

Automorphism-weighted ensembles from TQFT gravity

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ABSTRACT: We study the recent proposal of [1] which poses a precise holographic duality between a 3d TQFT summed over all topologies and a unitary ensemble of boundary 2d CFTs. In that proposal, the sum over topologies is obtained via genus reduction from topologies with a large genus boundary Riemann surface, while the boundary ensemble is given by all CFTs described by Lagrangian condensations of the bulk TQFT. The main result of this work is to show that each member of this ensemble is weighted by a symmetry factor given by the invertible symmetry group of its categorical symmetry relative to the bulk TQFT as its SymTFT. This is the natural — uniform up to isomorphism — measure on the groupoid of Lagrangian algebras that describe the boundary theories. We also write the sum over topologies more explicitly in terms of equivalence classes of Heegaard splittings of 3-manifolds with a given boundary and comment on their weights. The holographic duality in this framework can then be viewed as a generalization of the Siegel-Weil formula. We discuss the implications of the main result for non-compact TQFTs. In particular, for the Virasoro case, this implies an ensemble of all CFTs at a given central charge in which CFTs are weighted by their full invertible symmetry. Finally, we show how this TQFT gravity framework gives a natural construction of the baby universe Hilbert space.

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1 Introduction and summary of results

The quest for understanding the holographic duality of 3d gravity in asymptotically anti de-Sitter space (AdS) faces some interesting puzzles. First, in the standard AdS/CFT, we expect gravity to be dual to a single CFT on the boundary. However, in multi-boundary scenarios, contributions from Euclidean wormholes spoil the factorization of the partition function. One resolution of this puzzle is that gravity is dual to an ensemble rather than a single theory. This resolution seems to be backed up by the results of JT gravity [2]. The ensemble picture is an illustration of the early ideas considered by Coleman [3] and by Giddings and Strominger [4, 5] according to which the contribution of euclidean wormholes leads to disorder averaging. These ideas were recently recast in the context of asymptotically AdS boundaries by Marolf and Maxfield [6]. If 3d gravity is dual to an ensemble, the question remains how should we average over the space of CFTs. One can make progress by considering a random ensemble of CFT data, for which there are many attempts as in [7–12], but this still does not exactly tell us what kind of average over bona fide CFTs this is.

The other puzzle is that evaluating the path integral of semiclassical AdS₃ gravity with a single boundary by summing over saddles leads to some pathological results. The resulting density of states is continuous and negative in some part of the spectrum [13, 14]. The continuous spectrum is not really a problem in an ensemble interpretation, but the negativity problem is a serious blow to any possibility for a unitary interpretation. The negativity problem has been shown to be cured by including additional contributions to the classical saddles, such as off-shell contributions like Seifert manifolds [15] or conical defects [16]. This seems to tell us that even in the semiclassical limit one should be careful not to dismiss the sum over all 3-manifolds.

One way to tackle these issues is through the TQFT approach to gravity, which is now more precisely formulated in terms of Virasoro TQFT (VTQFT) [17], also known as Teichmüller TQFT [18]. Schematically, we have

$$\mathcal{Z}_{\text{grav}} = \sum_{\text{topologies}} \mathcal{Z}_{\text{VTQFT}}, \quad (1.1)$$

where we leave implicit the possibility of a nontrivial measure in this sum. Working with VTQFT has proved to be very fruitful [17, 19–22] but there are obviously certain limitations due to divergences, after all it is a noncompact TQFT, which would need some kind of regularization, see for example the recent attempts of [23–25]. One can instead try to answer these questions in a more tamed TQFT hoping eventually to learn something about the Virasoro case. There are many toy models of ensemble averaging in the literature, either directly or indirectly involving a bulk TQFT setup [26–32].

However, in most of these examples, the sum over topologies involves only handlebodies, which in many nonAbelian TQFT examples [33, 34] suffers from a negativity problem similar in spirit to pure 3d gravity, or attempts to sum over all manifolds in a setting involving Abelian TQFTs [35, 36] which are immune to the negativity problem and are limited as toy models of VTQFT.

To understand the holographic duality properly in a general TQFT setting, we need a way to construct the sum over all topologies and also figure out what kind of boundary ensemble we should get, i.e. what are the CFTs? and what are their weights? The picture proposed in [37], motivated by a connection between codes and CFTs, is that the sum over all topologies should be dual an ensemble of all Lagrangian condensations of the TQFT. A Lagrangian condensation here means a maximal gauging of a (generally non-invertible) 1-form symmetry of the TQFT reducing it to a trivial TQFT. These are related to states in the TQFT Hilbert space that describe topological boundary conditions, which we will denote as $|Z_\alpha\rangle$. So our general expectation now is of the form

$$\sum_{\text{topologies}} \mathbf{Z}_{\mathcal{T}} = \sum_{\alpha} w_{\alpha} Z_{\alpha}, \quad w_{\alpha} \geq 0, \quad (1.2)$$

where the left-hand-side (LHS) describes the “*TQFT gravity*” partition function, which is a sum over the partition function $\mathbf{Z}_{\mathcal{T}}$ of the TQFT $\mathcal{T}$ placed on different topologies, while the RHS describes an ensemble of CFTs whose partition functions are denoted by Z_{α} . This should be purely understood as a statement in the Hilbert space of the TQFT where the LHS is a sum over states prepared by putting the TQFT $\mathcal{T}$ on a particular topology, and the RHS is a sum over states corresponding to a fixed topology of a handlebody with insertion of line defects. The statement in (1.2) is not completely trivial since, in general, the states $|Z_{\alpha}\rangle$ do not form a complete basis in the space of modular invariant states.

This picture was made more precise by Dymarsky and Shapere in [1] with the motivation that the boundary ensemble should be fixed for all genera of the boundary Riemann surface, while the sum over topologies at a given genus should be consistent with reduction from its counterpart at a higher genus. With this idea one can bootstrap the ensemble by studying what happens in the large genus limit. In that limit, the duality simplifies drastically: the sum over topologies becomes just a sum over handlebodies, and the task of evaluating the weights of the boundary ensemble becomes much simpler as the states $|Z_{\alpha}\rangle$ can be shown to become orthogonal in that limit. For Abelian TQFTs, as was shown explicitly in [1], all Lagrangian condensations are weighted equally, but in a general TQFT, the relative weights will have the form

$$\frac{w_{\alpha}}{w_{\alpha'}} = \lim_{g \rightarrow \infty} \frac{\langle Z_{\alpha'} | Z_{\alpha'} \rangle}{\langle Z_{\alpha} | Z_{\alpha} \rangle}, \quad (1.3)$$

where the norms $\langle Z_\alpha | Z_\alpha \rangle$ of the states corresponding to Lagrangian condensations are generally different, leading to nontrivial weights. These weights can be calculated in some simple nonAbelian examples (see the End-Matter section of [37]) but the general structure of how to compute these weights or what they mean (if any) was still missing.

In this work, we calculate these weights explicitly for general 3d TQFTs based on semi-simple modular tensor categories and give them a physical interpretation in terms of symmetries. We also briefly shed some light on the sum over topologies, writing explicitly in terms of equivalences of Heegaard splittings, and also make a connection to the construction of the closed (baby) universe Hilbert space. Below we summarize the main result of the paper which is related to the weights of the ensemble.

1.1 Summary of the main result

The main result of this paper is that a 3d TQFT summed over all topologies as proposed in [1] is dual to an ensemble of boundary theories where each member of the ensemble is (inversely) weighted by a symmetry factor. The bulk TQFT naturally acts as a symmetry topological field theory (SymTFT) for each member, and so each member α has a certain categorical symmetry $\mathcal{C}^{(\alpha)}$ relative to the bulk TQFT. What we will show is that the weights are inversely related to the order of the group of invertible symmetries of this categorical symmetry, namely (up to an overall normalization)

$$w_\alpha = \frac{1}{|\text{Inv}(\mathcal{C}^{(\alpha)})|} \quad (1.4)$$

where $\text{Inv}(\mathcal{C})$ denotes the group of isomorphism classes of invertible objects in the category $\mathcal{C}$. In a given boundary theory, these are the invertible TDLs that commute with the vertex algebra associated with the bulk TQFT.

The key point in deriving (1.4) is to consider $\langle Z_\alpha | Z_\alpha \rangle$ as a partition function of a 2d TQFT with a categorical symmetry $\mathcal{C}^{(\alpha)}$, and in the large genus limit this partition function will effectively count the number of invertible topological line defects (TDLs) of $\mathcal{C}^{(\alpha)}$.

The weights in (1.4) are not only intrinsic to the TQFT holographic ensemble duality, but they are also — as we will argue — the natural weights for defining a uniform average up-to-isomorphism. Each member of the ensemble, being a Lagrangian condensation, is algebraically described by what is known as a *Lagrangian algebra* $\mathcal{A}$. We will show that our resulting ensemble average can be written as

$$\langle Z \rangle = \left(\sum_{\mathcal{A}} \frac{1}{|\text{Aut}(\mathcal{A})|} \right)^{-1} \sum_{\mathcal{A}} \frac{1}{|\text{Aut}(\mathcal{A})|} Z_{\mathcal{A}} \quad (1.5)$$

where $\text{Aut}(\mathcal{A})$ denotes the automorphism group of $\mathcal{A}$, which will be defined precisely in the main text. This is the natural way to average over the groupoid of Lagrangian

algebras of a given modular tensor category. The normalization sum $\sum_{\mathcal{A}} \frac{1}{|\text{Aut}(\mathcal{A})|}$ in (1.5), which defines the counting measure in the groupoid sense, is known as the groupoid cardinality [38]. It is the analog of the Smith–Minkowski–Siegel “mass” formula¹ for lattices [40–42], and as we will show later it will be given by the norm of the Hartle–Hawking (HH) state, which is the partition function of a closed universe

$$\langle \text{HH} | \text{HH} \rangle = \sum_{\mathcal{A}} \frac{1}{|\text{Aut}(\mathcal{A})|}. \quad (1.6)$$

The holographic duality we present then can be viewed as a generalization of the Siegel–Weil formula [43–46]. The Siegel–Weil formula relates the automorphism-weighted average over lattice Riemann theta functions (related to the partition functions of boundary free bosons) to an Eisenstein series (related to the sum over handlebodies), and explains the holographic duality of the ensemble average of Narain theories [26, 27]. In our case, we get an automorphism-weighted average over partition functions of Lagrangian algebras equivalent to a sum over all 3-manifolds instead of just handlebodies.

1.2 Outline

The paper is organized as follows. In section 2, we provide some preliminaries for 3d TQFTs, anyon condensation and SymTFT. This will serve as a brief review of the main ingredients of our setup as well as establishing the notation and conventions for later sections. In section 3, we first review the main derivation of TQFT gravity following [1] and elaborate a bit more on the sum over topologies, writing it explicitly in terms of equivalences of Heegaard splittings. We then give some arguments that the mapping class group invariant subspace of the infinite genus Hilbert space has a natural interpretation as the baby universe Hilbert space. Section 4 provides the main derivation for the weights of the ensemble in terms of symmetries and the relation to automorphisms of Lagrangian algebra. In section 5, we present some simple examples to illustrate the main result of getting the weights from symmetries. In section 6, we discuss the implications of the TQFT gravity framework for noncompact TQFTs. Finally, in section 7, we end with some discussion on the interpretation of this result in light of the principle of maximum ignorance as well as possible future directions.

2 Preliminaries

2.1 3d TQFTs

3d TQFTs are closely connected to 2d rational CFTs (RCFTs) and the concept of chiral algebras which more formally known as Vertex Operator Algebras (VOAs) [47–49]. In

¹See also [39] for upcoming work related to the mass formula and topological boundary conditions.

the TQFT/RCFT correspondence [50–52], the simple line defects (anyons) of the TQFT are in one to one correspondence with rational primary operators of the RCFT which are isomorphism classes of irreducible representations of some VOA. The algebraic structure of these anyons is captured by a unitary modular tensor category (MTC). Objects of this category are representations of the VOA and morphisms between objects are intertwiners. We refer the reader to appendix A for a brief review of the MTC structure and the related notation used in this section.

In what follows, we will denote the 3d TQFT as well as its category by $\mathcal{T}$ and its VOA as $\mathcal{V}_{\mathcal{T}}$, so we have $\mathcal{T} = \text{Rep}(\mathcal{V}_{\mathcal{T}})$.² While conventionally $\mathcal{V}_{\mathcal{T}}$ is taken as the chiral algebra of an RCFT, we will work in general with $\mathcal{V}_{\mathcal{T}}$ as the full non-chiral³ algebra of the RCFT. So our analysis is not only restricted to the standard case of $\mathcal{V} = \mathcal{V}_L \times \bar{\mathcal{V}}_L$, but also heterotic cases $\mathcal{V} = \mathcal{V}_L \times \bar{\mathcal{V}}_R$ and non-chiral algebra extensions. However, since our end goal is to compare to $\text{Vir} \times \overline{\text{Vir}}$, one should always keep in mind the standard case $\mathcal{V} = \mathcal{V}_L \times \bar{\mathcal{V}}_L$ as the main example throughout the text.

Hilbert space. The anyons form a natural basis in the Hilbert space of the torus. We can prepare such basis by inserting the corresponding anyon around the nonshrinkable cycle of the solid torus. The characters of the modules of the VOA $\mathcal{V}_{\mathcal{T}}$ can be viewed as wavefunctions corresponding to these states

$$\langle \tau | a \rangle \equiv \chi_a(\tau) \quad (2.1)$$

where τ is the modulus of the torus. The state $|\tau\rangle$ defines a gapless boundary condition similar to that used in the Chern Simons/Chiral Wess-Zumino-Witten (CS/WZW) correspondence. We will define $|\tau\rangle$ via (2.1) and call it a “holomorphic” boundary condition even though in the nonchiral case it will depend on both τ and $\bar{\tau}$.

One can do something similar on higher genus. Let us denote the Hilbert space of $\mathcal{T}$ on Riemann surface Σ_g as $\mathcal{H}_g$ where g denotes the genus of the Riemann surface. Any state in $\mathcal{H}_g$ can be prepared by appropriate line insertions inside a handlebody $S\Sigma_g$. A basis for $\mathcal{H}_{\Sigma_g}$ can be prepared by insertion of a spine network of lines labeled by the simple anyons (subject to fusion rules at the junctions) as shown in figure 1.

For convenience we will denote such basis states by the shorthand

$$|\vec{a}, \vec{b}, \vec{c}; \vec{\mu}\rangle \equiv |a_1 \dots a_g; b_{12} \dots b_{g-1, g}; c_2 \dots c_{g-1}; \mu_1 \dots \mu_{2g-2}\rangle \quad (2.2)$$

²Note that to define the TQFT from an MTC, one should specify the chiral central charge c_- since the category is only sensitive to $c_- = 0 \pmod{8}$.

³In this context we say the VOA is nonchiral if it contains $\text{Vir} \times \overline{\text{Vir}}$; see [53–55] for definitions of nonchiral VOAs

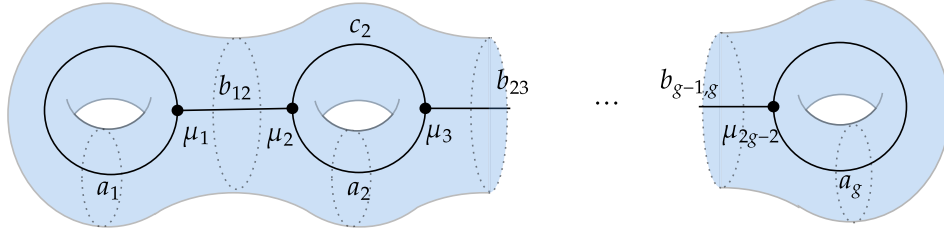


Figure 1. A choice of basis states in $\mathcal{H}_g$ corresponding to inserting the shown network of lines inside in a handlebody $S\Sigma_g$.

Using this basis, one can prepare wavefunctions corresponding to genus g conformal blocks⁴ of $\mathcal{V}_{\mathcal{T}}$ by doing the path integral with the corresponding insertion and using gapless boundary conditions analogous to our genus 1 case. We will denote such wavefunctions as

$$\langle \Omega | \vec{a}, \vec{b}, \vec{c}, \vec{\mu} \rangle \equiv \chi_{\vec{a}, \vec{b}, \vec{c}, \vec{\mu}}(\Omega) \quad (2.3)$$

where Ω is the modulus matrix of Σ_g . We note again that when $\mathcal{T}$ is nonchiral, e.g. $\mathcal{T} = \mathcal{T}_L \times \bar{\mathcal{T}}_L$, the blocks (2.3) will be nonholomorphic in Ω .

The dimension of $\mathcal{H}_g$ is given by

$$\dim \mathcal{H}_g = \sum_{\vec{a}, \vec{b}, \vec{c} \in \mathcal{T}} N_{a_1 a_1}^{b_{12}} N_{a_2 c_2}^{b_{12}} N_{a_2 c_2}^{b_{23}} \dots N_{a_{g-1} c_{g-1}}^{b_{g-1,g}} N_{a_g a_g}^{b_{g-1,g}}. \quad (2.4)$$

which can be simplified, using the Verlinde formula to diagonalize the fusion rules, into

$$\dim \mathcal{H}_g = \mathcal{D}_{\mathcal{T}}^{2g-2} \sum_a d_a^{2-2g}, \quad (2.5)$$

where d_a denotes the quantum dimensions the anyon a , and $\mathcal{D}_{\mathcal{T}} = \sqrt{\sum_a d_a^2}$ is the total quantum dimension of $\mathcal{T}$.

$\mathcal{H}_g$ furnishes a representation for the mapping class group (MCG) of Σ_g . The generators can be explicitly constructed from F, R matrices and the modular data as shown for example in [57, 58].

A note on basis normalization. In $\mathcal{H}_g$ we chose an orthogonal basis given by figure 1. For convenience, we will work with a normalized basis, i.e. we will normalize

⁴These can be defined from sewing torus n-point blocks [56].

the vacuum insertion

$$\langle 0_g | 0_g \rangle = 1, \quad (2.6)$$

such that the vacuum conformal block from the CFT perspective is normalized to unity (particularly on S^2). We want to illustrate the meaning of this choice from the TQFT perspective. The vacuum norm has a geometric interpretation as the partition function $\mathbf{Z}$ of the TQFT on a connected sum of $S^2 \times S^1$

$$\langle 0_g | 0_g \rangle = \mathbf{Z}_{\mathcal{T}}(\#^g S^2 \times S^1). \quad (2.7)$$

However, on a connected sum $M_1 \# M_2$, the TQFT partition function is given by [50, 57]

$$\mathbf{Z}_{\mathcal{T}}(M_1 \# M_2) = \frac{\mathbf{Z}_{\mathcal{T}}(M_1) \mathbf{Z}_{\mathcal{T}}(M_2)}{\mathbf{Z}_{\mathcal{T}}(S^3)}, \quad (2.8)$$

which leads to

$$\langle 0_g | 0_g \rangle = \mathcal{D}_{\mathcal{T}}^{g-1}, \quad (2.9)$$

where we used $\mathbf{Z}_{\mathcal{T}}(S^3) = \frac{1}{\mathcal{D}_{\mathcal{T}}}$ via modular S-matrix, and $\mathbf{Z}_{\mathcal{T}}(S^2 \times S^1) = \text{Tr}_{\mathcal{H}_{S^2}}(\mathbf{I}) = 1$.

The choice of normalization affects the values assigned to connected sums of closed manifolds versus disjointed manifolds. Choosing $\langle 0_g | 0_g \rangle = 1$ amounts to treating the two cases in the same way, while choosing $\langle 0_g | 0_g \rangle = \mathcal{D}_{\mathcal{T}}^{g-1}$ preserves the topological nature of arbitrary adding connected sums of S^3 without changing the assigned value of the manifold. The latter normalization is more natural when we do the sum over all manifolds.

2.2 Algebra objects and anyon condensation

We want to construct states in $\mathcal{H}_g$ that correspond to CFT partition functions when projected onto $\langle \Omega |$. In other words, this would be a linear combination of conformal blocks that are MCG invariant and satisfy certain physicality constraints (e.g. unique vacuum, positivity of spectrum and consistency with sewing/factorization). To get a true CFT, we should be able to construct such state for arbitrary genus. Such construction can be achieved by anyon condensation (or generalized gauging), specifically Lagrangian anyon condensation which reduces the TQFT to a trivial TQFT with the boundary theory being the CFT. The condensation amounts to inserting a specific network of line defects in the bulk manifold and hence condensed/gauged theory can be embedded in the Hilbert space of the parent TQFT. Below we briefly describe the algebraic construction of anyon condensation. We refer the reader to [59] for further mathematical details, see also [60] for a review of anyon condensation in the context of holography.

