

Property (T) group factors whose Jones index set equals all positive integers

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Abstract

Using a mélange of techniques at the rich intersection of deformation/rigidity theory, finite index subfactor theory, and geometric group theory, we prove the existence of a continuum of property (T) factors that are pairwise non-stably isomorphic and whose Jones index sets consist of all positive integers. These factors are realized as group von Neumann algebras $\mathcal{L}(G)$ associated with property (T) generalized wreath-like product groups $G \in \mathcal{WR}(A, B \curvearrowright I)$ introduced in [CIOS23b], where A is abelian, B is a non-parabolic subgroup of a relatively hyperbolic group with residually finite peripheral structure, and $B \curvearrowright I$ is a faithful action with infinite orbits. Integer index subfactors of $\mathcal{L}(G)$ are constructed from extensions of G . This result advances an open question of P. de la Harpe [dlH95].

1 Introduction

The theory of subfactors was initiated by Jones [Jon83], who introduced the notion of the Jones index $[\mathcal{M} : \mathcal{N}]$ for an inclusion of II_1 factors $\mathcal{N} \subseteq \mathcal{M}$. Defined as the Murray-von Neumann dimension of $L^2(\mathcal{M})$ as a left $\mathcal{N}$ -module, the index measures the “size” of $\mathcal{M}$ relative to $\mathcal{N}$. Jones’s landmark result [Jon83] establishes that, for any fixed II_1 factor $\mathcal{M}$, the collection $\mathcal{J}(\mathcal{M})$ of all finite Jones indices $[\mathcal{M} : \mathcal{N}]$ of subfactors $\mathcal{N} \subseteq \mathcal{M}$ is always contained in

$$\left\{ 4 \cos^2 \left(\frac{\pi}{n} \right) \mid n \in \mathbb{N}, n \geq 3 \right\} \cup [4, \infty). \quad (1.1)$$

He also showed that all of these values are realized as Jones indices of subfactors of the hyperfinite II_1 factor $\mathcal{R}$ [Jon83]. In the same work, Jones asked which of these values can occur as Jones indices of irreducible subfactors of $\mathcal{R}$. This problem has proved to be extremely difficult and remains wide open.

For some II_1 factors, their Jones index sets are identified. Using results from [IPP08], Vaes constructed in [Vae09] a II_1 factor $\mathcal{M}$ in which every finite-index subfactor $\mathcal{N} \subseteq \mathcal{M}$ is trivial; that is, every finite-index inclusion of factors $\mathcal{N} \subseteq \mathcal{M}$ is isomorphic to $\mathcal{N} \subseteq \mathcal{N} \otimes \mathbb{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$. As a result, the set $\mathcal{J}(\mathcal{M})$ of Jones indices of $\mathcal{M}$ is precisely $\mathbb{N}^2 = \{n^2 \mid n \in \mathbb{N}\}$, and the only finite index of an irreducible subfactor is 1.

The structure of the Jones index set remains largely mysterious for many natural classes of factors—one notable example being property (T) factors. Answering a question from [Jon83], Popa proved in [Pop86] via a general argument that for every property (T) II_1 factor $\mathcal{M}$, its Jones index set $\mathcal{J}(\mathcal{M})$ is countable. This, combined with the earlier results of Connes [Con80] on the countability of symmetries of property (T) II_1 factors, provides further evidence that such factors are highly rigid objects, illustrating, for instance, a stark contrast with the case of amenable factors. Motivated by these initial results, P. de la Harpe proposed for study in [dlH95, Problem 5] the following problem: *What are the possible values of the Jones index set $\mathcal{J}(\mathcal{L}(G))$ when G is an ICC property (T) group?*

Although the problem has been around for three decades, significant progress was made only recently. In particular, it was shown in [CIOS24, AMAKCK25] that for broad classes of property (T) wreath-like product groups and relatively hyperbolic groups G , the Jones index set $\mathcal{J}(\mathcal{L}(G))$ consists solely of integers. Motivated by these results, as well as recent advances on the Connes Rigidity Conjecture and its strong form by Popa [CDHK24, CIOS23a, CIOS23b, CIOS24, AMAKCK25], it was conjectured in [CIOS24] that $\mathcal{J}(\mathcal{L}(G)) \subseteq \mathbb{N}$ holds universally. Nevertheless, no *explicit* computation of $\mathcal{J}(\mathcal{L}(G))$ is known for any ICC property (T) group G . The primary goal of this paper is to present the first such computation.

1.1 Main results

To properly introduce our results we recall the concept of wreath-like product group introduced in [CIOS23a]. Let A, B be countable groups and let $B \curvearrowright I$ be an action on a countable set I . A group G is a *wreath-like product* of groups A and $B \curvearrowright I$, written as $G \in \mathcal{WR}(A, B \curvearrowright I)$, if G is an extension of the form

$$1 \rightarrow \bigoplus_{i \in I} A_i \hookrightarrow G \xrightarrow{\kappa} B \rightarrow 1, \quad (1.2)$$

where $A_i \cong A$ and the action of G on $A^{(I)} = \bigoplus_{i \in I} A_i$ by conjugation permutes the summands as follows

$$gA_i g^{-1} = A_{\kappa(g)i} \quad \text{for all } g \in G, i \in I.$$

In [CIOS23a], a natural quotienting procedure was introduced in the context of group-theoretic Dehn filling, producing many examples of wreath-like product groups, including numerous groups with property (T). Some classes of these groups provided the first known examples of property (T) groups that are entirely reconstructible from their von Neumann algebras; in particular, they satisfy the Connes Rigidity Conjecture [Con82]. Moreover, in [CIOS23b, Theorem 7.5], for an even broader class of wreath-like product groups G and H with abelian base, it was completely described all isomorphisms between $\mathcal{L}(G)$ and any amplification $\mathcal{L}(H)^t$ with $t > 0$ solely in terms of isomorphisms between the underlying groups G and H , together with multiplicative characters on G . As a result, all such groups give a positive answer to Jones' Outer Automorphisms Conjecture from [Jon00, Problem 8].

In this paper we establish the following nontrivial generalization of this result to all virtual $*$ -isomorphisms. Recall that two II_1 factors $\mathcal{N}$ and $\mathcal{M}$ are called *virtually isomorphic* if there exists $t > 0$ and an injective $*$ -homomorphism $\Theta : \mathcal{N} \rightarrow \mathcal{M}^t$ with finite index image, i.e., $[\mathcal{M}^t : \Theta(\mathcal{N})] < \infty$. The map Θ is called a *virtual $*$ -isomorphism*.

Theorem A. *Let A, C be non-trivial abelian groups. Let B, D be nonparabolic ICC subgroups of finitely generated groups which are hyperbolic relative to a finite family of residually finite groups.*

Let $G \in \mathcal{WR}(A, B \curvearrowright I)$ and $H \in \mathcal{WR}(C, D \curvearrowright J)$ be property (T) groups, where $B \curvearrowright I$ and $D \curvearrowright J$ are faithful actions with infinite orbits.

Let $t > 0$ be a scalar and let $\Theta : \mathcal{L}(G) \hookrightarrow \mathcal{L}(H)^t$ be any virtual $$ -isomorphism.*

Then $t \in \mathbb{N}$, and we can find $m \in \mathbb{N}$ and for every $1 \leq i \leq m$, a finite index subgroup $K_i \leq G$, an injective homomorphism $\gamma_i : K_i \rightarrow H$ with finite index image and a unitary representation $\rho_i : K_i \rightarrow \mathcal{U}_{s_i}(\mathbb{C})$, for some $s_i \in \mathbb{N}$, and a unitary $w \in \mathcal{L}(H)^t = \mathcal{L}(H) \otimes \mathbb{M}_t(\mathbb{C})$ such that $\sum_{i=1}^m [G : K_i] s_i = t$ and

$$w\Theta(u_g)w^* = \text{diag}(\text{Ind}_{K_1}^G(\pi_{\gamma_1, \rho_1})(g), \dots, \text{Ind}_{K_m}^G(\pi_{\gamma_m, \rho_m})(g)), \text{ for every } g \in G.$$

Here, $(u_g)_{g \in G} \subset \mathcal{L}(G)$ and $(v_h)_{h \in H} \subset \mathcal{L}(H)$ are the canonical unitaries, $\pi_{\gamma_i, \rho_i}(g) = v_{\gamma_i(g)} \otimes \rho_i(g)$ for all $g \in K_i$ and $\text{Ind}_{K_i}^G(\pi_{\gamma_i, \rho_i}) : G \rightarrow \mathcal{U}(\mathcal{L}(K_i) \otimes \mathbb{M}_{s_i[G:K_i]}(\mathbb{C}))$ is the canonical induced representation.

Moreover, we have the following formula for the index of the image of Θ

$$[\mathcal{L}(H)^t : \Theta(\mathcal{L}(G))] = t \sum_{i=1}^m s_i [H : \gamma_i(K_i)].$$

This theorem should also be compared with the recent embedding result obtained in [CIOS24, Theorem 5.6]. We note that, although our setting is more restrictive than the general embedding framework considered there, our result applies to a much broader class of wreath-like product groups. However, our proof still follows very closely the general methods introduced in [CIOS24, Theorem 5.6], but is complemented by an analysis involving finite index techniques.

By iteratively applying the basic construction [Chr79, Jon83], we observe that any subfactor $\mathcal{N} \subseteq \mathcal{L}(G)$ of finite Jones index $[\mathcal{L}(G) : \mathcal{N}] = t > 0$ gives rise to a $*$ -embedding $\mathcal{L}(G) \hookrightarrow \mathcal{L}(G)^t$ whose image has finite index. In particular, the previous theorem implies that for every property (T) wreath-like product group G as in the hypothesis, the Jones index set satisfies $\mathcal{J}(\mathcal{L}(G)) \subseteq \mathbb{N}$. Moreover, we show that for some of these groups, *every* positive integer $t \in \mathbb{N}$ can be realized as the Jones index of an irreducible finite index subfactor of $\mathcal{L}(G)$. The key insight is that such subfactors can be constructed from finite index extensions of the group

G . Extensions H are chosen that $\mathcal{L}(G) \subset \mathcal{L}(H)$ are irreducible subfactors. The irreducibility is shown by expressing the von Neumann algebras $\mathcal{L}(H)$ as cocycle crossed products and applying [Sut80B, Theorem 6.1]. Applying the downward basic construction in [Jon83, PP86] to $\mathcal{L}(G) \subset \mathcal{L}(H)$ yields irreducible subfactors of $\mathcal{L}(G)$ with index $[H : G]$.

Theorem B. *There is a continuum of ICC property (T) wreath-like product groups $(G_j)_{j \in J}$ as in the statement of Theorem A, such that their II_1 factors $\mathcal{L}(G_j)$ are pairwise not stably isomorphic, and $\mathcal{J}(\mathcal{L}(G_j)) = \mathbb{N}$. Moreover, all values in $\mathcal{J}(\mathcal{L}(G_j))$ can be realized as the Jones indices of irreducible subfactors of $\mathcal{L}(G_j)$.*

It would be very interesting to understand which other intermediate subsets $\mathbb{N}^2 \subseteq S \subseteq \mathbb{N}$ can arise as Jones index sets of property (T) group factors. In particular, one may ask whether there exist ICC property (T) groups G for which $\mathcal{J}(\mathcal{L}(G)) = \mathbb{N}^2$, and moreover, whether there are situations where such factors admit no non-canonical finite-index subfactors, as in [Vae09].

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2 Preliminaries

2.1 Cocycle crossed product von Neumann algebras

Let $(\mathcal{M}, \text{tr})$ be a tracial von Neumann algebra endowed with a normalized trace tr and $\mathcal{U}(\mathcal{M})$ and $\text{Aut}(\mathcal{M})$ be its unitary group and automorphism group. Each $w \in \mathcal{U}(\mathcal{M})$ induces a conjugation automorphism $\text{ad}(w) \in \text{Aut}(\mathcal{M})$ given by $\text{ad}(w)(x) = wxw^*$. Let Γ be a discrete group.

Definition 2.1. A *cocycle action* of Γ on $\mathcal{M}$ is denoted by $\Gamma \curvearrowright^{\alpha, \omega} \mathcal{M}$ and consist of maps $\alpha : \Gamma \rightarrow \text{Aut}(\mathcal{M})$ and $\omega : \Gamma \times \Gamma \rightarrow \mathcal{U}(\mathcal{M})$ satisfying the following relations:

1. $\alpha_1 = \text{id}_{\mathcal{M}}$,
2. $\alpha_g \alpha_h = \text{ad}(\omega_{g,h}) \alpha_{gh}$ for all $g, h \in \Gamma$, and
3. $\alpha_g(\omega_{h,k}) \omega_{g,hk} = \omega_{g,h} \omega_{gh,k}$ for all $g, h, k \in \Gamma$.

Note that α and ω are not necessarily group homomorphisms. The map ω is called a *2-cocycle* for α .

The following is a well-known fact:

Proposition 2.2. *If $\mathcal{M}$ is a II_1 factor and $\Gamma \curvearrowright^{\alpha, \omega} \mathcal{M}$ is a cocycle action as in Definition 2.1 then the following hold:*

1. $\omega_{1,1} \in \mathbb{C}1$.
2. $\omega_{1,1} = \omega_{1,k} = \omega_{g,1}$ for all $g, k \in \Gamma$.

A 2-cocycle ω is *normalized* if $\omega_{1,g} = \omega_{g,1} = 1$ for all $g \in \Gamma$. We can normalize 2-cocycle by replacing $\omega_{g,h}$ by $\omega_{1,1}^* \omega_{g,h}$. Thus, throughout the paper we will assume that all 2-cocycles are normalized unless otherwise stated.

Definition 2.3. Let $\Gamma \curvearrowright^{\alpha, \omega} \mathcal{M}$ be a cocycle action. Then the *cocycle crossed product algebra* $\mathcal{M} \rtimes_{\alpha, \omega} \Gamma$ is the von Neumann algebra generated in $\mathcal{B}(\ell^2(\Gamma, L^2(\mathcal{M}, \text{tr})))$ by the following operators:

1. elements $x \in \mathcal{M}$ with the action given by $x(\xi)(h) = \alpha_h^{-1}(x)\xi(h)$ for all $\xi \in \ell^2(\Gamma, L^2(\mathcal{M}))$ and $h \in \Gamma$,
2. unitaries v_g given by $v_g(\xi)(h) = \alpha_h^{-1}(\omega_{g,g^{-1}h})\xi(g^{-1}h)$ for all $g \in G$.

We notice these generators of $\mathcal{M} \rtimes_{\alpha, \omega} \Gamma$ satisfy the following relations:

$$v_g v_h = \omega_{g,h} v_{gh}, \quad v_g x v_g^* = \alpha_g(x), \quad v_1 = 1. \quad (2.1)$$

2.2 Finite index inclusions of von Neumann algebras

Given an inclusion of II_1 factors $\mathcal{N} \subseteq \mathcal{M}$ its Jones index [Jon83] is defined as the dimension $[\mathcal{M} : \mathcal{N}] = \dim_{\mathcal{N}}(L^2(\mathcal{M}))$. More generally, Pimsner and Popa discovered in [PP86] a more probabilistic definition of finite index which then generalizes the index to an inclusion $\mathcal{N} \subseteq \mathcal{M}$ of tracial von Neumann algebras. Specifically, consider

$$c := \inf \left\{ \frac{\|\mathbb{E}_{\mathcal{N}}(x)\|_2^2}{\|x\|_2^2} \mid x \in \mathcal{M}_+, x \neq 0 \right\}.$$

Then we define the index of the inclusion $\mathcal{N} \subseteq \mathcal{M}$ as $[\mathcal{M} : \mathcal{N}] = c^{-1}$ with the convention that $\frac{1}{0} = \infty$.

We also say the inclusion $\mathcal{N} \subseteq \mathcal{M}$ has a finite (right) Pimsner-Popa basis $x_1, \dots, x_n \in \mathcal{M}$ if these elements are $\mathcal{N}$ -orthogonal and $\mathcal{M} = \sum_i \mathcal{N}x_i$. This implies that for every $x \in \mathcal{M}$ we have $x = \sum_i \mathbb{E}_{\mathcal{N}}(xx_i^*)x_i$. In [PP86] it was proved that for II_1 factors $\mathcal{N} \subseteq \mathcal{M}$ the existence of a finite Pimsner-Popa basis for the inclusion is equivalent to $[\mathcal{M} : \mathcal{N}] < \infty$.

We continue by recording several basic facts from the literature concerning finite index inclusions of tracial von Neumann algebras which we will use in the proofs of some of our main results. Recall that a von Neumann algebra $\mathcal{M}$ is called *completely atomic* if 1 is an orthogonal sum of minimal projections in $\mathcal{M}$.

