

Feasibility and Single Parameter Scaling of Extinctions in Multispecies Lotka-Volterra Ecosystems

Philippe Jacquod

Department of Quantum Matter Physics, University of Geneva, Geneva, Switzerland

Andlinger Center for Energy and the Environment, Princeton University, Princeton, USA

School of Engineering, University of Applied Sciences of Western Switzerland HES-SO, Sion, Switzerland

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Multispecies ecosystems modelled by generalized Lotka-Volterra equations exhibit stationary population abundances, where large number of species often coexist. Understanding the precise conditions under which this is at all feasible and what triggers species extinctions is a key, outstanding problem in theoretical ecology. Using standard methods of random matrix theory, I show that distributions of species abundances are Gaussian at equilibrium, in the weakly interacting regime. One consequence is that feasibility is generically broken before stability, for large enough number of species. I further derive an analytic expression for the probability that $n = 0, 1, 2, \dots$ species go extinct and conjecture that a single-parameter scaling law governs species extinctions. These results are corroborated by numerical simulations in a wide range of system parameters.

Introduction. Many physical systems are so complex, that not only their state, but even their nature itself cannot be determined precisely. To deal with such systems, Wigner, Dyson and others constructed an extension of statistical mechanics, where either the Hamiltonian matrix, the time-evolution operator or, in the case of an open system, the scattering matrix belongs to an ensemble from which statistical properties can be evaluated [1, 2]. The resulting Random Matrix Theory (RMT) [3] has been very successful in describing the excited spectrum of nuclei [4, 5], of small metallic nanograins [6], of quantum chaotic systems [7] and of strongly correlated, many-body systems [8–10], in calculating quantum corrections to electronic transport in mesoscopic systems [11], and in characterizing the structure of excited states in complex atoms [12] and heavy nuclei [4, 5] among others.

A similar statistical approach has been advocated in theoretical ecology. Multispecies ecosystems are standardly modeled as dynamical systems, whose degrees of freedom represent abundances for each of S species [13]. Time-evolution is governed by coupled differential equations with at least $\propto S^2$ parameters – e.g. the carrying capacity for each species, as well as pairwise interspecies interactions [13] – and even more if higher-order interactions are taken into account [14, 15]. Determining all of them with appropriate accuracy is a hopeless task in large, biodiverse ecosystems. But even if all parameters were known, computing analytically the evolution of species abundances, determining the existence, nature and stability of attractors and anticipating species extinctions can only be delegated to numerical studies already for moderate S [13, 16–18]. RMT has proven to be an invaluable alternative in the study of multispecies ecosystems, by assuming that interspecies interactions have a random distribution defined by its average and variance. Under this assumption, May conjectured that equilibrium states of large ecosystems are generically linearly unstable – almost any small perturbation would push the dynamics away from them – unless the variance of

interspecies interactions goes down $\propto S^{-1}$ or faster [19].

May’s work initiated a new, RMT approach to theoretical ecology [19–25, 27–36]. Focusing on the generalized Lotka-Volterra model [See Eq. (1) below], RMT established in particular that, when interspecies interactions are weakly fluctuating, a single, globally stable equilibrium exists, while when fluctuations are sufficiently strong, the system enters a phase with multiple, coexisting equilibria [20, 27, 31]. How the transition happens between the two regimes depends on how species interact [21, 25, 31, 32]. In particular, mutualistic or competitive interactions have been found to be generally destabilizing, while predator-prey interactions tend to stabilize ecosystems [21, 32, 33]. These, and other RMT results in theoretical ecology have been reviewed in Refs. [35, 36]. May’s work re-opened the debate on the common wisdom that complexity – measured e.g. by biodiversity – begets stability, an important, still open question [37–41].

Stability is however only one side of the problem, and soon after May’s work, it was pointed out that, when investigating species coexistence in mathematical models, stability and feasibility – the condition that all species abundances are positive – must be considered on an equal footing [42]. Coexistence from the combined point of view of stability and feasibility has been investigated within RMT in Refs. [24, 30, 43–46]. A general, qualitative consensus seems to emerge, that feasibility is more restrictive than stability, though a general theory of feasibility is still lacking.

In this paper, I use RMT to calculate average, variance, skewness and kurtosis of the distribution $P(N_i^*)$ of species abundances for equilibrium fixed-points of generalized Lotka-Volterra equations, in the regime of weak interaction and for $S \gg 1$. I find that abundances follow a Gaussian distribution, with average $\bar{N}$ and variance Σ_N determined by the average, the variance and the cross-diagonal correlator of interspecies interactions [See Eqs. (2) below]. From this distribution, I further give an analytic expression for the probability that stationary solutions have $n = 0, 1, 2, \dots$ species with negative

abundances, as a function of the number S of species. When $n \geq 1$, the fixed-point is unfeasible, and the key question is whether this distribution has any connection to the extinction dynamics governed by the generalized Lotka-Volterra equations. I find numerically that, quite surprisingly, the dynamically obtained distribution of extinctions closely matches the probability distribution of species with negative abundances for fixed-points. I finally show that the ratio of average and standard deviation of the Gaussian distribution of abundances gives a single parameter which governs a scaling law for the number of extinctions in generalized Lotka-Volterra equations. The scaling law is confirmed numerically in a surprisingly large set of parameters, up to when half of the initially present species go extinct.

Model and Methodology. I consider large Lotka-Volterra models defined by the following S coupled non-linear ordinary differential equations [35, 36]

$$\dot{N}_i = N_i \left(k_i - N_i - \sum_{j=1}^S \mathbb{A}_{ij} N_j \right), \quad i = 1, \dots, S. \quad (1)$$

They generalize the Lotka-Volterra model [47, 48] to mixed ecosystems with multiple species. They determine the time-evolution of the abundance $N_i(t)$ for each species labeled i , as a function of (i) its carrying capacity/growth rates k_i and (ii) interspecies interactions that are encoded in the elements of the real $S \times S$ matrix $\mathbb{A}$. Following RMT [3], $\mathbb{A}$ belongs to an ensemble defined by the statistics of its matrix elements $\mathbb{A}_{ij}$, and I assume that the latter are normally distributed with average and second moment given by

$$\langle \mathbb{A}_{ij} \rangle = \mu / S, \quad (2a)$$

$$\langle \mathbb{A}_{ij} \mathbb{A}_{kl} \rangle = \sigma^2 (\delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}) / S + (\mu / S)^2. \quad (2b)$$

Beside μ and σ , Eq. (2b) introduces a third parameter γ in our theory, whose meaning is the following. The interaction matrix element $\mathbb{A}_{ij} > 0$ (< 0) represents the loss (growth) rate of species i due to its interaction with species j . Then, if $\mathbb{A}_{ij} > 0$ (< 0) and $\mathbb{A}_{ji} < 0$ (> 0) the two species i and j are in a predator-prey relationship. If, on the other hand $\mathbb{A}_{ij} > 0$ and $\mathbb{A}_{ji} > 0$ then the two species are in competition, while when $\mathbb{A}_{ij} < 0$ and $\mathbb{A}_{ji} < 0$, they are in mutualistic symbiosis. Thus $\gamma < 0$ (> 0) means that pairs with $\mathbb{A}_{ij}$ having the opposite (same) sign as $\mathbb{A}_{ji}$ dominate. The cross-diagonal covariance parameter $\gamma \in [-1, 1]$ tunes between exclusively predator-prey behaviors, $\gamma = -1$, to exclusively competitive and/or mutualistic behavior, $\gamma = 1$ [49].

