THERE IS NO UNIVERSAL SEPARABLE BANACH ALGEBRA

TOMASZ KANIA

Dedicated to W. B. Johnson on the occasion of his 80^{th} birthday

Abstract. We prove that no separable Banach algebra is universal for homomorphic embeddings of all separable Banach algebras, whether embeddings are merely bounded or required to be contractive. The same holds in the commutative category.

The proof uses the following scheme. To each bounded bilinear form β we attach a separable test algebra $A(\beta)$ whose multiplication records β. Any homomorphic embedding of $A(\beta)$ into a candidate B forces the linearisation of β to factor through the fixed separable space $B\widehat{\otimes}_\pi B$. Choosing β so that the associated operator fails to factor through $B\widehat{\otimes}_\pi B$, by the theorem of Johnson–Szankowski, yields a contradiction. In the commutative case, we take β symmetric so $A(\beta)$ is commutative.

1. Introduction

A natural question in the theory of Banach algebras concerns the existence of universal objects for certain classes thereof: does there exist a separable Banach algebra that contains (in a suitable sense) all separable Banach algebras from the given class? The answer may depend critically on both the algebraic structure considered and the notion of 'containment' employed.

For commutative C*-algebras, the situation is well understood. Since every compact metrisable space is a quotient of the Cantor set $\Delta := 2^{\mathbb{N}}$, the algebra $C(\Delta)$ of continuous functions on the Cantor set is universal for separable commutative C*-algebras: every such algebra embeds isometrically as a subalgebra of $C(\Delta)$. In contrast, there is *no* separable C*-algebra that is universal for *all* separable C*-algebras (commutative or not), as can be deduced from [4, Proposition 2.6].

For general Banach algebras without additional structure, it is folklore that there exists no separable commutative Banach algebra B such that every separable commutative Banach algebra admits an *isometric* embedding into B. A straightforward proof appears in [5]: if p and q are commuting projections in a Banach algebra, then $\|p - q\| \geqslant 1$, so any set of commuting projections is discrete; since projections in Banach algebras can have arbitrarily large norms, a commutative Banach algebra containing uncountably many commuting projections must be non-separable.

However, the question of universality for *bounded* (or *contractive*) *homomorphic embeddings*, *i.e.*, embeddings that are merely required to have closed range, appears to have been left open in the literature. This is the gap we address in the present work.

We work over the field of complex numbers however the proofs are valid for real Banach algebras too. A *homomorphism* between Banach algebras means a bounded algebra homomorphism. An *embedding* is an injective homomorphism that is a topological isomorphism onto its (closed)

Date: November 12, 2025.

2020 *Mathematics Subject Classification.* 46H10, 46H15, 46M05.

Key words and phrases. Banach algebra, projective tensor product, factorisation of operators, universal Banach algebra, Johnson–Szankowski theorem.

RVO: 67985840.

range. Sometimes we additionally require homomorphisms to be contractive; in the other we do not, however we always stress this out explicitly.

Our argument is unified and uses only the ordinary projective tensor product. Given a bounded bilinear form β, we build a separable test algebra $A(\beta)$ whose multiplication remembers β. Any homomorphic embedding $\phi : A(\beta) \to B$ forces the linearised form $\widetilde{\beta}$ to factor through the fixed separable space $G := B\widehat{\otimes}_\pi B$. Choosing β so that $\widetilde{\beta}$ (equivalently, a composition $S \circ \widetilde{\beta}_U$ defined below) does not factor through G, by Johnson–Szankowski [3], yields a contradiction. In the commutative case we pick β symmetric so that $A(\beta)$ itself is commutative, but we still linearise via $X\widehat{\otimes}_\pi X$ and avoid radicals and symmetric tensor products entirely.

We work over $\mathbb{C}$ however the results remain valid for real Banach algebras too. A *homomorphism* between Banach algebras means a bounded algebra homomorphism. An *embedding* is an injective homomorphism that is a topological isomorphism onto its closed range.

