ON A GRAUERT-RIEMENSCHNEIDER VANISHING THEOREM IN DIMENSION 3

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Abstract. Suppose R is an excellent ring of dimension 3 and has rational singularities. Let $\pi : X \longrightarrow \mathrm{Spec}\ R$ be a blow-up and $\phi : W \longrightarrow X$ be any projective, birational morphism such that X and W are both normal, Cohen-Macaulay and have pseudorational singularities in codimension 2. Then $\mathbf{R}^i\phi_*\omega_W = 0$ and $\mathbf{R}^i\pi_*\omega_X = 0$ for all $i > 0$ and X has rational singularities. We use this result to prove Lipman's vanishing conjecture in dimension 3 for arbitrary characteristics and provide a few applications.

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Date: October 27, 2025.

1. Introduction

Kodaira-type vanishing theorems are fundamental tools in birational algebraic geometry. A relative variant of Kodaira vanishing is the Grauert-Riemenschneider (GR) vanishing theorem [GR70] which states that for a projective birational morphism $f : Y \longrightarrow X$ where Y is a smooth projective variety over $\mathbb{C}$, we have $\mathbf{R}^i f_* \omega_Y = 0$ for all $i > 0$.

Rational singularities have been the gold standard for "mild" singularities in algebraic geometry. In characteristic 0, a scheme X is said to have *rational singularities* if for any resolution of singularities $f : Y \longrightarrow X$, we have $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$, (i.e, X is normal), and $\mathbf{R}^i f_*\mathcal{O}_X = 0$ for $i > 0$. Kempf, using Grauert-Riemenschneider vanishing, proved that a normal scheme X has rational singularities if and only if it is Cohen-Macaulay and for any resolution of singularities $f : Y \longrightarrow X$, we have $f_*\omega_Y \simeq \omega_X$, see [KKMSD73, page 50]. There is a close relationship between MMP singularities and rational singularities; in particular, Kawamata log terminal singularities are rational ([Elk81]) and Gorenstein rational singularities are canonical.

Unfortunately, Kodaira vanishing fails in positive characteristics ([Ray78]), and as a result, by the cone construction one sees that, the GR vanishing fails when the dimension is at least 3 ([HK15, Example 3.11]). On a positive note, Chatzistamatiou and Rülling proved in [CR15] that GR vanishing holds for excellent, regular schemes X and Y in any dimension. In dimension 3, positive and mixed characteristics ($p > 5$), GR vanishing holds for klt singularities, proving that they are rational, see [ABL22, BK23, Bhu25, HW19].

However, as the resolution of singularities is not known in dimensions greater than 3 in positive and mixed characteristics, what should be the "correct" definition of rational singularity remains an interesting question, see [Kov17, Lyu22, IY25]. Due to the example in [Cut90, Page 174], we can not simply assume Y to be normal, even when X is regular. Furthermore, as the statement involves the dualizing sheaf ω_Y and not the dualizing complex $\omega_Y^\bullet$, we need to assume Y is Cohen-Macaulay. But, by applying Brodmann's Macaulayfication ([Bro83, Corollary 1.4]) of Cutkosky's example, Ma showed that there exists a projective, birational morphism $f : Y \longrightarrow X = \mathbb{A}^3_{\mathbb{C}}$ with Y arithmetically Cohen-Macaulay, such that $\mathbf{R}^2 f_*\mathcal{O}_Y \neq 0$, see [IY25, Example 3.2] for details. However, we show in our main theorem that such (counter-)examples can not occur if we assume Y to be Cohen-Macaulay and pseudorational in codimension 2.

Theorem A. (cf. Theorems 3.1, 3.2 and, 3.4) *Let* $(R, \mathfrak{m})$ *be an excellent, local ring of dimension 3 with rational singularities. Let* $\pi : X \longrightarrow \operatorname{Spec} R$ *be a blow-up where* X *is normal, Cohen-Macaulay, and has pseudorational singularities in codimension 2. Let* $\phi : W \longrightarrow X$ *be any projective, birational morphism where* W *is normal, Cohen-Macaulay, and pseudorational in codimension 2. Then we have the following:*

(a) $\mathbf{R}^i \phi_* \omega_W = 0$ *for* $i > 0$,
(b) $\mathbf{R}^i \pi_* \omega_X = 0$ *for* $i > 0$,

(c) *X has rational singularities.*

Using this theorem, we prove Lipman's vanishing conjecture (see [Lip94a, Vanishing Conjecture (2.2)]) for arbitrary characteristics in dimension 3.