Population abundances N_i^* at the equilibrium solution of Eq. (1) are determined by $N_i^* = 0$ for extinct species and $k_i = N_i^* + \sum_{j=1}^S \mathbb{A}_{ij} N_j^*$ for surviving ones. Given a set of parameters $\{\vec{k}, \mathbb{A}\}$ it is hard to determine *a priori* which species will be extinct. A first procedure is to numerically time-evolve Eq. (1) until an equilibrium is reached. Strictly speaking, extinctions are reached only asymptotically, $t \rightarrow \infty$. In practice, they can be defined

consistently by introducing a low enough threshold and considering as extinct the species whose abundance go below that threshold [34, 50]. A second procedure is to iteratively solve the set of linear equations

$$\vec{N}^* = (\mathbb{1} + \mathbb{A})^{-1} \vec{k}, \quad (3)$$

by removing negative abundances at each iteration until a solution is reached with only positive abundances for $S_s \leq S$ surviving species. It is generally not clear that the two procedures lead to the same set of extinctions.

Below I compare the first procedure with a simplified version of the second, where extinctions are conjectured to be limited to the set $\mathcal{S}$ of species with negative abundances obtained by solving Eq. (3) for the full set of S species. This presupposes that, removing $\mathcal{S}$, i.e. solving Eq. (3) over the corresponding lower dimensional space will provide a feasible solution. That a solution exists is likely – the randomness of $\mathbb{A}$ ensures that $\mathbb{1} + \mathbb{A}$ is singular only at isolated points (σ, μ) . That it is feasible is however not guaranteed and needs to be checked. From the solution of Eq. (3) in the full S -dimensional species space, I calculate the four lower moments of species abundances over an ensemble of different interaction matrices $\mathbb{A}$ defined by Eqs. (2). The approach is to expand the matrix inverse in Eq. (3) in a Neumann series and calculate RMT ensemble averages term by term, keeping only the leading order terms that survive when $S \rightarrow \infty$. I will do this in two different ways. The first one expands Eq. (3) directly in the matrix $\mathbb{A}$. This expansion converges if the spectral radius of $\mathbb{A}$ satisfies $\rho_{\mathbb{A}} < 1$. From Refs. [21, 32, 51], one has $\rho_{\mathbb{A}} = \max[|\mu|, \sigma(1 + |\gamma|)]$. The approach is therefore valid for $|\mu| < 1$ and $\sigma(1 + |\gamma|) < 1$. The second one is to write $\mathbb{A} = \mathbb{M} + \delta\mathbb{A}$, with the matrix average $\mathbb{M}_{ij} = \langle \mathbb{A}_{ij} \rangle = \mu / S$, and to expand Eq. (3) in the matrix $\delta\mathbb{A}$. The two methods coincide for $\mu = 0$, and the second one is advantageous when $\mu \gg 1$, because its radius of convergence does not depend on μ and the condition for validity of the expansion is $\sigma(1 + |\gamma|) < 1$. In both cases, convergence of the expansion guarantees that one is in the regime $\sigma < (1 + \gamma)^{-1}$, of linear stability [19, 21].

Calculational details for both expansion are given in the Supplemental Material. I use the first expansion to calculate the first four moments of $P(N_i^*)$ up to order $\mathcal{O}(\mu^{p_1} \sigma^{p_2})$, $p_1 + p_2 \leq 6$. How it is applied in practice is illustrated with several example calculations in the Supplemental Material. The second expansion retains its validity for arbitrarily large values of μ , however, it is mathematically more intricate and calculations are limited to $\mathcal{O}(\sigma^4)$. The two approaches give compatible results where their respective regimes of validity overlap.

Abundance Statistics. I calculate the first four moments of the abundance distribution from Eq. (S2) using the two different expansions I just discussed. I assume that carrying capacities are distributed with average and variance given by $\langle k_i \rangle = \kappa$ and $\langle k_i k_j \rangle - \langle k_i \rangle^2 = \chi^2 \delta_{ij}$, i.e. $\chi = 0$ corresponds to equal carrying capacities for all species. I obtain the distribution average

$$\bar{N} = \kappa[(1 + \mu)^{-1} + (1 + \mu)^{-2}\gamma\sigma^2 + ((1 + \mu)^{-2} + (1 + \mu)^{-3})\gamma^2\sigma^4] + 5\gamma^3\sigma^6 + \mathcal{O}(\sigma^8, S^{-1}). \quad (4)$$

It is important to realize that the σ^6 -term is the zeroth order term in an expansion in powers of μ . All other terms, are obtained using either expansion, and resumming powers of μ . Therefore, Eq. (4) is valid for $|\mu| < 1$, if the σ^6 -term is taken into account, or for arbitrary values of μ if the σ^6 -term is neglected (in which case its accuracy vs. σ is reduced).

With the first Neumann expansion I further obtain the variance of abundances

$$\Sigma_N = \kappa^2 [(1 - 2\mu + 3\mu^2 - 4\mu^3 + 5\mu^4)\sigma^2 + (1 - 2\mu + 3\mu^2)\sigma^4 + (4 - 10\mu + 18\mu^2)\gamma\sigma^4 + (1 + 6\gamma + 14\gamma^2)\sigma^6] + \chi^2 [1 + (1 + 2\gamma)\sigma^2 + (1 + 4\gamma + 5\gamma^2)\sigma^4 + (1 + 6\gamma + 14\gamma^2 + 14\gamma^3)\sigma^6] + \mathcal{O}(\sigma^8, \sigma^6\mu^2, \dots S^{-1}). \quad (5)$$

It looks like the first three contributions in the term $\propto \kappa^2$ in Eq. (5) could be resummed as $(\sigma^2 + \sigma^4)/(1 + \mu)^2 + 2\gamma\sigma^4[(1 + \mu)^{-2} + (1 + \mu)^{-3}]$. However, extending the second Neumann expansion to the variance is rather intricate and I have been unable to confirm this analytically. The fact that contributions to $\bar{N}$ and Σ_N remain finite for $S \rightarrow \infty$ directly follow from the choice of S -dependance in Eq. (2). If necessary, further S -dependance can be attributed to σ and μ , e.g. reflecting increased or reduced interactions as the number of species increases.

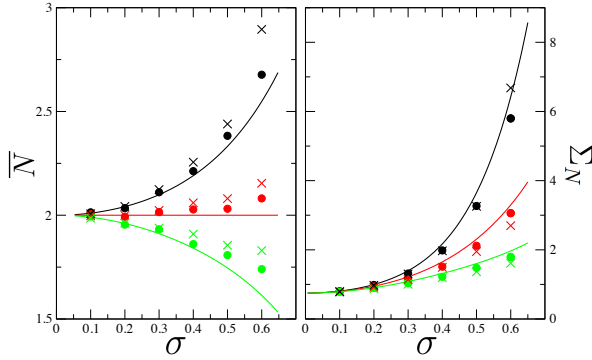


Figure 1. Average (left panel) and variance (right) of the abundance distribution at stable equilibria for Eq. (1), $\mu = 0$, and a random, homogeneous distribution $k_i \in [0.5, 3.5]$, i.e. with $\kappa = 2$ and $\chi = 0.75$, as a function of σ and for $\gamma = -0.5$ (green), 0 (red) and 0.5 (black). Solid lines give the RMT predictions of Eqs. (4) and (5). Data correspond to averages over all species when the equilibrium of Eq. (1) has been reached for 500 realizations of the random interaction matrix $\mathbb{A}$ with $S = 37$ ($\times$) and 157 (circles).

