Rainbow planar Turán numbers of cycles

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November 7, 2025

Abstract

The rainbow Turán number of a fixed graph H, denoted by $\mathrm{ex}^*(n, H)$, is the maximum number of edges in an n-vertex graph such that it admits a proper edge coloring with no rainbow H. We study this problem in planar setting. The rainbow planar Turán number of a graph H, denoted by $\mathrm{ex}_{\mathcal{P}}{}^*(n, H)$, is the maximum number of edges in an n-vertex planar graph such that it has a proper edge coloring with no rainbow H. We consider the rainbow planar Turán number of cycles. Since C_3 is complete, $\mathrm{ex}^*_{\mathcal{P}}(n, C_3)$ is exactly its planar Turán number, which is $2n - 4$ for $n \geq 3$. We show that $\mathrm{ex}^*_{\mathcal{P}}(n, C_4) = 3n - 6$ for $n = k^2 - 3k + 2$ where $k \geq 5$, and $\mathrm{ex}^*_{\mathcal{P}}(n, C_k) = 3n - 6$ for all $k \geq 5$ and $n \geq 3$.

1 Introduction

All graphs in this paper are finite and simple. Given a graph H, the classical *Turán number* of H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as subgraph. The study on the Turán number of graphs is a central topic in extremal graph theory. Motivated by extremal problems in additive number theory and graph coloring, Keevash, Mubayi, Sudakov and Verstraëte [14] initiated the study on the *rainbow Turán problems*. Given a graph G, a *proper edge-coloring* of G is an edge coloring of G such that no two adjacent edges of G receive the same color, and a *k-edge-coloring* of G for some integer k is a an edge coloring of G using k colors. An edge-colored graph H is called *rainbow* if no two edges of H receive the same color. For a fixed graph H, the *rainbow Turán number* of H, denoted by $\mathrm{ex}^*(n, H)$, is defined as the maximum number of edges in an n-vertex graph that has a proper edge-coloring with no rainbow copy of H. It is obvious that $\mathrm{ex}^*(n, H) \geq \mathrm{ex}(n, H)$. In [14], Keevash, Mubayi, Sudakov and Verstraëte determined $\mathrm{ex}^*(n, H)$ asymptotically for non-bipartite graphs H. In particular, they showed that for any fixed H and sufficiently large n,

$$\mathrm{ex}(n, H) \leq \mathrm{ex}^*(n, H) \leq \mathrm{ex}(n, H) + o(n^2).$$

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of colors that are needed to color all the vertices in G such that no two adjacent vertices of G receive the same color. A graph H is called *color-critical* if it contains an edge e such that $\chi(H \backslash e) = \chi(H) - 1$. For example, complete graphs and odd cycles are all color-critical. Keevash, Mubayi, Sudakov and Verstraëte in [14] proved that $\mathrm{ex}(n, H) = \mathrm{ex}^*(n, H)$ if H is color-critical. For even cycles, they [14] showed that for all $k \geq 2$, $\mathrm{ex}^*(n, C_{2k}) \geq cn^{1+1/k}$ for some $c > 0$ and proved that $\mathrm{ex}^*(n, C_{2k}) = O(n^{1+1/k})$ for $k \in \{2, 3\}$. They conjectured that the same asymptotic upper

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bound on $\mathrm{ex}^*(n, C_{2k})$ holds for all $k \geq 2$, which was recently confirmed by Janzer [11]. The rainbow Turán number of paths were recently studied in [12, 13, 3, 9, 8]. Further results on (generalized) rainbow Turán problems can be found in [11, 4] and reference therein.

In a more sparse setting, recently, there has been extensive research on Turán problems in planar graphs. For a fixed graph H, the *planar Turán number* of H, denoted by $\mathrm{ex}_{\mathcal{P}}(n, H)$, is defined as the maximum number of edges in an n-vertex planar graph without containing H as a subgraph. It is clear from Euler's formula that $\mathrm{ex}_{\mathcal{P}}(n, C_3) = 2n - 4$ for $n \geq 3$. Dowden [2] proved that $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}$ for all $n \geq 4$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$. Ghosh, Győri, Martin, Paulos and Xiao [5] showed that $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5n-14}{2}$ for all $n \geq 18$. All bounds above are tight for infinitely many n. In the same paper, Ghosh et al. [5] conjectured that $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq \frac{3(k-1)}{k} n - \frac{6(k+1)}{k}$ for all $k \geq 7$ and sufficiently large n, which was disproved by Cranston, Lidický, Liu and Shantanam [1] for all $k \geq 11$ (see also [16]). However, the conjecture of Gosh et al. [5] for $7 \leq k \leq 10$ may still hold and was recently verified to be true for $k = 7$ very recently by Shi, Walsh and Yu [18] and independently by Győri, Li and Zhou [6]. Confirming a conjecture of Cranston et al. [1], Shi, Walsh and Yu [17] proved an upper bound of $3n - 6 - Dn/\ell^{\log_2 3}$ for $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ for all $\ell, n \geq 4$, where D is some constant. For recent results on the planar Turán number of graphs other than cycles, see the recent survey by Lan, Shi and Song [15] and references therein.

