ON A VARIATION OF SELECTIVE SEPARABILITY USING IDEALS

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ABSTRACT. A space X is H-separable (Bella et al., 2009) if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and every nonempty open set of X intersects F_n for all but finitely many n. In this paper, we introduce and study an ideal variant of H-separability, called $\mathcal{I}$-H-separability.

Key words and phrases: Selective separable, M-separable, H-separable, $\mathcal{I}$-H-separable.

1. Introduction

In [18], Marion Scheepers introduced and studied the notion of selective separability (also called M-separability; see [6]) as a generalization of separability, and the study was later continued in [4, 6, 7]. It is worth noting that $C_p(X)$ and 2^κ played crucial roles in the study of selective separability.

A family $\mathcal{I} \subseteq 2^A$ of subsets of a nonempty set A is said to be an ideal on A if (i) $B, C \in \mathcal{I}$ imply $B \cup C \in \mathcal{I}$ and (ii) $B \in \mathcal{I}$ and $C \subseteq B$ imply $C \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of A further satisfies $\{a\} \in \mathcal{I}$ for each $a \in A$. Throughout the paper $\mathcal{I}$ stands for a proper admissible ideal on $\mathbb{N}$. The symbol Fin denotes the ideals of all finite subsets of $\mathbb{N}$.

As a variant of selective separability, the notion of H-separability was introduced and studied by Bella et al. [6]. Motivated by this, we generalize H-separability by introducing its ideal variant, $\mathcal{I}$-H-separability. We establish that if $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals on $\mathbb{N}$ with $\mathcal{I}_1 \leq_{\text{1-1}} \mathcal{I}_2$, then $\mathcal{I}_1$-H-separability implies $\mathcal{I}_2$-H-separability. We show that $\mathcal{I}$-H-separability lies strictly between H-separability and M-separability. We further show that $\mathcal{I}$-H-separability and H-separability are equivalent for a certain class of ideals. The $\mathcal{I}$-H-separability possesses well-behaved categorical properties: it remains invariant under open continuous mappings, closed irreducible mappings, dense subspaces, and open subspaces. However, it does not remain invariant under continuous mappings (see Example 3.17). It is observed that the product of two countable $\mathcal{I}$-H-separable spaces remains $\mathcal{I}$-H-separable provided that one of the factors has π-weight less than $\mathfrak{b}$. For a compact space we show that $\mathcal{I}$-H-separability is equivalent to π-weight of the space is countable.

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The cardinal number $\mathfrak{b}(\mathcal{I})$ and the space 2^κ are used to obtain certain fascinating observations. We begin by showing that, under a certain assumption, a space with π-weight less than $\mathfrak{b}(\mathcal{I})$ is $\mathcal{I}$-H-separable. As a consequence of this result, we establish that every countable subspace of 2^κ, where $\kappa < \mathfrak{b}(\mathcal{I})$, is $\mathcal{I}$-H-separable. Furthermore, we construct a countable dense non-$\mathcal{I}$-H-separable subspace of $2^{\mathfrak{b}(\mathcal{I})}$. This allows us to determine that the smallest π-weight of a countable non-$\mathcal{I}$-H-separable space is precisely $\mathfrak{b}(\mathcal{I})$. It is also demonstrated that 2^{ω_1} contains a countable dense $\mathcal{I}$-H-separable subspace.

Finally, we consider the function space $C_p(X)$ to further our investigation in this direction. To this end, we introduce the notions of $\mathcal{I}$-weakly Fréchet in the strict sense (abbreviated as $\mathcal{I}$WFS) and $\mathcal{I}$-weakly Fréchet in the strict sense with respect to dense subspaces (abbreviated as $\mathcal{I}$WFSD). On $C_p(X)$, $\mathcal{I}$-H-separability is characterized by the $\mathcal{I}$WFS and $\mathcal{I}$WFSD properties, as well as by the condition that the i-weight of X is countable and every finite power of X is $\mathcal{I}$-Hurewicz. If X is zero-dimensional, then we observe that $C_p(X)$ is $\mathcal{I}$-H-separable is equivalent to $C_p(X,2)$, $C_p(X,\mathbb{Z})$, and $C_p(X,\mathbb{Q})$ are $\mathcal{I}$-H-separable. Additionally, we obtain several interesting results concerning $C_p(X)$. Our investigation also leaves some problems for future study.