Results and Discussion. Fig. 1 compares Eqs. (4) and (5) to numerically obtained results for the time-evolution of Eq. (1) to equilibrium. The agreement is very good for moderate values of σ , becomes better at larger S and breaks down only at larger values of σ . For $\gamma > 0$, this is so because one reaches the radius of convergence, $\sigma_c \simeq (1 + \gamma)^{-1}$, of the Neumann expansions, while for $\gamma < 0$, terms $\propto \sigma^8$ and higher become important.

In the Supplemental Material I further show that the skewness vanishes and the kurtosis $\mathcal{K} = \langle (N_i^* - \bar{N})^4 \rangle / \Sigma_N^2 = 3 + \mathcal{O}(\sigma^8, S^{-1})$. Furthermore, even standardized moments of order $2p > 4$ have contributions already existing in lower moments, with combinatorial factors of

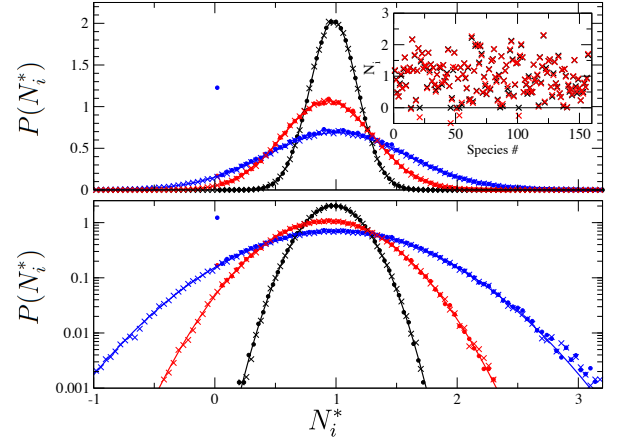


Figure 2. Normal (a) and semilog (b) plots of species abundance distributions for $\mu = 0$, $k_i \equiv 1$, and $\sigma = 0.2$, $\gamma = -0.5$ (black circles), $\sigma = 0.4$, $\gamma = -0.5$ (red), $\sigma = 0.5$, $\gamma = 0$ (blue). Solid lines give Gaussian distributions with average ($\bar{N} = 0.98, 0.93$ and 1.01) and variances ($\Sigma_N = 0.038, 0.138, 0.31$) given by Eqs. (4) and (5). Distributions are calculated from fixed point solutions to Eq. (1) (dots) and from solutions to Eq. (3) ($\times$), over 1000 realizations of the interaction matrix. The blue (red) dot at $N_i^* = 0$ correspond to $\simeq 6.5$ (0.8) extinctions per realization, in agreement with Eq. (6). Inset: Abundances N_i at a fixed point solution to Eq. (1) (black) and for Eq. (3) (red) for the same realization of $\mathbb{A}$, for $\mu = 0$, $k_i \equiv 1$, $\sigma = 0.5$ and $\gamma = 0$. There are six species with negative abundances at the fixed point and six species with abundances below 10^{-20} and still going down after 2×10^7 Runge-Kutta iterations of Eq. (3).

$(2p - 1)!!$, relating the $2p^{\text{th}}$ moment to the variance, that correspond to the Gaussian case in the spirit of the central limit theorem [52]. These results point toward a Gaussian distribution of abundances, which is confirmed in Fig. 2. Distributions obtained numerically from the time-evolution of Eq. (1) to equilibrium are equal to those obtained from Eq. (3) for $N_i^* > 0$. Both are furthermore Gaussian, with average and variance given by Eq. (4) and (5). This is true even when several species have negative abundances according to Eq. (3), in which case, they correspond to a peak at $N_i^* = 0$ in the distribution for the equilibrium of Eq. (1). (See the blue and red dots at $N_i^* = 0$ in Fig. 2, corresponding respectively to 6.5 and 0.8 extinctions per realization). Thus there is a one-to-

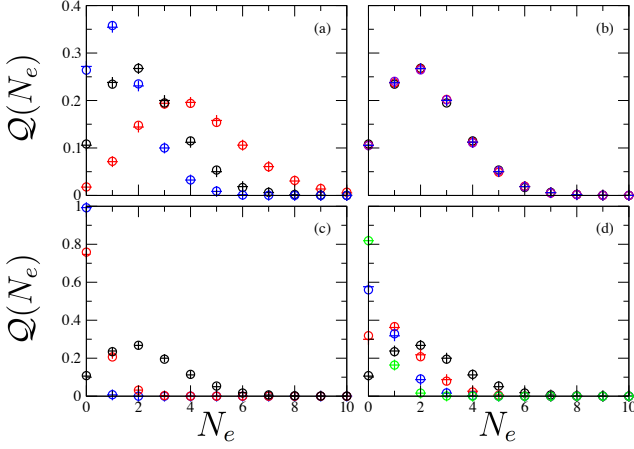


Figure 3. Distribution of number N_e of extinct species, from the species with negative abundances in Eq. (3) (+) and from Eq. (6) (dots), with $k_i \equiv 1$. (a) $S = 607$, $\mu = 0$, $\sigma = 0.35$ and $\gamma = -0.5$ (blue), 0 (black) and 0.5 (red); (b) $S = 607$, $\sigma = 0.35$, $\mu = 0$ and $\gamma = 0$ (black), 2 (red) and 4 (blue); (c) $S = 607$, $\mu = 0$, $\gamma = 0$ and $\sigma = 0.23$ (blue), 0.29 (red) and 0.35 (black); (d) $\mu = 0$, $\gamma = 0$, $\sigma = 0.35$ and $S = 57$ (green), 157 (blue), 307 (red) and 607 (black). All distributions are calculated over 1000 different realizations of the interaction matrix $\mathbb{A}$.

one correspondence between the extinct species obtained by letting Eq. (1) relax to equilibrium, and the species with negative abundances obtained from Eq. (3). This correspondence is further illustrated in the inset of Fig. 2 for an individual realization of $\mathbb{A}$ with six extinctions.

Species abundances at fixed-point equilibria are thus normally distributed as $P(N_i^*) = \exp[-(N_i^* - \bar{N})^2/2\Sigma_N]/\sqrt{2\pi\Sigma_N}$. The probability to have N_e species with negative abundances in an ecosystem of S species is therefore (see Supplemental Material)

$$Q(N_e) = \binom{S}{N_e} F_-^{N_e} F_+^{S-N_e}, \quad (6a)$$

$$F_{\pm} = \left(1 \pm \text{erf}\left[\bar{N}/\sqrt{2\Sigma_N}\right]\right)/2, \quad (6b)$$

with the error function $\text{erf}[x]$. Fig. 3 illustrates that $Q(N_e)$ matches the distribution of species with negative abundances obtained from Eq. (3). Furthermore, it shows that more negative abundances occur as σ and γ increase, but that $Q(N_e)$ is insensitive to μ . This is in agreement with Eqs. (4) and (5) which give $\bar{N}^2/\Sigma_N$ independent of μ to leading order. More importantly, increasing S , every other parameter being equal, systematically leads to more negative abundances [Fig. 3d]. This is so, because populating the left tail of the Gaussian distribution becomes more likely when S increases [see the binomial coefficient in Eq. (6a)].