Theorem 1.1. *There are no separable universal objects for the categories of separable Banach algebras, in either the commutative or the general (not-necessarily commutative) setting, for contractive or merely bounded homomorphic embeddings. More precisely,*

(A) *There is no separable* commutative *Banach algebra B such that every separable* commutative *Banach algebra A admits a* contractive *homomorphic embedding $A \hookrightarrow B$.*
(B) *There is no separable* commutative *Banach algebra B such that every separable* commutative *Banach algebra A admits a (bounded) homomorphic embedding $A \hookrightarrow B$.*
(C) *There is no separable Banach algebra B such that every separable Banach algebra A admits a* contractive *homomorphic embedding $A \hookrightarrow B$.*
(D) *There is no separable Banach algebra B such that every separable Banach algebra A admits a (bounded) homomorphic embedding $A \hookrightarrow B$.*

We present a single proof that covers *all four* cases. Contractivity plays no rôle beyond boundedness.

2. Preliminaries

We refer to [6, Chapter 2] for all facts related to the projective tensor product of Banach spaces and to [1, Chapter 2] for Banach-algebraic aspects thereof. For a Banach space X and an integer n, the (n-fold) *projective tensor product* $X^{\widehat{\otimes}_\pi n}$ (or simply $X\widehat{\otimes}_\pi X$ for $n = 2$) is defined as the completion of the algebraic tensor product $X^{\otimes n}$ under the projective norm.

Theorem 2.1 (Universality of the projective tensor product)**.** *Let X, Y be Banach spaces and $n \in \mathbb{N}$. For every continuous n-linear map $M\colon X^n \to Y$ there exists a unique bounded operator $L_M \in \mathcal{L}\big(X^{\widehat{\otimes}_\pi n}, Y\big)$ such that*

$$M = L_M \circ \otimes_\pi^n.$$

Moreover, the correspondence $M \mapsto L_M$ is an isometric linear isomorphism

$$\mathcal{L}(X^n; Y) \;\cong\; \mathcal{L}\big(X^{\widehat{\otimes}_\pi n}, Y\big),$$

so that $\|L_M\| = \|M\|$. Diagrammatically,

$$\begin{tikzcd} & X^n \arrow[dl, "\otimes_\pi^n"'] \arrow[dr, "M"] & \\ X^{\widehat{\otimes}_\pi n} \arrow[rr, "L_M"'] & & Y \end{tikzcd}$$

Lemma 2.2 (Functoriality of $\widehat{\otimes}_\pi$)**.** *Let $u : X \to X_1$ and $v : Y \to Y_1$ be bounded linear maps between Banach spaces. There is a unique bounded operator*

$$u \otimes_\pi v : \; X\widehat{\otimes}_\pi Y \;\longrightarrow\; X_1\widehat{\otimes}_\pi Y_1$$

such that $(u \otimes_\pi v)(x \otimes y) = u(x) \otimes v(y)$ for all $x \in X$, $y \in Y$, and $\|u \otimes_\pi v\| \leqslant \|u\|\,\|v\|$. In particular, if B is a Banach algebra with multiplication $\mu_B : B \times B \to B$ and linearisation $\widetilde{\mu}_B : B\widehat{\otimes}_\pi B \to B$, then for any $u_1, u_2 : X \to B$ we have

$$\widetilde{\mu}_B \circ (u_1 \otimes_\pi u_2) \;=\; L_{\mu_B \circ (u_1 \times u_2)},$$

the (unique) linearisation of the bilinear map $(x, x') \mapsto \mu_B(u_1(x), u_2(x'))$. An analogous statement holds for the symmetric projective tensor product.

