Random access Bell game by sequentially measuring the control of the quantum SWITCH

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Preserving quantum correlations such as Bell nonlocality in noisy environments remains a fundamental challenge for quantum technologies. We introduce the Random Access Bell Game (RABG), a task where an entangled particle propagates through a sequence of identical noisy blocks, and the ability to violate a Bell inequality is tested at a randomly chosen point (access node). We consider a scenario where each noisy block is composed of two complete erasure channels, an extreme entanglement-breaking channel with vanishing quantum and classical capacities. We investigate the performance of the Random Access Bell Game in this configuration and attempt to mitigate the effect of noise by coherently controlling the order of each channel in the noise using the quantum `SWITCH`. However, the quantum `SWITCH` in its canonical setup with a coherent state in the control fails to provide any advantage in the Random Access Bell Game. Our main contribution is a protocol that leverages initial entanglement between the target and control of the quantum `SWITCH` and employs sequential, unsharp measurements on the control system, showing that it is possible to guarantee a Bell violation after an arbitrarily large number of channel applications. Furthermore, our protocol allows for a near-maximal (Tsirelson bound) Bell violation to be achieved at any desired round, while still ensuring violations in all preceding rounds. We prove that this advantage is specific to generalized Greenberger-Horne-Zeilinger (GHZ) states, as the protocol fails for W-class states, thus providing an operational way to distinguish between these two fundamental classes of multipartite entanglement.

I. INTRODUCTION

Bell nonlocality stands as a cornerstone of quantum mechanics, revealing a stark departure from the classical principles of local realism [1–3]. Violation of local realism has conceptually altered the understanding of quantum mechanics and the nature of quantum correlations in general [4–7]. It has been demonstrated extensively in experiments [8–10] with recent breakthroughs in loophole-free Bell tests as well [11–13]. Apart from foundational interests, Bell inequality violations turn out to be the crucial ingredient for various quantum information processing tasks. A non-exhaustive list includes tasks that range from device-independent cryptography [14–17], random number generation [18, 19], to reducing the communication complexity of certain problems [20–22]. All these have motivated investigations of Bell inequality violation in physical platforms like continuous variable and quantum optical systems [23–26] and many-body systems [27–30] to make predictions that can be compared with actual experiments. Another line of fundamentally important research involves the investigation of the relationship between Bell inequality violation and other facets of nonclassicality, like entanglement [31–33]. However, the quantum correlations that enable nonclassical effects are notoriously fragile, readily degrading through interaction with a noisy environment [34, 35]. Understanding and mitigating [36, 37] the impact of noise is therefore paramount for the development of robust quantum technologies [38].

In this paper, we address this challenge by conceptualizing the task of a "Random Access Bell Game." In this task, an entangled state is shared between two parties, Alice and Bob. Bob's particle traverses a noisy environment, modelled as a series of discrete interactions with identical noisy quantum channels. A referee can, at any point, randomly request that Alice and Bob perform a Bell test on their shared state. The central question is to determine the maximum number of noisy interactions, $k_{\max}$, the system can endure while retaining the ability to violate a Bell inequality, regardless of which "access node" $k \leq k_{\max}$ the referee chooses. See Fig. 1 for a schematic of the game. For many types of noise, particularly entanglement-breaking channels [39, 40], the ability to win this game is severely limited, with $k_{\max}$ often being zero. Specifically, we consider an extreme scenario where the noise is characterized by complete erasure channels [41, 42], which we refer to as pin maps. They possess vanishing classical and quantum capacities [41].

The primary contribution of this work is a novel protocol that overcomes this limitation by leveraging the quantum `SWITCH` [43] in conjunction with sequential, unsharp measurements. Over the last decade, the quantum `SWITCH` has gathered a lot of attention in error mitigation providing advantages in various tasks ranging from quantum communication [44–46], quantum metrology [47], and quantum work extraction [48–50] to name a few. Many of these theoretical advantages have also been simulated in experiments as well [51–53]. At the same time, unsharp measurements have also gained a lot of popularity for their role in tasks that involve probing the system sequentially. Some of the prominent examples of such tasks involve sequential violation of Bell inequalities [54–57], steering [58, 59], witnessing entanglement [60–62], and even communication tasks like quantum teleportation [63].

In our protocol, we demonstrate that by repeatedly performing weak (unsharp) measurements on the control qubit that preshares entanglement with the target system of the `SWITCH`, one can certify a Bell violation at each stage, for multiple rounds while minimally disturbing the system, thereby preserving the quantum entanglement required for obtaining a violation in subsequent rounds. In particular, we identify that unlike projective measurements which yield maximal violation in a single deterministic round at the cost of all future nonlocality, uhsharp measurements introduce a trade-off of available Bell violation across access nodes. A weaker measurement at a given round yields a smaller violation but preserves more entanglement, enabling stronger violations in subsequent rounds. This trade-off can be utilized to control the available violation depending on the measurement sharpness.

The quantum `SWITCH` allows for the order of the noisy operations to be governed by a control qubit. Indeed, a key component in the success of our protocol comes from the structure of the effective map after the `SWITCH` action on two pin maps. While each pin map individually destroys all entanglement, the quantum `SWITCH` combines them into a single operation whose resulting channel is no longer entanglement-breaking. The crucial effect of this `SWITCH`ing action is the creation of a protected two-dimensional subspace. In our work, we consider the two erasure channels to pin every state to two orthogonal states that we label as $|0\rangle$ and $|1\rangle$ respectively. The quantum `SWITCH` with these two pin maps possesses a decoherence-free subspace spanned by the vectors $\{|00\rangle, |11\rangle\}$. This right away implies that we can expect best success for initial states whose support in Bob and the Control's is restricted to the $\{|00\rangle, |11\rangle\}$ subspace. To this end, we show that the SWITCH in the canonical form, i.e., with a coherent state in the control, is useless for the "Random Access Bell Game." Our analysis reveals that a necessary condition for obtaining success in the Random Access Bell Game is entanglement between the quantum particles in possession of Alice and Bob, with the control of the quantum SWITCH. The requirement of entanglement along with the decoherence-free subspace spanned by $\{|00\rangle, |11\rangle\}$ motivates us to choose the generalized GHZ state [64] as the prototypical initial state for which we demonstrate our results. Physically, unique multipartite correlations of a generalized GHZ state align perfectly with the protected subspace created by the `SWITCH`, allowing it to survive the noisy evolution.

In this paper, we derive the exact analytical form of the tripartite state after an arbitrary number of rounds, allowing us to precisely quantify the Bell violation as a function of the measurement sharpness. From this, we demonstrate that for an initial generalized GHZ state, a Bell violation can be maintained for an arbitrarily large number of rounds. We then show how to strategically exploit the identified trade-off by setting the measurement sharpness parameters in a geometric progression. This strategy allows a Bell violation arbitrarily close to the Tsirelson bound [4] to be achieved at any desired target round, while still guaranteeing a violation in all preceding rounds. We complement these analytical findings with a numerical analysis that quantifies the persistence of nonlocality, calculating the maximum number of rounds for which a specific, predefined threshold of Bell violation can be guaranteed. We finally contrast the success of the generalized GHZ state to the case where we consider the initial state between Alice, Bob and the control to be a W-state [65]. We prove that the protocol fails completely when the initial resource is a W state. The W state's distinct correlation structure is not preserved by the `SWITCH` action, causing the state to become

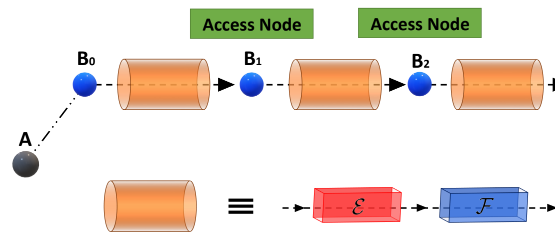

FIG. 1. **Schematic of the random access Bell game.** The subsystem at B repeatedly passes through identical noise blocks. The particle can be picked randomly from any access nodes and used to assess the Bell violation.

fully separable after the first round and thus providing a clear operational distinction between the two SLOCC (Stochastic Local Operations and Classical Communication) inequivalent classes of three qubit pure states [65].

The paper is organized as follows. After this introduction in Sec. I, we formally define the Random Access Bell Game in Sec. II. In this section, we also analyze preliminary strategies, demonstrating the limitations of canonical approaches that do not involve pre-shared entanglement with the control system. Our main protocol, which leverages the quantum `SWITCH` in conjunction with sequential unsharp measurements and a GHZ-type initial state, is presented in Sec. III. We then generalize our results to an arbitrary number of rounds in Sec. III A 3, where we provide the central proofs that Bell violation can be maintained indefinitely and that near-maximal violation is achievable at any chosen round. In Sec. III B, we highlight the necessity of GHZ-type entanglement by proving the protocol's failure for W states. Finally, we summarize our findings and discuss their broader implications in Sec. IV. Detailed proofs and calculations are deferred to the Appendices along with a brief section on preliminaries.

II. RANDOM ACCESS BELL GAME

The setting of the game involves a pre-shared entangled state ρ_{AB_0} between Alice (A) and Bob (B). The subscript 0 denotes the initial setting. Now the particle with Bob is transmitted through a series of identical noisy channels, say, Λ. The output state after passing through N iterations of the channel is denoted by $\rho_{AB_N} := (\mathbb{I} \otimes \Lambda)^{\otimes N}(\rho_{AB_0})$. The subsystem B_k can be accessed for measurement after any round k of channel actions. A third party, which we refer to as the referee R, *randomly* dictates after how many channel actions k the state can be extracted. We name the locations after each channel action from which the particle can be extracted as an *access node*. Now the central question is whether ρ_{AB_k} violates a Bell inequality. Or more generally, what is the maximum value $k_{\max}$ up to which violation is possible when the particle is randomly extracted from any access node

$k \leq k_{\max}$.

If Λ is a perfect (noiseless) channel, then $k_{\max}$ can be arbitrarily large, provided the initial state ρ_{AB_0} violates, in particular, the CHSH inequality [66]. This inequality tests for nonlocality by constraining the correlations between local measurements; a violation occurs when the measured Bell parameter $|\mathcal{B}|$ exceeds the classical bound of 2, which is impossible under local realism but permitted by quantum mechanics. For a detailed description of the formalism, please see Appendix A.

The situation becomes interesting when Λ is noisy. For example, when Λ is an entanglement-breaking channel, $k_{\max} = 0$ since $\mathbb{I} \otimes \Lambda(\rho_{AB_0}) = \rho_{AB_1}$ is a separable state and hence does not violate a Bell inequality. The particular noisy channel we consider is one where Λ is composed of entanglement-breaking pin maps, $\mathcal{E}$ and $\mathcal{F}$ whose actions are given as follows

$$\mathcal{E}(\rho) = |0\rangle\langle 0| \;\forall \rho, \quad \mathcal{F}(\rho) = |1\rangle\langle 1| \;\forall \rho. \tag{1}$$

The noisy channel is described by a probabilistic mixture (a causally separable configuration) of the channels $\mathcal{E}$ and $\mathcal{F}$

$$\Lambda_p = p\,\mathcal{E} \circ \mathcal{F} + (1-p)\,\mathcal{F} \circ \mathcal{E} = p\,\mathcal{E} + (1-p)\,\mathcal{F}. \tag{2}$$

Even if there were a control system C that possessed the power to tune the probability p, the output state of the channel would still be separable,

$$\mathbb{I} \otimes \Lambda_p(\rho_{AB_0}) = \rho_A \otimes \big(p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|\big), \tag{3}$$

where $\rho_A = \mathrm{Tr}_B(\rho_{AB_0})$ is the marginal state of A. Now the key point of investigation is to consider the case where C can coherently control the order of the channels $\mathcal{E}$ and $\mathcal{F}$. This setup, which we refer to as the SWITCH action (see Fig. 2), also turns out to be of limited utility in its usual configuration.

