

ON THE VIETORIS-RIPS COMPLEXES OF INTEGER LATTICES

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ABSTRACT. For a metric space X and $r \geq 0$, the Vietoris-Rips complex $\mathcal{VR}(X; r)$ is a simplicial complex whose simplices are finite subsets of X with diameter at most r . Vietoris-Rips complexes has applications in various places, including data analysis, geometric group theory, sensor networks, etc. Consider the integer lattice $\mathbb{Z}^n$ as a metric space equipped with the d_1 -metric (the Manhattan metric or standard word metric in the Cayley graph). Ziga Virk [Contractibility of the Rips complexes of integer lattices via local domination, Trans. Amer. Math. Soc. 378, no. 3, 1755–1770, 2025] proved that if either $r \geq n^2(2n - 1)$, or $n \in \{1, 2, 3\}$ and $r \geq n$, then the complex $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible, and posed a question if $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible for all $r \geq n$. Recently, Matthew Zaremsky [Contractible Vietoris–Rips Complexes of $\mathbb{Z}^n$, Proc. Amer. Math. Soc, 2025] improved Ziga’s result and proved that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible if $r \geq n^2 + n - 1$. Further, he conjectured that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible for all $r \geq n$. We prove Zaremsky’s conjecture for $n \leq 5$, i.e, we prove that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible if $n \leq 5$ and $r \geq n$. Further, we prove that $\mathcal{VR}(\mathbb{Z}^6; r)$ is contractible for $r \geq 10$.

We determine the homotopy type of $\mathcal{VR}(\mathbb{Z}^n; 2)$, and show that these complexes are homotopy equivalent to a wedge of countably infinite copies of $\mathbb{S}^3$. We also show that $\mathcal{VR}(\mathbb{Z}^n; r)$ is simply connected for $r \geq 2$.

1. INTRODUCTION

For a metric space (X, d) and $r \geq 0$, the Vietoris-Rips complex $\mathcal{VR}(X; r)$ is a simplicial complex on X , where a finite set $\sigma \subseteq X$ is a simplex if and only if diameter of σ is at most r , i.e., $d(x, y) \leq r$ for all $x, y \in \sigma$. The Vietoris-Rips complex was first discovered by Vietoris [27] to define a homology theory for metric spaces and independently re-discovered by E. Rips for studying hyperbolic groups, where it has been popularised as Rips-complex [18, 20]. One of the main motivations behind introducing these complexes was to create a finite simplicial model for metric spaces.

Vietoris-Rips complexes have been used in topological data analysis to probe the shape of a point cloud data using persistence homology [4, 8, 13, 35]. These complexes have been used heavily in computational topology, as a simplicial model for point-cloud data [9, 10, 11, 12] and as simplicial completions of communication links in sensor networks [16, 17, 24].

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In this article, we consider the Vietoris-Rips complexes of the integer lattice $\mathbb{Z}^n$ with the Manhattan metric d (standard word metric in the Cayley graph), *i.e.*, for any $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{Z}^n$, $d(x, y) = \sum_{i=1}^n |x_i - y_i|$. One of the main motivations for our results in this article has a connection to the world of geometric group theory and topological finiteness properties of groups. Recall that a group is of type F_n if it admits a geometric (that is, proper and cocompact) action on an $(n-1)$ -connected CW complex. A group is of type F_* if it admits a geometric action on a contractible CW complex. Zaremsky [33] pointed out that an adequate understanding of the Vietoris-Rips complexes of a group G with the word metric can reveal topological finiteness properties of G . Using the Brown's Criterion¹ [7], he [33, Lemma 3.6] proved that, G (with the word metric corresponding to some finite generating set) is of type F_n if and only if the filtration $(\mathcal{VR}(G; t))_{t \in \mathbb{R}}$ is essentially $(n-1)$ -connected. If some $\mathcal{VR}(G; t)$ is contractible, then G is of type F_* . Rips proved that if a hyperbolic group is equipped with a word metric, then for sufficiently large scale r , its Vietoris-Rips complex is contractible [6, Proposition III.Γ.3.23]. Beyond the hyperbolic case, the contractibility of its Vietoris-Rips complexes is quite hard to prove. It is clear from the definition that the Vietoris-Rips complex of bounded metric spaces is contractible for sufficiently large scale r . The contractibility of Vietoris-Rips complexes at large scales is less understood for unbounded metric spaces, even for simple examples such as integer lattices. For the group $\mathbb{Z}^n$, the question of contractibility of $\mathcal{VR}(\mathbb{Z}^n; r)$ was first posed by Zaremsky in 2018 [33]: *Are the Rips complexes of the free finitely generated Abelian groups (integer lattices in word metric) contractible for large scales?* This question remains open and had been attracting the attention of researchers for more than seven years. In [29], Virk introduced the local domination technique, and using it he proved that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible for $r \geq n^2(2n-1)$. In [34], the author applied Bestvina-Brady discrete Morse theory and improved the bound to $r \geq n(n+1)-1$. Using local domination McCarty [23, Theorem 3.1] showed that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible for $r \geq n(n+1)$. In [34], Zaremsky made the following conjecture (also posed as a question in [29, Section 6]).

Conjecture 1.1 (Zaremsky). For any $r \geq n$, the Vietoris-Rips complex $\mathcal{VR}(\mathbb{Z}^n; r)$ of $\mathbb{Z}^n$ with the Manhattan metric (standard word metric) is contractible.

The Conjecture 1.1 is known to be true for $n \leq 3$ (see [29, 30]). One of the main results of this article is that the Conjecture 1.1 is true for $n \leq 5$. We also prove that $\mathcal{VR}(\mathbb{Z}^6; r)$ is contractible for $r \geq 10$.

Theorem 1.2. (Theorem 3.15) For $n \leq 5$ and $r \geq n$, $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible.

Theorem 1.3. (Theorem 3.16) $\mathcal{VR}(\mathbb{Z}^6; r)$ is contractible for $r \geq 10$.

Virk proved the Conjecture 1.1 for $n = 1, 2$ using domination (see Definition 2.2) and remarked [29, Remark 3.3] that the domination cannot be used for $n \geq 3$. For $n = 3$, he used the local domination technique to prove the Conjecture 1.1. We observed that for $n \geq 3$, the domination cannot be used directly, but it can be used recursively in the links of the vertices chosen carefully in a certain order. To prove

¹If a group acts properly on an $(n-1)$ -connected CW complex X with an invariant cocompact filtration $(X_t)_{t \in \mathbb{R}}$, then the group is of type F_n if and only if this filtration is essentially $(n-1)$ -connected, meaning for all t there exists $s \geq t$ such that the inclusion $X_t \rightarrow X_s$ induces the trivial map in π_k for $k \leq n-1$.

Theorems 1.2 and 1.3, we prove that the $\mathcal{VR}(\{0, 1, \dots, M\}^n; r)$ is contractible for all M . We establish a series of lemmas (Lemmas 3.2 to 3.8 and 3.10 to 3.12), which are true for any $r \geq n \geq 2$ (except a few obvious lower bound conditions for n and r , and the condition $n \geq 5$ and $r \geq 10$ in Lemma 3.12). The idea is to reduce (without changing the homotopy type) the links of vertices (recursively by choosing an ordering of vertices) to a smaller induced subcomplex on the vertices, such that the sum of the absolute value of its coordinates has a fixed upper bound. In Lemma 3.12, we have deduced the condition that the sum of the absolute values of any of the 4 coordinates is $\leq r - 1$. We also believe that our proof strategy should work to fully settle Conjecture 1.1. In particular, if Lemma 3.12 can be generalized to $n - 2$ coordinates (see Conjecture 5.1), then we can prove the Conjecture 1.1 (see Section 5 for the proof assuming that Conjecture 5.1 is true.).

In [29], the author also noted that the bound given in Conjecture 1.1 is optimal in the sense that $\mathcal{VR}(\mathbb{Z}^n; r)$ is not contractible if $r < n$. In fact, it is shown in [3] that $\mathcal{VR}(\{0, 1\}^n; r)$ is not contractible for $r < n$, and in [28] showed that the inclusion $\{0, 1\}^n \rightarrow \mathbb{Z}^n$ induces an injection on the homology of Vietoris-Rips complexes at each scale r . The complex $\mathcal{VR}(\{0, 1\}^n; r)$ has been paid a lot of attention in recent years [1, 3, 5, 14, 26]. In [1], authors proved that $\mathcal{VR}(\{0, 1\}^n; 2)$ is homotopy equivalent to wedge of 3-dimensional spheres $\mathbb{S}^3$'s. Since, there is an injection $\tilde{H}_*(\mathcal{VR}(\{0, 1\}^n; 2)) \rightarrow \tilde{H}_*(\mathcal{VR}(\mathbb{Z}^n; 2))$, it is a natural question to ask whether $\mathcal{VR}(\mathbb{Z}^n; 2)$ is homotopy equivalent to wedge of $\mathbb{S}^3$'s. Motivated by this question, using (Forman's) discrete Morse theory, we prove the following.

Theorem 1.4. (Theorem 4.15) *For $n \geq 3$, $\mathcal{VR}(\mathbb{Z}^n; 2)$ is homotopy equivalent to wedge sum of countably infinite copies of $\mathbb{S}^3$'s.*

We also prove that $\mathcal{VR}(\mathbb{Z}^n; r)$ is simply connected for $r \geq 2$.

Theorem 1.5. (Theorem 4.14) *$\mathcal{VR}(\mathbb{Z}^n; r)$ is simply connected for $r \geq 2$.*

The article is organized as follows. In Section 2, we recall the necessary definitions and basic results used throughout the paper. Section 3 is devoted to the study of $\mathcal{VR}(\mathbb{Z}^n; r)$ for $r \geq n$, where we present several observations, establish key lemmas, and prove our main result on the contractibility Theorems 1.2 and 1.3. In Section 4, we first characterize the maximal simplices of $\mathcal{VR}(\mathbb{Z}^n; 2)$ for $n \geq 3$, and then determine its homotopy type using discrete Morse theory. Finally, in Section 5, we summarize our contributions and discuss directions for future research. In particular, we prove Conjecture 1.1 under the assumption of Conjecture 5.1. We propose questions and a conjecture that naturally arise from this work.

2. PRELIMINARIES

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set called the *vertex set*, and $E(G) \subseteq \binom{V(G)}{2}$ is a set of 2-element subsets of $V(G)$, called the *edge set* of G . A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *induced subgraph* of G on a subset $W \subseteq V(G)$, denoted by $G[W]$, is the graph with vertex set W and edge set $\{\{u, v\} \in E(G) : u, v \in W\}$. For more details on graph-related terminologies, we refer to [31].

An (*abstract*) *simplicial complex* Δ on a vertex set V is a collection of finite subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. If $\sigma \in \Delta$ and $\text{Card}(\sigma) = k + 1$, then σ is called a simplex of dimension k , or a k -simplex, here $\text{Card}(S)$ denote

the cardinality of the set S . We assume that every simplicial complex contains the empty set as the simplex of dimension -1 . The 0-dimensional simplices are referred to as the *vertices* of Δ , and the set of all vertices in Δ is denoted by $V(\Delta)$.

A *subcomplex* of a simplicial complex is a subcollection of simplices that also forms a simplicial complex. For a subset $S \subseteq V(\Delta)$, $\text{Ind}_\Delta(S) = \{\sigma \cap S : \sigma \in \Delta\}$ is called the *induced subcomplex* of Δ induced on the vertex set S .

If Δ is a simplicial complex and $\sigma \in \Delta$, then the *link* of σ is a subcomplex, defined as follows $\text{lk}(\sigma, \Delta) := \{\tau \in \Delta : \sigma \cup \tau \in \Delta \text{ and } \sigma \cap \tau = \emptyset\}$. The *deletion* of σ is defined as the subcomplex $\{\tau \in \Delta : \sigma \not\subseteq \tau\}$ and is denoted by $\text{del}(\sigma, \Delta)$.

In this article, we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization and details about terminologies related to simplicial complexes, we refer the reader to [22]. For details on homotopy theory and related topological concepts, we refer to [21].

Proposition 2.1 (Lemma 2.5, [2]). Let v be a vertex of a simplicial complex K . If the inclusion $\text{lk}(v, K) \hookrightarrow K \setminus v$ is null-homotopic, then we have

$$K \simeq (K \setminus v) \vee \Sigma \text{lk}(v, K).$$

Here, $K \setminus v = \text{del}(v, K)$ is the deletion of v , and ΣX denote the suspension of the space X .

Definition 2.2. (Domination) Let K be a simplicial complex and let $a, b \in V(K)$ with $a \neq b$. We say that the vertex a is *dominated* by b , if for every simplex $\sigma \in K$ containing a , the set $\sigma \cup \{b\}$ is also a simplex in K .

For a simplicial complex K and a vertex v , we define the *open neighborhood* of v as $N(v, K) = \{u \in V(K) : \{u, v\} \in K\}$, and the *closed neighborhood* as $N[v, K] = N(v, K) \cup \{v\}$.

A simplicial complex K is called *flag* if for any $\sigma \subseteq V(K)$, $\sigma \in K$ if and only if $\{u, v\} \in K$ for any $u, v \in \sigma$. Observe that if K is a flag simplicial complex and $N[a, K] \subseteq N[b, K]$, then a is dominated by b . From [25, Proposition 3.2], we have following.

Proposition 2.3. Let K be a simplicial complex and $a, b \in V(K)$ such that $a \neq b$. Let $N[a, K] \subseteq N[b, K]$. If K is a flag complex, then $K \simeq K \setminus a$.

Note that Vietoris-Rips complexes are flag simplicial complexes. Hence Proposition 2.3 is true for any Vietoris-Rips complex.

Let G be a finitely generated group with a finite symmetric generating set S (i.e., $S = S^{-1} := \{x^{-1} : x \in S\}$) such that the identity element $e \notin S$. The *Cayley graph* $\text{Cay}(G, S)$ is the graph whose vertex set is G , with an edge between $g, h \in G$ if and only if $g^{-1}h \in S$.

The *word metric* $d : G \times G \rightarrow \mathbb{N} \cup \{0\}$ on a group G associated with the generating set S is defined by

$$d(g, h) := \min \{n \in \mathbb{N} \cup \{0\} \mid g^{-1}h = s_1 s_2 \cdots s_n \text{ for some } s_i \in S\}.$$

This metric coincides with the graph distance (minimum path length distance) between g and h in the Cayley graph $\text{Cay}(G, S)$.

Let $\mathbb{Z}^n$ denote the n -dimensional integer lattice (free abelian group on n generators). For $x \in \mathbb{Z}^n$, we denote by x_i the i^{th} coordinate of x . We consider $\mathbb{Z}^n$ as a metric space equipped with Manhattan distance d , i.e., for any $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{Z}^n$, $d(x, y) = \sum_{i=1}^n |x_i - y_i|$. We also consider $\mathbb{Z}^n$ as a

graph, where any two vertices x and y are connected by an edge if and only if $d(x, y) = 1$. Note that the metric d in $\mathbb{Z}^n$ is the word metric associated with the standard generators $\{\pm e_1, \dots, \pm e_n\}$, where e_i is the element with i^{th} coordinate 1, and all other co-ordinates are 0.

3. THE COMPLEX $\mathcal{VR}(\mathbb{Z}^n; r)$, $r \geq n$

In this section, we prove Theorems 1.2 and 1.3. For a positive integer n , let $[n] = \{1, \dots, n\}$. For $m > 0$, let $\mathcal{G}_m^n$ denote the induced subgraph $\mathbb{Z}^n[\{0, \dots, m\}^n]$. Let $\Delta_m^{n,r} = \mathcal{VR}(\mathcal{G}_m^n; r)$. By Whitehead's theorem [32], if all homotopy groups of a CW complex X are trivial, then the unique map $X \rightarrow \star$ induces isomorphisms on all homotopy groups and is therefore a homotopy equivalence, here $\star$ denote the one-point space. Hence, X is contractible. Observe that any homotopy class of $\mathcal{VR}(\mathbb{Z}^n; r)$ is contained in the $\mathcal{VR}(\mathcal{G}_m^n; r)$ for some $m \in \mathbb{N}$. Thus, to prove Theorems 1.2 and 1.3, it suffices to show that for each positive integer m , the complex $\Delta_m^{n,r} = \mathcal{VR}(\mathcal{G}_m^n; r)$ is contractible.

We first fix some notations. Let $\text{sgn} : \mathbb{Z} \rightarrow \{-1, 1\}$ denote the sign function, *i.e.*, $\text{sgn}(x) = 1$ if $x > 0$ and $\text{sgn}(x) = -1$ if $x < 0$. Throughout this article, whenever we use $\text{sgn}(x)$ for some $x \in \mathbb{Z}$, the value of x is always nonzero, *i.e.*, either a positive or a negative integer.

Definition 3.1. For $x \in \mathbb{Z}^n$ and $S \subseteq [n]$, we define $\lambda^{[x;S]}$ in $V(\mathbb{Z}^n)$ as follows:

$$\lambda_j^{[x;S]} = \begin{cases} x_j - 1, & \text{if } j \in S \text{ and } x_j > 0, \\ x_j + 1, & \text{if } j \in S \text{ and } x_j < 0, \\ x_j, & \text{elsewhere.} \end{cases}$$

Moreover, if $S = \{i_1, \dots, i_r\} \subseteq [n]$, then as a notation, we write $\lambda^{[x;S]}$ as $\lambda^{[x;i_1 i_2 \dots i_r]}$.

Throughout this section, we fix a positive integer $n \geq 2$. Let $\prec$ denote the anti-lexicographic order on $V(\mathbb{Z}^n)$, *i.e.*, for any two distinct vertices $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in V(\mathbb{Z}^n)$, we have $x \prec y$ if and only if the largest index i at which $x_i \neq y_i$, $x_i < y_i$.

Fix a positive integer m . For $1 \leq \alpha \leq \text{Card}(V(\mathcal{G}_m^n))$, let $\mathcal{H}_m^{n,\alpha}$ denote the subset of $V(\mathcal{G}_m^n)$ after removing the first α elements (with respect to order $\prec$) from $V(\mathcal{G}_m^n)$. Let $\delta = (\delta_1, \dots, \delta_n)$ be the least element of $\mathcal{H}_m^{n,\alpha}$. Let $Y_m^{n,\alpha}$ be the induced subgraph of $\mathbb{Z}^n$ on the vertex set $\{x - \delta : x \in \mathcal{H}_m^{n,\alpha}\}$. Clearly, $(0, \dots, 0) \in V(Y_m^{n,\alpha})$ and it is the smallest element of $V(Y_m^{n,\alpha})$.

Let $\mathbf{0}$ denote the vertex $(0, \dots, 0) \in V(\mathbb{Z}^n)$. Let $\Gamma_n^{\alpha,r} = \text{lk}(\mathbf{0}, \mathcal{VR}(Y_m^{n,\alpha}; r))$. Since the link of a vertex in a flag complex is flag, it follows that, $\Gamma_n^{\alpha,r}$ is flag.

For any $r \geq 1$, we have

$$V(\Gamma_n^{\alpha,r}) \subseteq \underbrace{([-r, r] \cap \mathbb{Z}) \times \dots \times ([-r, r] \cap \mathbb{Z})}_{(n-1)\text{-times}} \times ([0, r] \cap \mathbb{Z}).$$

Observe that, for any $x \in V(\mathbb{Z}^n)$, if $\mathbf{0} \prec x$, $x + \delta \in \mathcal{H}_m^{n,\alpha}$ and $d(\mathbf{0}, x) \leq r$, then $x \in V(\Gamma_n^{\alpha,r})$. Further, if $z \in V(\Gamma_n^{\alpha,r})$, $\mathbf{0} \prec x \prec z$ and $d(\mathbf{0}, x) \leq r$, then $x \in V(\Gamma_n^{\alpha,r})$.

To show that $\Delta_m^{n,r}$ is contractible, we successively remove vertices from $\mathcal{G}_m^n$, using Proposition 2.1, without changing the homotopy type, until only one vertex remains in the complex. For this, it is sufficient to show that $\Gamma_n^{\alpha,r}$ is contractible for all $1 \leq \alpha \leq \text{Card}(V(\mathcal{G}_m^n)) - 1$. To prove the contractibility of $\Gamma_n^{\alpha,r}$, we first establish a series of lemmas.

Lemma 3.2. *Let $r \geq 2$ and let Δ be a subcomplex of $\Gamma_n^{\alpha,r}$. Let $x, y, z \in V(\Delta)$ be such that $z \in V(\text{lk}(x, \Delta))$ and $d(x, y) = 1$. Let $i \in [n]$ be such that $|y_i| = |x_i| - 1$ and $y_j = x_j$ for all $j \neq i$.*

- (i) *If $|z_i| < |x_i|$, then $d(z, y) \leq r - 1$.*
- (ii) *If $|z_i| \geq |x_i|$ and, $\text{sgn}(z_i x_i) = -1$, then $d(z, y) \leq r - 1$.*

Proof. Given $z \in V(\text{lk}(x, \Delta))$, it follows that $d(x, z) \leq r$. Observe that each of the hypotheses in (i) and (ii) leads to $|z_i - y_i| = |z_i - x_i| - 1$. Since $|z_j - y_j| = |z_j - x_j|$ for every $j \neq i$, we have $d(z, y) = |z_i - y_i| + \sum_{j \neq i} |z_j - y_j| = (|z_i - x_i| - 1) + \sum_{j \neq i} |z_j - x_j| = d(z, x) - 1 \leq r - 1$. This completes the proof. $\square$

Lemma 3.3. *Let $r \geq 2$. Then $\Gamma_n^{\alpha,r}$ is homotopy equivalent to the induced subcomplex of $\Gamma_n^{\alpha,r}$ on the vertex set $\{x \in V(\Gamma_n^{\alpha,r}) : |x_i| \leq \lfloor \frac{r}{2} \rfloor \text{ for all } 1 \leq i \leq n\}$.*

Proof. Without changing the homotopy type of $\Gamma_n^{\alpha,r}$, we remove all the vertices x such that $|x_i| > \lfloor \frac{r}{2} \rfloor$ for some i . Let $x \in V(\Gamma_n^{\alpha,r})$ be such that $|x_i| = r$ for some $i \in [n]$. Since $d(\mathbf{0}, z) = \sum_{i=1}^n |z_i| \leq r$ for all $z \in V(\Gamma_n^{\alpha,r})$ and $x \succ \mathbf{0}$, we get that $x_i = r$ and $x_j = 0$ for all $j \neq i$. Let us consider the vertex $\lambda^{[x;i]}$. Since $r \geq 2$, $\lambda_i^{[x;i]} > 0$ and $\lambda_j^{[x;i]} = 0$ for all $j \neq i$. This implies that $\lambda^{[x;i]} \succ \mathbf{0}$ and $\lambda^{[x;i]} \in V(\Gamma_n^{\alpha,r})$. Clearly, $d(\lambda^{[x;i]}, x) = 1$.

