ON A VARIATION OF SELECTIVE SEPARABILITY: S-SEPARABILITY

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ABSTRACT. A space X is M-separable (selectively separable) (Scheepers, 1999; Bella et al., 2009) if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $\cup_{n\in\mathbb{N}} F_n$ is dense in X. In this paper, we introduce and study a strengthening of M-separability situated between H- and M-separability, which we call S-separability: for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for each finite family $\mathcal{F}$ of nonempty open sets of X some n satisfies $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$.

Key words and phrases: Selective separable, M-separable, S-separable, H-separable, L-separable.

1. Introduction

In [18], Scheepers introduced and investigated a selective version of separability called selective separability (also known as M-separability; see [8]). This line of research was further developed in [6,8,9]. Certain variations of selective separability was also introduced and studied in [8]. Motivated by this, in this paper, we introduce a variation of selective separability, called S-separability. Note that S-separability lies between H-separability and M-separability.

We establish the basic theory of S-separability. It is preserved by dense and open subspaces, and by open continuous and closed irreducible images, but not by arbitrary continuous mappings. Products behave subtly: the product of two S-separable spaces need not be S-separable; nevertheless, if X and Y are countable H-separable spaces and $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is S-separable, and more generally, $\prod_{n\in\mathbb{N}} X_n$ is S-separable provided each finite initial product is S-separable. For compact spaces we obtain a complete characterization: a compact X is S-separable if and only if $\pi w(X) = \omega$. Consequences include permanence results for continuous images of separable compact spaces that are scattered or of countable tightness, and an example of a compact S-separable space with a non–S-separable continuous image.

We link S-separability to selection principles in function spaces. For Tychonoff X the following are equivalent: $C_p(X)$ is S-separable; $C_p(X)$ is M-separable; $iw(X) = \omega$ and every finite power of X is Menger. For zero-dimensional X the same equivalence holds for $C_p(X, \mathbb{Q})$, $C_p(X, \mathbb{Z})$, and $C_p(X, 2)$. On the level of small subspaces, the threshold $\mathfrak{d}$ is sharp: the smallest π-weight of a countable non-S-separable space is $\mathfrak{d}$. In particular, every countable subspace of 2^κ is S-separable for $\kappa < \mathfrak{d}$, whereas

2020 *Mathematics Subject Classification.* Primary: 54D65; Secondary: 54C35, 54D20, 54D99.

$2^{\mathfrak{d}}$ contains a countable dense non-S-separable subspace. We also show (in ZFC) that 2^{ω_1} contains a dense and hence a countable dense S-separable subspace. Finally, we define L-separability and compare it with separability, hereditary separability, and M-separability.

Several natural problems remain open, including whether in ZFC there exists a S-separable space that is not H-separable, and whether there exists a M-separable space that is not S-separable.

2. Preliminaries

By a space we always mean a topological space. All spaces are assumed to be Tychonoff; otherwise will be mentioned. For undefined notions and terminologies see [10]. $w(X)$ denotes the weight of a space X. A collection $\mathcal{B}$ of open sets of X is called π-base (respectively, π-base at $x \in X$) if every nonempty open set in X (respectively, every neighbourhood of x in X) contains a nonempty member of $\mathcal{B}$. $\pi w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base for } X\}$ denotes the π-weight of X. The minimum cardinality of a π-base at $x \in X$ is denoted by $\pi_\chi(x, X)$ and $\pi_\chi(X) = \sup\{\pi_\chi(x, X) : x \in X\}$. $d(X)$ denotes the minimal cardinality of a dense subspace of X and $\delta(X) = \sup\{d(Y) : Y \text{ is dense in } X\}$. For any space X, $d(X) \leq \delta(X) \leq \pi w(X) \leq w(X)$. The i-weight of a space (X, τ) is defined as

$$iw((X,\tau)) = \min\{\kappa : \text{there is a Tychonoff topology } \tau' \subseteq \tau \text{ such that } w((X,\tau')) = \kappa\}.$$

A space X has countable fan tightness [3] if for any $x \in X$ and any sequence (Y_n) of subsets of X with $x \in \cap_{n\in\mathbb{N}}\overline{Y_n}$ there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $x \in \overline{\cup_{n\in\mathbb{N}}F_n}$. X has countable fan tightness with respect to dense subspaces [8] if for any $x \in X$ and any sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $x \in \overline{\cup_{n\in\mathbb{N}}F_n}$. A space X has countable tightness (which is denoted by $t(X) = \omega$) if for each $x \in X$ and each $Y \subseteq X$ with $x \in \overline{Y}$ there exists a countable set $E \subseteq Y$ such that $x \in \overline{E}$. A space X is scattered if every nonempty subspace Y of X has an isolated point. For a Tychonoff space X, βX denotes the Stone-Čech compactification of X.

Let X be a Tychonoff space and $C(X)$ be the set of all continuous real valued functions. As usual $C_p(X)$ denotes the space $C(X)$ with pointwise convergence topology. Let $f \in C(X)$. Then a basic open set of f in $C_p(X)$ is of the form

$$B(f, F, \epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in F\},$$

where F is a finite subset of X and $\epsilon > 0$. For a separable metrizable space X, $C_p(X)$ is hereditarily separable. Also recall from [3, Theorem II.2.10] that $C_p(X)$ is Menger if and only if X is finite.

A space X is said to be selective separable (M-separable) [9] (see also [8]) if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $\cup_{n\in\mathbb{N}}F_n$ is dense in X. A space X is said to be H-separable [8] if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and every nonempty open set of X intersects F_n for all but finitely many n.

3. Main Results

3.1. **S-separability and basic observations on it.** We begin with the following definition of a class of spaces, also considered by Aurichi et al. [4].

Definition 3.1. *A space X is said to be S-separable if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a n such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$.*

Clearly every H-separable space is S-separable and every S-separable space is M-separable. In the following example we observe that the class of H-separable spaces is strictly contained in the class of S-separable spaces.

Example 3.2. *There exists a S-separable space which is not H-separable.*

Proof. Assume that $\mathfrak{b} < \mathfrak{d}$. By [8, Theorem 31], there exists a countable dense subspace Y of $2^{\mathfrak{b}}$ such that Y is not H-separable. Also by Corollary 3.18, every countable subspace of $2^{\mathfrak{b}}$ is S-separable. Thus Y is S-separable. $\square$

The above example is consistent with ZFC. So the following question can be asked.

Problem 3.3. *In ZFC, does there exist a S-separable space which is not H-separable?*

The following question also remains open.

Problem 3.4. *Does there exist a M-separable space which is not S-separable?*

Proposition 3.5. *Let X be a S-separable space. Then*
(1) *every dense subspace of X is S-separable.*
(2) *every open subspace of X is S-separable.*

Proof. (1). Let Y be a dense subspace of X. Pick a sequence (Y_n) of dense subspaces of Y. Obviously each Y_n is dense in X because Y is dense in X. Since X is S-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$. Observe that the sequence (F_n) witnesses for the sequence (Y_n) that Y is S-separable.