Proof. The well-definiteness of the operator $u_1 \otimes_\pi u_2$ is standard consequence of the universality of the projective tensor product and the norm estimate follows from reasonability of the projective crossnorm. For the identity, evaluate both sides on elementary tensors:

$$\big(\widetilde{\mu}_B \circ (u_1 \otimes_\pi u_2)\big)(x \otimes x') = \widetilde{\mu}_B\big(u_1(x) \otimes u_2(x')\big) = \mu_B\big(u_1(x), u_2(x')\big).$$

By uniqueness in Theorem 2.1, this determines the linearisation. □

Lemmas 2.2 and 2.3 provide the basic functorial calculus for linearisations that we shall use repeatedly. The key point is that if two bilinear maps M and N are related by precomposition with linear maps, their linearisations are related by the induced tensor product maps.

Lemma 2.3 (Compatibility of linearisation)**.** *Let $M : X_1 \times Y_1 \to Z$ be a bounded bilinear map and let $u : X \to X_1$, $v : Y \to Y_1$ be bounded linear maps. Define $N := M \circ (u \times v) : X \times Y \to Z$. Then*

$$\widetilde{N} \;=\; \widetilde{M} \circ (u \otimes_\pi v).$$

We include the following well-known result for the sake of completeness in order to identify the radical of the test algebras we are about to construct, even though irrelevant for the proof of the main theorem.

Lemma 2.4. *Let A be a Banach algebra and let $I \lhd A$ be a (two-sided) ideal with $I^n = 0$ for some $n \in \mathbb{N}$. Then $I \subseteq \operatorname{rad}(A)$. Moreover, if A/I is semisimple (equivalently $\operatorname{rad}(A/I) = 0$), then $\operatorname{rad}(A) \subseteq I$.*

Proof. If $a \in I$ then $(1 - a)(1 + a + \cdots + a^{n-1}) = 1$, so $1 - a$ is invertible and $a \in \operatorname{rad}(A)$. The quotient statement is standard; see *e.g.*, [1, §3.3]. □

Proposition 2.5. *Let X, Y be Banach spaces, and let $\beta : X \times X \to Y$ be a bounded bilinear map. Define a multiplication on the ℓ_1-direct sum*

$$A(\beta) := \mathbb{C} \oplus_1 X \oplus_1 Y$$

by

$$(\alpha, x, y) \cdot (\alpha', x', y') = \big(\alpha\alpha',\ \alpha x' + \alpha' x,\ \alpha y' + \alpha' y + \beta(x, x')\big).$$

Then $A(\beta)$ is a unital Banach algebra (after an equivalent renorming if desired). It is commutative whenever β is symmetric, and in all cases

$$\operatorname{rad}\big(A(\beta)\big) \;=\; X \oplus Y.$$

Proof. We equip $A(\beta)$ with the norm $\|(\alpha,x,y)\| = |\alpha| + \|x\| + \|y\|$.

First, we verify that the multiplication is bounded. Take $a = (\alpha,x,y)$ and $b = (\alpha',x',y')$ in $A(\beta)$. Then

$$\begin{aligned}\|ab\| &= |\alpha\alpha'| + \|\alpha x' + \alpha' x\| + \|\alpha y' + \alpha' y + \beta(x,x')\| \\ &\leqslant |\alpha\alpha'| + |\alpha|\,\|x'\| + |\alpha'|\,\|x\| + |\alpha|\,\|y'\| + |\alpha'|\,\|y\| + \|\beta\|\,\|x\|\,\|x'\|.\end{aligned}$$

Now observe that $|\alpha\alpha'| \leqslant |\alpha|\,\|b\|$ and $|\alpha|\,\|x'\| \leqslant \|a\|\,\|b\|$; similarly for the other terms. A routine calculation shows that

$$\|ab\| \leqslant (5 + \|\beta\|)\,\|a\|\,\|b\|.$$

If desired, one may renorm $A(\beta)$ by scaling the given norm by the constant factor $5 + \|\beta\|$ to make the multiplication contractive.

The element $e = (1,0,0)$ is clearly a two-sided identity for the multiplication.