We show that $N[x, \Gamma_n^{\alpha,r}] \subseteq N[\lambda^{[x;i]}, \Gamma_n^{\alpha,r}]$. Clearly, $x \in N[\lambda^{[x;i]}, \Gamma_n^{\alpha,r}]$. Let $z \in N[x, \Gamma_n^{\alpha,r}]$ with $z \neq x$. Since $z \succ \mathbf{0}$, $d(z, x) \leq r$ and $x_i = r$, we conclude that $|z_i| < r = x_i$. Therefore, from Lemma 3.2, $d(z, \lambda^{[x;i]}) \leq r - 1$. Thus, $z \in N[\lambda^{[x;i]}, \Gamma_n^{\alpha,r}]$, and hence $N[x, \Gamma_n^{\alpha,r}] \subseteq N[\lambda^{[x;i]}, \Gamma_n^{\alpha,r}]$. From Proposition 2.3, we conclude that $\Gamma_n^{\alpha,r} \simeq \text{Ind}_{\Gamma_n^{\alpha,r}}(V(\Gamma_n^{\alpha,r}) \setminus \{x\})$. Let $A = \{y \in V(\Gamma_n^{\alpha,r}) : y_i = r \text{ for some } i \in [n] \text{ and } y_j = 0 \text{ for all } j \neq i\}$. Repeating the above argument for each vertex $y \in A$, we get that

$$\Gamma_n^{\alpha,r} \simeq \text{Ind}_{\Gamma_n^{\alpha,r}}(V(\Gamma_n^{\alpha,r}) \setminus A).$$

If $r = 2$, then for every $z \in \text{Ind}_{\Gamma_n^{\alpha,r}}(V(\Gamma_n^{\alpha,r}) \setminus A)$, $|z_j| \leq 1$ for all $j \in [n]$ and thus we are done. Now, we assume that $r \geq 3$.

Inductively, assume that, without changing the homotopy type of $\Gamma_n^{\alpha,r}$, we have removed all the vertices y from $\Gamma_n^{\alpha,r}$ such that $r \geq |y_i| \geq \lfloor \frac{r}{2} \rfloor + t + 1$ for some $t \geq 1$ and $i \in [n]$. Let Γ' denote the resulting subcomplex of $\Gamma_n^{\alpha,r}$ after removal of all such vertices. Then Γ' is the induced subcomplex on the vertex set $\{x \in V(\Gamma_n^{\alpha,r}) : |x_i| \leq \lfloor \frac{r}{2} \rfloor + t \text{ for all } i \in [n]\}$ and $\Gamma' \simeq \Gamma_n^{\alpha,r}$.

Let $u \in V(\Gamma')$ be such that $|u_s| = \lfloor \frac{r}{2} \rfloor + t$ for some $s \in [n]$. Since $d(\mathbf{0}, u) = \sum_{j \in [n]} |u_j| \leq r$, we see that $\sum_{j \neq s} |u_j| \leq \lfloor \frac{r}{2} \rfloor - t$. Consider the element $\lambda^{[u;s]}$. Clearly, $d(\mathbf{0}, \lambda^{[u;s]}) \leq d(\mathbf{0}, u) - 1 \leq r - 1$ and $d(u, \lambda^{[u;s]}) = 1$. Since $r \geq 2$, and $\mathbf{0} \prec \lambda^{[u;s]} \prec u$, we get $\lambda^{[u;s]} \succ \mathbf{0}$ and $\lambda^{[u;s]} \in V(\Gamma_n^{\alpha,r})$. Further, since $|\lambda_j^{[u;s]}| < \lfloor \frac{r}{2} \rfloor + t$ for all $j \in [n]$, we have $\lambda^{[u;s]} \in V(\Gamma')$.

We now prove that $N[u, \Gamma'] \subseteq N[\lambda^{[u;s]}, \Gamma']$. Clearly, $u \in N[\lambda^{[u;s]}, \Gamma']$. Let $v \in N[u, \Gamma']$ with $v \neq u$. If $|v_s| < |u_s|$ or, $|v_s| \geq |u_s|$ and $\text{sgn}(v_s u_s) = -1$, then from Lemma 3.2, $d(v, \lambda^{[u;s]}) \leq r$. Thus, in this case, $v \in N[\lambda^{[u;s]}, \Gamma']$.

Suppose $|v_s| \geq |u_s|$ and $\text{sgn}(v_s u_s) = 1$. Since $|v_s| \leq \lfloor \frac{r}{2} \rfloor + t$, it follows that $|v_s| = \lfloor \frac{r}{2} \rfloor + t$ and $v_s = u_s$. Further, since $v \in V(\Gamma') \subseteq V(\Gamma_n^{\alpha,r})$, $d(\mathbf{0}, v) = \sum_{j \in [n]} |v_j| \leq r$.

Thus, we have $\sum_{j \neq s} |v_j| \leq \lceil \frac{r}{2} \rceil - t$. Therefore,

$$\begin{aligned} d(v, \lambda^{[u;s]}) &= |v_s - \lambda_s^{[u;s]}| + \sum_{j \neq s} |v_j - \lambda_j^{[u;s]}| \\ &= |v_s - u_s| + 1 + \sum_{j \neq s} |v_j - u_j| \leq 1 + \sum_{j \neq s} |u_j| + \sum_{j \neq s} |v_j| \\ &\leq 1 + 2\lceil \frac{r}{2} \rceil - 2t \leq r + 2 - 2t \leq r, \text{ as } t \geq 1. \end{aligned}$$

This implies that $N[u, \Gamma'] \subseteq N[\lambda^{[u;s]}, \Gamma']$. Therefore, from Proposition 2.3, $\Gamma' \simeq \text{Ind}_{\Gamma'}(V(\Gamma') \setminus \{u\})$. Let $B = \{x \in V(\Gamma') : |x_i| = \lfloor \frac{r}{2} \rfloor + t \text{ for some } i \in [n]\}$. Repeating the above argument for each vertex $y \in B$, we find that $\Gamma' \simeq \text{Ind}_{\Gamma'}(V(\Gamma') \setminus B)$. Consequently, $\Gamma_n^{\alpha, r} \simeq \text{Ind}_{\Gamma'}(V(\Gamma') \setminus B)$.

By induction, we conclude that $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex of $\Gamma_n^{\alpha, r}$ on the vertex set $\{x \in V(\Gamma_n^{\alpha, r}) : |x_i| \leq \lfloor \frac{r}{2} \rfloor \forall 1 \leq i \leq n\}$. $\square$

Lemma 3.4. *Let $r \geq 2$ and let Δ be a subcomplex of $\Gamma_n^{\alpha, r}$. Let $x, y, z \in V(\Delta)$ be such that $z \in V(\text{lk}(x, \Delta))$ and $d(x, y) = 2$. Let $|y_i| = |x_i| - 1$ and $|y_j| = |x_j| - 1$ for some distinct $i, j \in [n]$, and $y_k = x_k$ for all $k \neq i, j$.*

- (a) *If $|z_i| < |x_i|$ or $|z_j| < |x_j|$, then $d(z, y) \leq r$.*
- (b) *If $|z_i| \geq |x_i|$ and $\text{sgn}(z_i x_i) = -1$, or $|z_j| \geq |x_j|$ and $\text{sgn}(z_j x_j) = -1$, then $d(z, y) \leq r$.*

Proof. Suppose $|z_i| < |x_i|$. Then we have $|z_i - y_i| = |z_i - x_i| - 1$. Since $|y_j| = |x_j| - 1$, regardless of whether $|z_j| < |x_j|$ or $|z_j| \geq |x_j|$, we have $|z_j - y_j| \leq |z_j - x_j| + 1$. Therefore, $d(z, y) = |z_i - y_i| + |z_j - y_j| + \sum_{k \neq i, j} |z_k - y_k| \leq (|z_i - x_i| - 1) + (|z_j - x_j| + 1) + \sum_{k \neq i, j} |z_k - x_k| \leq r$. A similar argument applies if $|z_j| < |x_j|$. This proves part (a).

For part (b), note that if $|z_i| \geq |x_i|$ and $\text{sgn}(z_i x_i) = -1$, then $|z_i - y_i| = |z_i - x_i| - 1$. Thus, similar computations as above show that $d(z, y) \leq r$. A similar argument applies if $|z_j| \geq |x_j|$ and $\text{sgn}(z_j x_j) = -1$. This proves part (b). $\square$

Lemma 3.5. *Let $r \geq 3$. Then $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex of $\Gamma_n^{\alpha, r}$ on the vertex set $\{x \in \Gamma_n^{\alpha, r} : |x_i| \leq \lfloor \frac{r}{2} \rfloor \text{ and } |x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil \text{ for all } i, j, k \in [n], j \neq k\}$.*

Proof. From Lemma 3.3, $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex, say Δ , of $\Gamma_n^{\alpha, r}$ on the vertex set $\{y \in \Gamma_n^{\alpha, r} : |y_i| \leq \lfloor \frac{r}{2} \rfloor \text{ for all } 1 \leq i \leq n\}$.

Consider the dictionary order $\prec'$ on the set $\{(i, j) : i, j \in [n], i < j\}$. Then we have $(1, 2) \prec' (1, 3) \prec' \dots \prec' (1, n) \prec' (2, 3) \prec' \dots \prec' (n-1, n)$. Without changing the homotopy type of Δ , we remove all the vertices whose sum of the i -th and j -th entries exceeds $\lceil \frac{r}{2} \rceil$ for some $i, j \in [n]$. We remove such vertices in a sequence following the order $\prec'$.

We begin with the pair $(1, 2)$ and remove all vertices y from Δ such that $|y_1| + |y_2| > \lceil \frac{r}{2} \rceil$, using induction on the value $r - |y_1| - |y_2|$.

For the base case, let $x \in V(\Delta)$ with $|x_1| + |x_2| = r$. Since $d(\mathbf{0}, x) \leq r$ and $|x_1|, |x_2| \leq \lfloor \frac{r}{2} \rfloor$, we see that $x_1, x_2 \neq 0$ and $x_j = 0$ for all $3 \leq j \leq n$. If $r = 3$, then since $|x_1|, |x_2| \leq \lfloor \frac{3}{2} \rfloor = 1$, we get $|x_1| + |x_2| \leq 2$, contradicting the assumption. Thus we assume $r \geq 4$, which implies $|x_1|, |x_2| \geq 2$. Now, consider $\lambda^{[x; 12]}$. Since $r \geq 4$ and $\mathbf{0} \prec x$, it follows that $x_2 \geq 2$. Thus, from the definition of $\lambda^{[x; 12]}$ we

conclude that $\mathbf{0} \prec \lambda^{[x;12]}$ and $\lambda^{[x;12]} \in V(\Delta)$. Clearly, $d(x, \lambda^{[x;12]}) = 2 \leq r$ and therefore $x \in N[\lambda^{[x;12]}, \Delta]$. We show that $N[x, \Delta] \subseteq N[\lambda^{[x;12]}, \Delta]$.

Let $z \in N[x, \Delta]$ with $z \neq x$. If $|z_1| < |x_1|$ or $|z_2| < |x_2|$, then by Lemma 3.4, $d(z, \lambda^{[x;12]}) \leq r$. Similarly, if $|z_1| \geq |x_1|$ and $\text{sgn}(z_1 x_1) = -1$, or $|z_2| \geq |x_2|$ and $\text{sgn}(z_2 x_2) = -1$, then again $d(z, \lambda^{[x;12]}) \leq r$.

Now consider the case $|z_1| \geq |x_1|$, $\text{sgn}(z_1 x_1) = 1$ and $|z_2| \geq |x_2|$, $\text{sgn}(z_2 x_2) = 1$. Since $|x_1| + |x_2| = r$, it follows that $z_1 = x_1$ and $z_2 = x_2$, hence $d(z, \lambda^{[x;12]}) = 2 \leq r$. Therefore, $N[x, \Delta] \subseteq N[\lambda^{[x;12]}, \Delta]$. From Proposition 2.3, $\Delta \simeq \Delta \setminus x$.

By repeating the above argument for all vertices y such that $|y_1| + |y_2| = r$, we get that $\Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus \{y \in V(\Delta) : |y_1| + |y_2| = r\})$.

Assume that, without changing the homotopy type of Δ , we have removed all the vertices y from Δ such that $r \geq |y_1| + |y_2| \geq \lceil \frac{r}{2} \rceil + t + 1$ for some $t \geq 1$. Let Δ_1 denote the resulting subcomplex of Δ after removal of all such vertices. Then Δ_1 is the induced subcomplex on $\{a \in V(\Delta) : |a_1| + |a_2| \leq \lceil \frac{r}{2} \rceil + t, t \geq 1\}$ and $\Delta \simeq \Delta_1$.

Let $p \in V(\Delta_1)$ with $|p_1| + |p_2| = \lceil \frac{r}{2} \rceil + t$. By the definition of Δ_1 , we have $p \succ \mathbf{0}$, moreover, since $r \geq 3$ and $|p_1|, |p_2| \leq \lfloor \frac{r}{2} \rfloor$, we get $p_1, p_2 \neq 0$. Now, we have the following cases:

Case (i): Suppose that either $p_2 \geq 2$, or $p_2 \leq 1$ and there exists $k > 2$ such that $p_k > 0$. Without loss of generality, assume that when $p_2 \leq 1$, k is the largest such index. Since $p \succ \mathbf{0}$, $p_j = 0$ for all $j > k$.

Consider the element $\lambda^{[p;12]}$. Observe that $d(p, \lambda^{[p;12]}) = 2$, $|\lambda_1^{[p;12]} - p_1| = 1$, $|\lambda_2^{[p;12]} - p_2| = 1$, and $\lambda_j^{[p;12]} = p_j$ for all $j \geq 3$. In case of $p_2 \leq 1$, $\lambda_k^{[p;12]} > 0$ and $\lambda_j^{[p;12]} = 0$ for all $j > k$. Therefore, $\lambda^{[p;12]} \succ \mathbf{0}$. In case of $p_2 \geq 2$, there exists $s \geq 2$ such that $p_s > 0$ and s is the largest such index. Then from the definition of $\lambda^{[p;12]}$, $\lambda_s^{[p;12]} > 0$ and $\lambda_j^{[p;12]} = 0$, for all $j \geq s$. Therefore, $\lambda^{[p;12]} \succ \mathbf{0}$. Moreover, since $|\lambda_j^{[p;12]}| = |p_j| - 1$ for $j \in \{1, 2\}$, we have $|\lambda_1^{[p;12]}| + |\lambda_2^{[p;12]}| \leq \lceil \frac{r}{2} \rceil + t - 2$. Therefore, $\lambda^{[p;12]} \in V(\Delta_1)$.

We claim that $N[p, \Delta_1] \subseteq N[\lambda^{[p;12]}, \Delta_1]$. Let $q \in N[p, \Delta_1]$ with $q \neq p$. If $|q_1| < |p_1|$ or $|q_2| < |p_2|$, then $d(q, \lambda^{[p;12]}) \leq r$ by Lemma 3.4. Similarly, if $|q_1| \geq |p_1|$ and $\text{sgn}(q_1 p_1) = -1$, or $|q_2| \geq |p_2|$ and $\text{sgn}(q_2 p_2) = -1$, then $d(q, \lambda^{[p;12]}) \leq r$.

Now consider $|q_1| \geq |p_1|$, $\text{sgn}(q_1 p_1) = 1$, and $|q_2| \geq |p_2|$, $\text{sgn}(q_2 p_2) = 1$. Since $|q_1| + |q_2| \leq \lceil \frac{r}{2} \rceil + t$ and $|p_1| + |p_2| = \lceil \frac{r}{2} \rceil + t$, we get $q_1 = p_1$ and $q_2 = p_2$. Then $\sum_{i \geq 3} |q_i| \leq \lfloor \frac{r}{2} \rfloor - t$ and $\sum_{i \geq 3} |p_i| \leq \lfloor \frac{r}{2} \rfloor - t$. Thus,

$$\begin{aligned} d(q, \lambda^{[p;12]}) &= |q_1 - \lambda_1^{[p;12]}| + |q_2 - \lambda_2^{[p;12]}| + \sum_{i \geq 3} |q_i - \lambda_i^{[p;12]}| \\ &= |p_1 - \lambda_1^{[p;12]}| + |p_2 - \lambda_2^{[p;12]}| + \sum_{i \geq 3} |q_i - p_i| \\ &\leq 2 + \sum_{i \geq 3} |p_i| + \sum_{i \geq 3} |q_i| \leq 2 + 2 \lfloor \frac{r}{2} \rfloor - 2t \leq r + 2 - 2t \leq r. \end{aligned}$$

Therefore, $N[p, \Delta_1] \subseteq N[\lambda^{[p;12]}, \Delta_1]$. From Proposition 2.3, $\Delta_1 \simeq \Delta_1 \setminus p$.

Case (ii): $p_2 \leq 1$ and $p_k \leq 0$ for all $k > 2$.

Since $p \succ \mathbf{0}$ and $p_2 \neq 0$, we conclude that $p_2 = 1$ and $p_j = 0$ for all $j > 2$.

Using $|p_1| + |p_2| = \lceil \frac{r}{2} \rceil + t$, we get $|p_1| = \lceil \frac{r}{2} \rceil + t - 1$. Since $t \geq 1$, it follows that $|p_1| \geq \lceil \frac{r}{2} \rceil$. Further, since $|p_1| \leq \lfloor \frac{r}{2} \rfloor$, we get that r is even and $|p_1| = \frac{r}{2}$. Consider the element $\lambda^{[p;1]}$.

Note that $|\lambda_1^{[p;1]}| = |p_1| - 1$, and $\lambda_j^{[p;1]} = p_j$ for all $j \geq 2$. Since $|p_1| + |p_2| = \lceil \frac{r}{2} \rceil + t$ we get $|\lambda_1^{[p;1]}| + |\lambda_2^{[p;1]}| < \lceil \frac{r}{2} \rceil + t$. Using $p \succ \mathbf{0}$, $p_2 = \lambda_2^{[p;1]} = 1$ and $p_j = \lambda_j^{[p;1]} = 0$ for all $j > 2$, we see that $\lambda^{[p;1]} \succ \mathbf{0}$ and $\lambda^{[p;1]} \in V(\Delta_1)$.

Let $u \in N[p, \Delta_1]$ with $u \neq p$. If $|u_1| < |p_1|$, then $d(u, \lambda^{[p;1]}) \leq r$ by Lemma 3.2. Similarly, if $|u_1| \geq |p_1|$ and $\text{sgn}(u_1 p_1) = -1$, then $d(u, \lambda^{[p;1]}) \leq r$.

Now consider $|u_1| \geq |p_1|$, $\text{sgn}(u_1 p_1) = 1$. Since $|u_1| \leq \frac{r}{2}$, we must have $u_1 = p_1$. Then $\sum_{i \geq 2} |u_i| \leq \frac{r}{2}$ and $\sum_{i \geq 2} |p_i| = 1$. Thus,

$$\begin{aligned} d(u, \lambda^{[p;1]}) &= |u_1 - \lambda_1^{[p;1]}| + \sum_{i \geq 2} |u_i - \lambda_i^{[p;1]}| = |p_1 - \lambda_1^{[p;1]}| + \sum_{i \geq 2} |u_i - p_i| \\ &\leq 1 + \sum_{i \geq 2} |p_i| + \sum_{i \geq 2} |u_i| \leq 1 + 1 + \frac{r}{2} \leq r. \end{aligned}$$

Thus, $N[p, \Delta_1] \subseteq N[\lambda^{[p;1]}, \Delta_1]$, and from Proposition 2.3, $\Delta_1 \simeq \Delta_1 \setminus p$.

From above Case (i) and Case (ii), and using induction, we conclude that

$$\Gamma_n^{\alpha, r} \simeq \Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus \{x \in V(\Delta) : |x_1| + |x_2| > \lceil \frac{r}{2} \rceil\}).$$

Let Δ_2 denote the induced complex $\text{Ind}_\Delta(V(\Delta) \setminus \{x \in V(\Delta) : |x_1| + |x_2| > \lceil \frac{r}{2} \rceil\})$. Let $l_1, l_2 \in [n]$ such that $(l_1, l_2) \neq (1, 2)$. Assume that Δ_2 is homotopy equivalent to the induced subcomplex of Δ_2 on the vertex set $V(\Delta_2) \setminus \{u : |u_s| + |u_t| > \lceil \frac{r}{2} \rceil\}$ for all ordered pairs $(s, t) \prec (l_1, l_2)$.