(2). Let Y be an open subspace of X. Pick a sequence (Y_n) of dense subspaces of Y. Obviously for each n $Z_n = (X \setminus Y) \cup Y_n$ is dense in X. Since X is S-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of Z_n and for each finite collection $\mathcal{F}$ of nonempty open sets of X there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$. Now we can express each F_n as $F_n = A_n \cup B_n$, where A_n is a finite subset of $X \setminus Y$ and B_n is a finite subset of Y_n. Observe that the sequence (B_n) witnesses for (Y_n) that Y is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of Y. Since Y is open in X, $\mathcal{F}$ is also a finite collection of nonempty open sets of X. Then we get a $m \in \mathbb{N}$ such that $U \cap F_m \neq \emptyset$ for all $U \in \mathcal{F}$. Since $U \subseteq Y$ and $A_m \subseteq X \setminus Y$, $U \cap B_m \neq \emptyset$. Thus Y is S-separable. $\square$

Proposition 3.6. *For a space X the following assertions are equivalent.*
(1) *X is hereditarily S-separable.*
(2) *X is hereditarily separable and every countable subspace of X is S-separable.*

Proof. (2) $\Rightarrow$ (1). Let Y be a subspace of X. Pick a sequence (Y_n) of dense subspaces of Y. Since X is hereditarily separable, we get a sequence (Z_n) such that for each n Z_n is a countable dense subspace of Y_n. Choose $Z = \cup_{n\in\mathbb{N}} Z_n$. Clearly for each n Z_n is dense in Z and Z is S-separable as every countable subspace of X is S-separable. Apply the S-separability of Z to (Z_n) to obtain a sequence (F_n) such that for each n F_n is a finite subset of Z_n and for every finite collection $\mathcal{F}$ of nonempty open sets of Z there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$. We claim that the sequence (F_n) guarantees for (Y_n) that Y is S-separable. Let $\mathcal{G}$ be a finite collection of nonempty open sets of Y. Then $\mathcal{F} = \{U \cap Z : U \in \mathcal{G}\}$ is a finite collection of nonempty open sets of Z as Z is dense in Y. This gives a $k \in \mathbb{N}$ such that $V \cap F_k \neq \emptyset$ for all $V \in \mathcal{F}$, i.e. $U \cap F_k \neq \emptyset$ for all $U \in \mathcal{G}$. Thus Y is S-separable. □

Proposition 3.7. *If a space X has an open dense S-separable subspace, then X is S-separable.*

Proof. Let Y be an open dense S-separable subspace of X. Pick a sequence (Y_n) of dense subspaces of X. For each n $Z_n = Y_n \cap Y$ is dense in Y. Since Y is S-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of Z_n and for every finite collection $\mathcal{F}$ of nonempty open sets of Y there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$. Observe that (F_n) witnesses for (Y_n) that X is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. Define a finite collection $\mathcal{G}$ of nonempty open sets of Y as $\mathcal{G} = \{U \cap Y : U \in \mathcal{F}\}$. Then there exists a $m \in \mathbb{N}$ such that $V \cap F_m \neq \emptyset$ for all $V \in \mathcal{G}$, i.e. $U \cap Y \cap F_m \neq \emptyset$ for all $U \in \mathcal{F}$. Thus $U \cap F_m \neq \emptyset$ for all $U \in \mathcal{F}$ and hence X is S-separable. □

Recall that a continuous mapping $f : X \to Y$ is irreducible if the only closed subset C of X satisfying $f(C) = Y$ is $C = X$.

Proposition 3.8. *Let X be a S-separable space. Then*
(1) *every open continuous image of X is S-separable.*
(2) *every closed irreducible image of X is S-separable.*

Proof. (1). Let $f : X \to Y$ be an open continuous mapping from X onto a space Y. Let (Y_n) be a sequence of dense subspaces of Y. Now $(f^{-1}(Y_n))$ is a sequence of dense subspaces of X as dense subspaces are inverse invariant under open mappings. Since X is S-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of $f^{-1}(Y_n)$ and for every finite collection $\mathcal{F}$ of nonempty open sets of X there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$. For each n choose $A_n = f(F_n)$. Observe that the sequence (A_n) witnesses for (Y_n) that Y is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of Y. Then $f^{-1}(\mathcal{F}) = \{f^{-1}(V) : V \in \mathcal{F}\}$ is a finite collection of nonempty open sets of X. This gives a $m \in \mathbb{N}$ such that $f^{-1}(V) \cap F_m \neq \emptyset$ for all $V \in \mathcal{F}$. Clearly $V \cap A_m \neq \emptyset$ for all $V \in \mathcal{F}$. Thus Y is S-separable.

(2). Let $f : X \to Y$ be a closed irreducible mapping from X onto a space Y. Let (Y_n) be a sequence of dense subspaces of Y. Now $(f^{-1}(Y_n))$ is a sequence of dense subspaces of X as dense subspaces are inverse invariant under closed irreducible mappings. Since X is S-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of $f^{-1}(Y_n)$ and for every finite collection $\mathcal{F}$ of nonempty open sets of X there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{F}$. For each n choose $A_n = f(F_n)$. Observe that the sequence (A_n) witnesses for (Y_n) that Y is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of Y. Then $f^{-1}(\mathcal{F}) = \{f^{-1}(V) : V \in \mathcal{F}\}$ is a finite collection of nonempty open sets of X. This gives a $m \in \mathbb{N}$ such that $f^{-1}(V) \cap F_m \neq \emptyset$ for all $V \in \mathcal{F}$. Clearly $V \cap A_m \neq \emptyset$ for all $V \in \mathcal{F}$. Thus Y is S-separable. □

Theorem 3.9. *If X is a countable H-separable space and Y is a countable space with $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is H-separable.*