2. Preliminaries

By a space we always mean a topological space. All spaces are assumed to be Tychonoff; otherwise will be mentioned. For undefined notions and terminologies see [11]. $w(X)$ denotes the weight of a space X. A collection $\mathcal{B}$ of open sets of X is called π-base if every nonempty open set in X contains a nonempty member of $\mathcal{B}$. $\pi w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base for } X\}$ denotes the π-weight of X. $d(X)$ denotes the minimal cardinality of a dense subspace of X and $\delta(X) = \sup\{d(Y) : Y \text{ is dense in } X\}$. For any space X, $d(X) \leq \delta(X) \leq \pi w(X) \leq w(X)$. The i-weight of a space (X,τ) is defined as

$$iw((X,\tau)) = \min\{\kappa : \text{there is a Tychonoff topology } \tau' \subseteq \tau \text{ such that } w((X,\tau')) = \kappa\}.$$

A space X has countable fan tightness [2] if for any $x \in X$ and any sequence (Y_n) of subsets of X with $x \in \cap_{n\in\mathbb{N}}\overline{Y_n}$ there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $x \in \overline{\cup_{n\in\mathbb{N}}F_n}$. X has countable fan tightness with respect to dense subspaces [6] if for any $x \in X$ and any sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $x \in \overline{\cup_{n\in\mathbb{N}}F_n}$. A space X has countable tightness (which is denoted by $t(X) = \omega$) if for each $x \in X$ and each $Y \subseteq X$ with $x \in \overline{Y}$ there exists a countable set $E \subseteq Y$ such that $x \in \overline{E}$. A space X is scattered if every nonempty subspace Y of X has an isolated point. For a space X, βX denotes the Stone-Čech compactification of X.

Let X be a space and $C(X)$ be the set of all continuous real valued functions. As usual $C_p(X)$ denotes the space $C(X)$ with pointwise convergence topology. Let $f \in C(X)$. Then a basic open set of f in $C_p(X)$ is of the form

$$B(f,F,\epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in F\},$$

where F is a finite subset of X and $\epsilon > 0$. We use $\underline{1}$ to denote the function which takes the constant value 1 everywhere on X. For a separable metrizable space X, $C_p(X)$ is hereditarily separable.

A space X is said to be M-separable [7] (see also [6]) if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and $\cup_{n\in\mathbb{N}} F_n$ is dense in X. X is said to be H-separable [6] if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and every nonempty open set of X intersects F_n for all but finitely many n.

A space X is said to have the $\mathcal{I}$-Hurewicz property [9] (see also [10]) if for each sequence $(\mathcal{U}_n)$ of open covers of X there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \in \mathcal{I}$.

3. Main Results

3.1. **$\mathcal{I}$-H-separability.** We now introduce the main definition of the paper.

Definition 3.1. A space X is said to be $\mathcal{I}$-H-separable if for every sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every nonempty open set U of X, $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$.

For a function $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $\mathcal{A}_1, \mathcal{A}_2 \subseteq 2^{\mathbb{N}}$, $\mathcal{A}_1 \leq_\varphi \mathcal{A}_2$ implies that $\varphi^{-1}(A) \in \mathcal{A}_2$ for all $A \in \mathcal{A}_1$. $\mathcal{A}_1 \leq_{\text{1-1}} \mathcal{A}_2$ implies that there exists a one-one function $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{A}_1 \leq_\varphi \mathcal{A}_2$.

Theorem 3.2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two ideals on $\mathbb{N}$ such that $\mathcal{I}_1 \leq_{\text{1-1}} \mathcal{I}_2$. If X is $\mathcal{I}_1$-H-separable, then X is $\mathcal{I}_2$-H-separable.*

Proof. Since $\mathcal{I}_1 \leq_{\text{1-1}} \mathcal{I}_2$, there exists a one-one function $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that $\varphi^{-1}(A) \in \mathcal{I}_2$ for all $A \in \mathcal{I}_1$. Let (Y_n) be a sequence of dense subspaces of X. We define a sequence (Z_n) of dense subspaces of X by

$$Z_n = \begin{cases} Y_m, & \text{if } n = \varphi(m) \\ Y_1, & \text{otherwise.} \end{cases}$$

Since X is $\mathcal{I}_1$-H-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of Z_n and for every nonempty open set U of X, $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}_1$. It follows that for every nonempty open set U of X, $\varphi^{-1}(\{n \in \mathbb{N} : U \cap F_n = \emptyset\}) \in \mathcal{I}_2$.