We now establish associativity. Let $a = (\alpha,x,y)$, $b = (\alpha',x',y')$, and $c = (\alpha'',x'',y'')$. We have

$$ab = \big(\alpha\alpha',\, \alpha x' + \alpha' x,\, \alpha y' + \alpha' y + \beta(x,x')\big),$$

whence

$$(ab)c = \Big((\alpha\alpha')\alpha'',\, (\alpha\alpha')x'' + \alpha''(\alpha x' + \alpha' x),\, S\Big),$$

where

$$\begin{aligned}S &= (\alpha\alpha')y'' + \alpha''(\alpha y' + \alpha' y + \beta(x,x')) + \beta(\alpha x' + \alpha' x, x'') \\ &= \alpha\alpha' y'' + \alpha\alpha'' y' + \alpha'\alpha'' y + \alpha''\beta(x,x') + \alpha\beta(x',x'') + \alpha'\beta(x,x''),\end{aligned}$$

by the bilinearity of β. On the other hand,

$$bc = \big(\alpha'\alpha'',\, \alpha' x'' + \alpha'' x',\, \alpha' y'' + \alpha'' y' + \beta(x',x'')\big),$$

so that

$$a(bc) = \Big(\alpha(\alpha'\alpha''),\, \alpha(\alpha' x'' + \alpha'' x') + (\alpha'\alpha'')x,\, T\Big),$$

where

$$\begin{aligned}T &= \alpha(\alpha' y'' + \alpha'' y' + \beta(x',x'')) + (\alpha'\alpha'')y + \beta(x, \alpha' x'' + \alpha'' x') \\ &= \alpha\alpha' y'' + \alpha\alpha'' y' + \alpha\beta(x',x'') + \alpha'\alpha'' y + \alpha'\beta(x,x'') + \alpha''\beta(x,x').\end{aligned}$$

Comparing coordinates, we see that the first two coordinates of $(ab)c$ and $a(bc)$ coincide immediately, while the third coordinates S and T are manifestly equal (they consist of the same six terms). Thus the multiplication is associative. (We remark that only the bilinearity of β is used here; symmetry plays no rôle.)

If β is symmetric, commutativity follows. Indeed,

$$ab = \big(\alpha\alpha',\, \alpha x' + \alpha' x,\, \alpha y' + \alpha' y + \beta(x,x')\big) = \big(\alpha'\alpha,\, \alpha' x + \alpha x',\, \alpha' y + \alpha y' + \beta(x',x)\big) = ba.$$

Next, we determine the radical of $A(\beta)$. Set $J = X \oplus Y = \{(0,x,y) : x \in X,\, y \in Y\}$. It is straightforward to check that J is an ideal of $A(\beta)$. Moreover, J is nilpotent: for $(0,x,y) \in J$, we have

$$(0,x,y)^2 = (0,0,\beta(x,x)) \quad \text{and} \quad (0,x,y)^3 = 0.$$

Since $J^3 = \{0\}$, Lemma 2.4 gives $J \subseteq \operatorname{rad}(A(\beta))$. The quotient $A(\beta)/J \cong \mathbb{C}$ is semi-simple, so again by Lemma 2.4 we have $\operatorname{rad}(A(\beta)) \subseteq J$. Hence

$$\operatorname{rad}\big(A(\beta)\big) = X \oplus Y\ .$$

This argument uses only that J is nilpotent, so it holds whether or not β is symmetric. $\square$

The key factorisation identity holds without assuming B is commutative.

Lemma 2.6. *Let B be a (possibly noncommutative) Banach algebra, and let X, Y be Banach spaces. For a bounded bilinear map $\beta : X \times X \to Y$ let $\widetilde{\beta} : X \widehat{\otimes}_\pi X \to Y$ denote its linearisation. If $\phi : A(\beta) \to B$ is a bounded homomorphism and we set*

$$i := \phi|_X : X \to B, \qquad j := \phi|_Y : Y \to B,$$

then

$$j \circ \widetilde{\beta} \;=\; \widetilde{\mu}_B \circ (i \otimes_\pi i) \quad : \quad X \widehat{\otimes}_\pi X \longrightarrow B,$$

where $\mu_B : B \times B \to B$ is multiplication and $\widetilde{\mu}_B : B \widehat{\otimes}_\pi B \to B$ its linearisation. If ϕ is an embedding, then j is injective and bounded below.