Proof. Let $\kappa < \mathfrak{b}$ and $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ be a π-base of Y. Pick a sequence (Y_n) of dense subspaces of $X \times Y$. For each n we can choose $Y_n = \{x_m^{(n)} : m \in \mathbb{N}\}$. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projection mappings on X and Y respectively. Let $\alpha < \kappa$ be fixed. For each n let $Z_n^{(\alpha)} = p_1(Y_n \cap (X \times B_\alpha))$. Clearly each Z_n is dense in X. Since X is H-separable, there exists a sequence $(F_n^{(\alpha)})$ such that for each n $F_n^{(\alpha)}$ is a finite subset of $Z_n^{(\alpha)}$ and every nonempty open set of X intersects $F_n^{(\alpha)}$ for all but finitely many n. For each n choose a finite subset $G_n^{(\alpha)}$ of $Y_n \cap (X \times B_\alpha)$ such that $F_n^{(\alpha)} = p_1(G_n^{(\alpha)})$. Choose a $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $G_n^{(\alpha)} \subseteq \{x_m^{(n)} : m \leq f_\alpha(n)\}$. Thus we obtain a subset $\{f_\alpha : \alpha < \kappa\}$ of $\mathbb{N}^{\mathbb{N}}$. Since $\kappa < \mathfrak{b}$, there exists a $f \in \mathbb{N}^{\mathbb{N}}$ such that $f_\alpha \leq^* f$ for all $\alpha < \kappa$. For each n let $F_n = \{x_m^{(n)} : m \leq f(n)\}$. Observe that the sequence (F_n) witnesses for (Y_n) that $X \times Y$ is H-separable. Let U be a nonempty open set of $X \times Y$. Choose a nonempty open set V of X and a $\alpha < \kappa$ such that $V \times B_\alpha \subseteq U$. Then V intersects $F_n^{(\alpha)}$ for all but finitely many n. We can find a $n_0 \in \mathbb{N}$ such that $V \cap F_n^{(\alpha)} \neq \emptyset$ and $f_\alpha(n) \leq f(n)$ for all $n \geq n_0$. We claim that $(V \times B_\alpha) \cap F_n \neq \emptyset$ for all $n \geq n_0$. Choose a $n \geq n_0$ and a $x \in V \cap F_n^{(\alpha)}$. Now $x \in F_n^{(\alpha)}$ gives $x = p_1(x, y)$ for some $(x, y) \in G_n^{(\alpha)}$. It follows that $y \in B_\alpha$ and $(x, y) = x_m^{(n)}$ for some $m \leq f_\alpha(n)$. Since $f_\alpha(n) \leq f(n)$, from the construction of F_n we can say that $(x, y) \in F_n$. Also since $(x, y) \in V \times B_\alpha$, $(V \times B_\alpha) \cap F_n \neq \emptyset$. Thus $(V \times B_\alpha) \cap F_n \neq \emptyset$ for all $n \geq n_0$, i.e. $U \cap F_n \neq \emptyset$ for all $n \geq n_0$. Hence $X \times Y$ is H-separable. □

Corollary 3.10. *If X is a countable H-separable space and Y is a countable space with $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is S-separable.*

From [5, Corollary 2.5] we get two separable metrizable spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are M-separable but the product space $C_p(X) \times C_p(Y)$ is not M-separable (hence not S-separable). By Theorem 3.30, $C_p(X)$ and $C_p(Y)$ are S-separable. Thus we can say that the product of two S-separable spaces need not be S-separable. However, we obtain the following result.

Theorem 3.11. *Let $\{X_n : n \in \mathbb{N}\}$ be a family of spaces and $X = \prod_{n\in\mathbb{N}} X_n$. If each n $X'_n = \prod_{k=1}^{n} X_k$ is S-separable, then X is S-separable.*

Proof. Let (Y_n) be a sequence of dense subspaces of X. Let $\{N_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into disjoint infinite subsets. For each n let $p_n : X \to X'_n$ be the natural projection. Fix n. Apply the S-separability of X'_n to $(p_n(Y_k) : k \in N_n)$ to obtain a sequence $(F_k : k \in N_n)$ such that for each $k \in N_n$, F_k is a finite subset of Y_k and for every finite collection $\mathcal{F}$ of nonempty open sets of X'_n there exists a $k \in N_n$ such that $U \cap p_n(F_k) \neq \emptyset$ for all $U \in \mathcal{F}$. Thus we obtain a sequence (F_n) such that for each n F_n is a finite subset of Y_n. Observe that (F_n) witnesses for (Y_n) that X is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. For each $U \in \mathcal{F}$ choose a basic member $V(U)$ of X such that $V(U) \subseteq U$. Then for each $U \in \mathcal{F}$, $V(U) = \prod_{n\in\mathbb{N}} V(U)_n$ with $V(U)_n = X_n$ for all but finitely many n. We can find a $m \in \mathbb{N}$ such that $V(U)_n = X_n$ for all $n \geq m$ and for all $U \in \mathcal{F}$. There exists a $k \in N_m$ such that $p_m(V(U)) \cap p_m(F_k) \neq \emptyset$, i.e. $\prod_{i=1}^{m} V(U)_i \cap p_m(F_k) \neq \emptyset$ for all $U \in \mathcal{F}$. Now $\prod_{i=1}^{m} V(U)_i \cap p_m(F_k) \neq \emptyset$ gives a $x(U) = (x(U)_n) \in F_k$ such that $(x(U)_i : 1 \leq i \leq m) \in \prod_{i=1}^{m} V(U)_i$. Since $V(U)_n = X_n$ for all $n \geq m$, $x(U) \in V(U)$. Thus $V(U) \cap F_k \neq \emptyset$ for all $U \in \mathcal{F}$. Hence X is S-separable. $\square$

Theorem 3.12 ([11]). *If X is any compact Hausdorff space, then X has a dense subspace Y with $d(Y) = \pi w(X)$.*

Proposition 3.13. *A compact space X is S-separable if and only if $\pi w(X) = \omega$.*

Proof. Assume that X is S-separable. By Theorem 3.12, there exists a dense subspace Y of X such that $d(Y) = \pi w(X)$. Then by Proposition 3.5(1), Y is S-separable and so Y is separable, i.e. $d(Y) = \omega$. Thus $\pi w(X) = \omega$.

Conversely suppose that $\pi w(X) = \omega$. This gives $\delta(X) = \omega$. By Theorem 3.17, X is S-separable. $\square$

It is important to mention that compactness is essential in the above result. Indeed, the space $C_p([0,1])$ is S-separable because $iw([0,1]) = \omega$ and every finite power of $[0,1]$ is Menger (see Theorem 3.30). But $\pi w(C_p([0,1])) = \mathfrak{c}$.

Proposition 3.14. *Let X be a separable compact space. If X is either scattered or has countable tightness, then every continuous image of X is S-separable.*

Proof. Let Y be a continuous image of X. If X is scattered, then Y is scattered and separable. It follows that $\pi w(Y) = \omega$ and hence by Proposition 3.13, Y is S-separable.

Now assume that X has countable tightness, i.e. $t(X) = \omega$. This gives $t(Y) = \omega$ (see [1, 1.1.1]) and consequently $\pi_\chi(Y) = \omega$ (see [16]). From the separability of Y we can say that $\pi w(Y) = \omega$. Again by Proposition 3.13, Y is S-separable. $\square$

Corollary 3.15. *If a compact space X is either scattered or has countable tightness, then every separable subspace of X is S-separable.*

Proof. Suppose that X is scattered. Let Y be a separable subspace of X. Since every subspace of a scattered space is scattered, Y is scattered. It follows that $\pi w(Y) = \omega$ and hence by Proposition 3.13, Y is S-separable.

Now suppose that X has countable tightness, i.e. $t(X) = \omega$. This gives $t(Y) = \omega$ and consequently $\pi_\chi(Y) = \omega$ (see [16]). From the separability of Y we can say that $\pi w(Y) = \omega$. Again by Proposition 3.13, Y is S-separable. □

Thus if X is a countable non-S-separable space, then X cannot be embedded in a compact space of countable tightness.

In the following example we show that a continuous image of a compact S-separable space may not be S-separable.

