

Dynamics of actions of automorphisms on the space of one-parameter subgroups of a torus and applications

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04 November, 2025

Abstract

For a connected Lie group G , we study the dynamics of actions of automorphisms of G on certain compact invariant subspaces of closed subgroups of G in terms of distality and expansivity. We show that only the finite order automorphisms of G act distally on Sub_G^p , the smallest compact space containing all closed one-parameter subgroups of G , when G is any n -torus, $n \in \mathbb{N}$. This enables us to relate distality of the T -action on Sub_G^p with that of the T -action on G and characterise the same in terms of compactness of closed subgroups generate by T in the group $\text{Aut}(G)$, in case G is not a vector group. We also extend these results to the action of subgroups of automorphisms. We show that any n -torus G , $n \geq 2$, more generally, any connected Lie group G whose central torus has dimension at least 2, does not admit any automorphism which acts expansively on Sub_G^p . Our results generalise some results on distal actions by Shah and Yadav, and by Chatterjee and Shah, and some results on expansive actions by Prajapati and Shah.

Keywords and phrases: distal and expansive actions of automorphisms, Lie groups, n -torus, one-parameter subgroups, Chabauty topology.

2020 Mathematics Subject Classification: 37B05 (Primary), 22E15, 22D45 (Secondary)

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1 Introduction

Distal and expansive actions are two significant areas of study in topological dynamics. The notion of distality was introduced by David Hilbert to study non-ergodic actions on compact spaces. Distal actions have been studied by Ellis [12] and Furstenberg [14] on compact spaces, and Abels [1], Moore [18], Raja-Shah [24, 25] and Shah [27] on Lie groups, see the references cited in [25]. The notion of expansivity was introduced by Utz to study chaotic orbits. Since then it has been widely studied by many in different contexts, see Glöckner-Raja [15], Shah [28] and Chodhury-Raja [10] and the references cited therein.

For a Hausdorff topological space X , a homeomorphism T of X is said to be *distal* (equivalently, T acts *distally* on X) if for any pair of distinct elements $x, y \in X$, the closure of the double orbit $\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}$ in $X \times X$ does not intersect the diagonal, i.e. for $x, y \in X$ with $x \neq y$, $\overline{\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}} \cap \{(d, d) \mid d \in X\} = \emptyset$. If X is a compact metric space with the metric d , then T is distal if and only if for every pair $x, y \in X$ with $x \neq y$, $\inf\{d(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\} > 0$. If X is a topological group and T is an automorphism then T is distal if and only if for every $x \in G$ with $x \neq e$, $e \notin \overline{\{T^n(x)\} \mid n \in \mathbb{Z}}$.

Let X be a metrizable topological space with a metric d , a homeomorphism T of X is said to be *expansive* if there exists $\epsilon > 0$ such that the following hold: If $x, y \in X$ with $x \neq y$, then $\sup\{d(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\} > \epsilon$. Here, ϵ is called an expansive constant for T . It is known that the expansivity of a homeomorphism on any compact metric space is independent of the metric. For a locally compact topological group G , an automorphism T is expansive if $\bigcap_{n \in \mathbb{Z}} T^n(U) = \{e\}$ for some neighbourhood U of e . In particular, if T is expansive on G , then G is first countable and admits a left invariant metric and this definition agrees with the one given above for the metric space. It is also well known that for any infinite compact metric space, the class of distal homeomorphisms and the class of expansive homeomorphisms are

mutually disjoint (cf. [8], Theorem 2).

For a locally compact Hausdorff topological group G , let Sub_G denote the set of all closed subgroups of G equipped with Chabauty topology. Note that Sub_G is compact and Hausdorff, and if G is second countable then it is metrizable. Let $\text{Aut}(G)$ denote the group of all automorphisms of G (i.e. homomorphisms of G which are also homeomorphisms). There is a natural action of $\text{Aut}(G)$ on Sub_G ; namely, $(T, H) \mapsto T(H)$, $T \in \text{Aut}(G)$, $H \in \text{Sub}_G$. This action gives rise to a homomorphism from $\text{Aut}(G)$ to $\text{Homeo}(\text{Sub}_G)$. Motivated by the fact that the image of $\text{Aut}(G)$ yields a large class of homeomorphisms of Sub_G , we aim to explore the dynamics of this subclass, particularly in the context of distality and expansivity.

Let G be a connected Lie group and $T \in \text{Aut}(G)$. We say that T acts distally (resp. expansively) on Sub_G if the homeomorphism of Sub_G corresponding to T is distal (resp. expansive). Shah and Yadav [29] first studied such distal actions for connected Lie group G and they showed that for a large class of Lie group G , which does not have non-trivial compact connected central subgroup, an automorphism T acts distally on Sub_G^a , the space of closed abelian subgroups of G , if and only if T generates a compact subgroup in $\text{Aut}(G)$. Moreover they showed that if G does not have any nontrivial compact connected central subgroup, then T acts distally on Sub_G^a implies that T is distal (on G); see Corollary 3.7 in [29]. Shah and Prajapati first studied such expansive actions for locally compact second countable groups. They showed that if T acts expansively on Sub_G , then T is expansive on G , moreover a (nontrivial) connected Lie group G does not admit any automorphism which acts expansively on Sub_G^a . Shah, together with Palit [19], and with Palit and Prajapati [20], investigated the distal and expansive action of automorphisms on Sub_G^a for discrete groups G , where G is either polycyclic or a lattice in a connected Lie group. Note that Sub_G^a is very large for many groups G , in particular if G is abelian, then it is the same as Sub_G .

A natural question that arose from the above investigations was whether distal (resp. expansive) actions of automorphisms of G can be characterised (resp. exist) on other smaller invariant subspace of Sub_G^a . A recent work by Chatterjee and Shah [9] considers one such subspace: the class of (nontrivial) smallest closed connected abelian subgroups, namely closed one-parameter subgroup of G . Let Sub_G^p denote the smallest closed subset of Sub_G containing all closed one-parameter subgroups of G . Note that Sub_G^p is compact and it is invariant under the action of $\text{Aut}(G)$. In Theorem 1.1 of [9], the class of distal actions of automorphisms on the space Sub_G^p is characterised when G is a Lie group without central torus as follows; if G is abelian, i.e. G is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then T acts distally on Sub_G^p .

if and only if $T \in \mathcal{KD}$, where $\mathcal{K}$ is a compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ and $\mathcal{D}$ is the center of $\mathrm{GL}(n, \mathbb{R})$, and if G is not abelian, then T acts distally on Sub_G^p if and only if $T \in \mathcal{K}$, a compact subgroup of $\mathrm{Aut}(G)$. This was also generalised to characterise T which act distally on the maximal central torus, see Theorem 1.4 of [9].

We know that the maximal compact connected central subgroup of a connected Lie group (maximal central torus) is either trivial or isomorphic to $\mathbb{T}^n$, the n -torus (also known as the central torus of G), for some $n \in \mathbb{N}$. The work of Chatterjee and Shah [9] does not cover the case when G is a torus, or the more general case of connected Lie groups G without any condition on the T -action on the central torus. Moreover, the behaviour of distal and expansive actions on $\mathrm{Sub}_{\mathbb{T}^n}^p$ has not yet been explored. In this paper, we investigate these dynamical properties when $G = \mathbb{T}^n$. We get the following which characterises distal actions on $\mathrm{Sub}_{\mathbb{T}^n}^p$.

Theorem 3.1. *Let G be an n -torus for some $n \in \mathbb{N}$, and let T be an automorphism on G . Then T acts distally on Sub_G^p if and only if $T^m = \mathrm{Id}$, the identity map, for some $m \in \mathbb{N}$.*

The following relates the distal action of automorphism of G on Sub_G^p and distality of the automorphism and it generalises Theorem 1.2 of [9].

Theorem 3.3. *Let G be a connected Lie group which is not a vector group and let $T \in \mathrm{Aut}(G)$. If T acts distally on Sub_G^p , then T acts distally on G .*

For a Lie group G , let $\mathrm{Aut}(G)$ be endowed with the compact-open topology. It is a Lie group which is identified with a closed subgroup of the Lie algebra automorphisms of the Lie algebra of G . Let Sub_G^c denote the set of all discrete cyclic subgroups of G . Using Theorem 3.1 on the distal actions of automorphisms on $\mathrm{Sub}_{\mathbb{T}^n}^p$, we obtain a characterisation of distal actions of automorphisms on the space Sub_G^p for any connected Lie group G in the following.