Fig. 2 suggested that species with negative abundances from Eq. (3) are those that go extinct under the dynamics of Eq. (1). This is corroborated in the two left panels of Fig. 4, which compare $Q(N_e)$ of Eqs. (6) with the distribution of extinctions obtained from time-evolving Eq. (1)

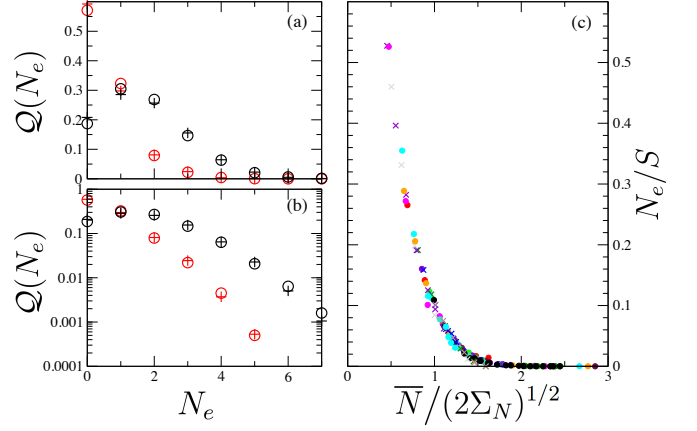


Figure 4. Normal (a) and semilog (b) plots of the distribution of number N_e of extinct species, for $S = 157$, $\mu = 0$, $k_i \equiv 1$, $\gamma = 0$, $\sigma = 0.35$ (red symbols) and 0.4 (black). Distributions are calculated from fixed point solutions obtained by time-evolving Eq. (1) (circles) and from solutions to Eq. (3) (+), over 1000 different realizations of the interaction matrix $\mathbb{A}$. (c) Scaling of the ratio of the average number of extinct species vs. ratio of the average over standard deviation of the abundance distribution, for 24 different sets of parameters, $\sigma \in [0, 0.6]$, $\mu \in [0, 1]$, $\gamma \in [-1, 1]$, $S \in [100, 300]$, and $k_i \equiv 1$ (circles) as well as $k_i \in [0.5, 3.5]$ (crosses). Averages are calculated over 100 to 500 realizations of the random matrix $\mathbb{A}$ for each set of parameter, by time-evolving Eq. (1) until a stationary solution is reached. Symbols of the same color correspond to sets varying either σ or γ , all other parameters being fixed. When time-evolving Eq. (1), extinctions are defined at a threshold $N_c = 10^{-20}$.

– the two distributions are the same, even when there are several extinctions on average and a $\simeq 15\%$ probability to have 3 or more extinctions. The two distributions remain the same at larger σ and γ (not shown). Note that values for $\bar{N}$ and Σ_N need to be adapted from Eqs. (4) and (5) when $\sigma \gtrsim 0.3 - 0.4$. These results show that species extinctions can be statistically predicted by Eqs. (6), together with Eqs. (4) and (5).

An important consequence of Eqs. (6) is that species extinctions depend only on $\bar{N}/\sqrt{2\Sigma_N}$. The right panel of Fig. 4 demonstrates the existence of a single-parameter scaling law governing the distribution of extinctions – the ratio N_e/S falls on a single curve for widely varying system parameters, when plotted as a function of the scaling parameter $\eta = \bar{N}/\sqrt{2\Sigma_N}$. This scaling is quite remarkable, as it applies even for $N_e/S \gtrsim 0.5$, when more than half of the species have gone extinct.

Conclusion. I have shown that RMT correctly predicts the distribution of species abundances for equilibrium solutions of multispecies Lotka-Volterra ecosystems. The theory predicts the emergence of unfeasibility and the distribution of species with negative abundances in solutions to Eq. (3). Its range of validity lies inside the stability boundary of fixed-point solutions, and therefore demonstrates that feasibility generically breaks down before stability. Because the theory is statistical, it can-

not rule out that the opposite occurs in rare, specific instances, however I have never observed this numerically.

I further made the point that species with negative abundances for Eq. (3) are those that get extinct under the dynamics of Eq. (1). There is no rigorous basis for this connection at this time, however numerics support it unambiguously. Deriving this connection rigorously is highly desirable. An important consequence of this connection and of the Gaussianity of the abundances distribution is the numerically confirmed conjecture that extinctions obey a single-parameter scaling law.

Compared to earlier works on feasibility [24, 30, 43, 44, 46], the present work goes one step further as it

includes all RMT parameters – in particular the cross-diagonal correlator γ can take any value, while earlier works considered specific choices $\gamma = 0$ [30] or $\gamma = 1$ with an heuristically extension to other values [46]. The presented theory directly calculated the full distribution of species abundances, and connected feasibility to extinctions. The theory is perturbative in σ and valid for $S \rightarrow \infty$, but makes no further assumption. I have found numerically that theoretical predictions work well already for moderately large ecosystems $S \approx 30$.

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Single Parameter Scaling of Species Extinction in Generalized Lotka-Volterra Systems Supplemental Material

In this Supplemental Material section I briefly sketch the two Neumann expansions I use to solve Eq (3). The general method is the following. Inserting the Neumann series $(\mathbb{1} + \mathbb{A})^{-1} = \sum_p (-1)^p \mathbb{A}^p$ into Eq. (3) leads to

$$N_i^* = \sum_{p=0}^{\infty} \sum_{j=1}^S (-1)^p [\mathbb{A}^p]_{ij} k_j. \quad (\text{S1})$$

Accordingly, the m^{th} moment $\langle (N_i^*)^m \rangle$ of the distribution $P(N_i^*)$ of abundances over the ensemble of random matrices $\mathbb{A}$ defined by Eqs. (2) reads

$$\langle (N_i^*)^m \rangle = \sum_{p_1, \dots, p_m} \sum_{j_1, \dots, j_m} (-1)^{\sum_n p_n} \left\langle \prod_{n=1}^m [\mathbb{A}^{p_n}]_{ij_n} k_{j_n} \right\rangle. \quad (\text{S2})$$

In this work, I assume that the distributions of $\mathbb{A}_{ij}$ and of k_j are uncorrelated. Under this assumption, I obtain $\langle \prod_{n=1}^m [\mathbb{A}^{p_n}]_{ij_n} k_{j_n} \rangle = \langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} \dots [\mathbb{A}^{p_m}]_{ij_m} \rangle \langle k_{j_1} k_{j_2} \dots k_{j_m} \rangle$. To calculate averages over the interaction matrix ensemble, I use Eqs. (2) and take the average over the carrying capacities as

$$\langle k_i \rangle = \kappa, \quad \langle k_i k_j \rangle = \chi^2 \delta_{ij} + \kappa^2. \quad (\text{S3})$$

S1. RMT CALCULATION OF THE AVERAGE ABUNDANCE

A. $\mu = 0$

From Eq. (S2), the average species abundance is given by

$$\bar{N} \equiv \langle N_i^* \rangle = \sum_p \sum_j (-1)^p \langle [\mathbb{A}^p]_{ij} k_j \rangle = \sum_p \sum_j (-1)^p \langle [\mathbb{A}^p]_{ij} \rangle \langle k_j \rangle = \kappa \sum_p \sum_j (-1)^p \langle [\mathbb{A}^p]_{ij} \rangle, \quad (\text{S4})$$

where the average is taken over both a RMT distribution of interaction matrices as well as a distribution of carrying capacities, under the assumption that the two averages are uncorrelated.