Example 3.16. *There exists a compact S-separable space which has a non-S-separable continuous image.*

Proof. Consider the space $\beta\mathbb{N}$. Since $\pi w(\beta\mathbb{N}) = \omega$ (see [13]), by Proposition 3.13, $\beta\mathbb{N}$ is S-separable. The Tychonoff cube $\mathbb{I}^{\mathfrak{c}}$ can be obtained as a continuous image of $\beta\mathbb{N}$, where $\mathbb{I} = [0, 1]$. $\mathbb{I}^{\mathfrak{c}}$ has a countable dense subspace which is not S-separable (see [9, Example 2.14] for explanation). By Proposition 3.5(1), $\mathbb{I}^{\mathfrak{c}}$ is not S-separable. □

Note that every continuous image of a compact space X is S-separable does not guarantee that $t(X) = \omega$. Indeed, there exists a separable compact scattered space X with $t(X) \geq \omega_1$ (see [9, Example 2.8]). By Proposition 3.14, every continuous image of X is S-separable.

3.2. **Further observations on S-separability.** For a set $Y \subseteq \mathbb{N}^\mathbb{N}$, maxfin$(Y)$ is defined as

$$\text{maxfin}(Y) = \{\max\{f_1, f_2, \dots, f_k\} : f_1, f_2, \dots, f_k \in Y \text{ and } k \in \mathbb{N}\},$$

where $\max\{f_1, f_2, \dots, f_k\}(n) = \max\{f_1(n), f_2(n), \dots, f_k(n)\}$ for all $n \in \mathbb{N}$.

Theorem 3.17. *If $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{d}$, then X is S-separable.*

Proof. Let $\mathcal{B} = \{U_\alpha : \alpha < \kappa\}$ be a π-base for X with $U_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Assume that $\kappa < \mathfrak{d}$. Let (Y_n) be a sequence of dense subspaces of X. Since $\delta(X) = \omega$, we get a sequence (Z_n) such that for each n $Z_n = \{x^{(n)}_m : m \in \mathbb{N}\}$ is a dense subspace of Y_n. For each $\alpha < \kappa$ we define a $f_\alpha \in \mathbb{N}^\mathbb{N}$ by $f_\alpha(n) = \min\{m \in \mathbb{N} : x^{(n)}_m \in U_\alpha\}$. Since $Y = \{f_\alpha : \alpha < \kappa\}$ has cardinality less than $\mathfrak{d}$, maxfin(Y) also has cardinality less than $\mathfrak{d}$. Then there exist a $g \in \mathbb{N}^\mathbb{N}$ and for each finite $F \subseteq \kappa$ a $n_F \in \mathbb{N}$ such that $f_F(n_F) < g(n_F)$ with $f_F \in$ maxfin(Y). We use the convention that if $F = \{\alpha\}$, $\alpha < \kappa$, then we write f_α instead of f_F. For each n let $F_n = \{x^{(n)}_m : m \leq g(n)\}$. Observe that the sequence (F_n) witnesses for (Y_n) that X is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. Then there exists a finite $F \subseteq \kappa$ such that each member of $\mathcal{F}$ contains a U_α for some $\alpha \in F$. We claim that $U \cap F_{n_F} \neq \emptyset$ for all $U \in \mathcal{F}$. Pick a $U \in \mathcal{F}$. Choose a $\beta \in F$ such that $U_\beta \subseteq U$. Now from the definition of f_β we get $x^{(n_F)}_{f_\beta(n_F)} \in U_\beta$. Since $f_F(n_F) < g(n_F)$ and $f_F = \max\{f_\alpha : \alpha \in F\}$, $f_\beta(n_F) < g(n_F)$. It follows that $x^{(n_F)}_{f_\beta(n_F)} \in F_{n_F}$ and hence $U_\beta \cap F_{n_F} \neq \emptyset$, i.e. $U \cap F_{n_F} \neq \emptyset$. Thus X is S-separable. □

Recall from [13] that for 2^κ (the Cantor cube of weight κ), $\pi w(2^\kappa) = \kappa = w(2^\kappa)$.

Corollary 3.18. *If $\kappa < \mathfrak{d}$, then every countable subspace of 2^κ is S-separable.*

Proof. Let X be a countable subspace of 2^κ. Then $\delta(X) = \omega$. Since $w(2^\kappa) = \kappa$ and weight is monotone (i.e. for every subspace Y of a space X, $w(Y) \leq w(X)$), $w(X) \leq \kappa$. Also since $\pi w(X) \leq w(X)$, $\pi w(X) \leq \kappa$. Hence the result. $\square$

Corollary 3.19. *Assume* $\mathrm{MA} + \neg\mathrm{CH}$. *Then every countable subspace of 2^{ω_1} is S-separable.*

Proof. Note that MA gives $\mathfrak{d} = \mathfrak{c}$ and so $\omega_1 < \mathfrak{d}$. Let X be a countable subspace of 2^{ω_1}. Then $\delta(X) = \omega$. Since $w(2^{\omega_1}) = \omega_1$, $w(X) \leq \omega_1$. Also since $\pi w(X) \leq w(X)$, $\pi w(X) \leq \omega_1 < \mathfrak{d}$. Thus X is S-separable. $\square$

Theorem 3.20 ([9, Theorem 2.18])**.** *The space $2^\mathfrak{d}$ contains a countable dense subspace which is not M-separable (hence not S-separable).*

Corollary 3.21. *The space $2^\mathfrak{d}$ contains a countable dense subspace which is not S-separable.*

Corollary 3.22. *The smallest π-weight of a countable non-S-separable space is $\mathfrak{d}$.*

Proof. Let X be a countable space. Then $\delta(X) = \omega$. If $\pi w(X) < \mathfrak{d}$, then by Theorem 3.17, X is S-separable.

Now by Corollary 3.21, $2^\mathfrak{d}$ contains a countable subspace Y such that $\pi w(Y) = \mathfrak{d}$ and Y is not S-separable. Thus the smallest π-weight of a countable non-S-separable space is $\mathfrak{d}$. $\square$

Let F be a subset of a space X. For each $x \in F$ choose an open set U_x of X containing x. By a neighbourhood of F we mean the collection $\{U_x : x \in F\}$ and we use $\mathcal{N}(F)$ to denote a neighbourhood of F. We now introduce the following notions.

(1) A space X is said to satisfy $(*_1)$ if for any finite $F \subseteq X$ and any sequence (Y_n) of subsets of X with $F \subseteq \cap_{n\in\mathbb{N}}\overline{Y_n}$ there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for each neighbourhood $\mathcal{N}(F)$ of F there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{N}(F)$.

(2) A space X is said to satisfy $(*_1)$ with respect to dense subspaces if for any finite $F \subseteq X$ and any sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for each neighbourhood $\mathcal{N}(F)$ of F there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{N}(F)$.

It is immediate that if X satisfies $(*_1)$ (respectively, $(*_1)$ with respect to dense subspaces), then X has countable fan tightness (respectively, countable fan tightness with respect to dense subspaces). Also if X satisfies $(*_1)$, then it satisfies $(*_1)$ with respect to dense subspaces, and if X is S-separable, then it satisfies $(*_1)$ with respect to dense subspaces.