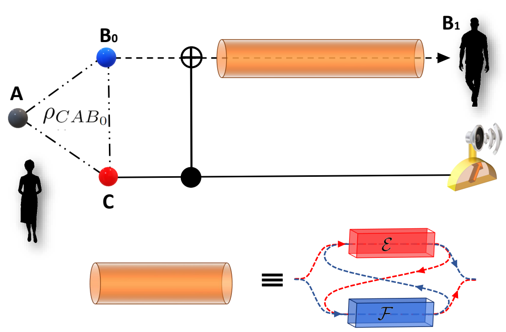

FIG. 2. **Random Access Bell Game using the quantum** SWITCH. The initial state ρ_{CAB_0} is passed through the coherently controlled noise, e.g., the SWITCH action, with C acting as the control. A measurement is then performed at C and the outcomes are communicated to Alice and Bob.

The SWITCH action channel which results from the quantum SWITCH supermap acting on channels $\mathcal{E}$ and $\mathcal{F}$ is elaborated below. The Kraus operators of the noisy entanglement-breaking channels $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{E}_1 = |0\rangle\langle 0|$, $\mathcal{E}_2 = |0\rangle\langle 1|$ and $\mathcal{F}_1 = |1\rangle\langle 0|$, $\mathcal{F}_2 = |1\rangle\langle 1|$. The quantum SWITCH supermap takes these as inputs and outputs the channel SWITCH$(\mathcal{E}, \mathcal{F})$ with Kraus operators $\mathcal{S}_{ij} = |0\rangle\langle 0| \otimes \mathcal{E}_i\mathcal{F}_j + |1\rangle\langle 1| \otimes \mathcal{F}_j\mathcal{E}_i$, while Alice's subsystem A remains unaffected. If we denote the combined channel over the three-party system as $\mathcal{K}$, we get the following Kraus operators:

$$\begin{aligned}
K_0 = \mathcal{S}_{11} \otimes \mathbb{I}_A &= |1\rangle\langle 1|_C \otimes \mathbb{I}_A \otimes |1\rangle\langle 0|_B\,, \\
K_1 = \mathcal{S}_{12} \otimes \mathbb{I}_A &= 0, \\
K_2 = \mathcal{S}_{21} \otimes \mathbb{I}_A &= |1\rangle\langle 1|_C \otimes \mathbb{I}_A \otimes |1\rangle\langle 1|_B \\
&\quad + |0\rangle\langle 0|_C \otimes \mathbb{I}_A \otimes |0\rangle\langle 0|_B\,, \\
K_3 = \mathcal{S}_{22} \otimes \mathbb{I}_A &= |0\rangle\langle 0|_C \otimes \mathbb{I}_A \otimes |0\rangle\langle 1|_B\,.
\end{aligned} \tag{4}$$

Any arbitrary three-party state ρ_{CAB} under the SWITCH action gets updated as $\mathcal{K}(\rho_{CAB}) = \sum_{i=0}^{3} K_i \rho_{CAB} K_i^\dagger$.

Lemma 1. *The quantum* SWITCH *with a canonical setup of employing a coherent state at the control is useless for the random access Bell game.*

Proof. We start with the most general configuration of the initial state, with a coherent control

$$\rho_{CAB_0} = |+\rangle_C \otimes |\phi_\alpha^+\rangle_{AB_0}\,, \tag{5}$$

where $|\phi_\alpha^+\rangle = \sqrt{\alpha}\,|00\rangle + \sqrt{1-\alpha}\,|11\rangle$. The output state after the SWITCH action is given by

$$\begin{aligned}
\rho_{CAB_1} = \frac{1}{2}\,|\mathrm{GHZ}_\alpha\rangle\langle \mathrm{GHZ}_\alpha| \;&+\; \frac{1-\alpha}{2}\,|010\rangle\langle 010| \\
&+\; \frac{\alpha}{2}\,|101\rangle\langle 101|),
\end{aligned}$$

where $|\mathrm{GHZ}_\alpha\rangle = \sqrt{\alpha}\,|000\rangle + \sqrt{1-\alpha}\,|111\rangle)$. After the control C is measured in the $|\pm\rangle$ basis, the un-normalized output states are computed to be

$$\widetilde{\rho}^{\pm}_{AB_1} = \frac{1}{4}\big(\,|\phi_\alpha^+\rangle\langle \phi_\alpha^+| + (1-\alpha)\,|10\rangle\langle 10| + \alpha\,|01\rangle\langle 01|\,\big). \tag{6}$$

Now, we show that $\widetilde{\rho}^{\pm}_{AB_1}$ is separable. Since $\widetilde{\rho}^{\pm}_{AB_1}$ we use the Postive Partial Transpose (PPT) criterion [67] which is necessary and sufficient for $2 \otimes 2$ and $2 \otimes 3$ systems. Now, it is straightforward to show

$$(\widetilde{\rho}^{\pm}_{AB_1})^{T_A} \geq 0 \;\;\forall \alpha, \tag{7}$$

since the eigenvalues of $(\widetilde{\rho}^{\pm}_{AB_1})^{T_A}$ are $\{1, 0, \alpha, 1-\alpha\}$. This, in turn, guarantees the separability of $\widetilde{\rho}^{\pm}_{AB_1}$. Hence the proof. $\square$

Lemma 2. *A GHZ state shared between CAB_0 can enable maximal violation of the CHSH inequality when the particle is deterministically/conclusively picked up from any access node k by B_k.*

Proof. It is evident from Eq. (4) that the SWITCH action preserves any state in the subspace of $\{|000\rangle, |111\rangle\}$. Therefore, initially starting with a GHZ state, $\rho_{CAB_0} = |\mathrm{GHZ}\rangle\langle \mathrm{GHZ}|$

from Eq. (4), after any arbitrary but pre-declared round k, we get

$$\rho_{CAB_k} = \mathcal{K}^{\otimes k}(\rho_{CAB_0}) = |\text{GHZ}\rangle\langle\text{GHZ}|\,. \quad (8)$$

After k iterations, the control C measures its part of the system in the $|\pm\rangle$ basis. Then, up to local unitaries, the state shared between A and B_k is $|\Phi^+\rangle$ that violates the CHSH inequality maximally. □

Things become different when the particle is picked up from a random node. Then, the quantity of interest becomes $k_{\max} =$: the maximal round up to which, if the particle is extracted, demonstrates violation of the CHSH inequality for all rounds till $k_{\max}$.

Observation 1. *The random access Bell game* ***fails*** *for any strategy involving projective measurements at C at any round m other than the randomly selected round.*

Proof. Projective measurements at the control C at any round m result in a product state across the $C : AB_m$ cut. Following Lemma 1, we conclude the absence of CHSH violation for all subsequent rounds. □

The objective of the paper is to construct a protocol that enables success in the random access Bell game, by demonstrating a finite number of rounds ($k > 1$) for which CHSH violation persists if the particle is picked from any access node $m \leq k$. Specifically, we want to investigate *what is the furthest access node $k_{\max}$ such that if the particle was picked from any access node $k \leq k_{\max}$, there is a guaranteed violation of the CHSH inequality between A and B_k.*

III. ADVANTAGES IN THE RANDOM ACCESS BELL GAME USING THE QUANTUM SWITCH

We now present a protocol that uses the quantum SWITCH and sequential unsharp measurements of the control to enable success in the random access Bell game. Mathematically, success implies that we can achieve $k_{\max} > 1$.

Protocol 1. (Sequential strategy.) *Alice and Bob initially share a tripartite entangled state, ρ_{CAB_0}. The protocol, which guarantees CHSH violation for all rounds $k \leq k_{\max}$, proceeds in the following steps.*

1. *The combined three qubit state undergoes the coherently controlled noise, i.e., the* SWITCH *action, with subsystem C set as the control.*

2. *After the* SWITCH *action, we obtain the state ρ_{CAB_1}. If $k_{\max} > 1$, Alice performs an unsharp measurement on the control system C described by the POVMs $\{E_\pm^{\lambda_k}\}$ to yield one of the post-measurement states $\rho_{CAB_1}^{E_\pm}$, where*

$$\begin{aligned} E_\pm^{\lambda_k} &= \lambda_k\,|\pm\rangle\!\langle\pm| + \frac{1-\lambda_k}{2}\mathbb{I} \\ &= \frac{1+\lambda_k}{2}\,|\pm\rangle\!\langle\pm| + \frac{1-\lambda_k}{2}\,|\mp\rangle\!\langle\mp|\,, \end{aligned} \quad (9)$$

and where λ_k is the sharpness parameter in round k. The sharpness parameter at any given round k is chosen to ensure Bell violation, i.e., $|\mathcal{B}(\rho_{AB_k})| \geq 2+\delta$ for some positive number $\delta \in (0, 2\sqrt{2}-2]$.

If $k_{\max} = 1$, C is measured projectively in the $\{|+\rangle, |-\rangle\}$ basis to maximize $\mathcal{B}(\rho_{AB_{k_{\max}}})$.

3. *Steps 1. and 2. are repeated. After k rounds, states ρ_{CAB_k} (post-*SWITCH *action) and $\rho_{CAB_k}^{E_\pm}$ (post-measurement) are obtained. This continues until the final round $k_{\max}th$, at which Alice performs a projective measurement on C and communicates the measurement outcome to Bob.*

In the subsequent section, we implement the protocol and evaluate the success metrics in the "Random Access Bell Game" when ρ_{CAB_0} is a generalized GHZ state.

A. Persistent violation for arbitrarily many rounds

Let us establish an instance of the above protocol where we start with a generalized GHZ state $\rho_{CAB_0} = |\text{GHZ}_\alpha\rangle\!\langle\text{GHZ}_\alpha|$, where $|\text{GHZ}_\alpha\rangle = \sqrt{\alpha}\,|000\rangle + \sqrt{1-\alpha}\,|111\rangle$.