I first calculate contributions up to order $\mathcal{O}(\sigma^6)$ for $\mu = 0$. They read

$$\bar{N} = \kappa \left(1 + \sum_{l,j} \langle \mathbb{A}_{il} \mathbb{A}_{lj} \rangle + \sum_{l_1, l_2, l_3, j} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 l_3} \mathbb{A}_{l_3 j} \rangle + \sum_{l_1, l_2, l_3, l_4, l_5, j} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 l_3} \mathbb{A}_{l_3 l_4} \mathbb{A}_{l_4 l_5} \mathbb{A}_{l_5 j} \rangle \right) + \mathcal{O}(\sigma^8). \quad (\text{S5})$$

In particular, only terms with even powers of $\mathbb{A}$ contribute to the RMT average. The quadratic term is straightforwardly calculated using Eq. (2),

$$\sum_{l,j} \langle \mathbb{A}_{il} \mathbb{A}_{lj} \rangle = (\sigma^2/S) \sum_{l,j} (\gamma \delta_{ij} + \delta_{il} \delta_{lj}) = \gamma \sigma^2 + \mathcal{O}(S^{-1}), \quad (\text{S6})$$

because only the first contribution has a sum surviving the Kronecker, which gives a factor S . To calculate the quartic and sextic contributions, one needs to identify the pairwise contractions of indices implied by Eq. (2b), that give the dominant order in S . The rule is that p^{th} order contributions have p sums and prefactors $\propto \sigma^p/S^{p/2}$. Each index contraction removes one sum and therefore only terms with at most $p/2$ independent index contractions give finite contributions in the limit $S \rightarrow \infty$. It turns out that $p/2$ is the minimal possible number of index contractions. The quartic term reads

$$\begin{aligned} \sum_{l_1, l_2, l_3, j} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 l_3} \mathbb{A}_{l_3 j} \rangle &= \sum_{l_1, l_2, l_3, j} \left(\langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \rangle \langle \mathbb{A}_{l_2 l_3} \mathbb{A}_{l_3 j} \rangle + \langle \mathbb{A}_{il_1} \mathbb{A}_{l_3 j} \rangle \langle \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 l_3} \rangle + \langle \mathbb{A}_{il_1} \mathbb{A}_{l_2 l_3} \rangle \langle \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_3 j} \rangle \right) \\ &= \gamma^2 (\sigma^4/S^2) \sum_{l_1, l_2, l_3, j} (\delta_{il_2} \delta_{l_2 j} + \delta_{l_1 l_3} \delta_{ij}) + \mathcal{O}(S^{-1}) = 2\gamma^2 \sigma^4 + \mathcal{O}(S^{-1}), \end{aligned} \quad (\text{S7})$$

where the third term on the right hand side of the first line requires at least three index contractions, and therefore gives a subdominant, $\mathcal{O}(S^{-1})$ contribution. Up to and including the quartic term, relevant contractions are easily identified. The procedure is similar for the sextic term, for which there are more contractions. I relied on a symbolic algebra code in parallel to the analytical calculation, to guarantee that all contributions are properly taken into account. For the sextic term in Eq. (S5), there are five relevant contractions and I obtain,

$$\sum_{l_1, l_2, l_3, l_4, l_5, j} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 l_3} \mathbb{A}_{l_3 l_4} \mathbb{A}_{l_4 l_5} \mathbb{A}_{l_5 j} \rangle = 5\gamma^3 \sigma^6 + \mathcal{O}(S^{-1}). \quad (\text{S8})$$

Putting Eqs. (S5–S8) together, I obtain

$$\bar{N} = \kappa \left(1 + \gamma \sigma^2 + 2\gamma^2 \sigma^4 + 5\gamma^3 \sigma^6 \right) + \mathcal{O}(\sigma^8, S^{-1}). \quad (\text{S9})$$

Fig. S1 illustrates the convergence of the series expansion, Eq. (S9), to the numerically obtained results, as orders in σ are added.

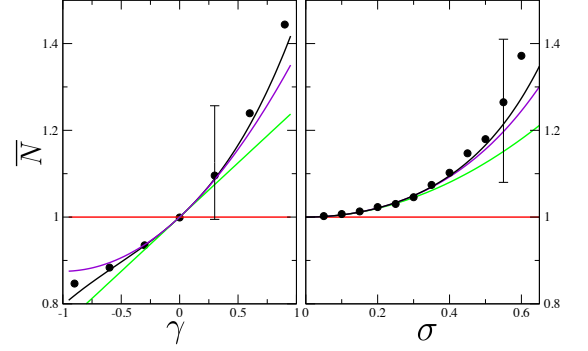


Figure S1. Convergence of the series expansion for $\mu = 0$ and $k_i \equiv \kappa = 1$. The solid curves give Eq. (S9) truncated at the zeroth (red), second (green), fourth (violet) and sixth (black) order in σ .

B. $\mu \neq 0$

I use two different approaches to extend this result to $\mu \neq 0$. Both are based on a Neumann expansion of Eq. (3) as

$$(\mathbb{X} + \mathbb{Y})^{-1} = (\mathbb{X})^{-1} \sum_{p=0}^{\infty} (-1)^p \left(\frac{\mathbb{Y}}{\mathbb{X}} \right)^p. \quad (\text{S10})$$

1. First expansion

The first Neumann expansion takes $\mathbb{X} = \mathbb{1}$ and $\mathbb{Y} = \mathbb{A} = \mathbb{M} + \delta\mathbb{A}$, with $\mathbb{M}_{ij} = \mu/S$, and $\delta\mathbb{A}$ containing the zero-average fluctuations of interspecies interactions, i.e. with $\langle \delta\mathbb{A}_{ij} \rangle = 0$ and $\langle \delta\mathbb{A}_{ij} \delta\mathbb{A}_{kl} \rangle = \sigma^2 (\delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk})/S$. One has

$$\bar{N} = \kappa \sum_p \sum_j (-1)^p \left\langle [(\mathbb{M} + \delta\mathbb{A})^p]_{ij} \right\rangle, \quad (\text{S11})$$