Using a similar argument as above, we get that

$$\Delta_2 \simeq \text{Ind}_{\Delta_2}(V(\Delta_2) \setminus \{u \in V(\Delta_2) : |u_s| + |u_t| > \lceil \frac{r}{2} \rceil \ \forall (s, t) \preceq (l_1, l_2)\}).$$

By induction, we conclude that

$$\Gamma_n^{\alpha, r} \simeq \Delta \simeq \Delta_2 \simeq \text{Ind}_{\Delta_2}(V(\Delta_2) \setminus \{u \in V(\Delta_2) : |u_s| + |u_t| > \lceil \frac{r}{2} \rceil \ \forall s, t \in [n]\}).$$

Hence, the result follows. $\square$

Lemma 3.6. *Let $r \geq 4$. Then $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex of $\Gamma_n^{\alpha, r}$ on $\{x \in \Gamma_n^{\alpha, r} : |x_i| \leq \lfloor \frac{r}{2} \rfloor, |x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil \text{ for all } i, j, k \in [n], j \neq k\}$.*

Proof. It follows from Lemma 3.5 that for $r \geq 4$, $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex of $\Gamma_n^{\alpha, r}$, say Δ , on the following set of vertices:

$$\{y \in V(\Gamma_n^{\alpha, r}) : |y_i| \leq \lfloor \frac{r}{2} \rfloor, |y_j| + |y_k| \leq \lceil \frac{r}{2} \rceil \text{ for all } i, j, k \in [n], j \neq k\}.$$

To complete the proof, it is sufficient to show that Δ is homotopy equivalent to the induced subcomplex of Δ on the vertex set

$$\{y \in V(\Delta) : |y_i| < \lfloor \frac{r}{2} \rfloor \text{ for all } i \in [n]\}.$$

We shall remove the vertices from Δ without changing the homotopy types in three steps. In step I, we remove the vertices of type $u \in \Delta$ such that $|u_i| = \lfloor \frac{r}{2} \rfloor$, and $\sum_{k \neq i} |u_k| \leq \lceil \frac{r}{2} \rceil - 2$ for some $i \in [n]$. In step II, we remove the vertices of type $v \in \Delta$ such that $|v_i| = \lfloor \frac{r}{2} \rfloor$, and $\sum_{k \neq i} |v_k| = \lceil \frac{r}{2} \rceil - 1$ for some $i \in [n]$. In the last step we remove the vertices of type $w \in \Delta$ such that $|w_i| = \lfloor \frac{r}{2} \rfloor$, and $\sum_{k \neq i} |w_k| = \lceil \frac{r}{2} \rceil$ for some $i \in [n]$.

Step I: Let x be a vertex in Δ such that $|x_s| = \lfloor \frac{r}{2} \rfloor$ and $\sum_{j \neq s} |x_j| \leq \lceil \frac{r}{2} \rceil - 2$. If $|x_k| > 1$ for some $k \neq s$, then $|x_s| + |x_k| > \lceil \frac{r}{2} \rceil$, which is a contradiction. Hence,

$|x_k| \leq 1$ for all $k \neq s$. Let us consider $\lambda^{[x;s]} \in V(\mathbb{Z}^n)$. Then, $d(x, \lambda^{[x;s]}) = 1$. Since $x \succ \mathbf{0}$ and $r \geq 4$, we conclude that $\lambda^{[x;s]} \succ \mathbf{0}$ and $\lambda^{[x;s]} \in V(\Delta)$.

We show that $N[x, \Delta] \subseteq N[\lambda^{[x;s]}, \Delta]$. For this, let $z \in N[x, \Delta]$, $z \neq x$. If $|z_s| < \lfloor \frac{r}{2} \rfloor$, or $|z_s| \geq \lfloor \frac{r}{2} \rfloor$ and $\text{sgn}(z_s x_s) = -1$, then by Lemma 3.2, $d(z, \lambda^{[x;s]}) \leq r-1$. So, assume that $|z_s| \geq \lfloor \frac{r}{2} \rfloor$ and $\text{sgn}(z_s x_s) = 1$. Now, $|z_s| \leq \lfloor \frac{r}{2} \rfloor$ implies that $|z_s| = \lfloor \frac{r}{2} \rfloor$. Then $z_s = x_s$, $|z_k| \leq 1$ for all $k \neq s$, and $\sum_{k \neq s} |z_k| \leq \lceil \frac{r}{2} \rceil$. Therefore,

$$\begin{aligned} d(z, \lambda^{[x;s]}) &= |z_s - \lambda_s^{[x;s]}| + \sum_{k \neq s} |z_k - \lambda_k^{[x;s]}| \\ &= 1 + \sum_{k \neq s} |z_k - x_k| \leq 1 + \sum_{k \neq s} |z_k| + \sum_{k \neq s} |x_k| \leq 1 + \lceil \frac{r}{2} \rceil + (\lceil \frac{r}{2} \rceil - 2) \leq r. \end{aligned}$$

Thus, $z \in N[\lambda^{[x;s]}, \Delta]$. Hence $N[x, \Delta] \subseteq N[\lambda^{[x;s]}, \Delta]$ and therefore $\Delta \simeq \Delta \setminus x$.

Let $\mathcal{A} = \{a \in \Delta : |a_i| = \lfloor \frac{r}{2} \rfloor \text{ and } \sum_{k \neq i} |a_k| \leq \lceil \frac{r}{2} \rceil - 2 \text{ for some } i \in [n]\}$. By repeating the above process for each $a \in \mathcal{A}$, we find that $\Delta \simeq \Delta_1$, where $\Delta_1 = \text{Ind}_\Delta(V(\Delta) \setminus \mathcal{A})$.

Step II: Let $p \in V(\Delta_1)$ such that $|p_s| = \lfloor \frac{r}{2} \rfloor$ and $\sum_{j \neq s} |p_j| = \lceil \frac{r}{2} \rceil - 1$. Since $r \geq 4$, there exists $j \neq s$ such that $|p_j| \neq 0$. If r is even, then $|p_s| + |p_j| > \lceil \frac{r}{2} \rceil = \frac{r}{2}$, which is a contradiction as $p \in \Delta$. Therefore, r must be odd and $r \geq 5$. Since $|p_s| + |p_k| \leq \lceil \frac{r}{2} \rceil$ for all $k \neq s$, it follows that $|p_k| \leq 1$ for all $k \neq s$. Since $r \geq 5$, there exists $j, l \in [n] \setminus \{s\}$ such that $j < l$ and $|p_j| = |p_l| = 1$.

Let us take $\lambda^{[p;s;j]} \in V(\mathbb{Z}^n)$. Clearly, $d(p, \lambda^{[p;s;j]}) = 2$. Let p_{i_0} be the last non-zero entry of p . Since $p \succ \mathbf{0}$, we must have $p_{i_0} > 0$. Since $|p_l| = 1$ and $l > j$, we have $i_0 > j$. Now, if $i_0 = s$, then, $\lambda_s^{[p;s;j]} > 0$ and $\lambda_s^{[p;s;j]}$ is the last non-zero entry of $\lambda^{[p;s;j]}$. If $i_0 \neq s$, then $p_{i_0} = \lambda_{i_0}^{[p;s;j]} > 0$. Therefore, we conclude that $\lambda^{[p;s;j]} \succ \mathbf{0}$. Since $|\lambda_s^{[p;s;j]}| \leq \lfloor \frac{r}{2} \rfloor - 1$, and $\sum_{k \neq s} |\lambda_k^{[p;s;j]}| < \lceil \frac{r}{2} \rceil - 1$, it follows that $\lambda^{[p;s;j]} \in V(\Delta_1)$. We first show that $N[p, \Delta_1] \subseteq N[\lambda^{[p;s;j]}, \Delta_1]$.

Let $q \in N[p, \Delta_1]$, $q \neq p$. If $|q_s| < |p_s|$ or $|q_j| < |p_j|$, then by Lemma 3.4, $d(q, \lambda^{[p;s;j]}) \leq r$. If $|q_s| \geq |p_s|$ and $\text{sgn}(q_s p_s) = -1$, or $|q_j| \geq |p_j|$ and $\text{sgn}(q_j p_j) = -1$, then again by Lemma 3.4, $d(q, \lambda^{[p;s;j]}) \leq r$.

Now, consider the case when $|q_t| \geq |p_t|$ and $\text{sgn}(q_t p_t) = 1$ for $t \in \{s, j\}$. Since $|q_s| + |q_j| \leq \lceil \frac{r}{2} \rceil = |p_s| + |p_j|$, we get $|q_s| = |p_s|$ and $|q_j| = |p_j|$. This implies that $q_s = p_s$ and $q_j = p_j$. Since $\sum_{k \neq s, j} |q_k| \leq \lfloor \frac{r}{2} \rfloor$, we get

$$\begin{aligned} d(q, \lambda^{[p;s;j]}) &= |q_s - \lambda_s^{[p;s;j]}| + |q_j - \lambda_j^{[p;s;j]}| + \sum_{k \neq s, j} |q_k - \lambda_k^{[p;s;j]}| \\ &= 1 + 1 + \sum_{k \neq s, j} |q_k| + \sum_{k \neq s, j} |\lambda_k^{[p;s;j]}| \leq 2 + \lfloor \frac{r}{2} \rfloor + (\lceil \frac{r}{2} \rceil - 2) \leq r. \end{aligned}$$

Hence, $q \in N[\lambda^{[p;s;j]}, \Delta_1]$. Thus $N[p, \Delta_1] \subseteq N[\lambda^{[p;s;j]}, \Delta_1]$. From Proposition 2.3, $\Delta_1 \simeq \Delta_1 \setminus p$. Let $B = \{a \in V(\Delta_1) : |a_s| = \lfloor \frac{r}{2} \rfloor \text{ and } \sum_{k \neq s} |a_k| = \lceil \frac{r}{2} \rceil - 1 \text{ for some } s \in [n]\}$. By repeating the same argument for every vertex $b \in B$, we get that $\Delta_1 \simeq \Delta_2$, where $\Delta_2 = \text{Ind}_{\Delta_1}(V(\Delta_1) \setminus B)$.

Step III: Let $w \in V(\Delta_2)$ with $|w_l| = \lfloor \frac{r}{2} \rfloor$ and $\sum_{k \neq l} |w_k| = \lceil \frac{r}{2} \rceil$. If r is even, then $|p_s| + |p_j| > \lceil \frac{r}{2} \rceil = \frac{r}{2}$ for some $j \neq s$, a contradiction as $p \in \Delta$. Thus, r must be odd and $r \geq 5$.

For each $k \neq l$, $|w_l| + |w_k| \leq \lceil \frac{r}{2} \rceil$ implies that $|w_k| \leq 1$. Let us consider $\lambda^{[w;l]} \in V(\mathbb{Z}^n)$. Let w_{j_0} be the last non-zero entry of w . Since $w \succ \mathbf{0}$, we have

$w_{j_0} > 0$. If $j_0 = l$, then clearly $\lambda_{j_0}^{[w;l]} > 0$ and $\lambda_{j_0}^{[w;l]}$ is the last non-zero entry. If $j_0 \neq l$, then $w_{j_0} = \lambda_{j_0}^{[w;l]}$. This implies that $\lambda_{j_0}^{[w;l]} > 0$ and $\lambda_{j_0}^{[w;l]}$ is the last non-zero entry. Thus, we conclude that $\lambda^{[w;l]} \succ \mathbf{0}$. Since $|\lambda_l^{[w;l]}| \leq \lfloor \frac{r}{2} \rfloor - 1$ and $\sum_{l \neq j} |\lambda_j^{[w;l]}| = \lceil \frac{r}{2} \rceil$, and $|\lambda_k^{[w;l]}| \leq 1$ for all $k \neq l$, we find that $\lambda^{[w;l]} \in V(\Delta_2)$. We claim that $N[w, \Delta_2] \subseteq N[\lambda^{[w;l]}, \Delta_2]$.

Let $v \in N[w, \Delta_2]$ with $v \neq w$. Note that if $|v_l| < \lfloor \frac{r}{2} \rfloor$, then $d(v, \lambda^{[w;l]}) \leq r - 1$, and hence $v \in N[\lambda^{[w;l]}, \Delta_2]$. So assume that $|v_l| = \lfloor \frac{r}{2} \rfloor$. Now, if $\text{sgn}(v_l w_l) = -1$, then $d(v, \lambda^{[w;l]}) \leq r$ and we are done.

Suppose $\text{sgn}(v_l w_l) = 1$. Then $v_l = w_l$. If $\sum_{j \neq l} |v_j| \leq \lceil \frac{r}{2} \rceil - 1$, then $v \notin \Delta_2$, so we must have $\sum_{k \neq l} |v_k| = \lceil \frac{r}{2} \rceil$.

Let $S = \{j \in [n] \setminus \{l\} : w_j, v_j \neq 0 \text{ and } \text{sgn}(v_j w_j) = -1\}$. Since for each $j \neq l$, $|v_j| \leq 1$, we have $\sum_{j \in S} |v_j - w_j| = 2 \cdot \text{Card}(S)$, and for $j \notin S \cup \{l\}$, either $w_j = 0$ or $v_j = 0$. Therefore,

$$\sum_{j \notin S \cup \{l\}} |w_j| = \sum_{j \notin S \cup \{l\}} |v_j| = \lceil \frac{r}{2} \rceil - \text{Card}(S).$$

Hence,

$$\begin{aligned} d(v, w) &= |v_l - w_l| + \sum_{j \in S} |v_j - w_j| + \sum_{j \notin S \cup \{l\}} |w_j - v_j| \\ &= 0 + 2 \cdot \text{Card}(S) + 2(\lceil \frac{r}{2} \rceil - \text{Card}(S)) = 2\lceil \frac{r}{2} \rceil = r + 1, \end{aligned}$$

which contradicts $v \in N[w, \Delta_2]$. Therefore, $\text{sgn}(v_l w_l)$ cannot be 1. Hence, we conclude that $N[w, \Delta_2] \subseteq N[\lambda^{[w;l]}, \Delta_2]$.

From Proposition 2.3, we have $\Delta_2 \simeq \Delta_2 \setminus w$. Let $C = \{c \in V(\Delta_2) : |c_i| = \lfloor \frac{r}{2} \rfloor \text{ and } \sum_{j \neq i} |c_j| = \lceil \frac{r}{2} \rceil \text{ for some } i \in [n]\}$. By repeating the same above arguments for each $c \in C$, we find that $\Delta_2 \simeq \text{Ind}_{\Delta_2}(V(\Delta_2) \setminus C)$.

Therefore, for $r \geq 4$, $\Gamma_n^{\alpha, r} \simeq \Delta \simeq \Delta_1 \simeq \Delta_2 \simeq \text{Ind}_{\Delta_2}(V(\Delta_2) \setminus C)$, and this completes the proof. $\square$

Lemma 3.7. *Let $r \geq 2$, and let Δ be a subcomplex of $\Gamma_n^{\alpha, r}$ that contains the vertex x . Then for any two vertices y and z in Δ , where $z \in V(\text{lk}(x, \Delta))$ and $|y_l| = |x_l| - 1$ for $l \in \{i, j, k\}$ and $y_l = x_l$ for $l \in [n] \setminus \{i, j, k\}$, we have the following:*

- (i) *If $|z_l| < |x_l|$ for at least two choices of $l \in \{i, j, k\}$, then $d(z, y) \leq r - 1$.*
- (ii) *Let $|z_k| < |x_k|$. If $|z_i| \geq |x_i|$ and $\text{sgn}(z_i x_i) = -1$, or $|z_j| \geq |x_j|$ and $\text{sgn}(z_j x_j) = -1$, then $d(z, y) \leq r$.*
- (iii) *If $|z_l| \geq |x_l|$ for all $l \in \{i, j, k\}$ and $\text{sgn}(z_s x_s) = -1$ for at least two choices of $s \in \{i, j, k\}$, then $d(z, y) \leq r$.*

Proof. (i) Without loss of generality, we assume that $|z_i| < |x_i|$ and $|z_j| < |x_j|$. Then, $|z_i - y_i| = |z_i - x_i| - 1$ and $|z_j - y_j| = |z_j - x_j| - 1$. Since $|y_k| = |x_k| - 1$, it follows that $|z_k - y_k| \leq |z_k - x_k| + 1$. Therefore, $d(z, y) = |z_i - y_i| + |z_j - y_j| + |z_k - y_k| + \sum_{l \neq i, j, k} |z_l - y_l|$, which gives $d(z, y) \leq (|z_i - x_i| - 1) + (|z_j - x_j| - 1) + (|z_k - x_k| + 1) + \sum_{l \neq i, j, k} |z_l - x_l| \leq d(z, x) - 1 \leq r - 1$.

(ii) Let $|z_k| < |x_k|$. Then $|z_k - y_k| = |z_k - x_k| - 1$. If $|z_i| \geq |x_i|$ and $\text{sgn}(z_i x_i) = -1$, then $|z_i - y_i| = |z_i - x_i| - 1$. Since $|y_j| = |x_j| - 1$, $|z_k - y_k| \leq |z_k - x_k| + 1$. Hence, a similar calculation as above shows that $d(z, y) \leq r$. Similarly, the result follows when $|z_j| \geq |x_j|$ and $\text{sgn}(z_j x_j) = -1$.

- (iii) It is given that $|z_l| \geq |x_l|$ for all $l \in \{i, j, k\}$. Now, assume that $\text{sgn}(z_i x_i) = \text{sgn}(z_j x_j) = -1$. Then $|z_i - y_i| = |z_i - x_i| - 1$, and $|z_j - y_j| = |z_j - x_j| - 1$. Since $|y_k| = |x_k| - 1$, regardless of whether $|z_k| < |x_k|$ or $|z_k| \geq |x_k|$, we find that $|z_k - y_k| \leq |z_k - x_k| + 1$. Now, a similar computation as in part (i) proves the result. $\square$

Lemma 3.8. *Let $r \geq n \geq 4$. Then $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex Δ' of $\Gamma_n^{\alpha, r}$, where every $x \in V(\Delta')$ satisfies the following: (i) $|x_i| < \lfloor \frac{r}{2} \rfloor$, $|x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil$ for all $i, j, k \in [n]$, $j \neq k$, and (ii) $|x_i| + |x_j| + |x_k| < r - 1$ for all distinct $i, j, k \in [n]$.*

Proof. From Lemma 3.6, $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex of $\Gamma_n^{\alpha, r}$, say, Δ , on the vertex set $\{x \in \Gamma_n^{\alpha, r} : |x_i| < \lfloor \frac{r}{2} \rfloor, |x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil \text{ for all } i, j, k \in [n], j \neq k\}$. Let $A = \{x \in \Delta : |x_i| + |x_j| + |x_k| \geq r - 1 \text{ for some distinct } i, j, k\}$. We show that $\Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus A)$.

Let $z \in A$. Then there exist distinct i, j, k such that $|z_i| + |z_j| + |z_k| \geq r - 1$. Without loss of generality, we assume that $i < j < k$. Since $z \in \Delta$, we have $|z_i| + |z_j| \leq \lceil \frac{r}{2} \rceil$ and $|z_k| < \lfloor \frac{r}{2} \rfloor$, which implies that $|z_i| + |z_j| + |z_k| \neq r$. Hence $|z_i| + |z_j| + |z_k| = r - 1$, and $\sum_{l \neq i, j, k} |z_l| \leq 1$.

If $|z_i| + |z_j| < \lceil \frac{r}{2} \rceil$, then $|z_k| < \lfloor \frac{r}{2} \rfloor$ implies that $|z_i| + |z_j| + |z_k| < r - 1$, which is a contradiction. Hence $|z_i| + |z_j| = \lceil \frac{r}{2} \rceil$ and $|z_k| = \lfloor \frac{r}{2} \rfloor - 1$. Similarly, we deduce that $|z_j| + |z_k| = \lceil \frac{r}{2} \rceil$, $|z_i| + |z_k| = \lceil \frac{r}{2} \rceil$, and $|z_i| = |z_j| = \lfloor \frac{r}{2} \rfloor - 1$. This implies that $3\lfloor \frac{r}{2} \rfloor - 3 = r - 1$ and hence $r \in \{4, 7\}$.

Case (i): $r = 4$.

Here, $|z_i| = |z_j| = |z_k| = 1$, and since $r \geq n \geq 4$, we have $n = 4$. Since $i < j < k$ and $z \succ \mathbf{0}$, it follows that $z_k = 1$, and $3 \leq k \leq 4$. Let Δ contains a vertex a such that $a_4 > 0$. In this case, let $\lambda \in V(\mathbb{Z}^n)$ be defined by $\lambda_4 = 1$ and $\lambda_l = 0$ for all $l \neq 4$. Then $\lambda \in \Delta$. Let $y \in N[z, \Delta]$. Then from the fact that $|y_l| < 2$ for any $l \in [4]$, we have $|y_l| \leq 1$ for all $l \in [4]$.

Therefore, $d(y, \lambda) \leq 4$. Hence, $y \in N[\lambda, \Delta]$. Thus, $N[z, \Delta] \subseteq N[\lambda, \Delta]$. From Proposition 2.3, $\Delta \simeq \Delta \setminus z$. By repeating the similar argument as above for each element of A , we conclude that $\Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus A)$.

Assume that there is no element $b \in \Delta$ such that $b_4 > 0$. This implies that $k = 3$, $|z_1| = |z_2| = 1$, $z_3 = 1$, and $z_4 = 0$.

Consider a vertex $\lambda' \in V(\mathbb{Z}^n)$ such that $\lambda'_3 = 1$ and $\lambda'_l = 0$ for $l \neq 3$. Then $\lambda' \in \Delta$, and for any vertex $u \in N[z, \Delta]$, $d(u, \lambda') \leq 4$. Thus, $u \in N[\lambda', \Delta]$. Hence $N[z, \Delta] \subseteq N[\lambda', \Delta]$. From Proposition 2.3, $\Delta \simeq \Delta \setminus z$. By repeating the same process for each $a \in A$, we get that $\Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus A)$.

Case (ii): $r = 7$.