Theorem B. (see [Lip94a, (2.2), Theorem A3.], [HV98, 2.7] and Proposition 4.1) *Let $(R, \mathfrak{m})$ be a three-dimensional rational singularity, and let $\mathcal{I} \subset R$ be an ideal. Consider the blow-up*

$$f : X = \operatorname{Proj}\ S = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}^n\right) \longrightarrow \operatorname{Spec} R$$

along $\mathcal{I}$. Assume that S is Cohen-Macaulay, X is normal, and has pseudo-rational singularities in codimension 2. Then for all integers $n \geq 0$ and all positive integers $i > 0$, we have

$$H^i(X, \mathcal{I}^n \omega_X) = 0.$$

In characteristic 0, this was proved by Cutkosky (see [Lip94a, Theorem A3.]) and by Hyry and Villamayor U. (see, [HV98, 2.7]) in all dimensions.
We end this short note by presenting some applications of Theorem A and Theorem B, see Corollary 3.3, Proposition 4.3 and Corollary 4.5.

2. Background

Definition 2.1. ([Kol13, Definition 2.76], [Lip69]) A local ring $(R, \mathfrak{m})$ is said to have *rational singularities* if

(a) R is normal, excellent, Cohen-Macaulay, admitting a dualising complex, and
(b) there exists a resolution of singularities $f : Y \longrightarrow \operatorname{Spec} R$ such that $\mathbf{R}^i f_* \mathcal{O}_Y = 0$ for all $i > 0$.

Remark 2.2. Thanks to [CR15], we have a well-defined notion of rational singularities in arbitrary characteristics, assuming (log) resolution of singularities (which exists, up to dimension 3, see [CP08], [CP09] and [CP19]).

A resolution-free definition of rational singularities was introduced by Lipman and Teissier.

Definition 2.3. ([LT81, Section 2]) A local ring $(R, \mathfrak{m})$ is said to have *pseudorational singularities* if

(a) R is normal and Cohen-Macaulay and, $\widehat{R}$ is reduced.
(b) For every proper birational morphism $\pi : W \longrightarrow \operatorname{Spec} R$ with W normal, if $E = \pi^{-1}(\mathfrak{m})$ is the closed fibre, then the canonical map

$$\delta_\pi : H^d_{\mathfrak{m}}(R) \longrightarrow H^d_E(\mathcal{O}_W)$$

is injective.

Remark 2.4. Kempf, using Grauert-Riemenschneider vanishing showed that in characteristic 0, a normal scheme X has rational singularities if and only if it has pseudorational singularities, see [KKMSD73, page 50]. Also, any regular ring is pseudorational ([LT81]).

Definition 2.5. A scheme X has *pseudorational singularities in codimension 2* if, for all $x \in X$ with $\dim \mathcal{O}_{X,x} = 2$, the ring $\mathcal{O}_{X,x}$ has pseudorational (or equivalently, rational, by [LT81, Page 103, Example (a)]) singularities.

Definition 2.6. ([Bli04, Definition 2], [BST17, Definition 2.1]) Suppose that X is either a normal, excellent $\mathbb{Q}$-scheme, or a normal, excellent scheme of dimension ≤ 3 and Δ is an effective $\mathbf{Q}$-Cartier divisor, $\mathfrak{a}$ is an ideal sheaf and $\lambda \geq 0$ is a real number. Let $\pi : Y \longrightarrow X$ be a proper birational morphism with Y normal such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-G)$ is invertible, and we assume that K_X and K_Y agree wherever π is an isomorphism. We assume that π is a log resolution of $(X, \Delta, \mathfrak{a})$. Then we define the *multiplier module* to be

$$\mathcal{J}(\omega_X, \Delta, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*\Delta - \lambda G \rceil) \subseteq \omega_X.$$

When $\Delta = 0$, we omit it, i.e, we write $\mathcal{J}(\omega_X, \mathfrak{a}^\lambda) := \mathcal{J}(\omega_X, 0, \mathfrak{a}^\lambda)$.

It is a general fact that these definitions are independent of the (log) resolution chosen, see [Laz04, Theorem 9.2.18] and [BST17].

Remark 2.7. Suppose that X is a reduced, equidimensional, and Cohen-Macaulay scheme and $\mathfrak{a}$ is an ideal sheaf on X. Then the pair $(X, \mathfrak{a}^\lambda)$ has rational singularities in the sense of [ST08, Definition 3.1] if and only if the multiplier submodule $\mathcal{J}(\omega_X, \mathfrak{a}^\lambda)$ is equal to ω_X, see [ST08, Corollary 3.8].

We recall the Sancho de Salas exact sequence ([SdS87]) for blow ups, which serves as a bridge connecting local cohomology to sheaf cohomology.