Very recently, motivated by the recent active developments on the rainbow Turán number and planar Turán number, Győri, Martin, Paulos, Tompkins and Varga [7] initiated the study on the *rainbow planar Turán problems*. Given a fixed graph H, the *rainbow planar Turán number* of H, denoted by $\mathrm{ex}^*_{\mathcal{P}}(n, H)$, is defined as the maximum number of edges in an n-vertex planar graph that has a proper edge-coloring with no rainbow copy of H. A *planar triangulation* is an edge-maximal planar graph such that every face of its plane embedding is bounded by a triangle. By Euler's formula, an n-vertex planar triangulation ($n \geq 3$) has exactly $3n-6$ edges. Note that for each $n \geq 3$, there exist planar triangulations on n vertices that can be properly edge-colored with at most 6 colors. For example, Figure 1 is a properly 6-edge-colored planar triangulation on even number vertices. Moreover, deleting the lower left vertex or the upper right vertex results in a properly 6-edge-colored planar triangulation on odd number vertices. It follows that if a graph H has more than six edges then $\mathrm{ex}^*_{\mathcal{P}}(n, H) = 3n - 6$ for $n \geq 3$. Győri et al. [7] made a systematic study on the rainbow planar Turán number pf paths. In particular, they observed that $\mathrm{ex}^*_{\mathcal{P}}(n, P_3) = \lfloor n/2 \rfloor$ and showed that $\mathrm{ex}^*_{\mathcal{P}}(n, P_4) = \mathrm{ex}^*_{\mathcal{P}}(n, P_5) = \lfloor 3n/2 \rfloor$ for all $n \geq 4$. We know that $\mathrm{ex}^*_{\mathcal{P}}(n, P_k) = 3n - 6$ for all $k \geq 8$ and $n \geq 3$. The cases for P_6, P_7 remain open. Győri et al. [7] conjectured that $2n - O(1) \leq \mathrm{ex}^*_{\mathcal{P}}(n, P_6) \leq 2n$ and $5n/2 - O(1) \leq \mathrm{ex}^*_{\mathcal{P}}(n, P_7) \leq 5n/2$. He and Liu in [10] considered the rainbow planar Turán number for some double stars, $S_{1,k}$ for all k except $k = 5$ and $S_{2,2}$, where $S_{s,k}$ denotes the graph obtained by taking an edge with s vertices joining one of its end vertices and k vertices joining the other end vertex. As $S_{1,5}$ has seven edges, we know that $\mathrm{ex}_{\mathcal{P}}(n, S_{1,5}) = 3n - 6$ for $n \geq 3$.

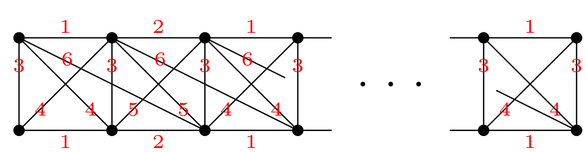

Figure 1: A properly 6-edge-colored planar triangulation.

In this paper, we investigate the rainbow planar Turán number of cycles, which is the next natural planar graph class to consider. For C_3, since a 3-cycle is a complete graph, we have that $\mathrm{ex}^*_{\mathcal{P}}(n, C_3) = \mathrm{ex}_{\mathcal{P}}(n, C_3) = 2n - 4$ for $n \geq 3$. The remaining open cases are C_4, C_5, C_6. It is surprising that there exists an n-vertex planar triangulation such that it has a proper edge-coloring with no rainbow C_4 for infinitely many n. For C_5 and C_6, we also find an n-vertex planar triangulation (see the even case in Figure 1) admitting a proper edge-coloring with no rainbow C_5 or C_6 for each $n \geq 3$.

Theorem 1. *For each integer $k \geq 5$ and $n = k^2 - 3k + 2$, $ex^*_{\mathcal{P}}(n, C_4) = 3n - 6$.*

While we cannot give constructions of planar triangulations having a proper edge-coloring with no rainbow C_4 for other values n, we have the following result for those graphs.

Theorem 2. *Let G be a planar triangulation on at least five vertices such that G has a proper edge-coloring containing no rainbow C_4. Then G has minimum degree 5 and G is 4-connected.*

Theorem 3. *For each $n \geq 3$ and each $k \geq 5$, $ex^*_{\mathcal{P}}(n, C_k) = 3n - 6$.*

This paper is organized as follows. In section 2, we first show Theorem 2, and then we describe an n-vertex planar triangulation with a proper edge-coloring containing no rainbow C_4 for $n = k^2 - 3k + 2$ with each $k \geq 5$. In section 3, we give an n-vertex triangulation with a proper edge-coloring such that it contains no rainbow C_5 or C_6, which implies Theorem 3.