Proposition 2.4. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of tracial von Neumann algebras with $[\mathcal{M} : \mathcal{N}] < \infty$. Then the following hold:*

1. [Jon83] If $p \in \mathcal{N}$ is a non-zero projection, then $[p\mathcal{M}p : p\mathcal{N}p] < \infty$.
2. [Pop95, Relation 1.1.2(ii)] If $\mathcal{N}$ is a factor and $r \in \mathcal{N}' \cap \mathcal{M}$ is a non-zero projection, then $[r\mathcal{M}r : \mathcal{N}r] < \infty$.
3. [Jon83] If $\mathcal{N}$ is a factor then $\dim_{\mathbb{C}}(\mathcal{N}' \cap \mathcal{M}) \leq [\mathcal{M} : \mathcal{N}] + 1$.
4. [Pop95, Relation 1.1.2(iv)] If $\mathcal{Z}(\mathcal{M})$ is completely atomic, then $\mathcal{Z}(\mathcal{N})$ is completely atomic.
5. [PP86] If $\mathcal{M}$ and $\mathcal{N}$ are factors, then there is a finite Pimsner-Popa basis for $\mathcal{N} \subseteq \mathcal{M}$.
6. [Jon83, PP86] If $\mathcal{N} \subseteq \mathcal{R} \subseteq \mathcal{M}$ is a von Neumann subalgebra, then $\max\{[\mathcal{M} : \mathcal{R}], [\mathcal{R} : \mathcal{N}]\} \leq [\mathcal{M} : \mathcal{N}] \leq [\mathcal{M} : \mathcal{R}][\mathcal{R} : \mathcal{N}]$. In particular, $[\mathcal{M} : \mathcal{R}], [\mathcal{R} : \mathcal{N}] < \infty$.
7. [Jon83, Lemmas 2.2.1-2] If $\mathcal{N}$ and $\mathcal{M}$ are II_1 factors then for every projection $p \in \mathcal{N}' \cap \mathcal{M}$ we have that

$$[p\mathcal{M}p : \mathcal{N}p] = [\mathcal{M} : \mathcal{N}] \text{tr}_{\mathcal{M}}(p) \text{tr}_{\mathcal{N}'}(p).$$

Moreover, for any partition of unity $(p_i) \subset \mathcal{N}' \cap \mathcal{M}$ by projections we have that

$$[\mathcal{M} : \mathcal{N}] = \sum_i \frac{1}{\text{tr}_{\mathcal{M}}(p_i)} [p_i\mathcal{M}p_i : \mathcal{N}p_i].$$

8. If $\mathcal{N} := \mathcal{P} \rtimes H \subseteq \mathcal{P} \rtimes G =: \mathcal{M}$ for an inclusion of discrete groups $H \leq G$ and trace preserving action $G \curvearrowright \mathcal{P}$ on tracial von Neumann algebra $\mathcal{P}$ then $[\mathcal{M} : \mathcal{N}] = [G : H]$.

2.3 Construction of finite index subfactors from outer automorphisms

In this subsection, we give a way to construct a finite index irreducible subfactor of an ICC group factor $\mathcal{L}(G)$ from a finite subgroup Γ_0 of the outer automorphism group $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$.

Theorem 2.5 ([Mac63, Theorem IV.9.1]). *Let G be a group with a trivial center and $\bar{\varphi} : \Gamma \rightarrow \text{Out}(G)$ be a homomorphism. Then there is an extension $1 \rightarrow G \rightarrow H \rightarrow \Gamma \rightarrow 1$ such that the induced homomorphism also denoted by $\bar{\varphi} : H/G \rightarrow \text{Out}(G)$ satisfies $\bar{\varphi}(hG) = \text{ad}(h) \text{Inn}(G)$.*

The description of H is given in [Mac63, Section IV.8]. We briefly summarize the construction. For each $k \in \Gamma$, we choose $\varphi_k \in \text{Aut}(G)$ whose image in $\text{Out}(G)$ is $\bar{\varphi}_k$. Then for each pair of elements $k_1, k_2 \in \Gamma$, we have $\varphi_{k_1}\varphi_{k_2} = \text{ad}(g_{k_1, k_2})\varphi_{k_1 k_2}$ for some g_{k_1, k_2} . Then H is given by

$$H = \langle G, \{h_k\}_{k \in \Gamma} \mid h_k g h_k^{-1} = \varphi_k(g) \text{ for all } g \in G \text{ and } k \in \Gamma, h_{k_1} h_{k_2} = g_{k_1, k_2} h_{k_1 k_2} \text{ for all } k_1, k_2 \in \Gamma \rangle.$$

Remark 2.6. 1. When G has trivial center and $\bar{\varphi}$ is injective, then H is isomorphic to the preimage of $\bar{\varphi}(\Gamma) \leq \text{Out}(G)$ in $\text{Aut}(G)$ [Mac63, Section IV.9].

2. The map $\Gamma \times \Gamma \rightarrow G$ given by $(k_1, k_2) \mapsto g_{k_1, k_2}$ is a G -valued 2-cocycle [Mac63, Section IV.8], [Sut80B, Section 3.1]. Thus, $\mathcal{L}(H)$ is a cocycle crossed product $\mathcal{L}(G) \rtimes \Gamma$. In particular, the 3-cocycle obstruction for the crossed product is trivial.

Theorem 2.7 ([Sut80B, Theorem 6.1]). *Any cocycle crossed product of a II_1 von Neumann algebra with a finite group is an ordinary crossed product.*

Corollary 2.8. *Let $\mathcal{M}$ be a II_1 factor with a normalized trace tr , Γ_0 be a finite group and $\Gamma_0 \curvearrowright^{\alpha, \omega} \mathcal{M}$ be a cocycle action. If α induces an injective homomorphism $\bar{\alpha} : \Gamma_0 \rightarrow \text{Out}(\mathcal{M})$ from Γ_0 to the outer automorphism group $\text{Out}(\mathcal{M}) = \text{Aut}(\mathcal{M})/\text{Inn}(\mathcal{M})$ of $\mathcal{M}$, then the following hold:*

1. [Sut80A, Sut80B] $\mathcal{M} \rtimes_{\alpha, \omega} \Gamma_0$ is a II_1 factor, its trace being given by

$$\text{tr}_{\mathcal{M} \rtimes_{\alpha, \omega} \Gamma_0} \left(\sum_{k \in \Gamma_0} x_k v_k \right) = \text{tr}_{\mathcal{M}}(x_1).$$

2. [OK90, KY92] $\mathcal{M} \subset \mathcal{M} \rtimes_{\alpha, \omega} \Gamma_0$ is an irreducible inclusion.

3. [OK90, KY92] $[\mathcal{M} \rtimes_{\alpha, \omega} \Gamma_0 : \mathcal{M}] = |\Gamma_0|$.

Corollary 2.8 follows directly from Theorem 2.7 and well-known properties about the ordinary crossed product. It can also be verified by direct computation.

We end this section with a canonical, yet very useful, procedure for constructing finite index subfactors using outer automorphisms of a factor.

Proposition 2.9. *Let G be an ICC group. Let $\Psi : \text{Aut}(G) \rightarrow \text{Aut}(\mathcal{L}(G))$ be the homomorphism defined by*

$$\Psi(\varphi)(u_g) = u_{\varphi(g)} \text{ for all } \varphi \in \text{Aut}(G) \text{ and } g \in G.$$

Furthermore, let $\bar{\Psi} : \text{Out}(G) \rightarrow \text{Out}(\mathcal{L}(G))$ be the canonical homomorphism induced by Ψ ,

$$\bar{\Psi}(\varphi \text{Inn}(G)) = \Psi(\varphi) \text{Inn}(\mathcal{L}(G)) \text{ for all } \varphi \in \text{Aut}(G).$$

Assume $\Gamma_0 \leq \text{Out}(G)$ is a finite subgroup and $\bar{\Psi}|_{\Gamma_0}$ is injective. Then there is an extension $1 \rightarrow G \rightarrow H \rightarrow \Gamma_0 \rightarrow 1$ such that $\mathcal{L}(G) \subseteq \mathcal{L}(H)$ is an irreducible subfactor with index $|\Gamma_0|$. In particular, H is ICC. Moreover, there exist irreducible subfactors $\mathcal{N} \subseteq \mathcal{L}(G)$ such that $[\mathcal{L}(G) : \mathcal{N}] = |\Gamma_0|$.

Proof. Since G is ICC, it has a trivial center. Consider the extension $1 \rightarrow G \rightarrow H \rightarrow \Gamma_0 \rightarrow 1$ as in Theorem 2.5. Then by Remark 2.6, $\mathcal{L}(H) \cong \mathcal{L}(G) \rtimes_{\alpha, \omega} \Gamma_0$ for some cocycle action $\Gamma_0 \curvearrowright^{\alpha, \omega} \mathcal{M}$. The injectivity of $\bar{\Psi}|_{\Gamma_0}$ and Corollary 2.8 implies that $\mathcal{L}(G) \subseteq \mathcal{L}(H)$ is an irreducible subfactor with index $|\Gamma_0|$. Performing the downward basic construction $\mathcal{N} \subseteq \mathcal{L}(G) \subseteq \mathcal{L}(H)$ gives a subfactor of index $|\Gamma_0|$. $\square$

2.4 Popa's intertwining techniques

In [Pop06] introduced the following powerful criterion for the existence of intertwining between von Neumann algebras.

Theorem 2.10. [Pop06] *Let $(\mathcal{M}, \text{tr})$ be a tracial von Neumann algebra and let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{M}$ be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

1. *There exist projections $p \in \mathcal{P}, q \in \mathcal{Q}$, a $*$ -homomorphism $\Theta : p\mathcal{P}p \rightarrow q\mathcal{Q}q$ and a partial isometry $0 \neq v \in \mathcal{M}$ such that $v^*v \leq p$, $vv^* \leq q$ and $\Theta(x)v = vx$, for all $x \in p\mathcal{P}p$.*
2. *For any group $\mathcal{G} \subset \mathcal{U}(\mathcal{P})$ such that $\mathcal{G}'' = \mathcal{P}$ there is no net $(u_n)_n \subset \mathcal{G}$ satisfying $\|\mathbb{E}_{\mathcal{Q}}(xu_ny)\|_2 \rightarrow 0$, for all $x, y \in \mathcal{M}$.*

3. There exist finitely many $x_i, y_i \in \mathcal{M}$ and $c > 0$ such that $\sum_i \|\mathbb{E}_{\mathcal{Q}}(x_i u y_i)\|_2^2 \geq c$, for all $u \in \mathcal{U}(\mathcal{P})$.

If one of the equivalent conditions from Theorem 2.10 holds, one says a corner of $\mathcal{P}$ embeds into $\mathcal{Q}$ inside $\mathcal{M}$, and writes $\mathcal{P} <_{\mathcal{M}} \mathcal{Q}$. If we moreover have that $\mathcal{P}p' <_{\mathcal{M}} \mathcal{Q}$, for any projection $0 \neq p' \in \mathcal{P}' \cap 1_{\mathcal{P}}\mathcal{M}1_{\mathcal{P}}$, then one writes $\mathcal{P} <_{\mathcal{M}}^s \mathcal{Q}$.

In the remaining part of the section we record a few technical intertwining results that will be used in the proofs of our main results. We start with the following well known result intertwining result in cocycle crossed product von Neumann algebras.

Proposition 2.11. *Let $\Gamma \curvearrowright^{\alpha, \omega} \mathcal{N}$ be a cocycle action on a tracial von Neumann algebra and let $\mathcal{M} = \mathcal{N} \rtimes_{\alpha, \omega} \Gamma$ the corresponding cocycle crossed product von Neumann algebra. Let $T < \Gamma$ be an infinite index subgroup and let $0 \neq p \in \mathcal{M}$ be a projection. Then for any finite index von Neumann subalgebra $\mathcal{P} \subseteq p\mathcal{M}p$ we have $\mathcal{P} \not<_{\mathcal{M}} \mathcal{N} \rtimes_{\alpha, \omega} T$.*

For further use, we will also record the following result from [CD18, Proposition 2.3]

Proposition 2.12 ([CD18]). *Let $\mathcal{N} \subseteq \mathcal{M}$ be II_1 factors such that $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}1$. Then $\mathcal{M} <_{\mathcal{M}} \mathcal{N}$ if and only if $[\mathcal{M} : \mathcal{N}] < \infty$.*

Next, we will continue with several intertwining results that will be used in an essential way in deriving the main results in the subsequent sections. Some of them may be well-known to the experts, but for readers' convenience we decided to include complete proofs.

Proposition 2.13. *Let $\mathcal{A} \subseteq \mathcal{B}$ be abelian von Neumann algebras. Then $\mathcal{B} <_{\mathcal{B}}^s \mathcal{A}$ if and only if there exists a countable set of mutually orthogonal projections $(r_n) \subset \mathcal{B}$ such that $\mathcal{B} = \oplus_n \mathcal{A}r_n$.*

Proof. We will prove only the forward implication as the converse is straightforward. In this direction we first show that for every projection $0 \neq q' \in \mathcal{B}$ there is a projection $0 \neq r \in \mathcal{B}q'$ such that $\mathcal{B}r = \mathcal{A}r$.

To see this, notice from assumptions, we can find nonzero projections $q \in \mathcal{B}q'$, $p \in \mathcal{A}$, a nonzero partial isometry $v \in \mathcal{B}$, and a unital $*$ -isomorphism on the image $\Theta : \mathcal{B}q \rightarrow \Theta(\mathcal{B}q) \subseteq \mathcal{A}p$ such that $vv^* \leq p$, $v^*v \leq q$, and $\Theta(x)v = vx$ for all $x \in \mathcal{B}q$. Since $\mathcal{B}$ is abelian, we further have $\Theta(x)vv^* = vxv^* = xvv^*$, for all $x \in \mathcal{B}q$. In particular, $\mathcal{B}vv^* = \Theta(\mathcal{B}q)vv^* \subseteq \mathcal{A}vv^*$. Since $\mathcal{A} \subseteq \mathcal{B}$ we further get $\mathcal{B}vv^* = \mathcal{A}vv^*$ and letting $r = vv^* \in \mathcal{B}$ we obtain the desired claim. Using Zorn's Lemma there exists $(r_n) \subset \mathcal{B}$ a (countable) maximal (under set inclusion) family of mutually orthogonal projections such that $\mathcal{B}r_n = \mathcal{A}r_n$ for all n . From the claim this family is nonempty.

Now, set $q' = 1 - \sum_n r_n$. If $q' \neq 0$ then one can find $0 \neq r \leq q'$ such that $\mathcal{B}r = \mathcal{A}r$. Adding r to the set (r_n) would contradict the maximality of the latter. Hence $q' = 0$ which completes our proof. $\square$

Proposition 2.14. *Let $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M}$ be von Neumann algebras such that $\mathcal{A} \subseteq \mathcal{N}$ is a MASA, $\mathcal{N} \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{A})''$, and the inclusion $\mathcal{N} \subseteq \mathcal{M}$ admits a finite Pimsner-Popa basis. Then the following hold:*

1. $\mathcal{A}' \cap \mathcal{M} <_{\mathcal{A}' \cap \mathcal{M}}^s \mathcal{A}$; in particular, $\mathcal{A}' \cap \mathcal{M}$ is type I;
2. If in addition $\mathcal{N}$ and $\mathcal{M}$ are II_1 factors then the inclusion $\mathcal{Z}(\mathcal{A}' \cap \mathcal{M}) \subseteq \mathcal{A}' \cap \mathcal{M}$ admits a finite Pimsner-Popa basis; and
3. There is countable set of mutually orthogonal projections $(r_n) \subset \mathcal{Z}(\mathcal{A}' \cap \mathcal{M})$ such that $\mathcal{Z}(\mathcal{A}' \cap \mathcal{M}) = \oplus_n \mathcal{A}r_n$.

Proof. 1. Let $\mathcal{G} \leq \mathcal{N}_{\mathcal{M}}(\mathcal{A})$ such that $\mathcal{G}'' = \mathcal{N}$. Since $\mathcal{G}$ normalizes $\mathcal{A}$ then it also normalizes $\mathcal{A}' \cap \mathcal{M}$. Let $\mathcal{Q} := \{\mathcal{G}\mathcal{U}(\mathcal{A}' \cap \mathcal{M})\}''$. Notice that $\mathcal{N} \subseteq \mathcal{Q} \subseteq \mathcal{M}$ and since $\mathcal{N} \subseteq \mathcal{M}$ has a finite Pimsner-Popa basis then so does $\mathcal{N} \subseteq \mathcal{Q}$. Pick $x_1, \dots, x_s \in \mathcal{Q}$ such a basis and notice that for every $x \in \mathcal{Q}$ we have that $x = \sum_i \mathbb{E}_{\mathcal{N}}(xx_i^*)x_i$. Approximating in $\|\cdot\|_2$ each element x_i by finite sums in $\mathcal{U}(\mathcal{A}' \cap \mathcal{M})\mathcal{G}$ and using the previous formula we get that for every $\varepsilon > 0$ there are finitely many $u_j \in \mathcal{U}(\mathcal{A}' \cap \mathcal{M})$ and $g_j \in \mathcal{G}$ and $c > 0$ such that for all $x \in (\mathcal{A}' \cap \mathcal{M})_1$ we have

$$\|x\|_2^2 - \varepsilon \leq c \sum_j \|\mathbb{E}_{\mathcal{N}}(xu_j g_j)\|_2^2 \leq c \sum_j \|\mathbb{E}_{\mathcal{N}}(xu_j)\|_2^2 = c \sum_j \|\mathbb{E}_{\mathcal{A}}(xu_j)\|_2^2.$$

In the last inequality above we used the commuting square property from [GdhHJ96, Theorem 4.3.1]. By Popa's intertwining techniques, this further implies that $\mathcal{A}' \cap \mathcal{M} <_{\mathcal{A}' \cap \mathcal{M}}^s \mathcal{A}$.