1. Round 1

SWITCH *Action.* Since the generalized GHZ state lies in the protected subspace of $\{|000\rangle, |111\rangle\}$ from Eq. (4), SWITCH action on the state $\rho_{CAB_0} = |GHZ_\alpha\rangle\!\langle GHZ_\alpha|$ leaves the state unchanged

$$\rho_{CAB_1} = \mathcal{K}(\rho_{CAB_0}) = |GHZ_\alpha\rangle\!\langle GHZ_\alpha|\,. \quad (10)$$

Unsharp Measurement. To evaluate the post-measurement states of the unsharp measurement, the POVM elements are expressed in terms of measurement operators $E_\pm^{\lambda_k} = \sqrt{E_\pm^{\lambda_k}}^\dagger\sqrt{E_\pm^{\lambda_k}}$, with

$$\sqrt{E_\pm^{\lambda_k}} = \sqrt{\frac{1\pm\lambda_k}{2}}\,|+\rangle\langle+| + \sqrt{\frac{1\mp\lambda_k}{2}}\,|-\rangle\langle-|\,. \quad (11)$$

With the GHZ state obtained in the first round, $\rho_{CAB_1} = |\text{GHZ}_\alpha\rangle\langle\text{GHZ}_\alpha|$, a post-measurement state $\rho_{CAB_1}^{E_\pm}$ for the combined system, depending on the measurement outcome, takes the form

$$\begin{aligned} \rho_{CAB_1}^{E_\pm} &= \frac{\left(\sqrt{E_\pm^{\lambda_1}}\otimes\mathbb{I}_{AB}\right)\rho_{CAB_1}\left(\sqrt{E_\pm^{\lambda_1}}\otimes\mathbb{I}_{AB}\right)^\dagger}{\mathrm{Tr}\Big[\rho_{CAB_1}(E_\pm^{\lambda_1}\otimes\mathbb{I}_{AB})\Big]} \\ &= \frac{1\pm\lambda_1}{2}\,|+\Phi_\alpha^+\rangle\langle+\Phi_\alpha^+| + \frac{1\mp\lambda_1}{2}\,|-\Phi_\alpha^-\rangle\langle-\Phi_\alpha^-| \\ &+ \frac{\sqrt{1-\lambda_1^2}}{2}\left(|+\Phi_\alpha^+\rangle\langle-\Phi_\alpha^-| + |-\Phi_\alpha^-\rangle\langle+\Phi_\alpha^+|\right), \end{aligned} \quad (12)$$

where the generalized Bell states are defined as

$$\begin{aligned} \left|\Phi^{\pm}_{\alpha}\right\rangle &= \sqrt{\alpha}\left|00\right\rangle \pm \sqrt{1-\alpha}\left|11\right\rangle, \\ \left|\Psi^{\pm}_{\alpha}\right\rangle &= \sqrt{\alpha}\left|01\right\rangle \pm \sqrt{1-\alpha}\left|10\right\rangle, \end{aligned} \tag{13}$$

and the sign $\pm$ depends on the measurement outcome. We also use the standard shorthand $|\Phi^+\rangle := \left|\Phi^+_{\alpha=\frac{1}{2}}\right\rangle$. The obtained state is a probabilistic mixture of two generalized Bell states with maximum mixing at $\lambda_1 = 0$, which decreases as λ_1 grows.

Bell Violation. We can now evaluate the Bell violation on the A:B cut. Tracing out subsystem C from the post-measurement state $\rho^{E\pm}_{CAB_1}$ yields

$$\rho^{E\pm}_{AB_1} = \frac{1\pm\lambda_1}{2}\left|\Phi^+_\alpha\right\rangle\left\langle\Phi^+_\alpha\right| + \frac{1\mp\lambda_1}{2}\left|\Phi^-_\alpha\right\rangle\left\langle\Phi^-_\alpha\right|.$$

Horodecki's criterion is particularly straightforward for generalized Bell states because of the simple form of the T_ρ matrix for these states and the fact that its components are linear in ρ. From the T_ρ matrices for such states,

$$\begin{aligned} T_{\Phi^+_\alpha} &= \mathrm{Diag}\left(2\sqrt{\alpha(1-\alpha)}, -2\sqrt{\alpha(1-\alpha)}, 1\right), \\ T_{\Phi^-_\alpha} &= \mathrm{Diag}\left(-2\sqrt{\alpha(1-\alpha)}, 2\sqrt{\alpha(1-\alpha)}, 1\right), \\ T_{\Psi^+_\alpha} &= \mathrm{Diag}\left(2\sqrt{\alpha(1-\alpha)}, 2\sqrt{\alpha(1-\alpha)}, -1\right), \\ T_{\Psi^-_\alpha} &= \mathrm{Diag}\left(-2\sqrt{\alpha(1-\alpha)}, -2\sqrt{\alpha(1-\alpha)}, -1\right), \end{aligned}$$

we get

$$T_{\rho^{E\pm}_{AB_1}} = \mathrm{Diag}\left(\pm 2\sqrt{\alpha(1-\alpha)}\lambda_1, \mp 2\sqrt{\alpha(1-\alpha)}\lambda_1, 1\right),$$

thus yielding an $M(\rho)$ value of $M(\rho) = 1 + 4\alpha(1-\alpha)\lambda_1^2$ and, therefore, a Bell violation in the first round of $|\mathcal{B}_1| = 2\sqrt{1+4\alpha(1-\alpha)\lambda_1^2}$, independent of which post-measurement outcome is obtained. As expected, the maximum violation occurs with GHZ and Bell states ($\alpha = 1/2$) and decays as α approaches 0 or 1. We conclude that a Bell violation will occur in the first round for all $\alpha \in (0,1)$ and unsharp measurements with $\lambda_1 > 0$.

2. *Round 2*

`SWITCH` *Action.* Starting with the post-measurement states $\rho^{E\pm}_{CAB_1}$ from round one, the second round of the protocol begins with an application of the `SWITCH` action, which results in the state

$$\begin{aligned} \rho_{CAB_2} &= \sum_{i=1}^{3}\mathcal{K}_i\rho^{E\pm}_{CAB_1}\mathcal{K}_i^\dagger \\ &= \frac{1+\sqrt{1-\lambda_1^2}}{2}\left|\mathrm{GHZ}_\alpha\right\rangle\left\langle\mathrm{GHZ}_\alpha\right| \\ &+ \frac{1-\sqrt{1-\lambda_1^2}}{4}\left(2(1-\alpha)\left|010\right\rangle\left\langle 010\right| + 2\alpha\left|101\right\rangle\left\langle 101\right|\right). \end{aligned}$$

Interestingly, the resulting state is independent of the first round's measurement outcome, as the `SWITCH` action maps both possible post-measurement states $\rho^{E+}_{CAB_1}$ and $\rho^{E-}_{CAB_1}$ to the same output state. As such, we do not have to keep track of the particular post-measurement outcomes. The result of performing unsharp measurement in the first round is now apparent in the form of states $|010\rangle\langle 010|$ and $|101\rangle\langle 101|$. Had we not performed the measurement (corresponding to the trivial case $\lambda_1 = 0$), the state would have remained a generalized GHZ state. Tracing out system C from this would result in a separable state for A and B, yielding no Bell violation.

Unsharp Measurement. After the `SWITCH` action, an unsharp measurement is performed on the control qubits C. Utilizing POVMs $E^{\lambda_2}_\pm$ and measurement operators $\sqrt{E^{\lambda_2}_\pm}$ leads to a post-measurement state

$$\begin{aligned} \rho^{E\pm}_{CAB_2} = &\left[\tfrac{1+\sqrt{1-\lambda_1^2}}{2}\right]\Big(\tfrac{1\pm\lambda_2}{2}\left|+\Phi^+_\alpha\right\rangle\left\langle+\Phi^+_\alpha\right| \\ &+ \tfrac{1\mp\lambda_2}{2}\left|-\Phi^-_\alpha\right\rangle\left\langle-\Phi^-_\alpha\right| \\ &+ \tfrac{\sqrt{1-\lambda_2^2}}{2}\left|+\Phi^+_\alpha\right\rangle\left\langle-\Phi^-_\alpha\right| \\ &+ \tfrac{\sqrt{1-\lambda_2^2}}{2}\left|-\Phi^-_\alpha\right\rangle\left\langle+\Phi^+_\alpha\right|\Big) \\ &+ \left[\tfrac{1-\sqrt{1-\lambda_1^2}}{4}\right]\Big(\tfrac{1\pm\lambda_2}{2}\left|+\Psi^+_\alpha\right\rangle\left\langle+\Psi^+_\alpha\right| \\ &+ \tfrac{1\pm\lambda_2}{2}\left|+\Psi^-_\alpha\right\rangle\left\langle+\Psi^-_\alpha\right| \\ &+ \tfrac{1\mp\lambda_2}{2}\left|-\Psi^+_\alpha\right\rangle\left\langle-\Psi^+_\alpha\right| \\ &+ \tfrac{1\mp\lambda_2}{2}\left|-\Psi^-_\alpha\right\rangle\left\langle-\Psi^-_\alpha\right|\Big) \\ &+ \left[\tfrac{1-\sqrt{1-\lambda_1^2}}{4}\tfrac{\sqrt{1-\lambda_2^2}}{2}\right]\Big((1-\alpha)\left|010\right\rangle\left\langle 010\right| \\ &+ \alpha\left|101\right\rangle\left\langle 101\right| - (1-\alpha)\left|110\right\rangle\left\langle 110\right| \\ &- \alpha\left|001\right\rangle\left\langle 001\right|\Big), \end{aligned} \tag{14}$$

where the sign $\pm$ depends on the particular post-measurement outcome, and $|\Phi^\pm_\alpha\rangle$, $|\Psi^\pm_\alpha\rangle$ are the appropriate generalized Bell states.

Bell Violation. Tracing out the system C to access the Bell violation on the A:B cut, we get

$$\rho_{AB_2}^{E_\pm} = \left[\frac{1+\sqrt{1-\lambda_1^2}}{2}\right]\left(\frac{1\pm\lambda_2}{2}\left|\Phi_\alpha^+\right\rangle\left\langle\Phi_\alpha^+\right| + \frac{1\mp\lambda_2}{2}\left|\Phi_\alpha^-\right\rangle\left\langle\Phi_\alpha^-\right|\right) + \left[\frac{1-\sqrt{1-\lambda_1^2}}{4}\right]\left(\left|\Psi_\alpha^+\right\rangle\left\langle\Psi_\alpha^+\right| + \left|\Psi_\alpha^-\right\rangle\left\langle\Psi_\alpha^-\right|\right). \tag{15}$$

Since the state is in the Bell basis, the $T_{\rho_{AB_2}^{E_\pm}}$ matrix is easily calculated:

$$T_{\rho_{AB_2}^{E_\pm}} = \text{Diag}\left(\pm\sqrt{\alpha(1-\alpha)}\left[1+\sqrt{1-\lambda_1^2}\right]\lambda_2, \mp\sqrt{\alpha(1-\alpha)}\left[1+\sqrt{1-\lambda_1^2}\right]\lambda_2, \sqrt{1-\lambda_1^2}\right), \tag{16}$$

that leads to a Bell violation of

$$|\mathcal{B}_2| = \begin{cases} 2\sqrt{2\alpha(1-\alpha)}\left[1+\sqrt{1-\lambda_1^2}\right]\lambda_2, \\ \quad \text{if } \sqrt{\alpha(1-\alpha)}\left[1+\sqrt{1-\lambda_1^2}\right]\lambda_2 \geq \sqrt{1-\lambda_1^2}, \\ 2\sqrt{\alpha(1-\alpha)\left[1+\sqrt{1-\lambda_1^2}\right]^2\lambda_2^2 + 1 - \lambda_1^2}, \\ \quad \text{if } \sqrt{\alpha(1-\alpha)}\left[1+\sqrt{1-\lambda_1^2}\right]\lambda_2 \leq \sqrt{1-\lambda_1^2}. \end{cases} \tag{17}$$

As in the first round, the violation is independent of which of the post-measurement outcomes was obtained. The violation decreases as α approaches 0 or 1, with the maximum obtained for the GHZ state ($\alpha = 1/2$). Note that setting $\alpha = 1/2$, $\lambda_2 = 1$, and $\lambda_1 = 0$ yields a maximal Bell violation of $2\sqrt{2}$ in the second round, but this would give no violation in the first round. There is a certain trade-off in the allowed violation since $|\mathcal{B}_1|$ increases with λ_1, while $|\mathcal{B}_2|$ decreases with λ_1 and increases with λ_2. Indeed, if we set $\lambda_1 = 1$ (a projective measurement), a maximum violation occurs in round 1, but no entanglement is preserved for subsequent rounds. As we decrease λ_1, the measurement becomes unsharp, leading to less violation in the first round and more violation in the second round. At $\lambda_1 = 0$, the measurement is maximally unsharp $E_+^{\lambda_1} = E_-^{\lambda_1} = \mathbb{I}/2$, yielding no information and thus no violation in the first round, while enabling a maximal violation in the second round. This creates a trade-off between the possible violations between the two rounds, more violation in one leads to lesser violation in the other.

