# Geometric inequalities related to fractional perimeter: fractional Poincaré, isoperimetric, and boxing inequalities in metric measure spaces

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**Abstract**

In the setting of a complete, doubling metric measure space $(X, d, \mu)$ supporting a $(1,1)$-Poincaré inequality, we show that for all $0 < \theta < 1$, the following fractional Poincaré inequality holds for all balls $B$ and locally integrable functions $u$,

$$\fint_B |u - u_B| d\mu \le C(1-\theta)\,\mathrm{rad}(B)^\theta \fint_{\tau B} \int_{\tau B} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x),$$

where $C \ge 1$ and $\tau \ge 1$ are constants depending only on the doubling and $(1,1)$-Poincaré inequality constants. Notably, this inequality features the scaling constant $(1-\theta)$ present in the Bourgain-Brezis-Mironescu theory characterizing Sobolev functions via nonlocal functionals.

From this inequality, we obtain a fractional relative isoperimetric inequality as well as global and local versions of a fractional boxing inequality, each featuring the same scaling constant $(1-\theta)$ and defined in terms of the fractional $\theta$-perimeter, and prove equivalences with the above fractional Poincaré inequality. We also show that $(X, d, \mu)$ supports a $(1,1)$-Poincaré inequality if and only if the above fractional Poincaré inequality holds for all $\theta$ sufficiently close to 1.

Under the additional assumption of lower Ahlfors $Q$-regularity of the measure $\mu$, we additionally use the aforementioned results to establish global inequalities, in the form of fractional isoperimetric and fractional Sobolev inequalities, which also feature the scaling constant $(1-\theta)$. Moreover, we prove that such inequalities are equivalent with the lower Ahlfors $Q$-regularity condition on the measure.

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# Contents



# 1 Introduction

In Euclidean spaces $\mathbb{R}^n$ with $n \geq 2$, the classical isoperimetric inequality states that all bounded sets $E \subset \mathbb{R}^n$ with smooth boundary satisfy

$$|E|^{\frac{n-1}{n}} \leq \frac{1}{n|B_1|^{\frac{1}{n}}} P(E).$$

Here, $|E|$ denotes the $n$-dimensional Lebesgue measure of $E$, $B_1$ is a unit ball in $\mathbb{R}^n$, and the perimeter of $E$, denoted as $P(E)$, coincides with the $(n-1)$-dimensional Hausdorff measure of the boundary $\partial E$ in this case. Notably, equality holds if and only if the set $E$ is a ball. The isoperimetric inequality is shown to be equivalent to the Gagliardo-Nirenberg-Sobolev embedding inequality when $p = 1$ with the sharp constant by Federer and Fleming [22], and Maz'ya [48]. The embedding inequality states that for any $f \in BV(\mathbb{R}^n)$,

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n|B_1|^{\frac{1}{n}}} \|Df\|_{L^1(\mathbb{R}^n)}.$$

Furthermore, a local version of the isoperimetric equality is given by the relative isoperimetric inequality. Let $B_r = B_r(x) \subset \mathbb{R}^n$ be an open ball centered at $x$ with radius $r$ and $E$ be as above, we have

$$\min\{|E \cap B_r|, |B_r \setminus E|\}^{\frac{n-1}{n}} \leq C(n) P(E, B_r),$$

where $P(E, B_r)$ denotes the perimeter of $E$ in $B_r$ and $C(n)$ is a constant depending only on the dimension. The relative isoperimetric inequality is known to be equivalent to the Sobolev-Poincaré inequality that states for any $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f - f_{B_r}\|_{L^{\frac{n}{n-1}}(B_r)} \leq C(n) \|Df\|(B_r).$$

Here, $f_{B_r} = \frac{1}{|B_r|}\int_{B_r} f dx = \fint_{B_r} f dx$ and $\|Df\|$ denotes the total variation measure of $f \in BV(\mathbb{R}^n)$ [21, 49, 61]. It is not hard to see that the relative isoperimetric inequality implies a global isoperimetric inequality with a non-sharp constant when taking $B_r$ sufficiently large and the reverse also holds [23]. The study of the above inequalities plays a fundamental role in many areas, including geometry, potential theory, partial differential equations, and probability. These inequalities have been extended to Riemannian manifolds and general metric measure spaces.

Motivated by the developments and applications in harmonic analysis, functional analysis, partial differential equations, and calculus of variation, the fractional Sobolev functions, BV functions and nonlocal fractional perimeter functionals are extensively studied in the Euclidean spaces, see [7, 8, 9, 50, 15, 10, 11, 17, 20, 45] and the references therein. The fractional isoperimetric inequality and fractional Sobolev-Poincaré inequality and their applications have been widely explored in the literature and we list only a selection of relevant work here [54, 25, 26, 24, 35, 18, 55, 1, 36, 57, 52].

Let $\theta \in (0,1)$ and we define the fractional $\theta$-perimeter of $E$ in an open set $\Omega \subset \mathbb{R}^n$ by

$$P_\theta(E,\Omega) = \int_{E\cap\Omega}\int_{\Omega\setminus E} \frac{1}{|x-y|^{n+\theta}} dydx = \frac{1}{2}[\chi_E]_{W^{\theta,1}(\Omega)}, \tag{1.1}$$

where $W^{\theta,1}(\Omega)$ is the fractional Sobolev seminorm. We write $P_\theta(E)$ when $\Omega = \mathbb{R}^n$. The fractional isoperimetric inequality with sharp cosnstant in the Eucliean space is proved by Frank and Seiringer [25] that for any Borel set $E \subset \mathbb{R}^n$ with finite Lebesgue measure

$$|E|^{(n-\theta)/n} \le C_{n,\theta} P_\theta(E), \tag{1.2}$$

and the equality holds if and only if $E$ is a ball [25, (4.2)]. Recall the fractional Poincaré inequality in the Euclidean space shown by Bourgain, Brezis and Mironescu [8, Theorem 1] states that for any cube $Q \subset \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\left(\fint_Q |f-f_Q|^{\frac{n}{n-\theta}} dx\right)^{\frac{n-\theta}{n}} \le C(n)(1-\theta)\mathrm{diam}(Q)^\theta \fint_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\theta}} dydx, \tag{1.3}$$

which easily implies that

$$\fint_Q |f-f_Q| dx \le C(n)(1-\theta)\mathrm{diam}(Q)^\theta \fint_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\theta}} dydx. \tag{1.4}$$

For any Borel set $E \subset \mathbb{R}^n$ with finite Lebesgue measure, one can deduce fractional relative isoperimetric inequalities by letting $f = \chi_E$ and $f = \chi_{E^c}$ that

$$\min\{|E\cap Q|, |Q\setminus E|\}^{\frac{n-\theta}{n}} \le C(1-\theta) P_\theta(E,Q), \tag{1.5}$$

and

$$\min\{|E\cap Q|, |Q\setminus E|\} \le C(1-\theta)\,\mathrm{diam}(Q)^\theta P_\theta(E,Q). \tag{1.6}$$

We note that these Euclidean fractional Poincaré and relative isoperimetric inequalities feature the coefficient $(1-\theta)$ on the right-hand side. In the celebrated work of Bourgain, Brezis, and Mironescu [7], it was shown that the Sobolev energy is explicitly recovered by multiplying this coefficient to the fractional Sobolev seminorm and letting $\theta \to 1^-$. In particular, for the case of the fractional perimeter, one has

$$\lim_{\theta \to 1^-}(1-\theta)P_\theta(E) = C(n)P(E), \quad \text{and} \quad \lim_{\theta\to 0^+}\theta P_\theta(E) = n|B_1||E|, \tag{1.7}$$

by the work of Dávila [15], and Mazýa and Shaposhnikova [50]. This scaling coefficient also allows one to recover the classical Poincaré inequality, as in (2.6), from the fractional Poincaré inequality by sending $\theta$ to 1. Without the coefficient $(1-\theta)$, the right-hand side of (1.3) and (1.4) blow up for all non-constant function [9].

In the past three decades, there has been significant progress in the study of various aspects of first-order analysis on metric measure spaces including the theory of Sobolev spaces, BV functions, and their relation to variational problems and partial differential equations, see [30, 51, 5, 34] and the references therein. In recent years, fractional Sobolev-type spaces have also undergone extensive study in the metric setting, in particular related to traces of first-order Sobolev spaces, construction of nonlocal operators, and the study of nonlocal minimal surfaces, for a sampling see [27, 47, 6, 12, 41]. In the spirit of [7], characterizations of BV and Sobolev spaces via nonlocal functionals have also recently attracted great attention in the metric setting. In particular, if $(X, d, \mu)$ is a complete, doubling metric measure space supporting a $(1, 1)$-Poincaré inequality, see Section 2.1 and Section 2.2, then it was shown in [16] that

$$P(E,X) \lesssim \liminf_{\theta\to 1^-}(1-\theta)P_\theta(E,X) \le \limsup_{\theta\to 1^-}(1-\theta)P_\theta(E,X) \lesssim P(E,X) \tag{1.8}$$

for every $\mu$-measurable $E \subset X$. Here, $P(E,X)$ is the perimeter measure of $E$ in $X$, see Section 2.2, and for an open set $\Omega \subset X$ the fractional perimeter $P_\theta(E,\Omega)$ of $E$ in $\Omega$ is defined by

$$P_\theta(E,\Omega) := \int_{\Omega\cap E}\int_{\Omega\setminus E} \frac{2}{d(x,y)^\theta[\mu(B(y,d(x,y)) + \mu(B(x,d(x,y)))]}d\mu(y)d\mu(x).$$

For generalizations of this result, see also [28, 33, 43, 44, 56]. From (1.8), we see that in this setting, the $\theta$-perimeter recovers the classical perimeter, up to comparability, as $\theta \to 1^-$, provided one rescales by the crucial $(1-\theta)$ coefficient. Since in the Euclidean setting, the fractional Poincaré inequalities (1.4) and (1.3), and consequently the fractional relative isoperimetric inequalities (1.6) and (1.5), hold with such a scaling coefficient [8], it is natural to ask whether such Bourgain-Brezis-Mironescu-type fractional inequalities also hold in the metric setting. The main goal of this paper is to establish these inequalities and explore the relationships between them in the setting of a complete, doubling metric measure space supporting a $(1, 1)$-Poincaré inequality. This setting encompasses a wide variety of spaces including Euclidean and Muckenhoupt weighted Euclidean spaces, a large class of domains in $\mathbb{R}^n$ satisfying appropriate conditions, Riemannian manifolds with non-negative Ricci curvature, Carnot groups, and Laakso spaces, for example.

The first of our main results is the following, which generalizes (1.3) and (1.4):

**Theorem 1.1.** *Let $(X, d, \mu)$ be a complete, doubling metric measure space supporting a $(1, 1)$-Poincaré inequality. Let $0 < \theta < 1$ and $1 \leq q \leq Q_d/(Q_d - \theta)$, where $Q_d > 1$ is the relative lower mass bound exponent* (2.1)*. Then there exists constants $C \geq 1$ and $\tau \geq 1$ so that for all balls $B := B(x, r) \subset X$ and all $u \in L^1_{\mathrm{loc}}(X)$, we have*

$$\left(\fint_B |u - u_B|^q d\mu\right)^{1/q} \leq C(1-\theta)r^\theta \fint_{\tau B}\int_{\tau B} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x). \tag{1.9}$$

*Here, the constants $C$ and $\tau$ depend only on the doubling and $(1, 1)$-Poincaré inequality constants.*

If (1.9) holds, then we say that $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-*Poincaré inequality*, see Definition 2.4. We use this notation to emphasize the presence of the $(1 - \theta)$ coefficient and to differentiate this inequality from versions without, such as those in [19, 14]. Our proof of the above theorem is inspired by the recent work by Myyryläinen, Pérez, and Weigt [52]. In particular, we establish the above result by first proving an inequality in Lemma 3.3 which acts as an improvement of the standard relative isoperimetric inequality. See Remark 3.4 for more details.

As a complete, doubling metric measure space supporting a $(1, 1)$-Poincaré inequality admits a geodesic metric which is bilipschitz equivalent with the original metric, see for example [34, Corollary 8.3.16], it suffices to prove Theorem 1.1 under the assumption that $(X, d)$ is geodesic, see the discussion at the beginning of Section 4. As the proof of Theorem 1.1 shows, we may take $\tau = 1$ in (1.9) when $(X, d, \mu)$ is additionally assumed to be geodesic. This mirrors the classical result for complete, doubling metric measure spaces supporting a $(1, p)$-Poincaré inequality [31].

As a direct application of the fractional Poincaré inequality (1.9), we obtain a fractional relative isoperimetric inequality:

**Corollary 1.2.** *Let $(X, d, \mu)$ be a complete, doubling metric measure space supporting a $(1, 1)$-Poincaré inequality. Let $0 < \theta < 1$ and $1 \leq q \leq Q_d/(Q_d - \theta)$, where $Q_d > 1$ is the relative lower mass bound exponent* (2.1)*. Then there exist constants $C \geq 1$ and $\tau \geq 1$, so that for all measurable sets $E \subset X$, and all balls $B := B(x, r) \subset X$, we have that*

$$\left(\frac{\min\{\mu(B \cap E),\, \mu(B \setminus E)\}}{\mu(B)}\right)^{1/q} \leq C(1-\theta)r^\theta \frac{P_\theta(E, \tau B)}{\mu(\tau B)}. \tag{1.10}$$

*Here the constants $C$ and $\tau$ depend only on the doubling and $(1, 1)$-Poincaré inequality constants.*

If (1.10) holds, then we say that $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-*relative isoperimetric inequality*, see Definition 2.4. There are various results of isoperimetric inequalities [59, 42, 13, 53] in metric spaces, but to our best knowledge, the fractional isoperimetric inequality has not been previously investigated in the general setting.

In [42], it was shown that a doubling metric measure space supports a $(q, 1)$-Poincaré inequality if and only if it supports a $(q, 1)$-relative isoperimetric inequality, see (2.7). We show here that the analogous result holds in the fractional case.

**Theorem 1.3.** *Let $0 < \theta < 1$ and $q \geq 1$. Then $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-Poincaré inequality if and only if it supports a $(\theta, q, 1)_{BBM}$-relative isoperimetric inequality.*

The assumption of a (non-fractional) $(1, p)$-Poincaré inequality imposes significant geometric restrictions on a metric measure space $(X, d, \mu)$. For example, such spaces are necessarily quasiconvex, see [34], and for any $p_0 \geq 1$, there are examples of metric measure spaces which support $(1, p)$-Poincaré inequalities for all $p > p_0$, but not $p = p_0$, see [46] or for a simple examples, consider the bowtie spaces. In contrast, fractional Poincaré inequalities, without the $(1 - \theta)$ coefficient, do not carry such geometric information and hold under rather mild assumptions on the space. For example, an analog of the fractional Poincaré inequality (1.3), without the $(1 - \theta)$ coefficient, follows immediately if $\mu$ is doubling, see [19, Lemma 2.2]. Furthermore, using a Maz'ya-type truncation argument, it was also shown in [19, Theorem 3.4] that an analog of (1.4), with $n$ replaced by the relative lower mass bound exponent from (2.1) and without the constant $(1 - \theta)$, holds when $\mu$ is both doubling and reverse doubling, see Section 2.1. See also [14] for formulations of this inequality on John domains in the metric measure space setting. In order to obtain the improved versions of these inequalities in Theorem 1.1, we additionally assume that $(X, d, \mu)$ supports a $(1, 1)$-Poincaré inequality. In fact, this assumption is necessary to obtain the $(1 - \theta)$ coefficients in the fractional Poincaré inequality:

**Theorem 1.4.** *Let $(X, d, \mu)$ be a complete, doubling metric measure space and let $1 \leq q < \infty$. Suppose there exist constants $C, \tau \geq 1$ and a sequence of numbers $\theta_i \in (0, 1)$ converging to $1$, such that for all balls $B := B(x, r) \subset X$, and all $u \in L^1_{\mathrm{loc}}(X)$, we have*

$$\left( \fint_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C(1 - \theta_i) r^{\theta_i} \fint_{\tau B} \int_{\tau B} \frac{|u(x) - u(y)|}{d(x, y)^{\theta_i} \mu(B(x, d(x, y)))} \, d\mu(x) \, d\mu(y).$$

*Then $(X, d, \mu)$ supports a $(q, 1)$-Poincaré inequality.*

In this manner, the $(\theta, q, 1)_{BBM}$-Poincaré inequalities encode the same geometric information as the $(1, 1)$-Poincaré inequality. Moreover, since a $(q, 1)$-Poincaré inequality is equivalent to a $(q, 1)$-relative isoperimetric inequality by [42], and the same holds for their fractional counterparts by Theorem 1.3 above, it follows by Theorem 1.4 that the $(q, 1)$-relative isoperimetric inequality is recovered by its fractional counterpart. That is, if a complete doubling metric measure space supports a $(\theta, q, 1)$-relative isoperimetric inequality for all $1 - \varepsilon < \theta < 1$, then it supports a $(q, 1)$-relative isoperimetric inequality. Theorem 1.4 is established in Section 5 as a corollary to Theorem 5.3, which is proven for more general kernels in the fractional Poincaré inequality.

Next, we apply the fractional Poincaré inequality (1.9) to deduce a fractional boxing inequality. The (non-fractional) boxing inequality in the Euclidean setting, first proved by Gustin [29], states that every open bounded set $U \subset \mathbb{R}^n$ can be covered by a collection of balls $U \subset \bigcup_{i=1}^{\infty} B_{r_i}$ such that $\sum_{i=1}^{\infty} r_i^{n-1} \leq C(n) P(U)$, where $C(n)$ depends only on the dimension. Such a boxing inequality was generalized to the setting of doubling metric measure spaces supporting a $(1, 1)$-Poincaré inequality by Kinnunen, Korte, Shanmugalingam and Tuominen [40]. For more on the extension of the boxing inequality to Banach spaces and Riemannian manifolds, see the recent work [4].

In the Euclidean setting, Ponce and Spector [55] established the following fractional boxing inequality, involving the $\theta$-perimeter: given an open bounded set $U \subset \mathbb{R}^n$, one can find a collection of balls covering $U$ such that

$$\sum_{i=1}^{\infty} r_i^{n-\theta} \leq C(n)\theta(1-\theta)P_\theta(U),$$

for every $\theta \in (0,1)$. As $\theta \to 1^+$, one recovers the Gustin boxing inequality. Applying the $(\theta,1,1)_{BBM}$-Poincaré inequality obtained in Theorem 1.1, we deduce the following fractional boxing inequality in the metric setting, using ideas from the Euclidean proof in [55] and the proof in [40] of the non-fractional boxing inequality in the metric setting:

**Theorem 1.5.** *Let $(X,d,\mu)$ be a complete, doubling metric measure space supporting a $(1,1)$-Poincaré inequality, such that $\mu(X)=\infty$. Then there exist constants $C \geq 1$ and $\tau \geq 1$ such that for any open bounded set $U \subset X$, there exists a countable collection of disjoint balls $\{B(x_i,r_i)\}_{i\in\mathbb{N}}$ such that*

$$U \subset \bigcup_{i\in\mathbb{N}} B(x_i, 5\tau r_i), \qquad \frac{1}{2C_\mu} \leq \frac{\mu(B(x_i,r_i)\cap U)}{\mu(B(x_i,r_i))} < \frac{1}{2}$$

*for each $i \in \mathbb{N}$, and for each $0<\theta<1$, we have*

$$\sum_{i\in\mathbb{N}} \frac{\mu(B(x_i,r_i))}{r_i^\theta} \leq C\theta(1-\theta)P_\theta(U,X). \tag{1.11}$$

*Here, the constants $C$ and $\tau$ depend only on the doubling and $(1,1)$-Poincaré inequality constants.*

By using the $(\theta,1,1)_{BBM}$-relative isoperimetric inequality, obtained in Corollary 1.2, we also establish in Proposition 6.3 a local version of the fractional boxing inequality above. In this result, $U$ may be taken to be an arbitrary measurable subset of a given ball $B$, contained up to a set of measure zero in the union of the desired collection of inflated balls, and the fractional perimeter in (1.11) is given by $P_\theta(U,B)$. This local fractional boxing inequality in turn implies the $(\theta,q,1)_{BBM}$-Poincaré inequality, as shown in Corollary 6.4.

In order to obtain a fractional isoperimetric inequality generalizing (1.2), as well as an equivalent fractional Sobolev-type embedding, we need a further assumption on the measure $\mu$. Namely, we assume the measure $\mu$ is *lower Ahlfors $Q$-regular*, that is, there exists a constants $c_0>0$ and $Q>0$ such that for every $x\in X$ and $0<r<2\,\mathrm{diam}(X)$, we have

$$\mu(B(x,r)) \geq c_0 r^Q. \tag{1.12}$$

Note that when $X$ is bounded and $\mu$ is doubling, the measure $\mu$ is lower Ahlfors $Q_d$-regular, where $Q_d$ is the relative lower mass bound exponent of $\mu$, see (2.1).