Theorem 3.4. *Let G be a connected Lie group and let $T \in \mathrm{Aut}(G)$. Consider the following statements.*

1. T acts distally on Sub_G^p .
2. T acts distally on $\overline{\mathrm{Sub}_G^c}$.
3. T acts distally on Sub_G^a .
4. T acts distally on Sub_G .
5. T is contained in a compact subgroup of $\mathrm{Aut}(G)$.

Then (2 – 5) are equivalent. If G is not a vector group, then (1 – 5) are equivalent.

Theorem 3.1 generalises several results from the work of Chatterjee and Shah [9], notably Theorems 1.1 (2), 1.2 and 1.4 of [9]. Note that when G is a vector group, Theorem 1.1 (1) of [9] has characterised the distal action of automorphisms on Sub_G^p . We also generalise Theorem 1.5 of [9] about the action of a subgroup of $\text{Aut}(G)$ on Sub_G^p , see Theorem 3.5.

For expansive actions, it is known that $\mathbb{T}^n$ does not admit any automorphism which acts expansively on $\text{Sub}_{\mathbb{T}^n}$ (cf. [22], Theorem 3.2). The question arises whether any automorphism of $\mathbb{T}^n$ acts expansively on a smaller invariant subspace of $\text{Sub}_{\mathbb{T}^n}$. The following shows that in case of $\text{Sub}_{\mathbb{T}^n}^p$ the answer is negative.

Theorem 4.2. *Let $G = \mathbb{T}^n$, the n -torus, for any $n \geq 2$. Then G does not admit any automorphism that acts expansively on Sub_G^p .*

As a consequence of the above theorem, we get the following for a larger class of connected Lie groups.

Theorem 4.5. *Let G be a connected Lie group such that it contains a central torus of dimension at least 2. Then G does not admit any automorphism that acts expansively on Sub_G^p .*

Theorem 4.5 generalises Theorem 3.1 of [22], in case G is a connected Lie group as above.

For many groups G , compact spaces Sub_G , Sub_G^a and Sub_G^c are identified (see Baik and Clavier [3, 4], Bridson et al. [7], Harmrouni and Kadri [16], and also Pourezza and Hubbard [21]); one can also identify Sub_G^p for some groups G , e.g. $\text{Sub}^p(\mathbb{R}^n)$ homeomorphic to $\mathbb{RP}^{n-1}$, the real projective space of dimension $n - 1$, $n \geq 2$. The study of the action of $\text{Aut}(G)$ on Sub_G and on its closed (compact) invariant subspaces leads to a better understanding of dynamics on these spaces.

For a subgroup H of G , let H^0 denote that connected component of the identity e in G and $\overline{H}$ is the closure of H in G ; both H^0 and $\overline{H}$ are subgroups of G . A k -torus is a compact connected abelian Lie group of dimension k , $k \in \mathbb{N}$. Any compact abelian subgroup of divisible if and only if it is connected (i.e. it is a torus). A maximal central torus (a maximal compact connected central subgroup) of G is characteristic in G . Any one-parameter subgroup in G is a continuous homomorphism from $\mathbb{R}$ to G . It is either closed and isomorphic to $\mathbb{R}$, $\{e\}$ or a 1-torus $\mathbb{T}^1$, or its closure is a k -torus for some $k \geq 2$ (cf. [9], Lemma 2.3). We may denote a one-parameter subgroup $\{x_t\}_{t \in \mathbb{R}}$ by just $\{x_t\}$.

In § 2 we discuss the topology of Sub_G and the action of $\text{Aut}(G)$ on Sub_G . We also derive some properties of $\text{Sub}_{\mathbb{T}^n}^p$ and prove some useful lemmas about its structure.

In § 3 we prove results on distal actions of automorphisms T on $\text{Sub}_{\mathbb{T}^n}^p$ and also that of automorphism groups of $\mathbb{T}^n$ on $\text{Sub}_{\mathbb{T}^n}^p$. In § 4 we prove a result on the orbits of subspaces of $\mathbb{R}^n$ and prove the main result about expansive actions.

2 Structure and properties of Sub_G and $\text{Sub}_{\mathbb{T}^n}^p$

For any locally compact (Hausdorff) group G , the space Sub_G , of all closed subgroups of G is endowed with the Chabauty topology, which is generated by a sub-basis

$$\{\mathcal{U}_1(K) \mid K \subset G \text{ is compact}\} \cup \{\mathcal{U}_2(U) \mid U \subset G \text{ is open}\},$$

where $\mathcal{U}_1(K) = \{H \in \text{Sub}_G \mid H \cap K = \emptyset\}$ and $\mathcal{U}_2(U) = \{H \in \text{Sub}_G \mid H \cap U \neq \emptyset\}$. Note that Sub_G is compact and Hausdorff, and it is metrizable if G is second countable (cf [5], Lemma E.1.1). Since we deal only with closed subgroups of a connected Lie group G , we have that Sub_G is metrizable. The following criteria of convergence in Sub_G is well-known (see e.g. Proposition E.1.2 in [5]).

Lemma 2.1. *Let G be a connected Lie group. A sequence $\{H_n\} \subset \text{Sub}_G$ converges to $H \in \text{Sub}_G$ if and only if the following hold:*

- (I) *For $g \in G$, if there exists a subsequence $\{H_{n_k}\}$ of $\{H_n\}$ with $h_k \in H_{n_k}$, $k \in \mathbb{N}$, such that $h_k \rightarrow g$ in G , then $g \in H$.*
- (II) *For every $h \in H$, there exists a sequence $\{h_n\}_{n \in \mathbb{N}}$ such that $h_n \in H_n$, $n \in \mathbb{N}$, and $h_n \rightarrow h$.*

Recall that for a connected Lie group G , there is a natural action of the space $\text{Aut}(G)$ of (bi-continuous) automorphisms of G on Sub_G ; namely, the map $H \mapsto T(H)$, $H \in \text{Sub}_G$, $T \in \text{Aut}(G)$. This is a continuous group action on Sub_G^p by homeomorphisms and it keeps the following subspaces invariant: Sub_G^a consisting of closed abelian subgroups, Sub_G^c consisting of discrete cyclic subgroups and its closure, the space of closed one-parameter subgroups and its closure Sub_G^p . For some basic structural properties of Sub_G and the action of $\text{Aut}(G)$ on it, we refer the readers to [29].

Note that $\text{Aut}(\mathbb{T}^n) = \text{GL}(n, \mathbb{Z})$, where $\text{GL}(n, \mathbb{Z})$ denotes the group of invertible $n \times n$ matrices of determinant ± 1 with integer entries. The compact open topology on $\text{GL}(n, \mathbb{Z})$ is discrete. Therefore, any compact subgroup of $\text{GL}(n, \mathbb{Z})$ is finite, in particular, an element of $\text{Aut}(\mathbb{T}^n)$ generates a relatively compact subgroup in $\text{Aut}(\mathbb{T}^n)$ if and only if it has finite order.

Any one-parameter subgroup in a torus $\mathbb{T}^n$ is either closed and isomorphic to the trivial subgroup or to $\mathbb{T}^1$, or its closure is isomorphic to an k -dimensional torus for $1 < m \leq n$ when $n \geq 2$. Moreover, $\text{Sub}_{\mathbb{T}^n}^p$ consists of all k -tori, $1 \leq k \leq n$ and the trivial subgroup (cf. [9], Lemma 2.3). If $n \neq 1$, $\text{Sub}_{\mathbb{T}^n}^p$ is infinite; in fact, the set of closed one-parameter subgroups is infinite as the set of roots of unity is dense in $\mathbb{T}^n$, and $\mathbb{T}^n$ is exponential. Note that $\text{Sub}_{\mathbb{T}^n}^p$ is countable as any k -torus in $\mathbb{T}^n$ corresponds to a vector subspace V of dimension k in the covering group $\mathbb{R}^n$ (of $\mathbb{T}^n$) such that V is generated by k (linearly independent) elements of $\mathbb{Z}^n$.

For the sake of convenience, we state these facts as a lemma about the structure of $\text{Sub}_{\mathbb{T}^n}^p$ which is essentially known.

Lemma 2.2. *For an n -torus $\mathbb{T}^n$, the following holds.*

1. $\text{Sub}_{\mathbb{T}^n}^p = \{\mathbb{T}^k \mid 1 \leq k \leq n\} \cup \{\{e\}\}.$
2. $\text{Sub}_{\mathbb{T}^n}^p$ is countable.
3. The trivial subgroup $\{e\}$ is isolated in $\text{Sub}_{\mathbb{T}^n}^p$.

Lemma 2.2 (3) is known (see e.g. Lemma 2.3 in [9]).