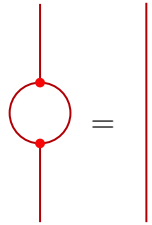
Condensable algebra. We want to construct an object in $\mathcal{T}$ that acts as the vacuum object in the condensed phase. This means we want an object that can freely fuse and braid with itself into itself in an arbitrary way. This can be reduced to the following diagrammatic conditions:

Fusion and splitting



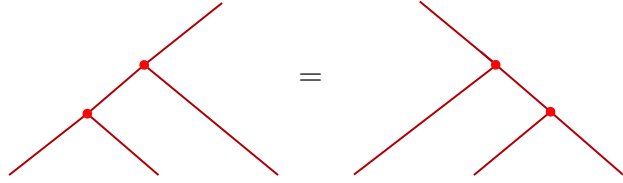
$$(2.10)$$

Bubble removal



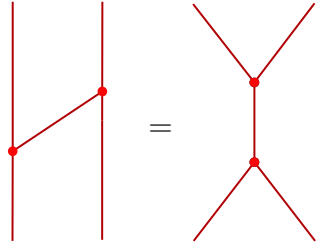
$$(2.11)$$

Associativity



$$(2.12)$$

Invariance under F -moves for fusion and splitting (crossing symmetry)



$$(2.13)$$

Invariance under braiding (commutativity)



$$(2.14)$$

In a unitary MTC, there is no simple object that satisfy these relations under fusion except the trivial/vacuum anyon; however, if we consider non-simple objects we will

have more freedom to consider junctions that could satisfy these conditions. Hence, we consider a non-simple object $\mathcal{A}$ with some junction morphisms $m \in \text{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$ and $m^\vee \in \text{Hom}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ that allow us to achieve such diagrammatics. We also require that $\mathcal{A}$ can be mapped to vacuum of the parent phase when we interface the two phases, so we must have a unit morphism $\eta \in \text{Hom}(0, \mathcal{A})$ and a co-unit $\eta^\vee \in \text{Hom}(\mathcal{A}, 0)$, shown graphically as

$$\begin{array}{c} \text{---} \circ \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \circ \\ | \\ \text{---} \end{array} \quad (2.15)$$

The unit and co-unit should be compatible with m and $m^\vee$, and hence we should also have the following diagrams

$$\begin{array}{c} \text{---} \\ | \\ \circ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \circ \text{---} \end{array} \quad (2.16)$$

Applying η and $\eta^\vee$ to the bubble removal (2.11) leads to the normalization

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} = \dim \mathcal{A} \quad (2.17)$$

where $\dim \mathcal{A}$ is the quantum dimension of the object $\mathcal{A}$. If we write $\mathcal{A}$ in terms of simple anyons as

$$\mathcal{A} = \bigoplus_a n_a a, \quad n_a \in \mathbb{Z}_{\geq 0}, \quad (2.18)$$

then

$$\dim \mathcal{A} = \sum_a n_a d_a \quad (2.19)$$

Finally, we require the uniqueness of the vacuum anyon when we condense, so we require $n_0 = 1$.

The above diagrams define what is known as a commutative connected special symmetric Frobenius algebra, and hence $\mathcal{A}$ is called an algebra object.

Given that $\mathcal{A}$ is a direct sum of simple anyons, we have morphisms to these simple objects denoted as

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \alpha \quad \alpha \in \text{Hom}(a, \mathcal{A}) \quad (2.20)$$

In these components, the product morphism is given by

$$= m_{a\alpha, c\beta}^{c\gamma; \mu} \quad (2.21)$$

and the diagrammatic conditions on $\mathcal{A}$ give us consistency conditions for the components $m_{a\alpha, a\beta}^{b\gamma; \mu}$ and $\check{m}_{a\alpha, a\beta}^{b\gamma; \mu}$. In a unitary theory, we can choose a gauge where $m^\vee = m^\dagger$ which is usually called unitary gauge.

Lagrangian condensation. To obtain a trivial TQFT after the condensation, the algebra object must have a maximal quantum dimension, i.e. $\dim \mathcal{A} = \mathcal{D}_{\mathcal{T}}$. Such an algebra is called a Lagrangian algebra. To implement the condensation in a manifold $\mathcal{M}$, we insert a fine mesh of $\mathcal{A}$ dual to the triangulation of $\mathcal{M}$. The diagrammatic properties satisfied by $\mathcal{A}$ ensures that this construction is independent of the choice of triangulation. On a handlebody Σ_g , this mesh can be reduced to insertion of the spine diagram labeled by $\mathcal{A}$ as shown in figure 2 which prepares for us a state in $\mathcal{H}_g$. In our chosen basis of anyons in $\mathcal{H}_g$, the vacuum state of the condensed phase is written as

$$|Z_{\mathcal{A}}\rangle = \sum_{\vec{\alpha}, \vec{\beta}, \vec{\gamma}} \sum_{\vec{a}, \vec{b}, \vec{c} \in \mathcal{A}} \sum_{\vec{\mu}} m_{a_1 \alpha_1, a_1 \check{\beta}_1}^{b_{12} \gamma_1; \mu_1} \check{m}_{a_2 \alpha_2, c_2 \beta_2}^{b_{12} \gamma_2; \mu_2} m_{a_2 c_2}^{b_{23}} \dots m_{a_{g-1} c_{g-1}}^{b_{g-1, g}} \check{m}_{a_g a_g^\vee}^{b_{g-1, g}} |\vec{a}, \vec{b}, \vec{c}; \vec{\mu}\rangle. \quad (2.22)$$

This state is invariant under the action of the mapping class group of Σ_g . We will normalize this state such that $\langle 0_g | Z_{\mathcal{A}} \rangle = \langle 0_g | 0_g \rangle = 1$, where $|0_g\rangle$ denotes the state corresponding the all vacuum insertion. The partition function of the CFT corresponding to this state is simply given by

$$Z_{\mathcal{A}}(\Omega) = \langle \Omega | Z_{\mathcal{A}} \rangle \quad (2.23)$$

On the torus, we simply have

$$\langle \tau | Z_{\mathcal{A}} \rangle = \sum_a n_a \chi_a(\tau). \quad (2.24)$$

If $\mathcal{T} = \mathcal{T}_L \times \bar{\mathcal{T}}_L$, there is always a canonical Lagrangian condensation which is just the diagonal condensation

$$\mathcal{A} = \bigoplus_{(a, \bar{a}) \in \mathcal{T}_L \times \bar{\mathcal{T}}_L} (a, \bar{a}). \quad (2.25)$$

In this case, one can always find a gauge where the product junctions can be written as $m_{(a, \bar{a})(b, \bar{b})}^{(c, \bar{c}); \mu} = 1$ [61].

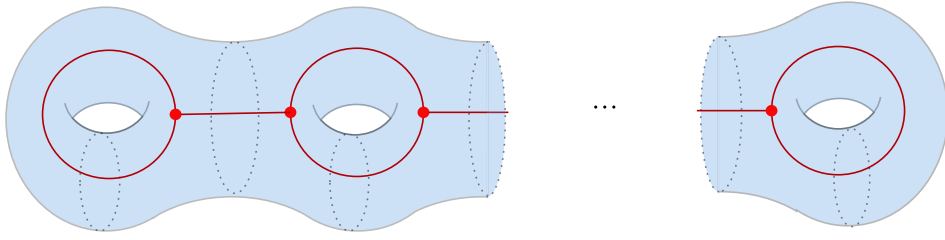


Figure 2. Condensation of $\mathcal{A}$ can be reduced to inserting the shown network of $\mathcal{A}$ lines in a handlebody. This prepares the state $|Z_{\mathcal{A}}\rangle$.

CFT perspective. One can think of the Lagrangian algebra as defining a CFT in the following sense. The direct summands of $\mathcal{A}$ give us the primary operators relative the vertex algebra $\mathcal{V}_{\mathcal{T}}$ while the product morphism m tells us their OPE structure. The associativity and the Frobenius conditions ensure that these OPEs satisfy crossing symmetry. The Lagrangian condition imposes completeness of the spectrum, which is equivalent to S modular invariance, and the commutativity condition ensures mutual locality of the given primaries. In other words, bootstrapping Lagrangian algebras for RCFTs can be viewed as a baby version of the full bootstrap program for general CFTs.

From the vertex algebra point of view, anyon condensation implements extension of the algebra by extra currents.⁵ Lagrangian anyon condensation is then a maximal extension into a vertex algebra with only one primary operator, with the character of that primary being the partition function itself. These vertex algebras are usually called self-dual vertex algebras or holomorphic VOAs in the chiral case [48].

2.3 SymTFT picture

Lagrangian algebras are in one to one correspondence with topological boundary conditions (gapped boundaries) in the TQFT [59]. Pictorially, we can fatten the network of $\mathcal{A}$ lines into a surface, then this surface will have a topological boundary condition [62]. This surface is an interface between the original TQFT and the trivial TQFT, so we can construct this surface by condensing in half the space. This gives us a SymTFT construction [63–66], also called the sandwich construction. In the sandwich construction, one starts from the TQFT on $\Sigma \times [0, 1]$ and places topological boundary condition on one boundary and a boundary condition that describes the physical theory on the other. The topological boundary, also called the symmetry boundary, hosts the symmetry line defects of the theory that lives at the physical boundary as shown in figure 3. Compactifying the interval brings us back to just the physical boundary theory

⁵Abelian anyons are known as simple currents in this language.

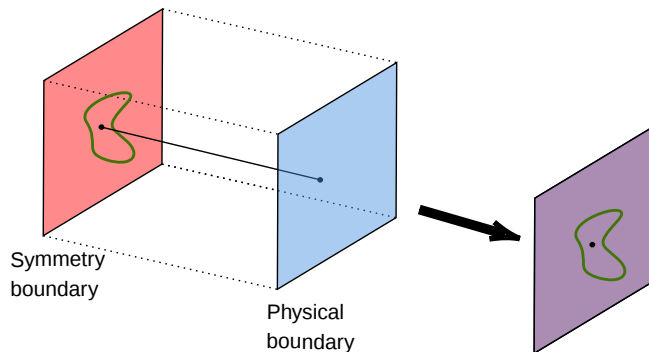


Figure 3. A depiction of the sandwich construction where the bulk TQFT is placed on an interval. The red boundary has a topological boundary condition with the green line as a defect living on the topological boundary. The blue boundary represents the physical boundary where the boundary QFT lives. We can have lines in the bulk extending between the two boundaries giving rise to local or defect operators in the boundary QFT. Upon collapsing the interval, we get the boundary QFT with possible insertions of topological defect lines and local or defect operators

with possible topological defects (and/or defect or local operators) insertions. In our case $|Z_{\mathcal{A}}\rangle$ is our topological boundary condition, $|\Omega\rangle$ is the physical boundary condition and the sandwich gives us the partition function of the boundary CFT as the overlap $\langle\Omega|Z_{\mathcal{A}}\rangle$.

For a given algebra object $\mathcal{A}$, the symmetry category at the corresponding topological boundary is the category of right $\mathcal{A}$ -modules which we will denote by $\mathcal{T}_{\mathcal{A}}$. Physically, we can think of this as the category of confined anyons in the condensed trivial phase where they live on the gapped domain wall between the two phases. For diagonal condensations, the topological defect lines will be the usual Verlinde lines [67]. Note that the confined anyons are not necessarily simple anyons of the bulk TQFT since the bulk anyons can split or be identified on the gapped boundary. Given the symmetry category $\mathcal{T}_{\mathcal{A}}$, the bulk TQFT is what is known as the Drinfeld center of that category [68–70] denoted as $\mathcal{Z}(\mathcal{T}_{\mathcal{A}})$ which physically we can think of as the analog of what a gauge theory is in the case of ordinary group symmetries.

Given a fixed boundary condition on the physical boundary, which we denoted previously by $|\Omega\rangle$, the boundary theories obtained from the different topological boundary conditions are all related to each other by generalized orbifold/gauging [63].

Note that the SymTFT construction is more general, the physical boundary can

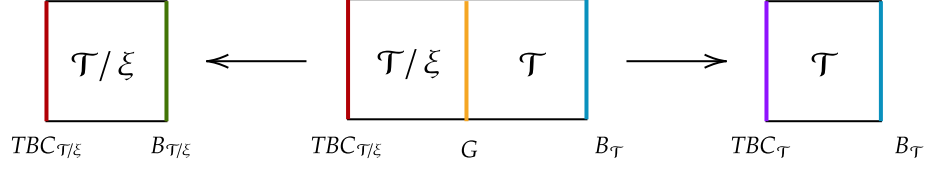


Figure 4. An illustration of the club sandwich construction. The middle figure shows a gapped interface G between theory $\mathcal{T}$ and a condensed phase $\mathcal{T}/\xi$. Collapsing G to the right gives us a sandwich construction for $\mathcal{T}$, while collapsing it to the left gives us a sandwich for $\mathcal{T}/\xi$. The theory at the physical boundary is the same for all these sandwiches, and the above diagram shows the relations between boundary conditions in the different scenarios.

correspond to any quantum field theory with a given symmetry category $\mathcal{C}$ such that $\mathcal{T}$ is given by the Drinfeld center $\mathcal{Z}(\mathcal{C})$, where all dynamical information is encoded in the physical boundary condition. For our holographic purposes we will be interested in gapless boundary conditions particularly ones associated with conformal blocks of the vertex algebra, which we have denoted before by the state $|\Omega\rangle$ since our goal is to mimic the Virasoro case but with a rational vertex algebra (see [23] for a similar discussion about boundary conditions). Other gapless boundary conditions can be thought of as starting from a bigger TQFT with the holomorphic boundary condition we discussed and then interfacing it with a condensed phase in half the sandwich, this construction is called the “club sandwich” [71]. Then one merges the physical boundary of the original sandwich with the interface giving us a sandwich construction with the bulk being the condensed TQFT and the boundary condition is now more general. This is depicted in figure 4.

3 TQFT gravity

3.1 Review of the main duality

The holographic proposal of [1] starts from a TQFT $\mathcal{T}$ summed over all topologies with a boundary Σ and shows that it is dual to an ensemble average of all CFTs constructed from condensation of Lagrangian algebras. Let us denote the state corresponding to condensing a given Lagrangian algebra $\mathcal{A}_\alpha$ by $|Z_\alpha\rangle$, where α for now is just a label for the different Lagrangian algebras (or different topological boundary conditions). The duality then reads

$$\sum_{\text{topologies}} \Psi_0(\Omega) = \sum_{\alpha} w_{\alpha} Z_{\alpha}(\Omega) \quad (3.1)$$

where in this notation $\Psi_0(\Omega)$ is the wavefunctional of $\mathcal{T}$ on a given manifold with boundary Σ_g and modulus Ω , and $Z_\alpha(\Omega) \equiv \langle \Omega | Z_\alpha \rangle$ as defined in (2.23). The sum over topologies in the left-hand-side (LHS) is obtained from starting from summing over handlebodies at a very high genus and then performing genus reduction, while the unnormalized coefficients w_α are given by

$$w_\alpha = \frac{1}{\langle Z_\alpha | Z_\alpha \rangle |_{g \rightarrow \infty}}. \quad (3.2)$$

In this section, we will review the derivation given in [1] in the context of a general non-Abelian TQFT.

We start by noting that algebra objects define surface operators (condensation defects) via higher gauging [72]. For Lagrangian algebras, these surface operators act as (un-normalized) projectors onto the respective topological boundary condition states. For an algebra object $\mathcal{A}_\alpha$, the corresponding surface operator is given by

$$\mathcal{S}_\alpha \equiv |Z_\alpha\rangle\langle Z_\alpha| \quad (3.3)$$

The fusion algebra of such surface operators can be written as

$$\mathcal{S}_\alpha \times \mathcal{S}_\alpha = \langle Z_\alpha | Z_\alpha \rangle \mathcal{S}_\alpha \quad (3.4)$$

$$\mathcal{S}_\alpha \times \mathcal{S}_{\alpha'} = \langle Z_\alpha | Z_{\alpha'} \rangle \mathcal{S}_{\alpha\alpha'} \quad (3.5)$$

where

$$\mathcal{S}_{\alpha\alpha'} \equiv |Z_{\alpha\alpha'}\rangle\langle Z_{\alpha\alpha'}| \quad (3.6)$$

The fusion coefficients $\langle Z_\alpha | Z_{\alpha'} \rangle$ are partition functions of 2d TQFTs as was argued in [72]. The above surface operators not only commute with the representation of the mapping class group operators, denoted below by $\mathcal{U}$, but they are also invariant under their action

$$\mathcal{U}_\gamma \mathcal{S} = \mathcal{S} \mathcal{U}_\gamma = \mathcal{S} \quad (3.7)$$

where $\gamma \in \text{MCG}(\Sigma_g)$ and $\text{MCG}(\Sigma_g)$ denotes the mapping class group (MCG) of Σ_g .

In what follows, for convenience, we will work with projectors built out of normalized states, so we define

$$P_\alpha \equiv \frac{1}{\langle Z_\alpha | Z_\alpha \rangle} |Z_\alpha\rangle\langle Z_\alpha|, \quad P_{\alpha\alpha'} \equiv \frac{1}{\sqrt{\langle Z_\alpha | Z_\alpha \rangle \langle Z_{\alpha'} | Z_{\alpha'} \rangle}} |Z_{\alpha\alpha'}\rangle\langle Z_{\alpha\alpha'}|, \quad (3.8)$$

which lead to

$$P_\alpha \times P_\alpha = P_\alpha, \quad P_\alpha \times P_{\alpha'} = \frac{\langle Z_\alpha | Z_{\alpha'} \rangle}{\sqrt{\langle Z_\alpha | Z_\alpha \rangle \langle Z_{\alpha'} | Z_{\alpha'} \rangle}} P_{\alpha\alpha'}. \quad (3.9)$$

At an arbitrarily large genus g , it is conjectured that all MCG invariant states are linear combinations of the physical invariants $|Z_\alpha\rangle$ defined by Lagrangian algebras. The motivation for this is that states corresponding to Lagrangian algebras can always be defined at any arbitrary genus due to the constraints satisfied by the algebra, while any linearly independent state at a particular genus g is not guaranteed to survive the MCG constraints at some higher genus $g+k$. Given this conjecture, if we show that

$$\lim_{g \rightarrow \infty} \frac{\langle Z_\alpha | Z_{\alpha'} \rangle}{\sqrt{\langle Z_\alpha | Z_\alpha \rangle \langle Z_{\alpha'} | Z_{\alpha'} \rangle}} = \delta_{\alpha\alpha'}, \quad (3.10)$$

then the projectors P_α will form a complete idempotent basis as $g \rightarrow \infty$. We will prove (3.10) in section 4. Using this, we can write the projector onto the MCG invariant subspace of $\mathcal{H}_g$ in terms of P_α at $g \rightarrow \infty$ as

$$\frac{1}{|\text{MCG}(\Sigma_g)|} \sum_{\gamma \in \text{MCG}(\Sigma_g)} \mathcal{U}_\gamma = \sum_\alpha P_\alpha \quad (3.11)$$

where from here on, for convenience, we will denote g as the regulator of the $g \rightarrow \infty$ limit and it will be implicit that the equality of (3.11) and other similar equations hold in the limit $g \rightarrow \infty$. Any actual finite genus will be denoted by $\tilde{g}$.

Acting by the projector (3.11) on the vacuum and capping it off from the left by $\langle \Omega_g |$ we get

$$\frac{1}{|\text{MCG}(\Sigma_g)/\Gamma^{(g)}|} \sum_{\gamma \in \text{MCG}(\Sigma_g)/\Gamma^{(g)}} \langle \Omega_g | \mathcal{U}_\gamma | 0_g \rangle = \sum_\alpha w_\alpha Z_\alpha(\Omega_g) \quad (3.12)$$

where $w_\alpha = \frac{1}{\langle Z_\alpha | Z_\alpha \rangle_g}$ and $\Gamma^{(g)}$ is the stabilizer group in $\text{MCG}(\Sigma_g)$ of the handlebody $S\Sigma_g$ associated with $|0_g\rangle$ which is known as the *handlebody group* [73, 74]. Note that the LHS is just the sum over handlebodies at genus g . Now we want to do genus reduction from genus g to genus $\tilde{g}$ by taking the pinching limit for Ω_g for $g - \tilde{g}$ cycles, effectively making them into $g - \tilde{g}$ independent tori and then sending the imaginary part of their modulus to infinity. In this limit, the partition functions in the RHS of (3.12) will degenerate to their counterpart at genus $\tilde{g}$. We can see this explicitly at the level of wavefunctions $\langle \Omega_g | Z \rangle$. Using the definition of $|\Omega_g\rangle$ in (2.3) and the fact that vacuum dominates at low temperature, we get

$$|\Omega_g\rangle \rightarrow |0_{g-\tilde{g}}, \Omega_{\tilde{g}}\rangle, \quad (3.13)$$

where the state $|0_{g-\tilde{g}}, \Omega_{\tilde{g}}\rangle$ is an element of the product subspace $\mathcal{H}_{g-\tilde{g}} \otimes \mathcal{H}_{\tilde{g}} \subset \mathcal{H}_g$.

Overlapping any ket state with $\langle 0_{g-\tilde{g}}, \Omega_{\tilde{g}} |$ then implements a projection map

$$\langle 0_{g-\tilde{g}}, \cdot | \equiv \Phi_{\tilde{g}} : \mathcal{H}_g \rightarrow \mathcal{H}_{\tilde{g}}, \quad (3.14)$$

$$\begin{array}{c}
\sum_{a,b,c} \Psi_{a_1,b_{12},a_2,c_2,b_{23},a_3} \left| \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle \\
\downarrow \langle 0_{g-1}, \cdot | \\
\sum_{a_3} \Psi_{0,0,0,0,0,a_3} \left| \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle
\end{array}$$

Figure 5. Illustration of projection via $\langle 0_{g-\tilde{g}}, \cdot |$ in a simple case for $g = 3$ and $\tilde{g} = 1$. The resulting state can be viewed as a state in $\mathcal{H}_{\tilde{g}=1}$.

where $\langle 0_{g-\tilde{g}}, \cdot |$ denotes overlapping with the first $g - \tilde{g}$ cycles with the vacuum insertion on those cycles leaving the final $\tilde{g}$ cycles intact.⁶ So starting from a state $|\Psi\rangle \in \mathcal{H}_g$, we effectively get a state $|\tilde{\Psi}\rangle \in \mathcal{H}_{\tilde{g}}$ given by $\langle 0_{g-\tilde{g}}, \cdot | \Psi \rangle$. This is illustrated for a simple case in figure 5. Hence, after genus reduction we get

$$\frac{1}{|\text{MCG}(\Sigma_g)/\Gamma^{(g)}|} \sum_{\gamma \in \text{MCG}(\Sigma_g)/\Gamma^{(g)}} \langle \Omega_{\tilde{g}} | \gamma \rangle_{\tilde{g}} = \sum_{\alpha} w_{\alpha} Z_{\alpha}(\Omega_{\tilde{g}}), \quad (3.15)$$

where we denote

$$|\gamma\rangle_{\tilde{g}} \equiv \langle 0_{g-\tilde{g}}, \cdot | \mathcal{U}_{\gamma} | 0_g \rangle. \quad (3.16)$$

What remains now is to show that the left hand side of (3.12) would give us wavefunctions corresponding to evaluating the TQFT path integral on all topologies ending on $\Sigma_{\tilde{g}}$ with boundary condition given by $|\Omega_{\tilde{g}}\rangle$. The argument in [1] was that $\langle 0_g | \mathcal{U}_{\gamma} | 0_g \rangle$ for all $\gamma \in \text{MCG}(\Sigma_g)$ will give us all possible Heegaard splittings of closed 3-manifolds, and so genus reduction should similarly produce all 3-manifolds with boundary. We will elaborate a bit more on this argument in the next subsection, showing explicitly that this will give us all 3-manifolds with $\Sigma_{\tilde{g}}$ boundary.