Here, $|z_i| = |z_j| = |z_k| = 2$. Consider the element $\lambda^{[z; ijk]} \in V(\mathbb{Z}^n)$. Clearly, $d(z, \lambda^{[z; ijk]}) = 3$ and $|z_l - \lambda_l^{[z; ijk]}| = 1$ for $l \in \{i, j, k\}$. Since $z \succ \mathbf{0}$, there exists $s \geq k$ such that $z_s > 0$ and $z_t = 0$ for all $t > s$. Since $|z_k| = 2$, $|\lambda_k^{[z; ijk]}| = |z_k| - 1$, and $\lambda_l^{[z; ijk]} = z_l$ for all $l > k$, we see that $\lambda_s^{[z; ijk]} > 0$ and $\lambda_t^{[z; ijk]} = 0$ for all $t > s$. Thus, $\lambda^{[z; ijk]} \succ \mathbf{0}$. Since $|\lambda_l^{[z; ijk]}| = |z_l| - 1$ for all $l \in \{i, j, k\}$, and $\sum_{l \neq i, j, k} |\lambda_l^{[z; ijk]}| \leq 1$, we see that $|\lambda_{i_0}^{[z; ijk]}| + |\lambda_{j_0}^{[z; ijk]}| + |\lambda_{k_0}^{[z; ijk]}| \leq 3$ for any $i_0, j_0, k_0 \in [n]$. Hence $\lambda^{[z; ijk]} \in \Delta$.

Claim 3.9. $N[z, \Delta] \subseteq N[\lambda^{[z; ijk]}, \Delta]$.

Proof of Claim 3.9: Clearly, $z \in N[\lambda^{[z;ijk]}, \Delta]$. Let $w \in N[z, \Delta]$ with $z \neq w$. On the contrary, suppose that $w \notin N[\lambda^{[z;ijk]}, \Delta]$. Then $d(w, \lambda^{[z;ijk]}) > 7$. From Lemma 3.7 (i), $|w_l| < |z_l|$ for at most one $l \in \{i, j, k\}$.

We first consider the case that there exists $l \in \{i, j, k\}$ such that $|w_l| < |z_l|$. If $|w_i| < |z_i|$, then from Lemma 3.7 (ii), we have $|w_j| \geq |z_j|$, $\text{sgn}(w_j z_j) = 1$, $|w_k| \geq |z_k|$, and $\text{sgn}(w_k z_k) = 1$. Since $|w_j| + |w_k| \leq 4$ and $|z_j| + |z_k| = 4$, we get that $w_j = z_j$ and $w_k = z_k$. Further, $d(\mathbf{0}, w) \leq 7$ implies that $\sum_{l \neq j, k} |w_l| \leq 3$. Therefore,

$$\begin{aligned} d(w, \lambda^{[z;ijk]}) &= |w_i - \lambda_i^{[z;ijk]}| + |w_j - \lambda_j^{[z;ijk]}| + |w_k - \lambda_k^{[z;ijk]}| + \sum_{l \neq i, j, k} |w_l - \lambda_l^{[z;ijk]}| \\ &= |w_i - z_i| - 1 + |w_j - z_j| + 1 + |w_k - z_k| + 1 + \sum_{l \neq i, j, k} |w_l - z_l| \\ &\leq |z_i| + 1 + \sum_{l \neq j, k} |w_l| + \sum_{l \neq i, j, k} |z_l| \leq 2 + 1 + 3 + 1 = 7. \end{aligned}$$

This implies that $d(w, \lambda^{[z;ijk]}) \leq 7$, which is a contradiction to our assumption. Similarly, if $|w_j| < |z_j|$ or $|w_k| < |z_k|$, then we arrive at a contradiction.

Now, assume that $|w_l| \geq |z_l|$ for all $l \in \{i, j, k\}$. Since $|z_i| = |z_j| = |z_k| = 2$ and $w \in \Delta$, we have $|w_l| = 2$ for $l \in \{i, j, k\}$. Consequently, it follows that $\text{sgn}(w_l z_l) = -1$ for at most one $m \in \{i, j, k\}$; otherwise, a similar computation as in part (iii) of Lemma 3.7 yields that $d(w, \lambda^{[z;ijk]}) \leq 7$, which is a contradiction.

Without loss of generality, we assume that $\text{sgn}(w_i z_i) = -1$, and $w_j = z_j$, $w_k = z_k$. Therefore, we have $|w_i - \lambda_i^{[z;ijk]}| = |w_i - z_i| - 1$, $|w_j - \lambda_j^{[z;ijk]}| = |w_j - z_j| + 1 = 1$, $|w_k - \lambda_k^{[z;ijk]}| = 1$, and $\sum_{l \neq i, j, k} |w_l| \leq 1$, implying that $d(w, \lambda^{[z;ijk]}) \leq 7$, which is a contradiction.

Thus, we left with the only case that $\text{sgn}(w_l z_l) = 1$, for all $l \in \{i, j, k\}$. This implies that $w_l = z_l$ for all $l \in \{i, j, k\}$. Then, clearly, $\sum_{l \neq i, j, k} |w_l| \leq 1$, and $\sum_{l \neq i, j, k} |z_l| \leq 1$ implies that $d(w, \lambda^{[z;ijk]}) \leq 5$. Hence, $d(w, \lambda^{[z;ijk]}) < 7$ and thus $w \in N[\lambda^{[z;ijk]}, \Delta]$. This is a contradiction.

Hence, $w \in N[\lambda^{[z;ijk]}, \Delta]$. Thus, $N[z, \Delta] \subseteq N[\lambda^{[z;ijk]}, \Delta]$. This completes the proof of Claim 3.9.

Using Claim 3.9 and Proposition 2.3, we get $\Delta \simeq \Delta \setminus z$. By using the similar argument for each $a \in A$, we get that $\Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus A)$. $\square$

Lemma 3.10. *Let $r \geq 4$, and let Δ be a subcomplex of $\Gamma_n^{\alpha, r}$ that contains the vertex x . Then for any two $y, z \in V(\Delta)$, where $z \in V(\text{lk}(x, \Delta))$ and $|y_t| = |x_t| - 1$ for $t \in \{i, j, k, l\}$ and $y_s = x_s$ for $s \in [n] \setminus \{i, j, k, l\}$, we have the following:*

- (i) *If $|z_s| < |x_s|$ for at least two choices of $s \in \{i, j, k, l\}$, then $d(z, y) \leq r$.*
- (ii) *Let $|z_s| < |x_s|$ for some $s \in \{i, j, k, l\}$. If there exists $t \in \{i, j, k, l\} \setminus \{s\}$ such that $|z_t| \geq |x_t|$ and $\text{sgn}(z_t x_t) = -1$, then $d(z, y) \leq r$.*
- (iii) *If $|z_t| \geq |x_t|$ for all $t \in \{i, j, k, l\}$, and $\text{sgn}(z_s x_s) = -1$ for at least two indices $s \in \{i, j, k, l\}$, then $d(z, y) \leq r$.*

Proof. (i) Without loss of generality, assume that $|z_i| < |x_i|$ and $|z_j| < |x_j|$. Thus, $|z_i - y_i| = |z_i - x_i| - 1$ and $|z_j - y_j| = |z_j - x_j| - 1$. Since $|y_t| = |x_t| - 1$ for $t \in \{i, j, k, l\}$, and $d(y, x) = 4$, we see that $|z_k - y_k| \leq |z_k - x_k| + 1$, and $|z_l - y_l| \leq |z_l - x_l| + 1$. Therefore, $d(z, y) \leq (|z_i - x_i| - 1) + (|z_j - x_j| -$

- 1) + $(|z_k - x_k| + 1) + (|z_l - x_l| + 1) + \sum_{t \neq i, j, k, l} |z_t - x_t| \leq d(z, x) = r$. This proves (i).
- (ii) Without loss of generality, assume that $|z_k| < |x_k|$, and $|z_i| \geq |x_i|$ and $\text{sgn}(z_i x_i) = -1$. Then, $|z_k - y_k| = |z_k - x_k| - 1$ and $|z_i - y_i| = |z_i - x_i| - 1$. Moreover, $|z_j - y_j| \leq |z_j - x_j| + 1$ and $|z_l - y_l| \leq |z_l - x_l| + 1$. Therefore, $d(z, y) \leq (|z_i - x_i| - 1) + (|z_j - x_j| + 1) + (|z_k - x_k| - 1) + (|z_l - x_l| + 1) + \sum_{t \neq i, j, k, l} |z_t - x_t| \leq d(z, x) = r$. This proves (ii).
- (iii) Let $|z_t| \geq |x_t|$ for all $t \in \{i, j, k, l\}$. Without loss of generality, we assume that $\text{sgn}(z_i x_i) = \text{sgn}(z_j x_j) = -1$. This implies that $|z_i - y_i| = |z_i - x_i| - 1$ and $|z_j - y_j| = |z_j - x_j| - 1$. Also, we have $|z_k - y_k| \leq |z_k - x_k| + 1$, and $|z_l - y_l| \leq |z_l - x_l| + 1$. Now, similar computations as above show that $d(z, y) \leq r$. This proves (iii). $\square$

Lemma 3.11. *Let $r \geq n \geq 4$. Then $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex Δ' of $\Gamma_n^{\alpha, r}$, where every $x \in V(\Delta')$ satisfies the following: (i) $|x_i| < \lfloor \frac{r}{2} \rfloor$, $|x_j| + |x_k| \leq \lfloor \frac{r}{2} \rfloor$ for all $i, j, k \in [n]$, $j \neq k$ (ii) $|x_i| + |x_j| + |x_k| < r - 1$ for all $\{i, j, k\} \subseteq [n]$, and (iii) $|x_i| + |x_j| + |x_k| + |x_l| \leq r - 1$ for all $\{i, j, k, l\} \subseteq [n]$.*

Proof. From Lemma 3.8, $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex Δ , where every $x \in V(\Delta)$ satisfies the following: (i) $|x_i| < \lfloor \frac{r}{2} \rfloor$, $|x_j| + |x_k| \leq \lfloor \frac{r}{2} \rfloor$ for all $i, j, k \in [n]$, $j \neq k$ and (ii) $|x_i| + |x_j| + |x_k| < r - 1$ for all $\{i, j, k\} \subseteq [n]$.

Let $x \in V(\Delta)$ be such that $|x_i| + |x_j| + |x_k| + |x_l| = r$ for some $i < j < k < l$. For any $t \in \{i, j, k, l\}$, if $|x_t| \leq 1$, then $\sum_{s \in \{i, j, k, l\} \setminus \{t\}} |x_s| \geq r - 1$, a contradiction. Hence $|x_t| \geq 2$ for all $t \in \{i, j, k, l\}$ and $x_s = 0$ for $s \notin \{i, j, k, l\}$. Hence $r \geq 8$.

Now, let us consider $\lambda^{[x; ijk l]} \in V(\mathbb{Z}^n)$. We see that $d(x, \lambda^{[x; ijk l]}) = 4$, and $|x_t - \lambda_t^{[x; ijk l]}| = 1$ for all $t \in \{i, j, k, l\}$. Since $x \succ \mathbf{0}$, $i < j < k < l$, and $x_t = 0$ for $t > l$, we get that $x_l > 0$. Now $x_l \geq 2$ implies that $\mathbf{0} \prec \lambda^{[x; ijk l]} \prec x$. Since $|\lambda_t^{[x; ijk l]}| < |x_t|$ for all $t \in \{i, j, k, l\}$, and $\lambda_t^{[x; ijk l]} = x_t$ for all $t \notin \{i, j, k, l\}$ it follows that $\lambda^{[x; ijk l]} \in \Delta$.

We show that $N[x, \Delta] \subseteq N[\lambda^{[x; ijk l]}, \Delta]$. Clearly, $x \in N[\lambda^{[x; ijk l]}, \Delta]$. Let $u \in N[x, \Delta]$, $u \neq x$. Suppose $u \notin N[\lambda^{[x; ijk l]}, \Delta]$, i.e., $d(u, \lambda^{[x; ijk l]}) > r$. First, assume that $|u_t| \geq |x_t|$ for all $t \in \{i, j, k, l\}$. Then from the fact $|x_i| + |x_j| + |x_k| + |x_l| = r$, we have $|u_t| = |x_t|$ for all $t \in \{i, j, k, l\}$ and $u_s = 0$ for $s \notin \{i, j, k, l\}$. Here, from Lemma 3.10 (iii), $\text{sgn}(u_t x_t) = -1$ for at most one $t \in \{i, j, k, l\}$. Now, $|u_t| = |x_t|$ for all $t \in \{i, j, k, l\}$ and $u \neq x$ implies that there exists exactly one $s \in \{i, j, k, l\}$ such that $\text{sgn}(u_s x_s) = -1$. Then $|u_s - \lambda_s^{[x; ijk l]}| = |u_s - x_s| - 1$ and $|u_t - \lambda_t^{[x; ijk l]}| = 1$ for $t \in \{i, j, k, l\} \setminus \{s\}$. Therefore, $d(u, \lambda^{[x; ijk l]}) = |u_i - \lambda_i^{[x; ijk l]}| + |u_j - \lambda_j^{[x; ijk l]}| + |u_k - \lambda_k^{[x; ijk l]}| + |u_l - \lambda_l^{[x; ijk l]}| = |u_s - x_s| - 1 + 1 + 1 + 1 = 2|x_s| + 2$. Since for any $t \in \{i, j, k, l\} \setminus \{s\}$, $|x_s| + |x_t| \leq \lfloor \frac{r}{2} \rfloor$ and $|x_t| \geq 2$, we get $|x_s| \leq \lfloor \frac{r}{2} \rfloor - 2$. Hence, $d(u, \lambda^{[x; ijk l]}) \leq 2(\lfloor \frac{r}{2} \rfloor - 2) + 2 = 2\lfloor \frac{r}{2} \rfloor - 2 < r$, which is a contradiction.

Now, we assume that there exists $p \in \{i, j, k, l\}$ such that $|u_p| < |x_p|$. For any $t \in \{i, j, k, l\} \setminus \{p\}$, if $|u_t| < |x_t|$, then by Lemma 3.10 (i), $d(u, \lambda^{[x; ijk l]}) \leq r$, a contradiction. Hence $|u_t| \geq |x_t|$ for $t \in \{i, j, k, l\} \setminus \{p\}$.

From Lemma 3.10 (ii), $\text{sgn}(u_t x_t) = 1$ for $t \in \{i, j, k, l\} \setminus \{p\}$. Write $|u_t| = |x_t| + a_t$ for some $a_t \geq 0$ and $t \in \{i, j, k, l\} \setminus \{p\}$. Then $\sum_{t \in \{i, j, k, l\} \setminus \{p\}} |u_t| = \sum_{t \in \{i, j, k, l\} \setminus \{p\}} (|x_t| + a_t)$. Hence $\sum_{t \notin \{i, j, k, l\} \setminus \{p\}} |u_t| \leq r - (\sum_{t \in \{i, j, k, l\} \setminus \{p\}} (|x_t| + a_t))$.

Since $|u_p| < |x_p|$, we have $|u_p - \lambda_p^{[x;ijkl]}| = |u_p - x_p| - 1$. Therefore,

$$\begin{aligned}
d(u, \lambda^{[x;ijkl]}) &= \sum_{t \in \{i,j,k,l\}} |u_t - \lambda_t^{[x;ijkl]}| + \sum_{t \notin \{i,j,k,l\}} |u_t| \\
&\leq |u_p - x_p| - 1 + \sum_{t \in \{i,j,k,l\} \setminus \{p\}} (|u_t - x_t| + 1) + \sum_{t \notin \{i,j,k,l\}} |u_t| \\
&\leq |u_p| + |x_p| + 2 + \sum_{t \in \{i,j,k,l\} \setminus \{p\}} a_t + \sum_{t \notin \{i,j,k,l\}} |u_t| \\
&\leq |x_p| + 2 + \sum_{t \in \{i,j,k,l\} \setminus \{p\}} a_t + \sum_{t \notin \{i,j,k,l\} \setminus \{p\}} |u_t| \\
&\leq |x_p| + 2 + \sum_{t \in \{i,j,k,l\} \setminus \{p\}} a_t + r - (\sum_{t \in \{i,j,k,l\} \setminus \{p\}} (|x_t| + a_t)) \\
&= r + 2 + |x_p| - \sum_{t \in \{i,j,k,l\} \setminus \{p\}} |x_t|.
\end{aligned}$$

Let $\{t_1, t_2, t_3\} = \{i, j, k, l\} \setminus \{p\}$. If $|x_{t_1}| + |x_{t_2}| < \lfloor \frac{r}{2} \rfloor$, then $|x_p| + |x_{t_3}| \leq \lceil \frac{r}{2} \rceil$, implies $|x_i| + |x_j| + |x_k| + |x_l| < r$, which is a contradiction. Hence, $|x_{t_1}| + |x_{t_2}| \geq \lfloor \frac{r}{2} \rfloor$. Also, since $|x_p| \leq \lceil \frac{r}{2} \rceil - 2$, we get: $d(u, \lambda^{[x;ijkl]}) \leq r + 2 + \lceil \frac{r}{2} \rceil - 2 - |x_{t_3}| - \lfloor \frac{r}{2} \rfloor$. Since $|x_{t_3}| \geq 2$, we conclude that $d(u, \lambda^{[x;ijkl]}) \leq r$, again contradicting our assumption that $d(u, \lambda^{[x;ijkl]}) > r$.

Thus, we conclude that $u \in N[\lambda^{[x;ijkl]}, \Delta]$. Since $x \in N[\lambda^{[x;ijkl]}, \Delta]$, $N[x, \Delta] \subseteq N[\lambda^{[x;ijkl]}, \Delta]$. From Proposition 2.3, $\Delta \simeq \Delta \setminus u$. Let $A = \{y \in \Delta : |y_i| + |y_j| + |y_k| + |y_l| = r \text{ for all } \{i, j, k, l\} \subseteq [n]\}$. By repeating the above process for each $y \in A$, we find that $\Delta \simeq \text{Ind}_\Delta(V(\Delta) \setminus A)$. This completes the proof. $\square$

Lemma 3.12. *Let $r \geq n \geq 5$ and $r \geq 10$. Then $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex Δ , where $x \in V(\Delta)$ satisfies the following: (i) $|x_i| < \lfloor \frac{r}{2} \rfloor$, $|x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil$ for all $i, j, k \in [n], j \neq k$ and (ii) there exists no $\{i, j, k, l\} \subseteq [n]$ such that $|x_i| + |x_j| + |x_k| + |x_l| \geq r - 1$ and $x_s = 0$ for all $s \notin \{i, j, k, l\}$.*

Proof. From Lemma 3.11, $\Gamma_n^{\alpha, r}$ is homotopy equivalent to the induced subcomplex Δ_1 of $\Gamma_n^{\alpha, r}$, where $x \in V(\Delta_1)$ satisfies the three conditions (i), (ii) and (iii), as given in Lemma 3.11.

Let $A = \{x \in V(\Delta_1) : \exists \{i, j, k, l\} \subseteq [n] \text{ such that } |x_i| + |x_j| + |x_k| + |x_l| = r - 1 \text{ and } x_s = 0 \text{ for all } s \notin \{i, j, k, l\}\}$. It is sufficient to show that $\Delta_1 \simeq \text{Ind}_{\Delta_1}(V(\Delta_1) \setminus \{A\})$.

Let $B = \{z \in V(\Delta_1) : \sum_{s \in \{i,j,k,l\}} |z_s| = r - 1, \text{ and } i < j < k < l, z_l \geq 2 \text{ for some } \{i, j, k, l\} \subseteq [n] \text{ and } z_s = 0 \text{ for } s \notin \{i, j, k, l\}\}$.

Let $C = \{z \in V(\Delta_1) \setminus B : \sum_{s \in \{i,j,k,l\}} |z_s| = r - 1 \text{ and } i < j < k < l, z_l = 1 \text{ for some } \{i, j, k, l\} \subseteq [n] \text{ and } z_s = 0 \text{ for } s \notin \{i, j, k, l\}\}$. Observe that $A = B \cup C$.

We shall remove all the elements of A from Δ_1 without changing the homotopy type of Δ_1 in two steps: in Step I, we shall remove the elements of B , and then in Step II, we shall remove the elements of C .

Step I: Let $x \in B$. Then, $|x_i| + |x_j| + |x_k| + |x_l| = r - 1$, where $\{i, j, k, l\} \subseteq [n]$, $x_l \geq 2$, $i < j < k < l$, and $x_s = 0$ for all $s \notin \{i, j, k, l\}$. If $|x_t| = 0$ for some $t \in \{i, j, k, l\}$, then $\sum_{s \in \{i,j,k,l\} \setminus \{t\}} |x_s| = r - 1$, which contradicts Lemma 3.8 (ii). Therefore, $|x_t| \geq 1$ for all $t \in \{i, j, k, l\}$.

Consider the vertex $\lambda^{[x;ijkl]} \in V(\mathbb{Z}^n)$. Clearly, $d(x, \lambda^{[x;ijkl]}) = 4$, and $|x_s - \lambda_s| = 1$ for all $s \in \{i, j, k, l\}$. Since $i < j < k < l$ and $\lambda_l^{[x;ijkl]} > 0$, and $\lambda_t^{[x;ijkl]} = 0$ for $t > l$, it follows that $\lambda^{[x;ijkl]} \succ \mathbf{0}$. On the other hand $|\lambda_s^{[x;ijkl]}| < |x_s|$ for all $s \in \{i, j, k, l\}$ implies that $\lambda^{[x;ijkl]} \in \Delta_1$.

Claim 3.13. $N[x, \Delta_1] \subseteq N[\lambda^{[x;ijkl]}, \Delta_1]$.

Proof of Claim 3.13: Since $r \geq 5$, $x \in N[\lambda^{[x;ijkl]}, \Delta_1]$. Let $y \in N[x, \Delta_1]$ with $x \neq y$. Suppose $y \notin N[\lambda^{[x;ijkl]}, \Delta_1]$. Then $d(y, \lambda^{[x;ijkl]}) > r$. From Lemma 3.10 (i), $|y_s| < |x_s|$ for at most one value of s in $\{i, j, k, l\}$, otherwise $d(y, \lambda^{[x;ijkl]}) \leq r$, which is a contradiction. We have the following cases:

Case 1.1: There exists $p \in \{i, j, k, l\}$ such that $|y_p| < |x_p|$.