The set $\text{Sub}_{\mathbb{T}}^p$ has only two elements, the trivial subgroup $\{e\}$ and $\mathbb{T}$. Moreover $\text{Aut}(\mathbb{T})$ consists of only two elements; namely, the identity map and the map $x \mapsto x^{-1}$, and the action of any of this maps on $\text{Sub}_{\mathbb{T}}^p$ is distal as well as expansive. If $n \geq 2$, $\mathbb{T}^n$ contains infinitely many subgroups which are isomorphic to $\mathbb{T}^k$ for each k , $0 < k < n$, as observed above. Since $\mathbb{T}^n$ is compact, the Chabauty topology on $\text{Sub}_{\mathbb{T}^n}$ is induced by the Hausdorff metric (cf. [5], Proposition E.1.3), and it allows us to use results from the theory of compact abelian metric groups related to the Hausdorff metric. We now state a useful lemma by Berend (cf. [6], Lemma 4.7).

Lemma 2.3. (Berend [6]) Let G be a compact abelian metric group and let Γ be the dual group. A sequence $\{G_m\}_{m=1}^\infty$ of closed subgroups of G satisfies $G_m \rightarrow G$ (in the Hausdorff metric) if and only if for every nonzero $\gamma \in \Gamma$ we have $\gamma \notin \text{Ann}(G_m)$ for sufficiently large m (where $\text{Ann}(H)$ denotes the annihilator in Γ of a closed subgroup H of G).

The following is a direct consequence of Theorem 2.3 ([6], Lemma 4.7).

Lemma 2.4. *Let $\mathbb{T}^n$ be the n -torus and let $\{G_m\}$ be a sequence of closed subgroups in $\mathbb{T}^n$. Then $\{G_m\}$ does not converges to $\mathbb{T}^n$ in $\text{Sub}_{\mathbb{T}^n}^p$ if and only if there exists a character γ of $\mathbb{T}^n$ such that $\gamma \in \text{Ann}(G_m)$ for infinitely many m .*

We know that $\{e\}$ is isolated in $\text{Sub}_{\mathbb{T}^n}^p$. We now show that any proper subtorus is isolated in the set of subtori of same dimension, or more generally, in the space of subtori of same or higher dimension. For $1 \leq k < n$, let $\mathfrak{S}_k$ be the space of all subtori of dimension k in $\mathbb{T}^n$ and let $\mathfrak{H}_k = \cup_{m=k}^{n-1} \mathfrak{S}_m \cup \{\mathbb{T}^n\}$. Then $\mathfrak{H}_k \subset \mathfrak{H}_{k-1}$ for $2 \leq k < n$ and

$$\text{Sub}_{\mathbb{T}^n}^p = \mathfrak{H}_1 \cup \{\{e\}\} = \cup_{k=1}^{n-1} \mathfrak{S}_k \cup \{\mathbb{T}^n\} \cup \{e\}.$$

It is easy to see that all $\mathfrak{S}_k$ and $\mathfrak{H}_k$ are T invariant for every $T \in \text{GL}(n, \mathbb{Z})$, for $1 \leq k \leq n-1$. As the set of k -dimensional subspaces in $\mathbb{R}^n$ is closed in $\text{Sub}_{\mathbb{R}^n}$, we get that each $\mathfrak{H}_k$ is closed in $\text{Sub}_{\mathbb{T}^n}^p$.

Lemma 2.5. *Let the notation be as above. For $n \geq 2$, $\mathfrak{H}_k$ is closed (compact) in $\text{Sub}_{\mathbb{T}^n}^p$. Moreover, every $H \in \mathfrak{S}_k$ is isolated in $\mathfrak{H}_k$.*

Proof. Note that since any subtorus is divisible and $\mathbb{T}^n$ is compact, the limit of a sequence of subtori is a subtorus or $\mathbb{T}^n$ (see also Theorem 2.2). Let $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the natural projection with $\ker \pi = \mathbb{Z}^n$. Then every $H \in \mathfrak{S}_k$ corresponds a k -dimensional vector subspace V of $\mathbb{R}^n$ such that $\pi(V) = H$. Since the set of all k -dimensional subspace of $\mathbb{R}^n$ is closed in $\text{Sub}_{\mathbb{R}^n}$, we get that if $H_m \rightarrow H$ in $\text{Sub}_{\mathbb{T}^n}^p$ for $\{H_m\} \subset \mathfrak{S}_k$, then $\dim(H) \geq k$. Hence $H \in \mathfrak{H}_k$. In particular, $\mathfrak{H}_k = \cup_{l=k}^{n-1} \mathfrak{S}_l \cup \{\mathbb{T}^n\}$ is closed. Thus the first assertion holds.

Step 1: Now we prove the second assertion. We first consider $k = n-1$. Let $H \in \mathfrak{S}_{n-1}$. We want to show that H is isolated in $\mathfrak{H}_{n-1}$. Suppose $H_m \in \mathfrak{H}_{n-1}$, $m \in \mathbb{N}$, is such that $H_m \rightarrow H$, then we may assume that $H_m \in \mathfrak{S}_{n-1}$ for all $m \in \mathbb{N}$. If possible, suppose $H_m \neq H$ for infinitely many m . Passing to a subsequence if necessary, we may assume that $H_m \neq H$ for all m . By Lemma 4.7 of [6] (see also Lemmas 2.3 or 2.4), there exists nonzero character ψ on $\mathbb{T}^n$ such that $\psi(H_m) = 0$, i.e. $H_m \subseteq (\ker \psi)^0$, for infinitely many m . Since $\dim(H_m) = n-1$, we get that $H_m = (\ker \psi)^0$ for infinitely many m , and hence $H = (\ker \psi)^0$. Therefore, $H_m = H$ for infinitely many m , which leads to a contradiction. Thus $H_m = H$ for all large m . Therefore, each $H \in \mathfrak{S}_{n-1}$ is isolated in $\mathfrak{S}_{n-1}$, and hence in $\mathfrak{H}_{n-1}$. In particular, the second assertion holds for $n = 2$.

Step 2: Now suppose $n \geq 3$. Suppose the second assertion holds for every torus with dimension $n-1$. Now we prove the assertion for $\mathbb{T}^n$. Suppose $1 \leq k \leq n-1$. If $k = n-1$, then the assertion follows from Step 1. Now suppose $1 \leq k \leq n-2$. Let $H \in \mathfrak{S}_k$. Suppose $H_m \rightarrow H$ for some $\{H_m\} \subset \mathfrak{H}_k$. If possible, suppose $H_m \neq H$ for infinitely many m . Passing to a subsequence if necessary, we may assume that that $H_m \neq H$ for all m . Since $H \neq \mathbb{T}^n$, arguing as in Step 1 using Lemma 4.7 of [6], we get

that there exists a character γ of $\mathbb{T}^n$ such that $\gamma(H_m) = 0$, i.e. $H_m \subset (\ker \gamma)^0 = K$ for infinitely many m , and hence it follows that $H \subset K$. Since $\dim(K) = n - 1$, by induction, H is isolated in the set (say) $\mathfrak{H}_k(K)$ of all subtori of K with dimension greater than or equal to k , and hence $H_m = H$ for infinitely many m , which leads to a contradiction. Therefore, H is isolated in $\mathfrak{H}_k$, and the assertion holds by induction for all n . $\square$

3 Distal actions of automorphisms of connected Lie groups G on Sub_G^p

In this section, we first prove Theorem 3.1 which extends Theorem 1.1 (2) of [9] to compact connected abelian Lie groups. Using Theorem 3.1 we generalise Theorems 1.1 (2), 1.2, 1.4 and 1.5 of [9].

