x_i) \in B(\nu, r)$ for all $1 \leq i \leq n$; that is,

$$\tilde{\mathcal{N}}_n^\delta(t, B(\nu, r)) = \sum_{x_1 < \dots < x_n \text{ in } F_n^\delta} \dots \sum \mathbb{1}_{\{\max_{1 \leq i \leq n} \|u(t, x_i) - \nu\| \leq r\}}.$$

For every $n \in \mathbb{N}$, $\delta \in (0, 1)$, $t \in [S, T]$, and $\nu \in \mathbb{R}^p$, we have

$$\begin{aligned} & \mathbb{P} \{ \mathcal{N}_n^\delta(t, B(\nu, 2^{-n})) \geq 2^{2\delta p n} \} \\ & \leq \mathbb{P} \left\{ \tilde{\mathcal{N}}_n^\delta(t, B(\nu, 2^{-n})) \geq \binom{\lceil 2^{2\delta p n} \rceil}{n} \right\} \leq \left(\binom{\lceil 2^{2\delta p n} \rceil}{n} \right)^{-1} \mathbb{E} \left[\tilde{\mathcal{N}}_n^\delta(t, B(\nu, 2^{-n})) \right], \end{aligned} \quad (6.2)$$

where the second inequality holds due to Chebyshev's inequality, and the expectation is given by

$$\mathbb{E} \left[\tilde{\mathcal{N}}_n^\delta(t, B(\nu, 2^{-n})) \right] = \sum_{x_1 < \dots < x_n \text{ in } F_n^\delta} \dots \sum \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|u(t, x_i) - \nu\| \leq 2^{-n} \right\}.$$

We estimate the last probability as follows. For all $x_1 < \dots < x_n$ in F_n^δ and $\varepsilon > 0$, we use (6.1) and apply the Anderson shifted-ball inequality [2] in order to see that

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|u(t, x_i) - \nu\| \leq \varepsilon \right\} &\leq \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|\sigma H(t, x_i)\| \leq \varepsilon \right\} \\ &\leq \left(\mathbb{P} \left\{ \max_{1 \leq i \leq n} |H_1(t, x_i)| \leq \varepsilon / \lambda_\sigma \right\} \right)^p, \end{aligned}$$

where $\lambda_\sigma > 0$ denotes the smallest singular value of σ . In the preceding, we have used the fact that the coordinates of H are i.i.d. Since the probability density function of a centered, real-valued Gaussian random variable Z is bounded above by $1/\sqrt{2\pi \text{Var}(Z)}$, we proceed iteratively by successive conditioning on the values of $H_1(t, x_1), \dots, H_1(t, x_i)$, as i varies from $n-1$ to 1, in order to see that uniformly for all $\varepsilon > 0$, $n \in \mathbb{N}$, $t \in [S, T]$ and $\nu \in \mathbb{R}^p$,

$$\begin{aligned} &\mathbb{P} \left\{ \max_{1 \leq i \leq n} \|u(t, x_i) - \nu\| \leq \varepsilon \right\} \\ &\leq \left(\frac{2\varepsilon}{\lambda_\sigma \sqrt{2\pi \text{Var}(H_1(t, x_1))}} \right)^p \prod_{i=2}^n \left(\frac{2\varepsilon}{\lambda_\sigma \sqrt{2\pi \text{Var}(H_1(t, x_i) \mid \mathcal{H}_{i-1})}} \right)^p \\ &\leq C^n \varepsilon^{pn} t^{-p/4} \prod_{i=2}^n |x_i - x_{i-1}|^{-p/2}, \end{aligned}$$

where $\mathcal{H}_i$ denotes the σ -algebra generated by $H_1(t, x_1), \dots, H_1(t, x_i)$. The last inequality follows from Lemma 3.4 and strong local nondeterminism in the form of Proposition 3.6 since $\sqrt{t} \geq \sqrt{S} \gtrsim |x_i - x_{i-1}|$ for all $i = 2, \dots, n$, and the number $C > 0$ depends only on (p, S, T, σ) . Whenever $x_0 \in F_n^\delta$,

$$\sum_{x \in F_n^\delta \setminus \{x_0\}} |x - x_0|^{-p/2} \leq \sum_{1 \leq j \leq 2^{2(1+\delta)n}} \frac{2}{(j2^{-2(1+\delta)n})^{p/2}}.$$

Therefore, the integral test of calculus implies that

$$\sum_{x \in F_n^\delta \setminus \{x_0\}} |x - x_0|^{-p/2} \lesssim C_p(n) = \begin{cases} 2^{p(1+\delta)n} & \text{if } p \geq 3, \\ n2^{2(1+\delta)n} & \text{if } p = 2, \end{cases}$$

where the implied constant depends only on (p, S, T, σ) . An iterative application of the preceding, applied with $\varepsilon = 2^{-n}$, implies that there exist constants $c_i = c_i(p, S, T, \sigma) > 0$ [$i = 1, 2$] such that, uniformly for all $n \in \mathbb{N}$, $t \in [S, T]$ and $\nu \in \mathbb{R}^p$,

$$\begin{aligned} &\mathbb{E} \left[\tilde{\mathcal{N}}_n^\delta(t, B(\nu, 2^{-n})) \right] \\ &\leq c_1^n 2^{-pn^2} \sum_{x_1 \in F_n^\delta} S^{-p/4} \sum_{x_2 \in F_n^\delta \setminus \{x_1\}} |x_2 - x_1|^{-p/2} \dots \sum_{x_n \in F_n^\delta \setminus \{x_{n-1}\}} |x_n - x_{n-1}|^{-p/2} \\ &\leq c_2^n 2^{-pn^2} [C_p(n)]^n. \end{aligned}$$

Plug this inequality into (6.2) and use Lemma A.5 to see that

$$\mathbb{P} \{ \mathcal{N}_n^\delta(t, B(\nu, 2^{-n})) \geq 2^{2\delta pn} \} \leq \begin{cases} c_3^n n^n 2^{-\delta pn^2} & \text{if } p \geq 3, \\ c_3^n n^{2n} 2^{-\delta pn^2} & \text{if } p = 2, \end{cases} \quad (6.3)$$

for some $c_3 > 0$. Thanks to Lemmas 4.4 and 5.2, for every $\delta \in (0, 1)$ and $R > 0$, there exist $K = K(\delta) > 0$ and $L = L(p, S, T, \sigma, \delta, R) > 0$ such that

$$\mathbb{P} \left\{ \sup_{t \in [S, T]} \sup_{\nu \in B(0, R)} n_n^\delta(t, B(\nu, 2^{-n})) \geq K 2^{2\delta p n} \right\} \leq L^n e^{-n^2/L} \quad \forall n \in \mathbb{N}.$$

Therefore, the Borel–Cantelli lemma implies that, almost surely,

$$\sup_{t \in [S, T]} \sup_{\nu \in B(0, R)} n_n^\delta(t, B(\nu, 2^{-n})) = \mathcal{O}(2^{2\delta p n}) \quad \text{as } n \rightarrow \infty. \quad (6.4)$$

Moreover, Lemma 4.4 implies that a.s.,

$$\sup_{t \in [S, T]} \sup_{\substack{x, x' \in \mathbb{T}: \\ x \neq x'}} \frac{\|u(t, x) - u(t, x')\|}{|x - x'|^{(1-\varepsilon)/2}} < \infty \quad \forall \varepsilon \in (0, 1). \quad (6.5)$$

With (6.4) and (6.5) in place, we can apply Lemma 5.1 to deduce that

$$\dim_{\mathbb{H}} u(\{t\} \times F) = 2 \dim_{\mathbb{H}} F \quad \forall \text{compact } F \subset \mathbb{T}, \quad t \in [S, T],$$

off a single P-null set. Part (i) follows since $S > 0$ and $T > 0$ are arbitrary.