The MCG average discussed in the derivation above should be viewed in a formal sense. For a generic nonAbelian TQFT, the representation of $\text{MCG}(\Sigma_g)$ for $g > 1$

⁶The intermediate link between the $g - \tilde{g}$ cycles and $\tilde{g}$ cycles must be vacuum otherwise the overlap will vanish.

usually has an infinite image [75] and thus the sum should be somehow regularized. There is no natural regularization for the MCG itself as it is known to be non-amenable [75, 76] and it is not clear whether the MCG image is amenable or not. Nonetheless, one can in principle go around this issue by summing over the distinct images of the vacuum under the action of MCG as in equation (3.12) which corresponds to summing over a coset instead of the full MCG similar to [13]. It would be interesting to test this in simple nonAbelian TQFTs with infinite MCG image like the Fibonacci and see if one could regularize the coset sum.

An important aspect of (3.12) is that topological boundary conditions that are related by an anyon permutation symmetry have equal weights. This is a consequence of the fact that anyon permutations commute with the action of the mapping class group.⁷

3.2 The sum over topologies

We now want to show how the states $|\gamma\rangle_{\tilde{g}}$ give rise to all manifolds with boundary $\Sigma_{\tilde{g}}$. The main idea is to show that any such state can be obtained by the following (generalized) Heegaard splitting procedure:

1. Start from a handlebody $S\Sigma_g$ corresponding to state $|0_g\rangle$ and carve out a handlebody $S\Sigma_{\tilde{g}}$ from inside $S\Sigma_g$. This gives us what is known as a *compression body* C with inner boundary $\partial_- C = \Sigma_{\tilde{g}}$ and outer boundary $\partial_+ C = \Sigma_g$. This is visualized in figure 6.
2. Get another handlebody related to the original $S\Sigma_g$ by a boundary mapping class group transformation γ , let us denote it as $S_\gamma\Sigma_g$.
3. Glue $S_\gamma\Sigma_g$ with C across $\partial_+ C$, i.e. $C \cup_{\Sigma_g} S_\gamma\Sigma_g$. This gluing can be also denoted simply as $C \cup_\gamma S\Sigma_g$.

The result of this procedure is some 3-manifold with a boundary $\Sigma_{\tilde{g}}$. For arbitrary $\gamma \in \text{MCG}(\Sigma_g)$ and arbitrarily large g , this procedure should be enough to generate all 3-manifolds with boundary $\Sigma_{\tilde{g}}$ (possibly many many times) based on the fact that all triangulated manifolds admit a (generalized) Heegaard splitting [77, 78]. We will elaborate on this point later in this section.

⁷Note that this statement is more general than the case at hand where the weights are related to the norms (which are clearly the same for two states related by anyon permutations).

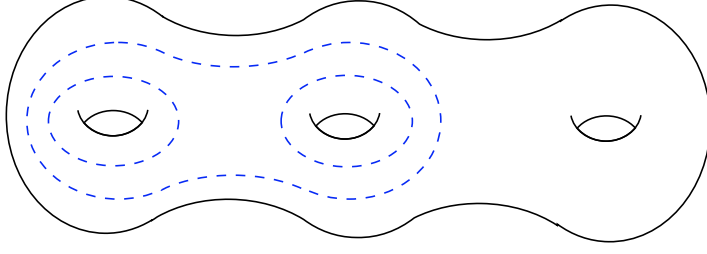


Figure 6. A visualization of a compression body with outer boundary $\partial_+ C = \Sigma_3$ and an inner boundary $\partial_- C = \Sigma_2$.

We can see that the procedure above is equivalent to the genus reduction by starting from the definition of $|\gamma\rangle_{\tilde{g}}$ and inserting resolution of the identity operator of $\mathcal{H}_g$

$$|\gamma\rangle = \langle 0_{g-\tilde{g}}, \cdot | \mathbf{I}_g \mathcal{U}_\gamma | 0_g \rangle \quad (3.17)$$

$$= \sum_{\vec{a}, \vec{b}, \vec{c}, \vec{\mu}} \langle 0_{g-\tilde{g}}, \cdot | \vec{a}, \vec{b}, \vec{c}; \vec{\mu} \rangle \langle \vec{a}, \vec{b}, \vec{c}; \vec{\mu} | U_\gamma | 0_g \rangle \quad (3.18)$$

$$\equiv C U_\gamma | 0_g \rangle \quad (3.19)$$

where

$$C \equiv \sum_{\vec{a}, \vec{b}, \vec{c}, \vec{\mu}} \langle 0_{g-\tilde{g}}, \cdot | \vec{a}, \vec{b}, \vec{c}; \vec{\mu} \rangle \langle \vec{a}, \vec{b}, \vec{c}; \vec{\mu} | \quad (3.20)$$

is our desired compression body, and we used the basis notation of eq. (2.2) corresponding to figure 1. The compression body can be viewed as a map $C : \mathcal{H}_g \rightarrow \mathcal{H}_{\tilde{g}}$ given by the composition $\Phi_{\tilde{g}} \circ \mathbf{I}_g$. Physically, this compression body is a wormhole between Σ_g and $\Sigma_{\tilde{g}}$ which can be obtained from the Euclidean wormhole $\Sigma_g \times I$ by degenerating Σ_g to $\Sigma_{\tilde{g}}$ on one end but not the other.

Let us illustrate this with a simple example of $g = 2$ and $\tilde{g} = 1$. We will drop the fusion junction label μ for notational convenience and label our basis states on Σ_2 by $|abc\rangle$ where, in the notation of figure 1, $a \equiv a_1$, $b \equiv b_{12}$ and $c \equiv c_2 = a_2$. First, let us construct the state $|\gamma\rangle_{\tilde{g}=1}$ explicitly from genus reduction. We get

$$U_\gamma | 000 \rangle = \sum_{abc} (\mathcal{U}_\gamma)_{(abc)(000)} | abc \rangle, \quad (3.21)$$

$$|\gamma\rangle_{\tilde{g}=1} = \sum_a (\mathcal{U}_\gamma)_{(00c)(000)} | a \rangle, \quad (3.22)$$

and we denote $(\mathcal{U}_\gamma)_{(abc)(a'b'c')} \equiv \langle abc | \mathcal{U}_\gamma | a'b'c' \rangle$. The Euclidean wormhole of $\Sigma_{g=2}$ is just the identity operator

$$\sum_{abc} | abc \rangle \langle abc |. \quad (3.23)$$

Degenerating to $\Sigma_{\tilde{g}=1}$ on one end gives us

$$C = \sum_c |c\rangle \langle 00c|. \quad (3.24)$$

where $|c\rangle$ denotes a state in $\mathcal{H}_{\tilde{g}=1}$. We can see that C acts as an identity operator when restricted to $\mathcal{H}_{\tilde{g}=1}$. Applying the gluing of Σ_g to $\partial_- C$ gives us our desired result as

$$C \cup_\gamma S\Sigma_2 = C\mathcal{U}_\gamma |000\rangle \quad (3.25)$$

$$= \sum_c |c\rangle \langle 00c| \sum_{a'b'c'} (\mathcal{U}_\gamma)_{(a'b'c')(000)} |a'b'c'\rangle \quad (3.26)$$

$$= \sum_c (\mathcal{U}_\gamma)_{(00c)(000)} |c\rangle \quad (3.27)$$

$$= |\gamma\rangle_{\tilde{g}=1}. \quad (3.28)$$

As we see, the genus reduction method is powerful enough to generate our desired sum over topologies at any arbitrary genus, which from the ensemble perspective calculates the (unnormalized) averaged partition function $\langle Z(\Omega_{\tilde{g}}) \rangle$. One can get multi-point moments of the partition function by taking the splitting limit of $\tilde{g}$ into disconnected surfaces of even lower genera. All of these can be obtained from (3.11) by applying the appropriate bra and ket states to the projector. For example for n -point torus moments this would amount to

$$\frac{1}{|\text{MCG}(\Sigma_g)|} \sum_{\gamma \in \text{MCG}(\Sigma_g)} \langle 0_{g-n}, \tau_1, \tau_2, \dots, \tau_n | \mathcal{U}_\gamma | 0_g \rangle = \sum_\alpha \frac{Z_\alpha(\tau_1) Z_\alpha(\tau_2) \dots Z_\alpha(\tau_n)}{\langle Z_\alpha | Z_\alpha \rangle |_g}, \quad (3.29)$$

where similar to our previous notation, $\langle 0_{g-n}, \tau_1, \tau_2, \dots, \tau_n |$ denotes a factorizable bra-state in $\mathcal{H}_{g-n} \otimes \mathcal{H}_1^{\otimes n} \subset \mathcal{H}_g$. More generally, we can include orientation reversal version of the tori

$$\begin{aligned} & \frac{1}{|\text{MCG}(\Sigma_g)|} \sum_{\gamma \in \text{MCG}(\Sigma_g)} \langle 0_{g-n}, \tau_1, \tau_2, \dots, \tau_n | \mathcal{U}_\gamma | 0_{g-m}, \bar{\tau}'_1, \bar{\tau}'_2, \dots, \bar{\tau}'_m \rangle \\ &= \sum_\alpha \frac{Z_\alpha(\tau_1) Z_\alpha(\tau_2) \dots Z_\alpha(\tau_n) Z_\alpha(\bar{\tau}'_1) Z_\alpha(\bar{\tau}'_2) \dots Z_\alpha(\bar{\tau}'_m)}{\langle Z_\alpha | Z_\alpha \rangle |_g} \end{aligned} \quad (3.30)$$

An important amplitude to consider is the vacuum to vacuum amplitude

$$\frac{1}{|\text{MCG}(\Sigma_g)|} \sum_{\gamma \in \text{MCG}(\Sigma_g)} \langle 0_g | \mathcal{U}_\gamma | 0_g \rangle = \sum_\alpha \frac{1}{\langle Z_\alpha | Z_\alpha \rangle |_g}. \quad (3.31)$$

Let us understand this sum a bit better and try to write it explicitly as sum over distinct homeomorphism classes of closed 3-manifolds. First, note that the distinct elements of

LHS are described by elements of the double-coset space $\Gamma^{(g)} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}$. Writing the sum only over the distinct elements, we get

$$\left(\sum_{\gamma \in \Gamma^{(g)} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}} \frac{1}{|\text{Aut}(\gamma)|} \right)^{-1} \sum_{\gamma \in \Gamma^{(g)} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}} \frac{1}{|\text{Aut}(\gamma)|} \mathbf{Z}_{\mathcal{T}}(M_\gamma) = \sum_{\alpha} \frac{1}{\langle Z_\alpha | Z_\alpha \rangle|_g}, \quad (3.32)$$

where M_γ is the closed 3-manifold obtained by the Heegaard gluing $S\Sigma_g \cup_\gamma S\Sigma_g$, $\mathbf{Z}_{\mathcal{T}}$ is the TQFT partition function, and $\text{Aut}(\gamma) \equiv \Gamma^{(g)} \cap \gamma \Gamma^{(g)} \gamma^{-1}$. Note that in going from (3.31) to (3.32), we have used the fact that the formal size of each double-coset $|\Gamma^{(g)} \gamma \Gamma^{(g)}|$ is given by

$$|\Gamma^{(g)} \gamma \Gamma^{(g)}| = \frac{|\Gamma^{(g)}|^2}{|\Gamma^{(g)} \cap \gamma \Gamma^{(g)} \gamma^{-1}|}, \quad (3.33)$$

and

$$|\text{MCG}(\Sigma_g)| = \sum_{\gamma \in \Gamma^{(g)} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}} |\Gamma^{(g)} \gamma \Gamma^{(g)}|. \quad (3.34)$$

Geometrically, $\text{Aut}(\gamma)$ is the group that preserves the Heegaard splitting $S\Sigma_g \cup S_\gamma \Sigma_g$ which is known as the *Goeritz group* $\mathcal{G}_g$ of the splitting [79–82]. It is the subgroup of the MCG of the splitting surface that extends to both handlebodies $S\Sigma_g$ and $S_\gamma \Sigma_g$, and hence can be expressed as the intersection of their handlebody groups, which are $\Gamma^{(g)}$ and $\gamma \Gamma^{(g)} \gamma^{-1}$ respectively in our case. To illustrate the Goeritz group in a simple example, consider the $g = 1$ case. The MCG of the torus is $\text{SL}(2, \mathbb{Z})$, and the handlebody group $\Gamma^{(1)}$ is the group generated by $\{-I, T\}$.⁸ The $g = 1$ splitting of S^3 involves the S transformation, so

$$\mathcal{G}_1(S^3) = \Gamma^{(1)} \cap S \Gamma^{(1)} S^{-1} = \{I, -I\} \simeq \mathbb{Z}_2, \quad (3.35)$$

while for $S^2 \times S^1$ it is just the identity map, hence

$$\mathcal{G}_1(S^2 \times S^1) = \Gamma^{(1)} \cap \Gamma^{(1)} = \Gamma^{(1)} \simeq \mathbb{Z} \times \mathbb{Z}_2. \quad (3.36)$$

The elements of the double-coset sum in (3.32) are unique Heegaard splittings of genus g but are they unique as 3-manifolds? At any finite genus, there could be two different Heegaard splittings that produce the same 3-manifold. However, in the $g \rightarrow \infty$ limit, this degeneracy will go away. This is a consequence of the Reidemeister-Singer

⁸Note that in this example we are working with the MCG itself rather than its action on the torus modulus τ or its TQFT representation. On the modulus τ , the MCG action is by $\text{PSL}(2, \mathbb{Z})$ since $-I$ acts trivially, and the stabilizer of $\chi_0(\tau)$ is just the group generated by T usually denoted as Γ_∞ . In the TQFT representation $-I$ acts as the charge conjugation matrix C and so the stabilizer of $|0\rangle$ state is the group generated by $\{C, T\}$.

theorem [83–85] which states that any two genus g Heegaard splittings of the same manifold become equivalent under a finite number of *stabilizations*, where a stabilization [86] means adding an extra handle, i.e. going from genus g to genus $g + 1$. Therefore, in the $g \rightarrow \infty$ limit, the set of double-cosets $\Gamma^{(g)} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}$ bijects onto the set of homeomorphism classes of closed orientable 3-manifolds [74, 87].

In a similar fashion to the closed-manifolds’ case, where now instead the Heegaard splitting involves a compression body and a handlebody, the sum over homeomorphism classes of compact orientable 3-manifolds with a boundary of genus $\tilde{g}$ should⁹ be described by a similar double-coset sum in the $g \rightarrow \infty$ limit, namely

$$\left(\sum_{\gamma \in \Gamma^{(g, \tilde{g})} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}} \frac{1}{|\text{Aut}(\gamma)|} \right)^{-1} \sum_{\gamma \in \Gamma^{(g, \tilde{g})} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}} \frac{1}{|\text{Aut}(\gamma)|} |\gamma\rangle_{\tilde{g}} = \sum_{\alpha} \frac{|Z_{\alpha}\rangle_{\tilde{g}}}{\langle Z_{\alpha} | Z_{\alpha} \rangle |g}, \quad (3.37)$$

where $\Gamma^{(g, \tilde{g})}$ (*the relative compression body group* [88]) is the subgroup of $\text{MCG}(\Sigma_g)$ that extends to the compression body C with $\partial_+ C = \Sigma_g$, $\partial_- C = \Sigma_{\tilde{g}}$ and leaves $\partial_- C$ invariant pointwise, while $\text{Aut}(\gamma) \equiv \Gamma^{(g, \tilde{g})} \cap \gamma \Gamma^{(g)} \gamma^{-1}$ is the Goeritz group of the generalized Heegaard splittings $C \cup_{\gamma} S\Sigma_g$.¹⁰ So each connected manifold M_{γ} with boundary $\partial M_{\gamma} = \Sigma_{\tilde{g}}$, labeled by $\gamma \in \Gamma^{(g, \tilde{g})} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}$, is weighted by

$$\mu_{\tilde{g}}(M_{\gamma}) = \left(\sum_{\gamma \in \Gamma^{(g, \tilde{g})} \backslash \text{MCG}(\Sigma_g) / \Gamma^{(g)}} \frac{1}{|\Gamma^{(g, \tilde{g})} \cap \gamma \Gamma^{(g)} \gamma^{-1}|} \right)^{-1} \frac{1}{|\Gamma^{(g, \tilde{g})} \cap \gamma \Gamma^{(g)} \gamma^{-1}|}, \quad g \rightarrow \infty \quad (3.38)$$

Note that in the case where $\tilde{g} \rightarrow g$ we get back the sum over only handlebodies in (3.12) as $\Gamma^{(g, g)}$ from our definition is just the identity element of $\text{MCG}(\Sigma_g)$.

What remains now is to understand the implications of this measure for the sum over 3-manifolds, which is related to the Goeritz group in the large genus limit. It is not a surprise that the sum over 3-manifolds we got is weighted by the Goeritz group, after all, we were averaging uniformly over all labeled Heegaard splittings and so restricting to the equivalence classes leads to an automorphism weighted average, with the Goeritz group being the natural automorphism group in this context. This is of course an ill-defined measure since the Goeritz group is infinite in this case for all manifolds and it is not obvious how to regularize it; however one would like to understand the meaning of using the large genus Goeritz group in the light of the relation between 3d gravity and VTQFT. We leave this task for future work.

⁹The equivalence between Heegaard splittings under stabilization should hold for manifolds with boundaries as well, see for example [87].

¹⁰More generally, in principle, one should be able to describe manifolds with disjoint boundaries using gluings of more general compression bodies and using the corresponding double-coset.

Finally, we want to note that even though we used the full MCG in our formal derivations, a given TQFT will not be sensitive to all manifolds since the representation of MCG is not faithful in general. In cases where the image of the of MCG is finite, as for example in Abelian TQFTs or the Ising, the TQFT will only distinguish a finite number of equivalence classes of manifolds at any finite genus. In these cases, the sum over topologies (regularized at a finite genus) is tractable, and the effective weights over the equivalence classes of manifolds will be finite. A very simple example is the $\mathbb{Z}_2$ gauge theory or toric code (TC) TQFT. At $g = 1$, the image of MCG is $\text{SL}(2, \mathbb{Z}_2)$ and hence, the set of all lens spaces will collapse to just two classes: $S^2 \times S^1$ and S^3 . The image ρ of their Goeritz group at the $g = 1$ splitting is $\rho(\mathcal{G}_1(S^2 \times S^1)) = \mathbb{Z}_2$ and $\rho(\mathcal{G}_1(S^3)) = \{I\}$. Hence, the (renormalized) sum over all $g = 1$ splittings of closed 3-manifolds in the toric code case can be simply reduced to

$$\frac{1}{|\mathbb{Z}_2|} \mathbf{Z}_{\text{TC}}(S^2 \times S^1) + \mathbf{Z}_{\text{TC}}(S^3) = \frac{1}{2}(1) + \frac{1}{2}. \quad (3.39)$$

3.3 The baby universe Hilbert space

In the previous subsection, we have seen that all possible moments of the ensemble are secretly encoded in the infinite genus g Hilbert space, or more precisely in the MCG invariant subspace which we will denote as $\mathcal{H}_g^{\text{MCG}}$. This suggests that there should be a natural interpretation of $\mathcal{H}_g^{\text{MCG}}$, which we interpret here as the baby universe Hilbert space.

To illustrate this, let us briefly revisit the construction of Marolf and Maxfield (MM) [6] of the baby universe Hilbert space. They start by considering the gravitational path integral on a manifold with multiple asymptotic boundaries given some arbitrary boundary conditions. They cut this path integral in a way such that all the connected components of the asymptotic boundaries are either entirely in the future or entirely in the past relative to the cut, hence defining a state in the baby (closed) universe Hilbert space $\mathcal{H}_{\text{BU}}$. In their notation, if we have n connected components of asymptotic boundaries with a set of boundary conditions labeled by $\{J_1, J_2, \dots, J_n\}$, then if we make a cut such that all these boundaries are the in the past, we get a state that we can denote by

$$|Z[J_1]Z[J_2]\dots Z[J_n]\rangle \in \mathcal{H}_{\text{BU}}. \quad (3.40)$$

Such a state if overlapped with the no-boundary (Hartle-Hawking) state $|\text{HH}\rangle$ defines the correlator $\langle Z[J_1]Z[J_2]\dots Z[J_n] \rangle$ of the boundary ensemble

$$\langle Z[J_1]Z[J_2]\dots Z[J_n] \rangle \equiv \langle \text{HH} | Z[J_1]Z[J_2]\dots Z[J_n] \rangle. \quad (3.41)$$

Now MM construct $\mathcal{H}_{\text{BU}}$ as the completion of all states of the form (3.40) while quotienting by null states. They then proceed to define an operator $\widehat{Z}[J]$ such that

$$|Z[J_1]Z[J_2]\dots Z[J_n]\rangle = \widehat{Z}[J_1]\widehat{Z}[J_2]\dots\widehat{Z}[J_n]|\text{HH}\rangle \quad (3.42)$$

which implies that all such $\widehat{Z}[J]$ operators commute. Hence, one can then define common eigenstates for all $\widehat{Z}[J]$, which are the so called α -states where factorization is restored

$$\widehat{Z}[J]|\alpha\rangle = Z_\alpha[J]|\alpha\rangle \quad (3.43)$$

and $|\alpha\rangle$ form an orthonormal basis in $\mathcal{H}_{\text{BU}}$. The α label is therefore a label for members of the ensemble as concluded in [6]. Using these states, one can write the normalized amplitudes in terms of $Z_\alpha[J]$

$$\langle Z[J_1]Z[J_2]\dots Z[J_n] \rangle \equiv \frac{\langle \text{HH} | \widehat{Z}[J_1]\widehat{Z}[J_2]\dots\widehat{Z}[J_n] | \text{HH} \rangle}{\langle \text{HH} | \text{HH} \rangle} \quad (3.44)$$

$$= \sum_{\alpha} p_{\alpha} Z_{\alpha}[J_1] Z_{\alpha}[J_2] \dots Z_{\alpha}[J_n] \quad (3.45)$$

where

$$p_{\alpha} \equiv \frac{|\langle \alpha | \text{HH} \rangle|^2}{\langle \text{HH} | \text{HH} \rangle} \quad (3.46)$$

are the probabilities of the ensemble properly normalized to add up to unity.