This behavior can be exploited. We can set a small value $\lambda_1 = \epsilon > 0$ arbitrarily close to zero and the maximum value $\lambda_2 = 1$. This ensures a finite violation in round one, $|\mathcal{B}_1| = 2\sqrt{1+4\alpha(1-\alpha)\epsilon^2} \approx 2 + 4\alpha(1-\alpha)\epsilon^2 > 2$, and a substantial violation in round two, $|\mathcal{B}_2| = 2\sqrt{\alpha(1-\alpha)[1+\sqrt{1-\epsilon^2}]^2 + 1 - \epsilon^2} \approx 2\sqrt{1+4\alpha(1-\alpha)} - \frac{1+2\alpha(1-\alpha)}{\sqrt{1+4\alpha(1-\alpha)}}\epsilon^2 > 2$. For $\alpha = 1/2$, the violation is near-maximal: $|\mathcal{B}_2| \approx 2\sqrt{2} - \frac{3}{2\sqrt{2}}\epsilon^2$. We conclude that unsharp measurements allow for near-maximal violations in round two while retaining at least some violation in round one; hence $k_{\max} \geq 2$. Now that we have proved the success of the random access Bell game by showing that $k_{\max}$ is indeed lower-bounded by two, we confront two important questions: 1) What is the value of $k_{\max}$? 2) What is the maximum possible violation in round k for $k < k_{\max}$? In what follows, we use the patterns observed so far to answer these questions.

3. *The Random Access Bell Game in the k-th Round*

In this subsection, we deduce theorems on the behavior of the random access Bell game. We elucidate the general forms of the states and the achievable Bell violation in an arbitrary k-th round before tackling the questions posed earlier. Before proceeding with the general case, we introduce some notation

$$R(k) = \prod_{i=1}^{k}\sqrt{1-\lambda_i^2} \quad \text{and} \quad S(k) = \prod_{i=1}^{k}\frac{1+\sqrt{1-\lambda_i^2}}{2}. \tag{18}$$

It is evident that $R(k) = R(k-1)\sqrt{1-\lambda_k^2}$ and $S(k) = S(k-1)\frac{1+\sqrt{1-\lambda_k^2}}{2}$. We also write $|\text{GHZ}_{\alpha+}\rangle = \sqrt{\alpha}\,|000\rangle + \sqrt{1-\alpha}\,|111\rangle$ and $|\text{GHZ}_{\alpha-}\rangle = \sqrt{\alpha}\,|000\rangle - \sqrt{1-\alpha}\,|111\rangle$. Using these conventions, we propose the general forms of the quantum states in the k-th round of the protocol.

Theorem 1. *The state ρ_{CAB_k}, obtained after running the protocol $k-1$ times and implementing the* `SWITCH` *action in the k-th round, has the form*

$$\begin{aligned}\rho_{CAB_k} &= \left[\frac{1+R(k-1)}{4} + \frac{S(k-1)}{2}\right]|GHZ_{\alpha+}\rangle\langle GHZ_{\alpha+}| \\ &+ \left[\frac{1+R(k-1)}{4} - \frac{S(k-1)}{2}\right]|GHZ_{\alpha-}\rangle\langle GHZ_{\alpha-}| \\ &+ \left[\frac{1-R(k-1)}{4}\right]\Big(2(1-\alpha)\,|010\rangle\langle 010| \\ &\qquad +2\alpha\,|101\rangle\langle 101|\Big) \end{aligned} \tag{19}$$

Proof. We prove this by induction. From Eqs. (10) and (14), it is easy to verify that ρ_{CAB_1} and ρ_{CAB_2} satisfy the above form. Assuming it is also true for some ρ_{CAB_k}, the induction step for $\rho_{CAB_{k+1}}$ can be derived by proceeding with the protocol. We prove the induction step in Appendix B. □

The state ρ_{CAB_k} is independent of which of the two outcomes was produced during the unsharp measurement at any of the preceding $k-1$ rounds. This is because, just as in the first two rounds, the `SWITCH` action maps possible post-measurement states to the same output state, as can be inferred from the proof in Appendix B. Using this form of the state, the post-measurement state obtained during the k-th round can be determined. The form is derived in Appendix B (Eq. (B2)).

The state shared between Alice and Bob after the unsharp measurement can be calculated from Eq. (B2) by simply tracing out the subsystem C. It is also evident that the derived form is a generalization of the states obtained in the first two rounds (Eqs. (14) and (14)).

Corollary 1. *The post-measurement state of the subsystem shared by Alice and Bob at the k-th round, $\rho^{E_\pm}_{AB_k}$, has the form*

$$\begin{aligned}
&\rho^{E_\pm}_{AB_k}\\
&= \left[\frac{1+R(k-1)}{4} + \frac{S(k-1)}{2}\right]\left(\frac{1\pm\lambda_k}{2}\left|\Phi^+_\alpha\right\rangle\left\langle\Phi^+_\alpha\right|\right.\\
&\qquad\left. + \frac{1\mp\lambda_k}{2}\left|\Phi^-_\alpha\right\rangle\left\langle\Phi^-_\alpha\right|\right)\\
&+ \left[\frac{1+R(k-1)}{4} - \frac{S(k-1)}{2}\right]\left(\frac{1\pm\lambda_k}{2}\left|\Phi^-_\alpha\right\rangle\left\langle\Phi^-_\alpha\right|\right.\\
&\qquad\left. + \frac{1\mp\lambda_k}{2}\left|\Phi^+_\alpha\right\rangle\left\langle\Phi^+_\alpha\right|\right)\\
&+ \left[\frac{1-R(k-1)}{4}\right]\left(\left|\Psi^+_\alpha\right\rangle\left\langle\Psi^+_\alpha\right| + \left|\Psi^-_\alpha\right\rangle\left\langle\Psi^-_\alpha\right|\right).
\end{aligned}\tag{20}$$

We can now evaluate the T_ρ matrix to measure the Bell violation on the A:B cut. Given that the state is always in the Bell basis, the form of the matrix is particularly convenient.

Theorem 2. *The matrix $T_{\rho^{E_\pm}_{AB_k}}$ for the post-measurement subsystem of Alice and Bob during the k-th round of the protocol has the form*

$$\begin{aligned}
T_{\rho^{E_\pm}_{AB_k}} &= Diag\Big(\pm 2\sqrt{\alpha(1-\alpha)}\,S(k-1)\lambda_k,\\
&\qquad \mp 2\sqrt{\alpha(1-\alpha)}\,S(k-1)\lambda_k, R(k-1)\Big)\\
&= Diag\Big(\pm 2\sqrt{\alpha(1-\alpha)}\prod_{i=1}^{k-1}\tfrac{1+\sqrt{1-\lambda_i^2}}{2}\lambda_k,\\
&\qquad \mp 2\sqrt{\alpha(1-\alpha)}\prod_{i=1}^{k-1}\tfrac{1+\sqrt{1-\lambda_i^2}}{2}\lambda_k, \prod_{i=1}^{k-1}\sqrt{1-\lambda_i^2}\Big).
\end{aligned}\tag{21}$$

where the signs $\pm$ depend upon the measurement outcome in the k-th round.

Proof. The proof is given in Appendix C. □

The matrix has particularly simple form in terms of the functions defined earlier. It is a direct generalization of the forms obtained in the first two rounds. Using the T_ρ matrix, it is straightforward to calculate the Bell violation after k rounds. This leads to our more important result.

Theorem 3. *The Bell violation $|\mathcal{B}_k|$ after k rounds is given by*

$$\begin{aligned}
|\mathcal{B}_k| &= \max\Bigg(2\sqrt{2}\,\sqrt{4\alpha(1-\alpha)}\,S(k-1)\lambda_k,\\
&\qquad 2\sqrt{4\alpha(1-\alpha)\,S(k-1)^2\lambda_k^2 + R(k-1)^2}\Bigg)\\
&= \max\Bigg(2\sqrt{2}\,\sqrt{4\alpha(1-\alpha)}\left[\prod_{i=1}^{k-1}\tfrac{1+\sqrt{1-\lambda_i^2}}{2}\right]\lambda_k,\\
&\qquad 2\sqrt{4\alpha(1-\alpha)\left[\prod_{i=1}^{k-1}\tfrac{1+\sqrt{1-\lambda_i^2}}{2}\right]^2\lambda_k^2 + \left[\prod_{i=1}^{k-1}(1-\lambda_i^2)\right]}\Bigg)
\end{aligned}\tag{22}$$

Proof. Starting with

$$\begin{aligned}
T_{\rho^{E_\pm}_{AB_k}} &= \mathrm{Diag}\Big(\pm 2\sqrt{\alpha(1-\alpha)}\,S(k-1)\lambda_k,\\
&\qquad \mp 2\sqrt{\alpha(1-\alpha)}\,S(k-1)\lambda_k, R(k-1)\Big),
\end{aligned}$$

the $U_{\rho^{E_\pm}_{AB_k}}$ matrix can be calculated as

$$\begin{aligned}
U_{\rho^{E_\pm}_{AB_k}} &= \left(T_{\rho^{E_\pm}_{AB_k}}\right)^T\left(T_{\rho^{E_\pm}_{AB_k}}\right)\\
&= \mathrm{Diag}\left(4\alpha(1-\alpha)S(k-1)^2\lambda_k^2,\right.\\
&\qquad \left.4\alpha(1-\alpha)S(k-1)^2\lambda_k^2,\ R(k-1)^2\right).
\end{aligned}\tag{23}$$

Now, the critical quantity $M\left(\rho^{E_\pm}_{AB_k}\right)$ is the sum of the two largest eigenvalues of the diagonal $U_{\rho^{E_\pm}_{AB_k}}$ matrix. Since, $|\mathcal{B}_k| = 2\sqrt{M(\rho^{E_\pm}_{AB_k})}$, we get the desired expression. □

Just like in the first two rounds, the Bell violation is independent of the chosen post-measurement outcomes of any of the unsharp measurements. All outcomes lead to the same Bell violation. The violation decays as α values get closer to 0 or 1 as expected.

Also, similar to the first two rounds,a trade-off is introduced by the unsharp measurements. The Bell violation in round k, $|\mathcal{B}_k|$, increases with the corresponding sharpness parameter λ_k but decreases with the sharpness parameters $\{\lambda_1, \ldots, \lambda_{k-1}\}$ of all preceding rounds. Consequently, a higher violation in an earlier round comes at the cost of the potential violation in a later one. This behavior can be exploited to generate desired Bell violations, which is the key to our next set of results, where we determine the value of $k_{\max}$ and the maximum achievable violation in any round $k \leq k_{\max}$.