Theorem 2.8. *(*[SdS87, Lip94b, HS03]*) The Sancho de Salas exact sequence for the Rees Algebra S is*

$$\cdots \longrightarrow H^i_{\mathfrak{m}_S}(S) \longrightarrow \bigoplus_{n \in \mathbb{N}} H^i_{\mathfrak{m}}(\mathfrak{a}^n) \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^i_Z(X, \mathcal{O}_X(-nE)) \longrightarrow H^{i+1}_{\mathfrak{m}_S}(S) \longrightarrow \ldots,$$

where $\mathfrak{m}_S = \mathfrak{m} \oplus \bigoplus_{n \geq 1} \mathfrak{a}^n$, $X = \mathrm{Proj}\ S \longrightarrow \mathrm{Spec}\ R$ is the blowup along $\mathfrak{a}$ so that $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-E)$ and $Z = X \times_{\mathrm{Spec}\, R} \mathrm{Spec}(R/\mathfrak{m})$ is the scheme-theoretic fiber over the closed point $\mathfrak{m}$ of $\mathrm{Spec}\, A$. We get, as graded R-modules, $H^{d+1}_{\mathfrak{m}_S}(S) \cong \oplus_{n<0} H^d_Z(X, \mathcal{O}_X(-nE))$ [HS03, 2.5.2 (1)]. *This is because the maps $H^d_m(\mathfrak{a}^n) \longrightarrow H^d_Z(X, \mathcal{O}_X(-nE))$ are surjective for all $n \geq 0$ (see,* [LT81, page 103]*). Hence, Matlis duality gives,*

$$\omega_{\mathcal{S}} \simeq \bigoplus_{n>0} H^0(X, \omega_X(-nE)).$$

A crucial ingredient in proving our main results is the following vanishing theorem of Lodh; and Ishii and Yoshida:

Theorem 2.9 ([IY25], [Lod24])**.** *Let Y be a Noetherian scheme of dimension $N \geq 1$. Assume, Y is locally quasi-unmixed, has pseudorational singularities in codimension two, and satisfies Serre's condition S_3. Let $\varphi : X \longrightarrow Y$ be a proper birational morphism of finite type with X normal. Then,*

$$\mathbf{R}^1\varphi_*\mathcal{O}_X = 0.$$

Lipman, by the following theorem, related Spec R having rational singularities with resolution being arithmetically Cohen-Macaulay in [Lip94b, Theorem (4.1)].

Theorem 2.10. *(*[Lip94b, Theorem 4.1]*) Let F be a filtration on a local ring $(R, \mathfrak{m})$ with Rees algebra R_F and $X = \operatorname{Proj}(R_F)$. Suppose:*

(a) *X is Cohen-Macaulay.*
(b) *The natural map $R \longrightarrow \mathbf{R}\Gamma(X, \mathcal{O}_X)$ is an isomorphism (i.e., $H^0(X, \mathcal{O}_X) = R$ and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$).*

Then for some $e > 0$, the Veronese subalgebra $R_{F^{(e)}}$ is Cohen-Macaulay. The converse also holds: if R_F is Cohen-Macaulay, then X is Cohen-Macaulay and $R \xrightarrow{\sim} \mathbf{R}\Gamma(X, \mathcal{O}_X)$.

3. Main Theorems

We now prove our main results.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be an excellent, local ring of dimension 3 that has rational singularities. Let $X = \operatorname{Proj} S \xrightarrow{\pi} \operatorname{Spec} R$ be a blow-up, where $S = \bigoplus_{n \geq 0} \mathfrak{a}^n$ for an ideal $\mathfrak{a} \subset R$. Assume that X is Cohen-Macaulay, normal, and has pseudorational singularities in codimension 2.*

Then

$$\mathbf{R}^i\pi_*\omega_X = 0 \quad \text{for all } i > 0.$$

Proof. We follow the strategy of [KK21] and proceed through six steps.

Step 1: Since R has rational singularities and is of dimension 3, there exists a dominating resolution of singularities $\phi : Y \longrightarrow X$.

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "f"] & \\ X \arrow[rr, "\pi"'] & & \operatorname{Spec} R \end{tikzcd}$$

By definition of rational singularities

$$\mathbf{R}^i(\pi \circ \phi)_*\mathcal{O}_Y = H^i(Y, \mathcal{O}_Y) = 0 \quad \text{for all } i > 0.$$

Step 2: We have two key properties of the resolution $\phi : Y \longrightarrow X$:

(a) Since ϕ is a resolution and X is normal, we have $\phi_*\mathcal{O}_Y = \mathcal{O}_X$.

(b) We prove that $\mathbf{R}^i\phi_*\mathcal{O}_Y$ has zero-dimensional support for $i > 0$, if they are non-zero. Let $x \in X$ be a point with $\dim \mathcal{O}_{X,x} = 2$. By assumption,

X has pseudorational singularities in codimension 2. By [LT81, Page 103, Example (a)], we have that pseudorational implies rational in dimension 2.