Proof of (ii). Suppose $p \geq 4$. The proof of (ii) is similar to that of case (i) above. Fix $0 < S < T$ and set $\alpha = \frac{1}{4}$. For any $n \in \mathbb{N}$, $\delta \in (0, 1)$, $x \in \mathbb{T}$, $\nu \in \mathbb{R}^p$ and $r > 0$, define

$$n_n^\delta(x, B(\nu, r)) = \#\{s \in F_n^\delta : u(s, x) \in B(\nu, r)\},$$

where $F_n^\delta = [S, T] \cap \{j 2^{-n(1+\delta)/\alpha}\}.$

Let $\tilde{n}_n^\delta(x, B(\nu, r))$ denote total the number of n -tuples $t_1 < \dots < t_n$ in F_n^δ such that $u(t_i, x) \in B(\nu, r)$ for all $1 \leq i \leq n$. Then, as in the proof of (6.3), we can use strong local nondeterminism (Proposition 3.6) to deduce that for every $n \in \mathbb{N}$, $\delta \in (0, 1)$, $x \in \mathbb{T}$, and $\nu \in \mathbb{R}^p$,

$$\begin{aligned} & \mathbb{P} \{n_n^\delta(x, B(\nu, 2^{-n})) \geq 2^{2\delta p n}\} \\ & \leq \binom{\lceil 2^{2\delta p n} \rceil}{n}^{-1} \sum_{t_1 < \dots < t_n \text{ in } F_n^\delta} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|u(t_i, x) - \nu\| \leq 2^{-n} \right\} \\ & \leq C_1^n n^n 2^{-2\delta p n^2 - p n^2} \sum_{t_1 \in F_n^\delta} S^{-p/4} \sum_{t_2 \in F_n^\delta \setminus \{t_1\}} |t_2 - t_1|^{-p/4} \dots \sum_{t_n \in F_n^\delta \setminus \{t_{n-1}\}} |t_n - t_{n-1}|^{-p/4} \\ & \leq \begin{cases} C_2^n n^n 2^{-\delta p n^2} & \text{if } p \geq 5, \\ C_2^n n^{2n} 2^{-\delta p n^2} & \text{if } p = 4, \end{cases} \end{aligned}$$

for some $C_i = C_i(p, S, T, \sigma) > 0$ [$i = 1, 2$]. Thanks to Lemmas 4.4 and 5.2 and the Borel–Cantelli lemma, the preceding implies that for every $\delta \in (0, 1)$ and $R > 0$,

$$\sup_{x \in \mathbb{T}} \sup_{\nu \in B(0, R)} n_n^\delta(x, B(\nu, 2^{-n})) = \mathcal{O}(2^{2\delta p n}) \quad \text{as } n \rightarrow \infty \text{ a.s.}$$

Also, by Lemma 4.4, there exists a P-null set off which

$$\sup_{x \in \mathbb{T}} \sup_{\substack{t, t' \in [S, T]: \\ t \neq t'}} \frac{\|u(t, x) - u(t', x)\|}{|t - t'|^{(1-\varepsilon)/4}} < \infty \quad \forall \varepsilon \in (0, 1).$$

Thanks to the preceding two displays, we may prove in the same way as in Lemma 5.1 (with $\alpha = 1/4$) – except that the roles of s and y are now reversed – that there is a single P-null set off which

$$\dim_{\mathbb{H}} u(F \times \{x\}) = 4 \dim_{\mathbb{H}} F \quad \forall \text{ compact } F \subset [S, T], \quad x \in \mathbb{T}.$$

Since $T > S > 0$ are arbitrary non-random numbers, this proves part (ii).

Proof of (iii). Suppose that $p \geq 6$. Fix $0 \leq S < T$. We will apply Lemma 5.3 with $I = [S, T] \times \mathbb{T} \cong [S, T] \times [-1, 1]$, $\alpha_1 = 1/4$, $\alpha_2 = 1/2$, and $Z(s) = u(s)$ for all $s \in I$. Let $\mathcal{N}_n^\delta(\nu, r)$ and F_n^δ as be defined in (5.11). In particular,

$$F_n^\delta = I \cap \{(j_1 2^{-4(1+\delta)n}, j_2 2^{-2(1+\delta)n}) : j_1, j_2 \in \mathbb{Z}\}.$$

In order to establish part (iii), we adopt an idea of Monrad and Pitt [18]. The key observation is that every set of n distinct points $s_1, \dots, s_n \in [S, T] \times [-1, 1]$ can be reordered so that

$$\rho(s_i, s_{i-1}) \leq \rho(s_i, s_j) \quad \forall 1 \leq j < i \leq n. \quad (6.6)$$

This can be done in at least n different ways – we pick in any manner we like a point as s_n first, then find s_{n-1} , then s_{n-2} , etc. By the Anderson shifted-ball inequality [2] and strong local nondeterminism (Proposition 3.6),

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|u(s_i) - \nu\| \leq \varepsilon \right\} &\leq \mathbb{P} \left\{ \max_{1 \leq i \leq n} |H_1(s_i)| \leq \varepsilon / \lambda_\sigma \right\}^p \\ &\leq \left(\frac{2\varepsilon}{\lambda_\sigma \sqrt{2\pi \operatorname{Var}(H_1(s_1))}} \right)^p \prod_{i=2}^n \left(\frac{2\varepsilon}{\lambda_\sigma \sqrt{2\pi \operatorname{Var}(H_1(s_i) | \mathcal{H}_{i-1})}} \right)^p \\ &\leq C^n \varepsilon^{pn} S^{-p/4} \prod_{i=2}^n \left[\min_{1 \leq j \leq i-1} \rho(s_i, s_j) \right]^{-p} \leq C^n \varepsilon^{pn} S^{-p/4} \prod_{i=2}^n \rho(s_i, s_{i-1})^{-p}, \end{aligned}$$

uniformly for all $\varepsilon > 0$, $n \in \mathbb{N}$ and $\nu \in \mathbb{R}^p$, and for all n distinct points $s_1, \dots, s_n \in F_n^\delta$ that satisfy (6.6). Here, $\mathcal{H}_i$ denotes the σ -algebra generated by $H_1(s_1), \dots, H_1(s_i)$, and $C = C(p, S, T, \sigma) > 0$ is a fixed number. Whenever $s_0 \in F_n^\delta$,

$$\sum_{s \in F_n^\delta \setminus \{s_0\}} \rho(s, s_0)^{-p} \leq \sum_{\substack{j \in \mathbb{Z}^2 \setminus \{0\}: \\ \rho(j, 0) \leq C 2^{(1+\delta)n}}} \frac{2^{p(1+\delta)n}}{(\rho(j, 0))^p} \lesssim 2^{p(1+\delta)n} \int_1^{C 2^{(1+\delta)n}} r^{5-p} dr \leq C_p(n),$$

where

$$C_p(n) = \begin{cases} C 2^{p(1+\delta)n} & \text{if } p \geq 7, \\ C n 2^{6(1+\delta)n} & \text{if } p = 6. \end{cases}$$