Now let us get back to our TQFT gravity setup. Our “gravitational” path integral is not the path integral of the TQFT because we want to glue the cuts with the MCG projector rather than the identity operator. However, the MCG projector is the identity operator on $\mathcal{H}_g^{\text{MCG}}$, so this is our “gravitational” Hilbert space for a closed universe. So we are prompted to identify $\mathcal{H}_{\text{BU}} \simeq \mathcal{H}_g^{\text{MCG}}$ (in the large g limit).

Let us illustrate this further. Our no-boundary state in $\mathcal{H}_g$ is given by $|0_g\rangle$. Projecting this onto $\mathcal{H}_g^{\text{MCG}}$ defines for us the Hartle-Hawking (HH) state as

$$|\text{HH}\rangle \equiv \lim_{g \rightarrow \infty} \frac{1}{|\text{MCG}(\Sigma_g)|} \sum_{\gamma \in \text{MCG}(\Sigma_g)} \mathcal{U}_{\gamma} |0_g\rangle \quad (3.47)$$

Since we are using a projector, the vacuum to vacuum amplitude in (3.31) is simply equal to $\langle \text{HH} | \text{HH} \rangle$ as expected. Similarly, all the amplitudes we calculated in the previous subsection can be defined this way, where now the states of the form $|0_{g-n}, \tau_1, \dots, \tau_n\rangle$ are analogs of the states corresponding to asymptotic boundaries but in $\mathcal{H}_g$. Projecting these onto $\mathcal{H}_g^{\text{MCG}}$ defines for us the MM states (3.40), for example

$$|Z[\tau_1]Z[\tau_2]\dots Z[\tau_n]\rangle \equiv \lim_{g \rightarrow \infty} \frac{1}{|\text{MCG}(\Sigma_g)|} \sum_{\gamma \in \text{MCG}(\Sigma_g)} \mathcal{U}_{\gamma} |0_{g-n}, \tau_1, \dots, \tau_n\rangle. \quad (3.48)$$

It should be now clear that the α -eigenstates that span $\mathcal{H}_{\text{BU}}$ should be matched with the normalized topological boundary condition states in the infinite genus limit

$$|\alpha\rangle \equiv \lim_{g \rightarrow \infty} \frac{|Z_\alpha\rangle}{\sqrt{\langle Z_\alpha | Z_\alpha \rangle}} \quad (3.49)$$

which now truly justifies our choice of label α . We remind the reader that so far we have only claimed in (3.10) that $|Z_\alpha\rangle$ are orthogonal, but we shall prove this in the next section. The probabilities in our case are then given by

$$p_\alpha = \lim_{g \rightarrow \infty} \left(\sum_\alpha \frac{1}{\langle Z_\alpha | Z_\alpha \rangle} \right)^{-1} \frac{1}{\langle Z_\alpha | Z_\alpha \rangle}. \quad (3.50)$$

One can also define an operator similar to $\widehat{Z[\mathcal{J}]}$ but there is no explicit need to do so in our case since we already know the α -states a priori and can define (3.40) directly in our TQFT Hilbert space.

To summarize, we see that the TQFT gravity approach gives a direct construction for $\mathcal{H}_{\text{BU}}$ as the MCG invariant subspace $\mathcal{H}_g^{\text{MCG}}$ in the large genus limit. Why does this construction work? One can argue that the need to go to the limit of large genus is to go to a Hilbert space that knows (through degeneration) about all possible topology changes, while the need to project to $\mathcal{H}_g^{\text{MCG}}$ is a manifestation of the mismatch between the partition function of gravity, which is MCG invariant, versus the partition function of the TQFT as noted in [17].

We see that the ensemble arises because we start in the no-boundary HH state in a baby universe Hilbert space with $\dim \mathcal{H}_{\text{BU}} > 1$. As expected in this baby-universe picture, starting from an α -state would lead to a factorizable answer. Note that the identification of topological boundary conditions (TBCs) with α -states from this factorization perspective is not new and has been mentioned in [23, 60, 89].¹¹ The nontrivial part that the TQFT gravity framework provides in this context of TBCs is answering the question of what is the baby-universe Hilbert space given the fact that states corresponding to TBCs are not orthogonal in general, nor they span the Hilbert space of the TQFT. These states only become orthogonal in the large g limit, and they span the MCG invariant subspace. Hence, we conclude that

$$\mathcal{H}_{\text{BU}} \simeq \lim_{g \rightarrow \infty} \mathcal{H}_g^{\text{MCG}} \quad (3.51)$$

¹¹See [89] for a different approach to the baby-universe than the approach discussed here.

4 Ensemble weights from symmetries

4.1 Main derivation

We want to evaluate the overlap between any two TBC states $\langle Z_{\alpha'} | Z_{\alpha} \rangle$ at an arbitrary genus g , which corresponds to the overlap of states of the form given in (2.22). For states corresponding to diagonal condensation, one can choose $m_{ab}^{c;\mu} = 1$ for all permissible junctions [61], and in that case we find that the norm of such state is the dimension of the chiral Hilbert space which can be evaluated using (2.5). The norm of course in this case is the trace of the identity operator in the chiral Hilbert space. For general condensations, the direct evaluation of such overlaps, or traces of products of surface operators in the chiral part, is generally difficult. Instead of this direct way, a more convenient and insightful approach is to use the SymTFT picture and the construction of $\mathcal{C}$ -symmetric 2d TQFTs [90, 91]. We can understand $\langle Z_{\alpha'} | Z_{\alpha} \rangle$ from the SymTFT picture as follows. Here we can put a gapped boundary at the physical boundary, which defines for us a 2d TQFT that has a symmetry category $\mathcal{C}^{(\alpha)}$. So we have a 2d TQFT with a partition function given by

$$\mathcal{Z}_{\text{2d TQFT}} = \langle Z_{\alpha'} | Z_{\alpha} \rangle. \quad (4.1)$$

Obviously here there is an ambiguity of which boundary to call the physical boundary versus the symmetry boundary, but one can choose any of them to play either role and get the same answer. In a general 2d TQFT, one needs to know local operators and their OPEs.¹² One can always find an idempotent basis that diagonalizes the OPEs such that [91]

$$\mathcal{O}_i \mathcal{O}_j = \frac{1}{\tilde{d}_i} \delta_{ij} \mathcal{O}_i \quad (4.2)$$

where $\tilde{d}_i \equiv \langle \mathcal{O}_i |_{S^2}$ is called the quantum dimension of $\mathcal{O}_i$. Such a basis corresponds to simple objects in the category of boundary conditions of the 2d TQFT where $\tilde{d}$ are their quantum dimensions. Using this basis, one can write the partition function on Σ_g as

$$\mathcal{Z}_{\text{2d TQFT}}(\Sigma_g) = \tilde{D}^{2g-2} \sum_i \tilde{d}_i^{2-2g}, \quad (4.3)$$

where $\tilde{D} = \sqrt{\sum_i \tilde{d}_i^2}$ and i runs over the simple local operators. Note that the partition function of a 2d TQFT is always defined up to an Euler counterterm λ^{2g-2} which is unphysical. Here we conveniently choose the Euler counterterm such that

¹²These OPE coefficients can be obtained from the product and coproduct junctions of the algebra objects associated with each gapped boundary as was shown in [92] for the case of using the same boundary condition on both sides.

$\mathcal{Z}_{2d \text{ TQFT}}(S^2) = 1$ to be consistent with our chosen 3d TQFT normalization and to easily determine the relative Euler counterterm between different $\mathcal{C}$ -symmetric TQFTs which is a physical quantity unlike the absolute version. One can always absorb the global Euler counterterm into the definition of $\tilde{d}$, but for our purposes we want to choose a normalization such that the quantum dimensions $\tilde{d} \geq 1$, and hence we are keeping the global $\tilde{\mathcal{D}}^{2g-2}$ factor explicit.

Let us first focus on the case where both boundaries of the SymTFT sandwich have the same boundary condition. This will give us what is called the *regular $\mathcal{C}$ -symmetric* 2d TQFT. In a regular $\mathcal{C}$ -symmetric TQFT, the quantum dimensions of the idempotent basis operators are simply given by the quantum dimensions of the topological lines of $\mathcal{C}$ themselves [91]. So by knowing $\mathcal{C}$, we can calculate the partition function directly from the quantum dimension data of $\mathcal{C}$. So we simply get

$$\mathcal{Z}_{\mathcal{C}} = \mathcal{D}_{\mathcal{C}}^{2g-2} \sum_{a \in \mathcal{C}} d_a^{2-2g}. \quad (4.4)$$

Thus, for a topological boundary condition $|Z\rangle$ that gives rise to a symmetry category $\mathcal{C}$, the norm of the state $\langle Z|Z\rangle|_{\Sigma_g}$ is given by (4.4), where now our Euler counterterm normalization reflects the fact that in our 3d TQFT we have $\langle Z|Z\rangle|_{\Sigma_0 \simeq S^2} = 1$.

Now as $g \rightarrow \infty$, the sum will be dominated by the invertible simple objects ($d = 1$) and hence we have up to leading order

$$\langle Z_{\alpha}|Z_{\alpha}\rangle|_{\Sigma_g} \approx \mathcal{D}_{\mathcal{T}}^{g-1} |\text{Inv}(\mathcal{C}^{(\alpha)})| \quad (4.5)$$

where we used the fact that $D_{\mathcal{C}}^2 = D_{\mathcal{T}}$ since $\mathcal{T}$ is the Drinfeld center of $\mathcal{C}$, and $\text{Inv}(\mathcal{C}^{(\alpha)})$ denotes the group of invertible simple objects in $\mathcal{C}^{(\alpha)}$.¹³ From this we get our main result

$$\mathcal{D}_{\mathcal{T}}^{g-1} w_{\alpha} = \frac{1}{|\text{Inv}(\mathcal{C}^{(\alpha)})|}. \quad (4.6)$$

Now we want to return back to the proof of (3.10) which leads to the orthogonality of the projectors P_i . This involves evaluating $\langle Z_{\alpha'}|Z_{\alpha}\rangle$ which from the above discussion can be considered a partition function of a $\mathcal{C}^{(\alpha)}$ -symmetric 2d TQFT albeit not the *regular* one. It was shown in [90, 91] that any $\mathcal{C}$ -symmetric 2d TQFT can be obtained from the regular one by gauging an algebra object in $\mathcal{C}$. If we denote the algebra object by A' , then the category of local operators in the gauged theory is given by the category of right A' modules, $\mathcal{C}_{A'}$. We can show that in our chosen normalization (see appendix B)

$$D_{\mathcal{C}}^2 = \dim A' \mathcal{D}_{\mathcal{C}_{A'}}^2. \quad (4.7)$$

¹³this is sometimes also known as the Picard group of $\mathcal{C}^{(\alpha)}$, see for example [93].

With this, we have the leading order in the large g limit as

$$\frac{\langle Z_\alpha | Z_{\alpha'} \rangle}{\sqrt{\langle Z_\alpha | Z_\alpha \rangle \langle Z_{\alpha'} | Z_{\alpha'} \rangle}} \approx \frac{1}{(\dim A')^{g-1}} \frac{\xi_{\mathcal{C}_{A'}}}{\sqrt{|\text{Inv}(\mathcal{C}^{(\alpha)})| |\text{Inv}(\mathcal{C}^{(\alpha')})|}} \quad (4.8)$$

where $\xi_{\mathcal{C}_{A'}}$ denotes the number of simple objects in $\mathcal{C}_{A'}$ with unit quantum dimension in our normalization.¹⁴ Since $\dim A' > 1$ for any nontrivial algebra object, (3.10) will be satisfied. This also tells us that at any finite g , the lowest bound on the suppression in g is $O(2^{1-g})$ since the lowest nontrivial algebra would have $\dim A' = 2$.

Finally, we note that so far we have been working with the normalized basis $\langle 0_g | 0_g \rangle = 1$, which leads to (4.6). Writing the holographic duality (3.1) with the schematic sum over topologies now takes the form

$$\mathcal{D}_{\mathcal{T}}^{g-1} \sum_{\text{topologies}} \Psi_0(\Omega) = \sum_{\alpha} \frac{1}{|\text{Inv}(\mathcal{C}^{(\alpha)})|} Z_{\alpha}(\Omega). \quad (4.9)$$

Absorbing the factor of $\mathcal{D}_{\mathcal{T}}^{g-1}$ into the TQFT wavefunction restores the natural normalization of the TQFT where $\langle 0_g | 0_g \rangle = \mathcal{D}_{\mathcal{T}}^{g-1}$ instead of $\langle 0_g | 0_g \rangle = 1$. Note that this is an overall normalization regardless of the genus reduction to any genus $\tilde{g}$ due to the fact that all our calculations are embedded in $\mathcal{H}_g$ for large fixed g . Hence, the natural normalization leads to

$$\sum_{M, \partial M = \Sigma_{\tilde{g}}} \mu(M) \mathbf{Z}_{\mathcal{T}}(M; \Omega_{\tilde{g}}) = \sum_{\alpha} \frac{1}{|\text{Inv}(\mathcal{C}^{(\alpha)})|} Z_{\alpha}(\Omega_{\tilde{g}}), \quad (4.10)$$

where the LHS is a sum over all connected orientable 3-manifolds M with boundary $\Sigma_{\tilde{g}}$ labeled by an element of the double-coset space we discussed in section 3.2, $\mathbf{Z}_{\mathcal{T}}(M; \Omega_{\tilde{g}})$ is the naturally normalized TQFT partition function on M , and the measure $\mu(M)$ is the Goeritz group measure (3.38).

4.2 Algebra automorphisms

As explained in section 2.3, for a given algebra object $\mathcal{A}$ in $\mathcal{T}$, the category of right $\mathcal{A}$ -modules $\mathcal{T}_{\mathcal{A}}$ is the category of topological defect lines (TDLs) that preserve the vertex algebra of $\mathcal{T}$. So our ensemble average when written as a sum over algebra objects can be written as

$$\sum_{\mathcal{A}} \frac{1}{|\text{Inv}(\mathcal{T}_{\mathcal{A}})|} Z_{\mathcal{A}}. \quad (4.11)$$

¹⁴The category $\mathcal{C}_{A'}$ in general is not necessarily a tensor category and so we cannot always speak of “invertible” objects. Instead, what we want is the objects of dimension 1 in our chosen normalization for the dimensions. When $\mathcal{C}$ is braided and A' is commutative special symmetric Frobenius algebra object then $\mathcal{C}_{A'}$. See appendix B for more details.

The group $\text{Inv}(\mathcal{T}_{\mathcal{A}})$ is known to be isomorphic to the automorphism group of the Lagrangian algebra $\mathcal{A}$ [94], and so the ensemble average can be written as

$$\sum_{\mathcal{A}} \frac{1}{|\text{Aut}(\mathcal{A})|} Z_{\mathcal{A}} \quad (4.12)$$

Before we discuss this formula further, let us elaborate on what $\text{Aut}(\mathcal{A})$ means here. Any given simple object a in the category $\mathcal{T}$, being an irreducible representation of the vertex algebra, has endomorphisms $\text{End}(a) \simeq \mathbb{C} \text{id}_a$. Since the object $\mathcal{A}$ in $\mathcal{T}$ is a direct sum of simple objects, $\mathcal{A} = \bigoplus_a n_a a$, its endomorphisms are $\text{End}(\mathcal{A}) \simeq \bigoplus_a \text{Mat}_{n_a}(\mathbb{C}) \text{id}_a$. The group $\text{Aut}(\mathcal{A})$ is then defined as the invertible endomorphisms of object $\mathcal{A}$ which preserve the algebra structure, i.e. it preserves the product morphism m and the unit morphism η . In other words, $\varphi \in \text{Aut}(\mathcal{A})$ if $m \circ \varphi = \varphi \circ m$ and $\eta \circ \varphi = \eta$ [95]. In the CFT picture, the simple anyons are primaries of the vertex algebra associated to $\mathcal{T}$, and so φ would correspond to a linear transformation on $\mathcal{H}_{\text{CFT}}$, acting on operators by conjugation, which preserves the vacuum primary and the fusion rules of these primary operators. This directly means that this is an invertible symmetry transformation that commutes with the vertex algebra.

For multiplicity free Lagrangian algebras, i.e. $n_a \in \{0, 1\}$, one needs to only know the allowed nonzero m_{ab}^c without explicitly knowing their value since in this case the actual value drops out of the conditions on candidate automorphism map φ . In other words, for a map that acts by

$$\varphi(\mathcal{A}) = \bigoplus_a \varphi_a a, \quad \varphi_a \in \text{U}(1), \quad \varphi_0 = 1 \quad (4.13)$$

the condition $m \circ \varphi = \varphi \circ m$ simply reads

$$\varphi_a \varphi_b = \varphi_c, \quad m_{ab}^c \neq 0. \quad (4.14)$$

The possible solutions to these equations are basically elements of the group characters of the Abelian subgroup of the fusion ring induced from the nonzero m_{ab}^c . For example, in diagonal Lagrangian algebras $\mathcal{A} = \bigoplus_a (a, \bar{a})$, the product junctions can be written as $m_{(a, \bar{a})(b, \bar{b})}^{(c, \bar{c})} = N_{ab}^c$ where N_{ab}^c are the fusion coefficients of the chiral part [61]. So elements of $\text{Aut}(\mathcal{A})$ in this case are given by the invertible lines of the chiral part. Let us label them by $\varphi^{(i)} \in \text{Aut}(\mathcal{A})$, then from the Verlinde formula the explicit solutions of (4.14) are given by

$$\varphi_a^{(i)} = \frac{S_{ia}}{S_{0a}}, \quad (4.15)$$

where i denotes an invertible anyon in the chiral part. We see that this is the action of the invertible Verlinde lines as expected in a diagonal RCFT.

Now let us return to eq. (4.12). This is a sum over isomorphism classes of Lagrangian algebras, so this is the natural measure on the groupoid of Lagrangian algebras of $\mathcal{T}$, and hence, this is a uniform measure up to isomorphisms. The holographic duality (4.10) then gives us a generalization of the Siegel-Weil formula where in this case $\mathcal{A}$ is the analog of the lattice and $Z_{\mathcal{A}}$ is the analog of the lattice Θ function. We will elaborate more on the Siegel-Weil formula in section 6.2. The mass formula, or in the groupoid language the cardinality of the groupoid, then gives us the vacuum to vacuum amplitude (3.32) as (if we restore the natural TQFT normalization $\langle 0_g | 0_g \rangle = \mathcal{D}^{g-1}$)

$$\langle \text{HH} | \text{HH} \rangle = \sum_{\mathcal{A}} \frac{1}{|\text{Aut}(\mathcal{A})|} \quad (4.16)$$

In some cases we can have more than one topological boundary condition giving rise to the same boundary CFT because of a hidden duality in the physical boundary condition $|\Omega\rangle$. In other words, the state $|\Omega\rangle$ can be invariant under a subgroup G of the anyon permutation symmetry and hence we can have $\langle \Omega | Z_{\mathcal{A}_I} \rangle = \langle \Omega | Z_{\mathcal{A}_J} \rangle$ for $\mathcal{A}_I = G \cdot \mathcal{A}_J$. If we want to restrict the sum in (4.12) to equivalence classes of $Z_{\mathcal{A}}$ we get

$$\sum_{[\mathcal{A}] \sim_G} \frac{1}{|\text{Aut}(Z_{\mathcal{A}})|} Z_{\mathcal{A}} \quad (4.17)$$

where $\text{Aut}(Z_{\mathcal{A}})$ is a group extension of $\text{Aut}_{\mathcal{T}}(\mathcal{A})$ by $\text{Aut}_G(\mathcal{A})$ where $\text{Aut}_{\mathcal{T}}(\mathcal{A})$ now denotes the automorphism group we discussed previously, and $\text{Aut}_G(\mathcal{A})$ is the automorphism group relative to the action of the permutation symmetry G on $\mathcal{A}$. From the CFT perspective, $\text{Aut}_{\mathcal{T}}(\mathcal{A})$ is the group of invertible topological defect lines that commute with the vertex algebra, while $\text{Aut}_G(\mathcal{A})$ corresponds to the group topological line defects that act on the vertex algebra by an outer-automorphism but leave the presentation of the theory in terms of the vertex algebra invariant (i.e. does not change the choice of the Lagrangian algebra object we choose to describe the CFT). For example if we take a rational compact boson with a partition function $\sum_{\lambda} |\chi_{\lambda}|^2$, where χ_{λ} are characters of $U(1)_k$ Chern Simons, the charge conjugation symmetry (reflection symmetry) changes this presentation into the T-dual presentation of the theory $\sum_{\lambda} \chi_{\lambda} \bar{\chi}_{-\lambda}$ so it is not part of the $\text{Aut}_G(\mathcal{A})$ group in this case.