We conclude this section with some terminology and notation. For any positive integer k let $[k] = \{1, 2, \ldots, k\}$, and for positive integers s, t with $s < t$ let $[s, t] = \{s, s + 1, \ldots, t\}$.

Let G be a graph. For $v \in V(G)$, we use $N_G(v)$ (respectively, $N_G[v]$) to denote the neighborhood (respectively, closed neighborhood) of v, and use $d_G(v)$ to denote $|N_G(v)|$. For distinct vertices u, v of G, we use $d_G(u, v)$ to denote the distance between u and v. (If there is no confusion we omit the reference to G.) For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and let $G - S = G[V(G) \backslash S]$. For a subgraph T of G, we often write $G - T$ for $G - V(T)$ and write $G[T]$ for $G[V(T)]$. A path (respectively, cycle) is often represented as a sequence (respectively, cyclic sequence) of vertices, with consecutive vertices being adjacent. A cycle C in a graph G is said to be *separating* if the graph obtained from G by deleting vertices in C is not connected.

Let G be a plane graph. For a cycle C in G, we use $\mathrm{Int}_G(C)$ and $\mathrm{Ext}_G(C)$ to denote the interior and the exterior of this cycle C in G, respectively.

2 Rainbow planar Turán number of C_4

Before we prove Theorem 2, we have the following observation for properly edge-colored planar triangulations with no rainbow C_4.

Observation 1. *Let G be a properly edge-colored planar triangulation on at least four vertices with no rainbow C_4. Then for any vertex v in G, $G[N(v)]$ has a rainbow cycle using all vertices of $N(v)$.*

Proof. Suppose G is a planar triangulation on at least four vertices and it has a proper edge-coloring c with no rainbow C_4. Let v be a vertex in G. Since G is edge-maximal and $G \neq K_3$, v has degree at least three and all neighbors of v are contained in a cycle. Assume that $d(v) = k$ for some $k \geq 3$ and $N(v) = \{v_1, v_2, \ldots, v_k\}$, where $C = v_1 v_2 \ldots v_k v_1$ is a cycle in G. To show $G[N(v)]$ has a rainbow cycle containing all vertices of $N(v)$, it suffices to show that C is rainbow.

We may now assume $d(v) = k \geq 4$ as a properly edge-colored triangle is always rainbow. Since G is properly edge-colored, we may assume that the edge vv_i is colored with the color i for each $i \in [k]$. For each $i \in [k]$, let $D_i := vv_iv_{i+1}v_{i+2}v$, where the indices are same under modulo k. Note that each D_i is a 4-cycle and G has no rainbow C_4. Since $D_1 = vv_1v_2v_3v$ is not rainbow, either $c(v_1v_2) = c(vv_3) = 3$ or $c(v_2v_3) = c(vv_1) = 1$. Without loss of generality, we may assume the edge v_2v_3 is colored with the color 1. Then observe that $D_2 = vv_2v_3v_4v$ is a 4-cycle with $c(vv_2) = 2, c(v_2v_3) = 1, c(vv_4) = 4$. It follows that the edge v_3v_4 has the same color as vv_2, i.e., $c(v_3v_4) = c(vv_2) = 2$. Similarly, we have that $c(v_iv_{i+1}) = i - 1$ for each $i \in [2, k-1]$ and $c(v_kv_1) = k - 1, c(v_1v_2) = k$. Thus C is rainbow, and this completes the proof. □

Proof of Theorem 2. Suppose G is a planar triangulation such that $n = |V(G)| \geq 5$ and it has a proper edge-coloring c with no rainbow C_4. It follows from Observation 1 that G has no degree 4 vertices. Since G has at least five vertices and exactly $3n - 6$ edges, the maximum degree of G is at least four. Hence, G has a vertex of degree at least 5. To show G has minimum degree 5, we need to show G has no vertex of degree 3. Suppose G has a degree 3 vertex, say u. Observe that all neighbors of u have degree at least 5 in G. Let v be a neighbor of u. Suppose $d(v) = k$ for some $k \geq 5$ and $N(v) = \{v_1, v_2, \ldots, v_k\}$ such that $C = v_1v_2 \ldots v_k$ is a cycle in G. Without loss of generality, we may assume $v_1 = u$ and so $v_kv_2 \in E(G)$. By Observation 1, we may assume that $c(vv_i) = c(v_{i+1}v_{i+2}) = i$ for each $i \in [k]$, where the indices are same under modulo k. We consider the color for the edge v_2v_k. Note that $c(vv_2) = 2, c(v_2v_3) = 1, c(v_kv_1) = k - 1$. Hence $c(v_2v_k) \notin \{1, 2, k-1\}$. This implies the 4-cycle $v_kv_1vv_2v_k$ is rainbow, giving a contradiction. Therefore, every vertex in G has degree at least 5. Since every planar graph has minimum degree at most 5, it follows that G has minimum degree 5.