Then $|y_s| \geq |x_s|$ for all $s \in \{i, j, k, l\} \setminus \{p\}$. If $\text{sgn}(y_s x_s) = -1$ for some $s \in \{i, j, k, l\} \setminus \{p\}$, then from Lemma 3.10 (ii), we get $d(y, \lambda^{[x;ijkl]}) \leq r$, which is a contradiction. Hence $\text{sgn}(y_s x_s) = 1$ for all $s \in \{i, j, k, l\} \setminus \{p\}$. Let $|y_s| = |x_s| + a_s$, where $a_s \geq 0$ for $s \in \{i, j, k, l\} \setminus \{p\}$. Since $\sum_s |y_s| \leq r$, we see that $\sum_{s \notin \{i, j, k, l\} \setminus \{p\}} |y_s| \leq r - (\sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s| + a_s)$. Therefore,

$$\begin{aligned} d(y, \lambda^{[x;ijkl]}) &= |y_p - \lambda_p^{[x;ijkl]}| + \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |y_s - \lambda_s^{[x;ijkl]}| + \sum_{s \notin \{i, j, k, l\}} |y_s - \lambda_s^{[x;ijkl]}| \\ &\leq |y_p - x_p| - 1 + \sum_{s \in \{i, j, k, l\} \setminus \{p\}} (|y_s - x_s| + 1) + \sum_{s \notin \{i, j, k, l\}} |y_s| \\ &\leq |x_p| + 2 + \sum_{s \in \{i, j, k, l\} \setminus \{p\}} a_s + \sum_{s \notin \{i, j, k, l\} \setminus \{p\}} |y_s| \\ &\leq |x_p| + 2 + \sum_{s \in \{i, j, k, l\} \setminus \{p\}} a_s + r - \sum_{s \in \{i, j, k, l\} \setminus \{p\}} (|x_s| + a_s) \\ &= r + 2 + |x_p| - \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s|. \end{aligned}$$

If $|x_p| - \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s| \leq -2$, then $d(y, \lambda^{[x;ijkl]}) \leq r$, thereby implying that $y \in N[\lambda^{[x;ijkl]}, \Delta_1]$. So, assume that $|x_p| - \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s| \geq -1$. On the other hand if $|x_p| - \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s| \geq 1$, then using the fact that $|x_p| + \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s| = r - 1$, we get that $|x_p| \geq \lfloor \frac{r}{2} \rfloor$, which is not possible. Thus, we have only two possibilities, either $|x_p| = \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s|$ or $|x_p| + 1 = \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s|$.

Now, if $|x_p| = \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s|$, then using equation $|x_p| + \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s| = r - 1$, we get that $|x_p| = \frac{r-1}{2}$. This implies that r must be odd and $|x_p| = \lfloor \frac{r}{2} \rfloor$, which is not possible. Therefore, we have $|x_p| + 1 = \sum_{s \in \{i, j, k, l\} \setminus \{p\}} |x_s|$. In this case, $|x_p| = \frac{r-2}{2}$, which implies that r is even. Since $r \geq 10$, there exists a $t \in \{i, j, k, l\} \setminus \{p\}$ such that $|x_t| \geq 2$. This implies that $|x_p| + |x_t| > \frac{r}{2}$. This is a contradiction.

Thus we conclude that $d(y, \lambda^{[x;ijkl]}) \leq r$, and therefore $y \in N[\lambda^{[x;ijkl]}, \Delta_1]$.

Case 1.2 $|y_s| \geq |x_s|$ for all $s \in \{i, j, k, l\}$.

From Lemma 3.10 (iii), $\text{sgn}(y_s x_s) = -1$ is possible for at most one value of s in $\{i, j, k, l\}$. If $|y_s| \geq |x_s|$, and $\text{sgn}(y_s x_s) = 1$ for all $s \in \{i, j, k, l\}$, then from $|y_i| + |y_j| + |y_k| + |y_l| \leq r - 1$, it follows that $x_s = y_s$ for all $s \in \{i, j, k, l\}$.

Consequently, $\sum_{s \notin \{i,j,k,l\}} |y_s| \leq 1$ implies that $d(y, \lambda^{[x;ijkl]}) \leq 5 < r$, which is a contradiction.

Thus, there exists a $q \in \{i, j, k, l\}$ such that $\text{sgn}(x_q y_q) = -1$. Then $\text{sgn}(x_s y_s) = 1$ for $s \in \{i, j, k, l\} \setminus \{q\}$. Since $|y_i| + |y_j| + |y_k| + |y_l| \leq r - 1$, it follows that $y_q = -x_q$ and $y_s = x_s$ for $s \in \{i, j, k, l\} \setminus \{q\}$. Then, we have $\sum_{s \notin \{i,j,k,l\}} |y_s| \leq 1$. Therefore,

$$\begin{aligned} d(y, \lambda^{[x;ijkl]}) &= |y_q - x_q| - 1 + \sum_{s \in \{i,j,k,l\} \setminus \{q\}} (|y_s - x_s| + 1) + \sum_{s \notin \{i,j,k,l\}} |y_s| \\ (3.1) \quad &\leq |y_q| + |x_q| + 3. \end{aligned}$$

If r is odd then $|x_q| = |y_q| \leq \lfloor \frac{r}{2} \rfloor - 1$. Thus, from Equation (3.1), $d(y, \lambda^{[x;ijkl]}) \leq \lfloor \frac{r}{2} \rfloor - 1 + \lfloor \frac{r}{2} \rfloor - 1 + 3 = r$, which is a contradiction. Therefore, r must be even. Further, if $|x_q| \leq \frac{r}{2} - 2$, then $d(y, \lambda^{[x;ijkl]}) \leq \frac{r}{2} - 1 + \frac{r}{2} - 2 + 3 = r$, which is a contradiction. From the bound $|x_q| < \frac{r}{2}$, we must have $|x_q| = \frac{r}{2} - 1$. Since $r \geq 10$, there exists a $t \in \{i, j, k, l\} \setminus \{q\}$ such that $|x_t| \geq 2$. This implies that $|x_q| + |x_t| > \frac{r}{2}$, which is a contradiction.

Thus we conclude that $d(y, \lambda^{[x;ijkl]}) \leq r$, and therefore $y \in N[\lambda^{[x;ijkl]}, \Delta_1]$. Hence, $N[x, \Delta_1] \subseteq N[\lambda^{[x;ijkl]}, \Delta_1]$. This proves Claim 3.13.

From Proposition 2.3, $\Delta_1 \simeq \text{Ind}_{\Delta_1}(V(\Delta_1) \setminus \{x\})$. Now, applying the same arguments for each $b \in B$, we find that $\Delta_1 \simeq \text{Ind}_{\Delta_1}(V(\Delta_1) \setminus B)$. Let $\Delta_2 = \text{Ind}_{\Delta_1}(V(\Delta_1) \setminus B)$. Then $\Delta_1 \simeq \Delta_2$.

Step II: Let $x \in C \subseteq \Delta_2$ be such that $|x_i| + |x_j| + |x_k| + |x_l| = r - 1$ where $\{i, j, k, l\} \subseteq [n]$, $i < j < k < l$, $x_l = 1$, and $x_s = 0$ for all $s \notin \{i, j, k, l\}$. If $|x_t| = 0$ for some $t \in \{i, j, k, l\}$, then $\sum_{s \in \{i,j,k,l\} \setminus \{t\}} |x_s| = r - 1$, which contradicts Lemma 3.11 (ii). Therefore, $|x_t| \geq 1$ for all $t \in \{i, j, k, l\}$.

Consider $\lambda^{[x;ijk]} \in V(\mathbb{Z}^n)$. Clearly, $d(x, \lambda^{[x;ijk]}) = 3$, and $|x_s - \lambda_s^{[x;ijk]}| = 1$ for $s \in \{i, j, k\}$. Since $x_l = 1$, and $x_s = 0$ for $s > l$, it follows that $\lambda_l^{[x;ijk]} = 1$ and $\lambda_s^{[x;ijk]} = 0$ for $s > l$. Thus, $\lambda \succ \mathbf{0}$. Moreover, since $|\lambda_s^{[x;ijk]}| < |x_s|$ for all $s \in \{i, j, k\}$, and $\lambda_s^{[x;ijk]} = p_s$ for $s \notin \{i, j, k\}$, we have $\lambda^{[x;ijk]} \in \Delta_2$.

Claim 3.14. $N[x, \Delta_2] \subseteq N[\lambda^{[x;ijk]}, \Delta_2]$.

Proof of Claim 3.14: Since $r > 3$, $x \in N[\lambda^{[x;ijk]}, \Delta_2]$. Let $y \in N[x, \Delta_2]$ with $x \neq y$. Suppose $y \notin N[\lambda^{[x;ijk]}, \Delta_2]$. Then $d(y, \lambda^{[x;ijk]}) > r$. From Lemma 3.7 (i), $|y_s| < |x_s|$ for at most one value of s in $\{i, j, k\}$, otherwise $d(y, \lambda^{[x;ijk]}) \leq r$, which is a contradiction. We have the following cases:

Case 2.1: There exists $p \in \{i, j, k\}$ such that $|y_p| < |x_p|$.

Then $|y_s| \geq |x_s|$ for all $s \in \{i, j, k\} \setminus \{p\}$. If $\text{sgn}(y_s x_s) = -1$ for some $s \in \{i, j, k\} \setminus \{p\}$, then from Lemma 3.7 (ii), we get $d(y, \lambda^{[x;ijk]}) \leq r$, which is a contradiction. Hence $\text{sgn}(y_s x_s) = 1$ for all $s \in \{i, j, k\} \setminus \{p\}$. Let $|y_s| = |x_s| + b_s$, where $b_s \geq 0$ for $s \in \{i, j, k\} \setminus \{p\}$. Since $\sum_s |y_s| \leq r$, we see that $\sum_{s \notin \{i,j,k\} \setminus \{p\}} |y_s| \leq r - (\sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| + b_s)$.

$$\begin{aligned}
d(y, \lambda^{[x;ijk]}) &= |y_p - \lambda_p^{[x;ijk]}| + \sum_{s \in \{i,j,k\} \setminus \{p\}} |y_s - \lambda_s^{[x;ijk]}| + \sum_{s \notin \{i,j,k\}} |y_s - \lambda_s^{[x;ijk]}| \\
&\leq |y_p - x_p| - 1 + \sum_{s \in \{i,j,k\} \setminus \{p\}} (|y_s - x_s| + 1) + \sum_{s \notin \{i,j,k\}} |y_s| + 1 \\
&\leq |x_p| + 2 + \sum_{s \in \{i,j,k\} \setminus \{p\}} b_s + \sum_{s \notin \{i,j,k\}} |y_s| + |y_p| \\
&\leq |x_p| + 2 + \sum_{s \in \{i,j,k\} \setminus \{p\}} b_s + \sum_{s \notin \{i,j,k\} \setminus \{p\}} |y_s| \\
&\leq |x_p| + 2 + \sum_{s \in \{i,j,k\} \setminus \{p\}} b_s + r - \sum_{s \in \{i,j,k\} \setminus \{p\}} (|x_s| + b_s) \\
&= r + 2 + |x_p| - \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s|.
\end{aligned}$$

If $|x_p| - \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| \leq -2$, then $d(y, \lambda^{[x;ijk]}) \leq r$, which is a contradiction. If $|x_p| - \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| \geq 1$, then using the fact that $|x_p| + \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| = r - 2$, we find that $|x_p| \geq \lfloor \frac{r}{2} \rfloor$, which is not possible. Thus, we have only two possibilities: either $|x_p| = \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s|$ or $|x_p| + 1 = \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s|$. If $|x_p| = \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s|$, then from the equation, $|x_p| + \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| = r - 2$, we get $|x_p| = \frac{r-2}{2}$. This implies that r is even. Since $r \geq 10$, there exists $t \in \{i, j, k\} \setminus \{p\}$ such that $|x_t| \geq 2$. Then $|x_p| + |x_t| > \frac{r}{2}$, which is a contradiction.

Now, we assume that $|x_p| + 1 = \sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s|$. Then $|x_p| = \frac{r-3}{2}$. This implies that r is odd and $|x_p| = \lfloor \frac{r}{2} \rfloor - 1$. Since $|x_l| = 1$, we get $\sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| = \lfloor \frac{r}{2} \rfloor$. If $|x_t| \geq 3$ for some $t \in \{i, j, k\} \setminus \{p\}$, then $|x_p| + |x_t| > \lfloor \frac{r}{2} \rfloor$, which is a contradiction. Therefore, $\sum_{s \in \{i,j,k\} \setminus \{p\}} |x_s| \leq 4$. This implies that $\lfloor \frac{r}{2} \rfloor \leq 4$ and thus, $r \leq 9$, which is a contradiction. Thus we conclude that $d(y, \lambda^{[x;ijk]}) \leq r$, and therefore $y \in N[\lambda^{[x;ijk]}, \Delta_2]$.

Case 2.2: $|y_s| \geq |x_s|$ for all $s \in \{i, j, k\}$.

From Lemma 3.7 (iii), $\text{sgn}(y_s x_s) = -1$ is possible for at most one value of s in $\{i, j, k\}$. If $|y_s| \geq |x_s|$, and $\text{sgn}(y_s x_s) = 1$ for all $s \in \{i, j, k\}$, then since from Lemma 3.11 (iii), $|y_i| + |y_j| + |y_k| \leq r - 2$, it follows that $x_s = y_s$ for all $s \in \{i, j, k\}$. Consequently, $\sum_{s \notin \{i,j,k\}} |y_s| \leq 2$ implying that $d(y, \lambda^{[x;ijk]}) \leq 6$, which contradicts our assumption that $y \notin N[\lambda^{[x;ijk]}, \Delta_2]$.

Thus, there exists a $q \in \{i, j, k\}$ such that $\text{sgn}(x_q y_q) = -1$. Then $\text{sgn}(x_s y_s) = 1$ for $s \in \{i, j, k\} \setminus \{q\}$. Since $|y_i| + |y_j| + |y_k| \leq r - 2$, it follows that $y_q = -x_q$, $y_s = x_s$, for $s \in \{i, j, k\} \setminus \{q\}$. Then, we have $\sum_{s \notin \{i,j,k\}} |y_s| \leq 2$. Therefore,

$$\begin{aligned}
d(y, \lambda^{[x;ijk]}) &= |y_q - x_q| - 1 + \sum_{s \in \{i,j,k\} \setminus \{q\}} (|y_s - x_s| + 1) + \sum_{s \notin \{i,j,k\}} |y_s - x_s| \\
&\leq |y_q| + |x_q| + 1 + \sum_{s \notin \{i,j,k\}} |y_s| + \sum_{s \notin \{i,j,k\}} |x_s| \\
(3.2) \quad &\leq |y_q| + |x_q| + 4 = 2|x_q| + 4.
\end{aligned}$$

If $|x_q| \leq \lfloor \frac{r}{2} \rfloor - 2$, then $d(y, \lambda^{[x;ijk]}) \leq r$, a contradiction to our assumption. So $|x_q| = \lfloor \frac{r}{2} \rfloor - 1$ (because $|x_q| < \lfloor \frac{r}{2} \rfloor$).

Suppose r is odd. Since $|x_l| = 1$, $\sum_{s \in \{i,j,k\} \setminus \{q\}} |x_s| = \lfloor \frac{r}{2} \rfloor$. If $|x_t| \geq 3$ for any $t \in \{i,j,k\} \setminus \{q\}$, then $|x_q| + |x_t| > \lceil \frac{r}{2} \rceil$, which is a contradiction. Therefore, $\sum_{s \in \{i,j,k\} \setminus \{q\}} |x_s| \leq 4$. This implies that $\lfloor \frac{r}{2} \rfloor \leq 4$ and thus, $r \leq 9$, a contradiction.

Suppose r is even. Since $r \geq 10$, there exists a $t \in \{i,j,k\} \setminus \{q\}$ such that $|x_t| \geq 2$. This implies that $|x_q| + |x_t| > \frac{r}{2}$. This is a contradiction.

Thus we conclude that $d(y, \lambda^{[x;ijk]}) \leq r$, and therefore $y \in N[\lambda^{[x;ijk]}, \Delta_2]$. Hence, $N[x, \Delta_1] \subseteq N[\lambda^{[x;ijk]}, \Delta_1]$. This proves Claim 3.14.

From Proposition 2.3, $\Delta_2 \simeq \Delta_2 \setminus x$. Now, applying the same arguments for each $c \in C$, we find that $\Delta_2 \simeq \text{Ind}_{\Delta_2}(V(\Delta_2) \setminus C)$. Moreover, since $A = B \cup C$ and $\Delta \simeq \Delta_1 \simeq \text{Ind}_{\Delta_1}(V(\Delta_1) \setminus B) = \Delta_2 \simeq \text{Ind}_{\Delta_2}(V(\Delta_2) \setminus C)$, it follows that $\Delta \simeq \text{Ind}_{\Delta}(V(\Delta) \setminus A)$. This completes the proof. $\square$

Theorem 3.15. *For $2 \leq n \leq 5$ and $r \geq n$, $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible.*

Proof. Recalling the discussion at the beginning of this section, to prove that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible, it is sufficient to show that $\Gamma_n^{\alpha,r}$ is contractible for all $1 \leq \alpha \leq \text{Card}(V(\mathcal{G}_m^n)) - 1$. We provide a proof for each particular value of n and considering α arbitrary.

Case (i): $n = 2$.

From Lemma 3.3, $\Gamma_2^{\alpha,r}$ is homotopy equivalent to the induced subcomplex, say Δ , of $\Gamma_2^{\alpha,r}$ on the vertex set $\{x \in V(\Gamma_2^{\alpha,r}) : |x_1|, |x_2| \leq \lfloor \frac{r}{2} \rfloor\}$. First, suppose Δ contains the vertex $e = (0, 1)$. Let $y \in V(\Delta)$. Since $y \succ (0, 0)$, $y_2 \geq 0$. Moreover, if $y_2 > 0$, then $d(y, e) = |y_1| + |y_2 - 1| \leq |y_1| + |y_2| - 1 \leq \lfloor \frac{r}{2} \rfloor + \lfloor \frac{r}{2} \rfloor - 1 \leq r - 1$, and if $y_2 = 0$, then $d(y, e) \leq |y_1| + 1 \leq \lfloor \frac{r}{2} \rfloor + 1 \leq r$. Therefore, Δ is a cone with apex e , and hence contractible. Therefore, $\Gamma_2^{\alpha,r}$ is contractible.

Now, consider the case when $(0, 1) \notin \Delta$. Then for every vertex $v \in \Delta$, $v_2 = 0$. If $V(\Delta) = \{(0, 0)\}$, then clearly Δ is contractible. So assume that $\text{Card}(V(\Delta)) \geq 2$. Clearly $(1, 0) \in \Delta$. Since $d(y, (1, 0)) \leq r$ for all $y \in V(\Delta)$. Therefore, Δ is a cone with apex at $(1, 0)$, and hence contractible. This completes the proof for $n = 2$.

Case (ii): $n = 3$.

From Lemma 3.5, $\Gamma_3^{\alpha,r}$ is homotopy equivalent to the induced subcomplex, say, X on the vertex set $\{x \in \Gamma_3^{\alpha,r} : |x_i| \leq \lfloor \frac{r}{2} \rfloor \text{ and } |x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil \text{ for all } i, j, k \in [3], j \neq k\}$.

If for every $y \in V(X)$, $y_3 = 0$, then X is an induced subcomplex on the vertex set $\{x \in \Gamma_3^{\alpha,r} : |x_1|, |x_2| \leq \lfloor \frac{r}{2} \rfloor, |x_1| + |x_2| \leq \lceil \frac{r}{2} \rceil \text{ and } x_3 = 0\}$. Now, by proceeding in the similar way as in Case (i) above, we conclude that X is contractible.

So, we assume that there exists an element in X whose third coordinate is non-zero. Then clearly $e' := (0, 0, 1) \in X$. Let $x \in V(X)$. If $x_3 \geq 1$, then $d(x, e') = |x_1| + |x_2| + |x_3 - 1| = |x_1| + |x_2| + |x_3| - 1 \leq r - 1$. If $x_3 = 0$, then $|x_1| + |x_2| \leq \lceil \frac{r}{2} \rceil \leq r - 1$, and hence $d(x, e') = |x_1| + |x_2| + 1 \leq r$. Thus, X is a cone with apex e' , hence contractible. Therefore, $\Gamma_3^{\alpha,r}$ is contractible. This completes the proof for $n = 3$.

Case (iii): $n = 4$.

From Lemma 3.8, $\Gamma_4^{\alpha,r}$ is homotopy equivalent to the induced subcomplex, say, $\mathcal{K}$ of $\Gamma_4^{\alpha,r}$, where every vertex $x \in V(\mathcal{K})$ satisfies the following: (i) $|x_i| < \lfloor \frac{r}{2} \rfloor, |x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil$ for all $i, j, k \in [4], j \neq k$ and (ii) $|x_i| + |x_j| + |x_k| < r - 1$ for all $\{i, j, k\} \subseteq [4]$.

If for every $y \in V(\mathcal{K})$, $y_4 = 0$, then by proceeding in the similar way as in Case (ii) above, we conclude that $\mathcal{K}$ is contractible.

Suppose that an element whose fourth coordinate is non-zero exists in $\mathcal{K}$. Then we consider the element $\gamma = (0, 0, 0, 1)$. Clearly, $\gamma \succ (0, 0, 0, 0)$, and $\gamma \in V(\mathcal{K})$. Let $z \in V(\mathcal{K})$. If $z_4 \geq 1$, then $d(z, \gamma) = |z_1| + |z_2| + |z_3| + |z_4| - 1 \leq r - 1$. If $z_4 = 0$, then $|z_1| + |z_2| + |z_3| \leq r - 2$, and hence $d(z, \gamma) = |z_1| + |z_2| + |z_3| + 1 \leq r - 1$. Thus, $\mathcal{K}$ is a cone with apex γ , and hence contractible. Therefore, $\Gamma_4^{\alpha, r}$ is contractible.

Case (iv): $n = 5$.