Let $M_n^\delta(\nu, r)$ denote the total number of all n -tuples of distinct points $s_1, \dots, s_n$ in F_n^δ that satisfy (6.6) and $u(s_i) \in B(\nu, r)$ for all $1 \leq i \leq n$. It follows that

$$\begin{aligned}
\mathbb{P} \{N_n^\delta(\nu, 2^{-n}) \geq 2^{2\delta pn}\} &\leq \mathbb{P} \left\{ M_n^\delta(\nu, 2^{-n}) \geq \binom{\lceil 2^{2\delta pn} \rceil}{n} \right\} \\
&\leq \binom{\lceil 2^{2\delta pn} \rceil}{n}^{-1} \sum_{\substack{\text{distinct } s_1, \dots, s_n \text{ in } F_n^\delta \\ \text{that satisfy (6.6)}}} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|u(s_i) - \nu\| \leq 2^{-n} \right\} \\
&\leq C^n n^n 2^{-2\delta pn^2 - pn^2} \sum_{s_1 \in F_n^\delta} S^{-p/4} \sum_{s_2 \in F_n^\delta \setminus \{s_1\}} \rho(s_2, s_1)^{-p} \cdots \sum_{s_n \in F_n^\delta \setminus \{s_{n-1}\}} \rho(s_n, s_{n-1})^{-p} \\
&\leq \begin{cases} c^n n^n 2^{-\delta pn^2} & \text{if } p \geq 7, \\ c^n n^{2n} 2^{-\delta pn^2} & \text{if } p = 6, \end{cases}
\end{aligned}$$

where $c = c(p, S, T, \sigma) > 0$. By Lemmas 4.4 and 5.4, and the Borel–Cantelli lemma, for every $\delta \in (0, 1)$ and $R > 0$, $\sup_{\nu \in B(0, R)} N_n^\delta(\nu, 2^{-n}) = \mathcal{O}(2^{2\delta pn})$ as $n \rightarrow \infty$, almost surely. Also, by Lemma 4.4, a.s.,

$$\sup_{s, s' \in [S, T] \times \mathbb{T}: s \neq s'} \frac{\|u(s) - u(s')\|}{(\rho(s, s'))^{1-\varepsilon}} < \infty \quad \forall \varepsilon \in (0, 1).$$

Thanks to the preceding two displays, we can apply Lemma 5.3 in order to see that there exists a $\mathbb{P}$ -null set off which

$$\dim_{\mathbb{H}} u(G) = \dim_{\mathbb{H}}^\rho G \quad \forall \text{compact } G \subset [S, T] \times \mathbb{T}.$$

Since $S > 0$ and $T > 0$ are arbitrary, part (iii) follows.

Part 2. Now, we consider the general case that the non-random function $b : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is Lipschitz continuous. Define

$$b_N(x) = \begin{cases} b(x) & \text{if } \|x\| \leq N, \\ b(Nx/\|x\|) & \text{if } \|x\| > N. \end{cases} \quad (6.7)$$

Then it is not hard to see that b_N is globally Lipschitz. In fact,

$$\text{Lip}(b_N) \leq \text{Lip}(b) \quad \forall N > 0. \quad (6.8)$$

Here is the short proof: If $b \in C_b^1(\mathbb{R}^p)$, then for all $N > 0$ we have $\partial_{v_i} b_N(v) = \partial_{v_i} b(v)$ when $\|v\| \leq N$ and

$$\partial_{v_i} b_N(v) = \partial_{v_i} b \left(\frac{vN}{\|v\|} \right) \frac{N}{\|v\|} \left(1 - \frac{v_i^2}{\|v\|^2} \right) \quad \text{when } \|v\| > N.$$

It follows that $\text{Lip}(b_N) = \|\nabla b_N\|_{L^\infty(\mathbb{R}^p)} \leq \|\nabla b\|_{L^\infty(\mathbb{R}^p)} = \text{Lip}(b)$ for every $N > 0$. In general, when we know only that $b \in \text{Lip}(\mathbb{R}^p)$, we write $b^t = \phi_t * b$ where ϕ_t was defined in (3.5). Direct inspection shows that $\text{Lip}(b^t) \leq \text{Lip}(b)$, whence $\text{Lip}(b_N^t) \leq \text{Lip}(b)$ thanks to the preceding argument. This means, among other things, that $\|b_N^t(x) - b_N^t(y)\| \leq \text{Lip}(b)\|x - y\|$ for every $t > 0$ and $x, y \in \mathbb{R}^p$. Send $t \downarrow 0$ to obtain (6.8).

Let u_N denote the solution to (1.1) where b is replaced by b_N and σ is a constant nonsingular $p \times p$ matrix. This equation can be rewritten as

$$\begin{cases} \partial_t u_N(t, x) = \partial_x^2 u_N(t, x) + \sigma [\sigma^{-1} b_N(u_N(t, x)) + \xi(t, x)] & \text{on } (0, \infty) \times \mathbb{T}, \\ u_N(0) = u_0 & \text{on } \mathbb{T}. \end{cases}$$

In other words, the mild formulation for the solution can be rewritten as

$$u_N(t, x) = (\mathcal{G}_t u_0)(x) + \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) \sigma [\sigma^{-1} b_N(u_N(s, y))] ds dy + \xi(ds dy).$$

Choose and fix $T > 0$. Since b_N is bounded and σ is nonsingular, Girsanov's theorem (see Lemma A.7) implies that $\zeta(t, x) = \sigma^{-1} b_N(u_N(t, x)) + \xi(t, x)$ $[(t, x) \in [0, T] \times \mathbb{T}]$ is a space-time white noise on the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$, where $\mathbb{Q}$ is mutually absolutely continuous with respect to $\mathbb{P}$. Under the measure $\mathbb{Q}$, the random field u_N solves $\partial_t u_N = \partial_x^2 u_N + \sigma \zeta$ on $(0, T) \times \mathbb{T}$ subject to $u_N(0) = u_0$ on $\mathbb{T}$. Therefore, Part 1 of this proof and the mutual absolute continuity of $\mathbb{P}$ and $\mathbb{Q}$ together yield

$$\mathbb{P}(\Omega_i(u_N, T)) = \mathbb{Q}(\Omega_i(u_N, T)) = 1 \quad \text{if } p \geq p_i \quad [i = 1, 2, 3],$$

where $p_1 = 2$, $p_2 = 4$, $p_3 = 6$, and the Ω_i s are the following three events:

$$\begin{aligned} \Omega_1(u_N, T) &= \{\dim_{\mathbb{H}} u_N(\{t\} \times A) = 2 \dim_{\mathbb{H}} A \ \forall \text{compact } A \subset \mathbb{T}, t \in (0, T)\}, \\ \Omega_2(u_N, T) &= \{\dim_{\mathbb{H}} u_N(B \times \{x\}) = 4 \dim_{\mathbb{H}} B \ \forall \text{compact } B \subset (0, T), x \in \mathbb{T}\}, \\ \Omega_3(u_N, T) &= \{\dim_{\mathbb{H}} u_N(C) = \dim_{\mathbb{H}}^{\rho} C \ \forall \text{compact } C \subset (0, T) \times \mathbb{T}\}. \end{aligned}$$

Since $T > 0$ is arbitrary, we may let $T \rightarrow \infty$ to see that $\mathbb{P}(\Omega_i(u_N, \infty)) = 1$ if $p \geq p_i$ $[i = 1, 2, 3]$.⁴ Define

$$T_N = \inf\{t > 0 : \sup_{x \in \mathbb{T}} \|u_N(t, x)\| \geq N\},$$

where $\inf \emptyset = \infty$. Every T_N is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of the noise ξ , and the uniqueness of the solution to (1.1) implies that

$$\mathbb{P}\{u_N(t) = u(t) \ \forall t \in (0, T_N)\} = 1.$$

It follows that $\mathbb{P}(\Omega_i(u, T_N)) = 1$ if $p \geq p_i$ $[i = 1, 2, 3]$, notation being clear from context. By continuity, u is bounded on $[0, t] \times \mathbb{T}$ for every $t > 0$, and hence $\lim_{N \rightarrow \infty} \mathbb{P}\{T_N > t\} = 1$ for every $t > 0$. Therefore, we may let $N \rightarrow \infty$ in order to see that $\mathbb{P}(\Omega_i(u, \infty)) = 1$ if $p \geq p_i$ $[i = 1, 2, 3]$. This concludes the proof of Theorem 2.1. $\square$

7. PROOF OF THEOREM 1.2

For each $n \in \mathbb{Z}_+$, $T > 0$, $t \in (0, T]$, $x \in \mathbb{T}$, $\nu \in \mathbb{R}^p$, $r > 0$, and $\delta > 0$, define

$$\mathcal{N}_n^\delta(t, B(\nu, r)) = \#\{x \in F_n^\delta : u(t, x) \in B(\nu, r)\},$$

where $F_n^\delta = \{j 2^{-2(1+\delta)n} \in \mathbb{T} : j \in \mathbb{Z} \cap [-2^{2(1+\delta)n}, 2^{2(1+\delta)n}]\}$.