Since ϕ is proper and Spec $\mathcal{O}_{X,x} \longrightarrow X$ is flat, by flat base change [Har77a],

$$(\mathbf{R}^i\phi_*\mathcal{O}_Y)_x \cong \mathbf{R}^i g_*\mathcal{O}_{Y_x}.$$

where $g : Y_x := Y \times_X \operatorname{Spec}\ \mathcal{O}_{X,x} \longrightarrow \operatorname{Spec}\mathcal{O}_{X,x}$ is a resolution of a two-dimensional rational singularity. So $\mathbf{R}^i g_*\mathcal{O}_{Y_x} = 0$ for $i > 0$, and hence, $(\mathbf{R}^i\phi_*\mathcal{O}_Y)_x = 0$ for all x with $\dim\mathcal{O}_{X,x} = 2$.

Therefore, $\operatorname{Supp}(\mathbf{R}^i\phi_*\mathcal{O}_Y) \subseteq \{x \in X : \dim\mathcal{O}_{X,x} \geq 3\}$. Since X is three-dimensional and excellent, this set is finite, and hence zero-dimensional.

Step 3: Consider the Leray spectral sequence for the composition $\pi \circ \phi$:

$$E_2^{p,q} = \mathbf{R}^q\pi_*(\mathbf{R}^p\phi_*\mathcal{O}_Y) \implies \mathbf{R}^{p+q}(\pi\circ\phi)_*\mathcal{O}_Y = H^{p+q}(Y, \mathcal{O}_Y).$$

From Step 2(b), we have that for $p > 0$, $\mathbf{R}^p\phi_*\mathcal{O}_Y$ has zero-dimensional support. Therefore, $H^i(X, \mathbf{R}^p\phi_*\mathcal{O}_Y) = 0$ for $i > 0$. Thus

$$E_2^{p,q} = \mathbf{R}^q\pi_*(\mathbf{R}^p\phi_*\mathcal{O}_Y) = H^q(X, \mathbf{R}^p\phi_*\mathcal{O}_Y) = 0$$

for $p > 0$ and $q > 0$.

The five-term exact sequence (see [GW23, Proposition F.105]) from this spectral sequence is

$$0 \longrightarrow \mathbf{R}^1\pi_*(\phi_*\mathcal{O}_Y) \longrightarrow \mathbf{R}^1(\pi\circ\phi)_*\mathcal{O}_Y \longrightarrow \pi_*(\mathbf{R}^1\phi_*\mathcal{O}_Y) \longrightarrow \mathbf{R}^2\pi_*(\phi_*\mathcal{O}_Y) \longrightarrow \mathbf{R}^2(\pi\circ\phi)_*\mathcal{O}_Y.$$

Using

- $\phi_*\mathcal{O}_Y = \mathcal{O}_X$ (from Step 2(a)),
- $\mathbf{R}^i(\pi\circ\phi)_*\mathcal{O}_Y = 0$ for $i = 1, 2$ (from Step 1),
- $\pi_*(\mathbf{R}^1\phi_*\mathcal{O}_Y) = H^0(X, \mathbf{R}^1\phi_*\mathcal{O}_Y)$,

we obtain

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0 \longrightarrow H^0(X, \mathbf{R}^1\phi_*\mathcal{O}_Y) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0.$$

We immediately get $H^1(X, \mathcal{O}_X) = 0$, and $H^0(X, \mathbf{R}^1\phi_*\mathcal{O}_Y) \cong H^2(X, \mathcal{O}_X)$.

Now, by Theorem 2.9 ([IY25]), we have $\mathbf{R}^1\phi_*\mathcal{O}_Y = 0$, implying $H^2(X, \mathcal{O}_X) = 0$ as well. Note that we already have $H^3(X, \mathcal{O}_X) = 0$ as the closed fibers of $\pi : X \longrightarrow \operatorname{Spec}\ R$ have dimension ≤ 2, see [Gro61, Corollaire (4.2.2)].

Step 5: After Step 4, we are in the situation of Theorem 2.10 ([Lip94b]). Therefore, there exists $N \gg 0$ such that the Veronese subring $S^{(N)} = \bigoplus_{n\geq 0}\mathfrak{a}^{nN}$ is Cohen-Macaulay. Since $X = \operatorname{Proj} S \cong \operatorname{Proj} S^{(N)}$, we may replace S by $S^{(N)}$ and assume henceforth that S itself is Cohen-Macaulay.