All terms with even powers of $\delta\mathbb{A}$ contribute to the RMT average, as before, and for $\mu \neq 0$, one additionally has terms with odd p and odd powers of $\mathbb{M}$. That each average interaction comes with S in the denominator – and not $S^{1/2}$ as the fluctuations of interaction – still leaves finite contributions that survive the $S \rightarrow \infty$ limit, because they are not associated with Kronecker pairing of indices. To illustrate this I calculate the cubic contribution to $\bar{N}$. I obtain

$$\begin{aligned} (-1)^3 \sum_{l_1, l_2, j} \langle (\mathbb{M} + \delta\mathbb{A})_{il_1} (\mathbb{M} + \delta\mathbb{A})_{l_1 l_2} (\mathbb{M} + \delta\mathbb{A})_{l_2 j} \rangle &= -\mu^3 - (\mu/S) \sum_{l_1, l_2, j} \left(\langle \delta\mathbb{A}_{l_1 l_2} \delta\mathbb{A}_{l_2 j} \rangle + \langle \delta\mathbb{A}_{il_1} \delta\mathbb{A}_{l_1 l_2} \rangle + \langle \delta\mathbb{A}_{il_1} \delta\mathbb{A}_{l_2 j} \rangle \right) \\ &= -\mu^3 - \gamma(\sigma^2/S)(\mu/S) \sum_{l_1, l_2, j} (\delta_{l_1 j} + \delta_{il_2}) + \mathcal{O}(S^{-1}) \\ &= -\mu^3 - 2\gamma\mu\sigma^2 + \mathcal{O}(S^{-1}), \end{aligned} \quad (\text{S12})$$

where the first line is obtained by realizing that only even powers of $\delta\mathbb{A}$ survive RMT average. Applying this procedure to all terms up to and including $p = 6$ in Eq. (S11), I obtain

$$\bar{N} = \kappa \left[1 - \mu + \mu^2 - \mu^3 + \mu^4 - \mu^5 + \mu^6 + \gamma\sigma^2(1 - 2\mu + 3\mu^2 - 4\mu^3 + 5\mu^4) + \gamma^2\sigma^4(2 - 5\mu + 9\mu^2) + 5\gamma^3\sigma^6 \right], \quad (\text{S13})$$

where terms $\mathcal{O}(\mu^{p_1} \sigma^{p_2}, S^{-1})$ with $p_1 + p_2 > 6$ have been neglected. It is straightforward to see that when considering only the average interaction in all terms in Eq. (S11), one gets $\sum_{p=0}^{\infty} (-1)^p \mu^p = (1 + \mu)^{-1}$. Still straightforward, terms $\propto \sigma^2$ that survive the $S \rightarrow \infty$ limit may acquire only a single Kronecker from the RMT average with Eq. (2). Therefore, RMT contractions can be taken only over pairs of consecutive matrices $\mathbb{M} + \delta\mathbb{A} \rightarrow \delta\mathbb{A}$ in Eq. (S11) that already share one index. There are $p - 1$ of them, so that the σ^2 contribution gives $\sum_p (-1)^p (p - 1) \mu^{p-2} \sigma^2 = \sigma^2 / (1 + \mu)^2$. All σ^4 terms can finally be resummed in a similar, though a bit more intricate way: First, counting arguments similar

to those after Eq. (S6) imply that only pairings with two Kroneckers in the RMT average with Eq. (2) survive in the limit $S \rightarrow \infty$. Such contributions correspond to two independent pairs of consecutive $\mathbb{M} + \delta\mathbb{A} \rightarrow \delta\mathbb{A}$ in Eq. (S11). Furthermore, when these pairs are consecutive, a multiplicative factor of 2 has to be taken into account, the origin of which is explained in Eq. (S7). Taking all this into account I end up with $p(p-3)/2$ contributions giving $\sum_p (-1)^p p(p-3)/2 \mu^{p-4} \sigma^4 = \sigma^4 [1/(1+\mu)^2 + 1/(1+\mu)^3]$. These calculations up to σ^4 suggest that similar infinite series in powers of μ exist for each power of σ , which can be resummed. I have however not been able to find a systematic (diagrammatic ?) expansion to prove this to all orders in σ . With the considerations just given, one has

$$\overline{N} = \kappa [(1+\mu)^{-1} + (1+\mu)^{-2} \gamma \sigma^2 + [(1+\mu)^{-2} + (1+\mu)^{-3}] \gamma^2 \sigma^4] + 5\gamma^3 \sigma^6 + \mathcal{O}(\sigma^8, S^{-1}), \quad (\text{S14})$$

which matches Eq. (S9) when $\mu = 0$. Care must be taken when taking into account the σ^6 -term, because it is the zeroth-order term in an expansion in μ . Including that term, Eq. (S14) is strictly valid only for $|\mu| \ll 1$.

2. Second expansion

The second Neumann expansion takes $\mathbb{X} = \mathbb{1} + \mathbb{M}$ and $\mathbb{Y} = \delta\mathbb{A}$ in Eq. (S10). I obtain, instead of Eqs. (S4) and (S11),

$$\overline{N} = \frac{\kappa}{1+\mu} \sum_{p=0}^{\infty} \sum_j (-1)^p \left\langle \left[\delta\mathbb{A} \left(\mathbb{1} - \frac{\mathbb{M}}{1+\mu} \right) \right]_{ij}^p \right\rangle, \quad (\text{S15})$$

where I used $(\mathbb{1} + \mathbb{M})^{-1} = (\mathbb{1} - \mathbb{M}/(1+\mu))$. Taking into account terms up to $p = 4$, I obtain

$$\overline{N} = \kappa [(1+\mu)^{-1} + (1+\mu)^{-2} \gamma \sigma^2 + [(1+\mu)^{-2} + (1+\mu)^{-3}] \gamma^2 \sigma^4] + \mathcal{O}(\sigma^6, S^{-1}). \quad (\text{S16})$$

Remarkably, I recover Eq. (S14) up to and including the σ^4 terms. Remember, however, that Eq. (S14) is derived under the assumption that $|\mu| < 1$ and $\sigma(1+|\gamma|) < 1$, while Eq. (S16) is valid for an extended range, $\sigma(1+|\gamma|) \lesssim 1$, i.e. with no restriction on μ . This further suggests that a resummation of the perturbation series obtained with the first approach may give a more compact and elegant expression valid for any value of μ , for $\sigma(1+|\gamma|) < 1$. At this stage, this statement is however only a conjecture.

S2. RMT CALCULATION OF THE VARIANCE OF THE ABUNDANCE

From Eq. (S2), the second moment of the abundance distribution is given by

$$\langle (N_i^*)^2 \rangle_{\mu=0} = \sum_{p_1, p_2} \sum_{j_1, j_2} (-1)^{p_1+p_2} \left\langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} k_{j_1} k_{j_2} \right\rangle = \sum_{p_1, p_2} \sum_{j_1, j_2} (-1)^{p_1+p_2} \left\langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} \right\rangle \langle k_{j_1} k_{j_2} \rangle \quad (\text{S17})$$

where, as before, the average is taken over both a RMT distribution of interaction matrices as well as a distribution of carrying capacities, under the assumption that the two are uncorrelated.