For the remaining part, notice this intertwining further implies that for every projection $0 \neq p \in \mathcal{A}' \cap \mathcal{M}$ there exists an abelian projection $0 \neq r \in \mathcal{A}$ such that $r \leq z(p)$, where $z(p)$ is the central carrier of p inside $\mathcal{A}' \cap \mathcal{M}$. Therefore $z(r) \leq z(p)$ and hence $0 \neq z(r) = z(r)z(p)$. By Comparison Theorem this further yields that r and p admit nonzero von Neumann equivalent subprojections. Since being abelian is preserved under both von Neumann equivalence and taking subprojections we conclude that p has a nonzero, abelian subprojection. As p was arbitrary it follows that $\mathcal{A}' \cap \mathcal{M}$ is type I.

2. This follows directly from Proposition 2.15.

3. Part 1. clearly implies that $\mathcal{Z}(\mathcal{A}' \cap \mathcal{M}) <_{\mathcal{A}' \cap \mathcal{M}}^s \mathcal{A}$. Combining this with [CFQT24, Proposition 3.2] we further have that $\mathcal{Z}(\mathcal{A}' \cap \mathcal{M}) <_{\mathcal{Z}(\mathcal{A}' \cap \mathcal{M})}^s \mathcal{A}$. Thus, the conclusion follows from Proposition 2.13. $\square$

Proposition 2.15. *Let $(\mathcal{M}, \text{tr})$ be a tracial von Neumann algebra such that $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}$ has finite index. Then $\mathcal{M}$ is type I and there exists a finite Pimsner-Popa basis for the inclusion $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}$.*

Proof. From the assumption there exists a constant $c > 0$ such that $\|\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(x)\|_2^2 \geq c\|x\|_2^2$ for all $x \in \mathcal{M}_+$. This further implies that for any projection $p \in \mathcal{M}$ we have

$$\text{tr}(\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(p)\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(p)) \geq c\text{tr}(p) = c\text{tr}(\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(p)). \quad (2.2)$$

Notice that $\mathcal{M} = \mathcal{M}r \oplus \mathcal{M}q$, for $r, q \in \mathcal{Z}(\mathcal{M})$ with $r + q = 1$ where $\mathcal{M}r$ is type I and $\mathcal{M}q$ is type II. Since $\mathcal{M}q$ is type II, for every $n \in \mathbb{N}$ one can find mutually orthogonal, von Neumann equivalent projections $q_i \in \mathcal{M}q$ for $1 \leq i \leq n$ such that $q = q_1 + \cdots + q_n$. Fixing $1 \leq i \leq n$ we see that $n\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q_i) = \mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q)$. Using this together with inequality (2.2) for $p = q_i$ we see that

$$0 \leq \text{tr}(q) = \text{tr}(\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q)) = n\text{tr}(\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q_i)) \leq \frac{n}{c}\text{tr}(\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q_i)\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q_i)) = \frac{1}{cn}\text{tr}(\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q)\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(q)).$$

Letting $n \nearrow \infty$ above and using the faithfulness of tr we get $q = 0$. Hence $\mathcal{M}$ is type I.

Therefore, after an isomorphism we can assume that $\mathcal{M} = \oplus_i \mathcal{A}_i \otimes \mathbb{M}_{n_i}(\mathbb{C})$ where $\mathcal{A}_i$ is abelian and (n_i) is a sequence of distinct positive integers. Next we show (n_i) is finite. Assume by contradiction, (n_i) is infinite. Thus we can assume that $n_i \nearrow \infty$ as $i \nearrow \infty$. Thus for every i there is a nonzero projection $z_i \otimes p_i \in \mathcal{A}_i \otimes \mathbb{M}_{n_i}(\mathbb{C})$ where $\text{tr}(p_i) = 1/n_i$. Since $\mathcal{Z}(\mathcal{M}) = \oplus_i \mathcal{A}_i$ we get that

$$\frac{\|z_i\|_2^2}{n_i^2} = \|z_i \otimes \mathbb{E}_{\mathcal{A}_i \otimes 1}(1 \otimes p_i)\|_2^2 = \|\mathbb{E}_{\mathcal{Z}(\mathcal{M})}(z_i \otimes p_i)\|_2^2 \geq c\|z_i \otimes p_i\|_2^2 = c\|z_i\|_2^2\|p_i\|_2^2 = \frac{c\|z_i\|_2^2}{n_i}.$$

However, this leads to a contradiction when n_i is sufficiently large. Since (n_i) is finite, using the structure of $\mathcal{M}$ we have that $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}$ has a finite Pimsner-Popa basis. $\square$

Proposition 2.16. *Let $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M}$ be tracial von Neumann algebras where $\mathcal{N} \subseteq \mathcal{M}$ is a finite index inclusion of II_1 factors and $\mathcal{A} \subseteq \mathcal{N}$ is a Cartan subalgebra. Consider a von Neumann subalgebra $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A}' \cap \mathcal{M}$ such that $\mathcal{B} \subseteq \mathcal{M}$ is a MASA. Then the following hold:*

1. *The inclusion $\mathcal{A} \subseteq \mathcal{A}' \cap \mathcal{M}$ has finite index and $\mathcal{A}' \cap \mathcal{M}$ is a type I. Moreover for every $\varepsilon > 0$ there is $z \in \mathcal{Z}(\mathcal{A}' \cap \mathcal{M})$ with $\text{tr}(z) \geq 1 - \varepsilon$ such that the inclusion $\mathcal{A}z \subseteq (\mathcal{A}' \cap \mathcal{M})z$ admits a finite Pimsner-Popa basis.*
2. *The center is completely atomic, i.e. $\mathcal{Z}(\mathcal{N}_{\mathcal{M}}(\mathcal{B})'') = \oplus_n \mathbb{C}z_n$. Also for every $n \in \mathbb{N}$ we have that $\mathcal{N}_{\mathcal{M}}(\mathcal{B})''z_n \subseteq z_n\mathcal{M}z_n$ is an irreducible, finite index inclusion of II_1 factors.*

Proof. 1. This follows directly from Proposition 2.14.

2. Next we prove the following

$$\mathcal{N}_{\mathcal{M}}(\mathcal{A}) \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{A}' \cap \mathcal{M}) \subseteq \mathcal{U}(\mathcal{A}' \cap \mathcal{M})\mathcal{N}_{\mathcal{M}}(\mathcal{B}). \quad (2.3)$$

The first inclusion follows directly from definitions. To see the second containment, fix $u \in \mathcal{N}_{\mathcal{M}}(\mathcal{A}' \cap \mathcal{M})$. Therefore, both $u\mathcal{B}u^*$ and $\mathcal{B}$ are MASAs in $\mathcal{A}' \cap \mathcal{M}$. Since by [Dix81, Theorem 6.5.5] in a type I von Neumann

algebra any two MASAs are unitarily conjugate, one can find $v \in \mathcal{U}(\mathcal{A}' \cap \mathcal{M})$ such that $u\mathcal{B}u^* = v\mathcal{B}v^*$. Hence $v^*u\mathcal{B}u^*v = \mathcal{B}$ and thus $w = v^*u \in \mathcal{N}_{\mathcal{M}}(\mathcal{B})$. As $u = vw$, we get the desired claim.

Denote by $\mathcal{Q} = \mathcal{N}_{\mathcal{M}}(\mathcal{B})''$. As $\mathcal{A}' \cap \mathcal{M}$ is type I, for every $\varepsilon > 0$ one can find a projection $q \in \mathcal{Z}(\mathcal{A}' \cap \mathcal{M})$ such that $\text{tr}(q) > 1 - \varepsilon$ and the inclusion $\mathcal{A}q \subseteq (\mathcal{A}' \cap \mathcal{M})q$ admits a finite Pimsner-Popa basis $x_1, \dots, x_n \in (\mathcal{A}' \cap \mathcal{M})q$; in other words, $(\mathcal{A}' \cap \mathcal{M})q = \sum_{i=1}^n x_i \mathcal{A}q$. This together with the containment (2.3) further implies that

$$L^2(q\mathcal{N}q) = L^2(q\mathcal{N}_{\mathcal{M}}(\mathcal{A})''q) \subseteq \overline{\sum_{i=1}^n x_i q \mathcal{Q} q}. \quad (2.4)$$

Since $\mathcal{N} \subseteq \mathcal{M}$ has finite index, by Proposition 2.4, so is $q\mathcal{N}q \subseteq q\mathcal{M}q$ and hence the latter inclusion admits a Pimsner-Popa basis $y_1, \dots, y_m \in q\mathcal{M}q$. This together with 2.4 show that $q\mathcal{M}q \subseteq \sum_{j=1, i=1}^{m, n} (y_j x_i) q \mathcal{Q} q$ and hence

$$L^2(q\mathcal{M}q) = \overline{\sum_{j=1, i=1}^{m, n} (y_j x_i) q \mathcal{Q} q}. \quad (2.5)$$

Thus $L^2(q\mathcal{M}q)$ is a finitely generated right $q\mathcal{Q}q$ -module. Using a Gram-Schmidt argument one can find finitely many $\xi_1, \dots, \xi_s \in L^2(q\mathcal{M}q)$ $q\mathcal{Q}q$ -orthogonal such that for any $y \in q\mathcal{M}q$ we have $y = \sum_{i=1}^s \xi_i \langle \xi_i, y \rangle$. In particular, we have $\|y\|_2^2 = \sum_i \|\mathbb{E}_{q\mathcal{Q}q}(\xi_i^* y)\|_2^2$. Applying this identity for all $y \in \mathcal{U}(q\mathcal{M}q)$, Popa's intertwining techniques further imply that

$$q\mathcal{M}q < q\mathcal{Q}q. \quad (2.6)$$

Passing to the relative commutants intertwining in (2.6) we get that $(q\mathcal{Q}q)' \cap q\mathcal{M}q < (q\mathcal{M}q)' \cap q\mathcal{M}q = \mathbb{C}q$. Since $\mathcal{B} \subset \mathcal{M}$ is a MASA one can check that $\mathcal{Q}' \cap \mathcal{M} = \mathcal{Z}(\mathcal{Q})$ and hence $\mathcal{Z}(\mathcal{Q})q < \mathbb{C}q$; in particular, every corner of $\mathcal{Z}(\mathcal{Q})$ has a nontrivial atomic subcorner. Thus one can find a countably family of orthogonal projections $(z_n)_n \subset \mathcal{Z}(\mathcal{Q})$ satisfying $\mathcal{Z}(\mathcal{Q}) = \bigoplus_n \mathbb{C}z_n$, yielding the first assertion of the conclusion. This further yields that $\mathcal{Q}z_n \subseteq z_n \mathcal{M} z_n$ is an irreducible inclusion of II_1 factors. Also using the intertwining (2.6) for $q = z_n$ we get, by Proposition 2.12 that $\mathcal{Q}z_n \subseteq z_n \mathcal{M} z_n$ is finite index. $\square$

3 Virtual isomorphisms between II_1 factors associated with property (T) wreath-like product groups

This section is mainly devoted to a result which describes the structure of all virtual $*$ -isomorphisms between II_1 factors associated with property (T) groups. The groups involved will be the wreath-like product groups introduced in [CIOS23a]. We start by recalling their definition.

Definition 3.1. Let A, B be arbitrary groups, I an abstract set, $B \curvearrowright I$ a (left) action of B on I . A group G is a *wreath-like product* of groups A and B corresponding to the action $B \curvearrowright I$ if G is an extension of the form

$$1 \rightarrow \bigoplus_{i \in I} A_i \hookrightarrow G \xrightarrow{\kappa} B \rightarrow 1, \quad (3.1)$$

where $A_i \cong A$ and the action of G on $A^{(I)} = \bigoplus_{i \in I} A_i$ by conjugation satisfies the rule

$$g A_i g^{-1} = A_{\kappa(g)i} \quad \text{for all } i \in I.$$

We call A the *base* of a wreath-like product $G \in \mathcal{WR}(A, B \curvearrowright I)$. We also call the quotient group homomorphism $G \xrightarrow{\kappa} B$ the *canonical epimorphism*.

If the action $B \curvearrowright I$ is regular (i.e., free and transitive), we say that G is a *regular wreath-like product* of A and B . The set of all wreath-like products of groups A and B corresponding to an action $B \curvearrowright I$ (respectively, all regular wreath-like products) is denoted by $\mathcal{WR}(A, B \curvearrowright I)$ (respectively, $\mathcal{WR}(A, B)$).

To introduce our statement we need one more definition. Two II_1 factors $\mathcal{M}$ and $\mathcal{N}$ are called *virtually isomorphic* if there is $t > 0$ and a $*$ -embedding $\Theta : \mathcal{M} \rightarrow \mathcal{N}^t$ whose image $\Theta(\mathcal{M}) \subseteq \mathcal{N}^t$ has finite index; such Θ is called a *virtual $*$ -isomorphism*.

The proof of this result closely follows that of [CIOS24, Theorem 5.1], incorporating certain elements from [CIOS23b]. We recommend that the reader consult the proof of [CIOS24, Theorem 5.1] and its corresponding preliminaries in advance. For completeness, however, we present all necessary details here, including repeating verbatim some of the arguments used in [CIOS24, Theorem 5.1].

Theorem 3.2. *Let A, C be non-trivial abelian groups. Let B, D be nonparabolic ICC subgroups of finitely generated groups which are hyperbolic relative to a finite family of residually finite groups.*

Let $G \in \mathcal{WR}(A, B \curvearrowright I)$ and $H \in \mathcal{WR}(C, D \curvearrowright J)$ be property (T) groups, where $B \curvearrowright I$ and $D \curvearrowright J$ are faithful actions with infinite orbits.

Let $t > 0$ be a scalar and let $\Theta : \mathcal{L}(G) \hookrightarrow \mathcal{L}(H)^t$ be any virtual $$ -isomorphism.*

Then $t \in \mathbb{N}$ and there are $t_1, \dots, t_m \in \mathbb{N}$ with $t_1 + \dots + t_m = t$, for some $m \in \mathbb{N}$, a finite index subgroup $K < G$, an injective homomorphism $\gamma_i : K \rightarrow H$ with finite index image $\gamma_i(K) \leq H$, and a unitary representation $\rho_i : K \rightarrow \mathcal{U}_{t_i}(\mathbb{C})$, for every $1 \leq i \leq m$, and a unitary $w \in \mathcal{L}(H)^t = \mathcal{L}(H) \overline{\otimes} \mathbb{M}_t(\mathbb{C})$ such that

$$w\Theta(u_g)w^* = \text{diag}(v_{\gamma_1(g)} \otimes \rho_1(g), \dots, v_{\gamma_m(g)} \otimes \rho_m(g)), \text{ for every } g \in K.$$

Proof. Let $\kappa_G : G \rightarrow B$ and $\kappa_H : H \rightarrow D$ be the canonical epimorphisms. For $b \in B$ and $d \in D$, fix $\hat{b} \in G$ and $\hat{d} \in H$ such that $\kappa_G(\hat{b}) = b$ and $\kappa_H(\hat{d}) = d$. Denote by $\mathcal{M} = \mathcal{L}(G)$, $\mathcal{P} = \mathcal{L}(A^{(I)})$, $\mathcal{N} = \mathcal{L}(H)$ and $\mathcal{Q} = \mathcal{L}(C^{(J)})$. Let n_t be the smallest integer such that $n_t \geq t$. Denote $\mathcal{N} = \mathcal{N} \overline{\otimes} \mathbb{M}_{n_t}(\mathbb{C})$ and $\tilde{\mathcal{Q}} = \mathcal{Q} \overline{\otimes} \mathbb{D}_{n_t}(\mathbb{C})$. Let $p \in \tilde{\mathcal{Q}}$ be a projection such that $(\text{tr} \otimes \text{Tr})(p) = t$ and identify $\mathcal{N}^t = p\mathcal{N}p$. For $1 \leq i \leq n_t$, let $e_i = \mathbf{1}_{\{i\}} \in \mathbb{D}_{n_t}(\mathbb{C})$. Let

$$\mathcal{R} := \Theta(\mathcal{P})' \cap p\mathcal{N}p.$$

Since H has property (T), so is D . By applying [CIOS23b, Lemma 3.31] to D , we find a short exact sequence

$$1 \rightarrow S \rightarrow D \xrightarrow{\kappa_D} L \rightarrow 1,$$

where S is either trivial or a nontrivial free product, $S = S_1 * S_2$ with $|S_1| \geq 2$ and $|S_2| \geq 3$, and L is a non-elementary subgroup of a hyperbolic group. Let $\kappa = \kappa_D \circ \kappa_H : H \rightarrow L$ and denote by $T = \ker(\kappa)$. Then $\kappa_H(T) = \ker(\kappa_D) = S$, $C^{(J)} < T < H$, and $T < H$ has infinite index.

Next, we establish the following:

Claim 3.3. $\Theta(\mathcal{P}) <_{\mathcal{N}}^s \mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C})$.