Theorem 4. *A Bell violation can be obtained for an arbitrarily large number of rounds; thus, $k_{\max} = \infty$.*

Proof. Suppose we wish to establish a Bell violation in round k and all preceding rounds. The observed trade-off suggests the following strategy. We select a scaling parameter

$q > 1$ and set the sharpness parameters as $\lambda_k = 1, \lambda_{k-1} = 1/q, \ldots, \lambda_m = 1/q^{k-m}, \ldots, \lambda_1 = 1/q^{k-1}$. For any round m, the sharpness parameter λ_m is significantly higher than those of all preceding rounds. This leads to a Bell violation in the m-th round of

$$|\mathcal{B}_m| = \max\left(\frac{2\sqrt{2}}{q^{k-m}}\sqrt{4\alpha(1-\alpha)}\left[\prod_{i=1}^{m-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right],\right.$$
$$\left. 2\sqrt{4\alpha(1-\alpha)\left[\prod_{i=1}^{m-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right]^2\frac{1}{q^{2(k-m)}}+\left[\prod_{i=1}^{m-1}1-q^{-2(k-i)}\right]}\right). \tag{24}$$

To prove $|\mathcal{B}_m| > 2$, it suffices to show that the term under the square root in the second argument is greater than 1. In Appendix D, we prove that by setting $q \geq \sqrt{2/[\alpha(1-\alpha)]}$, we have

$$4\alpha(1-\alpha)\left[\prod_{i=1}^{m-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right]^2\frac{1}{q^{2(k-m)}} + \prod_{i=1}^{m-1}(1-q^{-2(k-i)}) > 1, \tag{25}$$

and thus $|\mathcal{B}_m| > 2$, constituting a Bell violation at any arbitrary round $m \leq k$. Since k was also chosen arbitrarily, this proves that a violation can be observed for an arbitrarily large number of rounds, and thus $k_{\max} = \infty$. □

This result also shows that generalized GHZ states are a powerful resource for our protocol, since any initial state with non-trivial entanglement ($\alpha \neq 0, 1$) can provide violations for an arbitrary number of rounds. As we increase the scaling factor q, the drop off in sharpness parameter for all the rounds before k becomes much steeper. As the violation in round k increases with a decrease in the sharpness parameters of previous rounds, we expect the violation $|\mathcal{B}_k|$ to grow with the scaling factor q. This is indeed the case, by tweaking the scaling parameter, and α value, Bell violation arbitrary close to the Tsirelson bound of $2\sqrt{2}$ can be obtained in the kth round,while still retaining at least some violation in all preceding rounds.

Theorem 5. *A near-maximal Bell violation can be obtained in the k-th round of a successful Bell game, where k can be made arbitrarily large.*

Proof. Using the scaling factor q and setting the sharpness parameters to $\{\lambda_k = 1, \lambda_{k-1} = 1/q, \ldots, \lambda_1 = 1/q^{k-1}\}$, along with $\alpha = 1/2$, we evaluate the violation observed in the final round k from expression Eq. (24):

$$|\mathcal{B}_k| = \max\left(2\sqrt{2}\prod_{i=1}^{k-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2},\right.$$
$$\left. 2\sqrt{\left[\prod_{i=1}^{k-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right]^2+\prod_{i=1}^{k-1}(1-q^{-2(k-i)})}\right). \tag{26}$$

Since $\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}$ is the average of 1 and $\sqrt{1-q^{-2(k-i)}}$, it is always larger than the $\sqrt{1-q^{-2(k-i)}}$ term and thus

$$|\mathcal{B}_k| = 2\sqrt{2}\left[\prod_{i=1}^{k-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right], \tag{27}$$

It is evident that $|\mathcal{B}_k|$ grows with q. In the limit of arbitrarily large q we get

$$\lim_{q\to\infty}|\mathcal{B}_k| \approx 2\sqrt{2}\prod_{i=1}^{k-1}\left[1-\frac{1}{4q^{2(k-i)}}\right]$$
$$\approx 2\sqrt{2}-\frac{1}{\sqrt{2}q^2}+O\left(\frac{1}{q^4}\right). \tag{28}$$

This implies that we can set an arbitrarily high value for q to obtain any desired violation up to the Tsirelson bound. The guarantee of at least some violation in all previous rounds is ensured by Theorem 4. □

We have proved not only that violations can be obtained for any number of rounds but also that a near-maximal violation can be achieved at any round while retaining at least some violation in all preceding rounds. This demonstrates that a combination of the quantum `SWITCH` and unsharp measurements can be utilized effectively to preserve Bell violations against entanglement-breaking channels. Furthermore, as all post measurement outcomes lead to the same states and violations and successive rounds, we could set in our protocol that no classical information is transmitted between rounds.

While it is evident that at least some Bell violation is possible in as many rounds as needed, the trade-off behavior does put some constraints on the total available violation. For any given α, let us set a minimum threshold on the Bell violation value $|\mathcal{B}_{\min}|$. Now, we require the players to tweak the λ parameters in such a manner to attain at least $|\mathcal{B}_{\min}|$ at every round. From our analysis, we know that a successful violation at some round k requires the sharpness parameter of that round λ_k to be significantly higher than all the preceding rounds. At the same time the maximum λ_k value at any round can not go beyond 1. This suggests that with a fixed $|\mathcal{B}_{\min}|$ and α, there should only be a finite number of maximum rounds $N_{\max}(|\mathcal{B}_{\min}|, \alpha)$ such that a violation of at least $|\mathcal{B}_{\min}|$ is attained in every round up till $N_{\max}(|\mathcal{B}_{\min}|, \alpha)$. With the analytical expression of $|\mathcal{B}_k|$ with respect to λ parameters Eq. (22) available. We can easily evaluate $N_{\max}(|\mathcal{B}_{\min}|, \alpha)$ numerically by choosing minimum λ_k in successive rounds so that $|\mathcal{B}_{\min}|$ is obtained. We plot $N_{\max}(|\mathcal{B}_{\min}|, \alpha)$ to access this behavior.

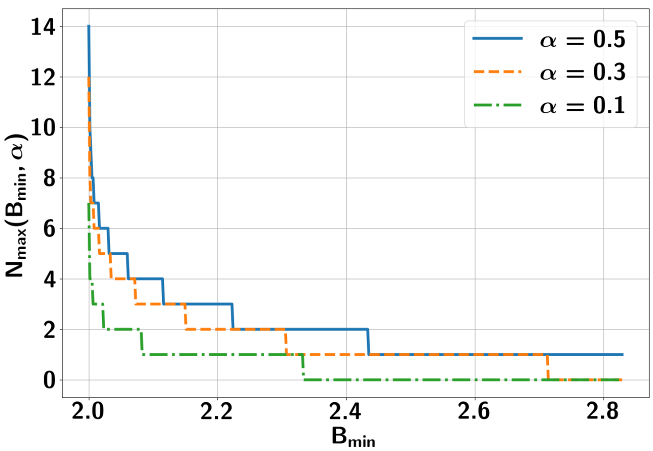

FIG. 3. **Persistence of Bell Violation**. Maximum rounds of violation $N_{\max}(|\mathcal{B}_{\min}|, \alpha)$ (ordinate) with respect to violation threshold $|\mathcal{B}_{\min}|$. The $N_{\max}$ value approaches ∞ as the threshold $|\mathcal{B}_{\min}|$ approaches 2, as predicted by our analysis. All axes are dimensionless.

Fig. 3 demonstrates how $N_{\max}$ scales with $|\mathcal{B}_{\min}|$. Higher the $|\mathcal{B}_{\min}|$, the fewer the rounds in which this level of violation can be observed. We also see that $N_{\max}$ increases as α approaches $1/2$ as expected. While, $N_{\max}$ value is nominal for higher threshold values, it tends towards infinity as the threshold tends to 2.

B. Random Access Bell Game with a W State

Our analysis has shown that a generalized GHZ state is a powerful resource for the proposed Bell game. It is instructive, therefore, to consider the case where the initial state is a W state, $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$, $\rho_{CAB_0} = |W\rangle\langle W|$ instead of a GHZ state, while keeping the rest of the protocol the same. We find that with a W state, no Bell violation is possible even in the first round. This shows that the GHZ type entanglement is pertinent to our protocol.

Theorem 6. *The random access Bell game fails for W states $\rho_{CAB_0} = |W\rangle\langle W|$, leading to $k_{\max} = 0$.*

Proof. Starting with the W state, the state ρ_{CAB_1} obtained after running the `SWITCH` action in the first round results in the state

$$\rho_{CAB_1} = \frac{1}{3}(|000\rangle\langle 000| + |101\rangle\langle 101| + |010\rangle\langle 010|). \tag{29}$$

A subsequent measurement on the control system C in the $|\pm\rangle$ basis yields

$$\begin{aligned}\rho^{E_\pm}_{CAB_1} = & \frac{1\pm 1}{6}|{+00}\rangle\langle{+00}| + \frac{1\mp 1}{6}|{-00}\rangle\langle{-00}| \\ & + \frac{1\pm 1}{6}|{+01}\rangle\langle{+01}| + \frac{1\mp 1}{6}|{-01}\rangle\langle{-01}| \\ & + \frac{1\pm 1}{6}|{+10}\rangle\langle{+10}| + \frac{1\mp 1}{6}|{-10}\rangle\langle{-10}|,\end{aligned} \tag{30}$$

which leads to the following joint state with Alice and Bob on the subsystem AB

$$\rho^{E_\pm}_{AB_1} = \frac{1}{3}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \tag{31}$$

This is a separable state with respect to the A:B partition and therefore cannot violate a Bell inequality. Since the protocol fails to provide a violation in the first round, we conclude that the W state is unsuitable for this game, and thus $k_{\max} = 0$. A proof for the general case of an arbitrary round k is provided in Appendix E. □

We conclude that generalized GHZ state or GHZ type entanglement is an important resource for our protocol. While any generalised GHZ state with $\alpha \in (0, 1)$ can provide Bell violations for an arbitrary number of rounds, the W state is unable to provide a violation in any round.

IV. CONCLUSION

In this paper, we introduced the Random Access Bell Game as a task for studying the resilience of nonlocality against repeated environmental interactions. We analyzed a scenario where the noise is composed of two distinct entanglement-breaking pin maps (complete erasure channels) and demonstrated that conventional strategies, including probabilistic compositions or a canonical quantum `SWITCH` setup, are insufficient to preserve nonlocality. Our main contribution is a protocol that synergistically combines three resources: the decoherence-free subspace created by the quantum `SWITCH`, the controllable disturbance of sequential unsharp measurements, and the specific correlation structure of GHZ-type entanglement. We proved that our protocol not only succeeds in the game but can preserve the ability to violate the CHSH inequality for an arbitrarily large number of rounds, effectively achieving $k_{\max} = \infty$. This result offers a strategy for mitigating environmental noise, demonstrating that quantum correlations can, in principle, be sustained indefinitely despite repeated interactions with entanglement-breaking environments.

Instead of correcting errors after they occur, our protocol utilizes the form of the system-environment interaction itself via indefinite causal order to create a decoherence-free subspace. We then use generalized measurements to manage the trade-off between information gain and disturbance to obtain success in the Random Access Bell Game. We have also shown that by strategically choosing the measurement sharpness parameters, one can achieve a near-maximal Bell violation at arbitrarily chosen round k, while still guaranteeing success in all preceding rounds. Interestingly, the protocol fails for W-states, thereby establishing the Random Access Bell Game as an operational task that can distinguish between GHZ and W states.

The findings open several avenues for future research. A natural next step is to investigate the protocol's performance against other, more general noise models, such as partial erasure channels, depolarizing, dephasing, or amplitude damping channels. The framework itself can be extended to protect other quantum resources beyond nonlocality, such as quantum steering or coherence, in similar sequential scenarios. Another possible route for generalization is to construct the game where each noisy block contains more than two channels, or an extension in a multipartite setting, involving more than two parties, possibly having higher dimensions. Finally, given that both the quantum `SWITCH` and unsharp measurements have been realized experimentally, assessing the feasibility of implementing this sequential protocol in a laboratory setting presents a tangible yet exciting challenge.

ACKNOWLEDGMENT

The authors acknowledge discussions with Pritam Halder and Tamal Guha. SR acknowledges discussions with Arkaprabha Ghoshal and Pratik Ghosal at YouQu 2023, HRI, Prayagraj, India.