By applying Theorem 1.5 with the additional assumption of lower Ahlfors $Q$-regularity, we obtain the following two inequalities, which we refer to as a $(\theta,\frac{Q}{Q-\theta},1)_{BBM}$*-isoperimetric inequality* and a $(\theta,\frac{Q}{Q-\theta},1)_{BBM}$*-Sobolev inequality* respectively, see Definition 2.4:

**Corollary 1.6.** *Let $(X, d, \mu)$ be a complete, doubling metric measure space supporting a $(1,1)$-Poincaré inequality. Assume that $\mu(X) = \infty$ and that $\mu$ satisfies the lower Ahlfors $Q$-regularity condition* (1.12) *for $Q > 1$. Then,*

*(i) there exists $C \geq 1$, depending only on the doubling, lower Ahlfors $Q$-regularity, and $(1,1)$-Poincaré inequality constants such that for all $0 < \theta < 1$ and all $\mu$-measurable $E \subset X$ such that $\mu(E) < \infty$, we have*

$$\mu(E)^{(Q-\theta)/Q} \leq C\theta(1-\theta)P_\theta(E, X). \tag{1.13}$$

*(ii) there exists $C \geq 1$, depending only on the doubling, lower Ahlfors $Q$-regularity, and $(1,1)$-Poincaré inequality constants such that for all $0 < \theta < 1$ and all $u \in L^1(X)$, we have*

$$\left(\int_X |u|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq C\theta(1-\theta)\int_X\int_X \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y)d\mu(x). \tag{1.14}$$

*Furthermore $(X, d, \mu)$ supports a $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-isoperimetric inequality if and only if it supports a $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-Sobolev inequality.*

**Remark 1.7.** Based on the asymptotic relation of fractional perimeter (1.7), it is clear that the order of $\theta$ is sharp in our Corollary 1.6. Note the sharp constant of this inequality proved by Frank and Seringer in the Euclidean space [25, (4.2)] is given by an integral formula and the order of $\theta$ near 0 and 1 is less obvious compared to our theorem. Then letting $\theta \to 1$ in (1.13), by (1.8), we have isoperimetric inequality in a metric measure space, that is, for every $\mu$-measurable $E \subset X$ with $\mu(E) < \infty$, one has

$$\mu(E)^{(Q-1)/Q} \leq CP(E, X), \tag{1.15}$$

where $C$ only depends on the doubling, lower Ahlfors $Q$-regularity, and $(1,1)$-Poincaré inequality constants. Moreover, (1.15) is the main result of Jiang and Koskela [37]. Under the conditions of Corollary 1.6, we find that $(X, d, \mu)$ satisfying local quantitative Lipschitz regularity for solutions to the Cheeger-Poisson equation, introduced by Jiang and Koskela, is not essential. For more details see [37].

If $\mu$ is additionally assumed to be lower Ahlfors $Q$-regular, then as an application of the local fractional boxing inequality Proposition 6.3, we also obtain versions of the $(\theta, q, 1)_{BBM}$-Poincaré and relative isoperimetric inequalities, with $q = Q/(Q-\theta)$, without the measure of the ball and radius terms, see Corollary 7.1. We refer to these inequalities respectively as *improved* $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$*-Poincaré* and *improved* $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$*-relative isoperimetric inequalities*, see Definition 2.4. Furthemore, these improved inequalities are also equivalent, by the arguments of Theorem 1.3.

It turns out that the lower Ahlfors $Q$-regularity condition is essential to obtain the $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-Sobolev inequality or the improved $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-Poincaré inequality (or their equivalent isoperimetric inequalities), as shown in the following theorem. This result is analogous to the results showing the equivalence of lower Ahlfors $Q$-regularity and the validity of Sobolev embedding theorems in Euclidean domains [32, 60], in metric spaces [2] and in quasi-metric spaces [3].

**Theorem 1.8.** *Let $(X, d, \mu)$ be a complete, doubling metric measure space supporting a $(1,1)$-Poincaré inequality such that $\mu(X) = \infty$. Then, the following conditions are equivalent.*

*(i) $\mu$ satisfies the lower Ahlfors $Q$-regularity condition (1.12) for $Q > 1$.*

*(ii) There exists a constant $C \geq 1$ such that for all $0 < \theta < 1$ and $u \in L^1(X)$, we have*

$$\left(\int_X |u|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq C\theta(1-\theta) \int_X \int_X \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

*(iii) There exist a constants $C \geq 1$ and $\tau \geq 1$ such that for all $0 < \theta < 1$, and all balls $B_0 := B(x_0, r_0) \subset X$ and $u \in L^1(X)$, we have*

$$\left(\int_{B_0} |u - u_{B_0}|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq C(1-\theta) \int_{\tau B_0} \int_{\tau B_0} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

*The lower Ahlfors $Q$-regularity constants in the implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) are independent of $\theta$, while the constants $C$ and $\tau$ in (ii) and (iii) may depend additionally on the doubling and $(1,1)$-Poincaré inequality constants for the implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii).*

A diagram of the main results proved in our paper on a complete doubling metric measure spaces is given below. The orange boxes contain the underlying conditions on the spaces, the green boxes contain functional inequalities while the red boxes include geometric inequalities.

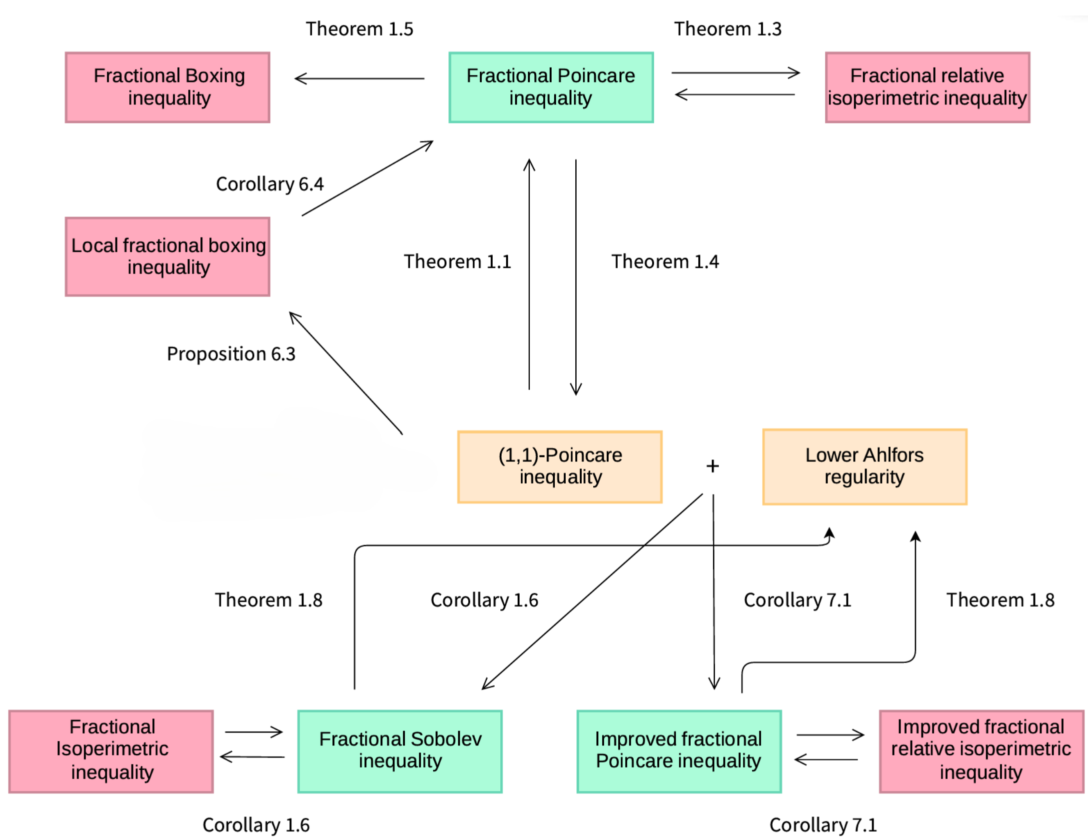


Figure 1: Inequality implications

The paper is organized as follows. We will review some background notions, definitions, and show some preliminary results in Section 2. In Section 3, we show several key lemmas which can be seen as an improved, nonlocal version of the classical relative isoperimetric inequality (2.7). In Section 4, we apply the key lemmas to show the fractional Poincaré inequality Theorem 1.1, fractional relative isoperimetric inequality Corollary 1.2, and the equivalence Theorem 1.3. The relationship between the fractional Poincaré inequality and $(1,1)$-Poincaré inequality given by Theorem 1.4 will be established in Section 5. The fractional boxing inequality Theorem 1.5 is proved in Section 6. Finally, we show the fractional isoperimetric and fractional Sobolev inequality Corollary 1.6, and the equivalence with the lower Ahlfors $Q$-regular condition Theorem 1.8 in Section 7.

# 2 Preliminaries

## 2.1 Doubling measures and length spaces

Throughout this paper, we assume that $(X,d,\mu)$ is a complete metric space equipped with a Borel regular outer measure $\mu$. We also assume that $\mu$ is *doubling*, that is, there exists a constant $C_\mu \geq 1$ such that for every $x \in X$ and $r > 0$, we have

$$0 < \mu(B(x,2r)) \leq C_\mu \, \mu(B(x,r)) < \infty.$$

By iterating the doubling condition, there exist constants $C \geq 1$ and $Q_d > 1$, depending only on $C_\mu$, such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C^{-1} \left(\frac{r}{R}\right)^{Q_d} \tag{2.1}$$

for all $x \in X$, $0 < r \leq R < \infty$, and $y \in B(x,R)$ [30, Lemma 4.7]. The constant $Q_d$ is referred to as the *relative lower mass bound exponent* of $\mu$.

A measure $\mu$ is said to be *reverse doubling* if there exists $s > 0$ and $C_s \geq 1$,

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq C_s \left(\frac{r}{R}\right)^{s} \tag{2.2}$$

for all $0 < r \leq R \leq 2 \operatorname{diam} X$, $x \in X$ and $y \in B(x,R)$. If $(X,d,\mu)$ is connected with $\mu$ doubling, then $\mu$ is also reverse doubling with $s$ and $C_s$ depending only on $C_\mu$, see for example [5, Corollary 3.8].

Throughout this paper, we at times let $C > 0$ denote a constant which, unless otherwise specified, depends only on the structural constants of the metric measure space, such as the doubling constant, or Poincaré inequality constants (see below), for example. Its precise value is not of interest to us, and may change with each occurrence, even within the same line. Furthermore, given quantities $A$ and $B$, we will often use the notation $A \simeq B$ to mean that there exists a constant $C \geq 1$ such that $C^{-1}A \leq B \leq CA$. Likewise, we use $A \lesssim B$ and $A \gtrsim B$ if the left and right inequalities hold, respectively.

A metric space $(X,d)$ is said to be *geodesic* if for any two points $x,y \in X$, there exists a curve $\gamma : [a,b] \to X$ joining $x$ to $y$ such that $\ell(\gamma) = d(x,y)$, where $\ell(\gamma)$ is the length of $\gamma$

with respect to $d$. A metric space $(X, d)$ is said to be a *length space* if for all $x, y \in X$, we have that

$$d(x, y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is over all curves joining $x$ to $y$.

Geodesic metric spaces equipped with a doubling measure satisfy the following Calderon-Zygmund-type decomposition for sets. This lemma is a generalization to the metric setting of [52, Lemma 2.1].

**Lemma 2.1.** *Let $(X, d, \mu)$ be a geodesic metric space, with $\mu$ a doubling measure. Let $B_0 := B(x_0, r_0) \subset X$ and $E \subset X$ be a $\mu$-measurable set. Assume that*

$$\frac{\mu(B_0 \cap E)}{\mu(B_0)} \leq \lambda$$

*holds for some $0 < \lambda < 1$. Then there exist countably many pairwise disjoint balls $B_i \subset B_0$, $i \in \mathbb{N}$ such that*

$$\begin{aligned}
&(i) \quad \mathrm{rad}(B_i) = 2^{-N_i} r_0 \textit{ for some } N_i \in \mathbb{N} \cup \{0\},\\
&(ii) \quad B_0 \cap E \subset \bigcup_i 5B_i \quad \textit{up to a set of } \mu\textit{-measure zero},\\
&(iii) \quad \lambda / C_\mu^2 \leq \frac{\mu(B_i \cap E)}{\mu(B_i)} \leq \lambda,
\end{aligned}$$

*where $C_\mu$ is the doubling constant of $\mu$.*

*Proof.* Let $B_0 := B(x_0, r_0) \subset X$. For any ball $B := B(x, r) \subset X$ and $k \in \mathbb{N}$, we define the collection of balls

$$\mathcal{D}_k(B) := \{B' \subset B : \mathrm{rad}(B') = 2^{-k} r_0\}.$$

We claim that for all $k \in \mathbb{N}$

$$B_0 = \bigcup_{B \in \mathcal{D}_k(B_0)} B. \tag{2.3}$$

To prove this claim, it suffices by induction to prove the case $k = 1$. To this end, let $x \in B_0$ and without loss of generality, assume that $d(x, x_0) \geq r_0/2$. Let $\delta := r_0 - d(x, x_0) \leq r_0/2$. Connecting $x$ to $x_0$ by a geodesic, there exists $y$ along this geodesic such that $d(x_0, y) = (r_0 - \delta)/2$. By the geodesic property, it follows that

$$d(x, y) = d(x, x_0) - d(x_0, y) = d(x, x_0) - (r_0 - \delta)/2 = d(x, x_0)/2 < r_0/2.$$

Thus, $x \in B(y, r_0/2)$. Moreover, if $z \in B(y, r_0/2)$, then by the triangle inequality, it follows that

$$d(z, x_0) \leq d(z, y) + d(y, x_0) < r_0/2 + (r_0 - \delta)/2 = r_0 - \delta/2 < r_0,$$

and so $B(y, r_0/2) \in \mathcal{D}_1(B_0)$. This proves the claim.

We now assume that $\mu(B_0 \cap E) < C_\mu^{-2} \lambda \mu(B_0)$, otherwise the collection consisting of only the ball $B_0$ satisfies the conclusion of the lemma. We define the following collection of balls

$$\mathcal{G}_1(B_0) := \{B \in \mathcal{D}_1(B_0) : \mu(B \cap E) \geq C_\mu^{-2} \lambda \mu(B)\}, \qquad \mathcal{B}_1(B_0) := \mathcal{D}_1(B_0) \setminus \mathcal{G}_1(B_0),$$

and set $\mathcal{G}_1 := \mathcal{G}_1(B_0)$ and $\mathcal{B}_1 := \mathcal{B}_1(B_0)$. As $\mu$ is doubling, it follows that for each $B \in \mathcal{G}_1$,

$$\mu(B \cap E) \leq \mu(B_0 \cap E) \leq C_\mu^{-2}\lambda\mu(B_0) \leq C_\mu^{-2}\lambda\mu(4B) \leq \lambda\mu(B).$$

Inductively, having defined $\mathcal{G}_k$ and $\mathcal{B}_k$, we define the following collection of balls for each $B \in \mathcal{B}_k$:

$$\mathcal{G}_{k+1}(B) := \{B' \in \mathcal{D}_{k+1}(B) : \mu(B' \cap E) \geq C_\mu^{-2}\lambda\mu(B')\}, \qquad \mathcal{B}_{k+1}(B) := \mathcal{D}_{k+1}(B) \setminus \mathcal{G}_{k+1}(B).$$

We then set $\mathcal{G}_{k+1} := \bigcup_{B \in \mathcal{B}_k} \mathcal{G}_{k+1}(B)$ and $\mathcal{B}_{k+1} := \bigcup_{B \in \mathcal{B}_k} \mathcal{B}_{k+1}(B)$. If $B' \in \mathcal{G}_{k+1}$, then there exists $B \in \mathcal{B}_k$ such that $B' \subset B$. By doubling, it then follows that

$$\mu(B' \cap E) \leq \mu(B \cap E) \leq C_\mu^{-2}\lambda\mu(B) \leq \lambda\mu(B'). \tag{2.4}$$

Let $\mathcal{G} := \bigcup_{k \in \mathbb{N}} \mathcal{G}_k$. We claim that $B_0 \cap E \subset \bigcup_{B \in \mathcal{G}} B$ up to a set of $\mu$-measure zero. Indeed, if $x \in (B_0 \cap E) \setminus \bigcup_{B \in \mathcal{G}} B$, then inductively using the claim (2.3) above, it follows that for each $k \in \mathbb{N}$, there exists $B_k \in \mathcal{B}_k$ such that $x \in B_k$. Since $\mu(B_k \cap E) < C_\mu^{-2}\lambda\mu(B_k)$, it follows that

$$\left| \fint_{B_k} \chi_E d\mu - 1 \right| = 1 - \frac{\mu(E \cap B_k)}{\mu(B_k)} > 1 - \frac{1}{C_\mu^2}$$

for every $k$. On the other hand, by the fact that $\mu$ is a doubling measure, one has

$$\left| \fint_{B_k} \chi_E d\mu - 1 \right| \lesssim \fint_{B(x, 2^{-k+1}r_0)} |\chi_E - 1| d\mu,$$

and the right hand side goes to zero as $k \to \infty$ whenever $x$ is a Lebesgue point of $E$ by Lebesgue differentiation theorem. This implies that $(B_0 \cap E) \setminus \bigcup_{B \in \mathcal{G}} B$ belongs to a null set and the claim is proved.

Therefore, by the 5-covering lemma, there exists a pairwise disjoint, countable subcollection $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{G}$ such that

$$B_0 \cap E \subset \bigcup_{i \in \mathbb{N}} 5B_i,$$

up to a set of $\mu$-measure zero, which gives us $(ii)$. Finally, the collection $\{B_i\}_{i \in \mathbb{N}}$ satisfies $(i)$ and $(iii)$ by construction and (2.4). □

Length spaces equipped with a doubling meausure satisfy the following annular decay property:

**Lemma 2.2.** *[34, Proposition 11.5.3] Let $(X, d)$ be a length space equipped with a doubling measure $\mu$. Then there exist constants $C_A \geq 1$ and $0 < \beta \leq 1$, depending only on $C_\mu$, such that*

$$\mu(B(x, r) \setminus \overline{B}(x, (1 - \varepsilon)r)) \leq C_A \varepsilon^\beta \mu(B(x, r))$$

*for every $x \in X$, $r > 0$, and $0 < \varepsilon \leq 1$.*

## 2.2 Sets of finite perimeter, Poincaré inequalities, and isoperimetric inequalities

A non-negative Borel function $g : X \to [0, \infty]$ is an *upper gradient* of a function $u : X \to \overline{\mathbb{R}}$ if the following holds for all non-constant, compact, rectifiable curves $\gamma : [a, b] \to X$:

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma g\, ds,$$

whenever $u(\gamma(a))$ and $u(\gamma(b))$ are both finite, and $\int_\gamma g\, ds = \infty$ otherwise.

Complete metric spaces equipped with a doubling measure are proper, that is, closed and bounded subsets are compact. Thus, for any open set $\Omega \subset X$, we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ to be the space of functions which are Lipschitz in any open $\Omega' \Subset \Omega$. By $\Omega' \Subset \Omega$, we mean that $\overline{\Omega}'$ is a compact subset of $\Omega$. We define other local spaces of functions similarly.

For an open set $\Omega \subset X$ and $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the *total variation* of $u$ in $\Omega$ by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_\Omega g_{u_i} d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\},$$

where each $g_{u_i}$ is an upper gradient of $u_i$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(\Omega) : \Omega \subset X \text{ open}, A \subset \Omega\}. \tag{2.5}$$

A $\mu$-measurable set $E \subset X$ is said to be a *set of finite perimeter* if $\|D\chi_E\|(X) < \infty$. The perimeter of $E$ in $\Omega$ is denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

If $P(E, \Omega) < \infty$, then $P(E, \cdot)$ is a finite Radon measure on $\Omega$ by [51, Theorem 3.4].

For $1 \leq p, q < \infty$, we say that $(X, d, \mu)$ supports a $(q, p)$-*Poincaré inequality* if there exist constants $C, \tau \geq 1$ such that for every ball $B \subset X$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$\left( \fint_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C \operatorname{rad}(B) \left( \fint_{\tau B} g^p \, d\mu \right)^{1/p}. \tag{2.6}$$

For $1 \leq q < \infty$, we say that $(X, d, \mu)$ supports a $(q, 1)$-*relative isoperimetric inequality* if there exist constants $C, \tau \geq 1$ such that for all balls $B \subset X$ and $\mu$-measurable sets $E \subset X$, we have

$$\left( \frac{\min\{\mu(B \cap E),\, \mu(B \setminus E)\}}{\mu(B)} \right)^{1/q} \leq C \operatorname{rad}(B) \frac{P(E, \tau B)}{\mu(\tau B)}. \tag{2.7}$$

By [42, Theorem 1.1], $(X, d, \mu)$ supports a $(q, 1)$-Poincaré inequality if and only if it supports a $(q, 1)$-relative isoperimetric inequality, with quantitatively related constants. Moreover, if $(X, d, \mu)$ is a doubling length space, supporting a $(q, 1)$-Poincaré inequality, then as shown in [31], we may take $\tau = 1$ to be the scaling constant in (2.6), and consequently in (2.7).

Furthermore, if $(X, d, \mu)$ is a doubling metric measure space supporting a $(1, 1)$-Poincaré inequality with scaling constant $\tau$, then by [5, Theorem 4.21] for example, $(X, d, \mu)$ also supports a $(\frac{Q}{Q-1}, 1)$-Poincaré inequality with scaling constant $2\tau$, where $Q := Q_d$ is the exponent from (2.1). Applying this inequality to Lipschitz approximating sequences, the following holds for all measurable sets $E \subset X$ and balls $B \subset X$:

$$\left(\frac{\min\{\mu(B\cap E),\, \mu(B\setminus E)\}}{\mu(B)}\right)^{(Q-1)/Q} \leq C\,\mathrm{rad}(B)\frac{P(E, 2\tau B)}{\mu(2\tau B)}. \tag{2.8}$$

Here, $C \geq 1$ depends only on the doubling constant and constants associated with the $(1,1)$-Poincaré inequality.