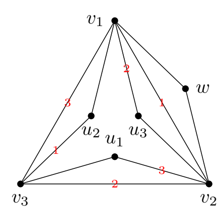

Figure 2: The separating triangle T

Next we show that G is 4-connected. Suppose not. Then G contains a separating triangle. Consider a plane embedding of G. We choose a separating triangle $T := v_1v_2v_3v_1$ of G such that its interior is vertex-minimum. We claim that v_1, v_2, v_3 cannot have a common neighbor in the interior of T. Suppose u is a common neighbor of v_1, v_2, v_3 in $\mathrm{Int}(T)$. Then we have that either u is a degree 3 vertex in G or there is a separating triangle with fewer vertices in its interior, a contradiction. Let u_{6-i-j} denote the common neighbor of v_i and v_j in $\mathrm{Int}(T)$ for i, j with $1 \leq i < j \leq 3$. Note that v_1, v_2 must have a common neighbor, say w, in $\mathrm{Ext}(T)$. We may assume that $c(v_1v_2) = 1, c(v_2v_3) = 2$ and $c(v_3v_1) = 3$. Since the 4-cycle $v_1u_3v_2v_3v_1$ is not rainbow, either $c(v_1u_3) = c(v_2v_3) = 2$ or $c(v_2u_3) = c(v_3v_1) = 3$. without loss of generality, we may assume that $c(v_1u_3) = c(v_2v_3) = 2$. Note that $v_1v_2v_3u_2v_1$ is not rainbow and $c(v_1u_2) \neq c(v_1u_3) = 2 = c(v_2v_3)$. It follows that $c(v_3u_2) = c(v_1v_2) = 1$. Similarly, we have that $c(v_2u_1) = 3$. Thus $c(v_1w) \notin \{1, 2, 3\}$ and $c(v_2w) \notin \{1, 2, 3\}$. This implies that the 4-cycle, $v_1wv_2v_3v_1$, is rainbow, a contradiction. Therefore, G has no separating triangle and so G is 4-connected. □

Now we want to give a planar triangulation such that it has a proper edge-coloring with no rainbow C_4. By Theorem 2, we know that this planar triangulation G should have minimum degree 5 and connectivity at least 4. If G is 5-connected, will this be helpful? We know that 5-connected planar triangulations have no separating 4-cycles, and so every 4-cycle in 5-connected planar triangulation is formed by two adjacent facial triangles. Hence G has exactly $3|V(G)|-6$ many 4-cycles for 5-connected G. This together with Observation 1 motivates us to give the following construction.

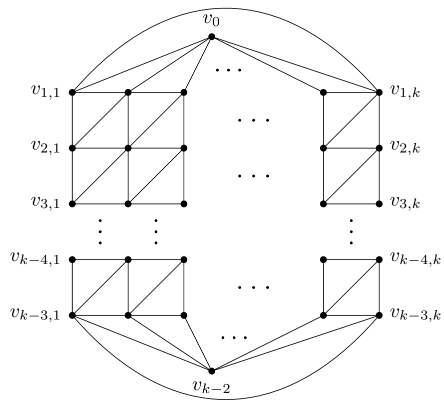

Figure 3: Part of H_k.

We define a planar triangulation H_k for each $k \geq 5$ such that H_k has $k^2 - 3k + 2$ vertices. Let

$$V(H_k) := \{v_0, v_{k-2}\} \cup \{v_{i,j} : i \in [k-3], j \in [k]\}.$$

Now we define the edges of H_k. We may assume that all numbers for the index j are same under modulo k. Let $v_{i,j}$ be adjacent to $v_{i,j+1}$ for every $i \in [k-3]$, i.e., the vertices $v_{i,1}, v_{i,2}, \ldots v_{i,k}$ form a k-cycle in H_k, denoted by D_i. Moreover, for each $i \in [k-4]$ we let $v_{i,j}v_{i+1,j} \in E(H_k)$ and $v_{i,j}v_{i+1,j-1} \in E(H_k)$. Let v_0 be adjacent to all vertices of D_1 and v_{k-2} be adjacent to all vertices of D_{k-3}. For convenience, let $v_{0,j} = v_0$ and $v_{k-2,j} = v_{k-2}$ for each $j \in [k]$. Hence, $v_{0,j}v_{1,j} = v_0 v_{1,j}$ and $v_{k-3,j}v_{k-2,j} = v_{k-3,j}v_{k-2}$.

Observation 2. *For each $k \geq 5$, H_k has the following properties.*

(i) H_k is a 5-connected planar triangulation on $(k-3)k+2$ vertices.

(ii) v_0, v_{k-2} both have degree k, $v_{i,j}$ has degree 5 for $i \in \{1, k-3\}$, and $v_{i,j}$ has degree 6 for $i \in [2, k-4]$ if $k \geq 6$.

Theorem 1 follows from the following result.