From Lemma 3.11, $\Gamma_5^{\alpha, r}$ is homotopy equivalent to a subcomplex, say, $\mathcal{L}$ of $\Gamma_5^{\alpha, r}$, where every vertex $x \in V(\mathcal{L})$ satisfies (i) $|x_i| < \lfloor \frac{r}{2} \rfloor$, $|x_j| + |x_k| \leq \lceil \frac{r}{2} \rceil$ for all $i, j, k \in [5], j \neq k$ (ii) $|x_i| + |x_j| + |x_k| < r - 1$ for all $\{i, j, k\} \subseteq [5]$, and (iii) $|x_i| + |x_j| + |x_k| + |x_l| \leq r - 1$ for all $\{i, j, k, l\} \subseteq [5]$.

If for every $w \in V(\mathcal{L})$, $w_5 = 0$, then by proceeding in the similar way as in Case (iii) above, we conclude that $\mathcal{L}$ is contractible.

Suppose that an element whose fifth coordinate is non-zero exists in $\mathcal{L}$. Then we consider the element $\beta = (0, 0, 0, 0, 1)$. Clearly, $\beta \succ (0, 0, 0, 0, 0)$, and $\beta \in V(\mathcal{L})$. Let $v \in V(\mathcal{L})$. If $v_5 \geq 1$, then $d(v, \beta) = |v_1| + |v_2| + |v_3| + |v_4| + |v_5| - 1 \leq r - 1$. If $v_5 = 0$, then $|v_1| + |v_2| + |v_3| + |v_4| \leq r - 1$, and hence $d(v, \beta) = |v_1| + |v_2| + |v_3| + |v_4| + 1 \leq r$. Thus, $\mathcal{L}$ is a cone with apex β , and hence contractible. Therefore, $\Gamma_5^{\alpha, r}$ is contractible. $\square$

Theorem 3.16. $\mathcal{VR}(\mathbb{Z}^6; r)$ is contractible for $r \geq 10$.

Proof. It is sufficient to show that $\Gamma_n^{\alpha, r}$ is contractible for all $1 \leq \alpha \leq \text{Card}(V(\mathcal{G}_m^n)) - 1$. From Lemma 3.12, $\Gamma_6^{\alpha, r}$ is homotopy equivalent to an induced subcomplex Δ such that there exists no vertex $u \in V(\Delta)$ with $|u_i| + |u_j| + |u_k| + |u_l| \geq r - 1$ and $u_s = 0$ for some $\{i, j, k, l\} \subset [6]$ and $s \notin \{i, j, k, l\}$.

If for every $y \in V(\Delta)$, $y_6 = 0$, then by proceeding in the similar way as in Case (iv) of Theorem 3.15 above, we conclude that $\mathcal{D}$ is contractible.

Suppose there exists an element in $V(\Delta)$ with a non-zero sixth coordinate. Let $p = (0, 0, 0, 0, 0, 1)$. Clearly $p \succ (0, 0, 0, 0, 0, 0)$ and $p \in V(\Delta)$.

First, suppose there exists an element in $V(\Delta)$ with a positive fifth coordinate. Then $q = (0, 0, 0, 0, 1, 0) \in V(\Delta)$. Let $w = (0, 0, 0, 0, 1, 1)$. We see that $w \succ (0, 0, 0, 0, 0, 0)$ and from Lemma 3.12, $w \in V(\Delta)$. Let $x \in V(\Delta)$. If $x_6 \geq 1$, then $d(x, w) = |x_1| + |x_2| + |x_3| + |x_4| + |x_5 - 1| + |x_6 - 1| \leq |x_1| + \dots + |x_5| + 1 + |x_6| - 1 \leq r$. If $x_6 = 0$ and $x_5 \neq 0$, then since $x \succ (0, 0, 0, 0, 0, 0)$, $x_5 \geq 1$, so $d(x, w) = |x_1| + \dots + |x_4| + |x_5 - 1| + 1 = |x_1| + \dots + |x_4| + |x_5| \leq r$. If $x_5 = x_6 = 0$, then since $|x_1| + \dots + |x_4| \leq r - 2$ for $r \geq 10$, hence $d(x, w) = |x_1| + \dots + |x_4| + 1 + 1 \leq r$ for $r \geq 10$. Thus, Δ is a cone with apex vertex w , and hence it is contractible. Therefore, $\Gamma_6^{\alpha, r}$ is contractible.

If there is no element in $V(\Delta)$ with a positive fifth coordinate, then for any $y \in V(\Delta)$, either $y_6 \geq 1$ or $y_5 = y_6 = 0$. Since we have $|y_1| + |y_2| + |y_3| + |y_4| \leq r - 2$, we conclude that $d(y, p) \leq r$. Therefore, $N[x, \Delta] \subseteq N(p, \Delta)$. Thus, using Proposition 2.3, we remove all the vertices in Δ except p . Hence, Δ is contractible, and thus $\Gamma_6^{\alpha, r}$ is contractible in $\mathcal{VR}(\mathbb{Z}^6; r)$. $\square$

4. THE COMPLEX $\mathcal{VR}(\mathbb{Z}^n; 2)$

In this section, we prove Theorems 1.4 and 1.5. We first characterize the maximal simplices of the complex $\mathcal{VR}(\mathbb{Z}^n; 2)$, and then use discrete Morse theory to determine the homotopy type of these complexes. We begin by defining a few notations that we will use throughout this section.

Recall that for a positive integer n , $[n] = \{1, 2, \dots, n\}$. Let $[-n] = \{-1, \dots, -n\}$ and $[n]^\pm = [n] \cup [-n]$. For $\{i_1, i_2, \dots, i_k\} \subseteq [n]^\pm$ such that $|i_s| \neq |i_t|$ for all $1 \leq s \neq t \leq k$, we define $x^{i_1, \dots, i_k} \in V(\mathbb{Z}^n)$ by

$$x^{i_1, \dots, i_k}(j) = \begin{cases} x(j) & \text{if } j \notin \{i_1, \dots, i_k\}, \\ x(j) + 1 & \text{if } j \in \{i_1, \dots, i_k\}, \\ x(j) - 1 & \text{if } -j \in \{i_1, \dots, i_k\}. \end{cases}$$

For $i \in [n]$ and $k \in \mathbb{Z}$, we define $x^{[i; k^+]} \in V(\mathbb{Z}^n)$ by

$$x^{[i; k^+]}(j) = \begin{cases} x(j) & \text{if } j \neq i \\ x(i) + k & \text{if } j = i. \end{cases}$$

Recall that $\mathbb{Z}^n$ is a graph, where any two elements x and y are connected by an edge if and only if $d(x, y) = \sum_{i=1}^n |x_i - y_i| = 1$. Define the *open neighborhood* of a vertex x in $\mathbb{Z}^n$ by $N(x, \mathbb{Z}^n) = \{y \in \mathbb{Z}^n : d(x, y) = 1\}$, and the *closed neighborhood* of x by $N[x, \mathbb{Z}^n] = N(x, \mathbb{Z}^n) \cup \{x\}$. For the simplicity of notation, we write $N(x)$ and $N[x]$ for the sets $N(x, \mathbb{Z}^n)$ and $N[x, \mathbb{Z}^n]$, respectively.

We first characterize the maximal simplices of $\mathcal{VR}(\mathbb{Z}^n; 2)$. The idea of the proof of the following lemma is similar to [26, Lemma 3.1].

Lemma 4.1. *Let $n \geq 3$ and τ be a maximal simplex of $\mathcal{VR}(\mathbb{Z}^n; 2)$. Then one of the following is true:*

- (i) $\tau = N[x]$ for some $x \in V(\mathbb{Z}^n)$.
- (ii) $\tau = \{x, x^{i_0}, x^{j_0}, x^{i_0, j_0}\}$ for some $x \in V(\mathbb{Z}^n)$ and $i_0, j_0 \in [n]^\pm$.
- (iii) $\tau = \{x, x^{i_0, j_0}, x^{j_0, k_0}, x^{i_0, k_0}\}$ for some $x \in V(\mathbb{Z}^n)$ and $i_0, j_0, k_0 \in [n]^\pm$.

Proof. We consider the following cases.

Case 1. There exists a $y \in \tau$ such that $N(y) \cap \tau \neq \emptyset$.

- Suppose $|N(y) \cap \tau| = 1$, and let $N(y) \cap \tau = \{x\}$. Since $x \in N(y)$, there exists $l \in [n]^\pm$ such that $y = x^l$. We first show that $N[x] \subseteq \tau$. If possible, let $s \in [n]^\pm$ such that $x^s \notin \tau$. Then there exists $v \in \tau$ such that $d(v, x^s) \geq 3$. Since $x \in \tau$, we have $d(v, x) \leq 2$. Furthermore, $d(x^s, x^t) = 2$ for every $t \in [n]^\pm \setminus \{s\}$ implies that $v \neq x^t$. Therefore, $d(x, v) = 2$, and hence $v = x^{i, j}$ for some $i, j \in [n]^\pm$, or there exists $k \in [n]$ such that $v \in \{x^{[k; 2^+]}, x^{[k; -2^+]}\}$.

Since $d(y, v) \leq 2$, for $v = x^{[k; 2^+]}$ we would have $l = k$ and for $v = x^{[k; -2^+]}$ we would have $l = -k$. However, in both of these situations $v \in N(y)$, which is a contradiction as $N(y) \cap \tau = \{x\}$. On the other hand, if $v = x^{i, j}$ for some $i, j \in [n]^\pm$, then $N(y) \cap \tau = \{x\}$ implies that $l \notin \{i, j\}$. Therefore, $d(v, y) = 3$, which contradicts the fact that $v, y \in \tau$. Hence, $N[x] \subseteq \tau$.

We now show that $\tau \subseteq N[x]$. Suppose, $w \in \tau \setminus N[x]$. Since $x, w \in \tau$, we have $d(x, w) = 2$. Therefore, either $w = x^{p, q}$ for some $p, q \in [n]^\pm$, or there exists $k \in [n]$ such that $w \in \{x^{[k; 2^+]}, x^{[k; -2^+]}\}$. If $w = x^{p, q}$, then $d(x^\alpha, w) = 3$ for $\alpha \in [n]^\pm \setminus \{p, q\}$, a contradiction as $x^\alpha \in \tau$. If $w \in \{x^{[k; 2^+]}, x^{[k; -2^+]}\}$, then $d(x^\beta, w) = 3$ for any $\beta \in [n] \setminus \{k\}$, which is again a contradiction. Thus, we conclude that $\tau = N[x]$ and it is of the type (i).

- Let $|N(y) \cap \tau| \geq 2$.

Then there exists $i_0, j_0 \in [n]^\pm$ such that $y^{i_0}, y^{j_0} \in \tau$. Thus $\{y, y^{i_0}, y^{j_0}\} \subseteq \tau$. Note that $|N(y^{i_0}) \cap \tau| \geq 1$, as $y \in N(y^{i_0}) \cap \tau$. If $|N(y^{i_0}) \cap \tau| = 1$, then it follows from the previous part that $\tau = N[y]$.

Let $|N(y^{i_0}) \cap \tau| \geq 2$. Then there exists $w \in \tau \setminus \{y\}$ such that $w \in N(y^{i_0})$. Then $w = (y^{i_0})^k$ for some $k \in [n]^\pm$. If $k = -i_0$, then $w = y$. Hence $k \neq -i_0$. If $j_0 = -i_0$, then $d(y^{j_0}, w) = 3$, a contradiction. Hence $j_0 \neq \pm i_0$. Since $d(y^{j_0}, w) \leq 2$, we get that $k = j_0$. Thus $w = y^{i_0, j_0}$ and therefore $\{y, y^{i_0}, y^{j_0}, y^{i_0, j_0}\} \subseteq \tau$.

Suppose there exists $u \in \tau \setminus \{y, y^{i_0}, y^{j_0}, y^{i_0, j_0}\}$. If $u \in N(y)$, then $u = y^i$ for some $i \in [n]^\pm \setminus \{i_0, j_0\}$. Here $d(u, y^{i_0, j_0}) = 3$, a contradiction. Hence $u \notin N(y)$, i.e., $d(y, u) = 2$. If $u \in \{y^{[l; 2^+]}, y^{[l; -2^+]}\}$ for some $l \in [n]$, then $d(u, v) = 3$ for some $v \in \{y, y^{i_0}, y^{j_0}, y^{i_0, j_0}\}$. Hence $u = y^{j, k}$ for some $j, k \in [n]^\pm$. If $\{i_0, j_0\} \cap \{j, k\} = \emptyset$, then $d(y^{i_0, j_0}, u) = 4$, a contradiction. Hence $\{i_0, j_0\} \cap \{j, k\} \neq \emptyset$. Without loss of generality, we assume that $i_0 \in \{j, k\}$. In this case $d(u, y^{j_0}) = 3$, a contradiction. Thus $\tau = \{y, y^{i_0}, y^{j_0}, y^{i_0, j_0}\}$. Hence τ is of the type (ii).

Case 2. $N(y) \cap \tau = \emptyset$ for all $y \in \tau$.

Let $y \in \tau$. Choose $v \in \tau$ such that $v \neq y$. Since $N(y) \cap \tau = \emptyset$ and $d(y, v) \leq 2$, we have $d(y, v) = 2$.

- (1) Let $v = y^{i_0, j_0}$ for some $i_0, j_0 \in [n]^\pm$, where $|i_0| \neq |j_0|$.

Then, $\{y, y^{i_0, j_0}\} \subseteq \tau$. Since $d(y^{i_0, t}, y) = 2 = d(y^{i_0, t}, y^{i_0, j_0})$ for every $t \in [n]^\pm \setminus \{\pm i_0, \pm j_0\}$, we see that $\{y, y^{i_0, j_0}, y^{i_0, t}\} \in \mathcal{VR}(\mathbb{Z}^n; 2)$. Thus $\{y, y^{i_0, j_0}\}$ is not a maximal simplex. Let $u \in \tau \setminus \{y, y^{i_0, j_0}\}$. By the assumption of $N(u) \cap \tau = \emptyset$, we have $d(y, u) = 2$. Therefore, either $u \in \{y^{[l; 2^+]}, y^{[l; -2^+]}\}$ for some $l \in [n]$ or $u = y^{i, j}$ for some $i, j \in [n]^\pm$.

- (1.a) Let $u \in \{y^{[l; 2^+]}, y^{[l; -2^+]}\}$ for some $l \in [n]$. Without loss of generality, let $u = y^{[l; 2^+]}$. Then, $\{y, y^{i_0, j_0}, y^{[l; 2^+]}\} \subseteq \tau$. If $l \notin \{i_0, j_0\}$, then $d(u, y^{i_0, j_0}) > 2$, and hence $l \in \{i_0, j_0\}$. Without loss of generality, assume that $l = i_0$. We show that $\tau = \{y, y^{[i_0; 2^+]}\} \cup \{y^{i_0, j} : j \in [n]^\pm \setminus \{\pm i_0\}\}$.

For every $j \in [n]^\pm \setminus \{\pm i_0, \pm j_0\}$, each of $d(y, y^{i_0, j}), d(y^{[i_0; 2^+]}, y^{i_0, j})$, and $d(y^{i_0, j_0}, y^{i_0, j})$ is 2. Therefore, $\{y, y^{i_0, j_0}, y^{[i_0; 2^+]}\}$ is not a maximal simplex. Let $x \in \tau \setminus \{y, y^{i_0, j_0}, y^{[i_0; 2^+]}\}$. Since $d(x, y) = 2 = d(x, y^{i_0, j_0}) = d(x, y^{[i_0; 2^+]})$, we have $x = y^{i_0, j}$ for some $j \in [n]^\pm \setminus \{\pm i_0, j_0\}$. Furthermore, for two distinct integers $j_1, j_2 \in [n]^\pm \setminus \{\pm i_0, j_0\}$, we have $d(y^{i_0, j_1}, y^{i_0, j_2}) = 2$. Since τ is maximal, we conclude that $\{y, y^{[i_0; 2^+]}\} \cup \{y^{i_0, j} : j \in [n]^\pm \setminus \{\pm i_0\}\} \subseteq \tau$. Now, for a vertex $z \in \{y, y^{[i_0; 2^+]}\} \cup \{y^{i_0, j} : j \in [n]^\pm \setminus \{\pm i_0\}\}$, we have $d(y^{i_0}, z) = 1$, and hence there exists some $\tilde{z} \in \tau \setminus \{y, y^{[i_0; 2^+]}\} \cup \{y^{i_0, j} : j \in [n]^\pm \setminus \{\pm i_0\}\}$. But the only choice for $\tilde{z}$ is y^{i_0} . Since $y^{i_0} \in N(y)$ and $N(y) \cap \tau = \emptyset$, we conclude that $y^{i_0} \notin \tau$. This contradicts the fact that τ is a maximal simplex.

Using a similar argument, if $u = y^{[l; -2^+]}$, then we get a contradiction. Hence, this case is not possible.

- (1.b) Let $u = y^{i, j}$ for some $i, j \in [n]^\pm$. If $\{i, j\} \cap \{i_0, j_0\} = \emptyset$, then $d(u, y^{i_0, j_0}) \geq 3$, a contradiction. Hence $\{i, j\} \cap \{i_0, j_0\} \neq \emptyset$. Without loss of generality, let $i = i_0$. Then $\{y, y^{i_0, j_0}, y^{i_0, j}\} \subseteq \tau$. Since

$N(y) \cap \tau = \emptyset$, we have $y^{i_0} \notin \tau$. Further, since τ is maximal, there exists $z \in \tau$ such that $d(z, y^{i_0}) \geq 3$. Clearly $d(y, z) = 2$. Observe that $z = y^{k,l}$ for some $k, l \in [n]^\pm$.

Since $d(z, y^{i_0}) \geq 3, i_0 \notin \{k, l\}$. Using the fact that $d(z, y^{i_0, j_0}) = 2 = d(z, y^{i_0, j})$, we conclude that $\{k, l\} = \{j_0, j\}$. Thus $\{y, y^{i_0, j_0}, y^{i_0, j}, y^{j_0, j}\} \subseteq \tau$. Suppose there exists a vertex $w \in \tau \setminus \{y, y^{i_0, j_0}, y^{i_0, j}, y^{j_0, j}\}$. Then $N(y) \cap \tau = \emptyset$ implies that $d(y, w) = 2$ and therefore $w = y^{s,t}$ for some $s, t \in [n]^\pm$. Since $d(w, y^{i_0, j_0}) = 2, \{i_0, j_0\} \cap \{s, t\} \neq \emptyset$. Further, $d(w, y^{i_0, j}) = 2$ implies that $\{i_0, j\} \cap \{s, t\} \neq \emptyset$ and $d(w, y^{j_0, j}) = 2$ implies that $\{j_0, j\} \cap \{s, t\} \neq \emptyset$, which is not possible. Hence $\tau = \{y, y^{i_0, j_0}, y^{i_0, j}, y^{j_0, j}\}$. Thus τ is of the type (iii).

(2) Let $v \in \{y^{[l; 2^+], y^{[l; -2^+]}}\}$ for some $l \in [n]$.

Without loss of generality, let $v = y^{[l; 2^+]}$. Since $d(y^{l,i}, y^{[l; 2^+]}) = 2 = d(y^{l,i}, y)$ for every $i \in [n] \setminus \{l\}$, τ is not a maximal simplex. Let $x \in \tau \setminus \{y, y^{[l; 2^+]}\}$. Since $d(x, y) = 2 = d(x, y^{[l; 2^+]})$, we get $x = y^{l, j_0}$ for some $j_0 \in [n]^\pm \setminus \{l, -l\}$. Hence $\{y, y^{[l; 2^+]}, y^{l, j_0}\} \subset \tau$. Using the same argument as in (1.a), we get a contradiction. Hence this case is not possible. $\square$

Fix an $m > 0$. Recall from Section 3, that $\mathcal{G}_m^n$ denote the induced subgraph $\mathbb{Z}^n[\{0, \dots, m\}^n]$ and $\Delta_m^{n,2} = \mathcal{VR}(\mathcal{G}_m^n; 2)$. The following Lemma is a consequence of Lemma 4.1.

Lemma 4.2. *Let $n \geq 2$, and let τ be a maximal simplex of $\Delta_m^{n,2}$. Then one of the following is true:*

- (i) $\tau = N[x] \cap V(\mathcal{G}_m^n)$ for some $x \in V(\mathbb{Z}^n)$.
- (ii) $\tau = \{x, x^{i_0}, x^{j_0}, x^{i_0, j_0}\} \cap V(\mathcal{G}_m^n)$ for some $x \in V(\mathbb{Z}^n)$ and $i_0, j_0 \in [n]^\pm$.
- (iii) $\tau = \{x, x^{i_0, j_0}, x^{j_0, k_0}, x^{i_0, k_0}\} \cap V(\mathcal{G}_m^n)$ for some $x \in V(\mathbb{Z}^n)$ and $i_0, j_0, k_0 \in [n]^\pm$.

We now give a brief description of Forman's discrete Morse theory [15]. For more detail, we refer to [22].

Definition 4.3. [22, Definition 11.1] A *partial matching* in a poset P is a subset $\mathcal{M}$ of $P \times P$ such that

- $(a, b) \in \mathcal{M}$ implies $b \gg a$, i.e. $a < b$ and $\nexists c$ such that $a < c < b$.
- Each element in P belongs to at most one element of $\mathcal{M}$.

If $\mathcal{M}$ is a partial matching on a poset P , then there exists $A \subset P$ and an injective map $f : A \rightarrow P \setminus A$ such that $f(x) \gg x$ for all $x \in A$.

Definition 4.4. An *acyclic matching* is a partial matching $\mathcal{M}$ on the poset P such that there does not exist a cycle

$$f(x_1) \gg x_1 \ll f(x_2) \gg x_2 \ll f(x_3) \gg x_3 \dots f(x_t) \gg x_t \ll f(x_1), t \geq 2.$$

For an acyclic partial matching on P , those elements of P that do not belong to the matching are called *critical*.