Proposition 7.1. *Suppose $p \geq 4$, $\mathcal{M}(b) < \infty$, $\mathcal{M}(\sigma) < \infty$ and $\inf_{v \in \mathbb{R}^p} \lambda(v) > 0$. Fix $T > 0$ and $\delta \in (0, 1)$. Then, there exists $L = L(p, \delta, T, b, \sigma, u_0) > 0$ such that*

$$\mathbb{P}\{\mathcal{N}_n^\delta(t, B(\nu, 2^{-n})) \geq 2^{2np\delta}\} \leq L^n n^{3pn} 2^{-\delta p n^2},$$

uniformly for all $\nu \in \mathbb{R}^p$, $t \in (0, T]$ and $n \in \mathbb{Z}_+$ such that $2^{-n} \leq t/2$.

⁴The proof of this included showing that the Ω_i s include measurable sets of $\mathbb{P}$ -mass one. Therefore, the Ω_i s are themselves measurable, thanks to the completeness of the underlying probability space.

Proof. Let $u_0 = h$ as in §4. Recall the definition of $I(t, x)$ in (4.3) and consider the p -dimensional random field

$$\mathcal{E}(t, x) = I(t, x) - \sigma(h(x))H(t, x) \quad \forall t \in (0, T], x \in \mathbb{T}.$$

First, we claim that if C_0 is a constant such that

$$\mathcal{H}_k(h) \leq C_0 \sqrt{k} \quad \forall k \in [2, \infty), \quad (7.1)$$

then there exists $L_1 = L_1(p, T, b, \sigma, C_0) > 0$ such that

$$\mathbb{E} \left(\sup_{x \in \mathbb{T}} \|I(t, x) - \sigma(h(x))H(t, x)\|^k \right) \leq L_1^k k^{3k/2} t^{k/2} |\log_+(1/t)|^{3k/2}, \quad (7.2)$$

uniformly for all $t \in (0, T]$ and $k \in [2, \infty)$. To see why this is the case, we first apply Lemma 4.5 and (7.1) to obtain

$$\mathbb{E} (\|\mathcal{E}(t, x)\|^k) \leq C^k k^{3k/2} t^{k/2} \quad \forall k \in [2, \infty).$$

This, together with Stirling's formula, implies the existence of some $c_1 > 0$ such that

$$\sup_{t, x} \mathbb{E} e^{c_1 [\|\mathcal{E}(t, x)\|/\sqrt{t}]^{2/3}} \leq \sum_{k=0}^{\infty} \frac{c_1^k}{k!} \sup_{t, x} \frac{\mathbb{E} (\|\mathcal{E}(t, x)\|^{2k/3})}{t^{k/3}} < \infty, \quad (7.3)$$

where " $\sup_{t, x} := \sup_{(t, x) \in [0, T] \times \mathbb{T}}$ " on both sides of the above. Next, we write

$$\mathcal{E}(t, x) = \mathcal{E}_1(t, x) + \mathcal{E}_2(t, x) - \mathcal{E}_3(t, x),$$

where

$$\begin{aligned} \mathcal{E}_1(t, x) &= \int_{(0, t) \times \mathbb{T}} G_{t-s}(x, y) b(u(s, y)) \, ds \, dy, \\ \mathcal{E}_2(t, x) &= \int_{(0, t) \times \mathbb{T}} G_{t-s}(x, y) \sigma(u(s, y)) \xi(ds \, dy), \quad \mathcal{E}_3(t, x) = \sigma(h(x))H(t, x). \end{aligned}$$

Recall the metric ρ defined in (1.10). Since $\mathcal{M}(b) < \infty$, we may deduce as in the proofs of Lemmas 4.2 and 4.3 that

$$\|\mathcal{E}_1(t, x) - \mathcal{E}_1(r, z)\|_k \lesssim \rho((t, x), (r, z)) \quad \forall k \in [2, \infty), t, r \in (0, T], x, z \in \mathbb{T}.$$

Similarly, since $\mathcal{M}(\sigma) < \infty$, we have

$$\|\mathcal{E}_2(t, x) - \mathcal{E}_2(r, z)\|_k \lesssim \sqrt{k} \rho((t, x), (r, z)) \quad \forall k \in [2, \infty), t, r \in (0, T], x, z \in \mathbb{T}.$$

Moreover, by the Gaussianity of H , Lemmas 3.1, 3.2, 3.4, and (7.1),

$$\begin{aligned} &\|\mathcal{E}_3(t, x) - \mathcal{E}_3(r, z)\|_k \\ &\leq \|\sigma(h(x))(H(t, x) - H(r, z))\|_k + \|(\sigma(h(x)) - \sigma(h(z)))H(r, z)\|_k \\ &\lesssim \mathcal{M}(\sigma) \sqrt{k} \|H(t, x) - H(r, z)\|_2 + \text{Lip}(\sigma) \mathcal{H}_k(h) |x - z|^{1/2} \sqrt{k} \|H(r, z)\|_2 \\ &\lesssim k \rho((t, x), (r, z)) \end{aligned}$$

uniformly for all $k \in [2, \infty)$, $t, r \in (0, T]$, $x, z \in \mathbb{T}$. Hence, we have

$$\|\mathcal{E}(t, x) - \mathcal{E}(r, z)\|_k \lesssim k \rho((t, x), (r, z)) \quad \forall k \in [2, \infty), t, r \in (0, T], x, z \in \mathbb{T}.$$

The preceding, together with a standard metric entropy argument, then yields the existence of a constant $c_2 > 0$ such that

$$\mathbb{E} \exp \left(c_2 \sup_{(t, x), (s, y)} \frac{\|\mathcal{E}(t, x) - \mathcal{E}(s, y)\|}{\rho((t, x), (s, y)) \log_+(1/\rho((t, x), (s, y)))} \right) < \infty, \quad (7.4)$$

where “ $\sup_{(t,x),(s,y)} := \sup_{(t,x),(s,y) \in [0,T] \times \mathbb{T}: (t,x) \neq (s,y)}$ ”

For every $m \in \mathbb{N}$, define $\mathbb{T}_m = \{i/m \in \mathbb{T} : i \in \mathbb{Z} \cap [-m, m]\}$. Let $\lambda_0 \geq e$ be a sufficiently large integer such that $a \mapsto a \log_+(1/a)$ is increasing on $(0, 1/\lambda_0]$. Then, we argue by interpolation, and use (7.3) and (7.4) together with Chebyshev's inequality, in order to deduce that