Proof of Claim 3.3. Fix a projection $0 \neq r \in \mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))' \cap p\mathcal{N}p$. Notice $\Theta(\mathcal{P})r$ is abelian and hence amenable. Moreover, its normalizer satisfies $\Theta(\mathcal{M})r \subseteq \mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))''r \subseteq r\mathcal{N}r$, and since $\Theta(\mathcal{M})r \subseteq r\mathcal{N}r$ has a finite index, it follows that $\mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))''r \subseteq r\mathcal{N}r$ has finite index as well. In particular, as $\mathcal{N}$ has property (T), so is $\mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))''r$ [Pop86, CI18]. Furthermore, since $T < H$ has infinite index, it follows from Proposition 2.11 $\mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))''r \not<_{\mathcal{N}} \mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C})$. Altogether, these in combination with [CIOS23b, Theorem 6.4 (a)] yield that $\Theta(\mathcal{P})r <_{\mathcal{N}} \mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C})$. Since this holds for all $0 \neq r \in \mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))' \cap p\mathcal{N}p$, using [DHI19, Lemma 2.4 (b)] we get the desired claim. ■

We continue with the following:

Claim 3.4. $\Theta(\mathcal{P}) <_{\mathcal{N}}^s \mathcal{Q} \otimes \mathbb{D}_{n_t}(\mathbb{C})$.

Proof of Claim 3.4. Fix a projection $0 \neq r \in \mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))' \cap p\mathcal{N}p = \mathcal{Z}(\mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P})))$. Note $\Theta(\mathcal{P}) \subset \Theta(\mathcal{M})$ is a Cartan subalgebra and $\Theta(\mathcal{M}) \subseteq p\mathcal{N}p$ has finite index. Consider a von Neumann subalgebra $\Theta(\mathcal{P}) \subseteq \mathcal{B} \subseteq \mathcal{R}$ such that $\mathcal{B} \subseteq p\mathcal{N}p$ is a MASA. Moreover, by Proposition 2.16 we have $\mathcal{Z}(\mathcal{N}_{p\mathcal{N}p}(\mathcal{B}))'' = \oplus_n \mathbb{C}z_n$ and the inclusion $\mathcal{N}_{p\mathcal{N}p}(\mathcal{B})''z_n \subseteq z_n\mathcal{N}z_n$ has finite index for all $n \in \mathbb{N}$. Since $\mathcal{Z}(\mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))) \subseteq \mathcal{B}$ and $\Theta(\mathcal{P}) \subseteq \mathcal{B}$ is finite index, by Proposition 2.4, so is $\Theta(\mathcal{P})rz_n \subseteq \mathcal{B}rz_n$. Thus, Claim 3.3 implies that $\mathcal{B}rz_n <_{\mathcal{N}} \mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C})$. [CIK15, Proposition 3.6] gives nonzero projections $e \in \mathcal{B}rz_n, q \in \mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C})$, a MASA $\mathcal{R} \subset q(\mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C}))q$, a projection $q' \in \mathcal{R}' \cap q\mathcal{N}q$ and $u \in \mathcal{U}(\mathcal{N})$ such that the inclusion $\mathcal{N}_{q(\mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C}))q}(\mathcal{R})'' \subset q(\mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C}))q$ has finite index and $u\mathcal{B}eu^* = \mathcal{R}q'$. Since $\kappa_H(T) = S$ is a nontrivial free product and $\ker(\kappa_H) = C^{(J)}$, [CIOS23b, Theorem 6.1 (c)] gives that $\mathcal{R} <_{\mathcal{L}(T) \otimes \mathbb{M}_{n_t}(\mathbb{C})}^s \mathcal{Q} \otimes \mathbb{M}_{n_t}(\mathbb{C})$. Thus, $\mathcal{R}q' <_{\mathcal{M}} \mathcal{Q} \otimes \mathbb{M}_{n_t}(\mathbb{C})$, which implies that $\Theta(\mathcal{P})r <_{\mathcal{N}} \mathcal{Q} \otimes \mathbb{M}_{n_t}(\mathbb{C})$. Since $\mathbb{D}_{n_t}(\mathbb{C}) \subseteq \mathbb{M}_{n_t}(\mathbb{C})$ has finite index we further have $\Theta(\mathcal{P})r <_{\mathcal{N}} \mathcal{Q} \otimes \mathbb{D}_{n_t}(\mathbb{C})$. Finally, since this holds for all projections $0 \neq r \in \mathcal{N}_{p\mathcal{N}p}(\Theta(\mathcal{P}))' \cap p\mathcal{N}p$, using [DHI19, Lemma 2.4 (b)] we get the desired claim. ■

Since $\Theta(\mathcal{M}) \subseteq p\mathcal{N}p$ is a finite index inclusion of II_1 factors, Proposition 2.16 implies that $\Theta(\mathcal{P}) \subseteq \mathcal{R}$ admits a finite Pimsner-Popa basis and also $\mathcal{R}$ is a type I von Neumann algebra. Thus for every projection $0 \neq r \in \mathcal{R}' \cap p\mathcal{N}p = \mathcal{L}(\mathcal{R})$ there is a subprojection $0 \neq r_0 \leq r$ we have that $\Theta(\mathcal{P})r_0 \subseteq \mathcal{R}r_0$ has finite Pimsner-Popa basis and hence $\mathcal{R}r_0 < \Theta(\mathcal{P})$. Combining this with Claim 3.4 we get that $\mathcal{R}r <_{\mathcal{N}} \tilde{\mathcal{Q}}$ and hence

$$\mathcal{R} <_{\mathcal{N}}^s \tilde{\mathcal{Q}}. \quad (3.2)$$

Since $\tilde{\mathcal{Q}} \subset \mathcal{N}$ is a Cartan subalgebra, by combining (3.2) with [CIOS23a, Lemma 3.7] (see also [Ioa11]), we get that after replacing Θ with $\text{ad}(w_0) \circ \Theta$, for some $w_0 \in \mathcal{U}(p\mathcal{N}p)$, we may assume that

$$\Theta(\mathcal{P}) \subset \tilde{\mathcal{Q}}p \subset \mathcal{R}. \quad (3.3)$$

If $b \in B$, then $\Theta(u_b)$ normalizes $\Theta(\mathcal{P})$ and thus $\mathcal{R}$. Denote $\beta'_b = \text{ad}(\Theta(u_b)) \in \text{Aut}(\mathcal{R})$. Then $\beta' = (\beta'_b)_{b \in B}$ defines an action of B on $\mathcal{R}$ which leaves $\Theta(\mathcal{P})$ invariant. The restriction of β' to $\Theta(\mathcal{P})$ is free since it is isomorphic to the conjugation action of B on $\mathcal{L}(A^{(I)})$, which is free by the hypothesis condition. Thus, [CIOS23a, Lemma 3.8] yields an action $\beta = (\beta_b)_{b \in B}$ of B on $\mathcal{R}$ satisfying

1. for every $b \in B$ we have $\beta_b = \beta'_b \circ \text{ad}(\omega_b) = \text{ad}(\Theta(u_b)\omega_b)$, for some $\omega_b \in \mathcal{U}(\mathcal{R})$, and
2. $\tilde{\mathcal{Q}}p$ is $\beta(B)$ -invariant and the restriction of β to $\tilde{\mathcal{Q}}p$ is free.

Our next goal is to apply [CIOS23a, Theorem 4.1]. Consider the action $\alpha = (\alpha_d)_{d \in D}$ of D on $\mathcal{Q}$ given by $\alpha_d = \text{ad}(v_{\hat{d}})$ for $d \in D$. Let $(\hat{X}, \hat{\mu})$ be the dual of C with its Haar measure. Let $(X, \mu) = (\hat{X}^J, \hat{\mu}^J)$ and $(\tilde{X}, \tilde{\mu}) = (X \times \mathbb{Z}/n_t\mathbb{Z}, \mu \times \mu_{n_t})$, where μ_{n_t} denotes the counting measure on $\mathbb{Z}/n_t\mathbb{Z}$. Identify $\mathcal{Q} = L^\infty(X)$ and $\tilde{\mathcal{Q}} = L^\infty(\tilde{X})$. Denote still by α the corresponding measure preserving action $D \curvearrowright^\alpha (X, \mu)$ and let $D \times \mathbb{Z}/n_t\mathbb{Z} \curvearrowright^{\tilde{\alpha}} (\tilde{X}, \tilde{\mu})$ be the action given by $(g, a) \cdot (x, m) = (g \cdot x, a + m)$. Let $X_0 \subset \tilde{X}$ be a measurable set such that $p = \mathbf{1}_{X_0}$. Since $\tilde{\mathcal{Q}}p = L^\infty(X_0)$ is $\beta(B)$ -invariant, we get a measure preserving action $B \curvearrowright^\beta (X_0, \tilde{\mu}|_{X_0})$.

Note that $\tilde{\mathcal{Q}} \subset \mathcal{N}$ is a Cartan subalgebra and the p.m.p. equivalence relation associated to the inclusion $\tilde{\mathcal{Q}} \subset \mathcal{N}$ [FM77] is equal to $\mathcal{R}(D \times \mathbb{Z}/n_t\mathbb{Z} \curvearrowright^{\tilde{\alpha}} \tilde{X})$. Since the restriction of β to $\tilde{\mathcal{Q}}p$ is implemented by unitaries in $p\mathcal{N}p$, we deduce that

$$\beta(B) \cdot \mathbf{m} \subset \tilde{\alpha}(D \times \mathbb{Z}/n_t\mathbb{Z}) \cdot \mathbf{m}, \text{ for almost every } \mathbf{m} \in X_0. \quad (3.4)$$

Since B has property (T) and α is built over $D \curvearrowright J$ as a consequence of [CIOS24, Lemma 3.4], applying Lemma [CIOS24, Lemma 3.10], we find a partition $X_0 = \sqcup_{i=1}^{n_0} X_i$ into non-null measurable sets, for some $n_0 \in \mathbb{N} \cup \{\infty\}$, and a finite index subgroup $B_i < B$ such that X_i is $\beta(B_i)$ -invariant and the restriction of $\beta|_{B_i}$ to X_i is weakly mixing, for all i .

Next we will prove the following

Claim 3.5. *There exists a sequence $(b_n) \subset B_i$ such that we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{\mu}(\{\mathbf{m} \in X_0 \mid \beta_{b_n}(\mathbf{m}) \in \tilde{\alpha}(\tilde{d}_1(\text{Stab}_D(j) \times \mathbb{Z}/n_t\mathbb{Z})\tilde{d}_2)(\mathbf{m})\}) &= 0, \text{ and} \\ \lim_{n \rightarrow \infty} \tilde{\mu}(\{\mathbf{m} \in X_0 \mid \beta_{b_n}(\mathbf{m}) \in \tilde{\alpha}(\tilde{d}_1(C_D(g) \times \mathbb{Z}/n_t\mathbb{Z})\tilde{d}_2)(\mathbf{m})\}) &= 0, \end{aligned} \quad (3.5)$$

for every $\tilde{d}_1, \tilde{d}_2 \in D \times \mathbb{Z}/n_t\mathbb{Z}$.

Proof of Claim 3.5. Let $H_0 = \kappa_H^{-1}(\text{Stab}_D(j))$ and $H_1 = \kappa_H^{-1}(C_D(d))$. Since the action $B \curvearrowright I$ has infinite orbits and B is an ICC nonparabolic subgroup of a relative hyperbolic group it follows that $H_0, H_1 < H$ are infinite index subgroups. Since $B_i \leq B$ is a finite index inclusion of groups and $\Theta(\mathcal{M}) \subset p\mathcal{N}p$ is a finite index inclusion of von Neumann algebras, Proposition 2.11 implies the existence of a sequence $(g_n) \subset \kappa_G^{-1}(B_i)$ such that for every $x, y \in \mathcal{N}$ we have

$$\lim_{n \rightarrow \infty} \|\mathbb{E}_{\mathcal{L}(H_0) \otimes \mathbb{M}_{n_t}(\mathbb{C})}(x\Theta(u_{g_n})y)\|_2 = 0 \text{ and } \lim_{n \rightarrow \infty} \|\mathbb{E}_{\mathcal{L}(H_1) \otimes \mathbb{M}_{n_t}(\mathbb{C})}(x\Theta(u_{g_n})y)\|_2 = 0. \quad (3.6)$$

We will show that $b_n = \kappa_G(g_n) \in B_i$ satisfy the assertion of the claim. Since $g_n^{-1}\hat{b}_n \in A^{(I)}$ and $\omega_{b_n} \in \mathcal{U}(\mathcal{R})$, we get that $\Theta(u_{\hat{b}_n})\omega_{b_n} \in \Theta(u_{g_n})\mathcal{U}(\mathcal{R})$. Thus, one can find $x_1, \dots, x_l$ such that $\Theta(u_{\hat{b}_n})\omega_{b_n} \in \sum_{i=1}^l \Theta(u_{g_n})(\tilde{\mathcal{Q}}p)_1 x_i$, for every $n \in \mathbb{N}$. As $\tilde{\mathcal{Q}}$ is regular in $\mathcal{N}$ and contained in $\mathcal{L}(H_0) \otimes \mathbb{M}_{n_t}(\mathbb{C})$ and $\mathcal{L}(H_1) \otimes \mathbb{M}_{n_t}(\mathbb{C})$, (3.6) implies that for every $x, y \in \mathcal{N}$

$$\lim_{n \rightarrow \infty} \|\mathbb{E}_{\mathcal{L}(H_0) \otimes \mathbb{M}_{n_t}(\mathbb{C})}(x\Theta(u_{\hat{b}_n})\omega_{b_n}y)\|_2 = 0 \text{ and } \lim_{n \rightarrow \infty} \|\mathbb{E}_{\mathcal{L}(H_1) \otimes \mathbb{M}_{n_t}(\mathbb{C})}(x\Theta(u_{\hat{b}_n})\omega_{b_n}y)\|_2 = 0. \quad (3.7)$$

On the other hand, we have that $\beta_{b_n} = \text{ad}(\Theta(u_{\hat{b}_n})\omega_{b_n})$ and $\alpha_h = \text{ad}(v_{\hat{h}})$, for every $h \in D$. Using these facts one can show that $\tilde{\mu}(\{\mathbf{m} \in X_0 \mid \beta_{b_n}(\mathbf{m}) \in \tilde{\alpha}(d_1 \text{Stab}_D(j)d_2 \times \mathbb{Z}/n_t\mathbb{Z})(\mathbf{m})\}) = \|\mathbb{E}_{\mathcal{L}(H_0) \otimes \mathbb{M}_{n_t}(\mathbb{C})}((u_{\hat{d}_1}^* \otimes 1)\Theta(u_{\hat{b}_n})\omega_{b_n}(u_{\hat{d}_2}^* \otimes 1))\|_2^2$. Thus, (3.7) proves the first assertion of Claim 3.5. The second assertion follows similarly. $\blacksquare$

Altogether, the prior relations show that all assumptions in [CIOS23a, Theorem 4.1] are satisfied and thus by the conclusion of this result we can find an injective group homomorphism $\bar{\gamma}_i : B_i \rightarrow D$ and $\varphi_i \in [\mathcal{R}(D \times \mathbb{Z}/n_t\mathbb{Z} \curvearrowright^{\tilde{\alpha}} \tilde{X})]$ such that $\varphi_i(X_i) = X \times \{i\} \equiv X$ and $\varphi_i \circ \beta_{b|X_i} = \alpha_{\bar{\gamma}_i(b)} \circ \varphi_{i|X_i}$, for all $b \in B_i$. In particular, $\tilde{\mu}(X_i) = 1$. Thus, $t = \tilde{\mu}(X_0) = \sum_{i=1}^{n_0} \tilde{\mu}(X_i) = n_0 \in \mathbb{N}$. Since n_t is the smallest integer with $n_t \geq t$, we get that $n_t = t = n_0$ and $p = 1_{\mathcal{N}}$.

For $1 \leq i \leq t$, let $p_i = \mathbf{1}_{X_i} \in \tilde{\mathcal{Q}}$ and $U_i \in \mathcal{N}(\tilde{\mathcal{Q}})$ such that $U_i y U_i^* = y \circ \varphi_i^{-1}$, for every $y \in \tilde{\mathcal{Q}}$. Then $U_i p_i U_i^* = 1 \otimes e_i \in \mathcal{N} \otimes \mathbb{M}_t(\mathbb{C})$ where $e_i \in \mathbb{M}_t(\mathbb{C})$ is the matrix that has 1 for the (i, i) -entry and 0 for the other entries. Since $\beta_b = \text{ad}(\Theta(u_{\hat{b}})\omega_b)$ we find $(\zeta_{i,b})_{b \in B_i} \subset \mathcal{U}(\mathcal{Q})$ with

$$U_i \Theta(u_{\hat{b}})\omega_b p_i U_i^* = \zeta_{i,b} v_{\bar{\gamma}_i(b)} \otimes e_i, \text{ for every } b \in B_i. \quad (3.8)$$

Under the previous notations, we now establish the following.

Claim 3.6. *The subgroup $\bar{\gamma}_i(B_i) \leq D$ has finite index for all $1 \leq i \leq t$.*

Proof of Claim 3.6. First, notice the relation (3.8) implies that $\mathcal{V} := U_i \tilde{\mathcal{Q}} p_i U_i^* \vee \{U_i \Theta(u_{\hat{b}})\omega_b p_i U_i^* \mid b \in B_i\}'' = \mathcal{L}(\kappa_H^{-1}(\bar{\gamma}_i(B_i))) \otimes e_i$. Observe that $\mathcal{Q} \otimes e_i = U_i \tilde{\mathcal{Q}} p_i U_i^* \subseteq \mathcal{V}$ is a MASA and $U_i \Theta(u_{\hat{b}})\omega_b p_i U_i^* \in \mathcal{N}_{U_i \mathcal{N} p_i U_i^*}(U_i \tilde{\mathcal{Q}} p_i U_i^*)$ for all $b \in B_i$.