Note that this extra equivalence between algebra objects — or topological boundary conditions — induced from the choice of gapless boundary condition is not a priori and so we will not deem it as fundamental unless there are no gapless boundary conditions that can distinguish such topological boundary conditions. In that case, from the baby universe Hilbert space, the equivalence becomes like a gauge redundancy where the operator $\widehat{Z[J]}$ will have a smaller subset of eigenstates than the set of topological boundary conditions, i.e. $\mathcal{H}_{\text{BU}}$ is a quotient of $\mathcal{H}_{g \rightarrow \infty}^{\text{MCG}}$ in that case. In the Virasoro

case this will not be an issue to worry about since Virasoro characters are distinct as modular functions unlike the case of some rational chiral algebras where we can have $\chi_a(\tau) = \chi_{G,a}(\tau)$ for some primaries.

5 Examples

5.1 Warm up: Abelian TQFTs

In [1], the general Abelian case was considered in detail and it was shown that the resulting ensemble has equal weights. In this section, we briefly review this result in light of the relation between weights and symmetries.

For Abelian TQFTs, all the anyons are invertible and hence form an Abelian group under fusion. Any Abelian TQFT can be described (although non-uniquely) by an Abelian Chern Simons theory with an action given by

$$S_{CS} = \frac{1}{4\pi} \int_{\mathcal{M}} K_{IJ} A^I \wedge dA^J \quad (5.1)$$

with D gauge fields with a torus gauge group $\mathbb{R}^D/\Lambda$, and K is a non-degenerate integer-valued bilinear form. The anyons are given by elements of the discriminant group $\mathcal{D} \equiv \Lambda^\perp/\Lambda$ where $\Lambda^\perp$ is the dual lattice with respect to the bilinear form K . The twist and braiding (monodromy) of anyons are all determined by K

$$\theta_a = e^{\pi i K^{IJ} a_I a_J}, \quad B_{ab} = e^{2\pi i K^{IJ} a_I b_J} \quad (5.2)$$

The bilinear form K induces a quadratic form $\mathbf{q} : \mathcal{D} \rightarrow \mathbb{Q}/\mathbb{Z}$, where $\mathbf{q}$ is given explicitly by $\frac{1}{2} K^{-1} \bmod 1$. Lagrangian anyon condensation corresponds to gauging a Lagrangian subgroup $\mathcal{L} \subset \mathcal{D}$ (with respect to the induced quadratic form on $\mathcal{D}$). This is related to the construction of code CFTs where $\mathcal{L}$ can be regarded as an even self-dual code over $\mathcal{D}$ as was shown in [37]. The corresponding Lagrangian algebra is just a direct sum of anyons of the Lagrangian subgroup

$$\mathcal{A}_{\mathcal{L}} = \bigoplus_{\ell \in \mathcal{L}} \ell. \quad (5.3)$$

The symmetry lines of the boundary theory are the confined anyons in condensed phase; these are the anyons that braid nontrivially with $\mathcal{L}$ modulo identifications by $\mathcal{L}$. So these are given by the group $\mathcal{D}/\mathcal{L} \simeq \hat{\mathcal{L}}$, where $\hat{\mathcal{L}} \equiv \text{Hom}(\mathcal{L}, U(1))$ is the Pontryagin dual of $\mathcal{L}$. This is indeed the same as $\text{Aut}(\mathcal{A}_{\mathcal{L}})$ from our discussion in section 4.2. This leads to an ensemble of equal weights.

As eluded to before, in some cases the different topological boundary conditions can lead to the same CFT on the boundary. In this case we would have equivalences

between Lagrangian subgroups, which are code equivalences in the code language. The resulting weights if we restrict ourselves to representatives of each equivalence class leads to

$$\langle Z \rangle = \sum_{[\mathcal{L}]} \frac{1}{|\text{Aut}(\mathcal{L})|} Z_{\mathcal{L}} \quad (5.4)$$

where $[\mathcal{L}]$ denotes an equivalence class for which $\mathcal{L}$ is a representative.

Let us illustrate this with a simple example: $U(1)_8 \times \overline{U(1)}_8$ Chern Simons. Let us label the anyons by the pair $(a, b) \in \mathbb{Z}_8 \times \mathbb{Z}_8$. The quadratic form is given by $\mathbf{q}(\lambda, \bar{\lambda}) = \frac{1}{16}(a^2 - b^2) \bmod 1$. There are three Lagrangian subgroups given by

$$\mathcal{L}_1 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\} \simeq \mathbb{Z}_8 \quad (5.5)$$

$$\mathcal{L}'_1 = \{(0, 0), (1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1)\} \simeq \mathbb{Z}_8 \quad (5.6)$$

$$\mathcal{L}_2 = \{(0, 0), (0, 4), (4, 0), (4, 4), (2, 2), (2, 6), (6, 2), (6, 6)\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \quad (5.7)$$

The full bulk anyon permutation symmetry is $\mathbb{Z}_2^L \times \mathbb{Z}_2^R$ where each $\mathbb{Z}_2$ is just the charge conjugation on the left and right copies of $U(1)_8$ respectively. Consider the boundary condition

$$|\tau, \xi\rangle = \sum_{(a,b)} \chi_a(\tau, \xi) \bar{\chi}_b(\bar{\tau}, \xi) |a, b\rangle, \quad (5.8)$$

where

$$\chi_a(\tau, \xi) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{k(n + \frac{a}{k})^2} e^{2\pi i \xi n}, \quad q \equiv e^{2\pi i \tau} \quad (5.9)$$

denotes the character of chiral algebra of $U(1)_k$ with flavor ξ denoting the value of the $U(1)$ gauge field on the boundary. In our case $\mathcal{L}_1$ and $\mathcal{L}'_1$ correspond to compact bosons of radii $R = 2$ and $R = \frac{1}{2}$ represented in terms of $\chi_a(\tau, \xi)$, while the $\mathcal{L}_2$ corresponds to the self-dual boson at $R = 1$. So our ensemble is given by

$$\langle Z(\tau, \xi) \rangle \sim Z_{R=2}(\tau, \xi) + Z_{R=1/2}(\tau, \xi) + Z_{R=1}(\tau, \xi). \quad (5.10)$$

Note that $R = 2$ and $R = \frac{1}{2}$ are T-dual radii (we are using the convention $R \leftrightarrow \frac{1}{R}$); however with nonzero flavor ξ , T-duality acts by

$$Z_R(\tau, \xi) = Z_{\frac{1}{R}}(\tau, -\xi), \quad (5.11)$$

or at the level of characters as

$$\chi_a(\tau, \xi) = \chi_{-a}(\tau, -\xi). \quad (5.12)$$

Hence, our ensemble with the boundary condition $|\tau, \xi\rangle$ distinguishes between partition functions for $\mathcal{L}_1$ and $\mathcal{L}'_1$. If instead we use a boundary condition with $\xi = 0$, $\mathcal{L}_1$ and $\mathcal{L}'_1$

will give the same partition function and so if we want to restrict to equivalence classes we would have two orbits, with representatives $\mathcal{L}_1$ and $\mathcal{L}_2$. Their automorphisms under $\mathbb{Z}_2^L \times \mathbb{Z}_2^R$ are $\text{Aut}(\mathcal{L}_1) = \mathbb{Z}_2$ and $\text{Aut}(\mathcal{L}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Writing the average over equivalence classes we evidently get

$$\langle Z \rangle \sim Z_{R=2} + \frac{1}{2} Z_{R=1}. \quad (5.13)$$

From the CFT perspective, both theories have the same size for the symmetry group that commutes with the $U(1)_8$ algebra. However, the $R = 2$ boson presentation in terms of the primaries of $U(1)_8$ changes under the $\mathbb{Z}_2$ charge conjugation (reflection) symmetry into its T-dual version, while the $R = 1$ boson is invariant. This is why T-duality contributes towards the weight of the self-dual boson here giving rise to a relative weight of $1/2$.

5.2 Illustrative example: $SU(2)_4 \times \overline{SU(2)_4}$

$SU(2)_4$ has 5 anyons L_λ labeled by the Dynkin labels $\lambda \in \{0, 1, 2, 3, 4\}$, where L_0 is the vacuum. The nontrivial fusion rules are given by

$$L_4 \otimes L_\lambda = L_{4-\lambda} \quad (5.14)$$

$$L_1 \otimes L_1 = L_0 \oplus L_2, \quad L_1 \otimes L_2 = L_1 \oplus L_3 \quad L_1 \otimes L_3 = L_2 \oplus L_4, \quad (5.15)$$

$$L_2 \otimes L_2 = L_0 \oplus L_2 \oplus L_4, \quad L_2 \otimes L_3 = L_1 \oplus L_3 \quad L_3 \otimes L_3 = L_0 \oplus L_2. \quad (5.16)$$

The quantum dimensions are $d_0 = d_4 = 1$, $d_1 = d_3 = \sqrt{3}$ and $d_2 = 2$, giving rise to a total quantum dimension $\mathcal{D}_{SU(2)_4} = \sqrt{12}$. The spins of the anyons are given by $h_\lambda = \frac{\lambda(\lambda+2)}{24}$.

The doubled $SU(2)_4$ Chern Simons has two gapped boundaries corresponding to the A-invariant ($SU(2)_4$ WZW CFT) Z_A and the D-invariant ($SU(3)_1$ WZW CFT) Z_D which corresponds to the conformal embedding $su(2)_4 \subset su(3)_1$. The VOA of the bulk TQFT is basically just left and right copies of $su(2)_4$. So according to our result, the weights should be inversely proportional to the invertible symmetry group that commutes with this VOA. For Z_A , these are just the invertible lines from the $su(2)_4$ Verlinde lines, which form a $\mathbb{Z}_2$ group. For Z_D , we have the $\mathbb{Z}_3$ from the Verlinde lines of $su(3)_1$ which also commutes with $su(2)_4 \subset su(3)_1$. We also have charge conjugation which does not commute with $su(3)_1$ because it exchanges the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations of $su(3)_1$ but commutes with $su(2)_4 \subset su(3)_1$ since $\mathbf{3}$ and $\bar{\mathbf{3}}$ project onto the same representation L_2 in $su(2)_4$. So our full group for Z_D is $S_3 \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_2$.

We now want to show how these appear explicitly from the symmetry category of each topological boundary condition. Let us label the topological boundary conditions

states in the TQFT Hilbert space as $|Z_A\rangle$ and $|Z_D\rangle$ respectively. The corresponding Lagrangian algebra objects are given by

$$\mathcal{A}_A = L_{0\bar{0}} \oplus L_{1\bar{1}} \oplus L_{2\bar{2}} \oplus L_{3\bar{3}} \oplus L_{4\bar{4}}, \quad (5.17)$$

$$\mathcal{A}_D = L_{0\bar{0}} \oplus L_{0\bar{4}} \oplus L_{4\bar{0}} \oplus L_{4\bar{4}} \oplus 2L_{2\bar{2}}. \quad (5.18)$$

Condensing the diagonal invariant algebra gives us the symmetry category as $\text{SU}(2)_4$ as expected from diagonal condensation. This gives us

$$\begin{aligned} \langle Z_A | Z_A \rangle |_{\Sigma_g} &= \mathcal{D}_{\text{SU}(2)_4}^{2g-2} \sum_{a \in \text{SU}(2)_4} d_a^{2-2g} \\ &= (12)^{g-1} (2 + 2(3)^{1-g} + 2^{2-2g}). \end{aligned} \quad (5.19)$$

For the D invariant, we get a symmetry category which is a $\mathbb{Z}_2$ extension of Tambara-Yamagami category $\text{TY}(\mathbb{Z}_3)$ [96], let us denote it by $\mathcal{C}_D$. This has 6 invertible anyons forming S_3 fusion rules, and two anyons σ_1 and σ_2 of quantum dimension $\sqrt{3}$. Let us denote the S_3 anyons by the following parametrization

$$S_3 = \langle r, s | r^3 = s^2 = (sr)^2 = 1 \rangle, \quad (5.20)$$

then the other key fusion rules are

$$r \otimes \sigma_i = \sigma_i, \quad \sigma_1 \otimes \sigma_2 = s \oplus sr \oplus sr^2, \quad (5.21)$$

$$s \otimes \sigma_1 = \sigma_2, \quad \sigma_i \otimes \sigma_i = 1 \oplus r \oplus r^2. \quad (5.22)$$

To briefly see how we get $\mathcal{C}_D$, we can do the full condensation of $\mathcal{A}_D$ in multiple Abelian steps while keeping track of the fusion category that includes the confined anyons, we will call this category the domain wall category for simplicity. We start by condensing $\text{SU}(2)_4$ to $\text{SU}(3)_1$ by condensing the L_4 anyon, which by the condensation rules of Bais and Slingerland [97] leads to identifications under L_4 fusion as $L_0 \sim L_4$, $L_1 \sim L_3$, and splits the fixed point L_2 into two anyons L_2^+ and L_2^- . These four sectors form the domain wall category $\text{TY}(\mathbb{Z}_3)$ which lives at the interface between $\text{SU}(2)_4$ and $\text{SU}(3)_1$. Hence, for $\text{SU}(2)_4 \times \overline{\text{SU}(2)}_4$ going to $\text{SU}(3)_1 \times \overline{\text{SU}(3)}_1$, we get $\text{TY}(\mathbb{Z}_3) \times \overline{\text{TY}(\mathbb{Z}_3)}$ as our domain wall category. Now we want to condense $\text{SU}(3)_1 \times \overline{\text{SU}(3)}_1$ (which is an Abelian phase) to the trivial phase by condensing the diagonal $\mathbb{Z}_3$. On the domain wall side, this leads to the quotient $\frac{\text{TY}(\mathbb{Z}_3) \times \overline{\text{TY}(\mathbb{Z}_3)}}{\mathbb{Z}_3^{\text{diag.}}}$, which again by the rules of [97] can be shown to be some $\mathbb{Z}_2$ extension of $\text{TY}(\mathbb{Z}_3)$. This must be a nontrivial $\mathbb{Z}_2$ extension in order for the Drinfeld center of this category to match with $\text{SU}(2)_4 \times \overline{\text{SU}(2)}_4$. One can verify that $\mathcal{Z}_{\text{Drinfeld}}(\mathcal{C}_D) = \text{SU}(2)_4 \times \overline{\text{SU}(2)}_4$ [98, 99] and hence our symmetry category is $\mathcal{C}_D$.

With this $\mathcal{C}_D$ symmetry category we get

$$\begin{aligned}\langle Z_D | Z_D \rangle |_{\Sigma_g} &= \mathcal{D}_{\mathcal{C}_D}^{2g-2} \sum_{a \in \mathcal{C}_D} d_a^{2-2g} \\ &= (12)^{g-1} (6 + 2(3)^{1-g}) \\ &= 2^{2g-1} (3^g + 1).\end{aligned}\tag{5.23}$$

One could also calculate this norm through the intermediate condensation to the Abelian phase $\text{SU}(3)_1 \times \overline{\text{SU}(3)}_1$ phase similar to the method utilized in [37]. This phase has $\mathbb{Z}_3 \times \mathbb{Z}_3$ fusion rules. There are two invariants related by charge conjugation, they correspond to the even self-dual codes $C_1 = \{00, 11, 22\}$ and $C_2 = \{00, 12, 21\}$. Both of these codes correspond to the D invariant in the parent phase (they are code equivalent). The induced norm in $\text{SU}(3)_1$ after condensing from $\text{SU}(2)_4$ is a (un-normalized) projector on the charge conjugation invariant space

$$2^{g-1} (I + \mathcal{S}_{\mathbb{Z}_2^{\text{c.c.}}}),\tag{5.24}$$

where $\mathcal{S}_{\mathbb{Z}_2^{\text{c.c.}}}$ is the surface operator that implements the charge conjugation 0-form symmetry. Applying this to our $\text{SU}(3)_1 \times \overline{\text{SU}(3)}_1$ phase, we get

$$\begin{aligned}\langle Z_D | Z_D \rangle |_{\Sigma_g} &= 2^{2g-2} \langle Z_{C_1} | I \otimes I + I \otimes \mathcal{S}_{\mathbb{Z}_2^{\text{c.c.}}} + \mathcal{S}_{\mathbb{Z}_2^{\text{c.c.}}} \otimes I + \mathcal{S}_{\mathbb{Z}_2^{\text{c.c.}}} \otimes \mathcal{S}_{\mathbb{Z}_2^{\text{c.c.}}} | Z_{C_1} \rangle \\ &= 2^{2g-2} (3^g + 1 + 1 + 3^g) \\ &= 2^{2g-1} (3^g + 1),\end{aligned}\tag{5.25}$$

which matches what we got from the 2d TQFT calculation.

With these results for $\langle Z_A | Z_A \rangle$ and $\langle Z_D | Z_D \rangle$, we see that as $g \rightarrow \infty$ we indeed get $w_A \sim \frac{1}{|\mathbb{Z}_2|}$ and $w_D \sim \frac{1}{|\mathbb{Z}_3|}$. Normalizing the weights to have unit norm, we get

$$\langle Z \rangle = \frac{3}{4} Z_A + \frac{1}{4} Z_D\tag{5.26}$$

For completeness, we give a demonstration of eq. (4.8) in this simple example. To get the overlap $\langle Z_D | Z_A \rangle$, we can start from the regular $\text{SU}(2)_4$ -symmetric 2d TQFT and gauge the algebra object $0 \oplus 4$. This exactly like the domain wall story in the condensation of L_4 in $\text{SU}(2)_4$, so the gauged 2d TQFT will have boundary conditions category given by $\text{TY}(\mathbb{Z}_3)$ which has $\mathcal{D}_{\text{TY}(\mathbb{Z}_3)}^2 = 6 = \frac{1}{2} \mathcal{D}_{\text{SU}(2)_4}^2$, and the full overlap is given by

$$\begin{aligned}\langle Z_D | Z_A \rangle |_{\Sigma_g} &= \mathcal{D}_{\text{TY}(\mathbb{Z}_3)}^{2g-2} \sum_{a \in \text{TY}(\mathbb{Z}_3)} d_a^{2-2g} \\ &= 6^{g-1} (3 + 3^{1-g}),\end{aligned}\tag{5.27}$$

which agrees with (4.8). Similarly, we could have started from the regular $\mathcal{C}_D$ -symmetric 2d TQFT and gauged the algebra corresponding to the $\mathbb{Z}_2$ reflection of the S_3 invertible part to get $\text{TY}(\mathbb{Z}_3)$.

5.3 $\mathbb{Z}_2$ orbifold of the compact boson

The chiral algebra of the $\mathbb{Z}_2$ orbifold of the compact boson of rational radius corresponds to $\text{O}(2)_{2N}$ Chern Simons theory which is obtained by gauging the charge conjugation 0-form symmetry in $\text{U}(1)_{2N}$. Let us start from the data of $\text{U}(1)_{2N}$. We have N Abelian lines forming $\mathbb{Z}_{2N}$ group under fusion, we can label them as ϕ_λ where $\lambda \in \mathbb{Z}_{2N}$. The spins are given by $h_\lambda = \frac{\lambda^2}{4N}$. This quadratic form is preserved by $\lambda \rightarrow \omega\lambda \pmod{2N}$ where

$$\omega^2 = 1 \pmod{4N}, \quad \omega \in \mathbb{Z}_{2N}. \quad (5.28)$$

These transformations form the group of anyon permutation symmetries of $\text{U}(1)_{2N}$. The modular invariants in the doubled theory are classified by factors of N . So for each factor δ , we get radius $R^2 = \frac{\delta^2}{N}$. These correspond to surface operators in the chiral $\text{U}(1)_{2N}$ [52, 72, 100]. Surface operators corresponding to δ such that $\gcd(\delta, \frac{N}{\delta}) \neq 1$ are non-invertible. Note that $\text{U}(1)_{2N}$ distinguishes between a boson and its T-dual radius since the charge conjugation permutes anyons of $\text{U}(1)_{2N}$ acting as $\lambda \leftrightarrow 2N - \lambda$. Let us also focus on the case where N is square free such that all modular invariants are in the same orbit with respect to the anyon permutation group. In other words, all the surface operators in $\text{U}(1)_{2N}$ in this case are invertible. The Ensemble for $\text{U}(1)_{2N}$ is then given by

$$\langle Z \rangle \sim \sum_{\substack{pq=N \\ \gcd(p,q)=1}} \frac{1}{|\mathbb{Z}_p \times \mathbb{Z}_q|} Z_{R^2=\frac{p}{q}}^{\text{circ.}} \quad (5.29)$$

where they all as expected share the same weight. Note that at this point we are keeping R and $\frac{1}{R}$ as distinct members since $\text{U}(1)_{2N}$ can tell the difference, i.e. they are different topological boundary conditions.

Now let us go to $\text{O}(2)_{2N}$ by gauging the charge conjugation 0-form symmetry in $\text{U}(1)_{2N}$. $\text{O}(2)_{2N}$ has $N+7$ anyons, let us explain how they are related to the $2N$ anyons of $\text{U}(1)_{2N}$. The $\text{U}(1)_{2N}$ anyons that transform nontrivially under charge conjugation will group together to form a single anyon of the $\text{O}(2)$ theory, so this gives us $N-1$ anyons denoted by ϕ_k where $k = 1, \dots, N-1$. While the invariant anyons, $\lambda = 0$ and $\lambda = N$, will split into even and odd sectors

$$\phi_0 \rightarrow 1 \oplus j, \quad (5.30)$$

$$\phi_N \rightarrow \phi_N^{(1)} \oplus \phi_N^{(2)}. \quad (5.31)$$

Finally, we have four twisted sectors σ_i, τ_i where $i = 1, 2$. The anyons, their spins, torus characters and quantum dimensions can be summarized as follows (using the notation of [101])

$$1 : \quad \chi = \frac{1}{2}\chi_0 + \frac{1}{2}\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \quad h = 0, \quad d = 1 \quad (5.32)$$

$$j : \quad \chi = \frac{1}{2}\chi_0 - \frac{1}{2}\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \quad h = 1, \quad d = 1 \quad (5.33)$$

$$\phi_N^{(i)} : \quad \chi = \frac{1}{2}\chi_N, \quad (i = 1, 2), \quad h = \frac{1}{4}N, \quad d = 1 \quad (5.34)$$

$$\phi_k : \quad \chi = \chi_k, \quad (k = 1, \dots, N-1), \quad h = \frac{k^2}{4N}, \quad d = 2 \quad (5.35)$$

$$\sigma_i : \quad \chi = \frac{1}{2}\vartheta \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + \frac{1}{2}\vartheta \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}, \quad (i = 1, 2), \quad h = \frac{1}{16}, \quad d = \sqrt{N} \quad (5.36)$$

$$\tau_i : \quad \chi = \frac{1}{2}\vartheta \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} - \frac{1}{2}\vartheta \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}, \quad (i = 1, 2), \quad h = \frac{9}{16}, \quad d = \sqrt{N}. \quad (5.37)$$

where

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(n+\alpha)^2} e^{2\pi i n \beta}, \quad (5.38)$$

and χ_a in the above equations denotes the $U(1)_k$ characters given in (5.9). Note that for $N = 1$, $O(2)$ is the just the same as $U(1)_8$ which we have dealt with in the Abelian section.