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Appendix A: Preliminaries

We review some preliminary concepts that form the foundation of our work. We begin by introducing the mathematical formalism of quantum channels to describe noisy dynamics. We then discuss the quantum `SWITCH` as a means to create superpositions of causal orders, followed by an overview of the CHSH inequality for certifying Bell nonlocality. Finally, we detail generalized and unsharp measurements.

1. Noisy Dynamics

Quantum systems are rarely perfectly isolated from their surroundings. The dynamics of a system interacting with an external environment, often referred to as noise, are described by the theory of open quantum systems. Within this framework, the evolution of the system's density matrix, ρ, is modeled by a quantum channel: a linear map $\mathcal{E}$ that is completely positive and trace-preserving (CPTP).

The action of any such channel can be expressed using the operator-sum representation (or Kraus representation):

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \tag{A1}$$

where the operators $\{E_i\}$, known as Kraus operators, describe the effect of the channel on the system's Hilbert space. The trace-preserving property of the channel, which ensures that the state remains normalized, imposes the condition $\sum_i E_i^\dagger E_i = \mathbb{I}$, where $\mathbb{I}$ is the identity operator. Physically, each term in the sum can be thought of as a possible evolutionary path the quantum system can take due to the environmental interaction.

2. The Quantum `SWITCH`

While quantum channels describe the evolution of a system, higher-order operations known as supermaps can transform these channels themselves. The quantum `SWITCH` is a prominent example of a supermap that takes two quantum channels, $\mathcal{E}$ and $\mathcal{F}$, as inputs and arranges them in a new configuration whose causal order is governed by a quantum degree of freedom [43, 68]. Specifically, the quantum `SWITCH` generates a new channel, $\mathcal{S}(\mathcal{E}, \mathcal{F})$, that acts on a target system and an ancillary control qubit. The order in which the channels $\mathcal{E}$ and $\mathcal{F}$ are applied to the target depends on the state of the control. If the control qubit is in the state $|0\rangle$, the channels are applied in the order $\mathcal{F} \circ \mathcal{E}$. If the control is in the state $|1\rangle$, the order is $\mathcal{E} \circ \mathcal{F}$. This conditional logic is captured by a single set of Kraus operators $\{S_{ij}\}$ acting on the joint target-control system:

$$S_{ij} = F_i E_j \otimes |0\rangle\langle 0| + E_j F_i \otimes |1\rangle\langle 1|\,, \tag{A2}$$

where $\{E_j\}$ and $\{F_i\}$ are the Kraus operators for the channels $\mathcal{E}$ and $\mathcal{F}$, respectively. The overall transformation is then given by

$$\mathcal{S}(\mathcal{E}, \mathcal{F})(\rho \otimes \rho_c) = \sum_{i,j} S_{ij} (\rho \otimes \rho_c) S_{ij}^\dagger, \tag{A3}$$

where ρ_c is the state of the control system. A remarkable feature of the quantum `SWITCH` arises when the control qubit is prepared in a superposition, such as $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. In this case, the target system evolves through a quantum superposition of the two alternative causal orderings, $\mathcal{F} \circ \mathcal{E}$ and $\mathcal{E} \circ \mathcal{F}$. The `SWITCH` supermap possesses an indefinite causal order; that is, it can not be written as a convex combination of quantum combs. This marks a fundamentally non-classical feature that has been shown to provide advantages in various quantum information tasks.

3. Bell Nonlocality and the CHSH Inequality

A key objective of our work is to certify quantum correlations that defy classical explanation. The primary tool for this is the Bell test, specifically the Clauser-Horn-Shimony-Holt (CHSH) inequality [2, 69]. The CHSH test considers two spatially separated parties, Alice and Bob, who share a quantum state ρ_{AB}. Alice can perform one of two measurements, labeled a and a', while Bob can choose between two measurements, b and b'. Each measurement has two possible outcomes, denoted ± 1.

The principles of local realism, which assume that measurement outcomes are predetermined by local hidden variables and are independent of spacelike separated events, impose a classical bound on the correlations between their outcomes. These correlations are captured by the Bell expression:

$$\mathcal{B} = E(a,b) + E(a',b) + E(a,b') - E(a',b'), \tag{A4}$$

where $E(x,y) = \langle A_x B_y \rangle_\rho$ is the expectation value of the product of Alice's outcome for setting x and Bob's for setting y. Local realism dictates that $|\mathcal{B}| \leq 2$. However, quantum mechanics predicts that this bound can be violated. For quantum systems, the maximum possible value is given by the Tsirelson bound, $|\mathcal{B}^Q|_{\max} = 2\sqrt{2}$ [70]. An observed violation of the classical bound, $|\mathcal{B}| > 2$, is a definitive signature of Bell nonlocality.

For any given two-qubit state ρ, the maximum possible CHSH violation can be calculated directly using Horodecki's criterion [71]. This involves constructing a 3×3 real matrix T_ρ with elements $[T_\rho]_{ij} = \mathrm{Tr}[\rho(\sigma_i \otimes \sigma_j)]$ for $i, j \in \{1, 2, 3\}$, where σ_i are the Pauli matrices. From this, one computes the matrix $U_\rho = T_\rho^T T_\rho$. If u_1 and u_2 are the two largest eigenvalues of U_ρ, then the maximal Bell violation for the state ρ is given by $|\mathcal{B}|_{\max} = 2\sqrt{u_1 + u_2}$. This provides a direct method to quantify the nonlocality of any two-qubit state.

4. Unsharp Measurements

The textbook concept of a projective measurement is an idealization. A more general and physically complete description of measurements in quantum mechanics is provided by the Positive Operator-Valued Measure (POVM) formalism. A POVM is a set of operators $\{E_l\}$ that satisfy two conditions: each operator E_l is positive semi-definite ($E_l \geq 0$), and the set forms a resolution of the identity ($\sum_l E_l = \mathbb{I}$).

When a measurement described by the POVM $\{E_l\}$ is performed on a system in state ρ, the probability of obtaining outcome l is given by Born's rule:

$$p(l) = \mathrm{Tr}(\rho E_l). \tag{A5}$$

An "unsharp measurement" is a type of POVM that smoothly interpolates between a sharp, projective measurement and a trivial measurement that extracts no information. In our protocol, we employ two-outcome unsharp measurements on the control qubit in the basis $\{|+\rangle, |-\rangle\}$. The corresponding POVM elements are

$$E^\lambda_\pm = \lambda\, |\pm\rangle\!\langle\pm| + \frac{1-\lambda}{2}\mathbb{I} = \frac{1+\lambda}{2}\, |\pm\rangle\!\langle\pm| + \frac{1-\lambda}{2}\, |\mp\rangle\!\langle\mp|\,, \tag{A6}$$

where the parameter $\lambda \in [0,1]$ controls the "sharpness" of the measurement. When $\lambda = 1$, this reduces to a standard projective measurement. When $\lambda = 0$, the POVM elements become $E_\pm = \mathbb{I}/2$, and the measurement provides no information about the state.

Crucially, upon obtaining a specific outcome, the state of the system undergoes a conditional evolution (a process known as state reduction or collapse). Each POVM element can be expressed as $E_l = M_l^\dagger M_l$, where M_l is the corresponding measurement operator. If outcome l is obtained, the state is updated according to the selective update rule:

$$\rho \xrightarrow{\text{outcome } l} \rho'_l = \frac{M_l \rho M_l^\dagger}{p(l)} = \frac{M_l \rho M_l^\dagger}{\mathrm{Tr}\left(M_l \rho M_l^\dagger\right)}. \tag{A7}$$

For the unsharp measurements in our protocol, the measurement operators can be chosen as $M_\pm = \sqrt{E^\lambda_\pm}$. This rule for updating the state conditioned on the measurement outcome is essential for analyzing the step-by-step evolution in our sequential protocol.

Appendix B: Proof of Theorem 1

The base cases for the induction, $k = 1, 2$, were verified in the main text. Here, we prove the inductive step of Theorem 1. Assuming that our hypothesis is true for some $k \geq 1$, thus

$$\begin{aligned}\rho_{CAB_k} &= \left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right| + \left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right| \\ &+ \left[\frac{1-R(k-1)}{4}\right]\Big(2(1-\alpha)\left|010\right\rangle\left\langle010\right| + 2\alpha\left|101\right\rangle\left\langle101\right|\Big) \end{aligned} \tag{B1}$$

We proceed with the protocol to derive the state $\rho_{CAB_{k+1}}$ and show that it indeed satisfies the above form for $k+1$. Now, ρ_{CAB_k} undergoes the unsharp measurement described by POVMs in Eq. (9) with measurement operators $\sqrt{E_{\pm}^{\lambda_k}} = \sqrt{\frac{1\pm\lambda_k}{2}}\left|+\right\rangle\left\langle+\right| + \sqrt{\frac{1\mp\lambda_k}{2}}\left|-\right\rangle\left\langle-\right|$. These produce the general post-measurement states

$$\begin{aligned}\rho_{CAB_k}^{E_\pm} &= \left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\Bigg(\frac{1\pm\lambda_k}{2}\left|+\Phi_\alpha^+\right\rangle\left\langle+\Phi_\alpha^+\right| + \frac{1\mp\lambda_k}{2}\left|-\Phi_\alpha^-\right\rangle\left\langle-\Phi_\alpha^-\right| \\ &\qquad + \frac{\sqrt{1-\lambda_k^2}}{2}\left|+\Phi_\alpha^+\right\rangle\left\langle-\Phi_\alpha^-\right| + \frac{\sqrt{1-\lambda_k^2}}{2}\left|-\Phi_\alpha^-\right\rangle\left\langle+\Phi_\alpha^+\right|\Bigg) \\ &+ \left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\Bigg(\frac{1\pm\lambda_k}{2}\left|+\Phi_\alpha^-\right\rangle\left\langle+\Phi_\alpha^-\right| + \frac{1\mp\lambda_k}{2}\left|-\Phi_\alpha^+\right\rangle\left\langle-\Phi_\alpha^+\right| \\ &\qquad + \frac{\sqrt{1-\lambda_k^2}}{2}\left|+\Phi_\alpha^-\right\rangle\left\langle-\Phi_\alpha^+\right| + \frac{\sqrt{1-\lambda_k^2}}{2}\left|-\Phi_\alpha^+\right\rangle\left\langle+\Phi_\alpha^-\right|\Bigg) \\ &+ \left[\frac{1-R(k-1)}{4}\right]\Bigg(\frac{1\pm\lambda_k}{2}\Big\{\left|+\Psi_\alpha^+\right\rangle\left\langle+\Psi_\alpha^+\right| + \left|+\Psi_\alpha^-\right\rangle\left\langle+\Psi_\alpha^-\right|\Big\} \\ &\qquad + \frac{1\mp\lambda_k}{2}\Big\{\left|-\Psi_\alpha^+\right\rangle\left\langle-\Psi_\alpha^+\right| + \left|-\Psi_\alpha^-\right\rangle\left\langle-\Psi_\alpha^-\right|\Big\} \\ &\qquad + \frac{\sqrt{1-\lambda_k^2}}{2}\Big\{(1-\alpha)\left|010\right\rangle\left\langle010\right| + \alpha\left|101\right\rangle\left\langle101\right| - (1-\alpha)\left|110\right\rangle\left\langle110\right| - \alpha\left|001\right\rangle\left\langle001\right|\Big\}\Bigg) \end{aligned} \tag{B2}$$

With this state at hand the referee R can ask to deduce Bell violation in the A:B cut. Alternatively, the state can be sent to the next round of protocol where the `SWITCH` action from Eq. (4) takes place to produce the desired state $\rho_{CAB_{k+1}}$.