Proposition 3.23. *A separable space X is S-separable if and only if X satisfies $(*_1)$ with respect to dense subspaces.*

Proof. Suppose that X satisfies $(*_1)$ with respect to dense subspaces. Let (Y_n) be a sequence of dense subspaces of X. Let $\{N_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite subsets. Since X is separable, we have a countable dense subspace Y of X. Choose $\mathrm{Fin}(Y) = \{G_n : n \in \mathbb{N}\}$. Fix $n \in \mathbb{N}$. Apply the property $(*_1)$ with respect to dense subspaces of X to $(Y_m : m \in N_n)$ to obtain a sequence $(F_m : m \in N_n)$ such that for each $m \in N_n$, F_m is a finite subset of Y_m and for each neighbourhood $\mathcal{N}(G_n)$ of G_n there exists a $m \in N_n$ such that $U \cap F_m \neq \emptyset$ for all $U \in \mathcal{N}(G_n)$. We claim that the sequence (F_n) witnesses for (Y_n) that X is S-separable. Let $\mathcal{F}$ be a finite collection of nonempty open sets of X. For each $U \in \mathcal{F}$ we can pick a $x_U \in U \cap Y$ and let G be the collection of all such x_U. Then $G = G_k$ for some $k \in \mathbb{N}$ and $\mathcal{F}$ is a neighbourhood of G_k. This gives a $m \in N_k$ such that $U \cap F_m \neq \emptyset$ for all $U \in \mathcal{F}$. Thus X is S-separable.

The other direction is trivial. □

Lemma 3.24. *If X satisfies $(*_1)$, then every subspace of X also satisfies $(*_1)$.*

Proof. Let Y be a subspace of X. Let F be a finite subset of Y and (Y_n) be sequence of subsets of Y such that $F \subseteq \overline{Y_n}^Y$ for all n. Clearly $F \subseteq \overline{Y_n}$ for all n. Since X satisfies $(*_1)$, there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every neighbourhood of $\mathcal{N}(F)$ of F in X there exists a $n \in \mathbb{N}$ such that $U \cap F_n \neq \emptyset$ for all $U \in \mathcal{N}(F)$. Observe that the sequence (F_n) witnesses for F and (Y_n) that Y satisfies $(*_1)$. Let $\mathcal{N}(F)$ be a neighbourhood of F in Y. Then for each $V \in \mathcal{N}(F)$ we can find an open $U(V)$ set in X such that $V = U(V) \cap Y$. Then $\{U(V) : V \in \mathcal{N}(F)\}$ is a neighbourhood of F in X. This gives a $m \in \mathbb{N}$ such that $U(V) \cap F_m \neq \emptyset$ for all $V \in \mathcal{N}(F)$. It follows that $V \cap F_m \neq \emptyset$ for all $V \in \mathcal{N}(F)$. Thus Y satisfies $(*_1)$. □

Theorem 3.25 ([2,8], [3, II.2.2. Theorem])**.** *For a space X the following assertions are equivalent.*

(1) *$C_p(X)$ has countable fan tightness.*
(2) *$C_p(X)$ has countable fan tightness with respect to dense subspaces.*
(3) *Every finite power of X is Menger.*

Theorem 3.26. *For a space X the following assertions are equivalent.*

(1) *$C_p(X)$ satisfies $(*_1)$.*
(2) *$C_p(X)$ satisfies $(*_1)$ with respect to dense subspaces.*
(3) *Every finite power of X is Menger.*
(4) *Every finite power of X is Scheepers.*

Proof. (2) ⇒ (3). If $C_p(X)$ satisfies $(*_1)$ with respect to dense subspaces, then $C_p(X)$ has countable fan tightness with respect to dense subspaces. By Theorem 3.25, every finite power of X is Menger.

(3) ⇒ (4). Let $k \in \mathbb{N}$ and $Y = X^k$. Since every finite power of X is Menger, every finite power of Y is also Menger. It follows that Y is Scheepers (see [12]). Thus every finite power of X is Scheepers.

(4) ⇒ (1). We will follow the technique of the proof of [3, Theorem II.2.2] to prove this implication. Let F be a finite subset of $C_p(X)$ and (Y_n) be a sequence of subsets of $C_p(X)$ such that $F \subseteq \cap_{n\in\mathbb{N}} \overline{Y_n}$. Let $k \in \mathbb{N}$ be fixed. Pick a $f \in F$.

Since $f \in \cap_{n\in\mathbb{N}}\overline{Y_n}$, for each $n \in \mathbb{N}$ and each $x = (x_1, x_2, \ldots, x_k) \in X^k$ there exists a $g_{f,x}^{(n,k)} \in Y_n$ such that $|g_{f,x}^{(n,k)}(x_i) - f(x_i)| < \frac{1}{k}$ for all $1 \leq i \leq k$. Since $g_{f,x}^{(n,k)}$ and f are continuous, for each $1 \leq i \leq k$ we can choose an open set $U_{f,i,x}^{(n,k)}$ of X containing x_i such that $|g_{f,x}^{(n,k)}(y) - f(y)| < \frac{1}{k}$ for all $y \in U_{f,i,x}^{(n,k)}$. Then $U_{f,x}^{(n,k)} = U_{f,1,x}^{(n,k)} \times U_{f,2,x}^{(n,k)} \times \ldots \times U_{f,k,x}^{(n,k)}$ is an open set in X^k containing x and $|g_{f,x}^{(n,k)}(y_i) - f(y_i)| < \frac{1}{k}$ for all $y = (y_1, y_1, \ldots, y_k) \in U_{f,x}^{(n,k)}$. Choose $U_x^{(n,k)} = \cap_{f\in F} U_{f,x}^{(n,k)}$. Clearly $|g_{f,x}^{(n,k)}(y_i) - f(y_i)| < \frac{1}{k}$ for all $y = (y_1, y_1, \ldots, y_k) \in U_x^{(n,k)}$ and for all $f \in F$. For each $n \in \mathbb{N}$, $\mathcal{U}_n^{(k)} = \{U_x^{(n,k)} : x \in X^k\}$ is an open cover of X^k. Apply the Scheepers property of X^k to $(\mathcal{U}_n^{(k)})$ to obtain a sequence $(\mathcal{V}_n^{(k)})$ such that for each n $\mathcal{V}_n^{(k)}$ is a finite subset of $\mathcal{U}_n^{(k)}$ and $\{\cup\mathcal{V}_n^{(k)} : n \in \mathbb{N}\}$ is an ω-cover of X^k. Then we can obtain a sequence $(A_n^{(k)})$ of finite subsets of X^k such that for each n $\mathcal{V}_n^{(k)} = \{U_x^{(n,k)} : x \in A_n^{(k)}\}$. For each $n \in \mathbb{N}$ and each $f \in F$, $F_{f,n}^{(k)} = \{g_{f,x}^{(n,k)} : x \in A_n^{(k)}\}$ is a finite subset of Y_n. For each $n \in \mathbb{N}$ and each $f \in F$ let $F_{f,n} = \cup_{k\leq n} F_{f,n}^{(k)}$. Also for each n let $F_n = \cup_{f\in F} F_{f,n}$. Thus we define a sequence (F_n) such that for each n F_n is a finite subset of Y_n. Observe that the sequence (F_n) witnesses for F and (Y_n) that $C_p(X)$ satisfies $(*_1)$. Let $\mathcal{N}(F) = \{U_f : f \in F\}$ be a neighbourhood of F in $C_p(X)$. Then we get a finite subset $F' = \{z_1, z_2, \ldots, z_m\}$ of X and an $\epsilon > 0$ such that $B(f, F', \epsilon) \subseteq U_f$ for all $f \in F$. We can assume that $\frac{1}{m} < \epsilon$. Since $z = (z_1, z_2, \ldots, z_m) \in X^m$, there exists a $p \geq m$ such that $z \in \cup\mathcal{V}_p^{(m)}$. If possible suppose that there exists a $f \in F$ such that $B(f, F', \epsilon) \cap F_{f,p} = \emptyset$. It follows that $B(f, F', \epsilon) \cap F_{f,p}^{(i)} = \emptyset$ for all $i \leq p$, i.e. $B(f, F', \epsilon) \cap F_{f,p}^{(m)} = \emptyset$. Subsequently $g_{f,x}^{(p,m)} \notin B(f, F', \epsilon)$ for all $x \in A_p^{(m)}$. Then for each $x \in A_p^{(m)}$, $|g_{f,x}^{(p,m)}(z_{i_x}) - f(z_{i_x})| \geq \epsilon > \frac{1}{m}$ for some $1 \leq i_x \leq m$. Consequently $z \notin U_x^{(p,m)}$ for all $x \in A_p^{(m)}$, i.e. $z \notin \cup\mathcal{V}_p^{(m)}$. Which is absurd. Thus $B(f, F', \epsilon) \cap F_{f,p} \neq \emptyset$ and so $B(f, F', \epsilon) \cap F_p \neq \emptyset$. It follows that $U_f \cap F_p \neq \emptyset$ for all $f \in F$. Hence $C_p(X)$ satisfies $(*_1)$. □