Lemma 4. *H_k has a proper edge-coloring with no rainbow C_4.*

Proof. We give a $(k+2)$-edge-coloring of H_k, $\sigma : E(H_k) \to [k] \cup \{a, b\}$, which is defined as follows.

$$\sigma(e) = \begin{cases} a & \text{if } e = v_{i,j}v_{i+1,j-1} \text{ and } i \in [k-4] \text{ is odd,} \\ b & \text{if } e = v_{i,j}v_{i+1,j-1} \text{ and } i \in [k-4] \text{ is even,} \\ t \text{ (where } t \in [k]) & \text{if } e = v_{i,t}v_{i,t+1} \text{ for } i \in [k-3] \text{ or } e = v_{i-1,t-i}v_{i,t-i} \text{ for } i \in [k-2]. \end{cases}$$

We claim that σ is a proper edge coloring of H_k and it has no rainbow C_4. To prove that σ is proper, it suffices to show that for each vertex v in H_k, all the edges incident with v receive distinct colors. We may assume the second indices for the vertices in H_k, as well as the colors labeled as integers, are same under modulo k. Suppose $v = v_0$. Recall that v_0 has degree k. By

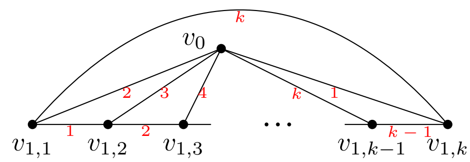

Figure 4: v_0 and its neighbors

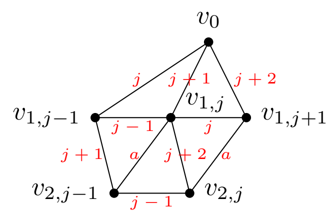

Figure 5: $v_{1,j}$ and its neighbors

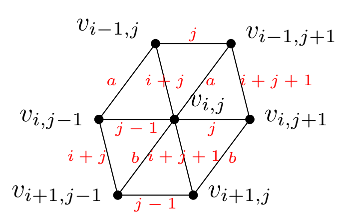

Figure 6: $v_{i,j}$ (for even $i \in [2, k-4]$) and its neighbors

the definition of σ, we have $\sigma(v_0v_{1,j}) = \sigma(v_{0,j}v_{1,j}) = j+1$ for every $j \in [k]$, and hence the edges incident with v_0 have distinct colors. We may now assume that $v = v_{1,j}$, which is a degree-5 vertex. Note that $\sigma(v_{1,j}v_{1,j+1}) = j, \sigma(v_{1,j}v_{1,j-1}) = j-1, \sigma_k(v_{1,j}v_0) = j+1, \sigma(v_{1,j}v_{2,j}) = j+2$, and $\sigma(v_{1,j}v_{2,j-1}) = a$. Observe that $j, j-1, j+1, j+2, a$ are pairwise distinct. Similarly, we can show that all edges incident with v_{k-2} or $v_{k-3,j}$ have distinct colors. Now we consider $v_{i,j}$ for $2 \leq i \leq k-4$ if $k \geq 6$. Note that $v_{i,j}$ is adjacent to $v_{i,j-1}, v_{i,j+1}, v_{i-1,j}, v_{i+1,j}, v_{i-1,j+1}, v_{i+1,j-1}$ and that all the six edges receive colors $j-1, j, i+j, i+j+1, a, b$. Since $2 \leq i \leq k-4$, the colors $j-1, j, i+j, i+j+1$ are pairwise distinct. This implies that σ is a proper edge-coloring.

Now we show that H_k contains no rainbow C_4 under σ. Since H_k is 5-connected, every 4-cycle of H_k is formed by two adjacent facial triangles and so it is corresponding to a unique edge of H_k. For each $v \in V(H_k)$ and each edge e incident with v, it is not hard to check that the 4-cycle determined by e is not rainbow. Hence, H_k has a proper edge-coloring with no rainbow C_4. $\square$

3 Rainbow planar Turán number of C_5, C_6

In this section, we define an n-vertex planar triangulation F_n for each $n \geq 4$. For each even n with $n \geq 4$, let $p := \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ and F_n be defined as follows.

$$V(F_n) = \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\},$$

and

$$F_n = P_u \cup P_v \cup Q_1 \cup Q_2,$$

where P_u, P_v, Q_1, Q_2 are paths and

$$P_u = u_1 u_2 \ldots u_p,$$

$$P_v = v_1 v_2 \ldots v_p,$$

$$Q_1 = u_1 v_1 u_2 v_2 \ldots u_{p-1} v_{p-1} u_p v_p,$$

$$Q_2 = v_2 u_1 v_3 u_2 \ldots v_{p-1} u_{p-2} v_p u_{p-1}.$$

It follows that for each $i \in [p]$, u_i is adjacent to u_j for $j \in [p]$ with $|j - i| = 1$ and v_j for $j \in [p]$ with $-1 \leq j - i \leq 2$; and v_i is adjacent to v_j for $j \in [p]$ with $|j - i| = 1$ and u_j for $j \in [p]$ with $-2 \leq j - i \leq 1$.