The fusion rules are slightly different for the case when N is even versus when it is odd. They are summarized in [101].

The modular invariants of $O(2)_{2N}$ are just all the compact boson theories we can construct from $U(1)_{2N}$ and their $\mathbb{Z}_2$ orbifolds. $O(2)_{2N}$ now does not distinguish between T-dual compact bosons so they are represented by one topological boundary condition, while on the other hand the $\mathbb{Z}_2$ orbifold theories correspond to two topological boundary conditions which are related by exchanging $\sigma_1 \leftrightarrow \sigma_2, \tau_1 \leftrightarrow \tau_2$. This is a 0-form symmetry for $O(2)_{2N}$ Chern Simons which on the CFT side corresponds to a $\mathbb{Z}_2$ shift symmetry for the orbifold boson [102].

Let us first focus on the case when N is prime. In this case there are three topological boundary conditions, let us label them as $|Z^{\text{orb.}}\rangle, |\tilde{Z}^{\text{orb.}}\rangle$ which correspond to the orbifold boson at $R = \sqrt{N}$ and $|Z^{\text{circ.}}\rangle$ which corresponds to the compact (circle) boson. The corresponding algebra objects are given by

$$\mathcal{A}^{\text{orb.}} = 1\bar{1} \oplus j\bar{j} \oplus \bigoplus_{k=1}^{N-1} (\phi_k \bar{\phi}_k) \oplus \sigma_1 \bar{\sigma}_1 \oplus \sigma_2 \bar{\sigma}_2 \oplus \tau_1 \bar{\tau}_1 \oplus \tau_2 \bar{\tau}_2 \quad (5.39)$$

$$\tilde{\mathcal{A}}^{\text{orb.}} = 1\bar{1} \oplus j\bar{j} \oplus \bigoplus_{k=1}^{N-1} (\phi_k \bar{\phi}_k) \oplus \sigma_1 \bar{\sigma}_2 \oplus \sigma_2 \bar{\sigma}_1 \oplus \tau_1 \bar{\tau}_2 \oplus \tau_2 \bar{\tau}_1 \quad (5.40)$$

$$\mathcal{A}^{\text{circ.}} = 1\bar{1} \oplus 1\bar{j} \oplus j\bar{1} \oplus j\bar{j} \oplus 2 \bigoplus_{k=1}^{N-1} (\phi_k \bar{\phi}_k) \quad (5.41)$$

For $Z^{\text{orb.}}$, being the diagonal invariant, the symmetry category is just given by the Verlinde lines which are just the anyons of the chiral part. The invertible ones correspond to $1, \phi_N^{(1)}, \phi_N^{(2)}$ and j , forming either $\mathbb{Z}_2 \times \mathbb{Z}_2$ for N even or $\mathbb{Z}_4$ when N is odd. The permutation invariant $\tilde{Z}^{\text{orb.}}$ has the same invertible lines (acting exactly in the same way as well) since the permuted anyons are the σ and τ sectors.

For $Z^{\text{circ.}}$, we can find the full category of line defects in a similar way to the $\text{SU}(2)_4$ examples. First, we can look at the domain wall defects that live on the interface between a chiral $\text{O}(2)$ and $\text{U}(1)$ by condensing j in $\text{O}(2)$. The category of domain wall defects $\mathcal{C}^{\text{U}(1)_{2N}}$ will have ϕ_λ forming the $\mathbb{Z}_{2N}$ unconfined lines of $\text{U}(1)_{2N}$ and two non-invertible defects σ_1 and σ_2 . The nontrivial fusion rules for the case of even N are given by

$$\sigma_1 \otimes \sigma_2 = \bigoplus_{\lambda \text{ odd}} \phi_\lambda, \quad \sigma_i \otimes \sigma_i = \bigoplus_{\lambda \text{ even}} \phi_\lambda \quad (5.42)$$

$$\phi_{\lambda \text{ even}} \otimes \sigma_i = \sigma_i, \quad \phi_{\lambda \text{ odd}} \otimes \sigma_1 = \sigma_2 \quad (5.43)$$

Now we combine the chiral and antichiral parts of the condensed $\text{U}(1)$ phases and gauge the diagonal $\mathbb{Z}_{2N}$. Similar to the $\text{SU}(2)_4$ case, we will get a symmetry category as a $\mathbb{Z}_2$ extension of $\mathcal{C}^{\text{U}(1)_{2N}}$ where the $\mathbb{Z}_2$ acts on the $\mathbb{Z}_{2N}$ by charge conjugation giving rise to the dihedral group D_{2N} as the invertible group while just trivially doubling the twist defects σ_i .¹⁵ Hence, we get

$$\langle Z \rangle \sim \frac{1}{4} Z_{R=\sqrt{N}}^{\text{orb.}} + \frac{1}{4} \tilde{Z}_{R=\sqrt{N}}^{\text{orb.}} + \frac{1}{4N} Z_{R=\sqrt{N}}^{\text{circ.}} \quad (5.44)$$

For $N = 1$, this reduces to our $\text{U}(1)_8$ example where $Z_{R=\sqrt{N}}^{\text{orb.}}$ is given. For $N = 2$, we have $\text{O}(2)_4 \simeq \text{Ising} \times \text{Ising}$ which was the example studied in [37].

Finally, the case where N is square free is very similar, but now we have an entire orbit for the circle branch and the orbifold branch where each theory in the same orbit shares the same weight. Thus in this case, restricting to the equivalence classes of CFTs,

$$\langle Z \rangle \sim \sum_{\substack{pq=N, p>q \\ \gcd(p,q)=1}} \left(\frac{1}{2} Z_{R^2=\frac{p}{q}}^{\text{orb.}} + \frac{1}{4N} Z_{R^2=\frac{p}{q}}^{\text{circ.}} \right). \quad (5.45)$$

¹⁵The rigorous reason why we got this $\mathbb{Z}_2$ extension in both of our examples is a special case of the general result discussed in [103].

As $N \rightarrow \infty$, we see that the orbifold branch dominates. This can be viewed as a regularized version of averaging over the full orbifold branch and the circle branch with their relative symmetry factor.

5.4 ADE classification: $SU(2)_k$ and minimal models

$SU(2)_k$ Chern Simons. The physical modular invariants of doubled $SU(2)_k$ Chern Simons theory follow the famous ADE classification [104, 105]. However, the number of linearly independent modular invariant combinations of left and right torus characters is given by the half the number of factors of $k + 2$, which is related to the classification problem for $U(1)_{2N}$ with $N = k + 2$ [105]. Hence, the general Poincare series does not always lead to a combination of only physical invariants as was shown in [34]. These unphysical modular invariants cannot be extended to all genera since they do not correspond to algebra objects in $SU(2)_k$, and there is evidence that they disappear completely in some cases at relatively small genus [106]. The TQFT gravity proposal cures this problem by construction and leads to a linear combination of the physical invariants with positive weights as shown before.

Let us briefly describe the ADE classification for $SU(2)_k$. The A series corresponds to the diagonal invariant, which is just the $SU(2)_k$ WZW. The D series corresponds to the $\mathbb{Z}_2$ orbifold of the $SU(2)_k$ WZW, which is possible only for k even since the $\mathbb{Z}_2$ symmetry is anomalous otherwise. The $D_{2\ell}$ series corresponds to a simple-current extended algebra invariant while the $D_{2\ell+1}$ is a permutation invariant. Finally, there are three exceptional cases: E_6 and E_8 are exceptional conformal embeddings corresponding to $su(2)_{10} \subset so(5)_1$ and $su(2)_{28} \subset (G_2)_1$ respectively, while the E_7 is an exceptional permutation of the D_{10} invariant of $su(2)_{16}$. They are related to their respective $SU(2)_k$ WZW CFT by generalized gauging of non-invertible symmetries (see for example [107]).

We now want to find the weights of the ADE invariants from their symmetries. The TDLs of the ADE theories arising from the $su(2)_k$ algebra are related to Ocneanu graph algebras [108–111], which are related to the ADE Dynkin graphs. Remarkably, the invertible symmetries of these TDLs are isomorphic to automorphisms of the Dynkin graphs.¹⁶ We thus get the general formula for any given level

$$\langle Z \rangle \sim \frac{1}{|\Gamma_A|} Z_A + \frac{1}{|\Gamma_D|} Z_D + \frac{1}{|\Gamma_E|} Z_E \quad (5.46)$$

where Γ denotes the automorphism group of the corresponding Dynkin graph. For example, in our $SU(2)_4$ example, the S_3 symmetry we got for the D invariant was

¹⁶This seems to be special to the ADE graphs of $su(2)_k$. For other affine lie algebras there are similar graphs analogous to the ADE, and in general the map between invertible symmetries of the CFT and graph symmetries is not one to one. I thank Noah Snyder for a discussion about this.

precisely the graph symmetry of D_4 . For any even k beyond $k = 4$, we get equal weights for the A and D invariants since their graph symmetries are just $\mathbb{Z}_2$.

Virasoro minimal models ($c < 1$ pure gravity). The unitary minimal models $\mathcal{M}(p, p+1)$ have ADE classification closely related to the $su(2)_k$ [104, 105]. This is related to the fact that they can be formulated in terms of a coset construction

$$\mathcal{M}(k+2, k+3) \simeq \frac{su(2)_k \otimes su(2)_1}{su(2)_{k+1}}, \quad (5.47)$$

which, in terms of TQFT language, amounts to starting from $SU(2)_k \times SU(2)_1 \times SU(2)_{-k-1}$ and gauging (condensing) the diagonal $\mathbb{Z}_2$ 1-form symmetry. So we have an ADE classification related to the two nontrivial copies of $SU(2)$: AA , AD (or DA), AE (or EA). The symmetries follow a similar pattern to the $SU(2)$ case (5.46), but now these are the full set of invertible symmetries of the CFTs since the chiral algebra here is $c < 1$ Virasoro. This gives a viable proposal for the boundary ensemble dual to pure $c < 1$ gravity generalizing the results of [33, 75] beyond the cases of Ising and Tricritical Ising, and solving the negativity issues that arise in the Poincare series [34].

6 Implications for noncompact TQFTs

So far our main results are strictly speaking valid for compact TQFTs based on semi-simple MTCs. Going beyond that regime, we have to be a bit schematic and use a more heuristic approach based on the lessons we have learned from the compact/rational case. What we have learned so far can be summarized as follows:

- We start from a TQFT based on the representation theory of the vertex algebra $\mathcal{V}_L \times \bar{\mathcal{V}}_R$.
- The dual ensemble consists of CFTs constructed from topological boundary conditions of the TQFT; these are all CFTs with $\mathcal{V}_L \times \bar{\mathcal{V}}_R$ as their common maximally extended vertex algebra.
- The CFTs are inversely weighted by the size of their automorphism group relative to $\mathcal{V}_L \times \bar{\mathcal{V}}_R$, which is the group of invertible topological defects that commute with $\mathcal{V}_L \times \bar{\mathcal{V}}_R$.

We can now try to apply these lessons to cases where $\mathcal{V}_L \times \bar{\mathcal{V}}_R$ is not rational. However, we will assume that the vacuum must appear in the physical spectrum of the CFTs akin to the rational case.

6.1 $\mathbb{R}$ Chern Simons

The holographic duality of the Narain ensemble average [26, 27] can be, at least formally, understood as a case of this TQFT gravity framework. Consider the Narain ensemble of D bosons. The bulk TQFT should be understood as a representation of left and right copies of the Heisenberg VOA $u(1)^D \times \bar{u}(1)^D$, which can be schematically taken to be $\mathbb{R}^{D,D}$ Chern Simons theory. This is an Abelian theory, so we should look for Lagrangian subgroups and use the analog of (5.4). As noted in [37, 60], the Lagrangian subgroups are given by even self-dual Lorentzian lattices $\Lambda \subset \mathbb{R}^{D,D}$, which gives us the symmetry $\mathbb{R}^{D,D}/\Lambda \simeq U(1)^D \times U(1)^D$.¹⁷ These are indeed the symmetries that commute with the $u(1)^D \times \bar{u}(1)^D$ current algebra. The moduli space of topological boundary conditions will be given by $O(D, D)/O(D) \times O(D)$ and since they all have the same symmetry as in any Abelian theory, namely $U(1)^D \times U(1)^D$ in this case, we should use the uniform measure on $O(D, D)/O(D) \times O(D)$ which is the Haar measure. Pushing this sum to the distinct Narain CFTs under the duality $O(D, D, \mathbb{Z})$ will give us the usual Narain average over $O(D, D, \mathbb{Z}) \backslash O(D, D)/O(D) \times O(D)$ with the Haar measure, which is the same as the Zamolodchikov measure. This leads to the $U(1)$ gravity result of [26, 27]. Even though the TQFT gravity sums over all topologies, we can argue that the bulk theory, which is schematically $\mathbb{R}$ Chern Simons, is only sensitive to handlebodies and that is why the bulk sum is a Poincare series. A regularized version of $U(1)$ gravity was studied in [113, 114] where the bulk TQFT was taken to be D copies of $\mathbb{Z}_k$ gauge theory for prime k and it was argued there that the full Narain moduli averaged with the Haar measure is reproduced in the limit $k \rightarrow \infty$. In that construction, the bulk theory was only sensitive to handlebodies. It was shown recently in [115] that this behavior persists for the non-square-free case of $k = p^2$ for prime p as $p \rightarrow \infty$.

6.2 Virasoro TQFT

For the Virasoro case, since our vertex algebra now is $\text{Vir}_c \times \overline{\text{Vir}_c}$, the ensemble should include all CFTs of central charge c , where each CFT is inversely weighted by the size of its full symmetry group.¹⁸ This tells us right away that highly symmetric theories will be heavily suppressed, for example theories with continuous symmetries will effectively drop out of the ensemble. This seems to be compatible with pure 3d gravity interpretation as $c \rightarrow \infty$. In the large c limit, the space of CFTs is very vast

¹⁷This is in line with recent work that relates $\mathbb{R}$ Chern-Simons theory to the SymTFT of $U(1)$ symmetries [112].

¹⁸The idea of averaging over CFTs with $1/|\text{Aut}|$ factor was speculated by Alex Maloney on multiple occasions given the results of $U(1)$ gravity (see for example [116]).

and has an enormous amount of highly symmetric theories as was argued in [117], for example we will have a proliferation of products of small c theories.. Luckily for us, these highly symmetric theories will crumble under their own weight. Hence, one might expect that a typical theory of such an ensemble would have a large gap of order c and sparse spectrum of light states. This is motivated by the average solution of the bootstrap at large c as shown in [8] as well as wisdom from the result of Narain average where a typical Narain theory at large c has a primary gap of $O(c)$ relative to the $u(1)$ vacuum even though we are unable to construct a single such theory.

To make more sense of this ensemble of all CFTs, we need to consider how to weigh families of CFTs that form a conformal manifold. A conformal manifold is generated by exactly marginal deformations, so all points share the same symmetry except for some special loci points with enhanced affine symmetry, which are points of zero measure on the manifold. Since we interpret our ensemble average result as uniform up to isomorphisms, the natural uniform measure to use on conformal manifolds is the Zamolodchikov measure. Each conformal manifold is then weighted by its respective symmetry factor, while the special loci points are weighted separately similar to isolated points in the space of CFTs. Motivated by the proposal of [117], we can try to motivate the following measure on the space of CFTs at a given central charge¹⁹

$$\sum_{\mathcal{M}} \frac{1}{|\text{Aut} C_{\mathcal{M}}|} \int d\mu_{\text{Zam.}} + \sum_C \frac{1}{|\text{Aut}(C)|}, \quad (6.1)$$

where $\mathcal{M}$ denotes a conformal manifold and $C_{\mathcal{M}}$ denotes a typical representative CFT from that manifold. This is in contrast to the proposed measure of [117] which weighs all CFTs with equal weights and does not agree with a pure gravity interpretation. We interpret the measure in (6.1) as the correct maximum ignorance measure where the equal weights are assigned to “labeled” CFTs which gives us a symmetry factor weight once we project onto isomorphism classes. This means that we should view the CFTs via some algebraic definition and then the space of CFTs should be a groupoid of some sort. We will comment on this aspect in the context of chiral CFTs later in this section.

To make sense of (6.1) as a normalizable measure, we need to introduce a cutoff on conformal manifolds with divergent volumes. This is motivated by the distance conjecture [118–120] where we expect CFTs at infinite distance on a conformal manifold to have a tower of light states and will not share the same symmetry common to all other members of the conformal manifold. So if we want to implement our $1/|\text{Aut}|$ measure properly they should be included separately and to account for that we should

¹⁹In our schematic setup here, we will work with a particular value of the central charge rather than a small window as done in [117] since our construction fixes a “bulk TQFT” based on the representation of the virasoro algebra with an a priori given central charge.

provide a cutoff on the integral over the manifold in (6.1). The inclusion of a cutoff can be implemented universally by defining the measure with a minimum gap as was done in [117].

We can now try to apply this for cases where we have at least a rough classification of CFTs beyond $c < 1$.

c=1 pure gravity. For $c = 1$, even though we do not have a proven classification, we sort of have a handle on what the space of (unitary) CFTs looks like. There are two main conformal manifolds: a compact boson branch, an orbifold branch, and then there are three isolated orbifold points of the compact boson at the self-dual radius corresponding to orbifolding by the three exceptional discrete subgroups of $SU(2)$: Tetrahedral (T), Octahedral (O) and Icosahedral (I) [121]. As we argued, the CFTs at infinite distance of these branches (the decompactified theories) should be treated separately, and in a suitable regularization their contribution to the ensemble will be negligible.²⁰ There is one more known noncompact theory to worry about: the Runkel-Watts (RW) theory [122] which is a $c \rightarrow 1$ limit theory of the AA minimal model and is the analog of Liouville theory at $c = 1$. This is a theory with continuous spectrum and no invertible symmetries. It is not obvious if we should include this theory in our ensemble because of pathologies like non-normalizability of the vacuum state and the fact that (similar to Liouville) the vacuum does not appear in physical OPEs. We will be more conservative and exclude such theories from the ensemble under the assumption that will not appear in a regularized version of the TQFT gravity framework for the virasoro case.

Just to demonstrate (6.1) in this case, the (unnormalized) average in the formal sense would look like

$$\begin{aligned} \langle Z \rangle = & \frac{1}{|D_8|} \int_{1+}^{R_{\max}} \frac{dR}{R} Z^{\text{orb.}}(R) + \frac{1}{|(U(1) \times U(1)) \rtimes \mathbb{Z}_2|} \int_{1+}^{R_{\max}} \frac{dR}{R} Z^{\text{circ.}}(R) \\ & + \frac{Z_{R=1}}{|(SU(2) \times SU(2))/\mathbb{Z}_2|} + \frac{Z_T}{|S_3|} + \frac{Z_O}{|\mathbb{Z}_2|} + \frac{Z_I}{|\mathbb{Z}_2|} \quad (6.2) \end{aligned}$$

where the symmetries of T, O and I can be deduced from their chiral algebra modular S-matrices which can be found in [102].²¹ As we can see, the compact boson branch effectively drops out.

²⁰For example, if one considers decompactification limit of the rational boson of radius $R^2 = N$ viewed as the diagonal invariant of $U(1)_{2N}$ Chern Simons, we find that that the relative symmetry is $\mathbb{Z}_N$, and since the partition function diverges as R , the symmetry factor will still win.

²¹We assume that they do not possess any other invertible symmetries beyond the automorphisms (outer and inner) of their chiral algebra.

Chiral gravity. We can consider the case of having only a chiral copy of Virasoro for which we get an ensemble of chiral CFTs at a given central charge, where we should consider $c = 24k$ for integer k in order to cancel the gravitational anomaly. On the gravity side, this corresponds to topologically massive gravity where we add a gravitational Chern Simons term with a particularly tuned coupling that kills the right-moving Virasoro asymptotic symmetry [123], and is thus known as chiral gravity. It was shown in [124] that the Poincare series of chiral gravity leads to the (chiral) extremal partition function conjectured by Witten [125], and hence evades the negativity problem that plagues the Poincare series of pure gravity. However, beyond $k = 1$, there is some evidence that there are no extremal CFTs [126], so chiral gravity with a just a handlebody sum over topologies seems to be unphysical beyond $k = 1$. The TQFT gravity framework would seem to solve this conundrum for $k > 1$ by posing that the true sum over topologies involves all manifolds (not just handlebodies) is dual to an ensemble of actual CFTs which will not be equal to the unphysical extremal partition function. Note that the ensemble is expected to be close to extremal, for example theories like k copies of the monster will be highly suppressed compared to other less symmetric theories.

For $k = 1$, there is the well-known classification proposed by Schellekens [127] which contains 71 theories, classified by their space of spin 1 currents denoted as V_1 . These theories were studied later and all theories, except for the Monster CFT [47], have been shown to be unique to their V_1 spaces [128–130]. It is highly conjectured that the Monster is the only theory with no continuous symmetries, but yet remains an open problem.