From Proposition 2.16 there is a projection $z \in \mathcal{Z}(\mathcal{R}) \subseteq \tilde{\mathcal{Q}}$ with $p_i z \neq 0$ such that $\Theta(\mathcal{P})p_i z \subseteq \mathcal{R}p_i z$ admits a finite Pimsner-Popa basis, $\xi_1, \dots, \xi_{l_1} \in \mathcal{R}p_i z$. Now fix $x \in \mathcal{P}$ and $b \in B_i$. Then one can see that

$$\begin{aligned} p_i z \Theta(u_{\hat{b}}x)p_i z &= p_i z \Theta(u_{\hat{b}})\Theta(x)p_i z = p_i z \Theta(u_{\hat{b}})\omega_{\hat{b}}\omega_b^* \Theta(x)p_i z \\ &= z \Theta(u_{\hat{b}})\omega_{\hat{b}}p_i \omega_b^* \Theta(x)p_i z = z \Theta(u_{\hat{b}})\omega_{\hat{b}}p_i \Theta(x)\omega_b^* p_i z \\ &= \sum_{j=1}^{l_1} z \Theta(u_{\hat{b}})\omega_{\hat{b}}p_i \Theta(x) \mathbb{E}_{\Theta(\mathcal{P})p_i z}(\omega_b^* p_i z \xi_j^*) \xi_j \in \sum_{j=1}^{l_1} z(U_i^* \mathcal{V} U_i) \xi_j. \end{aligned}$$

Denoting by $\mathcal{S} := \Theta(\mathcal{L}(\kappa_G^{-1}(B_i)))$ and using basic $\|\cdot\|_2$ -approximations, the prior formulae further imply

$$L^2(p_i z \mathcal{S} p_i z) \subseteq \overline{\sum_{j=1}^{l_1} z(U_i^* \mathcal{V} U_i) \xi_j}. \quad (3.9)$$

Since $B_i \leq B$ has finite index by Proposition 2.4 then so is $p_i z \mathcal{S} p_i z \subseteq p_i z \mathcal{N} p_i z$. Using Proposition 2.4 there is a Pimsner-Popa basis $\xi'_1, \dots, \xi'_{l_2} \in p_i z \mathcal{N} p_i z$ for this inclusion. Then (3.9) further implies that

$$L^2(p_i z \mathcal{N} p_i z) \subseteq \overline{\sum_{j_1=1}^{l_1} \sum_{j_2=1}^{l_2} z(U_i^* \mathcal{V} U_i) \xi_{j_1} \xi'_{j_2}}. \quad (3.10)$$

Thus the $zU_i^* \mathcal{V} U_i p_i z$ -bimodule $L^2(p_i z \mathcal{N} p_i z)$ is finitely generated, of length s , as a left $zU_i^* \mathcal{V} U_i z$ -module. Using [FGS11, Lemma 3.5] one can find a sequence of projections $(z_n)_n \subset (zU_i^* \mathcal{V} U_i z)' \cap zp_i \mathcal{N} zp_i$, SOT-convergent to $p_i z$ such that for every $n \in \mathbb{N}$ there exist $x_{1,n}, \dots, x_{s,n} \in zp_i \mathcal{N} zp_i$ that are $zU_i^* \mathcal{V} U_i z$ -orthogonal and satisfy

$$z_n y z_n = \sum_{j=1}^s \mathbb{E}_{zU_i^* \mathcal{V} U_i z}(y x_{j,n}^*) x_{j,n} \text{ for all } y \in p_i z \mathcal{N} p_i z. \quad (3.11)$$

Since $\|z_n - p_i z\|_2 \rightarrow 0$, for n large enough, (3.11) implies that for every $y \in \mathcal{U}(p_i z \mathcal{N} p_i z)$ we have $0 < \|p_i z\|^2/2 = \sum_{j=1}^s \|\mathbb{E}_z U_i^* \mathcal{V} U_i z(y x_{j,n}^*)\|_2^2$. By Popa's intertwining techniques, this implies $\mathcal{N} <_{\mathcal{N}} z U_i^* \mathcal{V} U_i z$. Thus $\mathcal{N} <_{\mathcal{N}} \mathcal{V} = \mathcal{L}(\kappa_H^{-1}(\bar{\gamma}_i(B_i))) \otimes e_i$. Proposition 2.11 further entails that $\bar{\gamma}_i(B_i) \leq D$ has finite index. ■

From here on our proof follows verbatim the proof of [CIOS24, Theorem 5.1], and we include it here only for reader's convenience.

After replacing B_i by $B_0 = \cap_{i=1}^t B_i$ we may assume that $B_i = B_0$, for all $1 \leq i \leq t$. We will prove the conclusion for $K = \kappa_G^{-1}(B_0)$. Let $\mathcal{M}_0 = \mathcal{L}(K)$. To prove the conclusion, it suffices to find a projection $p_0 \in \Theta(\mathcal{M}_0)' \cap \mathcal{N}$ with $(\text{tr} \otimes \text{Tr})(p_0) = t_1 \in \{1, \dots, t\}$, homomorphisms $\gamma : K \rightarrow H$, $\rho : K \rightarrow \mathcal{U}_{t_1}(\mathbb{C})$, $\tilde{U} \in \mathcal{U}(\mathcal{N})$ such that γ is injective, $\tilde{U} p_0 \tilde{U}^* = 1 \otimes (\sum_{i=1}^{t_1} e_i)$ and $\tilde{U} \Theta(u_g) p_0 \tilde{U}^* = v_{\gamma(g)} \otimes \rho(g)$, for all $g \in K$. Indeed, once we have this assertion, the conclusion will follow by a maximality argument. This concludes the first part of the proof.

The rest of the proof, which is divided between four claims (Claims 3.7-3.10), is devoted to proving the last assertion. We begin with the following:

Claim 3.7. *There are $1 \leq t_1 \leq t$ and a homomorphism $\bar{\gamma} : B_0 \rightarrow D$ such that, after renumbering, we have $p_0 = \sum_{i=1}^{t_1} p_i \in \Theta(\mathcal{M}_0)' \cap \mathcal{N}$ and can take $\bar{\gamma}_i = \bar{\gamma}$, for every $1 \leq i \leq t_1$.*

Proof of Claim 3.7. We start by showing that if $1 \leq i, j \leq t$ and $p_i \mathcal{R} p_j \neq \{0\}$, then $\bar{\gamma}_i$ and $\bar{\gamma}_j$ are conjugate. Let $x \in (\mathcal{R})_1$ with $p_i x p_j \neq 0$. Then $x_b = \text{ad}(\Theta(u_{\hat{b}}) \omega_b)(p_i x) = \beta_b(p_i x) \in (\mathcal{R})_1$ and $(\Theta(u_{\hat{b}}) \omega_b p_i) x p_j = x_b (\Theta(u_{\hat{b}}) \omega_b p_j)$, for every $b \in B_0$. Using (3.8) we derive that

$$(U_i^*(\zeta_{i,b} v_{\bar{\gamma}_i(b)} \otimes e_i) U_i) x p_j = x_b (U_j^*(\zeta_{j,b} v_{\bar{\gamma}_j(b)} \otimes e_j) U_j), \text{ for every } b \in B_0. \quad (3.12)$$

For every subset $F \subset D$, let P_F be the orthogonal projection from $L^2(\mathcal{N})$ onto the $\|\cdot\|_2$ -closed linear span of $\{v_h \otimes x \mid h \in \kappa_H^{-1}(F), x \in \mathbb{M}_t(\mathbb{C})\}$. Since $(\zeta_{i,b})_{b \in B_0}, (\zeta_{j,b})_{b \in B_0} \subset \mathcal{U}(\mathcal{Q})$ and $(x_b)_{b \in B_0} \subset (\mathcal{R})_1$, then using basic $\|\cdot\|_2$ -approximations and also (3.2) in combination with [Vae13, Lemma 2.5] (for b)), we can find finite $F \subset D$ so that for every $b \in B$ we have

$$\begin{aligned} \text{a)} \quad & \| (U_i^*(\zeta_{i,b} v_{\bar{\gamma}_i(b)} \otimes e_i) U_i) x p_j - P_{F\bar{\gamma}_i(b)F} ((U_i^*(\zeta_{i,b} v_{\bar{\gamma}_i(b)} \otimes e_i) U_i) x p_j) \|_2 < \frac{\|p_i x p_j\|_2}{2}, \text{ and} \\ \text{b)} \quad & \| x_b (U_j^*(\zeta_{j,b} v_{\bar{\gamma}_j(b)} \otimes e_j) U_j) - P_{F\bar{\gamma}_j(b)F} (x_b (U_j^*(\zeta_{j,b} v_{\bar{\gamma}_j(b)} \otimes e_j) U_j)) \|_2 < \frac{\|p_i x p_j\|_2}{2}. \end{aligned}$$

Combining a)-b) with (3.12), we derive that $F\bar{\gamma}_i(b)F \cap F\bar{\gamma}_j(b)F \neq \emptyset$, for every $b \in B_0$. Since $\bar{\gamma}_i$ is injective and $B_0 < B$ has finite index, Claim 3.6 further implies that $\bar{\gamma}_i(B_0) \leq D$ has finite index. Since for any $d \in D \setminus \{1\}$, $C_D(d) < D$ has infinite index then $\{\bar{\gamma}_i(b) d \bar{\gamma}_i(b)^{-1} \mid b \in B_0\}$ is infinite. By [BV14, Lemma 7.1] we find $d \in D$ such that $\bar{\gamma}_i(b) = d \bar{\gamma}_j(b) d^{-1}$, for every $b \in B_0$. This proves our assertion that $\bar{\gamma}_i$ and $\bar{\gamma}_j$ are conjugate.

Let $\bar{\gamma} = \bar{\gamma}_1 : B_0 \rightarrow D$. After renumbering, we may assume that $\bar{\gamma}_1, \dots, \bar{\gamma}_{t_1}$ are conjugate to $\bar{\gamma}$ and $\bar{\gamma}_{t_1+1}, \dots, \bar{\gamma}_t$ are not conjugate to $\bar{\gamma}$, for some $1 \leq t_1 \leq t$. The previous paragraph implies that $p_i \mathcal{R} p_j = \{0\}$, for every $1 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$. Thus, $p_0 = \sum_{i=1}^{t_1} p_i$ belongs to the center of $\mathcal{R}$. As p_0 commutes with $\Theta(u_{\hat{b}}) \omega_b$ and $\omega_b \in \mathcal{U}(\mathcal{R})$, then p_0 commutes with $\Theta(u_{\hat{b}})$, for every $b \in B_0$. Since $p_i \in \tilde{\mathcal{Q}} p \subset \mathcal{R} = \Theta(\mathcal{P})' \cap \mathcal{N}$, for every $1 \leq i \leq t$, $p_0 \in \Theta(\mathcal{P})' \cap \mathcal{N}$. Thus, p_0 also commutes with $\Theta(\mathcal{P})$. Since $\mathcal{P}$ and $(u_{\hat{b}})_{b \in B_0}$ generate $\mathcal{M}_0$, we get that $p_0 \in \Theta(\mathcal{M}_0)' \cap \mathcal{N}$. Moreover, if $1 \leq i \leq t_1$, there is $d_i \in D$ such that $\bar{\gamma}_i(h) = d_i \bar{\gamma}(h) d_i^{-1}$, for every $h \in B_0$. After replacing φ_i by $\alpha_{d_i^{-1}} \circ \varphi_i$ we may assume that $\bar{\gamma}_i = \bar{\gamma}$, for every $1 \leq i \leq t_1$. ■

Let $U = \sum_{i=1}^{t_1} U_i p_i$ and $e = \sum_{i=1}^{t_1} e_i$. Then U is a partial isometry, $U U^* = 1 \otimes e$, $U^* U = p_0$ and $U \tilde{\mathcal{Q}} p_0 U^* = \tilde{\mathcal{Q}}(1 \otimes e)$. If we let $\zeta_b = \sum_{i=1}^{t_1} \zeta_{i,b} \otimes e_i \in \mathcal{U}(\tilde{\mathcal{Q}}(1 \otimes e))$, then (3.8) gives that

$$U \Theta(u_{\hat{b}}) \omega_b p_0 U^* = \zeta_b (v_{\bar{\gamma}(b)} \otimes e), \text{ for every } b \in B_0. \quad (3.13)$$

Identify $\mathcal{N}_1 := (1 \otimes e) \mathcal{N} (1 \otimes e)$ and $\mathcal{Q}_1 := \tilde{\mathcal{Q}}(1 \otimes e)$ with $\mathcal{N} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})$ and $\mathcal{Q} \overline{\otimes} \mathbb{D}_{t_1}(\mathbb{C})$, respectively. Consider the unital $*$ -homomorphism $\Theta_1 : \mathcal{M}_0 \rightarrow \mathcal{N}_1$ given by $\Theta_1(x) = U \Theta(x) p_0 U^*$, for every $x \in \mathcal{M}_0$, and let $\mathcal{R}_1 = \Theta_1(\mathcal{P})' \cap \mathcal{N}_1$. Letting $w_b = U \omega_b p_0 U^* \in \mathcal{U}(\mathcal{R}_1)$, then (3.13) rewrites as

$$\Theta_1(u_{\hat{b}}) w_b = \zeta_b (v_{\bar{\gamma}(b)} \otimes 1), \text{ for every } b \in B_0. \quad (3.14)$$

By (3.3), $\mathcal{Q}_1 \subset \mathcal{R}_1$. Since $(\Theta_1(u_b)w_b)_{b \in B}$ normalizes $\mathcal{R}_1$ and $(\zeta_b)_{b \in B_0} \subset \mathcal{U}(\mathcal{Q}_1)$, (3.14) implies that $(v_{\widehat{\gamma(b)}} \otimes 1)_{b \in B_0}$ normalizes $\mathcal{R}_1$. Thus, $\eta_b = \zeta_b \text{ad}(v_{\widehat{\gamma(b)}} \otimes 1)(w_b^*) \in \mathcal{U}(\mathcal{R}_1)$ and

$$\Theta_1(u_b) = \eta_b(v_{\widehat{\gamma(b)}} \otimes 1), \text{ for every } b \in B_0. \quad (3.15)$$

Claim 3.8. $\mathcal{R}_1 = \mathcal{Q} \overline{\otimes} \mathcal{T}$, for a von Neumann subalgebra $\mathcal{T} \subset \mathbb{M}_{t_1}(\mathbb{C})$.

Proof of Claim 3.8. First, we show that $\mathcal{R}_1 \subset \mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})$. To this end, since $\overline{\gamma}(B_0) \leq D$ has finite index and $C_D(d)$ has infinite index in D then for every $d \in D \setminus \{1\}$, there is a sequence $(b_n) \subset B_0$ such that for every $d_1, d_2 \in D$ and $d' \in D \setminus \{1\}$ we have $d_1 \text{ad}(\overline{\gamma}(b_n))(d')d_2 \neq 1$, for every m large enough.

Next we claim that $\|\mathbb{E}_{\mathcal{Q}}(x_1 \text{ad}(v_{\widehat{\gamma(b_n)}})(b)x_2)\|_2 \rightarrow 0$, for every $x_1, x_2 \in \mathcal{N}$ and $y \in \mathcal{N} \ominus \mathcal{Q}$. We only have to check this for $x_1 = v_{h_1}, x_2 = v_{h_2}, y = v_k$, where $h_1, h_2 \in H, k \in H \setminus C^{(I)}$. In this case $\kappa_H(k) \neq 1$, thus $\kappa_H(h_1 \text{ad}(\overline{\gamma}(b_n))(k)h_2) = \kappa_H(h_1) \text{ad}(\overline{\gamma}(b_n))(\kappa_H(k))\kappa_H(h_2) \neq 1$ and therefore $\mathbb{E}_{\mathcal{Q}}(x_1 \text{ad}(v_{\widehat{\gamma(b_n)}})(y)x_2) = 0$, for every m large enough.

The previous paragraph further implies that

$$\|\mathbb{E}_{\mathcal{Q}_1}(x_1 \text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y)x_2)\|_2 \rightarrow 0, \text{ for every } x_1, x_2 \in \mathcal{N}_1 \text{ and } y \in \mathcal{N}_1 \ominus (\mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})). \quad (3.16)$$

Let $\mathcal{Z}(\mathcal{R}_1)$ be the center of $\mathcal{R}_1$. Since $\mathcal{Q}_1 \subset \mathcal{N}_1$ is a MASA we have $\mathcal{Z}(\mathcal{R}_1) \subseteq \mathcal{Q}_1 \subseteq \mathcal{R}_1$. Moreover, since $\Theta_1(\mathcal{M}_0) \subseteq \mathcal{N}_1$ has finite index then so does $\Theta_1(\mathcal{P}) \subseteq \mathcal{R}_1$. Thus, using Proposition 2.16 $\mathcal{Q}_1 \subseteq \mathcal{R}_1$ admits a finite Pimsner-Popa basis. Combining this with (3.16), basic approximations show that

$$\|\mathbb{E}_{\mathcal{R}_1}(\text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y))\|_2 \rightarrow 0, \text{ for every } y \in \mathcal{N}_1 \ominus (\mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})). \quad (3.17)$$

To see that $\mathcal{R}_1 \subset \mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})$, let $y_0 \in \mathcal{R}_1$. Put $y_1 = \mathbb{E}_{\mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})}(y_0)$ and $y_2 = y_0 - y_1$. Since $v_{\widehat{\gamma(b_n)}} \otimes 1$ normalizes $\mathcal{R}_1$, $\text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y_0) \in \mathcal{R}_1$ and thus $\|\mathbb{E}_{\mathcal{R}_1}(\text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y_0))\|_2 = \|y_0\|_2$, for every m . On the other hand, $\|\mathbb{E}_{\mathcal{R}_1}(\text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y_2))\|_2 \rightarrow 0$ by (3.17). Thus, we get that $\|\mathbb{E}_{\mathcal{R}_1}(\text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y_1))\|_2 \rightarrow \|y_0\|_2$. Since $\|\mathbb{E}_{\mathcal{R}_1}(\text{ad}(v_{\widehat{\gamma(b_n)}} \otimes 1)(y_1))\|_2 \leq \|y_1\|_2 \leq \|y_0\|_2$, we conclude that $\|y_1\|_2 = \|y_0\|_2$. Hence, $y_0 = y_1 \in \mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})$. This proves that $\mathcal{R}_1 \subset \mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})$.