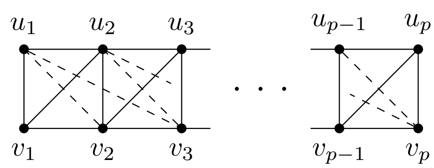

Figure 7: F_n for even n, where $p = \lfloor \frac{n}{2} \rfloor$

Now we use F_n, where $n \geq 4$ is even, to define F_{n+1}. We add a new vertex u_0 and join u_0 with u_1, v_1, v_2, i.e., insert u_0 to the face with boundary $u_1 v_1 v_2 u_1$ and join u_0 to those three vertices in the boundary. In fact, $F_{n-1} = F_n - \{v_1\}$ for even $n \geq 4$.

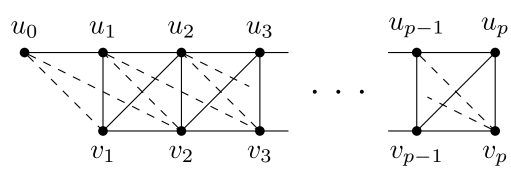

Figure 8: F_n for odd n, where $p = \lfloor \frac{n}{2} \rfloor$

We give a proper edge-coloring of F_n for each $n \geq 3$ with no rainbow C_5 or C_6 in the following lemma, which derives Theorem 3.

Lemma 5. *For each $n \geq 3$, F_n admits a proper edge-coloring with no rainbow C_5 or C_6.*

Proof. For each $n \geq 3$, let $p = \lfloor \frac{n}{2} \rfloor$. We use the same vertex labels as Figures 7 and 8. Note that $V(F_n) = \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$ for even n and $V(F_n) = \{u_0\} \cup \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$ for odd n. We give a proper 6-edge-coloring for F_n, $c : \mathrm{E}(F_n) \to [6]$ as follows.

$$c(e) = \begin{cases} 1 & \text{if } e = u_iu_{i+1} \text{ or } e = v_iv_{i+1} \text{ with odd } i, \\ 2 & \text{if } e = u_iu_{i+1} \text{ or } e = v_iv_{i+1} \text{ with even } i, \\ 3 & \text{if } e = u_iv_{i+1} \text{ or } e = v_iu_{i+1} \text{ with odd } i, \\ 4 & \text{if } e = u_iv_{i+1} \text{ or } e = v_iu_{i+1} \text{ with even } i, \\ 5 & \text{if } e = u_iv_i \text{ for every possible } i, \\ 6 & \text{if } e = u_iv_{i+2} \text{ for every possible } i. \end{cases}$$

It is obvious that the coloring c is proper. We claim that F_n with the coloring c has no rainbow C_5 or C_6. Suppose F_n contains a rainbow C_5, say D. We show that D must contain all colors in $\{3, 4, 5, 6\}$. Let $X_v = \{v_1, v_2, \ldots, v_p\}$ and $X_u = V(F_n) \backslash X_v$. Observe that all edges in $F_n[X_u]$ or $F_n[X_v]$ are colored 1 or 2, and so it follows that $V(D) \cap X_u \neq \emptyset$ and $V(D) \cap X_v \neq \emptyset$. Observe that D must have even number of edges with one end in X_u and one end in X_v, and hence D has exactly four such edges (as D is a rainbow C_5). This implies that D uses all the colors in $\{3, 4, 5, 6\}$ and only one color in $\{1, 2\}$. By the construction for F_n, we have that the distance between u_i and u_{i+3} is exactly two and $u_iv_{i+2}u_{i+3}$ is the unique u_iu_{i+3}-path of length two. Similarly, we obtain that $d(v_i, v_{i+3}) = 2$ with the unique v_iv_{i+3}-path $v_iu_{i+1}v_{i+3}$ having length two, $d(u_i, v_{i+3}) = 2$ with exactly two u_iv_{i+3}-paths $u_iv_{i+2}v_{i+3}, u_iu_{i+1}v_{i+3}$ of length two, and v_i, u_{i+3} have distance greater than two as there is no v_iu_{i+3}-path of length two in F_n. Moreover, for i, j with $|i - j| > 3$, we have that $d(u_i, u_j) > 2, d(v_i, v_j) > 2$, and $d(u_i, v_j) > 3$.