## 2.3 Fractional perimeter and fractional inequalities

Given $0 < \theta < 1$, a measurable set $E \subset X$, and an open set $\Omega \subset X$, we define the fractional perimeter of $E$ in $\Omega$ by

$$P_\theta(E,\Omega) := \int_{\Omega\cap E}\int_{\Omega\setminus E} \frac{2}{d(x,y)^\theta[\mu(B(x,d(x,y))) + \mu(B(y,d(x,y)))]} d\mu(y)d\mu(x). \tag{2.9}$$

Note that this definition agrees with the Euclidean definition given by (1.1), and the symmetry of the kernel yields $P_\theta(E,\Omega) = P_\theta(X\setminus E, \Omega)$. If $\mu$ is doubling then it also follows that

$$P_\theta(E,\Omega) \simeq \int_{\Omega\cap E}\int_{\Omega\setminus E} \frac{1}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y)d\mu(x).$$

We have the follow fractional coarea formula, see [58] and [55, Lemma 4.3] for such results the Euclidean setting:

**Lemma 2.3.** *Suppose that $\mu$ is $\sigma$-finite. Let $u \in L^1_{\mathrm{loc}}(X)$, and let $\Omega \subset X$ be an open set. Then for each $0 < \theta < 1$, we have that*

$$\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{d(x,y)^\theta[\mu(B(x,d(x,y))) + \mu(B(y,d(x,y)))]} d\mu(y)d\mu(x) = \int_{\mathbb{R}} P_\theta(\{u>t\},\Omega)dt.$$

*Proof.* For $(\mu\times\mu)$-a.e. $(x,y) \in \Omega\times\Omega$, we have that

$$|u(x)-u(y)| = \int_{\mathbb{R}} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|dt,$$

and so by Tonelli's theorem and the definition (2.9), we have that

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{d(x,y)^\theta[\mu(B(x,d(x,y))) + \mu(B(y,d(x,y)))]} d\mu(y)d\mu(x)\\
&\quad = \int_{\mathbb{R}}\int_\Omega\int_\Omega \frac{|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|}{d(x,y)^\theta[\mu(B(x,d(x,y))) + \mu(B(y,d(x,y)))]} d\mu(y)d\mu(x)dt\\
&\quad = \int_{\mathbb{R}} P_\theta(\{u>t\},\Omega)dt.
\end{aligned}$$

□

For the convenience of the reader, we conclude this section by listing all fractional Poincaré, isoperimetric, and Sobolev-type inequalities referenced in the introduction. These will be the principle objects of study in what follows.

**Definition 2.4.** *Let $0 < \theta < 1$ and $1 \le q < \infty$.*

1. *We say that $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-*Poincaré inequality *if there exists $C, \tau \ge 1$ such that for all balls $B := B(x, r) \subset X$ and $u \in L^1_{\mathrm{loc}}(X)$,*

$$\left( \fint_B |u - u_B|^q d\mu \right)^{1/q} \le C(1-\theta) r^\theta \fint_{\tau B} \int_{\tau B} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

2. *We say that $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-*relative isoperimetric inequality *if there exists $C, \tau \ge 1$ such that for all balls $B := B(x, r) \subset X$ and measurable $E \subset X$,*

$$\left( \frac{\min\{\mu(B \cap E),\, \mu(B \setminus E)\}}{\mu(B)} \right)^{1/q} \le C(1-\theta) r^\theta \frac{P_\theta(E, \tau B)}{\mu(\tau B)}.$$

3. *We say that $(X, d, \mu)$ supports an* improved $(\theta, q, 1)_{BBM}$-Poincaré inequality *if there exists $C, \tau \ge 1$ such that for all balls $B := B(x, r) \subset X$ and $u \in L^1_{\mathrm{loc}}(X)$,*

$$\left( \int_B |u - u_B|^q d\mu \right)^{1/q} \le C(1-\theta) \int_{\tau B} \int_{\tau B} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

4. *We say that $(X, d, \mu)$ supports an* improved $(\theta, q, 1)_{BBM}$-relative isoperimetric inequality *if there exists $C, \tau \ge 1$ such that for all balls $B := B(x, r) \subset X$ and measurable $E \subset X$,*

$$\min\{\mu(B \cap E),\, \mu(B \setminus E)\}^{1/q} \le C(1-\theta) P_\theta(E, \tau B).$$

5. *We say that $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-*Sobolev inequality *if there exists $C \ge 1$ such that for all $u \in L^1_{\mathrm{loc}}(X)$,*

$$\left( \int_X |u|^q d\mu \right)^{1/q} \le C\theta(1-\theta) \int_X \int_X \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

6. *We say that $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-*isoperimetric inequality *if there exists $C \ge 1$ such that for all measurable $E \subset X$ with $\mu(E) < \infty$,*

$$\mu(E)^{1/q} \le C\theta(1-\theta) P_\theta(E, X).$$

*In each of these definitions, the constants $C$ and $\tau$ can depend on structural data associated to $X$, but is independent of $\theta$, $u$, $B$, and $E$.*

# 3 Nonlocal improvement of the relative isoperimetric inequality

The main result of this section, Lemma 3.3, is a nonlocal, isoperimetric-type inequality, which will be used to prove the fractional Poincaré inequality Theorem 1.1 in the next section. As discussed in Remark 3.4, the inequality given in Lemma 3.3 acts as an improvement of the standard relative isoperimetric inequality (2.8). The key lemma used in establishing this inequality is Lemma 3.1, which allows us decompose a set near its boundary into a suitable collection of balls at a given, sufficiently small scale.

The lemmas of this section are metric setting analogs of [52, Lemmas 3.1, 3.2, 3.3], which were proven in the Euclidean setting and used to establish the corresponding fractional Poincaré inequalities there. Our proofs of these lemmas are similar, but key differences appear in the argument for Lemma 3.1. In our setting, we do not have access to a system of dyadic cubes (whose perimeters are controlled), and so we must take special care to control the overlap of certain collections of balls.

As mentioned in the introduction, it suffices to prove Theorem 1.1 under the additional assumption that $(X, d)$ is geodesic, see the discussion at the beginning of Section 4. As such, we will also additionally assume that $(X, d)$ is geodesic in order to establish the lemmas in this section. This assumption is quite useful, as it allows us to take the scaling constant of the $(1,1)$-Poincaré inequality, and hence $(1,1)$-relative isoperimetric ienquality, to be $\tau = 1$ [31], and it gives us access to the annular decay estimate Lemma 2.2, for example.

**Lemma 3.1.** *Let $(X, d, \mu)$ be a complete, geodesic metric measure space, with $\mu$ a doubling measure supporting a $(1,1)$-Poincaré inequality. Let $B_0 := B(x_0, r_0) \subset X$, and let $E \subset X$ be a measurable set such that*

$$\lambda \le \frac{\mu(B_0 \cap E)}{\mu(B_0)} \le 1 - \lambda$$

*for some $0 < \lambda \le 1/2$. Then there exist constants $C := C(C_\mu) \ge 1$ and $K := K(\lambda, C_\mu) \in \mathbb{N}$ so that for all $k \in \mathbb{N}$, $k \ge K$, there exists a countable collection of balls $\{B_i\}_{i\in\mathbb{N}}$, with $B_i \subset B_0$, $2^{-k} r_0 \le \operatorname{rad}(B_i) \le 2^{-k+4} r_0$, and $\{\frac{1}{60} B_i\}_{i\in\mathbb{N}}$ pairwise disjoint, such that*

$$(i) \quad \lambda/C \le \frac{\mu(B_i \cap E)}{\mu(B_i)} \le 1 - \lambda/C, \tag{3.1}$$

$$(ii) \quad 2^{-k}\mu(B_0) \lesssim \frac{1}{\lambda} \sum_i \mu(B_i), \tag{3.2}$$

*with the comparison constant in (ii) depending only on $C_\mu$ and the constants associated with the 1-Poincaré inequality. We may take $C = 16 C_\mu^{19}$ and*

$$K = \left\lceil \log_2 \left( 32 \, (4 C_A/\lambda)^{1/\beta} \right) \right\rceil,$$

*where $C_A \ge 1$ and $0 < \beta \le 1$ are the constants from Lemma 2.2, depending only on $C_\mu$.*

*Proof.* Let $k \in \mathbb{N}$ be such that $k \ge K$, with $K \in \mathbb{N}$ as defined above. Since $\mu$ is doubling, there exists a countable collection of balls $\{B(x_i, 2^{-k} r_0)\}_i$ covering $X$ such that $\{B(x_i, 2^{-k} r_0/5)\}_i$ is pairwise disjoint. For each $i$, there exists $2^{-k} r_0 \le \rho_i < 2^{-k+1} r_0$, by [39, Lemma 6.2], such that

$$P(B(x_i, \rho_i), X) \simeq \frac{\mu(B(x_i, \rho_i))}{2^{-k} r_0}. \tag{3.3}$$

Let $I := \{i \in \mathbb{N} : B(x_i, \rho_i) \subset B_0\}$ and denote $\mathcal{B}_k := \{B(x_i, \rho_i)\}_{i\in I}$. We note that the collection $\{\frac{1}{10}B\}_{B\in\mathcal{B}_k}$ is pairwise disjoint.

Let $C_0 := 16C_\mu'^{16}$, and define the subcollections

$$\mathcal{D}_k^1 := \left\{B \in \mathcal{B}_k : \frac{\mu(B\cap E)}{\mu(B)} \geq \lambda/C_0\right\}$$

and

$$\mathcal{D}_k^2 := \left\{B \in \mathcal{B}_k : \frac{\mu(B\setminus E)}{\mu(B)} \geq \lambda/C_0\right\}.$$

For ease of notation, we set

$$A_1 := \bigcup_{B\in\mathcal{D}_k^1} B, \quad A_2 := \bigcup_{B\in\mathcal{D}_k^2} B, \quad A_1' := \bigcup_{B\in\mathcal{D}_k^1\setminus\mathcal{D}_k^2} B, \quad A_2' := \bigcup_{B\in\mathcal{D}_k^2\setminus\mathcal{D}_k^1} B.$$

By the definition of $\mathcal{B}_k$, we have that $B(x_0, (1-2^{-k+2})r_0) \subset \bigcup_{B\in\mathcal{B}_k} B$. Thus, by Lemma 2.2 and by our choice of $K$ and $C_0$, we have that

$$\begin{aligned}
\mu\left(E\cap B_0\setminus A_1\right) &\leq \mu\left(B_0 \setminus \bigcup_{B\in\mathcal{B}_k} B\right) + \sum_{B\in\mathcal{B}_k\setminus\mathcal{D}_k^1} \mu(E\cap B)\\
&\leq \mu\left(B(x_0,r_0)\setminus B(x_0,(1-2^{-k+2})r_0)\right) + \lambda C_0^{-1}\sum_{B\in\mathcal{B}_k\setminus\mathcal{D}_k^1}\mu(B)\\
&\leq C_A(2^{-k+2})^\beta\mu(B_0) + \lambda C_\mu'^4 C_0^{-1}\sum_{B\in\mathcal{B}_k}\mu\left(\frac{1}{10}B\right)\\
&\leq (C_A(2^{-k+2})^\beta + \lambda C_\mu'^4 C_0^{-1})\mu(B_0)\\
&\leq \frac{\lambda}{2}\mu(B_0).
\end{aligned}$$

We also have for the complement $E^c$

$$\mu\left(E^c\cap B_0\setminus A_1\right) \leq \mu\left(E^c\cap B_0\right) \leq (1-\lambda)\,\mu(B_0),$$

and so combining these two estimates, we obtain

$$\mu\left(B_0\setminus A_1\right) \leq (1-\lambda/2)\,\mu(B_0).$$

Thus

$$\mu\left(A_1\right) \geq \frac{\lambda}{2}\mu(B_0). \tag{3.4}$$

By an analogous argument, replacing $\mathcal{D}_k^1$ with $\mathcal{D}_k^2$, we also have

$$\mu\left(A_2\right) \geq \frac{\lambda}{2}\mu(B_0). \tag{3.5}$$

In the case that

$$\mu\left(\bigcup_{B\in\mathcal{D}_k^1\cap\mathcal{D}_k^2} B\right) \geq \frac{\lambda}{8}\mu(B_0),$$

the collection of balls $\mathcal{D}_k^1 \cap \mathcal{D}_k^2$ satisfies the conclusion of the lemma with $C = 16C_\mu^{16}$, by the definition of $\mathcal{D}_k^1$ and $\mathcal{D}_k^2$. We therefore assume that

$$\mu\left(\bigcup_{B \in \mathcal{D}_k^1 \cap \mathcal{D}_k^2} B\right) < \frac{\lambda}{8}\mu(B_0).$$

*Claim:* The following holds for either $i = 1$ or $i = 2$:

$$\mu(A_i) \leq (1/2 + \lambda/4)\,\mu(B_0).$$

*Proof of claim:* Suppose that this inequality fails for both $A_1$ and $A_2$. We then have that

$$\begin{aligned}(1+\lambda/2)\mu(B_0) \leq \mu(A_1) + \mu(A_2) &\leq \mu(A_1') + \mu(A_2') + 2\mu\left(\bigcup_{B \in \mathcal{D}_k^1 \cap \mathcal{D}_k^2} B\right) \\ &\leq \mu(A_1' \cup A_2') + \mu(A_1' \cap A_2') + \frac{\lambda}{4}\mu(B_0) \\ &\leq (1+\lambda/4)\mu(B_0) + \mu(A_1' \cap A_2'). \end{aligned} \tag{3.6}$$

If $B \in \mathcal{D}_k^1 \setminus \mathcal{D}_k^2$ and $B' \in \mathcal{D}_k^2 \setminus \mathcal{D}_k^1$, then it follows that

$$\frac{\mu(B \cap E)}{\mu(B)} \geq (1 - \lambda/C_0) \quad \text{and} \quad \frac{\mu(B' \cap E)}{\mu(B')} < \lambda/C_0.$$

If $B \cap B' \neq \varnothing$, then $B' \subset 5B$, and so it follows from doubling that

$$\begin{aligned}\frac{\mu(B \cap B')}{\mu(B)} &= 1 - \frac{\mu(B \cap E)}{\mu(B)} + \frac{\mu(B \cap E \cap B')}{\mu(B)} - \frac{\mu(B \setminus (E \cup B'))}{\mu(B)} \\ &\leq \lambda/C_0 + \frac{\mu(B' \cap E)}{\mu(B)} \leq \lambda/C_0 + C_\mu^3 \frac{\mu(B' \cap E)}{\mu(B')} \leq 2\lambda C_\mu^3/C_0.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\mu(A_1' \cap A_2') \leq \sum_{B \in \mathcal{D}_k^1 \setminus \mathcal{D}_k^2} \mu(B \cap A_2') &\leq \sum_{B \in \mathcal{D}_k^1 \setminus \mathcal{D}_k^2} \sum_{B' \in \mathcal{D}_k^2 \setminus \mathcal{D}_k^1} \mu(B \cap B') \\ &\leq 2\lambda C_\mu^3 C_0^{-1} \sum_{B \in \mathcal{D}_k^1 \setminus \mathcal{D}_k^2} \sum_{B' \in \mathcal{D}_k^2 \setminus \mathcal{D}_k^1} \mu(B).\end{aligned}$$

Since $\mu$ is doubling and the collection $\{\frac{1}{10}B\}_{B \in \mathcal{B}_k}$ is pairwise disjoint, it follows that for each $B \in \mathcal{D}_k^1 \setminus \mathcal{D}_k^2$, there are at most $C_\mu^9$ balls in $\mathcal{D}_k^2 \setminus \mathcal{D}_k^1$ which intersect $B$. Thus, by our choice of $C_0$, we have that

$$\begin{aligned}\mu(A_1' \cap A_2') \leq 2\lambda C_\mu^{12} C_0^{-1} \sum_{\mathcal{D}_k^1 \setminus \mathcal{D}_k^2} \mu(B) \leq 2\lambda C_\mu^{16} C_0^{-1} \sum_{\mathcal{D}_k^1 \setminus \mathcal{D}_k^2} \mu\left(\frac{1}{10}B\right) &\leq 2\lambda C_\mu^{16} C_0^{-1} \mu(B_0) \\ &\leq \frac{\lambda}{8}\mu(B_0).\end{aligned}$$

Combining this inequality with (3.6) yields a contradiction, proving the claim.

We now assume that the claim holds with $i = 1$. In this case, we define the collection

$$\mathcal{C}_k := \{B \in \mathcal{D}_k^1 : \text{ there exists } B' \in \mathcal{B}_k \setminus \mathcal{D}_k^1 \text{ s.t. } 2B \cap B' \neq \varnothing,\, 6B \subset B_0\},$$

and we show that the collection $\{6B\}_{B\in\mathcal{C}_k}$ satisfies the conclusion of the lemma.

To this end, it follows from the definition of $\mathcal{C}_k$ that for all $B \in \mathcal{C}_k$, there exists $B' \in \mathcal{B}_k \setminus \mathcal{D}_k^1$ such that $B' \subset 6B$. Thus, by doubling, and the fact that $6B \subset 16B'$, we have that

$$\begin{aligned}\frac{\lambda}{C_\mu^3 C_0} \leq \frac{\mu(B\cap E)}{C_\mu^3 \mu(B)} \leq \frac{\mu(6B\cap E)}{\mu(6B)} &\leq \frac{\mu(6B\setminus B')}{\mu(6B)} + \frac{\mu(B'\cap E)}{\mu(B')} \\ &\leq \left(1 - \frac{\mu(B')}{\mu(6B)}\right) + \frac{\lambda}{C_0} \leq 1 - \frac{1}{C_\mu^4} + \frac{\lambda}{C_0} \leq 1 - \frac{1}{2C_\mu^4},\end{aligned} \tag{3.7}$$

where the last inequality follows from our choice of $C_0$. Hence, $\{6B\}_{B\in\mathcal{C}_k}$ satisfies $(i)$ with $C = 16C_\mu^{19}$.

By the relative isoperimetric inequality (2.7), it follows that

$$\min\{\mu((1-2^{-k+5})B_0 \cap A_1),\, \mu((1-2^{-k+5})B_0 \setminus A_1)\} \lesssim r_0 P\left(A_1, (1-2^{-k+5})B_0\right).$$

Note that as $(X, d)$ is geodesic, the scaling consant of the relative isoperimetric inequality (2.7) is taken here to be $\tau = 1$. If $x \in (\partial A_1) \cap (1-2^{-k+5})B_0$, then there exists $B \in \mathcal{D}_k^1$ such that $x \in \partial B$ and $6B \subset B_0$. Moreover, since $(1-2^{-k+2})B_0 \subset \bigcup_{B\in\mathcal{B}_k} B$, there exists $B' \in \mathcal{B}_k \setminus \mathcal{D}_k^1$ such that $x \in B'$. Therefore, $B \in \mathcal{C}_k$, and so we have that

$$(\partial A_1) \cap (1-2^{-k+5})B_0 \subset \bigcup_{B\in\mathcal{C}_k} \partial B.$$

From this fact, as well as (3.3), it follows from the previous inequality that

$$\begin{aligned}\min\{\mu((1-2^{-k+5})B_0 \cap A_1),\, \mu((1-2^{-k+5})B_0 \setminus A_1)\} &\lesssim r_0 \sum_{B\in\mathcal{C}_k} P(B, X) \\ &\lesssim r_0 \sum_{B\in\mathcal{C}_k} \frac{\mu(B)}{2^{-k} r_0} \\ &\leq 2^k \sum_{B\in\mathcal{C}_k} \mu(6B).\end{aligned} \tag{3.8}$$

By our choice of $K$ and (3.4), we have that

$$\begin{aligned}\mu((1-2^{-k+5})B_0 \cap A_1) \geq \mu(A_1) - \mu(B_0 \setminus (1-2^{-k+5})B_0) &\geq \frac{\lambda}{2}\mu(B_0) - C_A(2^{-k+5})^\beta \mu(B_0) \\ &\geq \frac{\lambda}{4}\mu(B_0).\end{aligned}$$

Likewise, by the claim and our choice of $K$, we have that

$$\begin{aligned}\mu((1-2^{-k+5})B_0 \setminus A_1) &\geq \mu(B_0 \setminus A_1) - \mu(B_0 \setminus (1-2^{-k+5})B_0) \\ &\geq (1/2 - \lambda/4)\mu(B_0) - C_A(2^{-k+5})^\beta \mu(B_0) \geq \frac{1}{4}\mu(B_0).\end{aligned}$$

Therefore, by (3.8), we have that

$$2^{-k}\mu(B_0) \lesssim \frac{1}{\lambda}\sum_{B\in\mathcal{C}_k}\mu(6B),$$

and so $\{6B\}_{B\in\mathcal{C}_k}$ satisfies $(ii)$.