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{x \in \mathbb{T}} \|\mathcal{G}(t, x)\| > z \right\} \\ & \leq \mathbb{P} \left\{ \max_{x \in \mathbb{T}_m} \|\mathcal{G}(t, x)\| > z/2 \right\} + \mathbb{P} \left\{ \sup_{x, y \in \mathbb{T}, |x-y| \leq 1/m} \|\mathcal{G}(t, x) - \mathcal{G}(t, y)\| > z/2 \right\} \\ & \lesssim m \exp \left(-c_1 \left[\frac{z}{2\sqrt{t}} \right]^{2/3} \right) + \exp \left(-\frac{c_2 z}{2m^{-1/2} \log_+(m^{1/2})} \right), \end{aligned}$$

uniformly for all $t, z > 0$ and $m \in \mathbb{N}$ with $m \geq \lambda_0^2$. Choose $m = \lambda_0^2 \lceil 1/t \rceil$ to find that there exists $c > 0$ such that for all $z > 0$ and $t \in (0, T]$,

$$\mathbb{P} \left\{ \sup_{x \in \mathbb{T}} \|\mathcal{G}(t, x)\| > z \right\} \lesssim \exp \left(\log_+(1/t) - \frac{cz^{2/3}}{t^{1/3}} \right) + \exp \left(-\frac{cz}{t^{1/2} \log_+(1/t)} \right).$$

It follows that

$$\begin{aligned} & \mathbb{E} \left(\sup_{x \in \mathbb{T}} \|I(t, x) - \sigma(h(x))H(t, x)\|^k \right) = \int_0^\infty kz^{k-1} \mathbb{P} \left\{ \sup_{x \in \mathbb{T}} \|\mathcal{G}(t, x)\| > z \right\} dz \\ & \lesssim k2^k c^{-3k/2} t^{k/2} |\log_+(1/t)|^{3k/2} \\ & \quad + \int_{2c^{-3/2}t^{1/2}|\log_+(1/t)|^{3/2}}^\infty kz^{k-1} \left(e^{\log_+(1/t) - \frac{cz^{2/3}}{t^{1/3}}} + e^{-\frac{cz}{t^{1/2} \log_+(1/t)}} \right) dz \\ & \leq k2^k c^{-3k/2} t^{k/2} |\log_+(1/t)|^{3k/2} \\ & \quad \left[1 + \int_1^\infty y^{k-1} \left(e^{\log_+(1/t) - 2^{2/3} \log_+(1/t) y^{2/3}} + e^{-2c^{-1/2} [\log_+(1/t)]^{1/2} y} \right) dy \right] \\ & \lesssim k2^k c^{-3k/2} t^{k/2} |\log_+(1/t)|^{3k/2} \left[1 + \int_1^\infty y^{k-1} e^{-(2^{2/3}-1)y^{2/3}} dy \right] \\ & \lesssim L^k t^{k/2} |\log_+(1/t)|^{3k/2} \Gamma(3k/2), \end{aligned}$$

uniformly for all $t \in (0, T]$ and $k \in [2, \infty)$, where $L > 0$ is a constant independent of t and k . This proves (7.2) under condition (7.1).

Next, let us write a subscript of t as follows to simplify the notation: $u_t(x) = u(t, x)$. This slightly abuses notation, since $u_1, \dots, u_p$ represent the respective coordinates of u , but it is consistent with standard probability nomenclature. To be sure, if we ever need to refer to the i th coordinate of $u_t(x)$, then we would write $u_{t,i}(x)$.

Consider a number $\eta \in (0, t)$, hold it fixed, and then apply the Markov property [4, Chapter 9] at time $t - \eta$ in order to see that the mild formulation (4.1) of the solution can be written as follows:

$$u_t(x) = (\mathcal{G}_\eta u_{t-\eta})(x) + \tilde{I}(\eta, x), \quad (7.5)$$

where

$$\begin{aligned}\tilde{I}(\eta, x) &= \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x, y) b(u_{t-\eta+s}(y)) \, ds \, dy \\ &\quad + \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x, y) \sigma(u_{t-\eta+s}(y)) \xi^{(t-\eta)}(ds \, dy),\end{aligned}$$

and $\xi^{(a)}$ denotes a space-time white noise that is independent of $\mathfrak{F}_a$. In fact, $\xi^{(a)}$ corresponds to a time shift by a units in the noise's time variable. Thanks to Lemma 4.2 and the assumption that $u_0 \in C^{1/2}(\mathbb{T})$ – see the Introduction – $\mathcal{H}_k(u_{t-\eta}) \leq C_0 \sqrt{k}$ uniformly for all $0 < \eta < t \leq T$ and $k \in [2, \infty)$, where C_0 is a positive number that depends only on u_0 . Therefore, we may apply (7.2), conditionally on $\mathfrak{F}_{t-\eta}$, in order to deduce that

$$\begin{aligned}\mathbb{E} \left(\sup_{x \in \mathbb{T}} \left\| \tilde{I}(\eta, x) - \sigma(u_{t-\eta}(x)) \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x, y) \xi^{(t-\eta)}(ds \, dy) \right\|^k \right) \\ \leq L_1^k k^{3k/2} \eta^{k/2} |\log_+(1/\eta)|^{3k/2},\end{aligned}$$

where $L_1 = L_1(p, T, b, \sigma, u_0) > 0$. This and (7.5) together yield

$$\begin{aligned}\mathbb{E} \left(\sup_{x \in \mathbb{T}} \left\| u(t, x) - (\mathcal{G}_\eta u_{t-\eta})(x) - \sigma(u_{t-\eta}(x)) \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x, y) \xi^{(t-\eta)}(ds \, dy) \right\|^k \right) \\ \leq L_1^k k^{3k/2} \eta^{k/2} |\log_+(1/\eta)|^{3k/2},\end{aligned}\tag{7.6}$$

once again with good parameter dependencies. Then, (7.6) and Chebyshev's inequality imply that, for every $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P} \left\{ \sup_{x \in \mathbb{T}} \left\| u(t, x) - (\mathcal{G}_\eta u_{t-\eta})(x) - \sigma(u_{t-\eta}(x)) \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x, y) \xi^{(t-\eta)}(ds \, dy) \right\| > \varepsilon \right\} \\ \leq (L_1/\varepsilon)^k k^{3k/2} \eta^{k/2} |\log_+(1/\eta)|^{3k/2}.\end{aligned}\tag{7.7}$$

Set $\mathcal{N}_n^\delta(v; t, \eta, B(\nu, r)) = \#\{x \in F_n^\delta : v(t, \eta, x) \in B(\nu, r)\}$, where

$$v(t, \eta, x) = (\mathcal{G}_\eta u_{t-\eta})(x) + \sigma(u_{t-\eta}(x)) \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x, y) \xi^{(t-\eta)}(ds \, dy),$$

and choose $\varepsilon = \varepsilon_n = 2^{-n}$. By the triangle inequality and (7.7),

$$\begin{aligned}\mathbb{P} \{ \mathcal{N}_n^\delta(t, B(\nu, 2^{-n})) \geq 2^{2\delta p n} \} \\ \leq \mathbb{P} \{ \mathcal{N}_n^\delta(v; t, \eta, B(\nu, 2^{-n+1})) \geq 2^{2\delta p n} \} + (L_1/\varepsilon)^k k^{3k/2} \eta^{k/2} |\log_+(1/\eta)|^{3k/2},\end{aligned}\tag{7.8}$$

uniformly for all $t \in (0, T]$, $\nu \in \mathbb{R}^p$, $n \in \mathbb{Z}_+$, $\eta \in (0, t)$, and $k \in [2, \infty)$. Let $\tilde{\mathcal{N}}_n^\delta(t, \eta, B(\nu, r))$ denote the total number of n -tuples $x_1 < \dots < x_n$ in F_n^δ such that $v(t, \eta, x_i) \in B(\nu, r)$ for all $1 \leq i \leq n$; namely,

$$\tilde{\mathcal{N}}_n^\delta(t, \eta, B(\nu, r)) = \sum_{x_1 < \dots < x_n \text{ in } F_n^\delta} \sum \mathbb{1}_{\{\max_{1 \leq i \leq n} \|v(t, \eta, x_i) - \nu\| \leq r\}}.$$