Our ensemble average for $k = 1$ is then just the ensemble of theories with $\dim V_1 = 0$ which is highly conjectured to be just the Monster. This seems to agree, at least at the level of the partition function with the Poincare series for $k = 1$. This poses an interesting puzzle with two possible resolutions. The first is that the extra topologies do not matter for some reason in this scenario, and it would be important to understand why if that is the case. The other is that the Poincare series could be giving an a priori different answer when viewed as a linear combination (with some negative coefficients) of $c = 24$ CFTs' partition functions than that of the sum over all topologies, but both agree at the level of the modular functions.²² To know for sure, one would in principle need to go to higher genus and compare the Poincare series to the sum over all topologies.

²²Note that this is possible regardless of the uniqueness of the Monster, since all $c = 24$ CFTs have the same partition function $J(\tau)$ as the monster up to an overall additive constant related to their $\dim V_1$, and the Poincare series does not guarantee an a priori positive-semidefinite linear combinations of the resulting partition functions.

Siegel-Weil formula for self-dual VOAs. Averaging over CFTs with symmetry factors tells us that we should think about CFTs from some algebraic definition rather than just averaging over OPE data. One way to do this is within the framework of VOAs. Chiral CFTs have been extensively studied in such framework where they correspond to self-dual VOAs (also known as holomorphic VOAs) which have only one primary. There are various approaches to generalize the framework of VOAs to the nonchiral case (see for example [54, 55, 131]); however, for concreteness, we will focus on the chiral case when discussing VOAs in this section.

As eluded to in section 4.2, the holographic duality of TQFT gravity gives a generalization for the Siegel-Weil formula. We will now discuss this more concretely in the context of VOAs. The usual Siegel-Weil formula can be cast in a VOA context by mapping the lattice to its corresponding lattice VOA via the known construction of [48]. In this case $|\text{Aut}(V_\Lambda)|$ is infinite for all V_Λ so the direct $\sum_\Lambda \frac{1}{|\text{Aut}(V_\Lambda)|} \Theta_\Lambda$ is ill defined. Instead, the meaningful statement is to extract the common continuous symmetry of $U(1)^c$ leaving us with the usual Siegel-Weil formula, which can be rewritten holographically as

$$\sum_\Lambda \frac{1}{|\text{Aut}(\Lambda)|} Z_\Lambda(\Omega_{\tilde{g}}) = \sum_{\gamma \in \Gamma(\tilde{g}) \backslash \text{Sp}(2\tilde{g}, \mathbb{Z})} \chi_0^{u(1)^c}(\gamma \cdot \Omega_{\tilde{g}}), \quad (6.3)$$

where Z_Λ is the partition function of the free-boson theory based on lattice Λ , $\chi_0^{u(1)^c}$ is the vacuum character of the $u(1)^c$ algebra and the coset sum is just the sum over handlebodies with boundary $\Sigma_{\tilde{g}}$. This is the same result that leads to $U(1)$ gravity (but chiral in this context) so there are no surprises there. Next, we want to consider a generalization of this beyond lattice VOAs.

The concept of a *genus of VOAs* was first proposed by Höhn [132] where he defined it as the set of all VOAs that have the same modular tensor category (MTC) as their representation category. If we consider holomorphic VOAs, where the associated MTC and TQFT are trivial, this definition is exactly the ensemble of all chiral CFTs at a given central charge. Höhn proposed a mass formula for a genus of VOAs $\text{gen}(V)$ as well as a corresponding average over the VOA partition functions (see also [133] for recent relevant work) as

$$\sum_{W \in \text{gen}(V)} \frac{1}{|\text{Aut}(W)|}, \quad \sum_{W \in \text{gen}(V)} \frac{Z_W}{|\text{Aut}(W)|} \quad (6.4)$$

where Z_W denotes the partition function of W . The TQFT gravity duality then suggests a physical interpretation of these quantities as partition functions of chiral Virasoro gravity, with the former being a sum over all closed 3-manifolds and the latter a sum

over 3-manifolds with boundaries akin to the Siegel-Weil formula, namely

$$\sum_{W \in \text{gen}(V)} \frac{1}{\text{Aut}(W)} \stackrel{?}{=} \lim_{g \rightarrow \infty} \sum_{\gamma \in \Gamma(g) \backslash \text{Sp}(2g, \mathbb{Z}) / \Gamma(g)} \mu_0(M_\gamma) \mathbf{Z}^{\text{Vir}}(M_\gamma), \quad \partial M_\gamma = \emptyset \quad (6.5)$$

$$\sum_{W \in \text{gen}(V)} \frac{Z_W(\Omega_{\tilde{g}})}{\text{Aut}(W)} \stackrel{?}{=} \lim_{g \rightarrow \infty} \sum_{\gamma \in \Gamma(g, \tilde{g}) \backslash \text{Sp}(2g, \mathbb{Z}) / \Gamma(g)} \mu_{\tilde{g}}(M_\gamma) \mathbf{Z}^{\text{Vir}}(M_\gamma; \Omega_{\tilde{g}}), \quad \partial M_\gamma = \Sigma_{\tilde{g}} \quad (6.6)$$

where $\mu_{\tilde{g}}(M)$ is defined as in (3.38).

7 Discussion

In this paper we have shown that a 3d TQFT summed over all topologies gives rise to an ensemble of boundary theories where each member is weighted by an appropriate symmetry factor corresponding to its invertible symmetry relative to the bulk. When viewed in terms of Lagrangian algebras, the ensemble average has a natural interpretation as the uniform-up-to-isomorphism average of boundary theories that can be constructed from the vertex algebra associated with the bulk TQFT. This gave us a generalization of the Siegel-Weil formula now applied to Lagrangian algebras.

As a toy model for holography, the duality we presented can be viewed as a confirmation of the principle of maximum ignorance [134] where semiclassical gravity is viewed as a maximally agnostic coarse graining of some fine grained microscopic description. The naive application of the principle of maximal ignorance would lead us to deduce that we should assign equal weights to all CFTs; however, as we have seen, if we define the CFTs algebraically then the correct maximally agnostic average that takes isomorphisms into account should include the appropriate automorphism/symmetry factors.

The coarse-grained interpretation of ensemble averaging in our context can be backed up by the following argument. The bulk calculation that we did involved no lines ending on the boundary. Information-wise, this amounts to having only knowledge about the asymptotic left and right vertex (chiral) algebra symmetry on the boundary and the identity operator. Maximal ignorance would then tell us that we should do a uniform (up to isomorphism) average on all theories that have this vertex algebra symmetry and the identity operator, and indeed the bulk gravity calculation involving the sum over all manifolds does exactly that. Having access to more knowledge on the boundary through, for example, specific local operators should lead us then to a more fine-grained average over just the theories that include these local operators. This can be reflected in a similar bulk calculation involving bulk lines that end on

the boundary, where now the bulk sum of TQFT gravity needs to be done with the mapping class group of the punctured boundary surface $\text{MCG}(\Sigma_{g,n})$, which will project us onto only theories where the boundary operators are local. In semiclassical gravity, we typically do not have access to fine grained boundary operators above the black hole threshold, we can at best know the energy and the spin of a primary operator since these are the macroscopic quantities of a BTZ black hole state. With only this knowledge, we will still get a coarse ensemble average due to the universality of the Cardy formula. However, if we somehow have access to some light operators below the threshold (which are sparse in the spectrum), we can fine grain our ensemble average since these operators are not expected to be universal (except for the identity operator of course). This seems to agree, at least in spirit, with the conclusion of Schlenker and Witten in [135] that there is no ensemble-averaging below the black hole threshold. In the full theory of quantum gravity, we should have access to black hole microstates and so our “ensemble” should be as fine-grained as it can get, meaning we should probably only have one microscopic theory. This would mean full knowledge about which α -state we are in, and as it is well known, in contrast with the no-boundary HH state, α -states lead to factorization.

We conclude this section with some open questions and possible future directions.

- While we have considered the implications of the TQFT gravity toy model to VTQFT, which seems to conform to the growing notion that VTQFT is the maximal SymTFT of all topological line defects in a CFT [23, 89], we need to actually try to implement our arguments explicitly in VTQFT to verify if these implications hold. It is not obvious how to do this, but an important step is to first understand the implications of the measure (3.38) of the sum over manifolds. Particularly, it would be interesting to understand if this measure resolves the tension between 3d gravity and VTQFT on off-shell manifolds [21]. Another direction to potentially understand these issues is by working in some regularized version of VTQFT as for example in recent Turaev-Viro approaches [23–25].
- Another direction is to study this approach in higher dimensions. Gravity is of course not topological beyond 3d and so the direct analog for our toy model is not a representative toy model for gravity in this case. Perhaps this could be part of the reason why pure gravity is not expected to be dual to an ensemble in $d > 3$. From the baby universe Hilbert space perspective, this would mean that $\dim \mathcal{H}_{\text{BU}} = 1$ in this case, i.e. the Hartle-Hawking state is the unique state as conjectured in [136] for $d > 3$. Nonetheless, one could study examples of ensembles in higher dimensions and see if the dual ensemble has the symmetry factor

structure similar to the one presented here. This is trivially true for Abelian TQFTs as for example the ensemble of 4d Maxwell theories considered in [115], so it would be more interesting to consider nonAbelian examples based on general higher fusion categories. A simple example would be to consider a $\mathbb{Z}_2$ charge conjugation gauging of the 5d Abelian BF theory considered in [115]. The resulting ensemble should include U(1) Maxwell theories with different couplings and their $\mathbb{Z}_2$ orbifolds the O(2) gauge theories. We might expect to find results similar to the $\mathbb{Z}_2$ orbifold of the compact boson considered in section 5.3.

- Finally, the holographic duality we considered gave us a Siegel-Weil formula for Lagrangian condensations. It is tempting then to ask if we one can similarly derive a Siegel-Weil formula for an average over condensable — but not Lagrangian — algebras that condense to the same phase. This would be the analog of a genus of non-self-dual VOAs, i.e. the average is not modular invariant but instead transforms in some representation of the mapping class group, which is the representation of the condensed phase. A prime example is the average over non-self-dual Narain lattices considered in [30]. It would be interesting to investigate if this could be derived in the large genus limit by considering a projection on a non-trivial representation of the mapping class group and then performing genus reduction.

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A Brief review of modular tensor categories

A simple object of the MTC is one that has endomorphisms proportional to the identity, i.e. $\text{End}(a) \simeq \mathbb{C}\text{id}_a$. A semisimple category is one where any object is a direct sum of simple objects. In semisimple tensor categories, we can take the tensor product (fusion) of two simple objects and express it in terms of a direct sum of simple objects

$$a \otimes b = \bigoplus_c N_{ab}^c c \tag{A.1}$$

The fusion coefficient N_{ab}^c is the dimension of the vector space of morphisms $V_{ab}^c \equiv \text{Hom}(a \otimes b, c)$. The fusion is associative, and in this case of MTCs it is also commutative due to the existence of braiding structure (to be explained below).

There always exists a unique trivial anyon (vacuum) 0 such that $a \otimes 0 = a$. For each object a there is a dual object $a^\vee$ such that

$$a \otimes a^\vee = 0 \oplus \dots \quad (\text{A.2})$$

This allows us to define dual morphisms $V_c^{ab} \equiv \text{Hom}(a \otimes b, c) \simeq V_{ab}^c$. We denote the morphisms graphically as

(A.3)

We also have orthogonality and completeness relations as follows:

$$\begin{array}{c} c' \\ | \\ \bullet \mu' \\ / \quad \backslash \\ a \quad b \\ \backslash \quad / \\ \bullet \mu \\ | \\ c \end{array} = \delta_{\mu\mu'} \delta_{cc'} \begin{array}{c} | \\ | \\ c \end{array} = \sum_c \sum_\mu \begin{array}{c} a \quad b \\ \backslash \quad / \\ \bullet \mu \\ | \\ c \\ | \\ \bullet \mu \\ / \quad \backslash \\ a \quad b \end{array} \quad (\text{A.4})$$

Using the isomorphism between V_{ab}^{cd} and $V_{ac}^{b\vee d}$, we can show that

$$\sum_e N_{ab}^e N_{cd}^e = \sum_f N_{ac^\vee}^f N_{b^\vee d}^f. \quad (\text{A.5})$$

Thus if we define the matrices $(\mathbf{N}_a)_{bc} := N_{ab}^c$, we get $[\mathbf{N}_a, \mathbf{N}_b] = 0$ and hence they can be simultaneously diagonalized. Since these matrices are positive semi-definite, one can use an analog of the Frobenius-Perron theorem to show that there is a unique vector $\mathbf{d}$ of maximal positive eigenvalues where we have

$$\mathbf{N}_a \mathbf{d} = d_a \mathbf{d}. \quad (\text{A.6})$$

The eigenvalues d_a are called the quantum dimensions (also known as Frobenius Perron dimensions). These can be defined as the trace of the identity morphism. Graphically, this is given by the unknot of an anyon

$$\text{a circle} = \text{Tr}(\mathbf{id}_a) = d_a \quad (\text{A.7})$$

From (A.6) we can see that the quantum dimensions are preserved under fusion

$$d_a d_b = \sum_c N_{ab}^c d_c. \quad (\text{A.8})$$

From d_a , we can define the total quantum dimension of the category as²³

$$\mathcal{D} = \sqrt{\sum_a d_a^2}. \quad (\text{A.9})$$

The associativity of the fusion rules is encoded in the fusion F matrix (6j symbols) defined as

$$\begin{array}{c} d \\ \diagup \\ \nu \\ \diagdown \\ a \quad b \quad c \end{array}
 \begin{array}{c} e \\ \diagup \\ \mu \\ \diagdown \\ b \quad c \end{array}
 = \sum_{f, \rho, \sigma} [F_d^{abc}]_{(e; \mu\nu)(f; \rho\sigma)}
 \begin{array}{c} d \\ \diagup \\ \sigma \\ \diagdown \\ a \quad b \quad c \end{array}
 \begin{array}{c} \rho \\ \diagup \\ f \\ \diagdown \\ a \quad b \end{array}
 \quad (\text{A.10})$$

The braiding structure is given by the braiding matrix R , defined as

$$\begin{array}{c} c \\ | \\ \mu \\ \curvearrowright \\ a \quad b \end{array}
 = \sum_{\nu} [R_{ab}^c]_{\mu\nu}
 \begin{array}{c} c \\ | \\ \nu \\ \diagdown \\ a \quad b \end{array}
 \quad (\text{A.11})$$

The F and R matrices satisfy consistency conditions called the pentagon and hexagon equations (Moore Seiberg conditions) [137]. F and R depend on the choice of basis in the vector spaces of morphisms V_{ab}^c , this is called a choice of “gauge”. The R matrix allows us to define the following gauge invariant quantities: the twist θ_a and the modular S matrix.

$$\begin{array}{c} | \\ \curvearrowright \\ a \end{array}
 = \theta_a
 \begin{array}{c} | \\ a \end{array}
 \quad (\text{A.12})$$

$$S_{ab} = \frac{1}{\mathcal{D}}
 \begin{array}{c} \bigcirc \quad \bigcirc \\ a \quad b \end{array}
 \quad (\text{A.13})$$

²³ $\mathcal{D}^2$ is also known as the Frobenius Perron dimension (FPdim) of the category.

where $\theta_a = e^{2\pi i s_a}$ and s_a is the spin of the anyon, which in the chiral case would be just chiral dimension h_a while in the nonchiral case it is $h - \bar{h}$.

The fusion rules are related to the S matrix via the Verlinde formula

$$N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} S_{cx}^*}{S_{0x}} \quad (\text{A.14})$$

so the S matrix diagonalizes the fusion matrices $\mathbf{N}_a$. The quantum dimensions and the total quantum dimension are also related to S by $d_a = \frac{S_{0a}}{S_{00}}$ and $\mathcal{D} = \frac{1}{S_{00}}$.

These twists define the matrix $T_{ab} = \theta_a \delta_{ab}$ which together with S give us a representation of $SL(2, \mathbb{Z})$. They satisfy

$$(ST)^3 = e^{2\pi i \frac{c_-}{8}} C, \quad S^2 = C, \quad C^2 = \mathbb{1} \quad (\text{A.15})$$

where C is the charge conjugation matrix which maps an anyon a to its dual $a^\vee$. The representation is projective in general, but it becomes linear when in the non-anomalous case $c_- = 0 \pmod{8}$.²⁴

The fusion ring can sometimes be preserved under permuting some anyons. We will call these anyon permutation symmetries. They preserve all the gauge invariant quantities like S and T but can act on F and R by a gauge transformation. The anyon permutation symmetries are 0-form symmetries of the TQFT. They do not correspond to the full 0-form symmetry of the TQFT though since one can have symmetries that do not permute anyons. These are like inner versus outer automorphisms of the underlying vertex algebra $\mathcal{V}_{\mathcal{T}}$.

B Normalization of quantum dimensions of module categories

We will start with the formalities and then explain the physics picture. Let $\mathcal{C}$ be a tensor category and A be a connected special symmetric Frobenius algebra object. The category of right A -modules $\mathcal{C}_A$ is called a module category [138]. We want to understand how to define quantum dimensions for simple objects in $\mathcal{C}_A$, where a quantum dimension here is a trace of the identity morphism of the simple object. If $\mathcal{C}_A$ is tensor, then one can define quantum dimensions from the fusion ring of $\mathcal{C}_A$. However, the module category $\mathcal{C}_A$ is not tensor in general, it is only tensor when $\mathcal{C}$ is braided and A is commutative. In that case there is no unique normalization for the quantum dimensions of simple objects [139]. We want to understand what choice of normalization that corresponds to our convention in eq. (4.3) corresponds to in this case. First, we

²⁴Note that the representation of the T matrix of the TQFT is related to that of the CFT by a phase $e^{2\pi i \frac{c_-}{24}}$.

start by writing the simple objects of $\mathcal{C}_A$ as objects in $\mathcal{C}$ and define their dimension as their dimension in $\mathcal{C}$. We will denote the dimensions by FPdim , and the simple objects of the module category $\mathcal{C}_A$ as M_i . In this case we get [138, Chapter 7].

$$\sum_i \text{FPdim}_{\mathcal{C}}(M_i)^2 = \text{FPdim}(A) \text{FPdim}(\mathcal{C}) \quad (\text{B.1})$$

where $\text{FPdim}(\mathcal{C}) = \sum_a \text{FPdim}(a)^2$ is the Frobenius-Perron dimension of the category $\mathcal{C}$ and a are the simple objects of $\mathcal{C}$. The category dimension $\text{FPdim}(\mathcal{C})$ what we denoted before as $\mathcal{D}_{\mathcal{C}}^2$, where $\mathcal{D}$ is the total quantum dimension. Now let us try to understand (B.1), the LHS is dimension of the module category $\mathcal{C}_A$ in some normalization. However, in this normalization the lowest quantum dimension, which corresponds to the image of object A , is equal to $\dim A$. Our chosen normalization in (4.7) was such that the lowest quantum dimension is unity (which was inherited from our 3d TQFT setup), so in that normalization we get

$$\sum_i \widetilde{\text{FPdim}}(M_i)^2 = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)} \quad (\text{B.2})$$

This is the natural normalization one would get in the case where $\mathcal{C}_A$ is tensor [131, 140].

In the physics picture, (B.1) can be derived by starting from the $\mathcal{C}$ -symmetric 2d TQFT on the sphere and gauging the algebra A . The partition function of the gauged theory $\mathcal{Z}_{\mathcal{C}_A}$ can be written as the partition function of the original theory $\mathcal{Z}_{\mathcal{C}}$ with insertion of a mesh of A lines. On the sphere, since there no non-contractible 1-cycles, the mesh can be reduced (by removing any bubbles) to the diagram in (2.17), which leads to

$$\mathcal{Z}_{\mathcal{C}_A}(S^2) = \dim A \mathcal{Z}_{\mathcal{C}}(S^2) \quad (\text{B.3})$$

where the simple objects of the module category $\mathcal{C}_A$, the category of boundary conditions of the gauged theory, have dimensions as objects in $\mathcal{C}$. Using the normalization of (B.2) is equivalent to choosing an Euler counterterm $(\dim A)^{2g-2}$ which is needed to normalize $\mathcal{Z}_{\mathcal{C}_A}(S^2)$ to unity as per our 3d TQFT normalization choice.

References

- [1] A. Dymarsky and A. Shapere, *TQFT gravity and ensemble holography*, [*JHEP* **02** \(2025\) 091](#) [[2405.20366](#)].
- [2] P. Saad, S.H. Shenker and D. Stanford, *JT gravity as a matrix integral*, [1903.11115](#).
- [3] S. Coleman, *Black holes as red herrings: Topological fluctuations and the loss of quantum coherence*, [*Nucl. Phys. B* **307** \(1988\) 88901101](#).