In the case that the claim holds with $i=2$, we define the collection $\mathcal{C}_k$ to be

$$\mathcal{C}_k := \{B \in \mathcal{D}_k^2 : \text{ there exists } B' \in \mathcal{B}_k \setminus \mathcal{D}_k^2 \text{ s.t. } 2B\cap B' \neq \varnothing,\, 6B \subset B_0\},$$

and similarly show that the collection $\{6B\}_{B\in\mathcal{C}_k}$ satisfies the conclusion of the lemma. Indeed, by the same argument used to obtain (3.7) in the previous case, as well as our choice of $C_0$, it follows that for all $B\in\mathcal{C}_k$,

$$\frac{1}{2C_\mu^4} \leq \frac{1}{C_\mu^4} - \frac{\lambda}{C_0} \leq \frac{\mu(6B\cap E)}{\mu(6B)} \leq 1 - \frac{\lambda}{C_\mu^3 C_0},$$

and so $\{6B\}_{B\in\mathcal{C}_k}$ satisfies $(i)$ with $C = 16C_\mu^{19}$. Likewise, by (3.5), the claim, and by applying the relative isoperimetric inequality as above with respect to $A_2$ rather than $A_1$, we obtain

$$2^{-k}\mu(B_0) \lesssim \frac{1}{\lambda}\sum_{B\in\mathcal{C}_k}\mu(6B),$$

again using our choice of $K$. Thus, $(ii)$ is satisfied in this case, concluding the proof. $\square$

**Lemma 3.2.** *Let $(X,d,\mu)$ be a geodesic metric measure space with $\mu$ a doubling measure. Let $B_0 := B(x_0,r_0)\subset X$, $a\leq r_0/2$ and $0<\varepsilon\leq 1/2$. Let $B_1 := B(x_1,r_1)\subset B_0$ with*

$$r_1 \leq \frac{a}{2(32C_\mu^4 C_A)^{1/\beta}}.$$

*Then for any measurable set $E\subset X$ with*

$$\varepsilon \leq \frac{\mu(B_1\cap E)}{\mu(B_1)} \leq 1-\varepsilon, \tag{3.9}$$

*we have*

$$\mu(B_1) \leq \frac{4}{\varepsilon}\int_{B_1}\left|\chi_E(x) - \frac{\mu(A(x)\cap E)}{\mu(A(x))}\right| d\mu(x),$$

*where $A(x) = B_0\cap B(x,a)\backslash B(x,a/2)$.*

*Proof.* We first claim that for each $x\in B_1$, we have

$$\left|\frac{\mu(A(x)\cap E)}{\mu(A(x))} - \frac{\mu(A(x_1)\cap E)}{\mu(A(x_1))}\right| \leq \frac{1}{4}. \tag{3.10}$$

To prove this, for each $x\in B_1$, set

$$\widetilde{A}(x) := B(x,a)\setminus B(x,a/2).$$

We then have that

$$
\begin{aligned}
|\mu(A(x)\cap E)-\mu(A(x_1)\cap E)| &= |\mu(A(x)\cap E\setminus A(x_1))-\mu(A(x_1)\cap E\setminus A(x)| \\
&\le \mu(A(x)\setminus A(x_1))+\mu(A(x_1)\setminus A(x)) \\
&\le \mu(\widetilde{A}(x)\setminus \widetilde{A}(x_1))+\mu(\widetilde{A}(x_1)\setminus \widetilde{A}(x)).
\end{aligned}
$$

Since

$$
\widetilde{A}(x)\setminus \widetilde{A}(x_1)\subset \Big(B(x,a/2+d(x,x_1))\setminus B(x,a/2)\Big)\cup\Big(B(x,a)\setminus B(x,a-d(x,x_1))\Big),
$$

it follows from Lemma 2.2 and our assumption on $r_1$ that

$$
\begin{aligned}
&\mu(\widetilde{A}(x)\setminus \widetilde{A}(x_1)) \\
&\qquad\le C_A\left(\frac{d(x,x_1)}{a/2+d(x,x_1)}\right)^{\beta}\mu(B(x,a/2+d(x,x_1))+C_A\left(\frac{d(x,x_1)}{a}\right)^{\beta}\mu(B(x,a)) \\
&\qquad\le 2C_A\left(\frac{2r_1}{a}\right)^{\beta}\mu(B(x,a)).
\end{aligned}
$$

By a similar argument and doubling, it also follows that

$$
\mu(\widetilde{A}(x_1)\setminus \widetilde{A}(x))\le 2C_A\left(\frac{2r_1}{a}\right)^{\beta}\mu(B(x_1,a))\le 2C_\mu C_A\left(\frac{2r_1}{a}\right)^{\beta}\mu(B(x,a)).
$$

Therefore, we have that

$$
\begin{aligned}
|\mu(A(x)\cap E)-\mu(A(x_1)\cap E)| &\le \mu(A(x)\setminus A(x_1))+\mu(A(x_1)\setminus A(x)) \\
&\le 4C_\mu C_A\left(\frac{2r_1}{a}\right)^{\beta}\mu(B(x,a)).
\end{aligned}
$$

From this inequality, we then obtain

$$
\begin{aligned}
&|\mu(A(x)\cap E)\mu(A(x_1))-\mu(A(x_1)\cap E)\mu(A(x))| \\
&\qquad\le |\mu(A(x)\cap E)\mu(A(x_1))-\mu(A(x_1)\cap E)\mu(A(x_1))| \\
&\qquad\qquad+|\mu(A(x_1)\cap E)\mu(A(x_1))-\mu(A(x_1)\cap E)\mu(A(x))| \\
&\qquad=|\mu(A(x)\cap E)-\mu(A(x_1)\cap E)|\,\mu(A(x_1))+|\mu(A(x_1))-\mu(A(x))|\mu(A(x_1)\cap E) \\
&\qquad\le 8C_\mu C_A\left(\frac{2r_1}{a}\right)^{\beta}\mu(B(x,a))\mu(A(x_1)).
\end{aligned}
$$

We note that there exists $y\in A(x)$ so that $B(y,a/4)\subset A(x)$. Indeed, if $x\in B(x_0,r_0/2)$, then this follows by connectedness of $X$ and since $B(x,a)\subset B_0$. If $x\in B_0\setminus B(x_0,r_0/2)$, then we join $x$ to $x_0$ by a geodesic, and choose $y$ to be a point on this geodesic such that $d(x,y)=3a/4$. Therefore, in either case, we have by doubling that

$$
\mu(B(x,a))\le\mu(B(y,2a))\le C_\mu^3\mu(B(y,a/4))\le C_\mu^3\mu(A(x)).
$$

Combining this with the previous inequality, we then obtain

$$|\mu(A(x)\cap E)\mu(A(x_1)) - \mu(A(x_1)\cap E)\mu(A(x))| \leq 8C_\mu^4 C_A \left(\frac{2r_1}{a}\right)^\beta \mu(A(x))\mu(A(x_1)).$$

Our assumption on $r_1$ then gives us (3.10), proving the claim.

If $\mu(A(x_1)\cap E)/\mu(A(x_1)) \geq 1/2$, then (3.10) implies that $\mu(A(x)\cap E)/\mu(A(x)) \geq 1/4$. In this case, we have from (3.9) that

$$\mu(B_1) \leq \frac{1}{\varepsilon}\mu(B_1 \setminus E) = \frac{1}{\varepsilon}\int_{B_1\setminus E} 1\, d\mu \leq \frac{4}{\varepsilon}\int_{B_1\setminus E}\left|\chi_E(x) - \frac{\mu(A(x)\cap E)}{\mu(A(x))}\right| d\mu(x).$$

Likewise, if $\mu(A(x_1)\cap E)/\mu(A(x_1)) < 1/2$, then (3.10) implies that $1-\mu(A(x)\cap E)/\mu(A(x)) \geq 1/4$, in which case (3.9) yields

$$\mu(B_1) \leq \frac{1}{\varepsilon}\mu(B \cap E) = \frac{1}{\varepsilon}\int_{B_1\cap E} 1\, d\mu \leq \frac{4}{\varepsilon}\int_{B_1\cap E}\left|\chi_E(x) - \frac{\mu(A(x)\cap E)}{\mu(A(x))}\right| d\mu(x).$$

Therefore, the conclusion of the lemma holds in either case. □

**Lemma 3.3.** *Let $(X, d, \mu)$ be a complete geodesic metric measure space, with $\mu$ a doubling measure supporting a $(1,1)$-Poincaré inequality. Let $B := B(x_0, r) \subset X$ be a ball, $E \in X$ a measurable set, and $k \in \mathbb{N}$ be such that*

$$\frac{1}{2^{kQ}} \leq \frac{\mu(B\cap E)}{\mu(B)} \leq \frac{1}{2},$$

*where $Q := Q_d > 1$ the relative lower mass bound exponent (2.1). Then there exists $C \geq 1$ such that*

$$\left(\frac{\mu(B\cap E)}{\mu(B)}\right)^{\frac{Q-1}{Q}} \leq C2^k \fint_B \fint_{B\cap B(x,2^{-k}r)\setminus B(x,2^{-k-1}r)} |\chi_E(x) - \chi_E(y)| d\mu(y) d\mu(x). \tag{3.11}$$

*Here $C$ depends only on the doubling and $(1,1)$-Poincaré inequality constants.*

*Proof.* Since $\mu(B\cap E)/\mu(B) \leq 1/2$, we apply Lemma 2.1 to obtain a countable collection of pairwise disjoint balls $B_i \subset B$ such that

$(i)$ $\quad r_i := \mathrm{rad}(B_i) = 2^{-M_i}r$ for some $M_i \in \mathbb{N}\cup\{0\}$,

$(ii)$ $\quad B\cap E \subset \bigcup_i 5B_i$ up to a set of $\mu$-measure zero,

$(iii)$ $\quad \dfrac{1}{2^{k_0 Q+1}} \leq \dfrac{\mu(B_i\cap E)}{\mu(B_i)} \leq \dfrac{1}{2},$

where $k_0$ is the smallest positive integer with $2^{-k_0 Q} \leq C_\mu^{-2}$. Denote by $k_1$ the smallest positive integer such that

$$2^{-k_1} \leq \frac{1}{32(32C_\mu^4 C_A)^{1/\beta}}, \tag{3.12}$$

and let
$$K := \left\lceil \log_2 \left( 32(4C_A/2^{k_0Q+1})^{1/\beta} \right) \right\rceil .$$
We apply Lemma 3.1 with $\max\{k+k_1-M_i, K\} \in \mathbb{N}$ for $E$ on each $B_i$ to obtain a collection $\{B_{i,j}\}_{j=1}^{N_i}$ of balls, with $B_{i,j} \subset B_i$ and $\{\frac{1}{60}B_{i,j}\}_{j=1}^{N_i}$ pairwise disjoint, such that
$$2^{-\max\{k+k_1-M_i,K\}} r_i \le r_{i,j} := \mathrm{rad}(B_{i,j}) \le 2^{-\max\{k+k_1-M_i,K\}+4} r_i, \tag{3.13}$$
and such that
$$\frac{1}{2^{k_0Q+1}(16C_\mu^{19})} \le \frac{\mu(B_{i,j} \cap E)}{\mu(B_{i,j})} \le 1 - \frac{1}{2^{k_0Q+1}(16C_\mu^{19})}. \tag{3.14}$$
From this lemma, we also have that
$$2^{-\max\{k+k_1-M_i,K\}} \mu(B_i) \lesssim \sum_j \mu(B_{i,j}).$$
Using these properties of $B_i$ and $B_{i,j}$, as well as doubling, it then follows that
$$\begin{aligned}
\left(\frac{\mu(B\cap E)}{\mu(B)}\right)^{\frac{Q-1}{Q}} &\lesssim \left(\frac{\mu(B\cap E)}{\mu(B)}\right)^{-\frac{1}{Q}} \mu(B)^{-1} \sum_i \mu(B_i) \\
&= \mu(B)^{-1} \min\left\{ \left(\frac{\mu(B\cap E)}{\mu(B)}\right)^{-\frac{1}{Q}},\ 2^k \right\} \sum_i \mu(B_i) \\
&\le \mu(B)^{-1} \sum_i \min\left\{ \left(\frac{\mu(B_i\cap E)}{\mu(B)}\right)^{-\frac{1}{Q}},\ 2^k \right\} \mu(B_i) \\
&\le \mu(B)^{-1} \sum_i \min\left\{ 2^{k_0+1/Q} \left(\frac{\mu(B_i)}{\mu(B)}\right)^{-\frac{1}{Q}},\ 2^k \right\} \mu(B_i) \\
&\lesssim \mu(B)^{-1} \sum_i \min\left\{ 2^{k_0+1/Q+M_i},\ 2^k \right\} \mu(B_i) \\
&= 2^{k_0+1/Q+k+k_1+K} \mu(B)^{-1} \sum_i \min\left\{ 2^{M_i-k-k_1-K},\ 2^{-k_0-1/Q-k_1-K} \right\} \mu(B_i) \\
&\le 2^{k_0+1/Q+k+k_1+K} \mu(B)^{-1} \sum_i 2^{-\max\{k+k_1-M_i,\,K\}} \mu(B_i) \\
&\lesssim 2^k \mu(B)^{-1} \sum_i \sum_j \mu(B_{i,j}),
\end{aligned} \tag{3.15}$$
where in the last inequality, we recall that $k_0$, $k_1$, and $K$ depend only on $C_\mu$.

Letting $a = 2^{-k}r$, it follows from (3.12) and (3.13) that
$$\mathrm{rad}(B_{i,j}) \le \frac{a}{2(32C_\mu^4 C_A)^{1/\beta}}.$$
Thus, from (3.14), we apply Lemma 3.2 on $B_{i,j}$, with $\varepsilon = (2^{k_0Q+1}(16C_\mu^{19}))^{-1}$, to obtain
$$\mu(B_{i,j}) \lesssim \int_{B_{i,j}} \left| \chi_E(x) - \frac{\mu(A(x)\cap E)}{\mu(A(x))} \right| d\mu(x),$$

where $A(x) := B \cap B(x, 2^{-k}r)\backslash B(x, 2^{-k-1}r)$. We then obtain

$$\begin{aligned}
\sum_i \sum_j \mu(B_{i,j}) &\lesssim \sum_i \sum_j \int_{B_{i,j}} \left| \chi_E(x) - \frac{\mu(A(x)\cap E)}{\mu(A(x))} \right| d\mu(x) \\
&= \sum_i \sum_j \int_{B_{i,j}} \left| \chi_E(x) - \fint_{A(x)} \chi_E(y) d\mu(y) \right| d\mu(x) \\
&\le \sum_i \sum_j \int_{B_{i,j}} \fint_{A(x)} |\chi_E(x) - \chi_E(y)|\, d\mu(y) d\mu(x) \\
&\lesssim \sum_i \int_{B_i} \fint_{A(x)} |\chi_E(x) - \chi_E(y)|\, d\mu(y) d\mu(x) \\
&\le \int_B \fint_{A(x)} |\chi_E(x) - \chi_E(y)| d\mu(y) d\mu(x).
\end{aligned}$$

Here, the second to last inequality follows since the collection $\{B_{i,j}\}_{j=1}^{N_i}$ has bounded overlap, as $\{\frac{1}{60}B_{i,j}\}_{j=1}^{N_i}$ is pairwise disjoint. The last inequality follows from pairwise disjointness of $\{B_i\}_i$. Combining this estimate with (3.15) yields the desired inequality. □

**Remark 3.4.** In [52, Remark 3.4], the authors showed that in the Euclidean setting, the inequality (3.11) acts as a nonlocal improvement of the standard relative isoperimetric inequality (2.8). In this remark, we adapt their argument to show that the same holds true in the metric setting under the assumptions of the previous lemma. In particular, we show that for all measurable $E \subset X$ and $B := B(x_0, r) \subset X$, the following holds for sufficiently large $k \in \mathbb{N}$:

$$\begin{aligned}
\left( \frac{\min\{\mu(B\cap E),\, \mu(B\setminus E)\}}{\mu(B)} \right)^{(Q-1)/Q} &\le C 2^k \fint_B \fint_{A_k(x)} |\chi_E(x) - \chi_E(y)| d\mu(y) d\mu(x) \\
&\le Cr \frac{P(E, (1+2^{-k+2})B)}{\mu(B)}, \qquad (3.16)
\end{aligned}$$

where $A_k(x) = B \cap B(x, 2^{-k}r)\backslash B(x, 2^{-k-1}r)$, and with $C \ge 1$ depending only on the doubling constant and constants associated with the $(1,1)$-Poincaré inequality.

To prove this, we first note that

$$\fint_B \fint_{A_k(x)} |\chi_E(x) - \chi_E(y)| d\mu(y) d\mu(x) = \fint_B \fint_{A_k(x)} |\chi_{E^c}(x) - \chi_{E^c}(y)| d\mu(y) d\mu(x)$$

and

$$P(E, (1+2^{-k+2})B) = P(E^c, (1+2^{-k+2})B).$$

Furthermore, if either $\mu(B\cap E) = 0$ or $\mu(B\setminus E) = 0$, then (3.16) holds trivially. Therefore, replacing $E$ with its complement $E^c$ if necessary, we may assume without loss of generality that

$$0 < \frac{\mu(B\cap E)}{\mu(B)} \le \frac{1}{2}.$$

Thus, there exists $K \in \mathbb{N}$ so that for all $k \in \mathbb{N}$ with $k \geq K$, we have that

$$\frac{1}{2^{kQ}} \leq \frac{\mu(B \cap E)}{\mu(B)} \leq \frac{1}{2}.$$

Applying Lemma 3.3 then yields the first inequality of (3.16).

To prove the second inequality in (3.16), define

$$f(z) := \int_{B \cap B(z, 2^{-k}r)} \fint_{A_k(x)} \frac{|\chi_E(x) - \chi_E(y)|}{\mu(B(x, 2^{-k}r))} d\mu(y) d\mu(x).$$

By Tonelli's theorem, it then follows that

$$\int_X f(z) d\mu(z) = \int_B \fint_{A_k(x)} |\chi_E(x) - \chi_E(y)| d\mu(y) d\mu(x). \tag{3.17}$$

Let $E_1 := \{f > 2^{-1}\}$, and for each $i \in \mathbb{N}$, $i \geq 2$, let

$$E_i := \{f > 2^{-i}\} \setminus \bigcup_{z \in E_1 \cup \cdots \cup E_{i-1}} B(z, 2^{-k+2}r).$$

For $z \in E_i$, it then follows that

$$\begin{aligned}
2^{-i} < f(z) &\lesssim \frac{1}{\mu(B(z, 2^{-k}r))^2} \int_{B \cap B(z, 2^{-k+1}r)} \int_{B \cap B(z, 2^{-k+1}r)} |\chi_E(x) - \chi_E(y)| d\mu(y) d\mu(x) \\
&= \frac{2}{\mu(B(z, 2^{-k}r))^2} \mu(B \cap B(z, 2^{-k+1}r) \cap E) \mu(B \cap B(z, 2^{-k+1}r) \setminus E) \\
&\leq \frac{2}{\mu(B(z, 2^{-k}r))} \min\{\mu(B \cap B(z, 2^{-k+1}r) \cap E),\, \mu(B \cap B(z, 2^{-k+1}r) \setminus E)\} \\
&\leq \frac{2}{\mu(B(z, 2^{-k}r))} \min\{\mu(B(z, 2^{-k+1}r) \cap E),\, \mu(B(z, 2^{-k+1}r) \setminus E)\}.
\end{aligned}$$

Here, to obtain the second inequality, we have used the fact that $B(x, 2^{-k}r) \subset B(z, 2^{-k+1}r)$ for all $x \in B \cap B(z, 2^{-k}r)$, as well as the assumption that $X$ is geodesic and doubling to ensure that $\mu(A_k(x)) \simeq \mu(B(x, 2^{-k}r)) \simeq \mu(B(z, 2^{-k}r))$ for such $x$. See the proof of (4.4) for the precise argument of this last statement. Applying the relative isoperimetric inequality (2.7), we then obtain

$$\mu(B(z, 2^{-k}r)) \lesssim 2^{-k+i} r P(E, B(z, 2^{-k+1}r)). \tag{3.18}$$

By definition of $E_i$, it follows that

$$X = \{f = 0\} \cup \bigcup_i \bigcup_{z \in E_i} B(z, 2^{-k+2}r),$$

and by the 5-covering lemma, there exists a countable, pairwise disjoint subcollection $\{B_{i,j}\}_{j \in \mathbb{N}} \subset \{B(z, 2^{-k+2}r)\}_{z \in E_i}$ satisfying

$$\bigcup_{z \in E_i} B(z, 2^{-k+2}r) \subset \bigcup_j 5B_{i,j}.$$

Furthermore, setting $\mathcal{B}_i := \{\frac{1}{2}B_{i,j}\}_{j\in\mathbb{N}}$, we see from the definition of $E_i$ that the collections $\mathcal{B}_i$ and $\mathcal{B}_{i'}$ are pairwise disjoint whenever $i \neq i'$. Using (3.18), it then follows that

$$\begin{aligned}\int_X f d\mu \le \sum_i \int_{\bigcup_{z\in E_i} B(z,2^{-k}r)} f d\mu &\le \sum_i 2^{-i+1}\mu\left(\bigcup_{z\in E_i} B(z,2^{-k+2}r)\right)\\ &\lesssim \sum_i 2^{-i}\sum_j \mu\left(\frac{1}{2}B_{i,j}\right)\\ &\lesssim 2^{-k}r\sum_i\sum_j P\Big(E,\frac{1}{2}B_{i,j}\Big)\\ &\le 2^{-k}rP(E,(1+2^{-k+2})B).\end{aligned}$$

Here, the last inequality follows from the pairwise disjointness of the collection $\{\frac{1}{2}B_{i,j}\}_{i,j\in\mathbb{N}}$, as well as the fact that $\frac{1}{2}B_{i,j} \subset (1+2^{-k+2})B$ by the definition of $f$ and $E_i$. The desired result (3.16) then follows from this estimate and (3.17).

# 4 Fractional Poincaré inequality

In this section, we prove Theorem 1.1, Corollary 1.2, and the equivalence given by Theorem 1.3. Our proof of Theorem 1.1 is inspired by the proof of [52, Theorem 4.1], established in the Euclidean setting. In particular, a key step in the argument is the inequality given by Lemma 3.3.

As mentioned in the introduction, it suffices to prove Theorem 1.1 under the additional assumption that $(X,d)$ is geodesic; we restate Theorem 1.1 with this assumption below in Theorem 4.1. Indeed, if $(X,d,\mu)$ is a complete metric measure space, with $\mu$ a doubling measure supporting a $(1,1)$-Poincaré inequality, then $(X,d)$ is $L$-bilipschitz equivalent to $(X,\tilde{d})$, where $\tilde{d}$ is the length metric induced from $d$, and the $L$ depends only on the doubling and $(1,1)$-Poincaré inequality constants, see for example [34, Corollary 8.3.16]. Moreover, completeness, the doubling property of $\mu$, and support of a $(1,1)$-Poincaré inequality are all preserved under a bilipschitz change of metric, with constants depending only on the original data, and consequently $(X,\tilde{d})$ is geodesic. Note that the relative lower mass bound exponent $Q_d$ given by (2.1) is also preserved under a bilipschitz change of metric. Hence, given a ball $B_d(x,r)\subset X$ (with respect to $d$), $u\in L^1_{\mathrm{loc}}(X)$, and $1\le q\le Q_d/(Q_d-\theta)$, it follows from doubling and Theorem 4.1 that

$$\begin{aligned}\left(\fint_{B_d(x,r)}|u-u_{B_d(x,r)}|^q d\mu\right)^{1/q} &\lesssim \left(\fint_{B_{\tilde{d}}(x,Lr)}|u-u_{B_{\tilde{d}}(x,Lr)}|^q d\mu\right)^{1/q}\\ &\lesssim (1-\theta)r^\theta\fint_{B_{\tilde{d}}(x,Lr)}\int_{B_{\tilde{d}}(x,Lr)}\frac{|u(y)-u(z)|}{\tilde{d}(y,z)^\theta\mu(B_{\tilde{d}}(y,\tilde{d}(y,z)))}d\mu(z)d\mu(y)\\ &\lesssim (1-\theta)r^\theta\fint_{B_d(x,L^2r)}\int_{B_d(x,L^2r)}\frac{|u(y)-u(z)|}{d(y,z)^\theta\mu(B_d(y,d(y,z)))}d\mu(z)d\mu(y).\end{aligned}$$

Thus, Theorem 1.1 follows from Theorem 4.1, with constants $C\ge 1$ and $\tau = L^2$ depending only on the original doubling and $(1,1)$-Poincaré inequality constants.