Step 6: Let $d = \dim X = 3$. Apply the SdS sequence from Theorem 2.8:

$$\cdots \longrightarrow H^i_{\mathfrak{m}_S}(S) \longrightarrow \bigoplus_{n\geq 0} H^i_{\mathfrak{m}}(\mathfrak{a}^n) \longrightarrow \bigoplus_{n\in\mathbb{Z}} H^i_Z(X, \mathcal{O}_X(n)) \longrightarrow H^{i+1}_{\mathfrak{m}_S}(S) \longrightarrow \cdots$$

where $\mathfrak{m}_S = \mathfrak{m} \oplus \bigoplus_{n\geq 1}\mathfrak{a}^n$ and $Z = \pi^{-1}(\mathfrak{m})$.

Since S is Cohen-Macaulay of dimension $d+1=4$, we have $H^i_{\mathfrak{m}_S}(S)=0$ for $i<4$. Thus, for $i<3$, we have,

$$0 \longrightarrow \bigoplus_{n\geq 0} H^i_{\mathfrak{m}}(\mathfrak{a}^n) \xrightarrow{\cong} \bigoplus_{n\in\mathbb{Z}} H^i_Z(X,\mathcal{O}_X(n)) \longrightarrow 0$$

For $n=0$ and $i<3$, $H^i_{\mathfrak{m}}(R)=0$ (since R is Cohen-Macaulay), so $H^i_Z(X,\mathcal{O}_X)=0$.

By Grothendieck duality (see [Lip94b, Lemma (4.2)],

$$H^i_Z(X,\mathcal{O}_X) \cong \operatorname{Hom}_R(H^{3-i}(X,\omega_X), E(R/\mathfrak{m})) = 0$$

This immediately gives us

$$H^{3-i}(X,\omega_X) = \mathbf{R}^{3-i}\pi_*\omega_X = 0,\ \forall i<3.$$

This completes the proof of Theorem 3.1. □

3.1. **Pseudorational Modification.** We now prove that the vanishing property is stable under further "pseudorational in codimension 2" modifications.

Theorem 3.2. *Let $Y \xrightarrow{\psi} X \xrightarrow{\pi} \operatorname{Spec} R$, where R is of dimension 3 and has rational singularities. Assume the following.*

(a) *X has pseudorational singularities in codimension 2;*
(b) *X is Cohen-Macaulay;*
(c) *Y is a dominating resolution of singularities;*
(d) *π is a blow-up of $\operatorname{Spec} R$.*

Then, X has rational singularities. In particular, $\mathbf{R}^i\psi_*\omega_Y=0$ for all $i>0$.

Proof. Since R has rational singularities and $\pi\circ\psi : Y \longrightarrow \operatorname{Spec} R$ is a resolution of singularities, we have:

$$\mathbf{R}^i(\pi\circ\psi)_*\mathcal{O}_Y = 0 \quad \text{for all } i>0.$$

Since $\operatorname{Spec} R$ is affine, [Har77b, Proposition III.8.1] implies that,

$$H^i(Y,\mathcal{O}_Y)=0 \quad \text{for all } i>0.$$

By repeating Step 2 in the proof of Theorem 3.1 we get the support of $\mathbf{R}^p\psi_*\mathcal{O}_Y$ is at most zero-dimensional. Thus,

$$H^q(X,\mathbf{R}^p\psi_*\mathcal{O}_Y)=0 \text{ for } \ p,q>0$$

We will show that for $p>0$, $\mathbf{R}^p\psi_*\mathcal{O}_Y=0$ by analyzing the Leray spectral sequence of ψ:

$$E_2^{p,q} = H^q(X,\mathbf{R}^p\psi_*\mathcal{O}_Y) \Rightarrow H^{p+q}(Y,\mathcal{O}_Y).$$

We know that $E_2^{p,q}=0$ for $p>0$ and $q>0$. On the other hand, $E_2^{0,q}=H^q(X,\psi_*\mathcal{O}_Y)=H^q(X,\mathcal{O}_X)$ for $p=0$ and $E_2^{p,0}=H^0(X,\mathbf{R}^p\psi_*\mathcal{O}_Y)$ for $q=0$ may be nonzero. Note that $H^{p+q}(Y,\mathcal{O}_Y)=0$ for $p+q>0$, so that $E_\infty^{p,q}=0$ for all $p+q>0$. To show vanishing, we will check that the spectral sequence degenerates at the 2nd page. We note that the differential of E_2 has bidegree $(-1,2)$, so the only possibly non-zero differential on the second page is

$$E_2^{1,0} = H^0(X, \mathbf{R}^1\psi_*\mathcal{O}_Y) \xrightarrow{d_2} H^2(X, \mathcal{O}_X) = E_2^{0,2}.$$