Theorem 3.1. *Let G be an n -torus for some $n \in \mathbb{N}$, and let T be an automorphism on G . Then T acts distally on Sub_G^p if and only if $T^m = \text{Id}$, the identity map, for some $m \in \mathbb{N}$.*

Proof. One way assertion is obvious. We prove the converse by the induction on the dimension n of the torus; i.e. we assume that T acts distally on Sub_G^p and show that $T^m = \text{Id}$ for some $m \in \mathbb{N}$. If $G = \mathbb{T}^1$, the one-dimensional torus, then $\text{Aut}(G)$ has only two elements, hence $T^2 = \text{Id}$. Thus the assertion holds trivially for $n = 1$. Now suppose the assertion holds for any torus G with $\dim G \leq n - 1$, where $n \geq 2$. Let $G = \mathbb{T}^n$. Let $H \in \mathfrak{S}_{n-1}$. Then $H \in \text{Sub}_G^p$ by Theorem 2.2 (see also Lemma 2.3 in [9]). As Sub_G^p is compact, there exists a strictly increasing sequence $\{m_k\}$ in $\mathbb{N}$, such that $T^{m_k}(H) \rightarrow L$ for some $L \in \text{Sub}_G^p$. As $H \in \mathfrak{S}_{n-1} \subset \mathfrak{H}_{n-1}$, and $\mathfrak{H}_{n-1}$ is compact and T -invariant, we have that $L \in \mathfrak{H}_{n-1}$. Since T acts distally on Sub_G^p and $T(G) = G$ and $H \neq G$, it follows that $L \neq G$. Therefore, $L \in \mathfrak{S}_{n-1}$. By Theorem 2.5, L is isolated in $\mathfrak{H}_{n-1}$, and hence $T^{m_k}(H) = L$ for all large k . Thus for some $k \in \mathbb{N}$, $T^{m_k}(H) = T^{m_k+i}(H)$ for all $i \in \mathbb{N}$. For $l = m_{k+1} - m_k$, we have that H is T^l -invariant. Since T^l acts distally on Sub_G^p , and also on Sub_H^p , and $\dim H = n - 1$, by the induction hypothesis we get that $(T|_H)^{lr} = \text{Id}$ for some $r \in \mathbb{N}$. Moreover, since G/H is isomorphic to $\mathbb{T}^1$, we have that $\overline{T}^2 = \text{Id}$, where $\overline{T}$ is the automorphisms of G/H induced by T . It follows that for $m = 2lr$, all the eigenvalues of T^m are equal to 1, and hence T^m is unipotent. Since T^m acts distally on Sub_G^p , by Theorem 1.3 of [9], $T^m = \text{Id}$. Thus the assertion holds for all n by induction. $\square$

We say that a subgroup $\mathcal{H}$ of homeomorphisms of X acts distally on a topological space X if for any $x, y \in X$ such that $x \neq y$, $\overline{\{(T(x), T(y)) \mid T \in \mathcal{H}\}} \cap \{(d, d) \mid d \in X\} = \emptyset$. Note that if $\mathcal{H}$ acts distally on X , then every element of $\mathcal{H}$ acts distally on X . The following corollary will be useful in the proof of Theorem 3.5.

Corollary 3.2. *Let $\mathcal{H}$ be a subgroup of $\text{Aut}(\mathbb{T}^n)$ for some $n \in \mathbb{N}$. Then the following are equivalent:*

1. *Every element of $\mathcal{H}$ acts distally on $\text{Sub}_{\mathbb{T}^n}^p$.*
2. *$\mathcal{H}$ acts distally on $\text{Sub}_{\mathbb{T}^n}^p$.*
3. *$\mathcal{H}$ is a finite group.*

Proof. The statements (3) $\implies$ (2) $\implies$ (1) are obvious. Now we show that (1) $\implies$ (3). This holds for $n = 1$ as $\text{Aut}(\mathbb{T})$ is a group of order 2. Now suppose $n \geq 2$. Suppose (1) holds. By Theorem 3.1, every element of $\mathcal{H}$ has finite order. Note that $\text{Aut}(\mathbb{T}^n) = \text{GL}(n, \mathbb{Z})$ is a discrete subgroup of $\text{GL}(n, \mathbb{R})$ as well of $\text{GL}(n, \mathbb{C})$. Then $\mathcal{H}$ is closed in $\text{GL}(n, \mathbb{C})$ and by Theorem 1.1 of [13], $\mathcal{H}$ is compact. As $\mathcal{H} \subset \text{GL}(n, \mathbb{Z})$ is discrete, we get that $\mathcal{H}$ is finite. Thus (3) holds, and hence (1 – 3) are equivalent. $\square$

In [9], for $T \in \text{Aut}(G)$ it is shown that T acts distally on Sub_G^p implies that it acts distally on G , where G is a connected non-abelian Lie group without any (nontrivial) central torus; more generally, if G is not a vector group and T acts distally on the maximal central torus. The following corollary generalises Theorem 1.2 of [9].

Corollary 3.3. *Let G be a connected Lie group which is not a vector group and let $T \in \text{Aut}(G)$. If T acts distally on Sub_G^p , then T acts distally on G .*

Proof. Suppose T acts distally on Sub_G^p for a connected Lie group G which is not a vector group. Let M be the largest compact connected central subgroup of G . If M is trivial, then the assertion follows from Theorem 1.1 (2) of [9]. Now suppose M is nontrivial. Then $M \cong \mathbb{T}^n$ for some $n \in \mathbb{N}$. Note that M is invariant under T and $T|_M$ acts distally on Sub_M^p . By Theorem 3.1 we get that some power of $T|_M$ is the identity map. In particular, T acts distally on M . As T acts distally on Sub_G^p , by Theorem 1.2 of [9], T acts distally on G . $\square$

The following theorem generalises Theorems 1.1 (2) and 1.4 of [9], and also a part of Theorem 4.1 of [29].

Theorem 3.4. *Let G be a connected Lie group and let $T \in \text{Aut}(G)$. Consider the following statements.*

1. *T acts distally on Sub_G^p .*
2. *T acts distally on $\overline{\text{Sub}_G^c}$.*
3. *T acts distally on Sub_G^a .*
4. *T acts distally on Sub_G .*
5. *T is contained in a compact subgroup of $\text{Aut}(G)$.*

Then (2 – 5) are equivalent. If G is not a vector group, then (1 – 5) are equivalent.

Proof. Statements (5) $\implies$ (4) $\implies$ (3) $\implies$ (2) $\implies$ (1) are trivial. Suppose G is not a vector group. We need to prove that (1) $\implies$ (5). Suppose (1) holds, i.e. T acts distally on Sub_G^p . Let M be the largest compact connected central subgroup of G . Then M is characteristic in G , i.e. $T(M) = M$. Then (1) implies that T acts distally on Sub_M^p . By Theorem 3.1, for some $m \in \mathbb{N}$, T^m acts trivially on M , in particular, T acts distally on M . Now by Theorem 1.4 of [9], T is contained in a compact subgroup of $\text{Aut}(G)$. Thus (5) holds, and (1 – 5) are equivalent if G is not a vector group.

Now suppose G is a vector group and suppose (2) holds. Then (2) implies that $T \in (\text{NC})$, i.e. the trivial subgroup $\{e\}$ is not a limit point of $\{T^n(C)\}_{n \in \mathbb{Z}}$ for any discrete closed cyclic subgroup C of G . Then by Theorem 4.1 of [29], T is contained in a compact subgroup of $\text{Aut}(G)$, i.e. (5) holds, and (2 – 5) are equivalent if G is a vector group. $\square$

The following generalises a part of Theorem 4.1 of [29] and Theorem 1.5 of [9] when G is not a vector group. Note that in case G is a vector group, the equivalence of (1) and (2), as well as that of (3 – 6) is shown in Theorem 1.5 of [9]. Note also that it is not possible to generalise Theorem 4.1 of [29] for all connected Lie groups G , for if G is an n -torus, $n \geq 2$, then $\text{Aut}(G) \cong \text{GL}(n, \mathbb{Z})$ and every $T \in \text{Aut}(G)$ belongs to (NC) .

Theorem 3.5. *Let G be a connected Lie group. Let $\mathcal{H}$ be a subgroup of $\text{Aut}(G)$. Consider the following statements:*

1. *Every element of $\overline{\mathcal{H}}$ acts distally on Sub_G^p .*
2. *$\mathcal{H}$ acts distally on Sub_G^p .*

- 3. $\mathcal{H}$ acts distally on $\overline{\text{Sub}_G^c}$.
- 4. $\mathcal{H}$ acts distally on Sub_G^a .
- 5. $\mathcal{H}$ acts distally on Sub_G .
- 6. $\overline{\mathcal{H}}$ is a compact group.

Then (1) and (2) are equivalent, and (3 – 6) are equivalent. If G is not a vector group, then (1 – 6) are equivalent.

Proof. It is obvious that (6) $\implies$ (5) $\implies$ (4) $\implies$ (3) $\implies$ (2). Also if H acts distally on Sub_G^p , then so does $\overline{\mathcal{H}}$ since Sub_G^p is compact (cf. [12], Theorem 1). Therefore (2) $\implies$ (1). Now suppose (1) holds. Let M be the largest compact connected central subgroup of G . Then M is characteristic in G and in particular, it is invariant under the action of $\overline{\mathcal{H}}$. If every element of $\overline{\mathcal{H}}$ acts distally on Sub_M^p , then by Theorem 3.2, $\{T|_M \mid T \in \overline{\mathcal{H}}\}$ is a finite group. Therefore $\overline{\mathcal{H}}$, and hence, $\mathcal{H}$ acts distally on M . If G is not a vector group, by Theorem 1.5 of [9], $\overline{\mathcal{H}}$ is compact. Thus (6) holds, and hence (1 – 6) are equivalent.