**Theorem 4.1.** *Let $(X, d, \mu)$ be a complete, geodesic metric measure space, with $\mu$ a doubling measure supporting a $(1,1)$-Poincaré inequality. Then there exists a constant $C \geq 1$ such that for all $0 < \theta < 1$, $1 \leq q \leq Q_d/(Q_d - \theta)$, all balls $B_0 := B(x_0, r_0) \subset X$, and all $u \in L^1_{\mathrm{loc}}(X)$, we have*

$$\left( \fint_{B_0} |u - u_{B_0}|^q d\mu \right)^{1/q} \leq C(1-\theta) r_0^\theta \fint_{B_0} \int_{B_0} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

*Here, $Q_d > 1$ is the relative lower mass bound exponent* (2.1)*, and the constant $C$ depends only on the doubling and $(1,1)$-Poincaré inequality constants.*

*Proof.* Let $Q := Q_d$, and let $q = Q_d/(Q_d - \theta)$. It suffices to prove the result for this value of $q$, as the other cases then follow by Hölder's ienquality. For $\lambda > 0$, let

$$\Omega_\lambda := \{x \in B_0 : |u - u_{B_0}| > \lambda\},$$

and note that

$$\lambda^{q-1} \mu(\Omega_\lambda)^{(q-1)/q} \leq \left( \int_0^\lambda \mu(\Omega_t)^{1/q} dt \right)^{q-1}.$$

By Cavalieri's principle, we then have that

$$\begin{aligned}
\left( \int_{B_0} |u - u_{B_0}|^q d\mu \right)^{1/q} &= \left( q \int_0^\infty \lambda^{q-1} \mu(B_0 \cap \{|u - u_{B_0}| > \lambda\}) d\lambda \right)^{1/q} \\
&= \left( q \int_0^\infty \lambda^{q-1} \mu(\Omega_\lambda)^{(q-1)/q} \mu(\Omega_\lambda)^{1/q} d\lambda \right)^{1/q} \\
&\leq \left( q \int_0^\infty \left( \int_0^\lambda \mu(\Omega_t)^{1/q} dt \right)^{q-1} \mu(\Omega_\lambda)^{1/q} d\lambda \right)^{1/q} \\
&\leq q^{1/q} \left( \int_0^\infty \mu(\Omega_t)^{1/q} dt \right)^{(q-1)/q} \left( \int_0^\infty \mu(\Omega_\lambda)^{1/q} d\lambda \right)^{1/q} \\
&\leq q^{1/q} \int_0^\infty \mu(\Omega_\lambda)^{1/q} d\lambda \\
&\leq q^{1/q} \left( \int_0^\infty \mu(B_0 \cap \{u - u_{B_0} > \lambda\})^{1/q} \, d\lambda + \int_0^\infty \mu(B_0 \cap \{u_{B_0} - u > \lambda\})^{1/q} \, d\lambda \right) \\
&\leq 2 \int_{u_{B_0}}^\infty \mu(B_0 \cap \{u > \lambda\})^{1/q} \, d\lambda + 2 \int_{-\infty}^{u_{B_0}} \mu(B_0 \cap \{u < \lambda\})^{1/q} \, d\lambda. \qquad (4.1)
\end{aligned}$$

If $u$ is replaced with $-u$, then the two terms on the right-hand side of the above equality switch roles, and so it suffices to estimate the first term.

Denoting

$$m_u := m_u(B_0) = \inf\{t \in \mathbb{R} : \mu(B_0 \cap \{u > t\}) < \mu(B_0)/2\},$$

we have that

$$\begin{aligned}
&\int_{u_{B_0}}^{\infty} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda \\
&\qquad = \int_{u_{B_0}}^{\max\{m_u, u_{B_0}\}} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda + \int_{\max\{m_u, u_{B_0}\}}^{\infty} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda \\
&\qquad =: I + II. \qquad (4.2)
\end{aligned}$$

To estimate $I$, we may assume that $u_{B_0} < m_u$. By definition of $u_{B_0}$, it follows that

$$\int_{u_{B_0}}^{\infty} \mu(B_0 \cap \{u > \lambda\}) d\lambda = \int_{-\infty}^{u_{B_0}} \mu(B_0 \cap \{u < \lambda\} d\lambda,$$

and by the definition of $m_u$, we have that

$$\int_{u_{B_0}}^{m_u} \mu(B_0 \cap \{u > \lambda\}) d\lambda \geq \frac{\mu(B_0)}{2}(m_u - u_{B_0}).$$

Hence, it follows that

$$\begin{aligned}
I \leq \int_{u_{B_0}}^{m_u} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda &\leq (m_u - u_{B_0})\mu(B_0)^{1/q} \\
&\leq \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{u_{B_0}}^{m_u} \mu(\{B_0 \cap \{u > \lambda\}) d\lambda \\
&\leq \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{u_{B_0}}^{\infty} \mu(\{B_0 \cap \{u > \lambda\}) d\lambda \\
&= \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{-\infty}^{u_{B_0}} \mu(\{B_0 \cap \{u < \lambda\}) d\lambda \\
&\leq \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{-\infty}^{m_u} \mu(\{B_0 \cap \{u < \lambda\}) d\lambda \\
&\leq 2 \int_{-\infty}^{m_u} \mu(B_0 \cap \{u < \lambda\})^{1/q}\, d\lambda, \qquad (4.3)
\end{aligned}$$

where we have again used the definition of $m_u$ to obtain the last inequality.

For $\lambda < m_u$, set

$$K_\lambda := \left\lceil \log_2 \left( \left( \frac{\mu(B_0)}{\mu(B_0 \cap \{u < \lambda\})} \right)^{1/Q} \right) \right\rceil.$$

For $\lambda < m_u$ and $k \geq K_\lambda$, we then have that

$$\frac{1}{2^{kQ}} \leq \frac{\mu(B_0 \cap \{u < \lambda\})}{\mu(B_0)} \leq \frac{1}{2},$$

and so applying Lemma 3.3, we obtain

$$\mu(B_0 \cap \{u < \lambda\})^{\frac{Q-1}{Q}} \lesssim \frac{2^k}{\mu(B_0)^{1/Q}} \int_{B_0} \fint_{A_k(x)} |\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)| d\mu(y) d\mu(x),$$

where

$$A_k(x) := B_0 \cap B(x, 2^{-k}r_0) \setminus B(x, 2^{-k-1}r_0).$$

Multiplying $2^{-k(1-\theta)}$ to both sides and summing over $k \geq K_\lambda$, it then follows that

$$\sum_{k \geq K_\lambda} 2^{-k(1-\theta)} \mu(B_0 \cap \{u < \lambda\})^{\frac{Q-1}{Q}} \lesssim \sum_{k \geq K_\lambda} \frac{2^{k\theta}}{\mu(B_0)^{1/Q}} \int_{B_0} \fint_{A_k(x)} |\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)| d\mu(y) d\mu(x).$$

By our choice of $K_\lambda$, we have that

$$\sum_{k \geq K_\lambda} 2^{-k(1-\theta)} = \frac{2^{-K_\lambda(1-\theta)}}{1 - 2^{-(1-\theta)}} \geq \frac{2^{-(1-\theta)}}{1 - 2^{-(1-\theta)}} \left( \frac{\mu(B_0 \cap \{u < \lambda\})}{\mu(B_0)} \right)^{\frac{1-\theta}{Q}}$$
$$\geq \frac{1}{2(1-\theta)} \left( \frac{\mu(B_0 \cap \{u < \lambda\})}{\mu(B_0)} \right)^{\frac{1-\theta}{Q}},$$

and so combining this with the previous expression, we obtain

$$\mu(B_0 \cap \{u < \lambda\})^{1/q} \lesssim \frac{(1-\theta)}{\mu(B_0)^{(q-1)/q}} \sum_{k \geq K_\lambda} 2^{k\theta} \int_{B_0} \fint_{A_k(x)} |\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)| d\mu(y) d\mu(x).$$

We claim that for all $x \in B_0$ and $y \in A_k(x)$,

$$\mu(A_k(x)) \simeq \mu(B(x, d(x,y))). \tag{4.4}$$

Indeed, if $x \in B(x_0, r_0/2)$, then $B(x, 2^{-k}r_0) \subset B_0$, and so, as $X$ is connected, there exists $z \in A_k(x)$ such that $d(x,z) = 3(2^{-k}r_0)/4$. Setting $\widetilde{B} := B(z, 2^{-k-2}r_0)$, we then have that $\widetilde{B} \subset A_k(x)$. If $x \in B_0 \setminus B(x_0, r_0/2)$, then joining $x$ to $x_0$ by a geodesic $\gamma$, there exists $z$ in $\gamma$ such that $d(x,z) = 3(2^{-k}r_0)/4$. Setting $\widetilde{B} := B(z, 2^{-k-2}r_0)$, it follows that $\widetilde{B} \subset A_k(x)$ in this case as well, since $\gamma$ is a geodesic. For all $y \in A_k(x)$, it follows from doubling that in either case, we have

$$\mu(\widetilde{B}) \leq \mu(A_k(x)) \leq \mu(B(x, 2d(x,y)) \lesssim \mu(B(x, d(x,y))) \leq \mu(8\widetilde{B}) \lesssim \mu(\widetilde{B}).$$

This proves the claim.

Applying this claim to the previous estimate, we have that

$$\begin{aligned}
\mu(B_0 \cap \{u < \lambda\})^{1/q} & \\
&\lesssim \frac{(1-\theta)}{\mu(B_0)^{(q-1)/q}} \sum_{k \geq K_\lambda} 2^{k\theta}(2^{-k}r_0)^\theta \int_{B_0} \int_{A_k(x)} \frac{|\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x) \\
&= (1-\theta) \frac{r_0^\theta}{\mu(B_0)^{(q-1)/q}} \sum_{k \geq K_\lambda} \int_{B_0} \int_{A_k(x)} \frac{|\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x) \\
&\leq (1-\theta) \frac{r_0^\theta}{\mu(B_0)^{(q-1)/q}} \int_{B_0} \int_{B_0} \frac{|\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).
\end{aligned}$$

From (4.3), we then obtain

$$
\begin{aligned}
I &\lesssim \frac{(1-\theta)r_0^\theta}{\mu(B_0)^{(q-1)/q}} \int_{B_0}\int_{B_0} \left(\int_{-\infty}^{m_u} |\chi_{\{u<\lambda\}}(y) - \chi_{\{u<\lambda\}}(x)|d\lambda\right) \frac{d\mu(y)d\mu(x)}{d(x,y)^\theta \mu(B(x,d(x,y)))} \\
&\leq (1-\theta)\frac{r_0^\theta}{\mu(B_0)^{-1/q}} \fint_{B_0}\int_{B_0} \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y)d\mu(x). \qquad (4.5)
\end{aligned}
$$

To estimate $II$, we first have that

$$
II \leq \int_{m_u}^{\infty} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda. \qquad (4.6)
$$

Using the same argument following (4.3) used to estimate $I$, but replacing $\{u < \lambda\}$ for $\lambda < m_u$ with $\{u > \lambda\}$ for $\lambda > m_u$, we similarly obtain

$$
II \lesssim (1-\theta)\frac{r_0^\theta}{\mu(B_0)^{-1/q}} \fint_{B_0}\int_{B_0} \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y)d\mu(x).
$$

Combining this estimate with (4.5), (4.2), and (4.1) completes the proof. □

As a corollary to Theorem 1.1, we obtain the following fractional relative isoperimetric inequality, which is a fractional analog of (2.7).

*Proof of Corollary 1.2.* The conclusion follows by setting $u := \chi_E$ and applying Theorem 1.1:

$$
\begin{aligned}
\left(\frac{\min\{\mu(B\cap E),\, \mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} &\leq \left(2\frac{\mu(B\cap E)}{\mu(B)}\frac{\mu(B\setminus E)}{\mu(B)}\right)^{1/q} \\
&= \left(\fint_B \fint_B |u(x)-u(y)|^q d\mu(y)d\mu(x)\right)^{1/q} \\
&\leq \left(2^{q+1}\fint_B |u-u_B|^q d\mu\right)^{1/q} \\
&\lesssim (1-\theta)r^\theta \fint_{\tau B}\int_{\tau B} \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y)d\mu(x) \\
&\simeq (1-\theta)r^\theta \frac{P_\theta(E,\tau B)}{\mu(\tau B)}.
\end{aligned}
$$

□

Finally, we give the other implication from the relative fractional perimeter inequality to the fractional Poincaré inequality, which completes the proof of Theorem 1.3.

*Proof of Theorem 1.3.* The Corollary 1.2 shows that if $(X,d,\mu)$ supports a $(\theta,q,1)_{BBM}$-Poincaré inequality, then it supports a $(\theta,q,1)_{BBM}$-relative isoperimetric inequality. To prove the other direction, we fix a ball $B := B(x,r) \subset X$ and $u \in L^1_{\mathrm{loc}}(X)$. As in the proof of Theorem 1.1, it suffices to estimate the sum

$$
I + II := \int_{u_B}^{\max\{m_u,u_B\}} \mu(B\cap\{u>\lambda\})^{1/q}\, d\lambda + \int_{\max\{m_u,u_B\}}^{\infty} \mu(B\cap\{u>\lambda\})^{1/q}\, d\lambda.
$$

Since $\mu(B\cap\{u<\lambda\})\le\mu(B)/2$ for $\lambda<m_u$, it follows from (4.3), inequality (1.10), and the fractional coarea formula Lemma 2.3 that

$$\begin{aligned}
I &\le 2\int_{-\infty}^{m_u}\mu(B\cap\{u<\lambda\})^{1/q}d\lambda\\
&\lesssim (1-\theta)\frac{r^\theta}{\mu(B)^{(q-1)/q}}\int_{-\infty}^{m_u}P_\theta(\{u<\lambda\},\tau B)\\
&\lesssim (1-\theta)\frac{r^\theta}{\mu(B)^{(q-1)/q}}\int_{\mathbb{R}}P_\theta(\{-u>\lambda\},\tau B)d\lambda\\
&= (1-\theta)\frac{r^\theta}{\mu(B)^{1/q}}\fint_{\tau B}\int_{\tau B}\frac{|u(x)-u(y)|}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x). \qquad (4.7)
\end{aligned}$$

To estimate $II$, we note that that $\mu(B\cap\{u>\lambda\})\le\mu(B)/2$ for $\lambda>m_u$. Similar to the estimate for $I$, we then obtain

$$II\lesssim(1-\theta)\frac{r_0^\theta}{\mu(B)^{1/q}}\fint_{\tau B}\int_{\tau B}\frac{|u(x)-u(y)|}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x). \qquad (4.8)$$

Combining (4.7) and (4.8) with (4.1) and (4.2) then yields

$$\left(\fint_B|u-u_B|^q d\mu\right)^{1/q}\lesssim(1-\theta)r^\theta\fint_{\tau B}\int_{\tau B}\frac{|u(x)-u(y)|}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x). \qquad \square$$

# 5 Equivalence between Poincaré inequalities

In this section, we prove Theorem 1.4, which shows that if $(X,d,\mu)$ supports a $(\theta,q,1)_{BBM}$-Poincaré inequality for $\theta\to1^-$, then it supports a $(q,1)$-Poincaré inequality. First we essentially repeat some arguments from [43].

Let $\Omega\subset X$ denote an open set. Let $u$ be a function defined on $\Omega$. We define

$$\operatorname{Lip}_r u(x):=\sup_{y\in\Omega\cap B(x,r)\setminus\{x\}}\frac{|u(y)-u(x)|}{d(y,x)},\quad x\in\Omega,\quad r>0, \qquad (5.1)$$

and

$$\operatorname{Lip} u(x):=\limsup_{r\to0}\operatorname{Lip}_r(x). \qquad (5.2)$$

We consider the following kernels. We consider a sequence of nonnegative $\mu\times\mu$-measurable functions $\{\rho_i(x,y)\}_{i=1}^\infty$, $x,y\in X$, and a fixed constant $1\le C_\rho<\infty$ satisfying the following conditions:

(1) For every $x,y\in X$ with $0<d(x,y)\le1$, we have

$$\sum_{j=1}^\infty d'_{i,j}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\le\rho_i(x,y)\le\sum_{j=1}^\infty d_{i,j}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))} \qquad (5.3)$$

for numbers $d'_{i,j},d_{i,j}\ge0$ for which $\sum_{j=1}^\infty d'_{i,j}\ge C_\rho^{-1}$ and $\sum_{j=1}^\infty d_{i,j}\le C_\rho$ for all $i\in\mathbb{N}$.

(2) For all $\delta > 0$, we have

$$\lim_{i\to\infty}\left(\sup_{y\in\Omega}\int_{\Omega\setminus B(y,\delta)}\frac{\rho_i(x,y)}{d(x,y)}\,d\mu(x)+\sup_{x\in\Omega}\int_{\Omega\setminus B(x,\delta)}\frac{\rho_i(x,y)}{d(x,y)}\,d\mu(y)\right)=0. \tag{5.4}$$

In particular, we can take

$$\rho_i(x,y)=\frac{1-\theta_i}{d(x,y)^{\theta_i-1}\mu(B(x,d(x,y)))},$$

for which we can choose $d'_{i,j}=(1-\theta_i)2^{-j(1-\theta_i)}$ and $d_{i,j}=C_\mu(1-\theta_i)2^{(-j+1)(1-\theta_i)}$ where $\theta_i\in(0,1)$ is a sequence converging to 1. Indeed, for every $x,y\in X$ with $0<d(x,y)\le 1$, we now have

$$\begin{aligned}\sum_{j=1}^{\infty}d'_{i,j}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}&\le\frac{1-\theta_i}{d(x,y)^{\theta_i-1}\mu(B(y,d(x,y)))}\\&\le\sum_{j=1}^{\infty}d_{i,j}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}.\end{aligned}$$

Here

$$\sum_{j=1}^{\infty}2^{-j(1-\theta_i)}=\frac{2^{\theta_i-1}}{1-2^{\theta_i-1}}\ge\frac{2^{\theta_i-1}}{1-\theta_i}\ge\frac{1/2}{1-\theta_i},$$

and so

$$\sum_{j=0}^{\infty}d'_{i,j}\ge 1/2,$$

satisfying the first inequality given after (5.3). Similarly,

$$\sum_{j=1}^{\infty}2^{(-j+1)(1-\theta_i)}\le 2\int_0^2 t^{(1-\theta_i)-1}dt=\frac{2^{2-\theta_i}}{1-\theta_i}\le\frac{4}{1-\theta_i},$$

and so

$$\sum_{j=0}^{\infty}d_{i,j}\le 4C_\mu,$$

satisfying the second inequality given after (5.3). Finally, we estimate

$$\begin{aligned}\frac{\rho_i(x,y)}{d(x,y)}&=(1-\theta_i)\frac{1}{d(x,y)^{\theta_i}\mu(B(x,d(x,y)))}\\&\le C_\mu(1-\theta_i)\sum_{j\in\mathbb{Z}}2^{j\theta_i}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))},\end{aligned}$$

and so (the notation $\sum_{j\le -\log_2\delta}$ means that we sum over integers $j$ at most $-\log_2\delta$)

$$\begin{aligned}\int_{X\setminus B(y,\delta)}\frac{\rho_i(x,y)}{d(x,y)}\,d\mu(x) &\le C_\mu(1-\theta_i)\sum_{j\le -\log_2\delta}2^{j\theta_i}\\ &\le C_\mu(1-\theta_i)\frac{\delta^{-\theta_i}}{1-2^{-\theta_i}}\\ &\to 0 \quad \text{as } \theta_i\to 1 \text{ when } i\to\infty,\end{aligned}$$

and so (5.4) holds.

**Lemma 5.1.** *Let $u\in \mathrm{Lip}(X)$ and $0<R\le 1$, and suppose $U\subset X$ is open. Suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers that satisfy (5.3). Then*

$$\int_U\int_{B(y,R)}\frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\le C_\rho\int_U \mathrm{Lip}\,u_R(y)\,d\mu(y) \tag{5.5}$$

*for every $i\in\mathbb{N}$.*

*Proof.* We have

$$\begin{aligned}\int_U\int_{B(y,R)}\frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y) &\le \int_U\int_{B(y,R)}\mathrm{Lip}\,u_R(y)\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\ &\le C_\rho\int_U \mathrm{Lip}\,u_R(y)\,d\mu(y)\end{aligned}$$

by (5.3). □

**Theorem 5.2.** *Suppose $\Omega\subset X$ is open and let $u\in\mathrm{Lip}(\Omega)$. Suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers that satisfy (5.3) and (5.4). Then*

$$\limsup_{i\to\infty}\int_\Omega\int_\Omega\frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\le C_\rho\int_\Omega \mathrm{Lip}\,u\,d\mu.$$

*Proof.* Consider $0<R\le 1$. By a standard extension, can assume that $u\in\mathrm{Lip}(X)$. We have

$$\begin{aligned}&\int_\Omega\int_\Omega\frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\ &\quad\le \int_\Omega\int_{\Omega\setminus B(y,R)}\frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\ &\qquad+\int_\Omega\int_{B(y,R)}\frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y).\end{aligned}$$

For the first term, we estimate

$$
\begin{aligned}
&\int_\Omega \int_{\Omega\setminus B(y,R)} \frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad\le 2\int_\Omega \int_{\Omega\setminus B(y,R)} \frac{|u(x)|+|u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad\le 2\int_\Omega |u(y)| \int_{\Omega\setminus B(y,R)} \frac{\rho_i(x,y)}{d(x,y)}\,d\mu(x)\,d\mu(y)\\
&\qquad + 2\int_\Omega |u(x)| \int_{\Omega\setminus B(x,R)} \frac{\rho_i(x,y)}{d(x,y)}\,d\mu(y)\,d\mu(x) \quad \text{by Fubini}\\
&\quad\le 2\int_\Omega |u|\,d\mu \left(\sup_{y\in\Omega}\int_{\Omega\setminus B(y,R)} \frac{\rho_i(x,y)}{d(x,y)}\,d\mu(x) + \sup_{x\in\Omega}\int_{\Omega\setminus B(x,R)} \frac{\rho_i(x,y)}{d(x,y)}\,d\mu(y)\right)\\
&\quad\to 0
\end{aligned}
$$

as $i\to\infty$ by (5.4).