Thus, $E_\infty^{p,q} = E_2^{p,q}$ for $(p,q) \neq (1,0),(0,2)$. Hence,

$$H^0(X, \mathbf{R}^2_*\psi_*\mathcal{O}_Y) = E_2^{2,0} = E_\infty^{2,0} = 0 \text{ and } H^0(X, \mathbf{R}^3_*\psi_*\mathcal{O}_Y) = E_2^{3,0} = E_\infty^{3,0} = 0,$$

so by Step 2 we have $\mathbf{R}^2\psi_*\mathcal{O}_Y = \mathbf{R}^3\psi_*\mathcal{O}_Y = 0$. Lastly, since X is Cohen-Macaulay, we conclude from Theorem 2.9 that $\mathbf{R}^1\psi_*\mathcal{O}_Y = 0$. Thus, the map $\mathcal{O}_X \longrightarrow \mathbf{R}\psi_*\mathcal{O}_Y$ is a quasi-isomorphism and X has rational singularities. By Grothendieck duality,

$$\mathbf{R}\psi_*\mathbf{R}\mathcal{H}om_Y(\mathcal{O}_Y, \omega_Y) \cong \mathbf{R}\mathcal{H}om_X(\mathbf{R}\psi_*\mathcal{O}_Y, \omega_X)$$

The left side is $\mathbf{R}\psi_*\omega_Y$. The right side is $\mathbf{R}\mathcal{H}om_X(\mathcal{O}_X, \omega_X) = \omega_X$ since $\mathbf{R}^i\psi_*\mathcal{O}_Y = 0$ for $i > 0$. Therefore $\mathbf{R}\psi_*\omega_Y \cong \omega_X$ i.e, $\mathbf{R}^i\psi_*\omega_Y = 0$ for all $i > 0$. $\square$

As a quick application, we get the following 3-dimensional analog of a result of Hara, Watanabe and Yoshida, see [HWY02, Theorem 3.5]).

Corollary 3.3. *([HWY02, Theorem 3.5]) Let $(R, \mathfrak{m})$ be an F-finite, local ring of characteristic p and dimension 3 that has rational singularities and $\mathfrak{a} \subset R$ is an $\mathfrak{m}$-primary ideal. Suppose, the blow up along $\mathfrak{a}$, $X = \operatorname{Proj} S \xrightarrow{\pi} \operatorname{Spec} R$ is a resolution of singularities, where $S = \bigoplus_{n \geq 0} \mathfrak{a}^n$. Then $S^{(N)}$ is F-rational for all $N \gg 0$.*

Proof. From Step 5 in the proof of Theorem 3.1 we get that $S^{(N)}$ is Cohen-Macaulay for all $N \gg 0$. We follow [HWY02, Proof of Theorem 3.5] for the rest. First note that, $\forall\, 0 \neq ft^N \in \mathfrak{a}^N t^N$, $S^{(N)}_{ft^N}$ is regular. So by [HH94], some power of f is a test element, giving $S^{(N)}_+ \subseteq \sqrt{\tau(S^{(N)})}$. Then, by [HWY02, Lemma 3.4], we just need to show that the Frobenius map acts injectively on $H^4_{\mathfrak{m}_{S^{(N)}}}(S^{(N)})$. Note that, by Theorem 2.8, we have

$$H^4_{\mathfrak{m}_{S^{(N)}}}(S^{(N)}) \cong \oplus_{n<0} H^3_E(X, \mathcal{O}_X(nN)).$$

So, we need to show that for all $n < 0$,

$$F : H^3_E(X, \mathcal{O}_X(nN)) \longrightarrow H^3_E(X, \mathcal{O}_X(pnN)) \text{ is injective.}$$

By [Har98, Proposition 3.5], injectivity follows if we can show the following two vanishings:

(a) $H^j_E(X, \Omega^i_X(\log\ E)(nN)) = 0$ for $i + j = 2$ and $i > 0$,
(b) $H^j_E(X, \Omega^i_X(\log\ E)(pnN)) = 0$ for $i + j = 3$ and $i > 0$

Now note that H^0_E of locally free sheaves are always 0, and in all other cases, we win by Serre vanishing as $N \gg 0$. So we get $S^{(N)}$ is F-rational for all $N \gg 0$. $\square$

Theorem 3.4. *Suppose R is of dimension 3 and has rational singularities. Let X be a blow-up of $\operatorname{Spec} R$ and X is normal, has pseudorational singularities in codimension 2 and is Cohen-Macaulay. Then for any projective, birational morphism $\phi : W \longrightarrow X$ with W normal, pseudorational in codimension 2 and Cohen-Macaulay, we have $\mathbf{R}^i\phi_*\omega_W = 0$ for $i > 0$.*