The calculation is similar as for the average of the abundance, with similar sum rules as above restricting the number of Kroneckers giving contributions surviving the $S \rightarrow \infty$ limit. The corresponding index contractions are similar to those discussed above, with however different sequences of indices in the terms of Eq. (S17) with $p_1 \neq 0 \neq p_2$. I calculated contributions up to order $\mathcal{O}(\sigma^6)$ for $\mu = 0$, however I give here only details up to $\mathcal{O}(\sigma^4)$. I have (factors of two count multiplicities of terms, e.g. in Eq. (S17), $p_1 = 2, p_2 = 0$ gives the same contribution as $p_1 = 0, p_2 = 2$),

$$\begin{aligned} \langle (N_i^*)^2 \rangle_{\mu=0} &= \langle k_i^2 \rangle + \sum_{j_1, j_2} \langle \mathbb{A}_{ij_1} \mathbb{A}_{ij_2} \rangle \langle k_{j_1} k_{j_2} \rangle + 2 \sum_{j, l} \langle \mathbb{A}_{il} \mathbb{A}_{lj} \rangle \langle k_j k_i \rangle + 2 \sum_{j, l_1, l_2, l_3} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 l_3} \mathbb{A}_{l_3 j} \rangle \langle k_j k_i \rangle \\ &\quad + 2 \sum_{j_1, j_2, l_1, l_2} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 l_2} \mathbb{A}_{l_2 j_1} \mathbb{A}_{ij_2} \rangle \langle k_{j_1} k_{j_2} \rangle + \sum_{j_1, j_2, l_1, l_2} \langle \mathbb{A}_{il_1} \mathbb{A}_{l_1 j_1} \mathbb{A}_{il_2} \mathbb{A}_{l_2 j_2} \rangle \langle k_{j_1} k_{j_2} \rangle + \mathcal{O}(\sigma^6) \\ &= (\kappa^2 + \chi^2) [1 + \sigma^2 + 2\gamma\sigma^2 + 4\gamma^2\sigma^4 + 4\gamma\sigma^4 + (1 + \gamma^2)\sigma^4] + \mathcal{O}(\sigma^6, S^{-1}), \end{aligned} \quad (\text{S18})$$

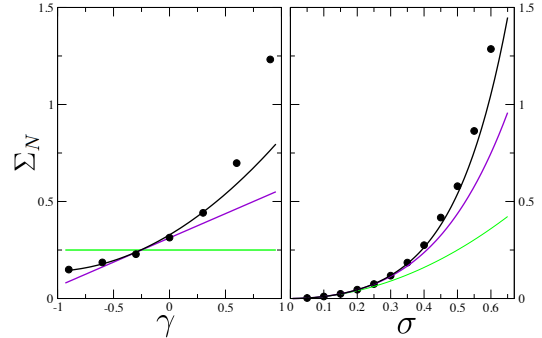


Figure S2. Convergence of the series expansion of Eq. (5) for $\mu = 0$ and $k_i \equiv \kappa = 1$ (hence $\chi = 0$). The solid curves give Eq. (5) truncated at the second (green), fourth (violet) and sixth (black) order in σ .

where in the square bracket in the last line, terms have been grouped to sequentially correspond to the six terms on the right hand side of the first equality. The variance of the abundance for $\mu = 0$, up to and including $\mathcal{O}(\sigma^6)$ is given by

$$\Sigma_N = \chi^2[1 + 2\gamma\sigma^2 + 5\gamma^2\sigma^4 + 14\gamma^3\sigma^6] + (\chi^2 + \kappa^2)[\sigma^2 + (1 + 4\gamma)\sigma^4 + (1 + 6\gamma + 14\gamma^2)\sigma^6] + \mathcal{O}(\sigma^8, S^{-1}). \quad (\text{S19})$$

The calculation of the variance of the abundance in the case $\mu \neq 0$ is tedious, though straightforward. Terms with no μ -dependence are multiplied by $\langle k_i^2 \rangle = \kappa^2 + \chi^2$, while those with a μ -dependence are multiplied by $\langle k_i \rangle^2 = \kappa^2$. I obtain Eq. (5) in the main text, using the expansion of Paragraph S1 B 1. Fig. S2 illustrates the convergence of the series expansion in Eq. (5) to the numerically obtained results, as orders in σ are added.

S3. RMT CALCULATION OF HIGHER MOMENTS

Skewness and kurtosis can be computed starting from the third and fourth moments of the abundance distribution,

$$\langle (N_i^*)^3 \rangle = \sum_{p_1, p_2, p_3} \sum_{j_1, j_2, j_3} (-1)^{p_1+p_2+p_3} \left\langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} [\mathbb{A}^{p_3}]_{ij_3} k_{j_1} k_{j_2} k_{j_3} \right\rangle, \quad (\text{S20a})$$

$$\langle (N_i^*)^4 \rangle = \sum_{p_1, p_2, p_3, p_4} \sum_{j_1, j_2, j_3, j_4} (-1)^{p_1+p_2+p_3+p_4} \left\langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} [\mathbb{A}^{p_3}]_{ij_3} [\mathbb{A}^{p_4}]_{ij_4} k_{j_1} k_{j_2} k_{j_3} k_{j_4} \right\rangle. \quad (\text{S20b})$$

Calculating them up to and including $\mathcal{O}(\sigma^6)$ is not more complicated than for the average and variance, though it becomes intricate with $\mu \neq 0$ and $\chi \neq 0$. To sketch the calculations, I therefore restrict myself to the case $\mu = 0$ and $k_i \equiv \kappa = 1$, $\chi = 0$, i.e.

$$\langle (N_i^*)^3 \rangle = \sum_{p_1, p_2, p_3} \sum_{j_1, j_2, j_3} \left\langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} [\mathbb{A}^{p_3}]_{ij_3} \right\rangle, \quad (\text{S21a})$$

$$\langle (N_i^*)^4 \rangle = \sum_{p_1, p_2, p_3, p_4} \sum_{j_1, j_2, j_3, j_4} \left\langle [\mathbb{A}^{p_1}]_{ij_1} [\mathbb{A}^{p_2}]_{ij_2} [\mathbb{A}^{p_3}]_{ij_3} [\mathbb{A}^{p_4}]_{ij_4} \right\rangle, \quad (\text{S21b})$$

where $\sum$ indicates that the sum is restricted to $\sum_i p_i = 2p$, i.e. the sum of powers is an even number.

For the third moment, and up to and including $\mathcal{O}(\sigma^6)$, we need to calculate terms with (p_1, p_2, p_3) such that $p_1 + p_2 + p_3 \leq 6$. As an example, the $(p_1, p_2, p_3) = (2, 1, 1)$ contribution to Eq. (S21a) reads, including its multiplicity of 3 [i.e. $(1, 2, 1)$ and $(1, 1, 2)$ give the same contribution],

$$3 \sum_{j_1, j_2, j_3} \sum_l \langle \mathbb{A}_{il} \mathbb{A}_{lj_1} \mathbb{A}_{ij_2} \mathbb{A}_{ij_3} \rangle = 3 \sum_{j_1, j_2, j_3} \sum_l \left[\langle \mathbb{A}_{il} \mathbb{A}_{lj_1} \rangle \langle \mathbb{A}_{ij_2} \mathbb{A}_{ij_3} \rangle + \langle \mathbb{A}_{il} \mathbb{A}_{ij_2} \rangle \langle \mathbb{A}_{lj_1} \mathbb{A}_{ij_3} \rangle + \langle \mathbb{A}_{il} \mathbb{A}_{ij_3} \rangle \langle \mathbb{A}_{lj_1} \mathbb{A}_{ij_2} \rangle \right]. \quad (\text{S22})$$