Combining Theorems 3.25 and 3.26 we obtain the following result.

Theorem 3.27. *For a space X the following assertions are equivalent.*
(1) *$C_p(X)$ satisfies $(*_1)$.*
(2) *$C_p(X)$ has countable fan tightness.*
(3) *$C_p(X)$ satisfies $(*_1)$ with respect to dense subspaces.*
(4) *$C_p(X)$ has countable fan tightness with respect to dense subspaces.*
(5) *Every finite power of X is Menger.*
(6) *Every finite power of X is Scheepers.*

Corollary 3.28. *If every finite power of X is Menger, then every separable subspace of $C_p(X)$ is S-separable.*

Proof. Let Y be a separable subspace of $C_p(X)$. Suppose that every finite power of X is Menger. Then by Theorem 3.27, $C_p(X)$ satisfies $(*_1)$. Also by Lemma 3.24, Y satisfies $(*_1)$. Thus by Proposition 3.23, Y is S-separable. □

From [1, 2.3.1] we now consider a space V called countable Fréchet-Urysohn fan. Since V has a dense subspace of isolated points, $\pi w(V) = \omega$. By Theorem 3.17, V is S-separable. But V does not have countable fan tightness, i.e. V does not satisfy $(*_1)$. However, we observe in Theorem 3.30 that $C_p(X)$ is S-separable implies that $C_p(X)$ satisfies $(*_1)$. We first recall the following well known result which will be used subsequently.

Theorem 3.29 ([15] (see also [3, Theorem I.1.5])). *For a Tychonoff space X, $iw(X) = d(C_p(X))$.*

Theorem 3.30. *For a Tychonoff space X the following assertions are equivalent.*
(1) *$C_p(X)$ is separable and satisfies $(*_1)$.*
(2) *$C_p(X)$ is separable and satisfies $(*_1)$ with respect to dense subspaces.*
(3) *$C_p(X)$ is S-separable.*
(4) *$C_p(X)$ is M-separable.*
(5) *$iw(X) = \omega$ and every finite power of X is Menger.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious. The implication (4) $\Rightarrow$ (5) was shown in [8, Theorem 21].

(2) $\Rightarrow$ (3). By Theorem 3.26, every finite power of X is Menger. Then by Corollary 3.28, $C_p(X)$ is S-separable.

(5) $\Rightarrow$ (1). By Theorem 3.27, $C_p(X)$ satisfies $(*_1)$. Since $iw(X) = \omega$, by Theorem 3.29, $d(C_p(X)) = \omega$. It follows that $C_p(X)$ is separable. ☐

Corollary 3.31. *If $C_p(X)$ is S-separable, then for each $n \in \mathbb{N}$, $C_p(X^n)$ is S-separable, and hence $(C_p(X))^\omega$ is S-separable.*

Proof. Let $n \in \mathbb{N}$ and $Y = X^n$. Since $C_p(X)$ is S-separable, $iw(X) = \omega$ and every finite power of X is Menger. It follows that every finite power of Y is also Menger and $iw(Y) = \omega$. Thus $C_p(Y)$ is S-separable. It follows that for each $n \in \mathbb{N}$, $C_p(X^n)$ is S-separable, and hence $(C_p(X))^\omega$ is S-separable. ☐

Corollary 3.32. *If X is a second countable space such that $C_p(X)$ is S-separable, then $C_p(X)$ is hereditarily S-separable.*

Proof. Let Y be a subspace of $C_p(X)$. Since X is a second countable space, it is separable metrizable. It follows that $C_p(X)$ is hereditarily separable and hence Y is separable. Since $C_p(X)$ is S-separable, $C_p(X)$ satisfies $(*_1)$. By Lemma 3.24, Y satisfies $(*_1)$ and so does $(*_1)$ with respect to dense subspaces. Also by Proposition 3.23, Y is S-separable. Thus $C_p(X)$ is hereditarily S-separable. ☐

Corollary 3.33. *If X is a second countable space with $|X| < \mathfrak{d}$, then $C_p(X)$ is hereditarily S-separable.*

Proof. Let Y be a subspace of $C_p(X)$. Since X is a second countable space, it is separable metrizable. It follows that $C_p(X)$ is hereditarily separable and hence Y is separable. Also every finite power of X is second countable and so every finite power of X is Lindelöf. Since $|X| < \mathfrak{d}$, every finite power of X is Menger (see [12]). It follows that $C_p(X)$ satisfies $(*_1)$ as $iw(X) = d(C_p(X)) = \omega$. Then Y satisfies $(*_1)$

and so does $(*_2)$ with respect to dense subspaces. Also by Proposition 3.23, Y is S-separable. Thus $C_p(X)$ is hereditarily S-separable. $\square$

Using Theorem 3.30 and [9, Corollary 2.13] the following result can be easily verified.