Proof. In $A(\beta)$, $(0, x, 0)(0, x', 0) = (0, 0, \beta(x, x'))$; applying ϕ gives $\mu_B(i(x), i(x')) = j(\beta(x, x'))$. Linearising via Theorem 2.1 and Lemma 2.2 yields the identity. If ϕ is an embedding with inverse ψ on its range, then $\|y\| = \|\psi(\phi(0, 0, y))\| \leqslant \|\psi\| \, \|j(y)\|$, so j is bounded below. ☐

Finally, we isolate the lifting step in a form that uses a *symmetric* β_U but always linearises through $X \widehat{\otimes}_\pi X$.

Lemma 2.7. *Let G be a separable Banach space, and let U, Y be Banach spaces. Suppose $S : U \to Y$ is a compact operator which does not factor through G. Set $X := U \oplus_1 \mathbb{C}$ and define the symmetric bilinear map*

$$\beta_U\big((u, \lambda), (u', \lambda')\big) := \lambda u' + \lambda' u.$$

Let $E := X \widehat{\otimes}_\pi X$ and let $\widetilde{\beta}_U : E \to U$ be the (unsymmetric) linearisation of β_U. Then $\widetilde{\beta}_U$ is surjective and admits a bounded right inverse $\sigma : U \to E$ given by $\sigma(u) := (u, 0) \otimes (0, 1)$. Consequently, for $T_0 := S \circ \widetilde{\beta}_U : E \to Y$ we have:

- *T_0 is compact; and*
- *T_0 does not factor through G (otherwise $S = T_0 \circ \sigma$ would factor through G).*

Proof. For $u \in U$, $\widetilde{\beta}_U\big((u, 0) \otimes (0, 1)\big) = \beta_U\big((u, 0), (0, 1)\big) = u$. Thus $\widetilde{\beta}_U$ is onto and σ is a right inverse with $\|\sigma(u)\|_\pi \leqslant \|(u, 0)\| \, \|(0, 1)\| = \|u\|$. The two bullet points follow immediately. ☐

3. Proof of the main theorem

Proof of Theorem 1.1. We present a unified proof that covers all four cases, (A)-(D). The argument proceeds by contradiction.

Let B be an arbitrary separable Banach algebra, which we suppose to be a universal object in one of the clauses described. If we are addressing the commutative categories (A) or (B), we assume B is commutative; otherwise, no such assumption is made. In all cases, set $G := B \widehat{\otimes}_\pi B$, which is a separable Banach space as B is. By the Johnson–Szankowski theorem [3], there exist separable Banach spaces U, Y and a compact operator $S : U \to Y$ which does not factor through G.

Our strategy is to construct a single separable test algebra A that cannot be embedded into B, regardless of whether B is commutative. To this end, form $X := U \oplus_1 \mathbb{C}$ and define the *symmetric* bilinear map

$$\beta_U\big((u, \lambda), (u', \lambda')\big) := \lambda u' + \lambda' u.$$

Let $E := X \widehat{\otimes}_\pi X$ and let $\widetilde{\beta}_U : E \to U$ be its (unsymmetric) linearisation. By Lemma 2.7, $\widetilde{\beta}_U$ is surjective with a bounded right inverse $\sigma : U \to E$, and the composite operator $T_0 := S \circ \widetilde{\beta}_U : E \to Y$ does not factor through G.

Now, define the bilinear map $\beta := S \circ \beta_U : X \times X \to Y$. Since β_U is symmetric, so is β. We then construct the test algebra $A := A(\beta) = \mathbb{C} \oplus_1 X \oplus_1 Y$ using the multiplication from Proposition 2.5. As β is symmetric, A is a *commutative* separable Banach algebra.