Theorem 4.5. [22, Theorem 11.13] (*Main theorem of Discrete Morse Theory*)

Let X be a simplicial complex and $\mathcal{A}$ be an acyclic matching on the face poset of X such that the empty set is not critical. Then, X is homotopy equivalent to a

cell complex which has a d -dimensional cell for each d -dimensional critical face of X together with an additional 0-cell.

The following remark is an immediate consequence of Theorem 4.5.

Remark 4.6. If an acyclic matching on a face poset of a simplicial complex Δ has critical faces only in a fixed dimension i , then Δ is homotopy equivalent to a wedge of spheres of dimension i .

Let X be a simplicial complex with vertex set $V(X) = \{v_1, v_2, \dots, v_n\}$. Assume that the vertices of X are linearly ordered as $v_1 < v_2 < \dots < v_n$. Let $\mathcal{P}(X)$ denote the face poset of X . We define an acyclic matching μ^X on $\mathcal{P}(X)$ as follows:

Let $S_1^X = \{\sigma \in \mathcal{P}(X) : v_1 \notin \sigma \text{ and } \sigma \cup \{v_1\} \in \mathcal{P}(X)\}$. Define

$$\mu_1^X : S_1^X \rightarrow \mathcal{P}(X) \setminus S_1^X \text{ by } \mu_1^X(\sigma) = \sigma \cup \{v_1\}.$$

Then observe that μ_1^X is an acyclic matching on $\mathcal{P}(X)$. Let $\mathcal{T}_1^X = \mathcal{P}(X) \setminus (S_1^X \cup \mu_1(S_1^X))$. For $2 \leq i \leq k$, define

$$\begin{aligned} S_i^X &= \{\sigma \in \mathcal{T}_{i-1}^X \mid v_i \notin \sigma \text{ and } \sigma \cup \{v_i\} \in \mathcal{T}_{i-1}^X\}, \\ \mu_i^X : S_i^X &\rightarrow \mathcal{T}_{i-1}^X \setminus S_i^X \text{ by } \mu_i^X(\sigma) = \sigma \cup \{v_i\} \text{ and} \\ \mathcal{T}_i^X &= \mathcal{T}_{i-1}^X \setminus (S_i^X \cup \mu_i^X(S_i^X)). \end{aligned}$$

By the above construction, $S_i^X \cap S_j^X = \emptyset$ for all $i \neq j$. Let $S^X = \bigcup_{i=1}^k S_i^X$. Define

$$(4.1) \quad \mu^X : S^X \rightarrow \mathcal{P}(X) \setminus S^X \text{ by } \mu^X(\sigma) = \mu_i^X(\sigma),$$

where i is the unique element such that $\sigma \in S_i^X$.

From [19, Proposition 3.2], the matching μ^X defined in Equation (4.1) is an acyclic matching.

Let $\mu^{\Delta_m^{n,2}}$ be the acyclic matching as defined in Equation (4.1) with respect to the anti-lexicographic order $\prec$ on vertices of $\mathcal{G}_m^n$. In the rest of the section, we consider the matching $\mu^{\Delta_m^{n,2}}$ on $\mathcal{P}(\Delta_m^{n,2})$, and for the convenience of notation, we denote the matching $\mu^{\Delta_m^{n,2}}$ simply by μ . Moreover, $S_i^X, \mathcal{T}_j^X$, and S^X will be denoted as $S_i, \mathcal{T}_j$, and S if the underlying simplicial complex X is clear from the context. We now characterize the critical cells corresponding to the matching μ .

Proposition 4.7. Let $\sigma \in \Delta_m^{n,2}$ be a simplex. If there exists a vertex x such that $x \prec y$ for all $y \in \sigma$, and $\sigma \cup \{x\}$ is a simplex, then σ is not a critical cell for the matching μ .

Proof. Let z be the smallest element such that $z \prec y$ for all $y \in \sigma$ and $\sigma \cup \{z\}$ is a simplex. Clearly, $z \notin \sigma$. Then σ and $\sigma \cup \{z\} \in \mathcal{T}_v$ for all $v \prec z$. Therefore, by the definition of μ , we get that $\mu(\sigma) = \mu_z(\sigma) = \sigma \cup \{z\}$. Hence, σ is not a critical cell. $\square$

Lemma 4.8. The matching μ yields no critical cells of dimension 0 and 1 in $\mathcal{P}(\Delta_m^{n,2})$ for $n \geq 3$.

Proof. Let $v \in \Delta_m^{n,2}$ be a vertex. If there exists $i \in [n]^-$ such that $v^i \in \Delta_m^{n,2}$, then $v^i \prec v$ and $\{v, v^i\}$ is a simplex. This implies that v is not a critical cell.

Now, suppose there is no $i \in [n]^-$ such that $v^i \in \Delta_m^{n,2}$. Then, $v = (0, \dots, 0)$, and therefore, $v = \mu_v(\emptyset)$. Thus, v is not a critical cell. Hence, the matching μ yields no critical cells of dimension 0 in $\Delta_m^{n,2}$.

Let $\gamma \in \Delta_m^{n,2}$ be a 1-simplex. Then γ must be one of the following four types: (i) $\gamma = \{v, v^i\}$ for some $v, v^i \in V(\mathcal{G}_m^n)$ and $i \in [n]$, (ii) $\gamma = \{v, v^{i,j}\}$ for some $v, v^{i,j} \in V(\mathcal{G}_m^n)$ and $i, j \in [n]$, (iii) $\gamma = \{v, v^{i,j}\}$ for some $v, v^{i,j} \in V(\mathcal{G}_m^n)$ with $i \in [n]$ and $j \in [n]^-$, and (iv) $\gamma = \{v, v^{[i;2^+]}\}$ for some $v \in V(\mathcal{G}_m^n)$ and $i \in [n]$.

Case (i): Let $\gamma = \{v, v^i\}$ for some $v, v^i \in V(\mathcal{G}_m^n)$ and $i \in [n]$. If there exists $l \in [n]^-$ such that $v^l \in V(\mathcal{G}_m^n)$, then $v^l \prec x$ for all $x \in \gamma$, and $\gamma \cup \{v^l\} \subseteq N[v]$. From Proposition 4.7, γ is not a critical cell. Suppose no such $l \in [n]^-$ exists. Then $v = (0, \dots, 0)$. Clearly, $\gamma = \mu_v(\{v^i\})$. Therefore, γ is not a critical cell.

Case (ii): Let $\gamma = \{v, v^{i,j}\}$ for some $v, v^{i,j} \in V(\mathcal{G}_m^n)$ and $i, j \in [n]$. Then $v \prec v^{i,j}$. If for some $x \prec v$, the set $\gamma \cup \{x\}$ forms a simplex, then $d(v, x) \leq 2$. Thus, $x = v^k$ for some $k \in [n]^-$ or $x = v^{t,l}$ for some $l \in [n]^-$ with $|t| < |l|$, or $x = v^{[k;-2^+]}$ for some $k \in [n]$. Clearly, if $x = v^k$ or $x = v^{[t;-2^+]}$ for some $k \in [n]^-$ and $t \in [n]$, then $\{x, v^{i,j}\}$ is not a 1-simplex in $\Delta_m^{n,2}$.

Now, if there exists an $s \in [n]^-$ such that $v^s \in \mathcal{G}_m^n$ and $|k| < |s|$ for some $k \in i, j$, then $\{v, v^{i,j}, v^{k,s}\}$ is a simplex and $v^{k,s} \prec y$ for all $y \in \gamma$. Thus, γ is not a critical cell.

If there is no $s \in [n]^-$ such that $v^s \in \mathcal{G}_m^n$ and $|k| < |s|$ for some $k \in \{i, j\}$, then there is no $y \prec v$ such that $\{y, v^{i,j}\}$ is a simplex. In this case, we conclude that neither γ nor $\gamma \setminus \{v\}$ belongs to $S_y \cup \mu_y(S_y)$ for any $y \prec v$. Hence, $\gamma = \mu_v(v^{i,j})$, which implies that γ is not a critical cell.

Case (iii): Let $\gamma = \{v, v^{i,j}\}$ for some $v, v^{i,j} \in V(\mathcal{G}_m^n)$ with $i \in [n]$ and $j \in [n]^-$. Since $v^{i,j} \in V(\mathcal{G}_m^n)$, we also have $v^j \in V(\mathcal{G}_m^n)$. It is clear that $v^j \prec v$ and $v^j \prec v^{i,j}$. Since $\{v, v^j, v^{i,j}\}$ is a 2-simplex in $\Delta_m^{n,2}$, it follows from Proposition 4.7 that γ is not a critical cell.

Case (iv): Let $\gamma = \{v, v^{[i;2^+]}\}$ for some $v \in V(\mathcal{G}_m^n)$ and $i \in [n]$. First, assume that $\gamma = \{v, v^{[i;2^+]}\}$ for some $i \in [n]$. If for some $k \in [n]^-$, where $|k| \neq i$, we have $v^k \in V(\mathcal{G}_m^n)$, then $\{v, v^{i,k}, v^{[i;2^+]}\}$ is a simplex and $v^{i,k} \prec y$ for $y \in \gamma$. Thus, γ is not a critical cell. On the other hand, if there is no $k \in [n]^-$, $|k| \neq i$ such that $v^k \in V(\mathcal{G}_m^n)$ then $v_t = 0$ for $t \neq i$. Thus, for any $x \prec v$, $\{x, v^{[i;2^+]}\}$ is not a simplex. Hence, we conclude that neither γ nor $\gamma \setminus \{v\}$ belongs to $S_y \cup \mu_y(S_y)$ for any $y \prec v$. Therefore, $\gamma = \mu_v(v^{[i;2^+]})$, which implies that γ is not a critical cell.

Hence the matching μ yields no critical cell of dimension 1 in $\Delta_m^{n,2}$. $\square$

Lemma 4.9. *The matching μ yields no critical cells of dimension 2 in $\mathcal{P}(\Delta_m^{n,2})$.*

Proof. Let $\gamma \in \Delta_m^{n,2}$ be a 2-simplex. Then γ is a face of a maximal simplex of the three types given in Lemma 4.2.

Case (a): Let γ be a face of a maximal simplex of type $\sigma = N[v]$ for some vertex v . Then there exist $i_0, j_0, k_0 \in [n]^\pm$ such that $\gamma = \{v, v^{i_0}, v^{j_0}\}$, or $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$. We have the following subcases:

Subcase (i): $\gamma = \{v, v^{i_0}, v^{j_0}\}$, or $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$, where $i_0, j_0, k_0 \in [n]$.

If $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$, then $v \prec x$ for all $x \in \gamma$ and $\gamma \cup \{v\} \subseteq N[v]$. Since $v \in V(\mathcal{G}_m^n)$, from Proposition 4.7, γ is not a critical cell.

Let us now assume that $\gamma = \{v, v^{i_0}, v^{j_0}\}$. If there exists a $l_0 \in [n]^-$ such that $v^{l_0} \in V(\mathcal{G}_m^n)$, then $v^{l_0} \prec x$ for all $x \in \gamma$ and $\gamma \cup \{v^{l_0}\} \subseteq N[v]$. From Proposition 4.7, γ is not a critical cell. Suppose there exists no $l_0 \in [n]^-$ such that $v^{l_0} \in V(\mathcal{G}_m^n)$.

Then $v = (0, \dots, 0)$, and it follows that $\gamma = \mu_v(\{v^{i_0}, v^{j_0}\})$. Therefore, γ is not a critical cell.

Subcase (ii): $\gamma = \{v, v^{i_0}, v^{j_0}\}$ or $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$, where $i_0 \in [n]^-$, $j_0, k_0 \in [n]$.

Here, $v^{i_0} \prec v^{j_0}, v^{k_0}$. If there exists an $l_0 < i_0$ such that $v^{l_0} \in V(\mathcal{G}_m^n)$, then $v^{l_0} \prec x$ for all $x \in \gamma$ and $\gamma \cup \{v^{l_0}\} \subseteq N[v]$. From Proposition 4.7, γ is not a critical cell. So, assume that there exists no $l_0 < i_0$ such that $v^{l_0} \in V(\mathcal{G}_m^n)$. This implies that $v(l) = 0$ for every $l > |i_0|$.

If $\gamma \setminus \{v^{i_0}\} \cup \{x\}$ is a simplex and $x \prec v^{i_0}$ for some $x \in V(\mathcal{G}_m^n)$, then $d(v, x) \leq 2$, and thus one of the following holds:

- $x(|i_0|) = v(|i_0|) - 1$, and $x(s) = v(s) - 1$ for some $s \in [n]$ with $s < |i_0|$ and $x(j) = v(j)$ for $j \notin \{i_0, s\}$,
- $x(|i_0|) = v(|i_0|) - 2$, and $x(j) = v(j)$ for $j \neq i_0$.

If $x(|i_0|) = v(|i_0|) - 1$ and $x(s) = v(s) - 1$ for some $s \in [n]$ with $s < |i_0|$ and $x(j) = v(j)$ for $j \notin \{i_0, s\}$, then $d(v^{j_0}, x) \geq 3$. Similarly, if $x(|i_0|) = v(|i_0|) - 2$ and $x(j) = v(j)$ for $j \neq i_0$, then also $d(v^{j_0}, x) \geq 3$. Hence for any $x \prec v^{i_0}$, $\gamma \setminus \{v^{i_0}\} \cup \{x\}$ is not a simplex. Thus, we conclude that both γ and $\gamma \setminus \{v^{i_0}\}$, do not belong to $S_y \cup \mu_y(S_y)$ for all $y \prec v^{i_0}$. Therefore, by definition, we get that $\gamma = \mu_{v^{i_0}}(\gamma \setminus \{v^{i_0}\})$. Hence γ is not a critical cell.

Subcase(iii): $\gamma = \{v, v^{i_0}, v^{j_0}\}$ or $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$, where $i_0, j_0, k_0 \in [n]^-$.

Observe that $v^{i_0, j_0} \in V(\mathcal{G}_m^n)$, $\{v, v^{i_0}, v^{j_0}, v^{i_0, j_0}\} \in \Delta_m^{n, 2}$, and $v^{i_0, j_0} \prec v, v^{i_0}, v^{j_0}$. Thus, from Proposition 4.7, $\{v, v^{i_0}, v^{j_0}\}$ is not a critical cell. Similarly, if $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$, then $v^{i_0, j_0, k_0} \in V(\mathcal{G}_m^n)$, $\{v^{i_0}, v^{j_0}, v^{k_0}\} \cup \{v^{i_0, j_0, k_0}\} \in \Delta_m^{n, 2}$, and $v^{i_0, j_0, k_0} \prec x$ for all $x \in \{v^{i_0}, v^{j_0}, v^{k_0}\}$. Therefore, $\{v^{i_0}, v^{j_0}, v^{k_0}\}$ is not a critical cell.

Subcase (iv): $\gamma = \{v^{i_0}, v^{j_0}, v^{k_0}\}$, where $i_0, j_0 \in [n]^-$ and $k_0 \in [n]$.

Without loss of generality, we assume that $|j_0| < |i_0|$. Then $v^{i_0} \prec v^{j_0}, v^{k_0}$. If there exists a vertex $v^{l_0} \in V(\mathcal{G}_m^n)$ for some $l_0 < i_0$, then $v^{l_0} \prec x$ for all $x \in \gamma$ and $\gamma \cup \{v^{l_0}\} \subseteq N[v]$. Hence, γ is not a critical cell by Proposition 4.7.

We now assume that there is no vertex in $V(\mathcal{G}_m^n)$ of type v^{l_0} for $l_0 < i_0$. This implies that for any $l > |i_0|$, $v(l) = 0$. Clearly, $v^{i_0, j_0, k_0} \in V(\mathcal{G}_m^n)$. If $k_0 < |j_0| < |i_0|$, then $v^{i_0, j_0, k_0} \prec x$ for all $x \in \gamma$ and $\gamma \cup \{v^{i_0, j_0, k_0}\}$ is a simplex in $\Delta_m^{n, 2}$. Therefore, from Proposition 4.7, γ is not a critical cell. Thus, we also assume that $k_0 \geq |j_0|$.

Now, we claim that $\gamma = \mu(\gamma \setminus \{v^{i_0}\})$. If $\gamma \setminus \{v^{i_0}\} \cup \{x\}$ is a simplex and $x \prec v^{i_0}$ for some $x \in V(\mathcal{G}_m^n)$, then one of the following holds:

- $x(|i_0|) = v(|i_0|) - 1$, and $x(s) = v(s) - t$ for some $s \in [n]$ with $s < |i_0|$ and $t \geq 1$,
- $x(|i_0|) = v(|i_0|) - r$ for some $r \geq 2$.

If $x(|i_0|) = v(|i_0|) - 1$ and $x(s) = v(s) - t$ for some $s \in [n]$ with $s < |i_0|$ and $t \geq 1$, then $d(v^{k_0}, x) \leq 2$ if and only if $x(k_0) = v(k_0) + 1$, $k_0 \neq s$, and $t = 1$. If $x(k_0) = v(k_0) + 1$, $k_0 \neq s$, and $t = 1$ then $k_0 < s$ and thus $|j_0| \neq s$. Therefore, $d(v^{j_0}, x) \geq 3$.

Similarly, if $x(|i_0|) = v(|i_0|) - r$ for some $r \geq 2$, then also $d(v^{k_0}, x) \leq 2$ if and only if $x(k_0) = v(k_0) + 1$, $k_0 \neq |i_0|$, and $r = 2$. If $x(k_0) = v(k_0) + 1$, $k_0 \neq |i_0|$, and $r = 2$ then from the fact that $|j_0| \leq k_0 < |i_0|$, we find that $d(v^{j_0}, x) \geq 3$.

Hence for any $x \prec v^{i_0}$, $\gamma \setminus \{v^{i_0}\} \cup \{x\}$ is not a simplex.

Thus, both γ and $\gamma \setminus \{v^{i_0}\}$, do not belong to $S_x \cup \mu_x(S_x)$ for all $x \prec v^{i_0}$. Therefore, by definition, $\gamma = \mu_{v^{i_0}}(\gamma \setminus \{v^{i_0}\}) = \mu(\gamma \setminus \{v^{i_0}\})$. Hence γ is not a critical cell.

Case (b): If γ is a face of a maximal simplex of type $\sigma = \{v, v^{i_0}, v^{j_0}, v^{i_0, j_0}\}$ for some vertex v and $i_0, j_0, k_0 \in [n]^\pm$, then the possible types of 2-simplices in σ are $\{v, v^{i_0}, v^{j_0}\}$, $\{v, v^{i_0}, v^{i_0, j_0}\}$, $\{v, v^{j_0}, v^{i_0, j_0}\}$, and $\{v^{i_0}, v^{j_0}, v^{i_0, j_0}\}$. Observe that $\{v, v^{i_0}, v^{j_0}\} \subseteq N[v]$, $\{v, v^{i_0}, v^{i_0, j_0}\} \subseteq N[v^{i_0}]$, $\{v, v^{j_0}, v^{i_0, j_0}\} \subseteq N[v^{j_0}]$ and $\{v^{i_0}, v^{j_0}, v^{i_0, j_0}\} \subseteq N[v^{i_0, j_0}]$. Thus, from case (a) above, γ is not a critical cell.

Case (c): If γ is a face of a maximal simplex of type $\sigma = \{v, v^{i_0, j_0}, v^{i_0, k_0}, v^{j_0, k_0}\}$ for some vertex v and $i_0, j_0, k_0 \in [n]^\pm$, then the possible types of 2-simplices in σ are $\{v, v^{i_0, j_0}, v^{i_0, k_0}\}$, $\{v, v^{i_0, j_0}, v^{j_0, k_0}\}$, $\{v, v^{i_0, k_0}, v^{j_0, k_0}\}$ and $\{v^{i_0, j_0}, v^{j_0, k_0}, v^{i_0, k_0}\}$. Observe that $\{v, v^{i_0, j_0}, v^{i_0, k_0}\} \subseteq N[v^{i_0}]$, $\{v, v^{i_0, j_0}, v^{j_0, k_0}\} \subseteq N[v^{j_0}]$, $\{v, v^{i_0, k_0}, v^{j_0, k_0}\} \subseteq N[v^{k_0}]$ and $\{v^{i_0, j_0}, v^{j_0, k_0}, v^{i_0, k_0}\} \subseteq N[v^{i_0, j_0, k_0}]$. Thus, from case (a) above, γ is not a critical cell. $\square$

Lemma 4.10. *The matching μ yields no critical cells of dimension 4 or more in $\mathcal{P}(\Delta_m^{n,2})$.*

Proof. Since $\text{Card}(N[x]) \leq 2n + 1$ for any $x \in V(\mathcal{G}_m^n)$, using Lemma 4.2, we get that for every $\gamma \in \Delta_m^{n,2}$, $\dim(\gamma) \leq 2n$. This implies that there is no critical cell of dimension $2n + 1$ or higher.

Let σ be a simplex in $\Delta_m^{n,2}$ with $4 \leq \dim(\sigma) \leq 2n$. Then, from Lemma 4.2, we have $\sigma \subseteq N[v]$ for some v . We claim that σ is not a critical cell.

Case (i): Let $v^{i_0} \notin \sigma$ for every $i_0 \in [n]^-$. If $v \notin \sigma$, then for any $x \in \sigma$, we have $v \prec x$ and $\sigma \cup \{v\} \subseteq N[v]$. Moreover, since $\dim(\sigma) \geq 4$, we have $v \in V(\mathcal{G}_m^n)$. Thus, it follows from Proposition 4.7 that σ is not a critical cell. Now, assume that $v \in \sigma$. If there exists a $l_0 \in [n]^-$ such that $v^{l_0} \in V(\mathcal{G}_m^n)$, then $v^{l_0} \prec x$ for all $x \in \sigma$ and $\sigma \cup \{v^{l_0}\} \subseteq N[v]$. From Proposition 4.7, σ is not a critical cell. Suppose there exists no $k \in [n]^-$ such that $v^k \in V(\mathcal{G}_m^n)$. Then $v = (0, \dots, 0)$. Clearly $\sigma = \mu(\sigma \setminus \{v\})$. Therefore, σ is not a critical cell.