If $G = \mathbb{R}^n$, a vector group, then the assertions (1) $\implies$ (2) and (3) $\implies$ (6) are proven in Theorem 1.5 in [9]. $\square$

4 Expansivity of actions of automorphisms of $\mathbb{T}^n$ on $\text{Sub}_{\mathbb{T}^n}^p$

In this section we prove that $\mathbb{T}^n$ does not have any automorphism that acts expansively on $\text{Sub}_{\mathbb{T}^n}^p$ for any $n \geq 2$ (see Theorem 4.2), and prove Theorem 4.5. We also prove that every element of $\text{GL}(n, \mathbb{Z})$ has infinitely many orbits consisting of $(n - 1)$ -dimensional rational subspaces (see Theorem 4.3).

We first state some well-known properties about expansive maps, as listed in Lemma 2.1 of [22].

Lemma 4.1 ([30], Corollary 5.22 & Theorem 5.26). *Let (X, d) be a compact metric space. Then the following hold for homeomorphisms of X :*

- (1) *Expansivity is a topological conjugacy invariant.*
- (2) *Expansivity of a homeomorphism is independent of the metric chosen as long as the metric induces the topology of X . However, expansivity constant may change.*

Moreover, the following hold for any homeomorphism ϕ of X :

- (3) ϕ^n is expansive for some $n \in \mathbb{Z} \setminus \{0\}$ if and only if ϕ^n is expansive for all $n \in \mathbb{Z} \setminus \{0\}$.
- (4) For any $n \in \mathbb{Z} \setminus \{0\}$, if ϕ is expansive then ϕ^n has only finitely many fixed points.
- (5) If ϕ is expansive and Y is a closed ϕ -invariant subset of X , the $\phi|_Y$ is also expansive.

Theorem 3.1 of [22] shows that a nontrivial connected Lie group does not admit any automorphism that acts expansively on Sub_G^a . As $\text{Sub}_G^p \subset \text{Sub}_G^a$, the following generalise Theorem 3.1 of [22] for the case when G is an n -torus, $n \geq 2$. (For $n = 1$, Sub_G^p has only two elements, and hence it holds trivially that every $T \in \text{Aut}(G)$ acts expansively on Sub_G^p .)

Theorem 4.2. *Let $G = \mathbb{T}^n$, the n -torus, for any $n \geq 2$. Then G does not admit any automorphism that acts expansively on Sub_G^p .*

Before proving the theorem, we define a notion of rational subspaces in $\mathbb{R}^n$ and discuss their orbits under the action of $\text{GL}(n, \mathbb{Z})$. A subspace W of $\mathbb{R}^n$ is said to be a *rational subspace*, if W is generated by $W \cap \mathbb{Z}^n$; equivalently, if $W \cap \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^k$, where $k = \dim(W)$. Note that W is a rational subspace if and only if the group $W + \mathbb{Z}^n$ is closed in $\mathbb{R}^n$. Each k -dimensional rational subspace W corresponds to a k -subtorus (k -dimensional compact connected subgroup) in $\mathbb{T}^n$; namely, the image of W in $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Any subspace generated by some integer points of $\mathbb{R}^n$ is a rational subspace. Note that if W is a rational subspace of $\mathbb{R}^n$, then $T(W)$ is also a rational subspace (of $\mathbb{R}^n$) with the same dimension as that of W , for all $T \in \text{GL}(n, \mathbb{Z})$. There are countably infinitely many k -dimensional rational subspaces in $\mathbb{R}^n$ for each k with $0 < k < n$. Let $\mathbb{H}_k$ (resp. R_k) denote the space of all k -dimensional subspaces (resp. rational subspaces) of $\mathbb{R}^n$. Then $\mathbb{H}_k$ is a closed (compact) proper subspace of $\text{Sub}_{\mathbb{R}^n}$, and $\mathbb{H}_k$ (resp. R_k) is invariant under the action of $\text{GL}(n, \mathbb{R})$ (resp. $\text{GL}(n, \mathbb{Z})$). It is easy to see that R_k is dense in $\mathbb{H}_k$, $0 \leq k \leq n$.

A linear map $T \in \text{GL}(n, \mathbb{R})$ is said to be *proximal* if it has a unique eigenvalue of maximum absolute value; such an eigenvalue is real. It is well-known that if T is proximal, then for any $L \in \text{Sub}_{\mathbb{R}^n}^p$, $T^n(L) \rightarrow L_\alpha$, where L_α in $\text{Sub}_{\mathbb{R}^n}^p$ is the one-dimensional eigenspace corresponding to the real eigenvalue α with maximum absolute value.

A linear map $T \in \text{GL}(n, \mathbb{R})$ is *distal* (i.e. it acts distally on $\mathbb{R}^n$) if and only if all its eigenvalues have absolute value 1; this is well-known and easy to prove (see

e.g. [18], [11] or [1]). Moreover, if $T \in \text{GL}(n, \mathbb{Z})$ is distal, then all its eigenvalues are roots of unity and T^m is unipotent for some $m \in \mathbb{N}$.

The following proposition will be useful for the proof of Theorem 4.2. Note that the condition below that T does not keep any nontrivial proper rational subspace of $\mathbb{R}^n$ invariant implies that T does not keep any proper subtorus of $\mathbb{T}^n$ invariant, and hence that either $T^m = \text{Id}$ for some $m \in \mathbb{N}$ or that T is ergodic on $\mathbb{T}^n$, $n \geq 2$; this follows from Theorems 2.3 or Theorem 3.15 of [17], (see also Proposition 2.1 of [23] or that of [25]), as there is a connected T -invariant subgroup H such that the T -action on H is ergodic and the corresponding automorphism $\bar{T}$ of G/H is distal. Now the condition implies that $G = H$ and T is ergodic or $G = \{e\}$ and T is distal, in the later case T^m is unipotent for some $m \in \mathbb{N}$, and hence $T^m = \text{Id}$, otherwise T would keep a proper rational subspace invariant. We will later see that Theorem 4.3 holds for all $T \in \text{GL}(n, \mathbb{Z})$, $n \geq 2$, without this extra condition on T (see Theorem 4.4).

Proposition 4.3. *Let $T \in \text{GL}(n, \mathbb{Z})$, $n \geq 2$. Suppose T does not keep any nonzero proper rational subspace of $\mathbb{R}^n$ invariant. Then there are infinitely many $(n - 1)$ -dimensional rational subspaces with disjoint T -orbits in $\text{Sub}_{\mathbb{R}^n}$.*

Proof. For $n \geq 2$, let $T \in \text{GL}(n, \mathbb{Z})$ be such that T does not keep any nonzero proper rational subspace of $\mathbb{R}^n$ invariant. This is equivalent to the condition that any nonzero proper T -invariant subspace is not contained in any proper rational subspace of $\mathbb{R}^n$. If V is a T -invariant subspace contained in a proper rational subspace (say) W of $\mathbb{R}^n$, then $V' = (\overline{V + \mathbb{Z}^n})^0$ is a T -invariant rational subspace of $\mathbb{R}^n$ and it is contained in W . Therefore, V' is proper and the condition on T as in the hypothesis implies that $V' = \{0\}$, and hence that $V = \{0\}$.

Now we prove the assertion that there are infinitely many $(n - 1)$ -dimensional rational subspaces with disjoint T -orbits. It is easy to see that the assertion holds for T if and only if it holds for T^m for any $m \in \mathbb{Z} \setminus \{0\}$. If $T^m = \text{Id}$ for some $m \in \mathbb{N}$, then all the rational subspaces are T^m -invariant, i.e. the assertion holds for T^m , and hence, for T . Now suppose $T^m \neq \text{Id}$ for any $m \in \mathbb{N}$. Note that T is not distal. For if T is distal, then for some $m \in \mathbb{N}$, T^m is unipotent, i.e. it has an eigenvalue 1, and its eigenspace V_1 is T -invariant, and V_1 is a nonzero proper rational subspace, which leads to a contradiction. Thus T is not distal, i.e. it has an eigenvalue with absolute value not equal to 1. Since $\det(T) = \pm 1$, T has at least one eigenvalue with absolute value greater than 1, and one with absolute value less than 1. Thus one can choose two distinct eigenvalues of T with different absolute values, say, α and β with $|\alpha| > |\beta|$.