Therefore, we have that $\mathcal{Q}_1 = \mathcal{Q} \overline{\otimes} \mathbb{D}_{t_1}(\mathbb{C}) \subseteq \mathcal{R}_1 \subseteq \mathcal{Q} \overline{\otimes} \mathbb{M}_{t_1}(\mathbb{C})$. Since $\mathcal{Q} = L^\infty(X, \mu)$, we can disintegrate $\mathcal{R}_1 = \int_X^\oplus \mathcal{T}_{\mathbf{m}} d\mu(\mathbf{m})$, where $(\mathcal{T}_{\mathbf{m}})_{\mathbf{m} \in X}$ is a measurable field of von Neumann subalgebras of $\mathbb{M}_{t_1}(\mathbb{C})$ containing $\mathbb{D}_{t_1}(\mathbb{C})$. We denote

$$\tilde{\alpha}_b = \text{ad}(v_{\widehat{\gamma(b)}}) \in \text{Aut}(\mathcal{Q}), \text{ for } b \in B_0.$$

Then $\tilde{\alpha} = (\tilde{\alpha}_b)_{b \in B_0}$ defines an action of B_0 on $\mathcal{Q}$. Since $(v_{\widehat{\gamma(b)}} \otimes 1)_{b \in B_0}$ normalizes $\mathcal{R}_1$, we have $\mathcal{T}_{\tilde{\alpha}_b(\mathbf{m})} = \mathcal{T}_{\mathbf{m}}$, for every $b \in B_0$ and almost every $\mathbf{m} \in X$. [CIOS23a, Lemma 3.4] implies that $\tilde{\alpha}$ is built over the action $B_0 \curvearrowright J$ given by $b \cdot j = \overline{\gamma}(b)j$. Since $\text{Stab}_D(j) < D$ has infinite index, $\overline{\gamma}$ is injective and $\overline{\gamma}(B_0) < D$ has finite index, the action $B_0 \curvearrowright J$ has infinite orbits. This implies that $\tilde{\alpha}$ is weakly mixing, hence ergodic. Thus, one can find a von Neumann subalgebra $\mathcal{T} \subset \mathbb{M}_{t_1}(\mathbb{C})$ such that $\mathcal{T}_{\mathbf{m}} = \mathcal{T}$, for almost every $\mathbf{m} \in X$, which proves the claim. $\blacksquare$

Next, let $\mathcal{U} = \mathcal{U}(\mathcal{T})/\mathcal{U}(\mathcal{Z}(\mathcal{T}))$, where $\mathcal{Z}(\mathcal{T})$ is the center of $\mathcal{T}$, and $\mathfrak{q} : \mathcal{U}(\mathcal{T}) \rightarrow \mathcal{U}$ be the quotient homomorphism. We continue with the following claim:

Claim 3.9. *There are maps $\bar{\rho} : B_0 \rightarrow \mathcal{U}(\mathcal{T})$ and $\nu : B_0 \rightarrow \mathcal{U}(\mathcal{Z}(\mathcal{R}_1))$ such that the map $B_0 \ni b \mapsto \mathfrak{q}(\bar{\rho}_b) \in \mathcal{U}$ is a homomorphism and after replacing Θ_1 by $\text{ad}(\sigma) \circ \Theta_1$, for some $\sigma \in \mathcal{U}(\mathcal{R}_1)$, we have that $\Theta_1(u_b) = \nu_b(v_{\widehat{\gamma(b)}} \otimes \bar{\rho}_b)$, for every $b \in B_0$.*

Proof of Claim 3.9. Note that $(\text{ad}(\Theta_1(u_b)))_{b \in B_0}$ and $(\tilde{\alpha}_b \otimes \text{Id}_{\mathcal{T}})_{b \in B_0}$ define actions of B_0 on $\mathcal{R}_1 = \mathcal{Q} \otimes \mathcal{T}$. Combining this observation with (3.15) gives that

$$\eta_{b_1 b_2}^* \eta_{b_1}(\tilde{\alpha}_{b_1} \otimes \text{Id})(\eta_{b_2}) \in \mathcal{Z}(\mathcal{R}_1) = \mathcal{Q} \otimes \mathcal{Z}(\mathcal{T}), \text{ for every } b_1, b_2 \in B_0. \quad (3.18)$$

Viewing every $\eta \in \mathcal{U}(\mathcal{R}_1)$ as a measurable function $\eta : X \rightarrow \mathcal{U}(\mathcal{T})$, (3.18) rewrites as $\eta_{b_1 b_2}(\mathbf{m})^* \eta_{b_1}(\mathbf{m}) \eta_{b_2}(\tilde{\alpha}_{b_1}^{-1} \mathbf{m}) \in \mathcal{U}(\mathcal{Z}(\mathcal{T}))$, for every $b_1, b_2 \in B_0$ and almost every $\mathbf{m} \in X$. Then $\psi_0 : B_0 \times X \rightarrow \mathcal{U}$ given by $\psi_0(b, \mathbf{m}) =$

$\mathfrak{q}(\eta_b(\mathbf{m}))$ is a 1-cocycle for $\tilde{\alpha}$. Since $\mathcal{U}(\mathcal{Z}(\mathcal{T}))$ is a closed central subgroup of the compact Polish group $\mathcal{U}(\mathcal{T})$, $\mathcal{U}$ is a compact Polish group with respect to the quotient topology. In particular, $\mathcal{U}$ is a $\mathcal{U}_{\text{fin}}$ group (see [Pop06, Lemma 2.7]). Since B_0 has property (T), $\tilde{\alpha}$ is built over $B_0 \curvearrowright J$ and $B_0 \curvearrowright J$ has infinite orbits, [CIOS24, Theorem 3.5] (see also [CIOS23a, Theorem 3.6]) implies that ψ_0 is cohomologous to a homomorphism $\psi : B_0 \rightarrow \mathcal{U}$.

Let $\sigma : X \rightarrow \mathcal{U}(\mathcal{T})$ be a measurable map satisfying $\mathfrak{q}(\sigma(\mathbf{m}))\psi_0(b, \mathbf{m})\mathfrak{q}(\sigma(\tilde{\alpha}_{b^{-1}}(\mathbf{m})))^{-1} = \psi_b$, for every $b \in B_0$ and almost every $\mathbf{m} \in X$. Let $\bar{\rho} : B_0 \rightarrow \mathcal{U}(\mathcal{T})$ be such that $\mathfrak{q}(\bar{\rho}_b) = \psi_b$. Thus, we find a measurable map $\nu_b : X \rightarrow \mathcal{U}(\mathcal{Z}(\mathcal{T}))$ such that $\sigma(\mathbf{m})\eta_b(\mathbf{m})\sigma(\tilde{\alpha}_{b^{-1}}(\mathbf{m}))^{-1} = \nu_b(\mathbf{m})\bar{\rho}_b$, for every $b \in B_0$ and almost every $\mathbf{m} \in X$. Equivalently, $\sigma \in \mathcal{U}(\mathcal{R}_1)$ and $\nu : B_0 \rightarrow \mathcal{U}(\mathcal{Z}(\mathcal{R}_1))$ satisfy $\sigma\eta_b(\tilde{\alpha}_b \otimes \text{Id})(\sigma)^* = \nu_b(1 \otimes \bar{\rho}_b)$, for every $b \in B_0$. This implies the claim. $\blacksquare$

We are now ready to finish the proof of the main assertion. Let $g \in K = \kappa_G^{-1}(B_0)$ and put $b = \kappa_G(g) \in B_0$. Then $a = \hat{g}\hat{b}^{-1} \in A$ and $\Theta_1(u_a) \in \mathcal{Z}(\mathcal{R}_1)$. Denoting $V_g = \Theta_1(u_a)\nu_b$ and using Claim 3.9, we get that $V_g \in \mathcal{U}(\mathcal{Z}(\mathcal{R}_1))$ and

$$\Theta_1(u_g) = V_g(v_{\widehat{\gamma(\kappa_G(g))}} \otimes \bar{\rho}_{\kappa_G(g)}), \text{ for every } g \in K. \quad (3.19)$$

Our final claim is the following.

Claim 3.10. *There are maps $\lambda : K \rightarrow C^{(I)}$, $z : K \rightarrow \mathcal{Z}(\mathcal{T})$ and $w \in \mathcal{U}(\mathcal{Z}(\mathcal{R}_1))$ such that $V_g = (v_{\lambda_g} \otimes z_g)w^*(\tilde{\alpha}_{\kappa_G(g)} \otimes \text{Id})(w)$, for every $g \in K$.*

Proof of Claim 3.10. We define $\Lambda : K \times K \rightarrow C^{(I)}$ by $\Lambda_{g,h} = \widehat{\gamma(\kappa_G(g))\gamma(\kappa_G(h))\gamma(\kappa_G(gh))}^*$ and $Z : K \times K \rightarrow \mathcal{U}(\mathcal{Z}(\mathcal{T}))$ by $Z_{g,h} = \bar{\rho}_{\kappa_G(gh)}\bar{\rho}_{\kappa_G(h)}^*\bar{\rho}_{\kappa_G(g)}^*$. Since $\Theta_1(u_g)\Theta_1(u_h) = \Theta_1(u_{gh})$, (3.19) gives

$$v_{\Lambda_{g,h}} \otimes 1 = (V_{gh}V_g^*(\tilde{\alpha}_{\kappa_G(g)} \otimes \text{Id})(V_h)^*)(1 \otimes Z_{g,h}), \text{ for every } g, h \in K. \quad (3.20)$$

Note that the map $K \times K \ni (g, h) \mapsto v_{\Lambda_{g,h}} \in \mathcal{U}(\mathcal{Q})$ is a 2-cocycle for the action $K \curvearrowright^{\tilde{\alpha} \circ \kappa_G} \mathcal{Q}$ and the map $K \times K \ni (g, h) \mapsto Z_{g,h} \in \mathcal{U}(\mathcal{Z}(\mathcal{T}))$ is a 2-cocycle for the trivial action. Since $\tilde{\alpha}$ is built over $B_0 \curvearrowright J$, $\tilde{\alpha} \circ \kappa_G$ is built over the action $K \curvearrowright J$ given by $k \cdot j = \kappa_G(k) \cdot j$, for every $k \in K$ and $j \in J$. Since the action $K \curvearrowright J$ has infinite orbits, $\tilde{\alpha} \circ \kappa_G$ is weakly mixing. If $n \in \mathbb{N}$, then the action $(\tilde{\alpha} \circ \kappa_G)^{\otimes n}$ is built over the diagonal product action $K \curvearrowright I^n$, which has infinite orbits. Since K has property (T), [CIOS24, Theorem 3.5] implies that $(\tilde{\alpha} \circ \kappa_G)^{\otimes n}$ is $\mathcal{U}_{\text{fin}}$ -cocycle superrigid. Altogether, we deduce that $\tilde{\alpha} \circ \kappa_G$ satisfies the hypothesis of [CIOS24, Theorem 4.1].

Let $\{f_1, \dots, f_l\}$ be the minimal projections of $\mathcal{Z}(\mathcal{T})$. Let $1 \leq i \leq l$. Since $Z_{g,h}f_i \in \mathbb{T}f_i$, for every $g, h \in K$, by using (3.20) and applying [CIOS24, Theorem 4.1] there are maps $\lambda : K \rightarrow C^{(I)}$ and $c^i : K \rightarrow \mathbb{T}$ such that $v_{\Lambda_{g,h}} = v_{\lambda_{gh}}v_{\lambda_g}^*\tilde{\alpha}_{\kappa_G(g)}(v_{\lambda_h})^*$ and $Z_{g,h}f_i = \overline{c_{gh}^i}c_g^ic_h^if_i$, for all $g, h \in K$. Define $c : K \rightarrow \mathcal{U}(\mathcal{Z}(\mathcal{T}))$ by letting $c_g = \sum_{i=1}^l c_g^if_i$, for every $g \in K$. Then $Z_{g,h} = c_{gh}^*c_gc_h$, for all $g, h \in K$. Using (3.20), we get that the map $K \ni g \mapsto (v_{\lambda_g}^* \otimes c_g^*)V_g \in \mathcal{U}(\mathcal{Z}(\mathcal{R}_1))$ is a 1-cocycle for the action $K \curvearrowright^{\tilde{\alpha} \circ \kappa_G \otimes \text{Id}} \mathcal{Z}(\mathcal{R}_1) = \mathcal{Q} \otimes \mathcal{Z}(\mathcal{T})$. Since $\tilde{\alpha} \circ \kappa_G$ is $\mathcal{U}_{\text{fin}}$ -cocycle superrigid, we deduce the existence of $w \in \mathcal{U}(\mathcal{Z}(\mathcal{R}_1))$ such that the claim holds. $\blacksquare$

Finally, (3.19) and Claim 3.10 imply that $\Theta_1(u_g) = w^*(v_{\lambda_g\widehat{\gamma(\kappa_G(g))}} \otimes c_g\bar{\rho}_{\kappa_G(g)})w$, for every $g \in K$. Then $\gamma : K \rightarrow H$ and $\rho : K \rightarrow \mathcal{U}(\mathcal{T}) \subset \mathcal{U}_{t_1}(\mathbb{C})$ given by $\gamma(g) = \lambda_g\widehat{\gamma(\kappa_G(g))}$ and $\rho(g) = c_g\bar{\rho}_{\kappa_G(g)}$ must be homomorphisms. Then $\Theta_1(u_g) = w^*(v_{\gamma(g)} \otimes \rho(g))w$, for every $g \in K$. By construction $A \subset K$ and $\gamma(A) \subset C^{(I)}$. Moreover, if $g \in \ker(\gamma)$, then we have that $\Theta_1(u_g) = w^*(1 \otimes \rho(g))w \in w^*(1 \otimes \mathbb{M}_{t_1}(\mathbb{C}))w$. This implies that $\ker(\gamma)$ must be finite. Since G and thus K are ICC it follows that γ is injective. This finishes the proof of the main assertion. $\square$

Theorem 3.2 leads to a complete description of all virtual *-isomorphisms $\Theta : \mathcal{L}(G) \rightarrow \mathcal{L}(H)^t$. To explain this, we assume the setting of Theorem 3.2 and introduce some terminology from [PV22, Section 2].

Let $K < G$ be a finite index subgroup. If $\gamma : K \rightarrow H$ and $\rho : K \rightarrow \mathcal{U}_s(\mathbb{C})$ are homomorphisms, for some $s \in \mathbb{N}$, we denote by $\pi_{\gamma,\rho} : K \rightarrow \mathcal{U}(\mathcal{L}(H) \otimes \mathbb{M}_s(\mathbb{C}))$ the homomorphism given by $\pi_{\gamma,\rho}(g) = v_{\gamma(g)} \otimes \rho(g)$ for $g \in K$. If $\pi : K \rightarrow \mathcal{U}(\mathcal{S})$ is a homomorphism, where $\mathcal{S}$ is a tracial von Neumann algebra, we denote by $\text{Ind}_K^G(\pi) : G \rightarrow \mathcal{U}(\mathcal{S} \otimes \mathbb{M}_{[G:K]}(\mathbb{C}))$ the induced homomorphism. Specifically, let $\chi : G/K \rightarrow G$ be a map such that $\chi(gK) \in gK$, for every $g \in G$, and define $c : G \times G/K \rightarrow K$ by letting $c(g, hK) = \chi(ghK)^{-1}g\chi(hK) \in K$,

for every $g, h \in G$. Identifying $\mathbb{M}_{[G:K]}(\mathbb{C}) = \mathbb{B}(\ell^2(G/K))$, we define $\text{Ind}_K^G(\pi)(g)(\xi \otimes \mathbf{1}_{hK}) = \pi(c(g, hK))\xi \otimes \mathbf{1}_{ghK}$. By [PV22, Defintion 2.1], a homomorphism $G \rightarrow \mathcal{U}(\mathcal{L}(H)^t)$ is called *standard* if it is unitarily conjugate to a direct sum of homomorphisms of the form $\text{Ind}_K^G(\pi_{\gamma, \rho})$ induced from finite index subgroups $K < G$. Theorem 3.2 implies that the restriction of any $*$ -homomorphism $\Theta : \mathcal{L}(G) \rightarrow \mathcal{L}(H)^t$ to $G = \{u_g\}_{g \in G}$ is standard.

We end this section with a result which is very similar in nature with [CIOS24, Theorem 5.6]. Also our proof largely follows the same argument verbatim, and we include it here only for reader's convenience.