$$\begin{aligned}&\rho_{CAB_{k+1}} = \\ &\left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\Bigg(\frac{1+\sqrt{1-\lambda_k^2}}{2}\left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right| + \frac{1-\sqrt{1-\lambda_k^2}}{4}\Big\{2(1-\alpha)\left|010\right\rangle\left\langle010\right| + 2\alpha\left|101\right\rangle\left\langle101\right|\Big\}\Bigg) \\ &+\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\Bigg(\frac{1+\sqrt{1-\lambda_k^2}}{2}\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right| + \frac{1-\sqrt{1-\lambda_k^2}}{4}\Big\{2(1-\alpha)\left|010\right\rangle\left\langle010\right| + 2\alpha\left|101\right\rangle\left\langle101\right|\Big\}\Bigg) \\ &+\left[\frac{1-R(k-1)}{4}\right]\Big(\alpha\left|000\right\rangle\left\langle000\right| + (1-\alpha)\left|010\right\rangle\left\langle010\right| + \alpha\left|101\right\rangle\left\langle101\right| + (1-\alpha)\left|111\right\rangle\left\langle111\right|\Big) \\ &+\left[\frac{1-R(k-1)}{4}\sqrt{1-\lambda_k^2}\right]\Big(-\alpha\left|000\right\rangle\left\langle000\right| + (1-\alpha)\left|010\right\rangle\left\langle010\right| + \alpha\left|101\right\rangle\left\langle101\right| - (1-\alpha)\left|111\right\rangle\left\langle111\right|\Big). \end{aligned} \tag{B3}$$

Now, notice that we can rewrite $2\alpha\left|000\right\rangle\left\langle000\right| + 2(1-\alpha)\left|111\right\rangle\left\langle111\right| = \left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right| + \left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right|$ to get

$$
\begin{aligned}
&\rho_{CAB_{k+1}} = \\
&\left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\left(\frac{1+\sqrt{1-\lambda_k^2}}{2}\left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right|+\frac{1-\sqrt{1-\lambda_k^2}}{4}\left\{2(1-\alpha)\left|010\right\rangle\left\langle 010\right|+2\alpha\left|101\right\rangle\left\langle 101\right|\right\}\right)\\
&+\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\left(\frac{1+\sqrt{1-\lambda_k^2}}{2}\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right|+\frac{1-\sqrt{1-\lambda_k^2}}{4}\left\{2(1-\alpha)\left|010\right\rangle\left\langle 010\right|+2\alpha\left|101\right\rangle\left\langle 101\right|\right\}\right)\\
&+\left[\frac{1-R(k-1)}{4}\right]\left(\frac{1}{2}\left|\mathrm{GHZ}_{\alpha}+\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right|+(1-\alpha)\left|010\right\rangle\left\langle 010\right|+\alpha\left|101\right\rangle\left\langle 101\right|+\frac{1}{2}\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right|\right)\\
&+\left[\frac{1-R(k-1)}{4}\sqrt{1-\lambda_k^2}\right]\left(-\frac{1}{2}\left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right|+(1-\alpha)\left|010\right\rangle\left\langle 010\right|+\alpha\left|101\right\rangle\left\langle 101\right|-\frac{1}{2}\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right|\right).
\end{aligned}
\tag{B4}
$$

Collecting similar terms together

$$
\begin{aligned}
\rho_{CAB_{k+1}} &= \left(\left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\frac{1+\sqrt{1-\lambda_k^2}}{2}+\frac{1-R(k-1)}{4}\left[\frac{1}{2}-\frac{\sqrt{1-\lambda_k^2}}{2}\right]\right)\left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right|\\
&+\left(\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\frac{1+\sqrt{1-\lambda_k^2}}{2}+\frac{1-R(k-1)}{4}\left[\frac{1}{2}-\frac{\sqrt{1-\lambda_k^2}}{2}\right]\right)\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right|\\
&+\left(\frac{1+R(k-1)}{2}\frac{1-\sqrt{1-\lambda_k^2}}{4}+\frac{1-R(k-1)}{4}\left[\frac{1}{2}+\frac{\sqrt{1-\lambda_k^2}}{2}\right]\right)\left(2(1-\alpha)\left|010\right\rangle\left\langle 010\right|+2\alpha\left|101\right\rangle\left\langle 101\right|\right),
\end{aligned}
\tag{B5}
$$

and simplifying the terms in the bracket

$$
\begin{aligned}
\left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\frac{1+\sqrt{1-\lambda_k^2}}{2}+\frac{1-R(k-1)}{4}\left[\frac{1}{2}-\frac{\sqrt{1-\lambda_k^2}}{2}\right] &= \frac{S(k)}{2}+\frac{1}{8}+\frac{R(k-1)}{8}+\frac{\sqrt{1-\lambda_k^2}}{8}+\frac{R(k)}{8}\\
&+\frac{1}{8}-\frac{R(k-1)}{8}-\frac{\sqrt{1-\lambda_k^2}}{8}+\frac{R(k)}{8}\\
&= \frac{1+R(k)}{4}+\frac{S(k)}{2},\\
\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\frac{1+\sqrt{1-\lambda_k^2}}{2}+\frac{1-R(k-1)}{4}\left[\frac{1}{2}-\frac{\sqrt{1-\lambda_k^2}}{2}\right] &= -\frac{S(k)}{2}+\frac{1}{8}+\frac{R(k-1)}{8}+\frac{\sqrt{1-\lambda_k^2}}{8}+\frac{R(k)}{8}\\
&+\frac{1}{8}-\frac{R(k-1)}{8}-\frac{\sqrt{1-\lambda_k^2}}{8}+\frac{R(k)}{8}\\
&= \frac{1+R(k)}{4}-\frac{S(k)}{2},\\
\frac{1+R(k-1)}{2}\frac{1-\sqrt{1-\lambda_k^2}}{4}+\frac{1-R(k-1)}{4}\left[\frac{1}{2}+\frac{\sqrt{1-\lambda_k^2}}{2}\right] &= \frac{1}{8}+\frac{R(k-1)}{8}-\frac{\sqrt{1-\lambda_k^2}}{8}-\frac{R(k)}{8}\\
&+\frac{1}{8}-\frac{R(k-1)}{8}+\frac{\sqrt{1-\lambda_k^2}}{8}-\frac{R(k)}{8}\\
&= \frac{1-R(k)}{4}.
\end{aligned}
\tag{B6}
$$

We get the final outcome for the state in the $k+1$th round

$$
\begin{aligned}
\rho_{CAB_{k+1}} &= \left[\frac{1+R(k)}{4}+\frac{S(k)}{2}\right]\left|\mathrm{GHZ}_{\alpha+}\right\rangle\left\langle\mathrm{GHZ}_{\alpha+}\right|+\left[\frac{1+R(k)}{4}-\frac{S(k)}{2}\right]\left|\mathrm{GHZ}_{\alpha-}\right\rangle\left\langle\mathrm{GHZ}_{\alpha-}\right|\\
&+\left[\frac{1-R(k)}{4}\right]\left(2(1-\alpha)\left|010\right\rangle\left\langle 010\right|+2\alpha\left|101\right\rangle\left\langle 101\right|\right),
\end{aligned}
\tag{B7}
$$

thus proving the induction step.

Appendix C: Proof of Theorem 2

We begin with the post-measurement quantum states of the AB subsystem during the kth round of the protocol

$$\begin{aligned}\rho_{AB_k}^{E_\pm} &= \left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\left(\frac{1\pm\lambda_k}{2}\left|\Phi_\alpha^+\right\rangle\left\langle\Phi_\alpha^+\right|+\frac{1\mp\lambda_k}{2}\left|\Phi_\alpha^-\right\rangle\left\langle\Phi_\alpha^-\right|\right)\\ &+\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\left(\frac{1\pm\lambda_k}{2}\left|\Phi_\alpha^-\right\rangle\left\langle\Phi_\alpha^-\right|+\frac{1\mp\lambda_k}{2}\left|\Phi_\alpha^+\right\rangle\left\langle\Phi_\alpha^+\right|\right)\\ &+\left[\frac{1-R(k-1)}{4}\right]\left(\left|\Psi_\alpha^+\right\rangle\left\langle\Psi_\alpha^+\right|+\left|\Psi_\alpha^-\right\rangle\left\langle\Psi_\alpha^-\right|\right).\end{aligned} \tag{C1}$$

With T_ρ matrix being linear on quantum states and the state being in the Bell basis the matrix can be calculated as

$$\begin{aligned}&T_{\rho_{AB_k}^{E_\pm}}\\ &=\left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\left(\frac{1\pm\lambda_k}{2}\mathrm{Diag}(2\sqrt{\alpha(1-\alpha)},-2\sqrt{\alpha(1-\alpha)},1)+\frac{1\mp\lambda_k}{2}\mathrm{Diag}(-2\sqrt{\alpha(1-\alpha)},2\sqrt{\alpha(1-\alpha)},1)\right)\\ &+\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\left(\frac{1\pm\lambda_k}{2}\mathrm{Diag}(-2\sqrt{\alpha(1-\alpha)},2\sqrt{\alpha(1-\alpha)},1)+\frac{1\mp\lambda_k}{2}\mathrm{Diag}(2\sqrt{\alpha(1-\alpha)},-2\sqrt{\alpha(1-\alpha)},1)\right)\\ &+\left[\frac{1-R(k-1)}{4}\right]\left(\mathrm{Diag}(2\sqrt{\alpha(1-\alpha)},2\sqrt{\alpha(1-\alpha)},-1)+\mathrm{Diag}(-2\sqrt{\alpha(1-\alpha)},-2\sqrt{\alpha(1-\alpha)},-1)\right)\\ &=\left[\frac{1+R(k-1)}{4}+\frac{S(k-1)}{2}\right]\mathrm{Diag}(\pm2\sqrt{\alpha(1-\alpha)}\lambda_k,\mp2\sqrt{\alpha(1-\alpha)}\lambda_k,1)\\ &+\left[\frac{1+R(k-1)}{4}-\frac{S(k-1)}{2}\right]\mathrm{Diag}(\mp2\sqrt{\alpha(1-\alpha)}\lambda_k,\pm2\sqrt{\alpha(1-\alpha)}\lambda_k,1)+\left[\frac{1-R(k-1)}{4}\right]\mathrm{Diag}(0,0,-2)\\ &=\mathrm{Diag}\Big(\pm2\sqrt{\alpha(1-\alpha)}S(k-1)\lambda_k,\ \mp2\sqrt{\alpha(1-\alpha)}S(k-1)\lambda_k,\ R(k-1)\Big)\end{aligned} \tag{C2}$$

Appendix D: Proof of Theorem 4

For any scaling factor q, with which we can set our sharpness parameters to be $\{\lambda_k = 1,\ \lambda_{k-1} = 1/q, \dots, \lambda_m = 1/q^{k-m}, \dots, \lambda_1 = 1/q^{k-1}\}$. We wish to show that

$$4\alpha(1-\alpha)\left[\prod_{i=1}^{m-1}\frac{1+\sqrt{1-1/q^{2(k-i)}}}{2}\right]^2\frac{1}{q^{2(k-m)}}+\left[\prod_{i=1}^{m-1}1-1/q^{2(k-i)}\right]\geq 1. \tag{D1}$$

Now, let $k > m \geq 1$ be integers and define

$$X_{m,q} = 4\alpha(1-\alpha)\left[\prod_{i=1}^{m-1}\frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right]^2, \qquad Y_{m,q} = \prod_{i=1}^{m-1}\left(1-q^{-2(k-i)}\right). \tag{D2}$$

We wish to show that $X_{m,q}\,q^{-2(k-m)} + Y_{m,q} \geq 1$, holds by induction on m.