The above and Chebyshev's inequality together imply that

$$\begin{aligned}
& \mathbb{P} \left\{ \mathcal{N}_n^\delta(v; t, \eta, B(\nu, 2^{-n+1})) \geq 2^{2\delta pn} \right\} \\
& \leq \mathbb{P} \left\{ \tilde{\mathcal{N}}_n^\delta(t, \eta, B(\nu, 2^{-n+1})) \geq \binom{\lceil 2^{2\delta pn} \rceil}{n} \right\} \\
& \leq \left(\binom{\lceil 2^{2\delta pn} \rceil}{n} \right)^{-1} \sum_{x_1 < \dots < x_n \text{ in } F_n^\delta} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|v(t, \eta, x_i) - \nu\| \leq 2^{-n+1} \right\}.
\end{aligned} \tag{7.9}$$

In order to estimate the last probability, let us consider an arbitrary but fixed n -tuple of distinct points $x_1 < \dots < x_n$ in F_n^δ , condition on $\mathcal{F}_{t-\eta}$, and notice that the quantity inside the $\|\dots\|$ in the event is conditionally a centered and continuous Gaussian process. Therefore, we may apply conditionally the Anderson shifted-ball inequality [2] in order to see that

$$\begin{aligned}
& \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|v(t, \eta, x_i) - \nu\| \leq 2\varepsilon \right\} \\
& \leq \mathbb{P} \left\{ \max_{1 \leq i \leq n} \left\| \sigma(u_{t-\eta}(x_i)) \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x_i, y) \xi^{(t-\eta)}(ds dy) \right\| \leq 2\varepsilon \right\} \\
& \leq \mathbb{P} \left\{ \max_{1 \leq i \leq n} \left\| \int_{(0, \eta) \times \mathbb{T}} G_{\eta-s}(x_i, y) \xi^{(t-\eta)}(ds dy) \right\| \leq 2\varepsilon / \inf_{v \in \mathbb{R}^p} \lambda(v) \right\} \\
& = \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|H(\eta, x_i)\| \leq 2\varepsilon / \inf_{v \in \mathbb{R}^p} \lambda(v) \right\} \\
& \leq \left(\mathbb{P} \left\{ \max_{1 \leq i \leq n} |H_1(\eta, x_i)| \leq 2\varepsilon / \inf_{v \in \mathbb{R}^p} \lambda(v) \right\} \right)^p,
\end{aligned}$$

where we recall $\lambda(v)$ denotes the smallest singular value of $\sigma(v)$ and H was defined in (3.1) and represents the solution to (1.1) with $\sigma = \text{identity matrix}$ and zero initial data. We have also used the facts that: (i) The law of $\xi^{(a)}$ does not depend on a ; and (ii) The coordinates of H are i.i.d. Since the probability density function of a centered, real-valued Gaussian random variable Z is bounded above by $1/\sqrt{2\pi \text{Var}(Z)}$, we proceed iteratively by successive conditioning on the values of $H_1(\eta, x_1), \dots, H_1(\eta, x_i)$ as i varies from $n-1$ to 1 in order to see that

$$\begin{aligned}
& \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|v(t, \eta, x_i) - \nu\| \leq 2\varepsilon \right\} \leq \left(\frac{2\varepsilon}{\inf_{v \in \mathbb{R}^p} \lambda(v) \sqrt{2\pi \text{Var}(H_1(\eta, x_1))}} \right)^p \times \\
& \quad \times \prod_{i=2}^n \left(\frac{2\varepsilon}{\inf_{v \in \mathbb{R}^p} \lambda(v) \sqrt{2\pi \text{Var}(H_1(\eta, x_i) | \mathcal{H}_{i-1})}} \right)^p \\
& \leq C^n \varepsilon^{np} \eta^{-p/4} \prod_{i=2}^n \left(\eta^{p/4} \wedge |x_i - x_{i-1}|^{p/2} \right)^{-1},
\end{aligned}$$

where $C > 0$ does not depend on the choice of $(t, x_1, x_2, \dots)$, $\mathcal{H}_i$ denotes the σ -algebra generated by $H_1(\eta, x_1), \dots, H_1(\eta, x_i)$ when $i = 1, \dots, n-1$, and the last inequality follows from Lemma 3.4 and strong local nondeterminism (Proposition

3.6). Whenever $x_0 \in F_n^\delta$,

$$\begin{aligned} \sum_{x \in F_n^\delta \setminus \{x_0\}} \left(\eta^{p/4} \wedge |x - x_0|^{p/2} \right)^{-1} &\leq \sum_{1 \leq j \leq \varepsilon^{-2(1+\delta)}} \frac{2}{\eta^{p/4} \wedge (j \varepsilon^{2(1+\delta)})^{p/2}} \\ &\lesssim \sum_{1 \leq j \leq \sqrt{\eta} \varepsilon^{-2(1+\delta)}} j^{-p/2} \varepsilon^{-p(1+\delta)} + \sum_{\sqrt{\eta} \varepsilon^{-2(1+\delta)} < j \leq \varepsilon^{-2(1+\delta)}} \eta^{-p/4} \\ &\lesssim \begin{cases} \varepsilon^{-p(1+\delta)} + \varepsilon^{-2(1+\delta)} \eta^{-p/4} & \text{if } p \geq 3, \\ \varepsilon^{-p(1+\delta)} \log(\sqrt{\eta} \varepsilon^{-2(1+\delta)}) + \varepsilon^{-2(1+\delta)} \eta^{-p/4} & \text{if } p = 2, \end{cases} \end{aligned}$$

where the implied constants depend only on p . Now let us suppose $p \geq 4$. In that case, we can optimize the preceding bound by choosing

$$\eta = \varepsilon^{(1+\delta)(4-(8/p))},$$

and deduce that $\sum_{x \in F_n^\delta \setminus \{x_0\}} (\eta^{p/4} \wedge |x - x_0|^{p/2})^{-1} \lesssim \varepsilon^{-p(1+\delta)}$, uniformly for all $n \in \mathbb{N}$ and $x_0 \in F_n^\delta$. It follows that

$$\begin{aligned} \mathbb{E} \left[\tilde{\mathcal{N}}_n^\delta(t, \eta, B(\nu, 2\varepsilon)) \right] &\leq \sum_{x_1 < \dots < x_n \text{ in } F_n^\delta} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|v(t, \eta, x_i) - \nu\| \leq 2\varepsilon \right\} \\ &\leq C_1^n \varepsilon^{pn} \sum_{x_1 \in F_n^\delta} \eta^{-p/4} \sum_{x_2 \in F_n^\delta \setminus \{x_1\}} \left(\eta^{p/4} \wedge |x_2 - x_1|^{p/2} \right)^{-1} \times \dots \\ &\quad \dots \times \sum_{x_n \in F_n^\delta \setminus \{x_{n-1}\}} \left(\eta^{p/4} \wedge |x_n - x_{n-1}|^{p/2} \right)^{-1} \\ &\leq C_2^n \varepsilon^{pn} \varepsilon^{-(1+\delta)pn} = C_2^n \varepsilon^{-\delta pn}, \end{aligned}$$

where $C_1, C_2 > 0$ do not depend on n . We plug this into (7.9), recall that $\varepsilon = 2^{-n}$, and then appeal to Lemma A.5 in order to see that

$$\mathbb{P} \left\{ \mathcal{N}_n^\delta(v; t, \eta, B(\nu, 2^{-n+1})) \geq 2^{2\delta pn} \right\} \leq c^n n^{2-\delta pn^2},$$

where c does not depend on n . Because $p \geq 4$, we also have $\eta^{1/2} \leq \varepsilon^{1+\delta}$. Therefore, we may select $k = pn$ in (7.8) to find that there exists $c_0, c_1 > 0$ such that

$$\begin{aligned} \mathbb{P} \left\{ \mathcal{N}_n^\delta(t, B(\nu, 2^{-n})) \geq 2^{2\delta pn} \right\} \\ \leq c_0^n n^{2-\delta pn^2} + L_1^{pn} (np)^{3pn/2} \varepsilon^{\delta pn} |\log(1/\eta)|^{3pn/2} \leq c_1^n n^{3pn} 2^{-\delta pn^2}, \end{aligned}$$

uniformly for all $t \in (0, T]$, $\nu \in \mathbb{R}^p$ and $n \in \mathbb{N}$. This completes the proof of Proposition 7.1. $\square$

We are ready to verify Theorem 1.2 and conclude the paper.