Case I: Suppose α and β as above are real, with $|\alpha| > |\beta|$. Let V_α (resp. V_β) be the eigenspace corresponding to α (resp. β). For any proper rational subspace W , $W \cap V_\alpha$ is a T -invariant subspace, and hence we have that $W \cap V_\alpha = \{0\}$. Since this holds in particular for any rational space $W \in R_{n-1}$, we have that $\dim(V_\alpha) = 1$. Similarly, $V_\beta \cap W = \{0\}$ for every $W \in R_{n-1}$ and $\dim(V_\beta) = 1$.

Let $V_{\alpha\beta} = V_\alpha + V_\beta$. Then $\dim(V_{\alpha\beta}) = 2$ and $T(V_{\alpha\beta}) = V_{\alpha\beta}$. Moreover, for any rational space $W \in R_{n-1}$, $\dim(W \cap V_{\alpha\beta}) = 1$ as $V_{\alpha\beta} \not\subset W$. Let $W \cap V_{\alpha\beta} = L_w$. Note that as $|\alpha| > |\beta|$, both $T|_{V_{\alpha\beta}}$ and $T^{-1}|_{V_{\alpha\beta}}$ are proximal. Then

$$T^m(L_w) \rightarrow V_\alpha \quad \text{and} \quad T^{-m}(L_w) \rightarrow V_\beta \quad \text{as } m \rightarrow \infty.$$

Now suppose $T^{m_k}(W) \rightarrow H$ for some unbounded sequence $\{m_k\} \subset \mathbb{Z}$. Then $V_\alpha \subset H$ (resp. $V_\beta \subset H$) if $m_k \rightarrow \infty$ (resp. if $m_k \rightarrow -\infty$). Let

$$\mathcal{A} = \{H \in \mathbb{H}_{n-1} \mid V_\alpha \subset H \text{ or } V_\beta \subset H\}.$$

Then $\mathcal{A}$ is a closed subset of $\mathbb{H}_{n-1}$, the space of all $(n-1)$ -dimensional subspaces of $\mathbb{R}^n$, and $\mathbb{H}_{n-1}$ is a proper closed (compact) subset of $\text{Sub}_{\mathbb{R}^n}$. Any rational subspace $W \in R_{n-1}$ does not belong to $\mathcal{A}$ as it contains neither V_α nor V_β . We can show that $\mathcal{A}$ is closed using Lemma 2.2 of [29]. Since $\mathbb{H}_{n-1}$ is a (compact) metric space and $\mathcal{A}$ is a proper closed subset of it, we can choose a neighbourhood U of $\mathcal{A}$ such that $\mathbb{H}_{n-1} \setminus U$ has nonempty interior, and hence $\mathbb{H}_{n-1} \setminus U$ contains infinitely many rational subspaces from R_{n-1} . Now U contains all but finitely many elements of the orbit $\{T^m(W)\}_{m \in \mathbb{Z}}$ of any rational subspace $W \in R_{n-1}$. Choose $W_1 \in R_{n-1}$ such that $W_1 \notin U$ and $T^m(W_1) \in U$ for all $m \in \mathbb{Z}$ with $|m| > l_1$ for some $l_1 \in \mathbb{N}$. Since $R_{n-1} \setminus U$ is infinite, we can choose $W_2 \in R_{n-1}$ such that $W_2 \notin U$ and $W_2 \neq T^m(W_1)$ if $|m| \leq l_1$. Now $T^m(W_2) \in U$ for all $m \in \mathbb{Z}$ with $|m| > l_2$, for some $l_2 \in \mathbb{N}$. Then the T -orbit of W_2 is disjoint from that of W_1 . For if $T^m(W_1) = W_2$ for some $m \in \mathbb{Z}$, then $|m| > l_1$, but then $T^m(W_1) \in U$ while $W_2 \notin U$, which leads to a contradiction.

For $k \geq 2$ and $1 \leq i \leq k$, suppose there are $W_i \in R_{n-1}$ such that $W_i \notin U$ with $T^m(W_i) \in U$ for all $m \in \mathbb{Z}$ with $|m| > l_i$ and $W_i \neq T^m(W_j)$ for any $1 \leq j < i$ and $|m| \leq l_j$. Since $R_{n-1} \setminus U$ is infinite, we can choose a rational subspace W_{k+1} as follows:

$$W_{k+1} \in R_{n-1}, \quad W_{k+1} \notin U \quad \text{and} \quad W_{k+1} \notin \cup_{i=1}^k \{T^m(W_i) \mid |m| \leq l_i\}.$$

Hence the T -orbit of W_{k+1} is disjoint from those of W_j , $1 \leq j \leq k$. Moreover, $T^m(W_{k+1}) \in U$ for all $m \in \mathbb{Z}$ with $|m| \geq l_{k+1}$ for some $l_{k+1} \in \mathbb{N}$. It is easy to see

that the T -orbit of W_{k+1} is disjoint from those of $W_1, \dots, W_k$. Thus by induction, there exist infinitely many $(n-1)$ -dimensional rational subspaces W_k , $k \in \mathbb{N}$, whose T -orbits are disjoint.

Case II: Suppose one of the eigenvalues α and β of T (as above) is real, and the other is complex. Replacing T by T^{-1} if necessary, we may assume that α is real with $|\alpha| > 1$, and β is complex with $|\beta| < 1$. Let V_α be the eigenspace for α , and V_β be the 2-dimensional vector subspace of $\mathbb{R}^n$ such that $T(V_\beta) = V_\beta$, and the eigenvalues of $T|_{V_\beta}$ are $\beta, \bar{\beta}$. Let $V_{\alpha\beta} = V_\alpha + V_\beta$. Then $\dim(V_{\alpha\beta}) = 3$, and T keeps $V_{\alpha\beta}$ invariant. There exist ϕ and S in $\text{GL}(V_{\alpha\beta})$ such that $T|_{V_{\alpha\beta}} = \phi S = S\phi$, $S|_{V_\alpha} = T|_{V_\alpha} = \alpha \text{Id}$, $S|_{V_\beta} = |\beta| \text{Id}$, $\phi|_{V_\alpha} = \text{Id}$, ϕ keeps V_β invariant and $\phi|_{V_\beta}$ is contained in a compact subgroup of $\text{GL}(V_\beta)$. In particular, ϕ is contained in a compact subgroup of $\text{GL}(V_{\alpha\beta})$. Since V_α and V_β are T -invariant, as noted above, neither V_α nor V_β is contained in any proper rational subspace.

For a rational vector subspace $W \in R_{n-1}$, let $L_w = W \cap V_\beta$. Then $V_\beta = L_w + L'_w$ for some one dimensional subspace L'_w of V_β . Here, $L'_w \not\subset W$ as $V_\beta \not\subset W$. Let $V_1 := V_\alpha + L'_w$. Then $\dim(V_1) = 2$ and V_1 is S -invariant. Now $\dim(W \cap V_1) = 1$ as neither V_α nor L'_w is contained in W . Let $S_1 := S|_{V_1}$. Then S_1 has two eigenvalues α and $|\beta|$ and both S_1 and S_1^{-1} are proximal as $|\alpha| > |\beta|$. Let $L_1 = W \cap V_1$. Then $S^m(L_1) = S_1^m(L_1) \rightarrow V_\alpha$ and $S^{-m}(L_1) = S_1^{-m}(L_1) \rightarrow L'_w$ in $\text{Sub}_{\mathbb{R}^n}^p$ as $m \rightarrow \infty$. Now for $m \in \mathbb{Z}$,

$$T^m(W \cap V_{\alpha\beta}) = \phi^m S^m(L_w + L_1) = \phi^m(S^m(L_w) + S^m(L_1)) = \phi^m(L_w) + \phi^m(S^m(L_1)).$$

As ϕ is contained in a compact group, all the limit points of $\{\phi^m\}$ keep V_α and V_β invariant. In particular, $\phi^m(L_w) \subset V_\beta$ for all m . Moreover, $\phi^m(S^m(L_1)) \rightarrow V_\alpha$ as $m \rightarrow \infty$. Thus all the limit points of $\{T^m(W \cap V_{\alpha\beta})\}$ contain V_α as $m \rightarrow \infty$.

The limit points of $\phi^{-m}(S^{-m}(L_1))$ are $\psi(L'_w)$ as $m \rightarrow \infty$, where ψ is any limit point of $\{\phi^m \mid -\infty < m \leq -1\}$. Thus the limit points of $\{T^{-m}(W \cap V_{\alpha\beta})\}$ are contained in V_β as $m \rightarrow \infty$. As $\dim(V_\beta) = 2 = \dim(W \cap V_{\alpha\beta})$, we get that $T^{-m}(W \cap V_{\alpha\beta}) \rightarrow V_\beta$ as $m \rightarrow \infty$.