Interactions give a prefactor σ^4/S^2 to all three contributions. There are four index sums, therefore, only contributions with two index contractions at most survive the $S \rightarrow \infty$ limit. The first contribution gives $\langle \mathbb{A}_{il} \mathbb{A}_{lj_1} \rangle \langle \mathbb{A}_{ij_2} \mathbb{A}_{ij_3} \rangle = \gamma(\sigma^4/S^2) \delta_{ij_1} \delta_{j_2 j_3}$. None of the last two contributions survives the $S \rightarrow \infty$ limit, because both have three index contractions. I obtain

$$3 \sum_{j_1, j_2, j_3} \sum_l \langle \mathbb{A}_{il} \mathbb{A}_{lj_1} \mathbb{A}_{ij_2} \mathbb{A}_{ij_3} \rangle = 3\gamma\sigma^4. \quad (\text{S23})$$

Other terms are calculated in a similar way. As before, the $\mathcal{O}(\sigma^6)$ terms have been computed in parallel by a symbolic algebra code and the just described analytical calculation. All terms contributing to the third moment, Eq. (S21a), and fourth moment, Eq. (S21b), are listed in Tables I and II, respectively.

Skewness and kurtosis are obtained from third and fourth moments as

$$\mathcal{S} = \langle (N_i^* - \overline{N})^3 \rangle / \Sigma_N^{3/2}, \quad \mathcal{K} = \langle (N_i^* - \overline{N})^4 \rangle / \Sigma_N^2. \quad (\text{S24})$$

Using the contributions listed in Table I, I obtain a vanishing skewness, $\mathcal{S} = \mathcal{O}(\sigma^8)$. With that result, and the contributions listed in Table II, I further obtain $\mathcal{K} = 3 + \mathcal{O}(\sigma^8, S^{-1})$. Therefore, up to and including terms $\mathcal{O}(\sigma^6)$, skewness and kurtosis indicate that we have a Gaussian distribution of species abundances.

This argument is easily extended to the case with distributed carrying capacities, $k_i \neq \kappa$, as well as to $\mu \neq 0$, with the same conclusion, that the abundance distribution is Gaussian up to order $\mathcal{O}(\sigma^6)$. I finally note that, as for the kurtosis, when calculating standardized higher even moments of the distribution of abundances, one obtains pairwise contractions already existing in lower moments, with combinatorial factors of $(2p-1)!!$, relating the $2p^{\text{th}}$ moment to the variance, that correspond to the Gaussian case [52].

Contributions to Eq. (S21a)	(p_1, p_2, p_3)	Multiplicity	Contribution	(p_1, p_2, p_3)	Multiplicity	Contribution
	(0,0,0)	1	1	(2,2,2)	1	$(3\gamma + \gamma^3)\sigma^6$
	(1,1,0)	3	$3\sigma^2$	(3,2,1)	6	$12\gamma^2\sigma^6$
	(2,0,0)	3	$3\gamma\sigma^2$	(3,3,0)	3	$3(1 + 4\gamma^2)\sigma^6$
	(2,1,1)	3	$3\gamma\sigma^4$	(4,1,1)	3	$6\gamma^2\sigma^6$
	(2,2,0)	3	$3(1 + \gamma^2)\sigma^4$	(4,2,0)	6	$6(3\gamma + 2\gamma^3)\sigma^6$
	(3,1,0)	6	$12\gamma\sigma^4$	(5,1,0)	6	$30\gamma^2\sigma^6$
	(4,0,0)	3	$6\gamma^2\sigma^4$	(6,0,0)	3	$15\gamma^3\sigma^6$

Table I. Contributions to the third moment, Eq. (S21a), up to and including $\mathcal{O}(\sigma^6)$. All contributions listed in the fourth and seventh columns already include their multiplicity.

Contributions to Eq. (S21b)	(p_1, p_2, p_3, p_4)	Multiplicity	Contribution	(p_1, p_2, p_3, p_4)	Multiplicity	Contribution
	(0,0,0,0)	1	1	(2,2,1,1)	6	$6(1 + \gamma^2)\sigma^6$
	(1,1,0,0)	6	$6\sigma^2$	(2,2,2,0)	4	$4(3\gamma + \gamma^3)\sigma^6$
	(2,0,0,0)	4	$4\gamma\sigma^2$	(3,1,1,1)	4	$24\gamma\sigma^6$
	(1,1,1,1)	1	$3\sigma^4$	(3,2,1,0)	24	$48\gamma^2\sigma^6$
	(2,1,1,0)	12	$12\gamma\sigma^4$	(3,3,0,0)	6	$6(1 + 4\gamma^2)\sigma^6$
	(2,2,0,0)	6	$6(1 + \gamma^2)\sigma^4$	(4,1,1,0)	12	$24\gamma^2\sigma^6$
	(3,1,0,0)	12	$24\gamma\sigma^4$	(4,2,0,0)	12	$12(3\gamma + 2\gamma^3)\sigma^6$
	(4,0,0,0)	4	$8\gamma^2\sigma^4$	(5,1,0,0)	12	$60\gamma^2\sigma^6$
				(6,0,0,0)	4	$20\gamma^3\sigma^6$

Table II. Contributions to the fourth moment, Eq. (S21b), up to and including $\mathcal{O}(\sigma^6)$. All contributions listed in the fourth and seventh columns include their multiplicity.

S4. FROM THE DISTRIBUTION OF ABUNDANCES TO THE PROBABILITY OF NEGATIVE ABUNDANCES

So far I have found that species abundances of fixed-point solutions of Eq. (3) are distributed according to the Gaussian distribution,

$$P(N_i^*) = \exp[-(N_i^* - \bar{N})^2 / 2\Sigma_N] / \sqrt{2\pi\Sigma_N}, \quad (\text{S25})$$

whose average and variance are given in the perturbation expansion of Eqs. (4) and (5). With S species, one has, by normalization of probabilities,

$$1 \equiv \left[\int_{-\infty}^0 P(N_i^*) dN + \int_0^{\infty} P(N_i^*) dN \right]^S = \sum_{N_e=0}^S \binom{S}{N_e} \left[\int_{-\infty}^0 P(N_i^*) dN \right]^{N_e} \left[\int_0^{\infty} P(N_i^*) dN \right]^{S-N_e}, \quad (\text{S26})$$

Each term in the sum on the right-hand side gives the probability to have N_e species with negative abundances. Noting that

$$\int_{-\infty}^0 P(N_i^*) dN = \left(1 - \text{erf}[\bar{N} / \sqrt{2\Sigma_N}] \right) / 2, \quad (\text{S27})$$

$$\int_0^{\infty} P(N_i^*) dN = \left(1 + \text{erf}[\bar{N} / \sqrt{2\Sigma_N}] \right) / 2, \quad (\text{S28})$$

I obtain Eq. (6) in the main text. The probability distribution of negative abundances depends on a single parameter only, $\eta = \bar{N} / \sqrt{2\Sigma_N}$. Together with the numerical observation that species with negative abundances are those that go extinct under the time-evolution of Eq. (1), this observation leads to the conjecture of a single-parameter scaling of species extinctions.