Theorem 3.11. *Let G, H be groups as in Theorem 3.2. Let $\Theta : \mathcal{L}(G) \rightarrow \mathcal{L}(H)^t$ be a virtual $*$ -isomorphism, for some $t > 0$. Then $t \in \mathbb{N}$ and we can find $m \in \mathbb{N}$ and for every $1 \leq i \leq m$, a finite index subgroup $K_i \leq G$, an injective homomorphism $\gamma_i : K_i \rightarrow H$ with finite index image and a unitary representation $\rho_i : K_i \rightarrow \mathcal{U}_{s_i}(\mathbb{C})$, for some $s_i \in \mathbb{N}$, and a unitary $w \in \mathcal{L}(H)^t = \mathcal{L}(H) \bar{\otimes} \mathbb{M}_t(\mathbb{C})$ such that $\sum_{i=1}^m [G : K_i] s_i = t$ and*

$$w\Theta(u_g)w^* = \text{diag}(\text{Ind}_{K_1}^G(\pi_{\gamma_1, \rho_1})(g), \dots, \text{Ind}_{K_m}^G(\pi_{\gamma_m, \rho_m})(g)), \text{ for every } g \in G.$$

Moreover, we have the following formula for the index of the image of Θ

$$[\mathcal{L}(H)^t : \Theta(\mathcal{L}(G))] = t \sum_{i=1}^m s_i [H : \gamma_i(K_i)].$$

Proof. By Theorem 3.2, there are $t_1, \dots, t_{m'} \in \mathbb{N}$ with $t_1 + \dots + t_{m'} = t$, for some $m' \in \mathbb{N}$, a finite index subgroup $K \leq G$, for every $1 \leq i \leq m'$, an injective homomorphism $\gamma'_i : K \rightarrow H$ with a finite index image, and a unitary representation $\rho'_i : K \rightarrow \mathcal{U}_{t_i}(\mathbb{C})$, and a unitary $w \in \mathcal{L}(H)^t = \mathcal{L}(H) \bar{\otimes} \mathbb{M}_t(\mathbb{C})$ such that

$$w\Theta(u_g)w^* = \text{diag}(v_{\gamma'_1(g)} \otimes \rho'_1(g), \dots, v_{\gamma'_{m'}(g)} \otimes \rho'_{m'}(g)), \text{ for every } g \in K.$$

Passing to a finite index subgroup we can assume without any loss of generality that $K \triangleleft G$ is a normal subgroup. After decomposing each ρ'_i into a direct sum of irreducible representations, we may assume ρ'_i is irreducible, for every $1 \leq i \leq m'$. Further, we can find $m_1, \dots, m_l, n_1, \dots, n_l \in \mathbb{N}$, for some $l \in \mathbb{N}$, with $m_1 n_1 + \dots + m_l n_l = t$, for every $1 \leq i \leq l$, an injective homomorphism $\gamma_i^0 : K \rightarrow H$ with a finite index image, an irreducible representation $\rho_i^0 : K \rightarrow \mathcal{U}_{m_i}(\mathbb{C})$, and a unitary $w_0 \in \mathcal{L}(H)^t$ such that for every $1 \leq i < j \leq l$, γ_i^0 is not conjugate to γ_j^0 or ρ_i^0 is not unitarily conjugate to ρ_j^0 , and after replacing Θ by $\text{ad}(w_0) \circ \Theta$ we have

$$\Theta(u_g) = \text{diag}(v_{\gamma_1^0(g)} \otimes \rho_1^0(g) \otimes I_{n_1}, \dots, v_{\gamma_l^0(g)} \otimes \rho_l^0(g) \otimes I_{n_l}), \text{ for every } g \in K. \quad (3.21)$$

Here, two homomorphisms $\gamma, \gamma' : K \rightarrow H$ are conjugate if $\gamma = \text{ad}(h) \circ \gamma'$ for some $h \in H$. We denote by $I_d \in \mathbb{M}_d(\mathbb{C})$ the identity matrix, for every $d \in \mathbb{N}$, and consider the natural unital embedding $\bigoplus_{i=1}^l (\mathcal{L}(H) \otimes \mathbb{M}_{m_i}(\mathbb{C}) \otimes \mathbb{M}_{n_i}(\mathbb{C})) \subset \mathcal{L}(H) \otimes \mathbb{M}_t(\mathbb{C})$.

We continue with the following claim:

Claim 3.12. *Let $\gamma, \gamma' : K \rightarrow H$ be injective homomorphisms with finite index images and $\rho : K \rightarrow \mathcal{U}_a(\mathbb{C})$, $\rho' : K \rightarrow \mathcal{U}_{a'}(\mathbb{C})$ be irreducible representations, for some $a, a' \in \mathbb{N}$. Assume there is $0 \neq x \in \mathcal{L}(H) \otimes \mathbb{M}_{a, a'}(\mathbb{C})$ such that $(v_{\gamma(g)} \otimes \rho(g))x = x(v_{\gamma'(g)} \otimes \rho'(g))$, for every $g \in K$. Then $a = a'$ and there exist $h \in H$ and $U \in \mathcal{U}_a(\mathbb{C})$ such that $\gamma(g)h = h\gamma'(g)$ and $\rho(g)U = U\rho'(g)$, for every $g \in K$. Moreover, $x = c(v_h \otimes U)$ for some $c \in \mathbb{C}$.*

Proof of Claim 3.12 Write $x = \sum_{h \in H} v_h \otimes x_h$, where $x_h \in \mathbb{M}_{a, a'}(\mathbb{C})$ and $\sum_{h \in H} \|x_h\|_2^2 = \|x\|_2^2 < \infty$. Then

$$x_{\gamma(g)h\gamma'(g)^{-1}} = \rho(g)x_h\rho'(g)^*, \text{ for every } g \in K, h \in H. \quad (3.22)$$

Let $\varepsilon > 0$ such that $F = \{h \in H \mid \|x_h\|_2 > \varepsilon\}$ is nonempty. Then (3.22) implies that F is a finite set such that $\gamma(g)F\gamma'(g)^{-1} = F$, for every $g \in K$. Thus, $\gamma(g)FF^{-1}\gamma(g)^{-1} = FF^{-1}$, for every $g \in K$. Assume by contradiction $FF^{-1} \neq \{1\}$. Thus, as F is finite there exist $1 \neq h \in FF^{-1}$ and a finite index subgroup $G_0 \leq G$ such that $\gamma(G_0) \leq C_H(h)$. As $\gamma(G) \leq H$ has finite index it follows that $C_H(h) \leq H$ has finite index. However, this contradicts that H is ICC. Thus $FF^{-1} = \{1\}$ and hence F consists of a single element. Since this holds for every small enough $\varepsilon > 0$, we conclude that $\{h \in H \mid x_h \neq 0\}$ has a single element. Thus, $x = v_h \otimes x_h$, for some $h \in H$. Hence, we have that $v_{\gamma(g)h} \otimes \rho(g)x_h = v_{h\gamma'(g)} \otimes x_h\rho'(g)$, which implies that

$\gamma(g)h = h\gamma'(g)$ and $\rho(g)x_h = x_h\rho'(g)$, for every $g \in K$. Since ρ, ρ' are irreducible, the latter relation implies that $a = a'$ and $x_h = cU$, for some nonzero $c \in \mathbb{C}$ and $U \in \mathcal{U}_a(\mathbb{C})$. This proves our assertion. $\blacksquare$

Let $[l] = \{1, \dots, l\}$. For every $i \in [l]$, let $f_i = 1 \otimes I_{m_i} \otimes I_{n_i}$. Combining (3.21) with Claim 3.12 we get that $\Theta(\mathcal{L}(K))' \cap \mathcal{L}(H)^t = \bigoplus_{i=1}^l (1 \otimes I_{m_i} \otimes \mathbb{M}_{n_i}(\mathbb{C}))$, and thus $\mathcal{Z}(\Theta(\mathcal{L}(K))' \cap \mathcal{L}(H)^t) = \bigoplus_{i=1}^l \mathbb{C}f_i$. Since $K < G$ is normal, $\Theta(u_g)$ normalizes $\Theta(\mathcal{L}(K))$ for all $g \in G$. Hence, there exists an action $G \curvearrowright [l]$ such that $\Theta(u_g)f_i\Theta(u_g)^* = f_{g \cdot i}$, for every $g \in G$ and $i \in [l]$. Let $J \subset [l]$ be a set which intersects every G -orbit exactly once.

Next, fix $i \in J$ and denote $K_i = \{g \in G \mid g \cdot i = i\}$. Then (3.21) implies that $K < K_i$. Let $h \in K_i$. If $g \in K$, then since $K < G$ is normal, $hgh^{-1} \in K$, and (3.21) gives that

$$\Theta(u_{hgh^{-1}}) = \text{diag}(v_{\gamma_1^0(hgh^{-1})} \otimes \rho_1^0(hgh^{-1}) \otimes I_{n_1}, \dots, v_{\gamma_l^0(hgh^{-1})} \otimes \rho_l^0(hgh^{-1}) \otimes I_{n_l}). \quad (3.23)$$

Since $\Theta(u_{hgh^{-1}})\Theta(u_h) = \Theta(u_h)\Theta(u_g)$, we get that $\Theta(u_{hgh^{-1}})(\Theta(u_h)f_i) = (\Theta(u_h)f_i)\Theta(u_g)$. By combining (3.21) and (3.23) we conclude that for every $g \in K$ we have

$$(v_{\gamma_i^0(hgh^{-1})} \otimes \rho_i^0(hgh^{-1}) \otimes I_{n_i})(\Theta(u_h)f_i) = (\Theta(u_h)f_i)(v_{\gamma_i^0(g)} \otimes \rho_i^0(g) \otimes I_{n_i}). \quad (3.24)$$

By applying the moreover part of Claim 3.12, we get that $\Theta(u_h)f_i = v_{\gamma_i(h)} \otimes \rho_i(h)$, for some $\gamma_i(h) \in H$ and $\rho_i(h) \in \mathcal{U}_{m_i n_i}(\mathbb{C})$. Then $\gamma_i : K_i \rightarrow H$ and $\rho_i : K_i \rightarrow \mathcal{U}_{m_i n_i}(\mathbb{C})$ must be homomorphisms such that $\gamma_i(g) = \gamma_i^0(g)$ and $\rho_i(g) = \rho_i^0(g) \otimes I_{n_i}$, for every $g \in K$, and $s_i = m_i n_i$ satisfies $\sum_{i \in J} [G : K_i] s_i = t$. Thus, in the notation introduced before this proof, we have that $\Theta(u_h)f_i = \pi_{\gamma_i, \rho_i}(h)$, for every $h \in K_i$.

Let $e_i = \sum_{j \in G \cdot i} f_j \in \mathcal{Z}(\Theta(\mathcal{L}(K))' \cap \mathcal{L}(H)^t)$. Then $e_i \in \Theta(\mathcal{L}(G))' \cap \mathcal{L}(H)^t$, hence $e_i \in \mathcal{Z}(\Theta(\mathcal{L}(G))' \cap \mathcal{L}(H)^t)$. Since $e_i = \sum_{g \in G/K_i} \Theta(u_g)f_i\Theta(u_g)^*$ and the projections $\{\Theta(u_g)f_i\Theta(u_g)^* \mid g \in G/K_i\}$ are pairwise orthogonal, the homomorphism $G \ni g \mapsto \Theta(u_g)e_i \in \mathcal{U}(e_i\mathcal{L}(H)^te_i)$ is unitarily conjugate to the induced homomorphism $\text{Ind}_{K_i}^G(\pi_{\gamma_i, \rho_i})$.

Since $\Theta(u_g) = \sum_{i \in J} \Theta(u_g)e_i$, for every $g \in G$, the conclusion follows.

Now we prove the moreover part. Let $\{h_l\}_l$ be the coset representatives of $\gamma_i(K_i) \leq H$. Then $\{\sqrt{s_i}v_{h_l} \otimes e_{j_1, j_2}\}_{j_1, j_2, l}$ is a Pimsner-Popa basis for $f_i\Theta(\mathcal{L}(K_i))f_i \subset f_i\mathcal{L}(H)^tf_i$. Thus,

$$[f_i\mathcal{L}(H)^tf_i : f_i\Theta(\mathcal{L}(K_i))f_i] = s_i^2[H : \gamma_i(K_i)].$$

Using the above and the local index formula, i.e. item 7. in Proposition 2.4, we have

$$[e_i\mathcal{L}(H)^te_i : e_i\Theta(\mathcal{L}(K_i))e_i] = [G : K_i] \frac{s_i^2[H : \gamma_i(K_i)]}{1/[G : K_i]} = s_i^2[G : K_i]^2[H : \gamma_i(K_i)].$$

Since $[\mathcal{L}(G) : \mathcal{L}(K_i)] = [G : K_i]$ we get

$$[e_i\mathcal{L}(H)^te_i : e_i\Theta(\mathcal{L}(G))e_i] = \frac{[e_i\mathcal{L}(H)^te_i : e_i\Theta(\mathcal{L}(K_i))e_i]}{[G : K_i]} = s_i^2[G : K_i][H : \gamma_i(K_i)].$$

Again, by Proposition 2.4 7. and the above, we have

$$[\mathcal{L}(H)^t : \Theta(\mathcal{L}(G))] = \sum_{i=1}^m \frac{[e_i\mathcal{L}(H)^te_i : e_i\Theta(\mathcal{L}(G))e_i]}{s_i[G : K_i]/t} = \sum_{i=1}^m \frac{s_i^2[G : K_i][H : \gamma_i(K_i)]}{s_i[G : K_i]/t} = t \sum_{i=1}^m s_i[H : \gamma_i(K_i)]. \quad \square$$

4 Jones index set for II_1 factors associated with property (T) wreath-like product groups

Let $\mathcal{M}$ be a II_1 factor. The *Jones index set* $\mathcal{J}(\mathcal{M})$ the collection of the Jones indices of all finite index subfactors of $\mathcal{M}$. In this section, we show that there are a continuum family of property (T) II_1 factors, whose Jones index set contains all the positive integers. To achieve this, we will use in an essential way the following result [CIOS23b, Corollary 2.12].

Theorem 4.1 ([CIOS23b]). *Let Γ be any countable group. Then one can find a continuum of ICC property (T) groups $(G_j)_{j \in J}$ such that the corresponding II_1 factors $(\mathcal{L}(G_j))_{j \in J}$ are pairwise non-stably isomorphic and satisfy $\text{Out}(\mathcal{L}(G_j)) \cong \Gamma$ and $\mathcal{F}(\mathcal{L}(G_j)) = \{1\}$ for every $j \in J$.*

We give a brief summary of the proof. First, they found a continuum family of ICC property (T) countable groups $(G_j)_{j \in J}$ with $\text{Out}(G_j) \cong \Gamma$ for every $j \in J$. The groups $(G_j)_{j \in J}$ satisfy the conditions in Theorem 3.2. i.e. G_j are in $\mathcal{WR}(A_j, B \curvearrowright I)$ where A_j are nontrivial abelian groups, B is a nonparabolic ICC subgroup of a finitely generated group that are hyperbolic relative to a finite family of residually finite groups, and $B \curvearrowright I$ is a faithful action. Then they introduced a homomorphism $\Psi^{(j)} : \text{Aut}(G_j) \rightarrow \text{Aut}(\mathcal{L}(G_j))$ defined by

$$\Psi^{(j)}(\varphi)(u_g) = u_{\varphi(g)} \quad (4.1)$$

for each $j \in J$. Next, they showed that for the above family of groups, $\Psi^{(j)}$ induces an isomorphism $\overline{\Psi}^{(j)} : \text{Out}(G_j) \rightarrow \text{Out}(\mathcal{L}(G_j))$ defined by

$$\overline{\Psi}^{(j)}(\varphi \text{ Inn}(G_j)) = \Psi^{(j)}(\varphi) \text{ Inn}(\mathcal{L}(G_j)).$$

The isomorphisms $\overline{\Psi}^{(j)}$ are key ingredients for the construction of property (T) factors whose Jones index sets are exactly the positive integers.

Theorem 4.2. *There is a continuum of ICC property (T) groups $(G_j)_{j \in J}$, such that their II_1 factors $\mathcal{L}(G_j)$ are pairwise not stably isomorphic, and $\mathcal{I}(\mathcal{L}(G_j)) = \mathbb{N}$. Moreover, all integer indices are realized as those of the irreducible subfactors of $\mathcal{L}(G_j)$.*

Proof. Let Γ be a countable group such that for all positive integer n there is a subgroup $\Gamma_0 \leq \Gamma$ with $|\Gamma_0| = n$. For instance, one can choose $\Gamma = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. By Theorem 4.1, we can take a continuum family of ICC property (T) groups $(G_j)_{j \in J}$ such that $(G_j)_{j \in J}$ satisfy the conditions in Theorem 3.2, the II_1 factors $(\mathcal{L}(G_j))_{j \in J}$ are pairwise not stably isomorphic and $\text{Out}(\mathcal{L}(G_j)) \cong \Gamma$ for every $j \in J$.