For the second term, from Lemma 5.1 we get for every $i\in\mathbb{N}$

$$
\int_\Omega \int_{B(y,R)} \frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y) \le C_\rho \int_\Omega \operatorname{Lip}_R u\,d\mu
$$

Passing first to the $\limsup_{i\to\infty}$ in the two terms and then sending $R\to 0$, we get, using also Lebesgue's dominated convergence theorem

$$
\limsup_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y) \le C_\rho \int_\Omega \operatorname{Lip} u\,d\mu.
$$

$\square$

**Theorem 5.3.** *Suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers that satisfy* (5.3) *and* (5.4)*, and suppose $\theta_i\in(0,1)$ with $\theta_i\to 1$ as $i\to\infty$, and let $1\le q<\infty$. Suppose there exist constants $C,\tau\ge 1$ such that for all $i\in\mathbb{N}$, all balls $B:=B(x,r)\subset X$, and all $u\in L^1_{\mathrm{loc}}(X)$, we have*

$$
\left(\fint_B |u-u_B|^q\,d\mu\right)^{1/q} \le C r^{\theta_i} \fint_{\tau B}\int_{\tau B} \frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y).
$$

*Then $X$ supports a $(q,1)$-Poincaré inequality.*

*Proof.* Let $u\in \operatorname{Lip}(X)$. Consider a ball $B(x,r)$. By Theorem 5.2 with the choice $\Omega = B(x,\tau r)$ we have

$$
\limsup_{i\to\infty} \fint_{B(x,\tau r)}\int_{B(x,\tau r)} \frac{|u(x)-u(y)|}{d(x,y)}\rho_i(x,y)\,d\mu(x)\,d\mu(y) \le C_\rho \fint_{B(x,\tau r)} \operatorname{Lip} u\,d\mu.
$$

It follows that

$$
\left(\fint_{B(x,r)} |u-u_{B(x,r)}|^q\,d\mu\right)^{1/q} \le C C_\rho r \fint_{B(x,\tau r)} \operatorname{Lip} u\,d\mu.
$$

Due to Keith [38, Theorem 2], it follows that $X$ supports a $(q,1)$-Poincaré inequality. $\square$

Theorem 1.4 is then proven with the following corollary:

**Corollary 5.4.** *Let $1 \le q < \infty$ and suppose there exist constants $C, \tau \ge 1$ and a sequence of numbers $\theta_i \in (0,1)$ converging to 1, such that for all balls $B := B(x,r) \subset X$, and all $u \in L^1_{\mathrm{loc}}(X)$, we have*

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \le C(1-\theta_i) r^{\theta_i} \fint_{\tau B} \int_{\tau B} \frac{|u(x) - u(y)|}{d(x,y)^{\theta_i} \mu(B(x, d(x,y)))} \, d\mu(x) \, d\mu(y).$$

*Then $X$ supports a $(q,1)$-Poincaré inequality.*

*Proof.* Apply Theorem 5.3 with the kernels

$$\rho_i(x,y) = \frac{1-\theta_i}{d(x,y)^{\theta_i - 1} \mu(B(x, d(x,y)))}$$

with $\theta_i \to 1$ as $i \to \infty$. The conditions (5.3) and (5.4) were verified for these kernels after they were introduced. □

# 6 Fractional boxing inequality

In this section we prove the fractional boxing inequality Theorem 1.5, as well as a local variant Proposition 6.3, under the assumptions of Theorem 1.1. Recall that as $(X,d,\mu)$ is doubling and connected under these assumptions, the reverse doubling condition holds. That is, there exist $s > 0$ and $C_s \ge 1$, depending only on the doubling constant, such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \le C_s \left(\frac{r}{R}\right)^s \tag{6.1}$$

for all $0 < r \le R \le 2 \operatorname{diam} X$, $x \in X$ and $y \in B(x,R)$.

We begin with the following lemma, which concerns the large-scale behavior of the fractional perimeter functional. This is a generalization to the metric setting of [55, Lemma 4.2].

**Lemma 6.1.** *Let $0 < \theta < 1$, and let $(X,d,\mu)$ be doubling and connected. Given $0 < \gamma < 1$, there exists a constant $C \ge 1$, depending only on $\gamma$ and the doubling constant such that*

$$\frac{\mu(B(x_0,R))}{R^\theta} \le C\theta \int_{B(x_0,R)\cap E} \int_{X \setminus E} \frac{1}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y) d\mu(x)$$

*for all measurable sets $E \subset X$, $x_0 \in X$, and $R > 0$ satisfying*

$$\frac{\mu(B(x_0, 2^{-1}R) \cap E)}{\mu(B(x_0, 2^{-1}R))} \ge \gamma \quad \text{and} \quad \frac{\mu(B(x_0, r) \cap E)}{\mu(B(x_0, r))} < \gamma \ \ \text{for all } r \ge R. \tag{6.2}$$

*Proof.* If $x \in B(x_0,R)$ and $y \in X \setminus B(x_0,2R)$, then $d(x,y) \le 2d(x_0,y)$. Thus, by (6.2) and doubling, we have that

$$\begin{aligned}
&\int_{B(x_0,R)\cap E}\int_{X\setminus E}\frac{1}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x)\\
&\qquad\ge \int_{B(x_0,R)\cap E}\int_{X\setminus(B(x_0,2R)\cup E)}\frac{1}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x)\\
&\qquad\gtrsim \mu(B(x_0,R)\cap E)\int_{X\setminus(B(x_0,2R)\cup E)}\frac{1}{d(x_0,y)^\theta\mu(B(x_0,d(x_0,y)))}d\mu(y)\\
&\qquad\ge \mu(B(x_0,R/2)\cap E)\int_{X\setminus(B(x_0,2R)\cup E)}\frac{1}{d(x_0,y)^\theta\mu(B(x_0,d(x_0,y)))}d\mu(y)\\
&\qquad\ge \gamma\mu(B(x_0,R/2))\int_{X\setminus(B(x_0,2R)\cup E)}\frac{1}{d(x_0,y)^\theta\mu(B(x_0,d(x_0,y)))}d\mu(y)\\
&\qquad\gtrsim \gamma\mu(B(x_0,R))\int_{X\setminus(B(x_0,2R)\cup E)}\frac{1}{d(x_0,y)^\theta\mu(B(x_0,d(x_0,y)))}d\mu(y).
\end{aligned}$$

Setting

$$\lambda_0 := \frac{1}{2}\left(\frac{1-\gamma}{2C_s}\right)^{1/s},$$

where $s, C_s$ come from the reverse doubling condition (6.1) and

$$A_i := B(x_0, 2R\lambda_0^{-i-1}) \setminus B(x_0, 2R\lambda_0^{-i})$$

for $i \in \mathbb{N}\cup\{0\}$, we then have

$$\begin{aligned}
&\int_{B(x_0,R)\cap E}\int_{X\setminus E}\frac{1}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x)\\
&\qquad\gtrsim \gamma\mu(B(x_0,R))\sum_{i=0}^\infty\int_{A_i\setminus E}\frac{1}{d(x_0,y)^\theta\mu(B(x_0,d(x_0,y)))}d\mu(y)\\
&\qquad\gtrsim \gamma R^Q\mu(B(x_0,R))\sum_{i=0}^\infty\frac{\lambda_0^{-iQ}}{\mu(B(x_0,2R\lambda_0^{-i-1}))}\int_{A_i\setminus E}\frac{1}{d(x_0,y)^{Q+\theta}}d\mu(y). \qquad (6.3)
\end{aligned}$$

Here, the last inequality follows from the fact that $d(x_0,y)\simeq R\lambda_0^{-i}$ for all $y \in A_i$.

Setting

$$f_i(y) := \frac{\chi_{A_i\setminus E}(y)}{d(x_0,y)},$$

it follows that for $\lambda > 0$,

$$\{y \in X : f_i(y) > \lambda\} = A_i \cap B(x_0, 1/\lambda)\setminus E.$$

Applying Cavalieri's principle, we obtain

$$\begin{aligned}
\int_{A_i\setminus E}\frac{1}{d(x_0,y)^{Q+\theta}}d\mu(y) &= (Q+\theta)\int_0^{\lambda_0^i/(2R)}\lambda^{Q+\theta-1}\mu(A_i\cap B(x_0,1/\lambda)\setminus E)d\lambda\\
&\ge Q\int_{\lambda_0^{i+1}/(2R)}^{\lambda_0^{i+1}/R}\lambda^{Q+\theta-1}\mu(B(x_0,1/\lambda)\setminus(B(x_0,2R\lambda_0^{-i})\cup E))d\lambda.
\end{aligned}$$

For $\lambda_0^{i+1}/(2R) \le \lambda \le \lambda_0^{i+1}/R$, it follows from (6.2), (6.1), and our choice of $\lambda_0$ that

$$\begin{aligned}
\mu(B(x_0,1/\lambda) \setminus (B(x_0,2R\lambda_0^{-i}) \cup E)) &\ge \mu(B(x_0,1/\lambda) \setminus B(x_0,2R\lambda_0^{-i}) - \mu(B(x_0,1/\lambda) \cap E) \\
&\ge \mu(B(x_0,1/\lambda)) \left(1 - \frac{\mu(B(x_0,2R\lambda_0^{-i}))}{\mu(B(x_0,1/\lambda))}\right) - \gamma\mu(B(x_0,1/\lambda)) \\
&\ge (1 - C_s(2R\lambda/\lambda_0^i)^s - \gamma)\mu(B(x_0,1/\lambda) \\
&\ge (1 - C_s(2\lambda_0)^s - \gamma)\mu(B(x_0,1/\lambda)) \\
&= \frac{1-\gamma}{2}\mu(B(x_0,1/\lambda)).
\end{aligned}$$

Therefore, by the previous expression, (6.3), and (6.1), we have that

$$\begin{aligned}
&\int_{B(x_0,R)\cap E} \int_{X\setminus E} \frac{1}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\
&\qquad \gtrsim \gamma(1-\gamma) R^Q \mu(B(x_0,R)) \sum_{i=0}^\infty \lambda_0^{-iQ} \int_{\lambda_0^{i+1}/(2R)}^{\lambda_0^{i+1}/(R)} \lambda^{Q+\theta-1} \frac{\mu(B(x_0,1/\lambda))}{\mu(B(x_0,2R\lambda_0^{-i-1}))} d\lambda \\
&\qquad \gtrsim \gamma(1-\gamma) R^Q \mu(B(x_0,R)) \sum_{i=0}^\infty \lambda_0^{-iQ} \int_{\lambda_0^{i+1}/(2R)}^{\lambda_0^{i+1}/(R)} \lambda^{Q+\theta-1} \left(\frac{\lambda_0^{i+1}}{2R\lambda}\right)^Q d\lambda \\
&\qquad \gtrsim \gamma(1-\gamma) \mu(B(x_0,R)) \sum_{i=0}^\infty \int_{\lambda_0^{i+1}/(2R)}^{\lambda_0^{i+1}/(R)} \lambda^{\theta-1} d\lambda \\
&\qquad = \gamma(1-\gamma)\lambda_0^\theta(1-2^{-\theta}) \frac{\mu(B(x_0,R))}{\theta R^\theta} \sum_{i=0}^\infty \lambda_0^{i\theta} \\
&\qquad = \gamma(1-\gamma)\lambda_0^\theta \frac{(1-2^{-\theta})}{(1-\lambda_0^\theta)} \frac{\mu(B(x_0,R))}{\theta R^\theta}.
\end{aligned}$$

This yields the desired result, since

$$\lim_{\theta\to 0} \frac{1-2^{-\theta}}{1-\lambda_0^\theta} = \frac{\log 2}{\log(1/\lambda_0)}. \qquad \square$$

We now prove the main result of this section, Theorem 1.5:

*Proof of Theorem 1.5.* Fix $x_0 \in U$. As $U$ is bounded and $\mu(X) = \infty$, there exists $R_0 > 0$ such that $U \subset B(x_0,R_0)$ and

$$\frac{\mu(B(x_0,R_0) \cap U)}{\mu(B(x_0,R_0))} \le 1/2.$$

For each $x \in U$, we have that $B(x_0,R_0) \subset B(x,2R_0)$. Therefore, for all $r \ge 2R_0$, we have that

$$\frac{\mu(B(x,r) \cap U)}{\mu(B(x,r))} \le \frac{\mu(B(x_0,R_0) \cap U)}{\mu(B(x_0,R_0))} \le 1/2.$$

Moreover, since $U$ is open, it follows that

$$\frac{\mu(B(x,r) \cap U)}{\mu(B(x,r))} = 1$$

for sufficiently small $r > 0$. Setting

$$\alpha_x := \sup\{r > 0 : \mu(B(x,r) \cap U) \geq \mu(B(x,r))/2\},$$

it then follows that $0 < \alpha_x \leq 2R_0$. We then choose $R_x > \alpha_x$ such that $R_x/2 \leq \alpha_x$ and

$$\frac{\mu(B(x, R_x/2) \cap U)}{\mu(B(x, R_x/2))} \geq 1/2. \tag{6.4}$$

Since $R_x > \alpha_x$, we also have that

$$\frac{\mu(B(x,r) \cap U)}{\mu(B(x,r))} < 1/2 \tag{6.5}$$

for all $r \geq R_x$. Note that $R_x \leq 4R_0$.

By doubling, (6.4), and (6.5), it follows that

$$\frac{1}{2} > \frac{\mu(B(x,R_x) \cap U)}{\mu(B(x,R_x))} \geq \frac{\mu(B(x, 2^{-1}R_x) \cap U)}{C_\mu \mu(B(x, 2^{-1}R_x))} \geq \frac{1}{2C_\mu}. \tag{6.6}$$

From this inequality and applying the $(\theta, 1, 1)_{BBM}$-relative isoperimetric inequality given by Corollary 1.2, we obtain

$$\begin{aligned}\frac{1}{2C_\mu}\mu(B(x,R_x)) \leq \mu(B(x,R_x) \cap U) &= \min\{\mu(B(x,R_x) \cap U),\, \mu(B(x,R_x) \setminus U)\}\\ &\leq C(1-\theta)R_x^\theta P_\theta(U, B(x, \tau R_x)),\end{aligned}$$

and so it follows that

$$\frac{\mu(B(x,R_x))}{R_x^\theta} \lesssim (1-\theta)\int_{B(x,\tau R_x)\cap U}\int_{B(x,\tau R_x)\setminus U}\frac{1}{d(z,y)^\theta \mu(B(z,d(z,y)))}d\mu(y)d\mu(z). \tag{6.7}$$

As the collection $\{B(x,\tau R_x)\}_{x\in U}$ has uniformly bounded radii, we apply the 5-covering lemma to obtain a countable, pairwise disjoint subcollection $\{B(x_i, \tau r_i)\}_{i\in\mathbb{N}}$ such that

$$U \subset \bigcup_{i\in\mathbb{N}} B(x_i, 5\tau r_i).$$

By pairwise disjointness of this subcollection and (6.7), we then obtain

$$\begin{aligned}\sum_{i\in\mathbb{N}}\frac{\mu(B(x_i,r_i))}{r_i^\theta} &\lesssim (1-\theta)\sum_{i\in\mathbb{N}}\int_{B(x_i,\tau r_i)\cap U}\int_{B(x_i,\tau r_i)\setminus U}\frac{1}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x)\\ &\leq (1-\theta)\int_U\int_{X\setminus U}\frac{1}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x).\end{aligned} \tag{6.8}$$

The estimates (6.4) and (6.5) also allow us to apply Lemma 6.1, with $E = U$, $x_0 = x_i$, $R = r_i$, and $\gamma = 1/2$, to obtain

$$\frac{\mu(B(x_i,r_i))}{r_i^\theta} \lesssim \theta\int_{B(x_i,r_i)\cap U}\int_{X\setminus U}\frac{1}{d(x,y)^\theta\mu(B(x,d(x,y)))}d\mu(y)d\mu(x).$$

Summing over $i \in \mathbb{N}$, we then have

$$\sum_{i \in \mathbb{N}} \frac{\mu(B(x_i, r_i))}{r_i^\theta} \lesssim \theta P_\theta(U, X).$$

Since $\min\{\theta, 1-\theta\} \leq 2\theta(1-\theta)$, combining this inequality with (6.8) yields (1.11). □

**Remark 6.2.** If we assume in addition that there exists $t > 0$ and $C_t \geq 1$ such that

$$C_t^{-1} R^t \leq \mu(B(x, R)) \tag{6.9}$$

for all $x \in X$ and $0 < R < 2\operatorname{diam}(X)$, i.e. $\mu$ is lower Ahlfors $t$-regular, then the conclusion of Theorem 1.5 holds under the weaker hypothesis that $\mu(U) < \infty$. Indeed, under this assumption, we have by (6.6) in the above proof that

$$R_x^t \leq C_t \mu(B(x, R_x)) \leq 2C_t C_\mu \mu(B(x, R_x) \cap U) \leq 2C_t C_\mu \mu(U) < \infty.$$

Thus, the radii $\{R_x\}_{x \in U}$ are uniformly bounded even if $U$ is unbounded. This allows us to apply the 5-covering lemma, and so the proof proceeds in the same manner. The assumption (6.9) is satisfied, for example, if in addition there exists a constant $c_0 > 0$ such that $\mu(B(x_0, 1)) \geq c_0$ for all $x_0 \in X$. In this case, as $(X, d, \mu)$ is doubling and connected (by the support of a 1-Poincaré inequality), $(X, d, \mu)$ is also reverse doubling, see (6.1). Applying this condition together with the lower bound on the measure of balls of unit radius gives (6.9).

If $(X, d, \mu)$ does not satisfy (6.9) and $U$ is unbounded, it may not be possible to cover $U$ with a bounded-overlap collection of balls $\{B_i\}_i$ for which $\mu(B_i \cap U)/\mu(B_i)$ is bounded away from one, as described in Theorem 1.5. For example, let $X \subset \mathbb{R}^2$ be the completion of the region of the upper half-plane bounded between the $x$-axis and the graph of $y = 1/x^2$, $x > 0$, and let $U = \{(x, y) \in X : x > 1\}$. Equipping $X$ with the Euclidean distance and $\mu = \mathcal{L}^2|_X$, we have that $U$ is unbounded, $\mu(U) < \infty$, and (6.9) does not hold. If $(x, y) \in U$ is contained in a ball $B$ such that $\mu(B \cap U)/\mu(B) < 1/2$, and if $x$ is sufficiently large, then $B$ must contain the set $\{(1, y) : 0 \leq y \leq 1\}$. Hence, it is not possible to obtain the cover described in Theorem 1.5.

Using the $(\theta, 1, 1)_{BBM}$-relative isoperimetric inequality obtained in Corollary 1.2, we now establish the following local version of the fractional boxing inequality:

**Proposition 6.3.** *Let $(X, d, \mu)$ be a complete metric measure space, with $\mu$ a doubling measure supporting a $(1,1)$-Poincaré inequality, and let $\kappa \in (0, 1)$. Then there exist constants $\tau \geq 1$ and $C \geq 1$ such that for any $\mu$-measurable set $U \subset X$ and $B_0 = B(x_0, r_0) \subset X$ satisfying*

$$\frac{\mu(U \cap B_0)}{\mu(B_0)} \leq \kappa,$$

*there exists a countable collection of pairwise disjoint balls $\{B_i\}_{i \in I \subset \mathbb{N}}$, with* $\operatorname{rad}(B_i) = 2^{-N_i} r_0$ *for some $N_i \in \mathbb{N} \cup \{0\}$, and a set $\mathcal{N} \subset U$ with $\mu(\mathcal{N}) = 0$ such that*

$$U \cap B_0 \subset \mathcal{N} \cup \bigcup_{i \in I} 5\tau B_i \quad \textit{and} \quad \bigcup_{i \in I} B_i \subset \tau B_0, \tag{6.10}$$

*and for all $0 < \theta < 1$, we have*

$$\sum_{i \in I} \frac{\mu(B_i)}{r_i^\theta} \le C(1-\theta) P_\theta(U \cap B_0, \tau B_0). \tag{6.11}$$

*Here, $\tau$ and $C$ depend only on $\kappa$, and the doubling and $(1,1)$-Poincaré inequality constants.*

*Proof.* We first prove the proposition under the additional assumption that $(X, d, \mu)$ is a geodesic metric measure space. Applying Lemma 2.1 with $E = U$, we obtain countably many pairwise disjoint balls $B_i \subset B_0$ , $i \in \mathbb{N}$ such that

$$\begin{aligned} &(i) \quad \mathrm{rad}(B_i) = 2^{-N_i} r_0 \text{ for some } N_i \in \mathbb{N} \cup \{0\}, \\ &(ii) \quad B_0 \cap U \subset \mathcal{N} \cup \bigcup_{i \in \mathbb{N}} 5B_i \\ &(iii) \quad \kappa / C_\mu^2 \le \frac{\mu(B_i \cap U)}{\mu(B_i)} \le \kappa, \end{aligned}$$

where $\mathcal{N} \subset U$ is a set of $\mu$-measure zero. Applying Corollary 1.2 on $B_i$ with $u = \chi_U$, it follows that

$$\begin{aligned} \frac{\mu(B_i)}{r_i^\theta} \le C_1(1-\theta) \max &\left\{ \frac{1}{(1-\kappa/C_\mu^2)\kappa/C_\mu^2}, \frac{1}{(1-\kappa)\kappa} \right\} \\ &\times \int_{B_i \cap U} \int_{B_i \setminus U} \frac{1}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x). \end{aligned}$$

Since the collection $\{B_i\}_{i \in \mathbb{N}}$ has uniformly bounded radii, by the 5-covering lemma, there exists a pairwise disjoint subcollection $\{B_i\}_{i \in I \subset \mathbb{N}}$ such that

$$B_0 \cap U \subset \mathcal{N} \cup \bigcup_{i \in I} 5B_i.$$

Therefore, by the previous inequality and pairwise disjointness, we have

$$\begin{aligned} \sum_{i \in I} \frac{\mu(B_i)}{r_i^\theta} &\le C(1-\theta) \sum_{i \in I} \int_{B_i \cap U} \int_{B_i \setminus U} \frac{1}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x) \\ &\le C(1-\theta) \int_{B_0 \cap U} \int_{B_0 \setminus U} \frac{1}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x), \end{aligned}$$

where $C = C_1 \max\left\{ \frac{1}{(1-\kappa/C_\mu^2)\kappa/C_\mu^2}, \frac{1}{(1-\kappa)\kappa} \right\}$. Thus by taking $\tau = 1$, we prove this proposition when $(X, d, \mu)$ is additionally assumed to be geodesic.