Since D is a 5-cycle, any two vertices in D have distance at most two in F_n. Then it follows that for any two vertices in D, the absolute value of the difference for their indices cannot exceed three. Next we claim that the difference between the maximum index and the minimum index of the vertices in D is exactly two. Suppose not and then their difference is three. Assume that $u_i, u_{i+3} \in V(D)$ for some i. Since $u_iv_{i+2}u_{i+3}$ is the unique u_iu_{i+3}-path of length two in F_n, it follows that $u_iv_{i+2}u_{i+3} \subseteq D$. We claim that $u_{i+3}v_{i+3} \notin E(D)$. Suppose $u_{i+3}v_{i+3} \in E(D)$. Then the other edge in D incident with v_{i+3} is either the edge $u_{i+2}v_{i+3}$ or the edge $u_{i+1}v_{i+3}$. Note that $c(u_{i+2}v_{i+3}) = c(v_{i+2}u_{i+3})$ and $c(u_{i+1}v_{i+3}) = c(u_iv_{i+2})$. Hence $u_{i+3}v_{i+3}$ is not contained in D, and so $u_{i+3}u_{i+2} \in E(D)$. Since D uses only one color in $\{1, 2\}$, this implies that $D = u_iv_{i+2}u_{i+3}u_{i+2}v_{i+1}u_i$ and the edges u_iv_{i+1} and $v_{i+2}u_{i+3}$ in D have the same color, contradicting that D is rainbow. Hence, both u_i, u_{i+3} cannot be contained in $V(D)$. By symmetry, we have that both v_i, v_{i+3} cannot be contained in $V(D)$. We may now assume that $u_i, v_{i+3} \in V(D)$ (as v_i, u_{i+3} have distance greater than three). It follows that $v_i \notin V(D)$ as $v_{i+3} \in V(D)$, and $u_{i+3} \notin V(D)$ as $u_i \in V(D)$. Suppose $u_iv_{i+2}v_{i+3} \subseteq D$. We claim that $u_iv_{i+1} \in E(D)$. Since $v_{i+2}v_{i+3} \in E(D)$ is colored by color 1, we have that $u_iu_{i+1} \notin E(D)$ and so $u_iv_{i+1} \in E(D)$. Hence, either $v_{i+1}u_{i+1}v_{i+3} \subseteq D$ or $v_{i+1}u_{i+2}v_{i+3} \subseteq D$. Observe that $c(u_{i+1}v_{i+3}) = c(u_iv_{i+2})$ and $c(u_{i+2}v_{i+3}) = c(u_iv_{i+1})$. Therefore, $u_iv_{i+2}v_{i+3}$ cannot be contained in D. Similarly, we have a contradiction if $u_iu_{i+1}v_{i+3} \subseteq D$. This implies that $u_i, v_{i+3} \in V(D)$ is not possible. Thus, the difference of the indices of any two vertices in D has absolute value at most two. It follows that either $|V(D) \cap X_u| = 3, |V(D) \cap X_v| = 2$ or $|V(D) \cap X_v| = 3, |V(D) \cap V(X_u)| = 2$. Without loss of generality, we may assume that $u_i, u_{i+1}, u_{i+2} \in V(D)$ for some i. Then D contains exactly two vertices in v_i, v_{i+1}, v_{i+2}. Observe that u_i, u_{i+1}, u_{i+2} are not consecutive in D as D cannot use two edges with colors 1 or 2. Hence, two vertices in $V(D) \cap X_v$ are not consecutive in D. It follows that either $u_iu_{i+1} \in E(D)$ or $u_{i+1}u_{i+2} \in E(D)$. Since D has all the colors in $\{3,4,5,6\}$, the only edge u_iv_{i+2} with color 6 must be contained in D. Suppose $u_iu_{i+1} \in E(D)$. This implies that $v_{i+2}u_{i+2} \in E(D)$ and $D = u_iv_{i+2}u_{i+2}v_{i+1}u_{i+1}u_i$, which is impossible as $u_{i+1}v_{i+1}, u_{i+2}v_{i+2} \in E(D)$ have the same color. We may now assume that $u_{i+1}u_{i+2} \in E(D)$. If $u_{i+1}v_{i+2} \in E(D)$ then $D = u_iv_{i+2}u_{i+1}u_{i+2}v_{i+1}u_i$ and $u_{i+1}v_{i+2}, v_{i+1}u_{i+2}$ have the same color, giving a contradiction. Thus $u_{i+2}v_{i+2} \in E(D)$ and $u_iv_{i+2}u_{i+2}u_{i+1} \subseteq D$. Then either $u_iv_iu_{i+1} \subseteq D$ or $u_iv_{i+1}u_{i+1} \subseteq D$, but both cases imply that there exist two edges in D with the same color 5. Therefore, F_n has no rainbow C_5 under the coloring c.