First, we show that $\mathcal{I}(\mathcal{L}(G_j)) \subset \mathbb{N}$. Set $\mathcal{M} = \mathcal{L}(G_j)$ and let $\mathcal{N} \subset \mathcal{M}$ be a finite index subfactor. Consider the iterated basic construction $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2$. Then $\mathcal{M}_2 \cong \mathcal{M}^{[\mathcal{M}:\mathcal{N}]}$ [PP86, Proposition 1.5] and this isomorphism induces an embedding $\mathcal{M} \hookrightarrow \mathcal{M}^{[\mathcal{M}:\mathcal{N}]}$. By Theorem 3.2, we have $[\mathcal{M}:\mathcal{N}] \in \mathbb{N}$.

Next, we show that $\mathbb{N} \subset \mathcal{I}(\mathcal{L}(G_j))$ for all $j \in J$. For each $n \in \mathbb{N}$, choose a subgroup $\Gamma_0 \leq \Gamma$ with $|\Gamma_0| = n$. Recall that the homomorphism $\Psi^{(j)}$ of (4.1) induces an isomorphism $\overline{\Psi}^{(j)} : \text{Out}(G_j) \rightarrow \text{Out}(\mathcal{L}(G_j))$. Therefore, applying Proposition 2.9 gives an irreducible subfactor $\mathcal{N} \subset \mathcal{L}(G_j)$ such that $[\mathcal{L}(G_j) : \mathcal{N}] = n$. Hence, $n \in \mathcal{I}(\mathcal{L}(G_j))$. $\square$

Appendices

A The basic construction of ICC group subfactors

In this section, we characterize the basic construction for the inclusion of von Neumann algebras of finite index ICC subgroups. This is well-known to experts, but we could not find a reference. Similar constructions can be found in [OK90, KY92, JS97].

A.1 Amplificational description of the basic construction

The following proposition can be shown simply by iterating [PP86, Proposition 1.5].

Proposition A.1 ([PP86, Proposition 1.5]). *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of II_1 factors with $t = [\mathcal{M}:\mathcal{N}] < \infty$, $\{x_i\}_{i \in I} \subset \mathcal{M}$ be a Pimsner-Popa basis where $|I| \leq [t] + 1$. Consider the Jones tower*

$$\mathcal{N} \subset \mathcal{M} \overset{e_1}{\subset} \mathcal{M}_1 \overset{e_2}{\subset} \mathcal{M}_2 \overset{e_3}{\subset} \dots$$

obtained by iterating the basic construction. Then

1. The set $\{x_i^{(k)}\}_{i \in I}$ of elements defined by $x_i^{(k)} = t^{k/2} e_k e_{k-1} \cdots e_1 x_i$ forms a Pimsner-Popa basis for the inclusion $\mathcal{M}_{k-1} \subset \mathcal{M}_k$.

For each pair $\mathbf{i}, \mathbf{j} \in I^k$ let $\Theta_{\mathbf{ij}}^{(k)} : \mathcal{M}_{2k} \rightarrow \mathcal{M}$ be defined by

$$\Theta_{\mathbf{ij}}^{(k)}(y) = \mathbb{E}_{\mathcal{M}} \left(x_{i_1}^{(2)} \mathbb{E}_{\mathcal{M}_2} \left(x_{i_2}^{(4)} \cdots \mathbb{E}_{\mathcal{M}_{2k-2}} \left(x_{i_k}^{(2k)} y \left(x_{j_k}^{(2k)} \right)^* \right) \cdots \left(x_{j_2}^{(4)} \right)^* \right) \left(x_{j_1}^{(2)} \right)^* \right)$$

and $\Theta^{(k)} : \mathcal{M}_{2k} \rightarrow \mathcal{M}^{t^k}$ by $(\Theta^{(k)}(y))_{\mathbf{ij}} = \Theta_{\mathbf{ij}}^{(k)}(y)$.

2. The map $\Theta^{(k)}$ is a $*$ -isomorphism with $\Theta^{(k)}(\mathcal{M}_{2k-1}) = \mathcal{N}^{t^k}$.
3. For every $y \in \mathcal{M}$ we have

$$\Theta_{\mathbf{ij}}^{(k)}(y) = \mathbb{E}_{\mathcal{N}} \left(x_{i_1} \mathbb{E}_{\mathcal{N}} \left(x_{i_2} \cdots \mathbb{E}_{\mathcal{N}} \left(x_{i_k} y \left(x_{j_k} \right)^* \right) \cdots \left(x_{j_2} \right)^* \right) \left(x_{j_1} \right)^* \right).$$

A.2 Inclusions of von Neumann algebras of ICC groups

Let $H \leq G$ be an inclusion of ICC groups with $t = [G : H] < \infty$. In this subsection, we describe the structure of the inclusion $\mathcal{L}(H) \subset \mathcal{L}(G)$ of their von Neumann algebras. In Section A.2.1, we describe the standard invariant of $\mathcal{L}(H) \subset \mathcal{L}(G)$. In Section A.2.2 we describe $\mathcal{L}(H)$ and $\mathcal{L}(G)$ as cocycle crossed product algebras.

A.2.1 The standard invariant

We describe the standard invariant of $\mathcal{L}(H) \subset \mathcal{L}(G)$ in terms of the representations of subgroups of G . We mimicked the description given in [JS97, Appendix A.4] for the subfactors induced by the finite group actions.

Let $I = \{1, 2, \dots, t\}$ and $\{g_i\}_{i \in I}$ be a full set of the right coset representatives of H , i.e. $G = \coprod_{i=1}^t H g_i$. For simplicity, we choose $g_1 = 1$. Then $\{u_{g_i}\}_{i=1}^t$ is a Pimsner-Popa basis for $\mathcal{L}(H) \subset \mathcal{L}(G)$. Let $k \in \mathbb{N}$ and $\Theta_{\mathbf{ij}}^{(k)} : \mathcal{L}(G) \rightarrow \mathcal{L}(H)$ be as in Proposition A.1. For $\mathbf{i} \in I^k$ let $g_{\mathbf{i}} = g_{i_1} g_{i_2} \cdots g_{i_k}$. Then for all $g \in G$

$$\Theta_{\mathbf{ij}}^{(k)}(u_g) = \begin{cases} u_{g_{\mathbf{i}} g g_{\mathbf{j}}^{-1}}, & g_{i_l} g_{i_{l+1}} \cdots g_{i_k} g \in H g_{j_l} g_{j_{l+1}} \cdots g_{j_k} \text{ for all } 1 \leq l \leq k, \\ 0, & \text{otherwise} \end{cases}. \quad (\text{A.1})$$

Note that for each $\mathbf{i}$ there is a unique $\mathbf{j}$ such that $\Theta_{\mathbf{ij}}^{(k)}(u_g) \neq 0$ and vice versa. Motivated by the above, we define the action $G \curvearrowright I^k$ by

$$g \cdot \mathbf{j} = \mathbf{i} \Leftrightarrow g_{j_l} g_{j_{l+1}} \cdots g_{j_k} g^{-1} \in H g_{i_l} g_{i_{l+1}} \cdots g_{i_k} \text{ for all } 1 \leq l \leq k. \quad (\text{A.2})$$

We compute the higher relative commutants of $\mathcal{L}(H) \subset \mathcal{L}(G)$. Let $g_0 \in G$ and $x = (x_{\mathbf{ij}})_{\mathbf{i}, \mathbf{j}} \in \mathcal{L}(G)^{t^k}$. Then

$$\begin{aligned} \left[x, \Theta^{(k)}(u_g) \right] = 0 &\Leftrightarrow x_{\mathbf{i}, (g_0 \cdot \mathbf{j})} u_{g_{(g_0 \cdot \mathbf{j})} g_0 g_{\mathbf{j}}^{-1}} = u_{g_{\mathbf{i}} g_0 g_{(g_0^{-1} \cdot \mathbf{i})}^{-1}} x_{(g_0^{-1} \cdot \mathbf{i}), \mathbf{j}} && \text{for all } \mathbf{i}, \mathbf{j} \in I^k \\ &\Leftrightarrow x_{(g_0 \cdot \mathbf{i}), (g_0 \cdot \mathbf{j})} u_{g_{(g_0 \cdot \mathbf{j})} g_0 g_{\mathbf{j}}^{-1}} = u_{g_{(g_0 \cdot \mathbf{i})} g_0 g_{\mathbf{i}}^{-1}} x_{\mathbf{i}, \mathbf{j}} && \text{for all } \mathbf{i}, \mathbf{j} \in I^k \end{aligned} \quad (\text{A.3})$$

Let $K = \bigcap_{i \in I} g_i^{-1} H g_i$. Then $K \triangleleft G$ and $[G : K] < \infty$. If $g_0 \in K$ then $g_0 \cdot \mathbf{i} = \mathbf{i}$ for all $\mathbf{i} \in I^k$ and the above condition becomes

$$x_{\mathbf{i}, \mathbf{j}} u_{g_{\mathbf{j}} g_0 g_{\mathbf{j}}^{-1}} = u_{g_{\mathbf{i}} g_0 g_{\mathbf{i}}^{-1}} x_{\mathbf{i}, \mathbf{j}} \quad \text{for all } \mathbf{i}, \mathbf{j} \in I^k. \quad (\text{A.4})$$

Suppose that $x \in \Theta(\mathcal{L}(K))' \cap \mathcal{L}(G)^{t^k}$. Let $x_{\mathbf{i}, \mathbf{j}} = \sum_{g \in G} c_g u_g$, $\varepsilon > 0$, and $F = \{g \in G : |c_g| > 0\}$. Since $\sum_{g \in G} |c_g|^2 = \|x_{\mathbf{i}, \mathbf{j}}\|^2 < \infty$, F is finite. Moreover, from (A.4), we have that $F = (g_{\mathbf{i}} g_0 g_{\mathbf{i}}^{-1}) F (g_{\mathbf{j}} g_0 g_{\mathbf{j}}^{-1})^{-1}$ for all $g_0 \in K$. Thus, $F F^{-1} = (g_{\mathbf{i}} g_0 g_{\mathbf{i}}^{-1}) F F^{-1} (g_{\mathbf{j}} g_0 g_{\mathbf{j}}^{-1})^{-1}$ for all $g_0 \in K$. Since G is ICC and $[G : K] < \infty$, we

have $FF^{-1} = \{1\}$. Hence, F is a singleton. Since this is true for all $\varepsilon > 0$, we have $x_{\mathbf{i}, \mathbf{j}} = c_{\mathbf{i}, \mathbf{j}} u_g$ for some $g \in G$ and some $c_{\mathbf{i}, \mathbf{j}} \in \mathbb{C}$. Suppose $c_{\mathbf{i}, \mathbf{j}} \neq 0$. Plugging this back to (A.4) yields $[g_{\mathbf{i}}^{-1} g g_{\mathbf{j}}, g_0] = 0$ for all $g_0 \in K$. Again, since G is ICC and $[G : K] < \infty$, $g_{\mathbf{i}}^{-1} g g_{\mathbf{j}} = 1$. Therefore,

$$x_{\mathbf{i}, \mathbf{j}} = c_{\mathbf{i}, \mathbf{j}} u_{g_{\mathbf{i}} g_{\mathbf{j}}^{-1}}. \quad (\text{A.5})$$

Let $H \leq G_0 \leq G$. We will describe the structures of the inclusions $\Theta^{(k)}(\mathcal{L}(G_0)) \subset \mathcal{L}(G)^{t^k}$ and $\Theta^{(k)}(\mathcal{L}(G_0)) \subset \mathcal{L}(H)^{t^k}$ by analyzing their relative commutants. Our main focus will be on the cases $G_0 = G, H$.

First, assume that $x \in \Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(G)^{t^k}$. Then by (A.3), we further have that

$$c_{(g_0 \cdot \mathbf{i}), (g_0 \cdot \mathbf{j})} = c_{\mathbf{i}, \mathbf{j}} \quad \text{for all } g_0 \in G_0 \text{ and } \mathbf{i}, \mathbf{j} \in I^k. \quad (\text{A.6})$$

Thus, the minimal central projections of $\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(G)^{t^k}$ correspond to the irreducible representations of G_0 with nonzero multiplicity in the permutation representation of G on $\mathbb{C}I^k$. The rank of the minimal central projections of $\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(G)^{t^k}$ is the multiplicity of the corresponding irreducible representation in $\mathbb{C}I^k$.

Next, we consider $\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(H)^{t^k}$. Notice that $\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(H)^{t^k} \subset \Theta^{(k)}(\mathcal{L}(K))' \cap \mathcal{L}(H)^{t^k}$. Suppose that $x \in \Theta^{(k)}(\mathcal{L}(K))' \cap \mathcal{L}(H)^{t^k}$. Then by (A.5), for all $\mathbf{i}, \mathbf{j} \in I^k$ there exists $c_{\mathbf{i}, \mathbf{j}} \in \mathbb{C}$ such that

$$x_{\mathbf{i}, \mathbf{j}} = \begin{cases} c_{\mathbf{i}, \mathbf{j}} u_{g_{\mathbf{i}} g_{\mathbf{j}}^{-1}}, & g_{\mathbf{i}} g_{\mathbf{j}}^{-1} \in H, \\ 0, & \text{otherwise} \end{cases}. \quad (\text{A.7})$$

Therefore,

$$\Theta^{(k)}(\mathcal{L}(K))' \cap \mathcal{L}(H)^{t^k} = \bigoplus_{m \in I} \{x : x_{\mathbf{i}, \mathbf{j}} = 0 \text{ if } g_{\mathbf{i}} \notin H g_m \text{ or } g_{\mathbf{j}} \notin H g_m\} \cong \bigoplus_{m \in I} \mathbb{M}_{I^{k-1}}(\mathbb{C}).$$

The algebra $\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(H)^{t^k}$ consists of elements satisfying (A.7) and (A.6). Let $J \subset I$ be a subset that contains exactly one element from each orbit of the action $G_0 \curvearrowright I$. For each $i \in J$, let $K_i \leq G_0$ be the stabilizer subgroup of i . Then

$$\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(H)^{t^k} \cong \bigoplus_{i \in J} (K_i' \cap M_{I^{k-1}}(\mathbb{C})).$$

Thus, $\Theta^{(k)}(\mathcal{L}(G_0))' \cap \mathcal{L}(H)^{t^k}$ is isomorphic to the direct sum of endomorphism spaces of copies of $\mathbb{C}I^{k-1}$, each of which is viewed as a representation of K_i . For $G_0 = G$, we may choose $J = \{1\}$ and $K_1 = H$. For $G_0 = H$, the orbits of $G_0 \curvearrowright I$ correspond to the double cosets of H , and J can be chosen that $G = \bigsqcup_{i \in J} H g_i H$. Moreover, for each $i \in J$ $K_i = H \cap g_i^{-1} H g_i$.

Proposition A.2. *Let $H \leq G$ be an inclusion of ICC groups with $[G : H] < \infty$. Denote the set of isomorphism classes of the finite-dimensional irreducible representations of a group G_0 as $\widehat{G}_0$. Then the principal graphs (Δ, Δ') of $\mathcal{L}(H) \subset \mathcal{L}(G)$ have the following description:*

1. Let $\{g_i\}_{i=1}^q$ be a full set of the double coset representatives of H and $K_i = H \cap g_i^{-1} H g_i$. Let $\tilde{\Delta}_0 = \bigsqcup_{i=1}^q \tilde{K}_i$ and $\tilde{\Delta}_1 = \widehat{H}$. Connect $\rho_0 \in \tilde{\Delta}_0$ and $\rho_1 \in \tilde{\Delta}_1$ with m edges if the multiplicity of ρ_0 in ρ_1 is m . Then Δ is the connected component of the resulting graph $\tilde{\Delta}$ that contains $1 \in \tilde{\Delta}_1$ with the designated vertex $1 \in \widehat{H} \subset \tilde{\Delta}_0$.
2. Let $\tilde{\Delta}'_0 = \widehat{G}$ and $\tilde{\Delta}'_1 = \widehat{H}$. Connect $\rho_0 \in \tilde{\Delta}'_0$ and $\rho_1 \in \tilde{\Delta}'_1$ with m edges if the multiplicity of ρ_1 in ρ_0 is m . Then $\tilde{\Delta}$ is the connected component of $\tilde{\Delta}'$ that has $1 \in \widehat{G} = \tilde{\Delta}'_0$ as the designated vertex.

A.2.2 Cocycle action description

Recall that $K = \bigcap_{i=1}^t g_i^{-1} H g_i \triangleleft G$ where $\{g_i\}_{i=1}^t$ is a full set of the right coset representatives of H . Let $\{\tilde{g}_j\}_j$ be a full set of right coset representatives of $K \subset G$ that contains a full set of right coset representatives of $K \subset H$. Define a map $\alpha : G/K \rightarrow \text{Aut}(\mathcal{L}(K))$ and a 2-cocycle $\omega : G/K \times G/K \rightarrow \mathcal{L}(K)$

$$\alpha_{K\tilde{g}_j} = \text{ad}(u_{\tilde{g}_j}), \quad \omega_{K\tilde{g}_i, K\tilde{g}_j} = g \text{ where } \tilde{g}_i \tilde{g}_j = g \tilde{g}_k \text{ for some } \tilde{g}_k \text{ and } g \in K.$$

Then $\mathcal{L}(K) \rtimes_{\alpha, \omega} G/K \rightarrow \mathcal{L}(G)$ defined by

$$u_g v_{K\tilde{g}_j} \mapsto u_{g\tilde{g}_j}$$

is a *-isomorphism and the restriction of this map to $\mathcal{L}(K) \rtimes_{\alpha, \omega} H/K$ has the image $\mathcal{L}(H)$.

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