Proposition 3.34. *The space $C_p(C_p(X))$ is S-separable if and only if X is finite.*

Remark 3.35.
(1) Recall that the Baire space $\mathbb{N}^{\mathbb{N}}$ can be condensed onto a compact space X. It follows that $C_p(X)$ can be densely embedded in $C_p(\mathbb{N}^{\mathbb{N}})$. Now $C_p(X)$ is separable, i.e. $d(C_p(X)) = \omega$ and so $iw(X) = \omega$ (see Theorem 3.29). Since X is compact, every finite power of X is Menger and hence by Theorem 3.30, $C_p(X)$ is S-separable. Let Y be a countable dense subspace of $C_p(X)$. Then Y is S-separable. Since $C_p(X)$ is densely embeddable in $C_p(\mathbb{N}^{\mathbb{N}})$, Y is a countable dense S-separable subspace of $C_p(\mathbb{N}^{\mathbb{N}})$. Also since $C_p(\mathbb{N}^{\mathbb{N}})$ can be densely embedded in the Tychonoff cube $\mathbb{I}^{\mathfrak{c}}$, Y is a countable dense S-separable subspace of $\mathbb{I}^{\mathfrak{c}}$. Thus $\mathbb{I}^{\mathfrak{c}}$ has a countable dense S-separable subspace.
(2) Moreover, we have already mentioned in Example 3.16 that $\mathbb{I}^{\mathfrak{c}}$ has a countable dense non-S-separable subspace.

Theorem 3.36. *For a zero-dimensional space X the following assertions are equivalent.*
(1) $C_p(X, \mathbb{Q})$ *is S-separable.*
(2) $C_p(X, \mathbb{Z})$ *is S-separable.*
(3) $C_p(X, 2)$ *is S-separable.*
(4) $iw(X) = \omega$ *and every finite power of X is Menger.*

Proof. (1) $\Rightarrow$ (2). For each $n \in \mathbb{Z}$ pick an irrational $x_n \in (n, n+1)$. Choose a $\psi \in \mathbb{Z}^{\mathbb{Q}}$ with $\psi(x) = n$ if $x \in (x_{n-1}, x_n)$. Let us define a mapping $\Phi : C_p(X, \mathbb{Q}) \to C_p(X, \mathbb{Z})$ by $\Phi(f) = \psi \circ f$ for all $f \in C_p(X, \mathbb{Q})$. Clearly Φ is open continuous from $C_p(X, \mathbb{Q})$ onto $C_p(X, \mathbb{Z})$. By Proposition 3.8(1), $C_p(X, \mathbb{Z})$ is S-separable.

(2) $\Rightarrow$ (3). Pick a $\phi \in 2^{\mathbb{Z}}$ satisfying

$$\phi(n) = \begin{cases} 0, & \text{if } n < 0 \\ 1, & \text{otherwise.} \end{cases}$$

Let us define a mapping $\Psi : C_p(X, \mathbb{Z}) \to C_p(X, 2)$ by $\Psi(f) = \phi \circ f$ for all $f \in C_p(X, \mathbb{Z})$. Now Ψ is open continuous from $C_p(X, \mathbb{Z})$ onto $C_p(X, 2)$. By Proposition 3.8(1), $C_p(X, 2)$ is S-separable.

(3) $\Rightarrow$ (4). Since $C_p(X, 2)$ is S-separable, it is M-separable. By [8, Proposition 25], $iw(X) = \omega$ and every finite power of X is Menger.

(4) $\Rightarrow$ (1). By Theorem 3.30, $C_p(X)$ is S-separable. Since X is zero-dimensional, $C_p(X, \mathbb{Q})$ is dense in $C_p(X)$. By Proposition 3.5(1), $C_p(X, \mathbb{Q})$ is S-separable. $\square$

With the help of [8, Proposition 25] the following corollary can be obtained.

Corollary 3.37. *For a zero-dimensional space X the following assertions are equivalent.*

(1) $C_p(X, \mathbb{Q})$ *is S-separable.*
(2) $C_p(X, \mathbb{Q})$ *is M-separable.*
(3) $C_p(X, \mathbb{Z})$ *is S-separable.*
(4) $C_p(X, \mathbb{Z})$ *is M-separable.*
(5) $C_p(X, 2)$ *is S-separable.*
(6) $C_p(X, 2)$ *is M-separable.*
(7) $iw(X) = \omega$ *and every finite power of* X *is Menger.*

Corollary 3.38. *Let* (X, τ) *be condensed onto a separable metrizable space* (X, τ')*. If every finite power of* (X, τ') *is Menger, then there exists a dense S-separable subspace of* $C_p(X, \tau')$*.*

Proof. We can think of $C_p(X, \tau')$ as subspace of $C_p(X, \tau)$. Clearly $C_p(X, \tau')$ is dense in $C_p(X, \tau)$. By Theorem 3.26, $C_p(X, \tau')$ satisfies $(*_1)$ with respect to dense subspaces. Since (X, τ') is separable metrizable, $C_p(X, \tau')$ is hereditarily separable. It follows that $C_p(X, \tau')$ is S-separable (see Proposition 3.23). $\square$

In the following example we observe that separability plays a crucial role in the validity of the above result.

Example 3.39. *There exists a metrizable space* X *such that* $C_p(X)$ *contains a dense S-separable subspace but* $C_p(X)$ *is not S-separable.*

Proof. Let $D(\mathfrak{c})$ be the discrete space of cardinality $\mathfrak{c}$. Now $C_p(D(\mathfrak{c})) = \mathbb{R}^{\mathfrak{c}}$. We interpret $\mathbb{R}^{\mathfrak{c}}$ as $\mathbb{R}^{2^\omega}$. Let $Y = C_p(2^\omega)$. Then Y is dense in $\mathbb{R}^{2^\omega}$. Note that $iw(2^\omega) = \omega$ and 2^ω is compact, i.e. every finite power of 2^ω is Menger. By Theorem 3.30, Y is S-separable. Thus Y is a dense S-separable subspace of $C_p(D(\mathfrak{c})) = \mathbb{R}^{\mathfrak{c}}$. Again by Theorem 3.30, $C_p(D(\mathfrak{c})) = \mathbb{R}^{\mathfrak{c}}$ is not S-separable as $D(\mathfrak{c})$ is not Menger. $\square$

It is natural to ask the following question.

Problem 3.40. *If* $C_p(X)$ *contains a dense S-separable subspace, then can* X *be condensed onto a second countable space every finite power of which is Menger?*

Theorem 3.41 ([8, Theorem 46])**.** *The space* $2^{\mathfrak{c}}$ *contains a countable dense H-separable subspace.*

Corollary 3.42. *The space* $2^{\mathfrak{c}}$ *contains a countable dense S-separable subspace.*

Thus in ZFC, $2^{\mathfrak{c}}$ has a countable dense S-separable subspace. Under the assumption MA + ¬CH, every countable subspace of 2^{ω_1} is S-separable (see Corollary 3.19). It is natural to ask the following question.

Problem 3.43. *In ZFC, does* 2^{ω_1} *contain a countable dense S-separable subspace?*

We now give an affirmative answer to the above problem.