Proof. For the first statement, take a common resolution

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Y \arrow[dr, "\psi"'] & & W \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

where Z is a resolution of singularities dominating both Y and W. By Theorem 3.2, we have $\mathbf{R}^i\psi_*\omega_Y = 0$ and $\mathbf{R}^i\alpha_*\omega_Z = 0$. Then, again by repeatedly applying Theorem 3.2, we get,

$$\omega_X = \mathbf{R}\psi_*\omega_Y = \mathbf{R}\psi_*(\mathbf{R}\alpha_*\omega_Z) = \mathbf{R}(\psi\circ\alpha)_*\omega_Z = \mathbf{R}(\phi\circ\beta)_*\omega_Z = \mathbf{R}\phi_*(\mathbf{R}\beta_*\omega_Z) = \mathbf{R}\phi_*\omega_W$$

Hence $\mathbf{R}\phi_*\omega_W \simeq \omega_X$, so $\mathbf{R}^i\phi_*\omega_W = 0$ for $i > 0$. □

4. Applications

4.1. **Lipman's Vanishing Conjecture in Dimension Three.** We now apply our Theorem 3.1 to establish a slight generalization of Lipman's vanishing conjecture (see [Lip94a, Vanishing Conjecture (2.2)]) for arbitrary characteristics in dimension 3. In characteristic 0 this was proved by Cutkosky (see [Lip94a, Theorem A3.]) and by Hyry and Villamayor U. (see, [HV98, 2.7]) for all dimensions. Our proof follows the same strategy as in [HV98, 2.7].

Proposition 4.1. *(see* [Lip94a, (2.2), Theorem A3.], [HV98, 2.7]*) Let $(R, \mathfrak{m})$ be a three-dimensional rational singularity, and let $\mathcal{I} \subset R$ be an ideal. Consider the blow-up*

$$f : X = \operatorname{Proj}\ S = \operatorname{Proj}\left(\bigoplus_{n\geq 0}\mathcal{I}^n\right) \longrightarrow \operatorname{Spec} R$$

along $\mathcal{I}$. Assume that S is Cohen-Macaulay, and X is normal, and has pseudorational singularities in codimension 2. Then for all integers $n \geq 0$ and all positive integers $i > 0$, we have

$$H^i(X, \mathcal{I}^n\omega_X) = 0.$$

Proof. We follow [HV98, Remark 2.8] closely and use Grothendieck's natural construction as in [HS03, 6.2.1]. Consider the graded S-algebra defined by:

$$S^{\#} := S \oplus S_{\geq 1} \oplus S_{\geq 2} \oplus \cdots ,$$

where $S_{\geq n} = \bigoplus_{m\geq n} \mathcal{I}^m$ for each integer $m \geq 0$. Define the scheme

$$Y := \operatorname{Proj}(S^{\#}).$$

The scheme Y admits two geometric interpretations:

(a) The algebra $S^{\#}$ is naturally isomorphic to the Rees algebra of the ideal $S_{\geq 1} \subset S$. So, the natural projection

$$\theta : Y \longrightarrow \operatorname{Spec} S$$

is the blow-up of $\operatorname{Spec} S$ along $S_{\geq 1}$.

(b) There exists a canonical isomorphism of X-schemes

$$Y \cong \operatorname{Spec}_X \left(\bigoplus_{m\geq 0} \mathcal{O}_X(m) \right).$$

This isomorphism identifies Y with the total space of the line bundle $\mathcal{O}_X(-1)^\vee$, which is the dual of the tautological line bundle associated with the blow-up $X = \operatorname{Proj} S$ and $\mathcal{O}_X(1) \simeq \mathcal{I} \cdot \mathcal{O}_X$. Let $\eta : Y \longrightarrow X$ denote the corresponding bundle map.

We have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\theta"] \arrow[d, "\eta"'] & \operatorname{Spec} S \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & \operatorname{Spec} R \end{tikzcd}$$

where $\pi : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is the natural affine morphism induced by the inclusion $R \hookrightarrow S$.
Since X has rational singularities by Theorem 3.2 and $\eta : Y \longrightarrow X$ is a smooth affine morphism, it follows that Y also has rational singularities. The morphism $\eta : Y \longrightarrow X$ is smooth of relative dimension 1. The relative cotangent bundle satisfies $\Omega_{Y/X} \cong \eta^*\mathcal{O}_X(1)$. So, we have

$$\omega_Y \cong \eta^*\omega_X \otimes \Omega_{Y/X} \cong \eta^*\omega_X \otimes \eta^*\mathcal{O}_X(1) \cong \eta^*(\omega_X(1)).$$

Since η is affine

$$\eta_*\omega_Y \cong \eta_* \left(\eta^*(\omega_X(1))\right) \cong \omega_X(1) \otimes \eta_*\mathcal{O}_Y.$$