Suppose, towards a contradiction, that there exists a bounded homomorphic embedding $\phi : A \hookrightarrow B$. (Note that if a contractive embedding exists, it is also a bounded one, so this assumption covers all cases.) Write $i := \phi|_X$ and $j := \phi|_Y$. By Lemma 2.6, the fact that ϕ is a homomorphism forces the following identity on the linearisations:

$$j \circ \widetilde{\beta} \;=\; \widetilde{\mu}_B \circ (i \otimes_\pi i) \quad : \quad E \longrightarrow B.$$

Recalling that $\widetilde{\beta} = S \circ \widetilde{\beta}_U = T_0$, this means $j \circ T_0$ factors through G. Since $\widetilde{\beta}_U$ has a bounded right inverse σ, we can compose with it to find that

$$j \circ S \;=\; (j \circ T_0) \circ \sigma$$

also factors through G. This factorisation is illustrated below.

$$\begin{tikzcd} & U \arrow[dl, "\widetilde{\mu}_B\circ(i\otimes_\pi i)\circ\sigma"'] \arrow[dr, "j\circ S"] & \\ G \arrow[rr, "\text{some bounded operator}"'] & & B \end{tikzcd}$$

As ϕ is an embedding, the map j is a topological isomorphism onto its range, so it has a bounded inverse $(j|_{\operatorname{ran} j})^{-1} : \operatorname{ran} j \to Y$. Pre-composing the factorisation of $j \circ S$ with this inverse shows that S itself must factor through G. This contradicts our choice of S.

The contradiction implies that no such embedding $\phi : A \to B$ can exist.

- In cases (A) and (B), we assumed B was a universal separable *commutative* Banach algebra. We have constructed a separable commutative Banach algebra A that does not embed into B, a contradiction.
- In cases (C) and (D), we assumed B was a universal separable Banach algebra for *all* separable Banach algebras. We have constructed a specific separable (and, as it happens, commutative) Banach algebra A that does not embed into B, which again is a contradiction.

Since our choice of B was arbitrary, no such universal object exists in any of the specified categories. The argument relied only on the boundedness of the embedding, so it holds equally for contractive embeddings. This completes the proof of (A)–(D). □

Remark 3.1. The compactness of the operator $S : U \to Y$ is not essential for the argument. It is used only so that we may appeal directly to the Johnson–Szankowski theorem in its classical form, which states that no separable Banach space is universal for the factorisation of compact operators. The proof requires merely that S be a bounded operator between separable Banach spaces which does not belong to the operator ideal of maps factoring through $G := R\widehat{\otimes}_{\pi,\mathrm{s}}R$. In particular, one could take $S = \mathrm{id}_U$ for a separable Banach space U which does not embed complementably into G; this again relies on the earlier result of Johnson and Szankowski [2], asserting that no separable Banach space contains complemented copies of all separable Banach spaces. We retain the compact case for definiteness.

Acknowledgements. Funding received from NCN Sonata-Bis 13 (2023/50/E/ST1/00067) is acknowledged with thanks.

References

[1] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series, vol. 24, The Clarendon Press, Oxford University Press, New York, 2000.

[2] W. B. Johnson and A. Szankowski, *Complementably universal Banach spaces*, Studia Math. **58** (1976), 91–97.

[3] W. B. Johnson and A. Szankowski, *Complementably universal Banach spaces, II*, J. Funct. Anal. **257** (2009), 3395–3408.

[4] M. Junge and G. Pisier, *Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$*, Geom. Funct. Anal. **5** (1995), 329–363.

[5] T. Kania, *There is no separable commutative Banach algebra that contains isometric copies of all separable commutative Banach algebras*, Math. Stack Exchange, `https://math.stackexchange.com/a/1381066/17929` (posted August 23, 2012).

[6] R. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.