Proof of Theorem 1.2. Suppose $p \geq 4$. Choose and fix $0 < S < T < \infty$ throughout. It suffices to prove that, off a single P-null set,

$$\dim_{\mathbb{H}} u(\{t\} \times F) = 2 \dim_{\mathbb{H}} F \quad \forall \text{ compact } F \subset \mathbb{T}, t \in [S, T]. \quad (7.10)$$

The proof of (7.10) is divided in two parts. In the first part, we verify (7.10) under the additional hypothesis that

$$\mathcal{M}(b) < \infty, \mathcal{M}(\sigma) < \infty, \text{ and } \inf_{v \in \mathbb{R}^p} \lambda(v) > 0. \quad (7.11)$$

The second part of the proof is concerned with removing (7.11).

Part 1. Suppose that (7.11) holds. It is well known that, with probability one, u is Hölder continuous with any fixed index $< \frac{1}{2}$ in its space variable locally uniformly in time and off a single P-null set. More precisely, there exists a P-null set off which

$$\sup_{t \in [0, T]} \sup_{\substack{x, z \in \mathbb{T}: \\ x \neq z}} \frac{\|u(t, x) - u(t, z)\|}{|x - z|^{(1-\varepsilon)/2}} < \infty \quad \forall \varepsilon \in (0, 1), \quad T > 0; \quad (7.12)$$

see for example Lemma 4.4. Also, thanks to Lemma 4.4 and Proposition 7.1, we can apply Lemma 5.2 to see that, for every $\delta \in (0, 1)$ and $R > 0$, there exist $K = K(\delta) > 0$ and $L = L(p, S, T, b, \sigma, u_0, \delta, R) > 0$ such that

$$\mathbb{P} \left\{ \sup_{t \in [S, T]} \sup_{\nu \in B(0, R)} n_n^\delta(t, B(\nu, 2^{-n})) \geq K 2^{2\delta p n} \right\} \leq L^n e^{-n^2/L} \quad \forall n \in \mathbb{N}.$$

By the Borel–Cantelli lemma, there exists a P-null set off which

$$\sup_{t \in [S, T]} \max_{\nu \in B(0, R)} N_n(t, B(\nu, 2^{-n})) = \mathcal{O}(2^{2np\delta}) \quad \text{as } n \rightarrow \infty. \quad (7.13)$$

With (7.12) and (7.13) in place, we can then apply Lemma 5.1 to obtain (7.10).

Part 2. We now apply a truncation argument to prove the theorem without assuming (7.11). The truncation argument is somewhat delicate and leads to the assumptions of Theorem 1.2, which are all we assume from now on.

Define, for every $N > 0$, a function b_N via (6.7). Recall from (6.8) that b_N is globally Lipschitz with

$$\text{Lip}(b_N) \leq \text{Lip}(b) \quad \forall N > 0. \quad (7.14)$$

According to Lemma 2.5, the function λ is continuous, where λ was the minimum singular-value function associated to σ ; see Definition 2.2. With (7.14) in place, we begin our truncation argument. Define $\Lambda(r) := \lambda^{-1}[0, r] = \{v \in \mathbb{R}^p : \lambda(v) \leq r\}$ to be the level set of λ at r for every $r > 0$, and set

$$\sigma_r(v) = \begin{cases} \sigma(v) & \text{if } v \in \Lambda(r), \\ \sigma(v) + d_r(v) \mathbf{I} & \text{if } v \in \Lambda(r), \end{cases} \quad (7.15)$$

where $\mathbf{I} = (\delta_0(i - j))_{i,j=1}^p$ denotes the $p \times p$ identity matrix and d_r is defined to be the “internal distance function to the restriction of the boundary $\partial\Lambda(r)$ ”; that is,

$$d_r(x) = \begin{cases} \inf \{\|x - y\| : y \in \partial\Lambda(r)\} & \text{if } x \in \Lambda(r), \\ 0 & \text{otherwise.} \end{cases}$$

The function d_r is Lipschitz continuous whenever $\partial\Lambda(r) \neq \emptyset$ (see Lemma A.1 below), and this is the case when $r > 0$ is sufficiently small. In this case, σ_r is Lipschitz continuous. Next, set

$$\sigma_{r,N}(v) = \begin{cases} \sigma_r(v) & \text{if } \|v\| \leq N, \\ \sigma_r(vN/\|v\|) & \text{if } \|v\| > N. \end{cases} \quad (7.16)$$

As in (6.8), $\text{Lip}(\sigma_{r,N}) \leq \text{Lip}(\sigma_r)$ for all $r, N > 0$. Therefore, $\sigma_{r,N}$ is bounded and Lipschitz continuous provided that $r \ll 1$ and $N \gg 1$.

Let $\lambda(v; r)$ and $\lambda(v; N, r)$ respectively denote the smallest singular value of σ_r and $\sigma_{r,N}$. That is,

$$\lambda(v; r) = \inf_{x \in \mathbb{R}^p : \|x\|=1} \|\sigma_r(v)x\|^2, \quad \lambda(v; N, r) = \inf_{x \in \mathbb{R}^p : \|x\|=1} \|\sigma_{r,N}(v)x\|^2.$$

Thanks to (7.15) and (7.16),

$$\lambda(v; N, r) = \begin{cases} \lambda(v) & \text{if } v \in B(0, N) \setminus \Lambda(r), \\ \lambda(v) + d_r(v)^2 & \text{if } v \in B(0, N) \cap \Lambda(r), \\ \lambda(vN/\|v\|; r) & \text{if } \|v\| > N. \end{cases}$$

Apply Lemma 2.5 with σ replaced by $\sigma_{r,N}$ to see that $v \mapsto \lambda(v; N, r)$ is continuous. By virtue of its construction, $\lambda(v; N, r) > 0$ everywhere in $B(0, N)$. Since $\inf_{\|v\| > N} \lambda(v; N, r) = \inf_{\|v\| = N} \lambda(v; N, r)$, compactness yields

$$\inf_{v \in \mathbb{R}^p} \lambda(v; N, r) = \inf_{v \in B(0, N)} \lambda(v; N, r) > 0. \quad (7.17)$$

With the above observations in place, let us write $u_{r,N}$ for the solution to (1.1) where b is replaced by b_N and σ is replaced by $\sigma_{r,N}$. Since $\mathcal{M}(b_N) = \sup_{\|v\| \leq N} \|b(v)\| < \infty$ and $\mathcal{M}(\sigma_{r,N}) = \sup_{\|v\| \leq N} \|\sigma_r(v)\| < \infty$, and because of (7.17), we may apply Part 1 of this proof to $u_{r,N}$ in place of u to see that

$$\mathbb{P} \{ \dim_{\mathbb{H}} u_{r,N}(\{t\} \times F) = 2 \dim_{\mathbb{H}} F \ \forall \text{ compact } F \subset \mathbb{T}, t > 0 \} = 1.$$