We now prove the result without the geodesic assumption. By the discussion at the beginning of Section 4, $(X, d)$ is $L$-bilipschitz equivalent to the geodesic space $(X, \tilde{d})$. That is, for all $x, y \in X$,

$$L^{-1} d(x,y) \le \tilde{d}(x,y) \le L d(x,y). \tag{6.12}$$

From (6.12), it directly follows that for all $x \in X$ and $r > 0$,

$$\tilde{B}(x,r) \subset B(x,Lr) \subset \tilde{B}(x,L^2 r), \tag{6.13}$$

where $\tilde{B}(x,r)$ denotes the ball $\tilde{B}(x,r) = \{y \in X : \tilde{d}(x,y) < r\}$. For any $\mu$-measurable set $U \subset X$ and $B_0 = B(x_0, r_0) \subset X$ satisfying

$$\frac{\mu(U \cap B_0)}{\mu(B_0)} \leq \kappa$$

for $0 < \kappa < 1$, we have by (6.13) that

$$\begin{aligned}\frac{\mu(U \cap \tilde{B}(x_0, Lr_0))}{\mu(\tilde{B}(x_0, Lr_0))} &\leq \frac{\mu(U \cap B_0) + \mu(\tilde{B}(x_0, Lr_0) \setminus B_0)}{\mu(B_0) + \mu(\tilde{B}(x_0, Lr_0) \setminus B_0)} \\ &= \frac{\mu(U \cap B_0)/\mu(B_0) + \mu(\tilde{B}(x_0, Lr_0) \setminus B_0)/\mu(B_0)}{1 + \mu(\tilde{B}(x_0, Lr_0) \setminus B_0)/\mu(B_0)} \\ &\leq \frac{\kappa + \mu(\tilde{B}(x_0, Lr_0) \setminus B_0)/\mu(B_0)}{1 + \mu(\tilde{B}(x_0, Lr_0) \setminus B_0)/\mu(B_0)} \leq \frac{\kappa + CL^{2Q_d}}{1 + CL^{2Q_d}} < 1,\end{aligned}$$

where the last inequality follows from that $f(x) = \frac{\kappa+x}{1+x}$ is an increasing function and

$$\frac{\mu(\tilde{B}(x_0, Lr_0) \setminus B_0)}{\mu(B_0)} \leq \frac{\mu(L^2 B_0)}{\mu(B_0)} \leq CL^{2Q_d}$$

by (2.1). By the first part of the proof under the geodesic assumption, it follows that there exists a countable collection of pairwise disjoint balls $\{\tilde{B}_i := \tilde{B}(x_i, 2^{-N_i} L r_0)\}_{i \in I \subset \mathbb{N}}$ and a set $\mathcal{N} \subset U$ with $\mu(\mathcal{N}) = 0$ such that

$$U \cap \tilde{B}(x_0, Lr_0) \subset \mathcal{N} \cup \bigcup_{i \in I} 5\tilde{B}_i, \qquad \bigcup_{i \in I} \tilde{B}_i \subset \tilde{B}(x_0, Lr_0),$$

and such that

$$\sum_{i \in I} \frac{\mu(\tilde{B}_i)}{\operatorname{rad}(\tilde{B}_i)^\theta} \leq C(1-\theta) P_\theta(U, \tilde{B}(x_0, Lr_0)).$$

For each $i \in I$, let $B_i := B(x_i, 2^{-N_i} r_0)$. Then by (6.13), it follows that $\{B_i\}_{i \in I}$ is pairwise disjoint, and that

(i) $\displaystyle\bigcup_{i \in I} B_i \subset L^2 B_0,$

(ii) $\displaystyle U \cap B_0 \subset \mathcal{N} \cup \bigcup_{i \in I} 5\tilde{B}_i \subset \mathcal{N} \cup \bigcup_{i \in I} 5L^2 B_i,$ *and*

(iii) $\displaystyle\sum_{i \in I} \frac{\mu(B_i)}{\operatorname{rad}(B_i)^\theta} \lesssim \sum_{i \in I} \frac{\mu(\tilde{B}_i)}{\operatorname{rad}(\tilde{B}_i)^\theta} \lesssim (1-\theta) P_\theta(U, \tilde{B}(x_0, Lr_0)) \leq (1-\theta) P_\theta(U, L^2 B_0).$

Taking $\tau = L^2$ completes the proof. $\square$

We see that the local fractional boxing inequality given in Proposition 6.3 actually implies the $(\theta, q, 1)_{BBM}$-Poincaré inequality.

**Corollary 6.4.** *Let $(X, d, \mu)$ be a doubling metric measure space, with relative lower mass bound exponent $Q := Q_d > 1$ (2.1), and let $0 < \theta < 1$. Suppose that there exist constants $C \geq 1$ and $\tau \geq 1$ such that for any $\mu$-measurable set $U \subset X$ and $B_0 = B(x_0, r_0) \subset X$ satisfying*

$$\frac{\mu(U \cap B_0)}{\mu(B_0)} \leq 1/2,$$

*there exists a countable collection of pairwise disjoint balls $\{B_i\}_{i \in I \subset \mathbb{N}}$, with $\mathrm{rad}(B_i) = 2^{-N_i} r_0$ for some $N_i \in \mathbb{N} \cup \{0\}$, and a set $\mathcal{N} \subset U$ with $\mu(\mathcal{N}) = 0$ such that*

$$U \cap B_0 \subset \mathcal{N} \cup \bigcup_{i \in I} 5\tau B_i \quad \text{and} \quad \bigcup_{i \in I} B_i \subset \tau B_0,$$

*and such that*

$$\sum_{i \in I} \frac{\mu(B_i)}{r_i^\theta} \leq C(1-\theta) P_\theta(U, \tau B_0).$$

*Then $(X, d, \mu)$ supports a $(\theta, q, 1)_{BBM}$-Poincaré inequality for all $1 \leq q \leq Q/(Q-\theta)$, with constants depending only on $C$, $\tau$, and the doubling constant.*

*Proof.* As in the proof of Theorem 1.1, with $q = \frac{Q}{Q-\theta}$, it suffices to estimate the sum

$$I + II := \int_{u_{B_0}}^{\max\{m_u, u_{B_0}\}} \mu(B_0 \cap \{u > \lambda\})^{1/q} \, d\lambda + \int_{\max\{m_u, u_{B_0}\}}^{\infty} \mu(B_0 \cap \{u > \lambda\})^{1/q} \, d\lambda.$$

Similar to (4.3), we have

$$I \leq 2 \int_{-\infty}^{m_u} \mu(B_0 \cap \{u < \lambda\})^{1/q} \, d\lambda.$$

Since $\mu(B_0 \cap \{u < \lambda\}) \leq \mu(B_0)/2$ for $\lambda < m_u$, we apply the hypothesis of the corollary with $U = \{u < \lambda\}$ to obtain a collection of balls $\{B_i\}_{i \in I \subset \mathbb{N}}$ satisfying (6.10) and (6.11). Using also (2.1), it therefore follows that

$$\begin{aligned}
\mu(B_0 \cap \{u < \lambda\})^{1/q} \leq \sum_{i \in I} \mu(5\tau B_i)^{1/q} &\lesssim \sum_{i \in I} \frac{\mu(B_i)}{r_i^\theta} \frac{r_i^\theta}{\mu(B_i)^{(q-1)/q}} \\
&= \frac{r_0^\theta}{\mu(B_0)^{(q-1)/q}} \sum_{i \in I} \frac{\mu(B_i)}{r_i^\theta} 2^{-N_i\theta} \left( \frac{\mu(B_0)}{\mu(B_i)} \right)^{(q-1)/q} \\
&\lesssim \frac{r_0^\theta}{\mu(B_0)^{(q-1)/q}} \sum_{i \in I} \frac{\mu(B_i)}{r_i^\theta} \\
&\leq C(1-\theta) \frac{r_0^\theta}{\mu(B_0)^{(q-1)/q}} P_\theta(\{u < \lambda\}, \tau B_0).
\end{aligned}$$

By the fractional coarea formula, Lemma 2.3, it then follows that

$$I \leq C(1-\theta) \frac{r_0^\theta}{\mu(B_0)^{(q-1)/q}} \int_{\tau B_0} \int_{\tau B_0} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

The estimation of $II$ is the same as $I$. Therefore, the proof is finished. $\square$

# 7 Improved and global inequalities, and lower Ahlfors regularity

In this section, we show that the additional assumption of lower Ahlfors $Q$-regularity (1.12) allows us to establish the improved $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-Poincaré and relative isoperimetric inequalities, see Corollary 7.1 below, as well the $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-isoperimetric and Sobolev inequalities in Corollary 1.6. These follow essentially as applications of the local and global fractional boxing inequalities given by Proposition 6.3 and Theorem 1.5, respectively. Finally, we prove Theorem 1.8, showing that these inequalities are all equivalent with lower Ahlfors $Q$-regularity of the measure $\mu$.

**Corollary 7.1.** *Suppose that the assumptions of Theorem 1.1 hold and, in addition, $\mu$ satisfies the lower Ahlfors $Q$-regular condition* (1.12) *with $Q > 1$. Then,*

*(i) there exists $C \geq 1$ and $\tau \geq 1$, depending only on $Q$, the doubling, lower Ahlfors regularity, and $(1,1)$-Poincaré inequality constants, such that for all $0 < \theta < 1$, and for all balls $B_0 \subset X$ and $u \in L^1_{\mathrm{loc}}(X)$, we have*

$$\left(\fint_{B_0} |u - u_{B_0}|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq C(1-\theta) \int_{\tau B_0} \int_{\tau B_0} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} d\mu(y) d\mu(x).$$

*(ii) there exists $C \geq 1$ and $\tau \geq 1$, depending only on $Q$, the doubling, lower Ahlfors regularity, and $(1,1)$-Poincaré inequality constants, such that for all $0 < \theta < 1$, and for all balls $B_0 \subset X$ and measurable sets $E \subset X$, we have*

$$\min\{\mu(B_0 \cap E),\, \mu(B_0 \setminus E)\}^{\frac{Q-\theta}{Q}} \leq C(1-\theta) P_\theta(E, \tau B_0).$$

*Furthermore, $(X, d, \mu)$ supports an improved $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-Poincaré inequality if and only if it supports an improved $(\theta, \frac{Q}{Q-\theta}, 1)_{BBM}$-relative isoperimetric inequality.*

*Proof.* We note that$(ii)$ follows from applying $(i)$ with $u = \chi_E$, and $(i)$ follows from $(ii)$ by the same argument used in the proof of Theorem 1.3. Thus, it remains to prove (i). The proof is similar to the proof of Corollary 6.4, with minor adjustments. We include the argument for sake of completeness.

As in the proof of Theorem 1.1, with $q = \frac{Q}{Q-\theta}$, it suffices to estimate the sum

$$I + II := \int_{u_{B_0}}^{\max\{m_u, u_{B_0}\}} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda + \int_{\max\{m_u, u_{B_0}\}}^{\infty} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda.$$

Similar to (4.3), we have

$$
\begin{aligned}
I \le \int_{u_{B_0}}^{m_u} \mu(B_0 \cap \{u > \lambda\})^{1/q}\, d\lambda &\le (m_u - u_{B_0})\mu(B_0)^{1/q} \\
&\le \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{u_{B_0}}^{m_u} \mu(\{B_0 \cap \{u > \lambda\})d\lambda \\
&\le \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{u_{B_0}}^{\infty} \mu(\{B_0 \cap \{u > \lambda\})d\lambda \\
&= \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{-\infty}^{u_{B_0}} \mu(\{B_0 \cap \{u < \lambda\})d\lambda \\
&\le \frac{2}{\mu(B_0)^{(q-1)/q}} \int_{-\infty}^{m_u} \mu(\{B_0 \cap \{u < \lambda\})d\lambda \\
&\le 2\int_{-\infty}^{m_u} \mu(B_0 \cap \{u < \lambda\})^{1/q}\, d\lambda.
\end{aligned}
$$

Since $\mu(B_0 \cap \{u < \lambda\}) \le \mu(B_0)/2$ for $\lambda < m_u$, we apply Proposition 6.3 with $\kappa = 1/2$ and $U = \{u < \lambda\}$, to obtain a collection of balls $\{B_i\}_{i\in I\subset\mathbb{N}}$ satisfying (6.10) and (6.11). Therefore, it follows that

$$
\begin{aligned}
\mu(B_0 \cap \{u < \lambda\})^{1/q} \le \sum_{i\in I} \mu(5\tau B_i)^{1/q} &\lesssim \sum_{i\in I} \frac{\mu(B_i)}{r_i^\theta} \frac{r_i^\theta}{\mu(B_i)^{\frac{\theta}{Q}}} \\
&\lesssim \sum_{i=0}^{\infty} \frac{\mu(B_i)}{r_i^\theta} \le C(1-\theta)P_\theta(\{u<\lambda\}, \tau B_0).
\end{aligned}
$$

By Lemma 2.3, it follows that

$$
\begin{aligned}
I &\lesssim (1-\theta)\int_{-\infty}^{m_u} P_\theta(\{u<\lambda\}, \tau B_0)d\lambda \\
&\lesssim (1-\theta)\int_{\tau B_0}\int_{\tau B_0} \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y)d\mu(x).
\end{aligned}
$$

The estimation of $II$ is the same as $I$. Therefore, the proof is finished. □

*Proof of Corollary 1.6.* We first prove $(i)$. Let $E \subset X$ be measurable with $\mu(E) < \infty$. From the boxing inequality Theorem 1.5 and Remark 6.2, there exists a countable collection of disjoint balls $\{B_i := B(x_i, r_i)\}_{i\in\mathbb{N}}$ such that

$$
E \subset \bigcup_{i\in\mathbb{N}} B(x_i, 5\tau r_i), \qquad \frac{1}{2C_\mu} \le \frac{\mu(B(x_i,r_i)\cap E)}{\mu(B(x_i,r_i))} < \frac{1}{2}
$$

for each $i \in \mathbb{N}$, and

$$
\sum_{i\in\mathbb{N}} \frac{\mu(B(x_i,r_i))}{r_i^\theta} \le C\theta(1-\theta)P_\theta(E,X).
$$

Hence,

$$\mu(E)^{\frac{Q-\theta}{Q}} \leq \sum_{i=0}^{\infty} \mu(5\tau B_i)^{\frac{Q-\theta}{Q}} \lesssim \sum_{i=0}^{\infty} \frac{\mu(B_i)}{r_i^\theta} \frac{r_i^\theta}{\mu(B_i)^{\frac{\theta}{Q}}}$$
$$\lesssim \sum_{i=0}^{\infty} \frac{\mu(B_i)}{r_i^\theta} \leq C\theta(1-\theta)P_\theta(E,X).$$

To prove $(ii)$, let $u \in L^1(X)$. For a.e. $\lambda > 0$, we have $\mu(\{|u| > \lambda\}) < \infty$. Applying Cavalieri's principle and computing as in (4.1), one has

$$\left(\int_X |u|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq 2\int_0^\infty \mu(\{|u| > \lambda\})^{\frac{Q-\theta}{Q}} d\lambda.$$

By (i) above and Lemma 2.3, we have that

$$\left(\int_X |u|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq 2\int_0^\infty \mu(\{|u| > \lambda\})^{\frac{Q-\theta}{Q}} d\lambda$$
$$\lesssim \theta(1-\theta)\int_0^\infty P_\theta(\{|u| > \lambda\}, X) d\lambda$$
$$\lesssim \theta(1-\theta)\int_X \int_X \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y) d\mu(x).$$

The proof of $(ii)$ shows that $(ii)$ follows from $(i)$. Applying $(ii)$ with $u = \chi_E$ also shows that $(i)$ follows from $(ii)$, and so these inequalities are equivalent. □

We now prove Theorem 1.8. As with Theorem 1.1, the conditions of the theorem persist under a bilipschitz change in metric, and so it suffices to prove the result under the additional assumption that $(X,d)$ is geodesic, see the discussion at the beginning of Section 4. We restate and prove this theorem here, under this additional assumption.

**Theorem 7.2.** *Let $(X,d,\mu)$ be a complete, geodesic metric measure space, with $\mu$ a doubling measure supporting a $(1,1)$-Poincaré inequality such that $\mu(X) = \infty$. Then, the following conditions are equivalent.*

*(i) $\mu$ satisfies the lower Ahlfors $Q$-regularity condition* (1.12) *for $Q > 1$.*

*(ii) There exists a constant $C \geq 1$, such that for all $0 < \theta < 1$ and $u \in L^1(X)$, we have*

$$\left(\int_X |u|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq C\theta(1-\theta)\int_X \int_X \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y) d\mu(x).$$

*(iii) There exists a constant $C \geq 1$ such that for all $0 < \theta < 1$, and all balls $B_0 := B(x_0,r_0) \subset X$ and $u \in L^1(X)$, we have*

$$\left(\int_{B_0} |u-u_{B_0}|^{\frac{Q}{Q-\theta}} d\mu\right)^{\frac{Q-\theta}{Q}} \leq C(1-\theta)\int_{B_0} \int_{B_0} \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} d\mu(y) d\mu(x).$$

*For the implications* $(ii) \implies (i)$ *and* $(iii) \implies (i)$*, the Ahlfors* $Q$*-regularity constants are independent of* $\theta$*. For the implications* $(i) \implies (ii)$ *and* $(i) \implies (iii)$*, the constants* $C$ *depend additionally on the doubling and* $(1,1)$*-Poincaré inequality constants.*

*Proof.* Note that $(i) \Rightarrow (ii)$ follows from Corollary 1.6 while $(i) \Rightarrow (iii)$ follows from Corollary 7.1. We will complete the proof by showing $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$.