Now we have that all the limit points of $\{T^m(W)\}$ contain V_α as $m \rightarrow \infty$, and contain V_β as $m \rightarrow -\infty$, i.e. for any rational subspace $W \in R_{n-1}$, if $T^{m_k}(W) \rightarrow H$ then $V_\alpha \subset H$ when $m_k \rightarrow \infty$, and $V_\beta \subset H$ when $m_k \rightarrow -\infty$. Consider the set $\mathcal{A} = \{H \in \mathbb{H}_{n-1} \mid V_\alpha \subset H \text{ or } V_\beta \subset H\}$. Then $\mathcal{A}$ is a proper closed (compact) subspace of $\mathbb{H}_{n-1}$. As R_{n-1} is dense in $\mathbb{H}_{n-1}$, using similar arguments as in Case I, we can find infinitely many rational subspaces in R_{n-1} whose T -orbits are disjoint.

Case III: Now suppose both the eigenvalues α and β of T (as above) are complex

with $|\alpha| > 1$ and $|\beta| < 1$. Let V_α (resp. V_β) be a T -invariant 2-dimensional subspace such that $T|_{V_\alpha}$ (resp. $T|_{V_\beta}$) have α and $\bar{\alpha}$ (resp. β and $\bar{\beta}$) as eigenvalues. Let $V_{\alpha\beta} = V_\alpha + V_\beta$. Then $\dim(V_{\alpha\beta}) = 4$ and T keeps $V_{\alpha\beta}$ invariant. Now we have that $T|_{V_{\alpha\beta}} = \phi S = S\phi$ where, $S|_{V_\alpha} = |\alpha| \text{Id}$, $S|_{V_\beta} = |\beta| \text{Id}$, and each of the maps $\phi|_{V_\alpha}$ and $\phi|_{V_\beta}$ generate a relatively compact group in $\text{GL}(V_\alpha)$ and $\text{GL}(V_\beta)$ respectively. In particular ϕ is contained in a compact subgroup of $\text{GL}(V_{\alpha\beta})$.

For any rational subspace $W \in R_{n-1}$, $V_\alpha \not\subset W$ and $V_\beta \not\subset W$. Let $L_{\alpha_1} = W \cap V_\alpha$ and let $L_{\beta_1} = W \cap V_\beta$. Then $V_\alpha = L_{\alpha_1} + L_{\alpha_2}$ and $V_\beta = L_{\beta_1} + L_{\beta_2}$ for some one-dimensional subspaces L_{α_2} in V_α and L_{β_2} in V_β . Let $V_2 = L_{\alpha_2} + L_{\beta_2}$. Then $\dim(V_2) = 2$ and V_2 is S -invariant. Let $S_2 := S|_{V_2}$. Then $S_2 \in \text{GL}(V_2)$ and S_2 and S_2^{-1} are both proximal. Let $L_2 = W \cap V_2$. Then $\dim(L_2) = 1$ as neither L_{α_2} nor L_{β_2} is contained in W . As S_2 is proximal, we get that $S^m(L_2) \rightarrow L_{\alpha_2}$ and $S^{-m}(L_2) \rightarrow L_{\beta_2}$ in $\text{Sub}_{\mathbb{R}^n}^p$ as $m \rightarrow \infty$. Now

$$S^m(W \cap V_{\alpha\beta}) = S^m(L_{\alpha_1} + L_{\beta_1} + L_2) = L_{\alpha_1} + L_{\beta_1} + S^m(L_2).$$

Therefore, as $m \rightarrow \infty$,

$$\begin{aligned} S^m(W \cap V_{\alpha\beta}) &\rightarrow L_{\alpha_1} + L_{\beta_1} + L_{\alpha_2} = V_\alpha + L_{\beta_1}, \text{ and} \\ S^{-m}(W \cap V_{\alpha\beta}) &\rightarrow L_{\alpha_1} + L_{\beta_1} + L_{\beta_2} = L_{\alpha_1} + V_\beta. \end{aligned}$$

As ϕ keeps V_α and V_β invariant, so does every limit point of $\{\phi^m\}_{m \in \mathbb{Z}}$. As $\{\phi^m\}_{m \in \mathbb{Z}}$ is relatively compact and $T^m(W \cap V_{\alpha\beta}) = \phi^m S^m(W \cap V_{\alpha\beta})$ for all $m \in \mathbb{Z}$, we have that limit points of $\{T^m(W \cap V_{\alpha\beta}) \mid m \in \mathbb{N}\}$ contain V_α and limit points of $\{T^{-m}(W \cap V_{\alpha\beta}) \mid m \in \mathbb{N}\}$ contain V_β . Thus limit points of $\{T^m(W) \mid m \in \mathbb{N}\}$ contain V_α and $\{T^{-m}(W) \mid m \in \mathbb{N}\}$ contain V_β for every rational subspace $W \in R_{n-1}$.

Let $\mathcal{A} = \{H \in \mathbb{H}_{n-1} \mid V_\alpha \subset H \text{ or } V_\beta \subset H\}$. Then $\mathcal{A}$ is a proper closed (compact) subset of $\mathbb{H}_{n-1}$. As R_{n-1} is dense in $\mathbb{H}_{n-1}$, using similar arguments as in Case I, we can find infinitely many rational subspaces in R_{n-1} whose T -orbits are disjoint. $\square$

Now we prove Theorem 4.2 where we will use Theorem 4.3 for a particular class of automorphisms.

Proof of Theorem 4.2. Let $G = \mathbb{T}^n$, the n -torus, for any $n \geq 2$ and let $T \in \text{Aut}(G)$. We want to show that the T -action on Sub_G^p is not expansive. If $T^m = \text{Id}$ for some $m \in \mathbb{N}$, then by Theorem 4.1, the T -action on Sub_G^p is not expansive as Sub_G^p is infinite. Now suppose that $T^m \neq \text{Id}$ for any $m \in \mathbb{N}$.

Recall that $\mathfrak{S}_{n-1}$ is the collection of all $(n-1)$ -dimensional subtori of $G = \mathbb{T}^n$, and $\mathfrak{H}_{n-1} = \mathfrak{S}_{n-1} \cup \{G\}$, both of these sets are T -invariant subsets of Sub_G^p , and $\mathfrak{H}_{n-1}$ is closed in Sub_G^p , and hence it is compact. Moreover, every $H \in \mathfrak{S}_{n-1}$ is isolated in $\mathfrak{H}_{n-1}$ (cf. Theorem 2.5); in particular, $\{H\}$ is open in $\mathfrak{H}_{n-1}$. Note that for every H in $\mathfrak{S}_{n-1}$, $T^m(H) \rightarrow G$, if $m \rightarrow \pm\infty$ unless $T^m(H) = H$ for some $m \in \mathbb{N}$. However, since Sub_G^p is countable, we are not able to use Theorem 1 of [26], even if $T^m(H) \neq H$, $m \in \mathbb{N}$, for infinitely many $H \in \mathfrak{S}_{n-1}$.

Step 1: Let $\mathbf{d}$ be the metric on Sub_G^p . If possible, suppose that the T -action on Sub_G^p is expansive. Then the T -action on $\mathfrak{H}_{n-1}$ is also expansive; suppose $\epsilon > 0$ is an expansive constant for this action. For any $H \in \mathfrak{S}_{n-1}$, there exists $m \in \mathbb{Z}$ such that $\mathbf{d}(T^m(H), G) > \epsilon$. Let $B(G, \epsilon)$ be the ball of radius ϵ centered at G in Sub_G^p . Consider the collection $\{\{H\} \mid H \in \mathfrak{S}_{n-1}\} \cup \{B(G, \epsilon)\}$; it is an open cover of $\mathfrak{H}_{n-1}$. As $\mathfrak{H}_{n-1}$ is compact, there exist $H_1, \dots, H_k$ in $\mathfrak{S}_{n-1}$ such that $\mathfrak{H}_{n-1} = B(G, \epsilon) \cup \{H_1\} \cup \dots \cup \{H_k\}$. As the T -action on Sub_G^p is expansive with an expansive constant ϵ , for any $H \in \mathfrak{S}_{n-1}$, there exist $m \in \mathbb{Z}$ (which depends on H) such that $T^m(H) = H_i$ for some $i \in \{1, \dots, k\}$. This implies that T has finitely many orbits in $\mathfrak{S}_{n-1}$. We will now show that T has infinitely many disjoint orbits in $\mathfrak{S}_{n-1}$, which would contradict the expansivity of T .