Let

$$T_{r,N} = \inf \left\{ t > 0 : \sup_{x \in \mathbb{T}} \|u(t, x)\| \geq N \right\} \wedge \inf \{ t > 0 : \lambda(u(t, x)) \leq r \},$$

where $\inf \emptyset = \infty$. Every $T_{r,N}$ is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of the noise ξ , and (7.16) and the uniqueness of the solution to (1.1) together imply that

$$T_{r,N} = \inf \left\{ t > 0 : \sup_{x \in \mathbb{T}} \|u_{r,N}(t, x)\| \geq N \right\} \wedge \inf \{ t > 0 : \lambda(u_{r,N}(t, x)) \leq r \},$$

and $\mathbb{P}\{u_{r,N}(t) = u(t) \text{ for all } t < T_{r,N}\} = 1$. Also, off a single $\mathbb{P}$ -null set,

$$\dim_{\mathbb{H}} u(\{t\} \times F) = 2 \dim_{\mathbb{H}} F \quad \forall \text{ compact } F \subset \mathbb{T}, t \in (0, T_{r,N}). \quad (7.18)$$

Because u is bounded on space-time compacta and $\{\lambda = 0\}$ is polar for u [Assumption 2.3], $\lim_{N \rightarrow \infty, r \rightarrow 0} \mathbb{P}\{T_{r,N} > t\} = 1$ for every $t > 0$. Therefore, (7.18) implies the result and concludes the proof. $\square$

APPENDIX A. A MISCELLANY OF RELATED RESULTS

This appendix contains a few technical results that are used in the body of the paper. The following is well known. We include a short proof for the sake of completeness.

Lemma A.1. *Let A be a nonempty subset of $\mathbb{R}^p$ and $d_A : \mathbb{R}^p \rightarrow \mathbb{R}$ be the distance function defined by $d_A(x) = \inf\{\|x - z\| : z \in A\}$. Then d_A is Lipschitz continuous function with Lipschitz constant 1.*

Proof. If $x, y \in \mathbb{R}^p$, then the triangle inequality yields $d_A(x) \leq \|x - z\| \leq \|x - y\| + \|y - z\|$ for all $z \in A$. Take infimum over $z \in A$ to see that $d_A(x) - d_A(y) \leq \|x - y\|$. Interchange the roles of x and y to conclude. $\square$

The following simple fact is found by splitting the sum according to whether or not $n > \lambda^{-1/p}$, particularly relevant only when $\lambda \in (0, 1)$.

Lemma A.2. *If $p > 1$, then $\sum_{n=1}^{\infty} n^{-2} \min(1, \lambda n^p) \lesssim \lambda^{1/p}$ uniformly for all $\lambda > 0$.*

The following is a basic form of the Poisson summation formula [20].

Lemma A.3. $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n) \quad \forall f \in \mathcal{S}(\mathbb{R})$.

The following is a multidimensional version of the Burkholder-Davis-Gundy inequality for stochastic convolutions. When $p = 1$, this inequality can be found for example in [13, Proposition 4.4]. The proof in the general case follows a similar route. We include it since the precise constants below might require some justification.

Lemma A.4 (BDG inequality). *Whenever $Z = \{Z(s, y)\}_{s>0, y \in \mathbb{T}}$ is a predictable random field with values in the space of $p \times p$ matrices,*

$$\left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) Z(s, y) \xi(ds dy) \right\|_k^2 \leq 4kp \int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \|Z(s, y)\|_k^2,$$

for every $t > 0$, $x \in \mathbb{T}$, and $k \in [2, \infty)$.

Proof. There is nothing to prove when $\int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \|Z(s, y)\|_k^2$ is infinite. Therefore, we assume throughout that the integral is finite. Because of that fact it follows that

$$M_0 = 0, \quad M_t = \int_{(0,t) \times \mathbb{T}} G_{T-s}(x, y) \sum_{j=1}^p Z_{i,j}(s, y) \xi_j(ds dy) \quad [0 < t \leq T]$$

defines a continuous L^2 -martingale for every $T > 0$. If X is random variable with values in $\mathbb{R}^p$, then

$$\|X\|_k^2 = [\mathbb{E}(\|X\|^k)]^{2/k} \leq \sum_{i=1}^p \| |X_i|^2 \|_{k/2} = \sum_{i=1}^p \|X_i\|_k^2. \quad (\text{A.1})$$

For every $i = 1, \dots, p$, the quadratic variation of $\{M_t\}_{t \in [0, T]}$ is

$$\langle M \rangle_t = \int_0^t ds \int_{\mathbb{T}} dy [G_{T-s}(x, y)]^2 \sum_{j=1}^p |Z_{i,j}(s, y)|^2 \quad \forall t \in (0, T].$$

Therefore, we apply (A.1) and the BDG inequality for stochastic convolutions [13]Prop. 4.4 with $t = T$ to see that

$$\begin{aligned} & \left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) Z(s, y) \xi(ds dy) \right\|_k^2 \\ & \leq \sum_{i=1}^p \left\| \int_{(0,t) \times \mathbb{T}} G_{t-s}(x, y) \sum_{j=1}^p Z_{i,j}(s, y) \xi_j(ds dy) \right\|_k^2 \\ & \leq 4k \sum_{i=1}^p \left\| \int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \sum_{j=1}^p |Z_{i,j}(s, y)|^2 \right\|_{k/2} \\ & \leq 4k \sum_{i=1}^p \int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \left\| \sum_{j=1}^p |Z_{i,j}(s, y)|^2 \right\|_{k/2} \\ & \leq 4kp \int_0^t ds \int_{\mathbb{T}} dy [G_{t-s}(x, y)]^2 \| \|Z(s, y)\|_k^2 \|_{k/2}. \end{aligned}$$

This implies the lemma. $\square$

We will need the following particular application of Stirling's formula.

Lemma A.5. *For every $\alpha > 0$, $\binom{\lceil 2^{\alpha n} \rceil}{n} \sim (2\pi n)^{-1/2} 2^{\alpha n^2} (e/n)^n$ as $n \rightarrow \infty$.*

And the following is a well-known consequence of a direct covering argument; see for example Falconer [8, Proposition 2.3].

Lemma A.6. *If $\exists \alpha > 0$ such that $f : \mathbb{T} \rightarrow \mathbb{R}^p$ satisfies $\|f(x) - f(z)\| \lesssim |x - z|^\alpha$ uniformly for all $x, z \in \mathbb{T}$, then $\dim_{\mathbb{H}} f(F) \leq \alpha^{-1} \dim_{\mathbb{H}} F$ for every Borel set $F \subset \mathbb{T}$.*

Finally, the following is an infinite-dimensional extension of the well-known Girsanov theorem. See Allouba [1] for a proof in the case that $p = 1$ and Da Prato and Zabczyk [4] for a quite general, abstract version.

Lemma A.7 (Girsanov's theorem). *Choose and fix a number $T > 0$, and a predictable random field $\{Z(t, x)\}_{(t,x) \in [0,T] \times \mathbb{T}}$, with values in $\mathbb{R}^p$, that satisfies $\mathbb{E} \exp(\frac{1}{2} \|Z\|_{L^2([0,T] \times \mathbb{T})}^2) < \infty$. Then, $\zeta(dt dx) = Z(t, x)dt dx + \xi(dt dx)$ is a p -dimensional space-time white noise on $(\Omega, \mathcal{F}_T, \mathbb{Q})$, where*

$$d\mathbb{Q}/d\mathbb{P} = \exp\left(-M_T - \frac{1}{2} \langle M \rangle_T\right),$$

for $M_t = \int_{[0,t] \times \mathbb{T}} Z \cdot d\xi$ for every $t \in [0, T]$.

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