- [4] S.B. Giddings and A. Strominger, *Loss of incoherence and determination of coupling constants in quantum gravity*, *Nucl. Phys. B* **307** (1988) 88901095.
- [5] S.B. Giddings and A. Strominger, *Baby Universes, Third Quantization and the Cosmological Constant*, *Nucl. Phys. B* **321** (1989) 481.
- [6] D. Marolf and H. Maxfield, *Transcending the ensemble: Baby universes, spacetime wormholes, and the order and disorder of black hole information*, *JHEP* **08** (2020) 044 [[2002.08950](#)].
- [7] J. Cotler and K. Jensen, *AdS₃ gravity and random CFT*, *JHEP* **04** (2021) 033 [[2006.08648](#)].
- [8] J. Chandra, S. Collier, T. Hartman and A. Maloney, *Semiclassical 3D gravity as an average of large-*c* CFTs*, *JHEP* **12** (2022) 069 [[2203.06511](#)].
- [9] G. Di Ubaldo and E. Perlmutter, *AdS₃/RMT₂ duality*, *JHEP* **12** (2023) 179 [[2307.03707](#)].
- [10] A. Belin, J. de Boer, D.L. Jafferis, P. Nayak and J. Sonner, *Approximate CFTs and random tensor models*, *JHEP* **09** (2024) 163 [[2308.03829](#)].
- [11] D.L. Jafferis, L. Rozenberg and G. Wong, *3d gravity as a random ensemble*, *JHEP* **02** (2025) 208 [[2407.02649](#)].
- [12] J. Boruch, G. Di Ubaldo, F.M. Haehl, E. Perlmutter and M. Rozali, *Modular-Invariant Random Matrix Theory and AdS₃ Wormholes*, *Phys. Rev. Lett.* **135** (2025) 121602 [[2503.00101](#)].
- [13] A. Maloney and E. Witten, *Quantum gravity partition functions in three dimensions*, *JHEP* **02** (2010) 029 [[0712.0155](#)].
- [14] C.A. Keller and A. Maloney, *Poincare Series, 3D Gravity and CFT Spectroscopy*, *JHEP* **02** (2015) 080 [[1407.6008](#)].
- [15] H. Maxfield and G.J. Turiaci, *The path integral of 3D gravity near extremality; or, JT gravity with defects as a matrix integral*, *JHEP* **01** (2021) 118 [[2006.11317](#)].
- [16] N. Benjamin, S. Collier and A. Maloney, *Pure Gravity and Conical Defects*, *JHEP* **09** (2020) 034 [[2004.14428](#)].
- [17] S. Collier, L. Eberhardt and M. Zhang, *Solving 3d gravity with virasoro TQFT*, *SciPost Phys.* **15** (2023) 151 [[2304.13650](#)].
- [18] J. Ellegaard Andersen and R. Kashaev, *A TQFT from Quantum Teichmüller Theory*, *Commun. Math. Phys.* **330** (2014) 887 [[1109.6295](#)].
- [19] S. Collier, L. Eberhardt and M. Zhang, *3d gravity from Virasoro TQFT: Holography, wormholes and knots*, *SciPost Phys.* **17** (2024) 134 [[2401.13900](#)].

- [20] B. Post and I. Tsiaras, *A non-rational Verlinde formula from Virasoro TQFT*, *JHEP* **04** (2025) 015 [[2411.07285](#)].
- [21] C. Yan, *Puzzles in 3D off-shell geometries via VTQFT*, *JHEP* **09** (2025) 104 [[2502.16686](#)].
- [22] J. Chandra, *Statistics in 3d gravity from knots and links*, *arXiv:2508.10864 [hep-th]* (2025) [[2508.10864](#)].
- [23] N. Bao, L.-Y. Hung, Y. Jiang and Z. Liu, *QG from SymQRG: AdS_3/CFT_2 Correspondence as Topological Symmetry-Preserving Quantum RG Flow*, [2412.12045](#).
- [24] T. Hartman, *Conformal Turaev-Viro Theory*, [2507.11652](#).
- [25] T. Hartman, *Triangulating quantum gravity in AdS_3* , [2507.12696](#).
- [26] A. Maloney and E. Witten, *Averaging Over Narain Moduli Space*, *JHEP* **10** (2020) 187 [[2006.04855](#)].
- [27] N. Afkhami-Jeddi, H. Cohn, T. Hartman and A. Tajdini, *Free partition functions and an averaged holographic duality*, *JHEP* **01** (2021) 130 [[2006.04839](#)].
- [28] J. Dong, T. Hartman and Y. Jiang, *Averaging over moduli in deformed WZW models*, *JHEP* **09** (2021) 185 [[2105.12594](#)].
- [29] V. Meruliya and S. Mukhi, *AdS_3 gravity and RCFT ensembles with multiple invariants*, *JHEP* **08** (2021) 098 [[2104.10178](#)].
- [30] M. Ashwinkumar, M. Dodelson, A. Kidambi, J.M. Leedom and M. Yamazaki, *Chern-Simons invariants from ensemble averages*, *JHEP* **08** (2021) 044 [[2104.14710](#)].
- [31] J. Raeymaekers, *A note on ensemble holography for rational tori*, *JHEP* **12** (2021) 177 [[2110.08833](#)].
- [32] J. Raeymaekers and P. Rossi, *Wormholes and surface defects in rational ensemble holography*, *JHEP* **01** (2024) 104 [[2312.02276](#)].
- [33] A. Castro, M.R. Gaberdiel, T. Hartman, A. Maloney and R. Volpato, *The Gravity Dual of the Ising Model*, *Phys. Rev. D* **85** (2012) 024032 [[1111.1987](#)].
- [34] V. Meruliya, S. Mukhi and P. Singh, *Poincaré Series, 3d Gravity and Averages of Rational CFT*, *JHEP* **04** (2021) 267 [[2102.03136](#)].
- [35] T. Nicosanti and P. Putrov, *Summing over homology groups of 3-manifolds*, [2508.02457](#).
- [36] N. Angelinos, *Abelian 3D TQFT gravity, ensemble holography and stabilizer states*, [2509.26052](#).
- [37] A. Barbar, A. Dymarsky and A.D. Shapere, *Global Symmetries, Code Ensembles, and Sums Over Geometries*, *Phys. Rev. Lett.* **134** (2025) 151603 [[2310.13044](#)].

- [38] J.C. Baez and J. Dolan, *From finite sets to feynman diagrams*, in *Mathematics Unlimited — 2001 and Beyond*, B. Engquist and W. Schmid, eds., (Berlin, Heidelberg), pp. 29–50, Springer Berlin Heidelberg (2001), [DOI](#).
- [39] A. Dymarsky and A. Shapere, *Mass formula for topological boundary conditions from TQFT gravity*, in progress.
- [40] H.J.S. Smith, *III. On the orders and genera of quadratic forms containing more than three indeterminates. — Second notice*, *Proceedings of the Royal Society of London* **16** (1868) 197.
- [41] H. Minkowski, *Untersuchungen über quadratische Formen: I. Bestimmung der Anzahl verschiedener Formen, welche ein gegebenes Genus enthält*, *Acta Mathematica* **7** (1885) 201.
- [42] C.L. Siegel, *über die analytische theorie der quadratischen formen*, *Annals of Mathematics* **36** (1935) 527.
- [43] C.L. Siegel, *Indefinite quadratische formen und funktionentheorie I*, *Mathematische Annalen* **124** (1951) 17.
- [44] H. Maass and T.P. Srinivasan, *Lectures on Siegel’s Modular Functions*, Lectures on Mathematics and Physics, Tata Institute of Fundamental Research (1954).
- [45] A. Weil, *Sur certains groupes d’opérateurs unitaires*, *Acta Mathematica* **111** (1964) 143.
- [46] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, *Acta Mathematica* **113** (1965) 1.
- [47] I.B. Frenkel, J. Lepowsky and A. Meurman, *A natural representation of the Fischer-Griess Monster with the modular function J as character*, *Proceedings of the National Academy of Sciences* **81** (1984) 3256.
- [48] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, vol. 134, Academic press (1989).
- [49] V.G. Kac, *Vertex Algebras for Beginners*, vol. 10, American Mathematical Soc. (1998).
- [50] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351.
- [51] G.W. Moore and N. Seiberg, *Taming the Conformal Zoo*, *Phys. Lett. B* **220** (1989) 422.
- [52] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators 1. Partition functions*, *Nucl. Phys. B* **646** (2002) 353 [[hep-th/0204148](#)].
- [53] A. Kapustin and D. Orlov, *Vertex algebras, mirror symmetry, and D-branes: The Case of complex tori*, *Commun. Math. Phys.* **233** (2003) 79 [[hep-th/0010293](#)].

- [54] M. Rosellen, *OPE-algebras and their modules*, [math/0312313](#).
- [55] R.K. Singh and M. Sinha, *Non-chiral vertex operator algebra associated to Lorentzian lattices and Narain CFTs*, *SciPost Phys.* **17** (2024) 047 [[2312.16296](#)].
- [56] G.W. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Commun. Math. Phys.* **123** (1989) 177.
- [57] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, De Gruyter, Berlin, Boston (2016), [doi:10.1515/9783110435221](#).
- [58] W. Bloomquist and Z. Wang, *On topological quantum computing with mapping class group representations*, *J. Phys. A* **52** (2018) 015301 [[1805.04622](#)].
- [59] L. Kong, *Anyon condensation and tensor categories*, *Nucl. Phys. B* **886** (2014) 436 [[1307.8244](#)].
- [60] F. Benini, C. Copetti and L.D. Pietro, *Factorization and global symmetries in holography*, *SciPost Phys.* **14** (2023) 019 [[2203.09537](#)].
- [61] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, *Correspondences of ribbon categories*, *Adv. Math.* **199** (2006) 192 [[math/0309465](#)].
- [62] J. Kaidi, Z. Komargodski, K. Ohmori, S. Seifnashri and S.-H. Shao, *Higher central charges and topological boundaries in 2+1-dimensional TQFTs*, *SciPost Phys.* **13** (2022) 067 [[2107.13091](#)].
- [63] D. Gaiotto and J. Kulp, *Orbifold groupoids*, *JHEP* **02** (2021) 132 [[2008.05960](#)].
- [64] W. Ji and X.-G. Wen, *Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions*, *Phys. Rev. Res.* **2** (2020) 033417 [[1912.13492](#)].
- [65] D.S. Freed, G.W. Moore and C. Teleman, *Topological symmetry in quantum field theory*, *arXiv:2209.07471 [hep-th]* (2022) [[2209.07471](#)].
- [66] F. Apruzzi, F. Bonetti, I. García Etxebarria, S.S. Hosseini and S. Schafer-Nameki, *Symmetry TFTs from String Theory*, *Commun. Math. Phys.* **402** (2023) 895 [[2112.02092](#)].
- [67] E.P. Verlinde, *Fusion Rules and Modular Transformations in 2D Conformal Field Theory*, *Nucl. Phys. B* **300** (1988) 360.
- [68] V.G. Drinfeld, *Quantum groups*, *Zap. Nauchn. Semin.* **155** (1986) 18.
- [69] A. Joyal and R. Street, *Braided tensor categories*, *Advances in Mathematics* **102** (1993) 20.
- [70] C. Kassel and V. Turaev, *Double construction for monoidal categories*, *Acta Mathematica* **175** (1995) 1.

- [71] L. Bhardwaj, L.E. Bottini, D. Pajer and S. Schafer-Nameki, *The Club Sandwich: Gapless Phases and Phase Transitions with Non-Invertible Symmetries*, [*SciPost Phys.* **18** \(2025\) 156 \[2312.17322\]](#).
- [72] K. Roumpedakis, S. Seifnashri and S.-H. Shao, *Higher gauging and non-invertible condensation defects*, [*Commun. Math. Phys.* **401** \(2023\) 3043 \[2204.02407\]](#).
- [73] S. Suzuki, *On homeomorphisms of a 3-dimensional handlebody*, [*Canadian Journal of Mathematics* **29** \(1977\) 111](#).
- [74] S. Hensel, *A primer on handlebody groups*, *Handbook of group actions. V* **48** (2020) 143.
- [75] C.-M. Jian, A.W.W. Ludwig, Z.-X. Luo, H.-Y. Sun and Z. Wang, *Establishing strongly-coupled 3D AdS quantum gravity with Ising dual using all-genus partition functions*, [*JHEP* **10** \(2020\) 129 \[1907.06656\]](#).
- [76] I. Romaidis and I. Runkel, *CFT correlators and mapping class group averages*, [*Commun. Math. Phys.* **405** \(2024\) 247 \[2309.14000\]](#).
- [77] E.E. Moise, *Affine structures in 3-manifolds: V. The triangulation theorem and hauptvermutung*, *Annals of Mathematics* **56** (1952) 96.
- [78] M. Scharlemann, J. Schultens and T. Saito, *Lecture Notes on Generalized Heegaard Splittings*, World Scientific (2016).
- [79] L. Goeritz, *Die abbildungen der brezelfläche und der vollbrezel vom geschlecht 2*, in *Abhandlungen Aus Dem Mathematischen Seminar Der Universität Hamburg*, vol. 9, pp. 244–259, Springer, 1933.
- [80] J. Johnson and H. Rubinstein, *Mapping class groups of Heegaard splittings*, [math/0701119](#).
- [81] S. Cho and Y. Koda, *The genus two Goeritz group of $S^2 \times S^1$* , [1303.7145](#).
- [82] N. Sekino, *The Goeritz group of a Heegaard splitting of genus two of a Seifert manifold whose base orbifold is sphere with three exceptional points of sufficiently complex coefficients*, [2202.05064](#).
- [83] K. Reidemeister, *Zur dreidimensionalen Topologie*, [*Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **9** \(1933\) 189](#).
- [84] J.W. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, *Transactions of the American Mathematical Society* **35** (1933) 88.
- [85] F. Lei, *On stability of Heegaard splittings*, in *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 129, pp. 55–57, Cambridge University Press, 2000.

- [86] J. Hass, A. Thompson and W. Thurston, *Stabilization of heegaard splittings*, *Geometry & Topology* **13** (2009) 2029.
- [87] M. Scharlemann, *Heegaard splittings of compact 3-manifolds*, [math/0007144](#).
- [88] U. Oertel, *Mapping class groups of compression bodies and 3-manifolds*, [math/0607444](#).
- [89] E. Torres and X. Yu, *SymTFT entanglement and holographic (non)-factorization*, [2510.06319](#).
- [90] L. Bhardwaj and Y. Tachikawa, *On finite symmetries and their gauging in two dimensions*, *JHEP* **03** (2018) 189 [[1704.02330](#)].
- [91] T.-C. Huang, Y.-H. Lin and S. Seifnashri, *Construction of two-dimensional topological field theories with non-invertible symmetries*, *JHEP* **12** (2021) 028 [[2110.02958](#)].
- [92] C. Cordova and D. García-Sepúlveda, *Topological cosets via anyon condensation and applications to gapped QCD₂*, [2412.01877](#).
- [93] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators. 3. Simple currents*, *Nucl. Phys. B* **694** (2004) 277 [[hep-th/0403157](#)].
- [94] A. Davydov, L. Kong and I. Runkel, *Invertible defects and isomorphisms of rational CFTs*, *Adv. Theor. Math. Phys.* **15** (2011) 43 [[1004.4725](#)].
- [95] J. Fuchs, I. Runkel and C. Schweigert, *Ribbon categories and (unoriented) CFT: Frobenius algebras, automorphisms, reversions*, *Contemp. Math.* **431** (2007) 203 [[math/0511590](#)].
- [96] D. Tambara and S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, *Journal of Algebra* **209** (1998) 692.
- [97] F.A. Bais and J.K. Slingerland, *Condensate induced transitions between topologically ordered phases*, *Phys. Rev. B* **79** (2009) 045316 [[0808.0627](#)].
- [98] R. Vanhove, L. Lootens, H.-H. Tu and F. Verstraete, *Topological aspects of the critical three-state Potts model*, *J. Phys. A* **55** (2022) 235002 [[2107.11177](#)].
- [99] Y. Ren, N. Tantivasadakarn and D.J. Williamson, *Efficient preparation of solvable anyons with adaptive quantum circuits*, *Phys. Rev. X* **15** (2025) 031060 [[2411.04985](#)].
- [100] A. Kapustin and N. Saulina, *Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory*, [1012.0911](#).
- [101] R. Dijkgraaf, C. Vafa, E.P. Verlinde and H.L. Verlinde, *The Operator Algebra of Orbifold Models*, *Commun. Math. Phys.* **123** (1989) 485.
- [102] R. Thorngren and Y. Wang, *Fusion category symmetry. Part II. Categoriosities at $c = 1$ and beyond*, *JHEP* **07** (2024) 051 [[2106.12577](#)].

- [103] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, *Nucl. Phys. B* **763** (2007) 354 [[hep-th/0607247](#)].
- [104] A. Cappelli, C. Itzykson and J.B. Zuber, *Modular invariant partition functions in two dimensions*, *Nucl. Phys. B* **280** (1987) 87901556.
- [105] A. Cappelli, C. Itzykson and J.B. Zuber, *The ADE classification of minimal and $A_1(1)$ conformal invariant theories*, *Commun. Math. Phys.* **113** (1987) 1.
- [106] J. Korinman, *Decomposition of some Witten-Reshetikhin-Turaev Representations into Irreducible Factors*, *SIGMA* **15** (2019) 011 [[1406.4389](#)].
- [107] O. Diatlyk, C. Luo, Y. Wang and Q. Weller, *Gauging non-invertible symmetries: Topological interfaces and generalized orbifold groupoid in 2d QFT*, *JHEP* **03** (2024) 127 [[2311.17044](#)].
- [108] V.B. Petkova and J.B. Zuber, *The Many faces of Ocneanu cells*, *Nucl. Phys. B* **603** (2001) 449 [[hep-th/0101151](#)].
- [109] V.B. Petkova and J.B. Zuber, *Generalized twisted partition functions*, *Phys. Lett. B* **504** (2001) 157 [[hep-th/0011021](#)].
- [110] R. Coquereaux and G. Schieber, *Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries*, *J. Geom. Phys.* **42** (2002) 216 [[hep-th/0107001](#)].
- [111] C.H.O. Chui, C. Mercat and P.A. Pearce, *Integrable and conformal twisted boundary conditions for $sl(2)$ A-D-E lattice models*, *J. Phys. A* **36** (2003) 2623 [[hep-th/0210301](#)].
- [112] A. Antinucci and F. Benini, *Anomalies and gauging of $U(1)$ symmetries*, *Phys. Rev. B* **111** (2025) 024110 [[2401.10165](#)].
- [113] O. Aharony, A. Dymarsky and A.D. Shapere, *Holographic description of Narain CFTs and their code-based ensembles*, *JHEP* **05** (2024) 343 [[2310.06012](#)].
- [114] A. Dymarsky, J. Henriksson and B. McPeak, *Holographic duality from Howe duality: Chern-Simons gravity as an ensemble of code CFTs*, [2504.08724](#).
- [115] A. Barbar, A. Dymarsky and A. Shapere, *Holographic description of 4d Maxwell theories and their code-based ensembles*, [2510.03392](#).
- [116] A. Maloney, *Quantum gravity as an average*, ([Slides for Strings and Fields 2021, Yukawa Institute for Theoretical Physics, Kyoto University](#)).
- [117] A. Belin, A. Maloney and F. Seefeld, *A measure on the space of CFTs and pure 3D gravity*, [2509.04554](#).
- [118] E. Perlmutter, L. Rastelli, C. Vafa and I. Valenzuela, *A CFT distance conjecture*, *JHEP* **10** (2021) 070 [[2011.10040](#)].

- [119] F. Baume and J. Calderón-Infante, *On higher-spin points and infinite distances in conformal manifolds*, *JHEP* **12** (2023) 163 [[2305.05693](#)].
- [120] H. Ooguri and Y. Wang, *Universal bounds on CFT distance conjecture*, *JHEP* **12** (2024) 154 [[2405.00674](#)].
- [121] P.H. Ginsparg, *Curiosities at $c = 1$* , *Nucl. Phys. B* **295** (1988) 153.
- [122] I. Runkel and G.M.T. Watts, *A Nonrational CFT with $c = 1$ as a limit of minimal models*, *JHEP* **09** (2001) 006 [[hep-th/0107118](#)].
- [123] W. Li, W. Song and A. Strominger, *Chiral gravity in three dimensions*, *JHEP* **04** (2008) 082 [[0801.4566](#)].
- [124] A. Maloney, W. Song and A. Strominger, *Chiral gravity, log gravity and extremal CFT*, *Phys. Rev. D* **81** (2010) 064007 [[0903.4573](#)].
- [125] E. Witten, *Three-dimensional gravity revisited*, [0706.3359](#).
- [126] Y.-H. Lin and D. Pei, *Holomorphic cfts and topological modular forms*, *Commun. Math. Phys.* **401** (2023) 325 [[2112.10724](#)].
- [127] A.N. Schellekens, *Meromorphic $C = 24$ conformal field theories*, *Commun. Math. Phys.* **153** (1993) 159 [[hep-th/9205072](#)].
- [128] J. van Ekeren, S. Möller and N.R. Scheithauer, *Construction and classification of holomorphic vertex operator algebras*, *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2020** (2017) 61.
- [129] J. van Ekeren, S. Möller and N.R. Scheithauer, *Dimension formulae in genus zero and uniqueness of vertex operator algebras*, *International Mathematics Research Notices* **2020** (2018) 2145.
- [130] J. van Ekeren, C.H. Lam, S. Möller and H. Shimakura, *Schellekens' list and the very strange formula*, *Advances in Mathematics* **380** (2021) 107567.
- [131] A. Kirillov and V. Ostrik, *On a q -Analogue of the McKay correspondence and the ADE classification of $\widehat{sl}_2$ conformal field theories*, *Advances in Mathematics* **171** (2002) 183.
- [132] G. Hoehn, *Genera of Vertex Operator Algebras and three dimensional Topological Quantum Field Theories*, *Fields Inst. Commun.* **39** (2003) 89 [[math/0209333](#)].
- [133] S. Möller and B.C. Rayhaun, *Equivalence relations on vertex operator algebras, I: Genus*, [2408.07117](#).
- [134] J. de Boer, D. Liska, B. Post and M. Sasieta, *A principle of maximum ignorance for semiclassical gravity*, *JHEP* **2024** (2024) 003.
- [135] J.-M. Schlenker and E. Witten, *No ensemble averaging below the black hole threshold*, *JHEP* **07** (2022) 143.

- [136] J. McNamara and C. Vafa, *Baby universes, holography, and the swampland*, [2004.06738](#).
- [137] G.W. Moore and N. Seiberg, *Polynomial Equations for Rational Conformal Field Theories*, *Phys. Lett. B* **212** (1988) 451.
- [138] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, American Mathematical Soc. (July, 2015).
- [139] G. Schaumann, *Traces on module categories over fusion categories*, *Journal of Algebra* **379** (2013) 382.
- [140] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, *Annals of mathematics* (2005) 581.