Step 2: Suppose that all the eigenvalues of T have absolute value 1, i.e. T is distal. As $T \in \text{GL}(n, \mathbb{Z})$, some power of T is unipotent (see e.g. Lemma 2.5 of [2]). The statement that there are infinitely many $(n-1)$ -dimensional subtori with disjoint T -orbits is equivalent to the statement that there are infinitely many $(n-1)$ -dimensional subtori with disjoint T^m -orbits for any $m \in \mathbb{N}$. Without loss of any generality, we may assume that T is unipotent and that $T \neq \text{Id}$. By Proposition 3.10 of [29], there exist nontrivial closed connected T -invariant subgroups $\{e\} = K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_{l+1} = G$ of $\mathbb{T}^n$ such that T acts trivially on K_i/K_{i-1} , $1 \leq i \leq l+1$ and T does not act trivially on K_i/K_{i-2} , $2 \leq i \leq l+1$ (see also Lemma 2.5 of [2]). Note that $l \neq 0$ as $T \neq \text{Id}$. If $\dim(K_l) \leq n-2$, then $\dim(G/K_l) \geq 2$, and we can choose infinitely many distinct tori (closed connected subgroups) B_m of co-dimension 1 in G/K_l , $m \in \mathbb{N}$. Let H_m be a subtorus in G containing K_l such that $H_m/K_l = B_m$ for each m . Then $\dim(H_m) = n-1$ and H_m is T -invariant for every m . Moreover, H_m 's are distinct as B_m 's are so and $\{H_m\}$ is a T -orbit in $\mathfrak{S}_{n-1}$ for each m .

Now suppose $\dim(K_l) = n-1$. There exists a closed one-parameter subgroup $C_0 = \{x_t\}$ such that $G = C_0 \times K_l$. Now $T(x_t) = x_t y_t$, $t \in \mathbb{R}$, for some nontrivial closed one-parameter subgroup $\{y_t\} \subset K_l$. For $m \in \mathbb{N}$, let $C_m := \{x_t y_{t/m}\}$.

Suppose $\dim(K_l) = 1$. Then $l = 1$ and $T|_{K_l} = \text{Id}$. Then $T(\{x_t\}) = \{x_t y_t\}$ and $T^k(\{x_t\}) = \{x_t y_t^k\} = \{x_t y_{kt}\}$ and the T -orbit of C_0 is $\{\{x_t y_{kt}\} \mid k \in \mathbb{Z}\}$. Moreover,

$T^k(C_m) = \{x_t y_{(k+1/m)t}\}$ and the T -orbit of C_m is $\{\{x_t y_{(k+1/m)t}\} \mid k \in \mathbb{Z}\}$. Observe that the T -orbits of C_m 's are disjoint.

Now suppose $\dim(K_l) \geq 2$. We can choose a T -invariant subtorus H such that $K_{l-1} \subset H \subset K_l$ and $K_l = \{y_t\} \times H$. Here, $H = K_{l-1}$ if $\dim(K_l/K_{l-1}) = 1$. Note that $T(y_t) \in y_t K_{l-1} \subset y_t H$, $t \in \mathbb{R}$. Let $H_m := C_m H$, $m \in \mathbb{N}$. Then $\dim(H_m) = n-1$ and $H_m \in \mathfrak{S}_{n-1}$, $m \in \mathbb{N}$. As H is T -invariant, it is easy to see that the T -orbits of H_k and H_m are disjoint if $k \neq m$. Thus, if T is not distal then T has infinitely many disjoint orbits in $\mathfrak{S}_{n-1}$.

Step 3: Now suppose T admits an eigenvalue with absolute value other than 1, i.e. T is not distal. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the natural projection with $\ker \pi = \mathbb{Z}^n$. Then we have $T \in \text{GL}(n, \mathbb{Z})$ as a linear automorphism of $\mathbb{R}^n$ with $\pi \circ T = T \circ \pi$. Recall that R_{n-1} , the set of all $(n-1)$ -dimensional rational subspace of $\mathbb{R}^n$, is a T -invariant subspace of $\text{Sub}_{\mathbb{R}^n}$, then $\pi(R_{n-1}) = \mathfrak{S}_{n-1}$, and the map R_{n-1} to $\mathfrak{S}_{n-1}$ induced by π is bijective.

Suppose T does not keep any nontrivial proper rational subspace of $\mathbb{R}^n$ invariant. Then by Theorem 4.3 in this case, there are infinitely many rational subspaces in R_{n-1} with disjoint T -orbits. Then there are infinitely many $(n-1)$ -dimensional tori in $\text{Sub}_{\mathbb{T}^n}^P$ with disjoint T -orbits.

Suppose $n = 2$, i.e. $\dim(G) = 2$. Then since T does not have an eigenvalue of absolute value 1, both the eigenvalues of T are real with absolute value other than 1. Thus T does not keep any nontrivial proper rational subspace of $\mathbb{R}^2$ invariant (as such a space would have dimension 1, and it would mean that eigenvalues of T have absolute value 1). Now by Theorem 4.3, T has infinitely many 1-dimensional rational subspaces with disjoint T -orbits. Thus the assertion that T has infinitely many co-dimension one subtori with disjoint T -orbits holds for all $T \in \text{Aut}(G)$ for $n = 2$.

Suppose $n \geq 3$. Suppose the assertion that every $T \in \text{Aut}(G)$ has infinitely many disjoint orbits in $\mathfrak{S}_{k-1}$ holds for any torus G with dimension k such that $1 < k < n$. Now suppose G is such that $\dim(G) = n$. If T is distal, then the assertion holds as shown in Step 2. If T does not keep any nontrivial proper rational subspace of $\mathbb{R}^n$ invariant, equivalently, if T does not keep any proper subtori invariant, then the assertion holds as shown above.

Now suppose that T keeps a nonzero proper rational subspace of $\mathbb{R}^n$ invariant. Then there is a proper subtorus H on G such that $T(H) = H$. Then $\dim(H) < n$. Let $\bar{T} \in \text{Aut}(G/H)$ be the automorphism induced by T . Suppose the dimension of G/H is greater than or equal to 2. Since $\dim(G/H) < n$, by the induction hypothesis, G/H has infinitely many subtori (say) B_m of co-dimension 1 with disjoint

$\overline{T}$ -orbits. Let H_m be the subtori of G containing H such that $H_m/H = B_m$. Then $\dim(H_m) = n - 1$ and H_m 's have disjoint T -orbits.

Now suppose $\dim(G/H) = 1$. Then the corresponding automorphism $\overline{T}$ of G/H , and hence, T has a real eigenvalue which is ± 1 . We can choose a one-dimensional subtorus, say, M of G which is T -invariant. Now $\dim(G/M) = n - 1 \geq 2$. Thus arguing as above for M instead of H , we have that G has infinitely many subtori of dimension $n - 1$ with disjoint T -orbits. By induction, the assertion that there are infinitely many subtori of co-dimension 1 in G with disjoint T -orbits holds.

As noted at the end of Step 1, it follows that the T -action on Sub_G^p is not expansive. $\square$

The proof of Theorem 4.2 actually proves a stronger statement that the T -action on $\mathfrak{H}_{n-1}$ is not expansive, where $\mathfrak{H}_{n-1}$ consists of the whole group G and all the $(n - 1)$ -dimensional subtori of G .

Remark 4.4. As shown in the proof of Theorem 4.2, any $T \in \text{GL}(n, \mathbb{Z})$ admits infinitely many subtori of co-dimension 1 in $\mathbb{T}^n$ with disjoint T -orbits. This is equivalent to the statement that T has infinitely many disjoint T -orbits in R_{n-1} , the space of $(n - 1)$ -dimensional rational subspaces of $\mathbb{R}^n$. Thus Theorem 4.3 holds for any $T \in \text{GL}(n, \mathbb{Z})$ without the condition on T mentioned there.

The following corollary follows easily from Theorem 4.2. We give a proof for the sake of completeness.

Corollary 4.5. *Let G be a connected Lie group such that it contains a central torus of dimension at least 2. Then G does not admit any automorphism that acts expansively on Sub_G^p .*

Proof. Let C be the maximal central torus in G . By the hypothesis, the dimension of C is at least 2. Let $T \in \text{Aut}(G)$. Then $T(C) = C$ and by Theorem 4.2, T does not act expansively on Sub_C^p . As Sub_C^p is a closed subspace of Sub_G^p , by Theorem 4.1 (5), we get that T does not act expansively on Sub_G^p . $\square$

Acknowledgments. We thank Amala Bhawe for valuable discussions. Debamita Chatterjee acknowledges the UGC Non-NET research fellowship, and Himanshu Lekharu acknowledges the CSIR-JRF research fellowship from CSIR, Govt. of India.

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