As Y is the total space of $\mathcal{O}_X(-1)$, we have

$$\eta_*\mathcal{O}_Y \cong \bigoplus_{m\geq 0} \mathcal{O}_X(m).$$

Therefore

$$\eta_*\omega_Y \cong \bigoplus_{m\geq 0} \omega_X(1) \otimes \mathcal{O}_X(m) \cong \bigoplus_{m\geq 0} \omega_X(m+1) \cong \bigoplus_{m\geq 0} \mathcal{I}^{m+1}\omega_X. \tag{1}$$

Now, if we view Y as

$$\theta : Y \longrightarrow \operatorname{Spec} S$$

the blow-up of $\operatorname{Spec} S$ along $S_{\geq 1}$, then we get the associated graded ring $\bigoplus_{n\geq 0} \frac{S_{\geq n}}{S_{\geq n+1}} \simeq S$ which is Cohen-Macaulay by assumption. Also note that as Y has rational singularities, by [SdS87, Theorem 1.4], we get that $H^i(Y, \omega_Y) = 0$ for all $i > 0$. Since η is affine, we get

$$0 = H^i(Y, \omega_Y) \cong H^i(X, \eta_* \omega_Y) \cong \bigoplus_{m\geq 0} H^i(X, \mathcal{I}^{m+1}\omega_X) \quad \text{for all } i \geq 0.$$

For the case $n = 0$, the result follows from Theorem 3.1. This finishes the proof. □

Proposition 4.2. *Let $(R, \mathfrak{m})$ be a three-dimensional rational singularity and $f : X \longrightarrow \operatorname{Spec} R$ be any projective, birational morphism. Assume that X is normal, and has pseudorational singularities in codimension 2. Let $D \subset X$ be an f-ample divisor such that the section ring $R(X, D)$ is Cohen-Macaulay, and generated in degree 1. Then for all integers $n \geq 0$ and all positive integers $i > 0$, we have*

$$H^i(X, \omega_X(nD)) = 0.$$

Proof. The proof is immediate from the previous proof after using [Har77a, Theorem II.7.17] (and [Sta, Tag 01VK]), see [Smi97, Section 1.3] for details. □

We apply Proposition 4.1 to get the following theorem of E. Hyry.

Proposition 4.3. *([Hyr99, Theorem 3.2]) Let $(R, \mathfrak{m})$ be a 3-dimensional local ring with rational singularities. Let $X = \operatorname{Proj} S \xrightarrow{\pi} \operatorname{Spec} R$ be a blow-up of an ideal $\mathcal{I} \subset R$, with X normal. If $\mathcal{J}(\omega_R, \mathcal{I}) = \omega_R$, then X has rational singularities.*

Proof. The proof follows [Hyr99, Theorem 3.2] verbatim by using Proposition 4.1 and noting that $\mathcal{I}^{d-2} = \mathcal{I}$. So we do not include it here. □

Remark 4.4. Hyry proved this result in [Hyr99, Theorem 3.2], for all dimensions, under the assumption that R is a regular local ring, essentially of finite type over a field of characteristic 0. Our result is for any 3-dimensional, excellent, local ring with rational singularities, as in 2.1.

4.2. **Briançon-Skoda-type result.** Here we briefly mention another quick application of Proposition 4.1. Most probably, this was the original motivation of Lipman to frame his Vanishing Conjecture ([Lip94a, Vanishing Conjecture (2.2)]).

Corollary 4.5. *([Lip94a, Conjecture 1.6]) Let $(R, \mathfrak{m})$ be a 3-dimensional local ring with rational singularities and $\mathfrak{a}$ be an ideal of R with analytic spread l. Then,*

$$\mathcal{J}(\omega_R, \mathfrak{a}^{n+1}) = \mathfrak{a} \cdot \mathcal{J}(\omega_R, \mathfrak{a}^n) \text{ for all } n \geq l - 1.$$

Proof. The exact proof in [Lip94a, (2.3)] works verbatim. □

Question 4.6. It would be interesting know if one can prove [HV98, Theorem 2.9 and Theorem 2.12] in dimension 3 for arbitrary characteristics.

5. Acknowledgment

I would like to thank my advisors, Christopher Hacon and Karl Schwede, for their constant encouragement, unwavering support, inspiring teachings, and infinite patience. I am grateful to Manoj Kummini for extremely helpful discussions about [KK21], which played a crucial role in this note. Finally, I would like to thank Daniel Apsley, Shikha Bhutani, Harold Blum, Mircea Mustaţă, and Joseph Sullivan for providing valuable feedback which improved the exposition. I was partially supported by NSF research grant DMS-2301374 and by a grant from the Simons Foundation SFI-MPS-MOV-00006719-07 while working on this project.

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