Case (ii): Assume that, there exists $i_0 \in [n]^-$ such that $v^{i_0} \in \sigma$. Let $v^{j_0} \in \sigma$ be the minimal such vertex in σ , i.e., $v^{j_0} \prec x$ for all $x \in \sigma \setminus \{v^{j_0}\}$, where $j_0 \in [n]^-$.

Suppose there exists a vertex $v^{l_0} \in V(\mathcal{G}_m^n)$ with $l_0 < j_0$ such that $\sigma \cup \{v^{l_0}\} \subseteq N[v]$. Then $v^{l_0} \prec x$ for all $x \in \sigma$. Thus, from Proposition 4.7, σ is not a critical cell. So, we assume that there is no $l_0 < j_0$ with $v^{l_0} \in V(\mathcal{G}_m^n)$ and $\sigma \cup \{v^{l_0}\} \subseteq N[v]$. This implies that $v(k) = 0$ for all $k > |j_0|$. We claim that $\sigma = \mu_{v^{j_0}}(\sigma \setminus \{v^{j_0}\})$.

Suppose $\sigma \setminus \{v^{j_0}\} \cup \{x\}$ is a simplex and $x \prec v^{j_0}$ for some $x \in V(\mathcal{G}_m^n)$. Then one of the following holds:

- $x(|j_0|) = v(|j_0|) - 1$, and $x(s) = v(s) - t$ for some $s \in [n]$ with $s < |j_0|$ and $t \geq 1$,
- $x(|j_0|) = v(|j_0|) - r$ for some $r \geq 2$.

Let $x(|j_0|) = v(|j_0|) - 1$ and $x(s) = v(s) - t$ for some $s \in [n]$ with $s < |j_0|$ and $t \geq 1$. Since $\sigma \subseteq N[v]$ and $4 \leq \dim(\sigma) \leq 2n$, there exists a vertex $v^p \in \sigma$, where $p \in [n]^\pm \setminus \{j_0, s\}$. Suppose $d(v^p, x) \leq 2$. Then $t = 1$ and $x(|p|) = v(|p|) \pm 1$, according to the sign of p . However, since $\text{Card}(\sigma) \geq 5$, there exists a $q \in [n]^\pm \setminus \{j_0, p, s\}$ such that $v^q \in \sigma$. Then $d(v^q, x)$ must be at least 3.

Similarly, if $x(|j_0|) = v(|j_0|) - r$ for some $r \geq 2$, then $d(v^p, x) \leq 2$ for some $v^p \in \sigma$ with $p \in [n]^\pm \setminus \{j_0\}$ implies that $x(|p|) = v(|p|) \pm 1$, according to the sign of p , and $r = 2$. However, since $\text{Card}(\sigma) \geq 5$, there exists a $q \in [n]^\pm \setminus \{j_0, p\}$ such that $v^q \in \sigma$. Then $d(v^q, x)$ must be at least 3. Hence, for any $x \prec v^{j_0}$, $\sigma \setminus \{v^{j_0}\} \cup \{x\}$ is not a simplex.

Thus, both σ and $\sigma \setminus \{v^{i_0}\}$, do not belong to $S_x \cup \mu_x(S_x)$ for all $x \prec v^{i_0}$. Therefore, by definition, $\sigma = \mu_{v^{j_0}}(\sigma \setminus \{v^{j_0}\}) = \mu(\sigma \setminus \{v^{j_0}\})$. Hence, σ is not a critical cell. This completes the proof. $\square$

Lemma 4.11. *Let $m \geq 3$. The matching μ yields at least $(m-2)^3$ critical cells of dimension 3 in $\mathcal{P}(\Delta_m^{3,2})$.*

Proof. Let $(k_1, k_2, k_3) \in \mathcal{G}_m^3$ be such that $k_1, k_2, k_3 \geq 2$. Then $\sigma = \{(k_1, k_2, k_3), (k_1, k_2 - 1, k_3), (k_1 - 1, k_2, k_3), (k_1 - 1, k_2 - 1, k_3)\}$ is a simplex of dimension three in $\Delta_m^{3,2}$. We show that σ is a critical cell. From Lemma 4.2 (ii), σ is a maximal simplex in $\Delta_m^{3,2}$, and thus $\sigma \cup \{x\} \notin \Delta_m^{3,2}$ for any $x \notin \sigma$. Thus, the only possibility for σ to not be a critical cell is that $\sigma = \mu(\sigma \setminus \{v\})$ for some vertex $v \in \sigma$. We now have the following cases:

Case (i): $v = (k_1 - 1, k_2 - 1, k_3)$. In this case, we find that $\sigma \setminus \{(k_1 - 1, k_2 - 1, k_3)\} \cup \{(k_1, k_2, k_3 - 1)\}$ is a simplex, and $(k_1, k_2, k_3 - 1) < y$ for all $y \in \sigma$.

Case (ii): $v = (k_1 - 1, k_2, k_3)$. In this case, $\sigma \setminus \{(k_1 - 1, k_2, k_3)\} \cup \{(k_1, k_2 - 1, k_3 - 1)\}$ is a simplex, and $(k_1, k_2 - 1, k_3 - 1) < y$ for all $y \in \sigma$.

Case (iii): $v = (k_1, k_2 - 1, k_3)$. In this case, $\sigma \setminus \{(k_1, k_2 - 1, k_3)\} \cup \{(k_1 - 1, k_2, k_3 - 1)\}$ is a simplex, and $(k_1 - 1, k_2, k_3 - 1) < y$ for all $y \in \sigma$.

Case (iv): $v = (k_1, k_2, k_3)$. In this case, $\sigma \setminus \{(k_1, k_2, k_3)\} \cup \{(k_1 - 1, k_2 - 1, k_3 - 1)\}$ is a simplex, and $(k_1 - 1, k_2 - 1, k_3 - 1) < y$ for all $y \in \sigma$.

Therefore, there is no $v \in \sigma$ such that $\sigma = \mu_v(\sigma \setminus \{v\})$. Hence, σ is a critical cell. Since number of such 3-tuples in $V(\mathcal{G}_m^3)$ is $(m-2)^3$, the result follows. $\square$

Proposition 4.12. For each $n \geq 3$, there exists a retraction $r : \Delta_m^{n,2} \rightarrow \Delta_m^{3,2}$.

Proof. Define $r_1 : V(\mathcal{G}_m^n) \rightarrow V(\mathcal{G}_m^3)$ by $r_1((v_1, \dots, v_n)) = (v_1, v_2, v_3)$. We extend the map r_1 to $r : \Delta_m^{n,2} \rightarrow \Delta_m^{3,2}$ by $r(\sigma) := \{r_1(v) : v \in \sigma\}$ for all $\sigma \in \Delta_m^{n,2}$. Since $d(r_1(v), r_1(w)) \leq d(v, w)$ for all $v, w \in V(\mathcal{G}_m^n)$, the map r_1 is a surjective simplicial map. Hence r is a retraction map. This completes the proof. $\square$

Theorem 4.13. For $m \geq 3$, $\Delta_m^{n,2} \simeq \bigvee^{\nu_m} \mathbb{S}^3$, where $\nu_m \geq (m-2)^3$.

Proof. Using Theorem 4.5, and Lemmas 4.8, 4.9, and 4.10, we obtain $\tilde{H}_i(\Delta_m^{n,2}; \mathbb{Z}) = 0$ if $i \neq 3$. Further, using Proposition 4.12 and Lemma 4.11, we have $\tilde{H}_3(\Delta_m^{n,2}; \mathbb{Z}) \neq 0$. From Remark 4.6, we conclude that $\Delta_m^{n,2} \simeq \bigvee^{\nu_m} \mathbb{S}^3$, where ν_m is the number of 3-dimensional critical cells corresponding to the matching μ defined above. Then the rank of $\tilde{H}_3(\Delta_m^{n,2}; \mathbb{Z})$ is ν_m . Now, the result follows from Proposition 4.12 and Lemma 4.11. $\square$

Theorem 4.14. The complex $\mathcal{VR}(\mathbb{Z}^n; r)$ is simply connected for all $r \geq 2$.

Proof. Let $\sigma : S^1 \rightarrow \mathcal{VR}(\mathbb{Z}^n; r)$ be a closed path in Δ_n . Since Δ_n is a simplicial complex, σ is homotopic to a closed path $c = x_1, x_2, \dots, x_1$, where $\{x_i, x_{i+1}\} \in \Delta_n$ for each i . If for some i , $d(x_i, x_{i+1}) = k \geq 2$, then there exist vertices $z_1, \dots, z_{k-1} \in \mathbb{Z}^n$ such that $d(x_i, z_1) = 1 = d(z_1, z_2) = \dots = d(z_{k-1}, x_{i+1})$. Clearly, the path $c_1 = x_1, \dots, x_i, z_1, \dots, z_{k-1}, x_{i+1}, \dots, x_1$ is homotopic to σ . Hence, by inserting a new vertices between each such pair of vertices of distance ≥ 2 , we can assume that $d(x_i, x_{i+1}) = 1$ for all i . Using the compactness of S^1 , we see that c_1 consists of finitely many edges of $\mathbb{Z}^n$. Hence c_1 is a closed edge path in $\Delta_m^{n,2}$ for some sufficiently large m . Result follows from Theorem 4.13. $\square$

Theorem 4.15. *For $n \geq 3$, $\mathcal{VR}(\mathbb{Z}^n; 2)$ is homotopy equivalent to the wedge sum of countably infinite copies of $\mathbb{S}^3$'s.*

Proof. Since any homology class of $\mathcal{VR}(\mathbb{Z}^n; 2)$ lies in $\Delta_m^{n,2}$ for sufficiently large m , we conclude that $\tilde{H}_i(\mathcal{VR}(\mathbb{Z}^n; 2), \mathbb{Z}) \neq 0$ if and only if $i = 3$. Further, using Theorem 4.13, we see that $\tilde{H}_3(\mathcal{VR}(\mathbb{Z}^n; 2), \mathbb{Z})$ is free abelian and is of countably infinite rank. Suppose rank of $\tilde{H}_3(\mathcal{VR}(\mathbb{Z}^n; 2), \mathbb{Z})$ is ν . Then $\tilde{H}_3(\mathcal{VR}(\mathbb{Z}^n; 2), \mathbb{Z}) \cong \mathbb{Z}^\nu$.

For an abelian group G and a positive integer k , let $M(G, k)$ denote a Moore space, i.e., $\tilde{H}_k(M(G, k); \mathbb{Z}) \cong G$ and $\tilde{H}_i(M(G, k); \mathbb{Z}) = 0$ for $i \neq k$ (for more details on Moore space, see [21]). Then $\mathcal{VR}(\mathbb{Z}^n; 2)$ is a Moore space $M(\mathbb{Z}^\nu, 3)$. Since $\mathcal{VR}(\mathbb{Z}^n; 2)$ is simply connected (Theorem 4.14), and the fact that $\bigvee^\nu \mathbb{S}^3$ is a $M(\mathbb{Z}^\nu, 3)$, by uniqueness (upto homotopy equivalence for simply connected CW complexes) of Moore space, we conclude that $\mathcal{VR}(\mathbb{Z}^n; 2) \simeq \bigvee^\nu \mathbb{S}^3$. $\square$

5. CONCLUSION AND FUTURE DIRECTIONS

In this article, we investigated the Vietoris-Rips complex of the Cayley graph (with respect to the standard generator) of the abelian group $\mathbb{Z}^n$ with the word metric. Building on earlier work, we confirmed Zaremsky's conjecture (Conjecture 1.1), for $n \leq 5$ and established the contractibility of $\mathcal{VR}(\mathbb{Z}^6; r)$ for $r \geq 10$. Using discrete Morse theory, we further characterized the homotopy type of $\mathcal{VR}(\mathbb{Z}^n; 2)$ for $n \geq 3$, proving that it is homotopy equivalent to the wedge sum of countably infinite copies of 3-spheres.

These results contribute to the growing understanding of Vietoris-Rips complexes beyond hyperbolic groups, highlighting the rich combinatorial and topological structure of $\mathbb{Z}^n$ under the word metric. Our findings not only extend the validity of Zaremsky's conjecture but also provide new insights into the homotopy types of Vietoris-Rips complexes arising from discrete spaces.

Several questions remain open, particularly the Conjecture 1.1, which is still open for $n \geq 7$. We believe that the Lemmas 3.8, 3.11, and 3.12 can be generalized for more coordinates. For a fix $m > 0$, let $\mathcal{G}_m^n$, $\Delta_m^{n,r}$, and $\Gamma_n^{\alpha,r}$ be the same as defined in the beginning of Section 3. Then, we propose the following conjecture.

Conjecture 5.1. Let $r \geq n \geq 2$. Then $\Gamma_n^{\alpha,r}$ is homotopy equivalent to the induced subcomplex Δ of $\Gamma_n^{\alpha,r}$, where every $x \in V(\Delta)$ satisfies the following: (i) $|x_i| < \lfloor \frac{r}{2} \rfloor$ for all $i \in [n]$ and (ii) for any set $S \subseteq [n]$ such that $\text{Card}(S) = n - 2$, $\sum_{i \in S} |x_i| \leq r - 2$.

Assuming that the above conjecture is true, we can prove Conjecture 1.1 in the following way.

Proof of Conjecture 1.1: Recall that to establish that $\mathcal{VR}(\mathbb{Z}^n; r)$ is contractible, it is sufficient to prove that $\Gamma_n^{\alpha,r}$ is contractible. From Theorem 3.15, $\Gamma_2^{\alpha,r}$ is contractible for $r \geq 2$. From Conjecture 5.1, $\Gamma_n^{\alpha,r}$ is homotopy equivalent to the induced subcomplex Δ_n of $\Gamma_n^{\alpha,r}$, where every $x \in V(\Delta_n)$ satisfies the following: $|x_i| < \lfloor \frac{r}{2} \rfloor$ for all $i \in [n]$, and for any set $S \subseteq [n]$ with $\text{Card}(S) = n - 2$, we have $\sum_{i \in S} |x_i| \leq r - 2$.

Case 1: There exists an element in $V(\Delta_n)$ with a positive n -th coordinate.

Let $p = (0, 0, \dots, 0, 1) \in \mathbb{Z}^n$. Then observe that $p \in \Delta_n$. Suppose there exists an element in $V(\Delta_n)$ with a positive $(n-1)$ -th coordinate. Then $q = (0, 0, \dots, 0, 1, 0) \in V(\Delta_n)$. Consider $w = (0, 0, \dots, 0, 1, 1)$. We have $w \succ \mathbf{0}$ and $w \in V(\Delta_n)$. Let $x \in V(\Delta_n)$.

- If $x_n \geq 1$, then $d(x, w) = \sum_{i \in [n-2]} |x_i| + |x_{n-1} - 1| + |x_n - 1| \leq \sum_{i \in [n-2]} |x_i| + |x_{n-1}| + |x_n| \leq r$.
- If $x_n = 0$ and $x_{n-1} \neq 0$, then since $x \succ \mathbf{0}$, we have $x_{n-1} \geq 1$. Thus, $d(x, w) = \sum_{i \in [n-2]} |x_i| + |x_{n-1} - 1| + 1 \leq r$.
- If $x_{n-1} = x_n = 0$, then $\sum_{i \in [n-2]} |x_i| \leq r - 2$. Hence, $d(x, w) = \sum_{i \in [n-2]} |x_i| + 1 + 1 \leq r$.

Therefore, $d(x, w) \leq r$ for all $x \in V(\Delta_n)$. Thus, Δ_n is a cone with apex w . This implies that Δ_n is contractible and hence $\Gamma_n^{\alpha, r}$ is contractible.

If there is no element in $V(\Delta_n)$ with a positive $(n-1)$ -th coordinate, then for any $y \in V(\Delta_n)$, either $y_n \geq 1$ or $y_{n-1} = y_n = 0$. Since $\sum_{i \in [n-2]} |y_i| \leq r - 2$, we conclude that $d(y, p) \leq r$. Therefore, Δ_n is a cone with apex p . Hence, Δ_n is contractible, and thus $\Gamma_n^{\alpha, r}$ is contractible.

Case 2: For every $y \in V(\Delta_n)$, $y_n = 0$.

In this case, Δ_n is homotopy equivalent to an induced subcomplex Δ_{n-1} of $\Gamma_{n-1}^{\beta, r}$, for some $1 \leq \beta \leq \text{Card}(V(\mathcal{G}_m^{n-1}))$, where every $x \in V(\Delta_{n-1})$ satisfies the following: $|x_i| < \lfloor \frac{r}{2} \rfloor$ for all $i \in [n-1]$, and for any set $S \subseteq [n-1]$ with $\text{Card}(S) = n-3$, we have $\sum_{i \in S} |x_i| \leq r - 2$.

Now, we have the following subcases:

Subcase (i): There exists an element in $V(\Delta_{n-1})$ with a positive $(n-1)$ -th coordinate.

Suppose there exists an element in $V(\Delta_{n-1})$ with a positive $(n-2)$ -th coordinate. Let $u = (0, 0, \dots, 0, 1) \in V(\Delta_{n-1})$. Consider $v = (0, 0, \dots, 0, 1, 1) \in \mathbb{Z}^{n-1}$. We have $v \succ \mathbf{0}$ and $v \in V(\Delta_{n-1})$. Using a similar computation as in Case (1), we get that Δ_{n-1} is a cone with apex at v . Therefore, Δ_{n-1} is contractible. Since $\Gamma_n^{\alpha, r} \simeq \Delta_n \simeq \Delta_{n-1}$, $\Gamma_n^{\alpha, r}$ is contractible.

If there is no element in $V(\Delta_{n-1})$ with a positive $(n-2)$ -th coordinate, then for any $y \in V(\Delta_{n-1})$, either $y_{n-1} \geq 1$ or $y_{n-2} = y_{n-1} = 0$. Thus, we conclude that $d(y, u) \leq r$ for all $y \in V(\Delta_{n-1})$. Therefore, Δ_{n-1} is a cone with apex at u . Hence, Δ_{n-1} is contractible, and thus $\Gamma_n^{\alpha, r}$ is contractible.

Subcase (ii): For every $z \in V(\Delta_{n-1})$, $z_{n-1} = 0$.

In this subcase, we proceed similarly to Case (2) again. We get that Δ_{n-1} is homotopy equivalent to an induced subcomplex Δ_{n-2} of $\Gamma_{n-2}^{\gamma, r}$, for some $1 \leq \gamma \leq \text{Card}(V(\mathcal{G}_m^{n-2}))$, where every $x \in V(\Delta_{n-2})$ satisfies the following: $|x_i| < \lfloor \frac{r}{2} \rfloor$ for all $i \in [n-2]$, and for any set $S \subseteq [n-2]$ with $\text{Card}(S) = n-4$, we have $\sum_{i \in S} |x_i| \leq r - 2$.

By repeatedly applying Case (1) and Case (2) a finite number of times, we obtain

$$\Gamma_n^{\alpha, r} \simeq \Delta_n \simeq \Delta_{n-1} \simeq \dots \simeq \Delta_2,$$

Where, for every $x \in V(\Delta_2)$, we have $|x_i| < \lfloor \frac{r}{2} \rfloor$ for all $i \in [2]$. Since Δ_2 is contractible from Case (i) of the proof of Theorem 3.15, Δ_n is contractible, and thus $\Gamma_n^{\alpha, r}$ is contractible. This completes the proof.

Theorem 4.15 is a generalization from $\mathcal{VR}(\{0, 1\}^n; 2)$ to $\mathcal{VR}(\mathbb{Z}^n; 2)$. It establishes that while $\mathcal{VR}(\{0, 1\}^n; 2)$ is homotopy equivalent to a wedge sum of finitely many $\mathbb{S}^3$'s ([1]), the complex $\mathcal{VR}(\mathbb{Z}^n; 2)$ is homotopy equivalent to a wedge sum of countably infinite copies of $\mathbb{S}^3$'s.

An extension from $\mathcal{VR}(\{0, 1\}^n; 2)$ to $\mathcal{VR}(\{0, 1\}^n; 3)$ was carried out in [26, 14]. The authors in [14] proved that $\mathcal{VR}(\{0, 1\}^n; 3)$ is homotopy equivalent to a wedge sum of finite copies of $\mathbb{S}^4$'s and $\mathbb{S}^7$'s. This naturally leads to the following questions.

Question 5.2. Fix $n \geq 4$. Is $\mathcal{VR}(\mathbb{Z}^n; 3)$ homotopy equivalent to a wedge sum of countably infinite copies of $\mathbb{S}^4$'s and $\mathbb{S}^7$'s?

Question 5.3. Let $2 < r < n$. Is $\mathcal{VR}(\mathbb{Z}^n; r)$ homotopy equivalent to wedge sum of spheres ?

It is known that the inclusion $\{0, 1\}^n \hookrightarrow \mathbb{Z}^n$ induces an injective homomorphism $\tilde{H}_i(\mathcal{VR}(\{0, 1\}^n; r); \mathbb{Z}) \longrightarrow \tilde{H}_i(\mathcal{VR}(\mathbb{Z}^n; r); \mathbb{Z})$, and $\tilde{H}_i(\mathcal{VR}(\{0, 1\}^n; r); \mathbb{Z}) \neq 0$ for $r < n$. We conjecture the following:

Conjecture 5.4. $\tilde{H}_i(\mathcal{VR}(\mathbb{Z}^n; r); \mathbb{Z}) \neq 0$ if and only if $\tilde{H}_i(\mathcal{VR}(\{0, 1\}^n; r); \mathbb{Z}) \neq 0$.

Since the complexes $\mathcal{VR}(\mathbb{Z}^n; r)$ are simply connected for $r \geq 2$ (Theorem 1.5), and the complexes $\mathcal{VR}(\{0, 1\}^n; r)$ are contractible for $r \geq n$ (they are just a simplex), the Conjecture 5.4 implies Conjecture 1.1.

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