Theorem 3.44. *The space* 2^{ω_1} *contains a dense S-separable subspace.*

Proof. Assume that $\omega_1 < \mathfrak{b}$. Then by Corollary 3.18, every countable subspace of 2^{ω_1} is S-separable. Thus we can find a countable dense S-separable subspace of 2^{ω_1}.

Next assume that $\omega_1 = \mathfrak{b}$. From [7, Theorem 10] we can say that there exists a zero-dimensional metrizable space X with cardinality $\mathfrak{b}$ such that every finite power of X is Menger. Clearly $iw(X) = \omega$. By Theorem 3.36, $C_p(X, 2)$ is a S-separable subspace of 2^{ω_1}. Also $C_p(X, 2)$ is dense in 2^{ω_1}. □

Corollary 3.45. *The space 2^{ω_1} contains a countable dense S-separable subspace.*

Proof. By Theorem 3.44, we get a dense S-separable subspace Y of 2^{ω_1}. Let Z be a countable dense subspace of Y. Then Z is S-separable. Clearly Z is dense in 2^{ω_1}. □

3.3. **L-separability.**

Definition 3.46. *A space X is said to be L-separable if for every dense subset D of X there exists a countable subset C of D such that C is dense in X (or equivalently, if every dense subspace of X is separable).*

Verify that a space X is L-separable if and only if $\delta(X) = \omega$. It is clear that every M-separable space is L-separable. However, the converse does not hold in general. Indeed, there exist L-separable spaces which are not M-separable. For instance, since every countable space is L-separable, any countable space that fails to be M-separable provides a counterexample. Such examples exist in ZFC (see Theorem 3.20). Therefore, M-separability is a strictly stronger property than L-separability. However, since a space X is L-separable if and only if $\delta(X) = \omega$, every L-separable space with π-weight less than $\mathfrak{d}$ is S-separable (and hence M-separable) (see Theorem 3.17).

Also note that L-separability implies separability. We now observe that the converse is not true.

Example 3.47. *There exists a Tychonoff separable space which is not L-separable.*

Proof. Let I be an index set with $|I| = \mathfrak{c}$, and consider the product $X = [0, 1]^I$ with the product topology. Then X is separable as the product of at most continuum many separable spaces is separable. Let

$$\Sigma(X) = \{x \in X : \mathrm{supp}(x) = \{i \in I : x(i) \neq 0\} \text{ is countable}\}.$$

We first show that $\Sigma(X)$ is dense in X. Let U be a basic open cylinder determined by finitely many coordinates $F \subseteq I$. One can choose $y \in \Sigma(X)$ that matches the prescribed values on F and is 0 elsewhere, so $y \in U \cap \Sigma(X)$.

We claim that $\Sigma(X)$ is not separable. Indeed, fix any countable $A \subseteq \Sigma(X)$ and set $J = \bigcup_{a\in A} \mathrm{supp}(a)$, which is countable. Choose $j' \in I \setminus J$ and a non-degenerate open interval $V \subseteq (0, 1]$. Consider the cylinder $U = \{x \in X : x(j') \in V\}$. Then $W = U \cap \Sigma(X)$ is a nonempty open subset of the subspace $\Sigma(X)$. For every $a \in A$ we have $a(j') = 0$, hence $A \cap U = \emptyset$. If $x \in W$, then the basic neighborhood of x that restricts only the j'-coordinate to lie in V is disjoint from A; therefore $x \notin \overline{A}^X$, and so $x \notin \overline{A}^{\Sigma(X)} = \overline{A}^X \cap \Sigma(X)$. Thus $W \cap \overline{A}^{\Sigma(X)} = \emptyset$, showing that no countable $A \subseteq \Sigma(X)$ is dense in $\Sigma(X)$. Hence $\Sigma(X)$ is a dense non-separable subspace of the separable space X.

Since X has a dense non-separable subspace $\Sigma(X)$, it follows that X is not L-separable. $\square$

Let $N(X) = (X \times \{0\}) \cup (\mathbb{R} \times (0, \infty))$ be the Niemytzki plane on a set $X \subseteq \mathbb{R}$. The topology on $N(X)$ is defined as follows. $\mathbb{R} \times (0, \infty)$ has the Euclidean topology and the set $X \times \{0\}$ has the topology generated by all sets of the form $\{(x, 0)\} \cup U$, where $x \in X$ and U is an open disc in $\mathbb{R} \times (0, \infty)$ which is tangent to $X \times \{0\}$ at the point $(x, 0)$. The topology on $N(X)$ is also called Niemytzki's tangent disk topology. It is to be noted that Niemytzki originally defined $N(\mathbb{R})$ (see [14]).

Recall that a space X is called hereditarily separable if every subspace of X is separable. Clearly if a space X is hereditarily separable, then X is L-separable. Using the following lemma we will show in Example 3.49 that the converse fails in general.

Lemma 3.48. *If X has an open dense subspace Y that is second countable, then X is S-separable.*

Proof. Let (Y_n) be a sequence of dense subspaces of X. Since Y is open and dense in X, each $Y_n \cap Y$ is dense in Y. Fix a countable base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the subspace Y.

For each $n \in \mathbb{N}$ choose a finite set $F_n \subseteq Y_n \cap Y$ with $F_n \cap B_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$ (possible since $Y_n \cap Y$ is dense in Y). We claim that (F_n) witnesses S-separability. Let $\mathcal{F}$ be any finite family of nonempty open subsets of X. Since Y is dense and open, $U \cap Y \neq \emptyset$ for each $U \in \mathcal{F}$, and $U \cap Y$ is open in Y. For each $U \in \mathcal{F}$ pick $B_{i(U)} \in \mathcal{B}$ with $B_{i(U)} \subseteq U \cap Y$. Let $n_0 = \max\{i(U) : U \in \mathcal{F}\}$. By construction, F_{n_0} meets every $B_{i(U)}$, hence $F_{n_0} \cap U \neq \emptyset$ for all $U \in \mathcal{F}$. Thus X is S-separable. $\square$

Example 3.49. *There exists a Tychonoff S-separable (hence L-separable) space which is not hereditarily separable.*

Proof. Consider the Niemytzki plane $N(\mathbb{R})$ on $\mathbb{R}$. Choose $H = \mathbb{R} \times (0, \infty)$ and $L = \mathbb{R} \times \{0\}$. $N(\mathbb{R})$ is not hereditarily separable because L, being an uncountable discrete subspace of it, is not separable. On the other hand, H is open and dense in $N(\mathbb{R})$, and the subspace topology on H is the usual Euclidean topology, so H is second countable. By Lemma 3.48, it follows that $N(\mathbb{R})$ is S-separable. $\square$

For separable metrizable spaces, separability, hereditary separability, and L-separability coincide.

However, hereditary separability does not imply M-separability (see Theorem 3.20). Also M-separability does not imply hereditary separability as the Niemytzki plane $N(\mathbb{R})$ on $\mathbb{R}$, being a separable Fréchet space, is M-separable (see [6, Theorem 2.9]) but not hereditary separability (see Example 3.49).

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