Base case ($m = 1$). When $m = 1$, both products are empty, so $X_{1,q} = 4\alpha(1-\alpha),\ \ Y_{1,q} = 1$. Thus

$$X_{1,q}\,q^{-2(k-1)} + Y_{1,q} = 1\cdot 4\alpha(1-\alpha)q^{-2(k-1)} + 1 = 1 + 4\alpha(1-\alpha)q^{-2(k-1)} \geq 1. \tag{D3}$$

Hence the claim holds for $m = 1$.

Inductive step. Assume the statement holds for some $1 \leq m \leq k$:

$$X_{m,q}\,q^{-2(k-m)} + Y_{m,q} \geq 1. \tag{D4}$$

We must show it for $m+1$. Setting $t = q^{-2(k-m)}, \quad s = \sqrt{1-t}$. Then by definition

$$X_{m+1\,q} = X_{m,q}\Big(\tfrac{1+s}{2}\Big)^2, \quad Y_{m+1} = Y_m(1-t), \quad q^{-2(k-(m+1))} = q^2 t. \tag{D5}$$

Since

$$\Big(\tfrac{1+s}{2}\Big)^2 = \frac{1+2s+s^2}{4} = \frac{2+2s-t}{4} = \frac{1+s}{2} - \frac{t}{4}, \tag{D6}$$

we obtain

$$X_{m+1,q}\, q^{-2(k-(m+1))} = X_m\big[q^2(1+s)t/2 - q^2t^2/4\big]. \tag{D7}$$

Hence

$$\begin{aligned} \mathrm{LHS}_{m+1,q} &= X_{m+1,q}\, q^{-2(k-(m+1))} + Y_{m+1,q} \\ &= X_{m,q}\big[q^2(1+s)t/2 - q^2t^2/4\big] + Y_{m,q}(1-t) \\ &= X_{m,q}\big[t + t\big(q^2/2 + q^2s/2 - q^2t/4 - 1\big)\big] + Y_{m,q}(1-t) \\ &= \underbrace{\big[X_{m,q}\,t + Y_{m,q}\big]}_{\geq 1 \text{ by IH}} + t\big[X_{m,q}\,(q^2/2 + q^2s/2 - q^2t/4 - 1) \;-\; Y_{m,q}\big]. \end{aligned} \tag{D8}$$

Call the second bracket Δ. Then $\Delta = t\big[X_{m,q}\,(q^2/2 + q^2s/2 - q^2t/4 - 1) \;-\; Y_{m,q}\big]$. Because $t > 0$, it suffices to show $4\alpha(1-\alpha)\left[\prod_{i=1}^{m-1} \frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right]^2 (q^2/2 + q^2s/2 - q^2t/4 - 1) \;\geq\; \prod_{i=1}^{m-1}\big(1-q^{-2(k-i)}\big)$. We note that

$$\begin{aligned} & 4\alpha(1-\alpha)(q^2/2 + q^2s/2 - q^2t/4 - 1) \geq 1 \\ \iff\; & q^2(2+2s-t)\alpha(1-\alpha) \geq 2 \\ \iff\; & q^2 \geq \frac{2}{(2+2s-t)\alpha(1-\alpha)} \end{aligned} \tag{D9}$$

Since $s = \sqrt{1-t} \quad\Longrightarrow\quad 2+2s-t \geq 1$ Indeed $t = q^{-2(k-m)} \leq 1$ and at $t=1$, $2+2s-t = 1$. Thus setting

$$q \;\geq\; \sqrt{\frac{2}{a(1-\alpha)}}\ \{0 < a < 1\}$$

$$\text{we get } q^2 \;\geq\; \frac{2}{a(1-\alpha)} \geq \frac{2}{(2+2s-t)\alpha(1-\alpha)} \tag{D10}$$

Furthermore, as each factor satisfies

$$\Big(\tfrac{1+\sqrt{1-q^{-2(k-i)}}}{2}\Big) \;\geq\; \sqrt{1-q^{-2(k-i)}}, \tag{D11}$$

hence $\left[\prod_{i=1}^{m-1} \frac{1+\sqrt{1-q^{-2(k-i)}}}{2}\right]^2 \;\geq\; \prod_{i=1}^{m-1}\big(1-q^{-2(k-i)}\big)$. Therefore

$$X_{m,q}(q^2/2 + q^2s/2 - q^2t/4 - 1) - Y_{m,q} \geq 0, \tag{D12}$$

so

$$\Delta = t\big[X_{m,q}\,(q^2/2 + q^2s/2 - q^2t/4 - 1) \;-\; Y_{m,q}\big] \;\geq\; 0. \tag{D13}$$

Combining,

$$\mathrm{LHS}_{m+1,q} \;=\; \big[X_{m,q}\,t + Y_{m,q}\big] + \Delta \;\geq\; 1 + 0 \;=\; 1, \tag{D14}$$

which completes the inductive step.

Appendix E: Failure of W state for round k

We prove by induction that when the initial resource is a W state, the state ρ_{CAB_k} obtained after running the protocol for $k-1$ rounds and implementing the SWITCH action in the k-th round has the form

$$\rho_{CAB_k} = \frac{1}{3}(|000\rangle\langle 000| + |101\rangle\langle 101|) + \left[\frac{1+R(k-1)}{6}\right]|010\rangle\langle 010| + \left[\frac{1-R(k-1)}{6}\right]|111\rangle\langle 111| \tag{E1}$$

Let us first verify if the induction hypothesis is true for $k=1$. Starting with the W state. The state after the SWITCH action is

$$\rho_{CAB_1} = \mathcal{K}(|W\rangle\langle W|) = \frac{1}{3}\left(|000\rangle\langle 000| + |010\rangle\langle 010| + |101\rangle\langle 101|\right).$$

Thus, the induction hypothesis is true at $k=1$. Assuming our hypothesis is true for some $k \geq 1$, we prove the inductive step by deriving the form of $\rho_{CAB_{k+1}}$. For this we perform the unsharp measurement Eq. (9) with measurement operators $\sqrt{E_{\pm}^{\lambda_k}} = \sqrt{\frac{1\pm\lambda_k}{2}}\,|+\rangle\langle+| + \sqrt{\frac{1\mp\lambda_k}{2}}\,|-\rangle\langle-|$ in the control qubit C, to obtain post measurement state

$$\begin{aligned}
\rho_{CAB_k}^{E_\pm} = {} & \frac{1}{6}\left((1\pm\lambda_k)|{+}00\rangle\langle{+}00| + (1\mp\lambda_k)|{-}00\rangle\langle{-}00| + \sqrt{1-\lambda_k^2}\,|{+}00\rangle\langle{-}00| + \sqrt{1-\lambda_k^2}\,|{-}00\rangle\langle{+}00|\right)\\
& + \frac{1}{6}\left((1\pm\lambda_k)|{+}01\rangle\langle{+}01| + (1\mp\lambda_k)|{-}01\rangle\langle{-}01| - \sqrt{1-\lambda_k^2}\,|{+}01\rangle\langle{-}01| - \sqrt{1-\lambda_k^2}\,|{-}01\rangle\langle{+}01|\right)\\
& + \left[\frac{1+R(k-1)}{6}\right]\left(\frac{1\pm\lambda_k}{2}|{+}10\rangle\langle{+}10| + \frac{1\mp\lambda_k}{2}|{-}10\rangle\langle{-}10| + \frac{\sqrt{1-\lambda_k^2}}{2}|{+}10\rangle\langle{-}10| + \frac{\sqrt{1-\lambda_k^2}}{2}|{-}10\rangle\langle{+}10|\right)\\
& + \left[\frac{1-R(k-1)}{6}\right]\left(\frac{1\pm\lambda_k}{2}|{+}11\rangle\langle{+}11| + \frac{1\mp\lambda_k}{2}|{-}11\rangle\langle{-}11| - \frac{\sqrt{1-\lambda_k^2}}{2}|{+}11\rangle\langle{-}11| - \frac{\sqrt{1-\lambda_k^2}}{2}|{-}11\rangle\langle{+}11|\right).
\end{aligned} \tag{E2}$$

This state is then sent for the next round of the protocol where the SWITCH action takes place to obtain $\rho_{CAB_{k+1}}$.

$$\begin{aligned}
\rho_{CAB_{k+1}} = {} & \frac{1+\sqrt{1-\lambda_k^2}}{6}|000\rangle\langle 000| + \frac{1-\sqrt{1-\lambda_k^2}}{6}|101\rangle\langle 101|\\
& + \frac{1-\sqrt{1-\lambda_k^2}}{6}|000\rangle\langle 000| + \frac{1+\sqrt{1-\lambda_k^2}}{6}|101\rangle\langle 101|\\
& + \left[\frac{1+R(k-1)}{6}\right]\left(\frac{1+\sqrt{1-\lambda_k^2}}{2}|010\rangle\langle 010| + \frac{1-\sqrt{1-\lambda_k^2}}{2}|111\rangle\langle 111|\right)\\
& + \left[\frac{1-R(k-1)}{6}\right]\left(\frac{1-\sqrt{1-\lambda_k^2}}{2}|010\rangle\langle 010| + \frac{1+\sqrt{1-\lambda_k^2}}{2}|111\rangle\langle 111|\right).
\end{aligned} \tag{E3}$$

Collecting similar terms together

$$\begin{aligned}
\rho_{CAB_{k+1}} = {} & \left[\frac{1+\sqrt{1-\lambda_k^2}}{6} + \frac{1-\sqrt{1-\lambda_k^2}}{6}\right]|000\rangle\langle 000|\\
& + \left[\frac{1+\sqrt{1-\lambda_k^2}}{6} + \frac{1-\sqrt{1-\lambda_k^2}}{6}\right]|101\rangle\langle 101|\\
& + \left[\frac{1+R(k-1)}{6}\frac{1+\sqrt{1-\lambda_k^2}}{2} + \frac{1-R(k-1)}{6}\frac{1-\sqrt{1-\lambda_k^2}}{2}\right]|010\rangle\langle 010|\\
& + \left[\frac{1+R(k-1)}{6}\frac{1-\sqrt{1-\lambda_k^2}}{2} + \frac{1-R(k-1)}{6}\frac{1+\sqrt{1-\lambda_k^2}}{2}\right]|111\rangle\langle 111|.
\end{aligned} \tag{E4}$$

which is simplified to

$$\rho_{CAB_{k+1}} = \frac{1}{3}(|000\rangle\langle 000| + |101\rangle\langle 101|) + \left[\frac{1+R(k)}{6}\right]|010\rangle\langle 010| + \left[\frac{1-R(k)}{6}\right]|111\rangle\langle 111|, \tag{E5}$$

and thus the induction step is proved. Much like the generalized GHZ state, the state is independent of whichever post-measurement state was selected in all the previous rounds. Performing an unsharp measurement on the control qubit C yields the post-measurement state Eq. (E2). The control of this post-measurement state is then traced out to obtain the joint state on subsystems AB:

$$\rho_{AB_k}^{E_\pm} = \frac{1}{3}(|00\rangle\langle 00| + |01\rangle\langle 01|) + \left[\frac{1+R(k-1)}{6}\right]|10\rangle\langle 10| + \left[\frac{1-R(k-1)}{6}\right]|11\rangle\langle 11|. \tag{E6}$$

This state is a separable state with respect to the A:B partition and therefore cannot violate any Bell inequality. We conclude that the W state cannot provide any Bell violation in any round, and thus $k_{\max} = 0$.