Now we show that F_n under the coloring c contains no rainbow C_6. Suppose D is a rainbow 6-cycle in F_n. We claim that the difference of the indices of any two vertices in D has absolute value at most three. Suppose $x_i, x_j \in V(D)$, where $j - i \geq 4$ and $x_i \in \{u_i, v_i\}, x_j \in \{u_j, v_j\}$. Note that the distance between x_i, x_j is at least three in F_n. Since every two vertices in a 6-cycle have distance at most three, we know that $d(x_i, x_j) = 3$ and D has two internally vertex-disjoint x_ix_j-paths of length three. Since $j - i \geq 4$, every x_ix_j-path of length three must contain an edge u_sv_{s+2} for some s. Since all the edges u_tv_{t+2} are colored by the color 6, it follows that D has two edges with the same color 6, giving a contradiction. Thus the difference between the maximum index and the minimum index of the vertices in D is at most three. Suppose the difference is exactly two. Then $V(D) = \{u_i, u_{i+1}, u_{i+2}, v_i, v_{i+1}, v_{i+2}\}$ for some i. Observe that $u_iv_{i+2} \in E(D)$ as this is the only one edge in $F_n[D]$ with color 6. We claim that $v_{i+1}v_{i+2} \notin E(D)$. Assume that $v_{i+1}v_{i+2} \in E(D)$. This implies that the two edges incident with u_{i+2} in D are the edges $u_{i+1}u_{i+2}$ and $v_{i+1}u_{i+2}$. Since $v_{i+1}v_{i+2}$ and $u_{i+1}u_{i+2}$ have the same color, this gives a contradiction and so $v_{i+1}v_{i+2} \notin E(D)$. Suppose $u_{i+1}v_{i+2} \in E(D)$. Similarly, we have $u_{i+1}u_{i+2}, v_{i+1}u_{i+2} \in E(D)$, and observe that $u_{i+1}v_{i+2}$ have the same color as $v_{i+1}u_{i+2}$. Thus we have that $u_{i+2}v_{i+2} \in E(D)$. By symmetry, we know that $u_iv_i \in E(D)$, which has the same color 5 as the edge $u_{i+2}v_{i+2} \in E(D)$. Therefore, we may assume that the minimum index and the maximum index of the vertices in D are i and $i+3$, respectively, for some i. Note that the edges $u_iv_{i+2}, u_{i+1}v_{i+3}$ are the only two edges with color 6 in the subgraph of F_n induced by $\{u_i, u_{i+1}, u_{i+2}, u_{i+3}, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$. By symmetry, we may assume that $u_iv_{i+2} \in E(D)$. We first show that $v_i \notin V(D)$. Suppose $v_i \in V(D)$. By a similar argument, we know that $u_iv_i \in E(D)$. Assume that $v_iu_{i+1} \in E(D)$. It follows that $u_{i+1}u_{i+2} \in E(D)$. Since D must contain one vertex in $\{u_{i+3}, v_{i+3}\}$, we have $u_{i+2}v_{i+3}v_{i+2} \subseteq D$ or $u_{i+2}u_{i+3}v_{i+2} \subseteq D$. As $u_{i+2}v_{i+3}, v_{i+2}u_{i+3}$ have the same color as the edge $v_iu_{i+1} \in E(D)$, the edge v_iu_{i+1} is not contained in D and so $v_iv_{i+1} \in E(D)$. This implies that $v_{i+1}u_{i+2} \in E(D)$. Similarly, we have $u_{i+2}v_{i+3}v_{i+2} \subseteq D$ or $u_{i+2}u_{i+3}v_{i+2} \subseteq D$. Since the edges $u_{i+2}u_{i+3}, v_{i+2}v_{i+3}$ have the same color as the edge $v_iv_{i+1} \in E(D)$, it follows that $v_iv_{i+1} \notin E(D)$ and so $v_i \notin V(D)$. We next show that $u_{i+3} \notin V(D)$. Assume that $u_{i+3} \in V(D)$. If $v_{i+2}u_{i+3} \notin E(D)$, then $u_{i+2}u_{i+3}v_{i+3} \subseteq D$. It implies that $v_{i+3}u_{i+1} \in E(D)$ or $v_{i+3}v_{i+2} \in E(D)$. Both give a contradiction as $u_iv_{i+2} \in E(D)$ and $u_{i+2}u_{i+3} \in E(D)$. Thus we may assume $v_{i+2}u_{i+3} \in E(D)$. Then it follows that $u_iv_{i+1} \notin E(D)$ and $u_iu_{i+1} \in E(D)$ as $v_i \notin V(D)$. Hence $u_{i+2}u_{i+3} \notin E(D)$ since it has the same color as u_iu_{i+1}, and then $u_{i+3}v_{i+3} \in E(D)$. Note all the other edges incident with v_{i+3}, which are $u_{i+2}v_{i+3}, u_{i+1}v_{i+3}, v_{i+2}v_{i+3}$, cannot be contained in D. Therefore, $u_{i+3} \notin V(D)$ and $V(D) = \{u_i, u_{i+1}, u_{i+2}, v_{i+1}, v_{i+2}, v_{i+3}\}$. Since $u_{i+1}v_{i+3} \notin E(D)$, we have that $v_{i+2}v_{i+3}u_{i+2} \subseteq D$. Note that either u_iv_{i+1} or u_iu_{i+1} is contained in $E(D)$. But u_iv_{i+1} has the same color as $u_{i+2}v_{i+3} \in E(D)$, and u_iu_{i+1} has the same color as $v_{i+2}v_{i+3} \in E(D)$. Hence, F_n cannot contain a rainbow C_6 under the coloring c. This completes the proof. □

Acknowledgments

The author wants to thank Zi-Xia Song for proposing this problem, early discussions and helpful suggestions.

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