$(ii) \Rightarrow (i)$. For a given ball $B(z,r)$ and $r > t$, we define a function $u(x)$ on $X$ by setting

$$u(x) = \begin{cases} 1, & \text{if } x \in B(z,t), \\ \frac{r-d(x,z)}{r-t}, & \text{if } x \in B(z,r) \setminus B(x,t), \\ 0, & \text{if } x \in X \setminus B(z,r). \end{cases}$$

*Claim:* There exists a constant $C > 0$ independent of $z, t, r$ such that

$$\int_X \int_X \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x)\, d\mu(y) \le C \left( \frac{1}{\theta} + \frac{1}{1-\theta} \right) \frac{\mu(B(z,r))}{(r-t)^\theta}.$$

*Proof of claim:* To this end, notice $u = 0$ on $X \setminus B(z,r)$. Then for the above double integral, we write

$$\begin{aligned} &\int_X \int_X \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x)\, d\mu(y) \\ &\quad \lesssim \int_{B(z,r)} \int_{X \setminus B(z,r)} \frac{|u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x)\, d\mu(y) \\ &\qquad + \int_{B(z,r)} \int_{B(z,r)} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x)\, d\mu(y) \\ &\quad =: H_1 + H_2. \end{aligned}$$

For $y \in B(z,r)$, we have

$$\begin{aligned} &\int_{X \setminus B(z,r)} \frac{1}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x) \\ &\quad \le \int_{X \setminus B(y, r-d(z,y))} \frac{1}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x) \\ &\quad = \sum_{m=0}^\infty \int_{B(y,2^{m+1}(r-d(z,y))) \setminus B(y,2^m(r-d(z,y)))} \frac{1}{d(x,y)^\theta \mu(B(x,d(x,y)))} \, d\mu(x) \\ &\quad \lesssim (r-d(z,y))^{-\theta} \sum_{m=0}^\infty (2^m)^{-\theta} \\ &\quad = (r-d(z,y))^{-\theta} \frac{1}{1-2^{-\theta}} \\ &\quad \lesssim \frac{1}{\theta} (r-d(z,y))^{-\theta}. \end{aligned}$$

Therefore

$$
\begin{aligned}
H_1 &\lesssim \frac{1}{\theta}\int_{B(z,r)\setminus B(z,t)}\left(\frac{r-d(y,z)}{r-t}\right)\frac{1}{(r-d(z,y))^{\theta}}d\mu(y)\\
&\quad+\frac{1}{\theta}\int_{B(z,t)}\frac{1}{(r-d(z,y))^{\theta}}d\mu(y)\\
&\lesssim \frac{1}{\theta}\frac{\mu(B(z,r))}{(r-t)^{\theta}}. \qquad (7.1)
\end{aligned}
$$

To estimate $H_2$, we write

$$
\begin{aligned}
H_2 &\le \int_{B(z,r)}\int_{B(x,r-t)}\frac{|u(x)-u(y)|}{d(x,y)^{\theta}\mu(B(x,d(x,y)))}\,d\mu(y)\,d\mu(x)\\
&\quad+\int_{B(z,r)}\int_{B(z,r)\setminus B(x,r-t)}\frac{|u(x)-u(y)|}{d(x,y)^{\theta}\mu(B(x,d(x,y)))}\,d\mu(y)\,d\mu(x)\\
&=: H_{2,1}+H_{2,2}.
\end{aligned}
$$

It is easy to know that $u$ is a Lipschitz function with Lipschitz constant $1/(r-t)$. Then we have

$$|u(x)-u(y)|\le \frac{1}{r-t}d(x,y).$$

Then

$$
\begin{aligned}
&\int_{B(x,r-t)}\frac{|u(x)-u(y)|}{d(x,y)^{\theta}\mu(B(x,d(x,y)))}\,d\mu(y)\\
&\qquad\le \frac{1}{r-t}\int_{B(x,r-t)}\frac{1}{d(x,y)^{\theta-1}\mu(B(x,d(x,y)))}\,d\mu(y)\\
&\qquad\le \frac{1}{r-t}\int_{B(x,r-t)}\sum_{j\ge -\log_2(r-t)}^{\infty}C_{\mu}2^{-j(1-\theta)}\frac{\chi_{B(x,2^{-j+1})\setminus B(x,2^{-j})}(y)}{\mu(B(x,2^{-j+1}))}\,d\mu(y) \qquad \text{by (5.3)}\\
&\qquad\lesssim \frac{(r-t)^{1-\theta}}{(r-t)(1-\theta)}. \qquad (7.2)
\end{aligned}
$$

Thus

$$H_{2,1}\lesssim \frac{1}{1-\theta}\frac{\mu(B(z,r))}{(r-t)^{\theta}}. \qquad (7.3)$$

Observing $0\le u\le 1$ and

$$\int_{X\setminus B(x,r-t)}\frac{1}{d(x,y)^{\theta}\mu(B(x,d(x,y)))}\,d\mu(y)\lesssim \frac{1}{\theta}(r-t)^{-\theta},$$

we also have

$$H_{2,2}\lesssim \frac{1}{\theta}\frac{\mu(B(z,r))}{(r-t)^{\theta}}. \qquad (7.4)$$

Combining (7.1), (7.3) and (7.4), we prove this claim.

Now let $z \in X$ and $r \in (0, \infty)$. By geodesic property, there exists a $b$ such that

$$\mu(B(z, br)) = \frac{1}{2}\mu(B(z, r)).$$

Taking $t = br$ and by the claim, we have

$$\begin{aligned}
\mu(B(z, br))^{\frac{Q-\theta}{Q}} &\le \|u\|_{L^{Q/(Q-\theta)}(X)} \\
&\lesssim \theta(1-\theta)\int_X\int_X \frac{|u(x)-u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))}\, d\mu(x)\, d\mu(y) \\
&\lesssim \frac{\mu(B(z,r))}{(r-br)^\theta}.
\end{aligned}$$

We can rewrite the above inequality as

$$r - br \lesssim \mu(B(z,r))^{\frac{1}{Q}}.$$

When then let $b_0 = 1$ and choose $b_j \in (0,1)$ satisfying the following equality

$$\mu(B(z, b_i r)) = \frac{1}{2}\mu(B(z, b_{j-1}r)) = \frac{1}{2^j}\mu(B(z,r)).$$

Obviously, $b_j \to 0$ as $j \to \infty$ and therefore

$$\begin{aligned}
r &= \sum_{j=1}^{\infty}(b_{j-1}r - b_j r) \\
&\lesssim \sum_{j=1}^{\infty} \mu(B(z, b_{j-1}r))^{1/Q} \\
&\lesssim \sum_{j=1}^{\infty} 2^{-jQ}\mu(B(z,r))^{1/Q} \\
&\lesssim \mu(B(z,r))^{1/Q}.
\end{aligned}$$

Now we have completed the proof of $(ii) \Rightarrow (i)$.

$(iii) \Rightarrow (i)$: the proof of $(iii) \Rightarrow (i)$ is similar to $(ii) \Rightarrow (i)$. For completeness, we include the proof. Now taking any $z \in X$ and $r \in (0,\infty)$. Letting $b_1 = 1$ and by geodesic property, there exists $b_0$ and $\{b_j\}_{j=2}^{\infty}$ only depending on $X$ such that

$$\mu(B(z, b_0 r)) = 2\mu(B(z,r)) \text{ and } \mu(B(z, b_j r)) = \frac{1}{2}\mu(B(z, b_{j-1}r)) = \frac{1}{2^{j-1}}\mu(B(z,r)).$$

Let $K \in \mathbb{N}$ be the smallest positive integer such that

$$C_K := \left(\frac{1}{2C_s}\right)^{1/s} - \left(\frac{C}{2^K}\right)^{1/Q} > 0. \tag{7.5}$$

Here $C \geq 1$ is the constant from (2.1) and $s$ and $C_s$ are as in (2.2). Then, for every $j \in \mathbb{N}$, we have

$$1 > \frac{b_j - b_{j+K-1}}{b_{j-1}} \geq \left(\frac{1}{2C}\right)^{1/Q} - \left(\frac{1}{2^K C_s}\right)^{1/s}.$$

Fix $j \in \mathbb{N}$, we define a function $u(x)$ on $B(z, b_{j-1}r)$ by setting

$$u(x) = \begin{cases} 1, & \text{if } x \in B(z, b_{j+K-1}r), \\ \frac{b_j r - d(x,z)}{(b_j - b_{j+K-1})r}, & \text{if } x \in B(z, b_j r) \setminus B(z, b_{j+K-1}r), \\ 0, & \text{if } x \in B(z, b_{j-1}r) \setminus B(z, b_j r). \end{cases}$$

*Claim:* There exists a constant $C > 0$ independent of $z, \{b_j\}, r$ and $\theta$ such that

$$\int_{B(z,b_{j-1}r)} \int_{B(z,b_{j-1}r)} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} \, d\mu(y) \, d\mu(x) \leq C \left(\frac{1}{1-\theta}\right) \frac{\mu(B(z, b_j r))}{(b_j r - b_{j+K-1}r)^\theta}.$$

*Proof of claim:* To this end, we also rewrite

$$\begin{aligned}
&\int_{B(z,b_{j-1}r)} \int_{B(z,b_{j-1}r)} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} \, d\mu(y) \, d\mu(x) \\
&\quad = \int_{B(z,b_{j-1}r)} \int_{B(z,b_{j-1}r) \cap B(x,(b_j - b_{j+K-1})r)} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} \, d\mu(y) \, d\mu(x) \\
&\qquad + \int_{B(z,b_{j-1}r)} \int_{B(z,b_{j-1}r) \setminus B(x,(b_j - b_{j+K-1})r)} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} \, d\mu(y) \, d\mu(x) \\
&\quad =: L_{2,1} + L_{2,2}.
\end{aligned}$$

Obviously, $u$ is a Lipschitz function with Lipschitz constant $1/(b_j - b_{j+K-1})r$. Then, by (7.2), we also have

$$\begin{aligned}
&\int_{B(z,b_{j-1}r) \cap B(x,(b_j - b_{j+K-1})r)} \frac{|u(x) - u(y)|}{d(x,y)^\theta \mu(B(x, d(x,y)))} \, d\mu(y) \\
&\quad \leq \frac{1}{(b_j - b_{j+k-1})r} \int_{B(z,b_{j-1}r) \cap B(x,(b_j - b_{j+K-1})r)} \frac{1}{d(x,y)^{\theta-1} \mu(B(x, d(x,y)))} \, d\mu(y) \\
&\quad \leq \frac{1}{(b_j - b_{j+k-1})r} \int_{B(x,(b_j - b_{j+K-1})r)} \frac{1}{d(x,y)^{\theta-1} \mu(B(x, d(x,y)))} \, d\mu(y) \\
&\quad \lesssim \frac{((b_j - b_{j+K-1})r)^{1-\theta}}{((b_j - b_{j+K-1})r)(1-\theta)}.
\end{aligned}$$

Thus

$$L_{2,1} \lesssim \frac{1}{1-\theta} \frac{\mu(B(z, b_{j-1}r))}{((b_j - b_{j+K-1})r)^\theta}. \tag{7.6}$$

Since $u$ is a Lipschitz function with Lipschitz constant $1/((b_j - b_{j+K-1})r)$, similar to (7.2) and recalling (7.5), we can obtain

$$
\begin{aligned}
\frac{1}{(b_j-b_{j+K-1})r}&\int_{B(x,2b_{j-1}r)\setminus B(x,(b_j-b_{j+K-1})r)}\frac{1}{d(x,y)^{\theta-1}\mu(B(x,d(x,y)))}\,d\mu(y)\\
&\lesssim \frac{1}{(1-\theta)((b_j-b_{j+K-1})r)}\left((2b_{j-1}r)^{1-\theta}-((b_j-b_{j+K-1})r)^{1-\theta}\right)\\
&\le \frac{1}{(1-\theta)((b_j-b_{j+K-1})r)}((2C_K^{-1})^{1-\theta}-1)((b_j-b_{j+K-1})r)^{1-\theta}.
\end{aligned}
$$

Therefore, we also have

$$
L_{2,2}\lesssim \frac{1}{1-\theta}\frac{\mu(B(z,b_{j-1}r))}{((b_j-b_{j+k-1})r)^{\theta}}. \tag{7.7}
$$

Combining (7.6) and (7.7), the proof of the claim is finished.

Notice that if $u_{B(z,b_{j-1}r)}>1/2$, then $|u-u_{B(z,b_{j-1}r)}|\ge 1/2$ on $B(z,b_{j-1}r)\setminus B(z,b_jr)$, and if $u_{B(z,b_{j-1}r)}<1/2$, then $|u-u_{B(z,b_{j-1}r)}|\ge 1/2$ on $B(z,b_{j+K-1}r)$. Thus by the claim, we can obtain

$$
\begin{aligned}
\left(\frac{1}{2^K}\right)^{\frac{Q-\theta}{Q}}\mu(B(z,b_jr))^{\frac{Q-\theta}{Q}}&\le \left\|u-u_{B(z,b_{j-1}r)}\right\|_{L^{Q/(Q-\theta)}(B(z,b_{j-1}r))}\\
&\lesssim (1-\theta)\int_{B(z,b_{j-1}r)}\int_{B(z,b_{j-1}r)}\frac{|u(x)-u(y)|}{d(x,y)^{\theta}\mu(B(x,d(x,y)))}\,d\mu(y)\,d\mu(x)\\
&\lesssim \frac{\mu(B(z,b_jr))}{(b_jr-b_{j+K-1}r)^{\theta}}.
\end{aligned}
$$

It follows that

$$
b_jr-b_{j+K-1}r\lesssim \mu(B(z,b_jr))^{\frac{1}{Q}}.
$$

The rest of the poof is the same as in $(ii)\Rightarrow(i)$ and we completed the proof of $(iii)\Rightarrow(i)$. $\square$

# References


[1] A. Alberico, A. Cianchi, L. Pick, and L. Slaváková, *Fractional Orlicz–Sobolev embeddings.* J. Math. Pures Appl. 149, 216–253 (2021)

[2] R. Alvarado, P. Górka, and P. Hajłasz, *Sobolev embedding for $M^{1,p}$ spaces is equivalent to a lower bound of the measure.* J. Funct. Anal. 279 (2020), no. 7, 108628, 39 pp.

[3] R. Alvarado, D. Yang, and W. Yuan, *Optimal embeddings for Triebel-Lizorkin and Besov spaces on quasi-metric measure spaces.* Math. Z. 307 (2024), no. 3, Paper No. 50, 59 pp.

[4] S. Avvakumov, and A. Nabutovsky, *Boxing inequalities in Banach spaces and Riemannian manifolds.* Preprint. (2023) arXiv:2304.02709.

[5] A. Björn, and J. Björn, *Nonlinear potential theory on metric spaces.* EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.

[6] A. Björn, J. Björn, and N. Shanmugalingam. *Extension and trace results for doubling metric measure spaces and their hyperbolic fillings.* J. Math. Pures Appl. (9) **159** (2022), 196–249.

[7] J. Bourgain, H. Brezis, and P. Mironescu, *Another look at Sobolev spaces*, Optimal Control and Partial Differential Equations, IOS Press, 2001, pp. 439–455.

[8] J. Bourgain, H. Brezis, and P. Mironescu, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications.* J. Anal. Math., 87 (2002), 77–101.

[9] H. Brezis, *How to recognize constant functions. A connection with Sobolev spaces.* Uspekhi Mat. Nauk 57 (2002), 59–74 (Russian); English transl., Russian Math. Surveys 57 (2002), 693–708.

[10] L. Caffarelli, J. M. Roquejoffre, and O. Savin, *Non-local minimal surfaces.* Comm. Pure Appl. Math. 63 (2010) 1111–1144.

[11] L. Caffarelli, and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces.* Calc. Var. Partial Differential Equations 41 (2011), no. 1-2, 203–240.

[12] L. Capogna, J. Kline, R. Korte, N. Shanmugalingam, and M. Snipes, *Neumann problems for p-harmonic functions, and induced nonlocal operators in metric measure spaces.* (To appear) Amer. J. Math. (2025)

[13] F. Cavalletti, and A. Mondino, *Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds.* Invent. Math. 208 (2017), no. 3, 803–849.

[14] E. Cejas, I. Drelichman, and J. C. Martínez-Perales, *Improved fractional Poincaré type inequalities on John domains.* Ark. Mat. 57:2, 2019, 285–316

[15] J. Dávila, *On an open question about functions of bounded variation.* Calc. Var. Partial Differential Equations 15 (2002), 519–527.

[16] S. Di Marino, and M. Squassina, *New characterizations of Sobolev metric spaces*, J. Funct. Anal. 276 (2019), no. 6, 1853–1874.

[17] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces.* Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[18] B. Dyda, L. Ihnatsyeva, and A. V. Vähäkangas. *On improved fractional Sobolev-Poincaré inequalities* Ark. Mat. 54, no. 2 (2016): 437–54.

[19] B. Dyda, J. Lehrbäck, and A. V. Vähäkangas, *Fractional Poincaré and localized Hardy inequalities on metric spaces.* Adv. Calc. Var. 16 (2023), no. 4, 867–884.

[20] D. E. Edmunds, and W. D. Evans, *Fractional Sobolev Spaces and Inequalities.* Cambridge Tracts in Mathematics, vol. 230. Cambridge University Press, Cambridge (2023)

[21] L. C. Evans, and R. F. Gariepy, *Measure theory and fine properties of functions*. Textb. Math. CRC Press, Boca Raton, FL, 2015, xiv+299 pp.

[22] H. Federer, and W. H. Fleming, *Normal and integral currents*, Ann. Math. 72 (1960) 458–520.

[23] F. Ferrari, Fausto, B. Franchi, and G. Lu, *On a relative Alexandrov-Fenchel inequality for convex bodies in Euclidean spaces*. Forum Math. 18 (2006), no. 6, 907–921.

[24] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Comm. Math. Phys. 336 (1) (2015) 441–507.

[25] R. Frank, and R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities.* J. Funct. Anal., 255 (2008), no. 12, 3407-3430.

[26] N. Fusco, V. Millot, and M. Morini, *A quantitative isoperimetric inequality for fractional perimeters*. J. Funct. Anal. 261(3), 697–715 (2011)

[27] A. Gogatishvili, P. Koskela, and N. Shanmugalingam, *Interpolation properties of Besov spaces defined on metric spaces*. Math. Nach. 283(2), 215-231 (2010)

[28] W. Górny, *Bourgain-Brezis-Mironescu approach in metric spaces with Euclidean tangents*. J. Geom. Anal. 32(4), 128 (2022)

[29] W. Gustin, *Boxing inequalities*. J. Math. Mech. 9 (1960), 229–239.

[30] P. Hajłasz, *Sobolev spaces on metric-measure spaces*. Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[31] P. Hajłasz, and P. Koskela. *Sobolev met Poincaré.* Mem. Amer. Math. Soc. 145 (2000)

[32] P. Hajłasz, P. Koskela, and H. Tuominen, *Sobolev embeddings, extensions and measure density condition.* J. Funct. Anal. 254 (2008), no. 5, 1217–1234.

[33] B. Han, *On the asymptotic behaviour of the fractional Sobolev seminorms: a geometric approach*. J. Funct. Anal. 287 (2024), no. 9, Paper No. 110608, 25 pp.

[34] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev Spaces on Metric Measure Spaces. An Approach Based on Upper Gradients.* New Math. Monogr. 27. Cambridge University Press, Cambridge, 2005.

[35] R. Hurri-Syrjänen, and A. Vähäkangas, *On fractional Poincaré inequalities*. J. Anal. Math. 120 (2013), 85–104.

[36] R. Hurri-Syrjänen, J. Martínez-Perales, C. Pérez, and A. Vähäkangas, *On the BBM-phenomenon in fractional Poincaré-Sobolev inequalities with weights*. Int. Math. Res. Not. IMRN 2023, no. 20, 17205–17244.

[37] R. Jiang, and P. Koskela, *Isoperimetric inequality from the Poisson equation via curvature*. Comm. Pure Appl. Math. 65 (2012), no. 8, 1145-1168.

[38] S. Keith, *Modulus and the Poincaré inequality on metric measure spaces*, Math. Z. 245 (2003), no. 2, 255–292.

[39] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen. *A characterization of Newtonian functions with zero boundary values.* Calc. Var. Partial Differential Equations 43 (2012) 507–528.

[40] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen. *Lebesgue points and capacities via boxing inequality in metric spaces.* Indiana Univ. Math. J. 57 No. 1 (2008) 401–430.

[41] J. Kline, *Regularity of Sets of Finite Fractional Perimeter and Nonlocal Minimal Surfaces in Metric Measure Spaces.* J. Geom. Anal. 35 (2025), no. 6, Paper No. 182.

[42] R. Korte, and P. Lahti, *Relative isoperimetric inequalities and sufficient conditions for finite perimeter on metric spaces.* Ann. Inst. H. Poincaré C Anal. Non Linéaire 31 (2014), no. 1, 129–154.

[43] P. Lahti, A. Pinamonti, and X. Zhou, *A characterization of BV and Sobolev functions via nonlocal functionals in metric spaces.* Nonlinear Anal. 241 (2024), Paper No. 113467, 14 pp.

[44] P. Lahti, A. Pinamonti, and X. Zhou, *BV functions and nonlocal functionals in metric measure spaces.* J. Geom. Anal. 34 (2024), no. 10, Paper No. 318, 34 pp.

[45] G. Leoni, *A First Course in Fractional Sobolev Spaces.* Grad. Stud. Math., 229 American Mathematical Society, Providence, RI, 2023. xv+586 pp.

[46] J. M. Mackay, J. T. Tyson, and K. Wildrick, *Modulus and Poincaré Inequalities on Non-Self-Similar Sierpiński Carpets.* Geom. Funct. Anal. 23, (2013) 985–1034.

[47] L. Malý, *Trace and extension theorems for Sobolev-type functions in metric spaces.* Preprint. (2017) https://arxiv.org/abs/1704.06344

[48] V. G. Maz'ya, *Classes of domains and imbedding theorems for function spaces*, Dokl. Akad. Nauk SSSR 133 , 527–530; translation in Soviet Math. Dokl. 1 (1960), 882–885.

[49] V. G. Maz'ya *Sobolev spaces.* Springer Ser. Soviet Math. Springer-Verlag, Berlin, 1985, xix+486 pp.

[50] V. G. Maz'ya, and T. O. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces.* J. Funct. Anal. 195 (2002), 230–238. Erratum: J. Funct. Anal. 201 (2003), 298–300.

[51] M. Miranda, *Functions of bounded variation on "good" metric spaces.* J. Math. Pures Appl. (9) 82 (2003), no. 8, 975–1004.

[52] K. Myyryläinen, C. Pérez, and J. Weigt, *Weighted fractional Poincaré inequalities via isoperimetric inequalities.* Calc. Var. Partial Differential Equations, 63 (2024), no. 8, Paper No. 205, 32 pp.

[53] H. Nakajima, *Isoperimetric inequality on a metric measure space and Lipschitz order with an additive error*. J. Geom. Anal. 32 (2022), no. 1, Paper No. 35, 43 pp.

[54] A. Ponce, *An estimate in the spirit of Poincaré's inequality.* J. Eur. Math. Soc., 6 (2004), no. 1, 1–15.

[55] A. Ponce, and D. Spector, *A boxing inequality for the fractional perimeter.* Ann. Sc. Norm. Super. Pisa Cl. Sci., (5) 20 (2020), no. 1, 107–141.

[56] R. Shimizu, *Characterizations of Sobolev functions via Besov-type energy functionals in fractals*, arXiv:2410.15317

[57] F. Sk, *Characterization of fractional Sobolev-Poincaré and (localized) Hardy inequalities.* J. Geom. Anal. 33 (2023), no. 7, Paper No. 223, 20 pp.

[58] A. Visintin, *Generalized coarea formula and fractal sets.* Japan J. Indust. Appl. Math. 8 (1991), 175-201

[59] S. Wenger, *Isoperimetric inequalities of Euclidean type in metric spaces.* Geom. Funct. Anal. 15 (2005), no. 2, 534–554.

[60] Y. Zhou, *Fractional Sobolev extension and imbedding*. Trans. Amer. Math. Soc. 367 (2015), no. 2, 959–979.

[61] W. Ziemer, *Weakly differentiable functions.* Graduate Texts in Mathematics 120 (Springer, 1989).

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