For each n let $F'_n = F_{\varphi(n)}$. Clearly for each n F'_n is a finite subset of Y_n. We claim that the sequence (F'_n) witnesses that X is $\mathcal{I}_2$-H-separable. Let U be a nonempty open set of X. Then $\varphi^{-1}(\{n \in \mathbb{N} : U \cap F_n = \emptyset\}) \in \mathcal{I}_2$. We now show that $\{n \in \mathbb{N} : U \cap F'_n = \emptyset\} \subseteq \varphi^{-1}(\{n \in \mathbb{N} : U \cap F_n = \emptyset\})$. Let $k \in \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$. Then $U \cap F'_k = \emptyset$, i.e. $U \cap F_{\varphi(k)} = \emptyset$. This gives us $\varphi(k) \in \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$, i.e. $k \in \varphi^{-1}(\{n \in \mathbb{N} : U \cap F_n = \emptyset\})$. Thus $\{n \in \mathbb{N} : U \cap F'_n = \emptyset\} \subseteq \varphi^{-1}(\{n \in \mathbb{N} : U \cap F_n = \emptyset\})$. It follows that $\{n \in \mathbb{N} : U \cap F'_n = \emptyset\} \in \mathcal{I}_2$ and hence X is $\mathcal{I}_2$-H-separable. ☐

It is clear that every H-separable space is $\mathcal{I}$-H-separable and every $\mathcal{I}$-H-separable space is M-separable. However, the converses do not hold, as demonstrated in the next two examples (Examples 3.3 and 3.4). To proceed, we first recall the following result of [8] (see [8, Theorem 2.1]). There are models of ZFC in which $\mathfrak{b} \ll \mathfrak{d}$, and every regular cardinal κ between them arises as $\kappa = \mathfrak{b}(\mathcal{I})$ for some maximal ideal $\mathcal{I}$.

Example 3.3. *There exists an $\mathcal{I}$-H-separable space which is not H-separable.*

Proof. Let $\mathcal{I}$ be a maximal ideal such that $\mathfrak{b} < \mathfrak{b}(\mathcal{I})$. By [6, Theorem 31], there exists a countable subspace Y of $2^{\mathfrak{b}}$ such that Y is not H-separable. Since Y is countable, $\delta(Y) = \omega$. Also $\pi w(Y) = \mathfrak{b} < \mathfrak{b}(\mathcal{I})$. By Theorem 3.18, Y is $\mathcal{I}$-H-separable. □

Example 3.4. *There exists a M-separable space which is not $\mathcal{I}$-H-separable.*

Proof. Let $\mathcal{I}$ be a maximal ideal such that $\mathfrak{b}(\mathcal{I}) < \mathfrak{d}$. By Theorem 3.20, there exists a countable subspace Y of $2^{\mathfrak{b}(\mathcal{I})}$ such that Y is not $\mathcal{I}$-H-separable. Since Y is countable, $\delta(Y) = \omega$. Also $\pi w(Y) = \mathfrak{b}(\mathcal{I}) < \mathfrak{d}$. By [6, Theorem 11] (see also [7]), Y is M-separable. □

The above two examples are consistent with ZFC. So it is natural to ask the following questions.

Problem 3.5. *In ZFC, does there exist an $\mathcal{I}$-H-separable space which is not H-separable?*

Problem 3.6. *In ZFC, does there exist a M-separable space which is not $\mathcal{I}$-H-separable?*

We say that a set C is almost contained in a set D if $C \setminus D$ is finite and we use $C \subseteq^* D$ to denote it. Let $\mathcal{C}$ be a family of subsets of $\mathbb{N}$. A subset D of $\mathbb{N}$ is said to be a pseudounion of $\mathcal{C}$ if $\mathbb{N} \setminus D$ is infinite and $C \subseteq^* D$ for every $C \in \mathcal{C}$.

Clearly Fin-H-separable = H-separable. We now show that $\mathcal{I}$-H-separability and H-separability are equivalent for certain class of ideals.

Theorem 3.7. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ having a pseudounion. Then a space X is H-separable if and only if it is $\mathcal{I}$-H-separable.*

Proof. For $\mathcal{I} = \text{Fin}$, the result follows immediately. Assume that $\mathcal{I} \neq \text{Fin}$. We only prove the non-trivial part. Suppose that X is an $\mathcal{I}$-H-separable space. Let D be a pseudounion of $\mathcal{I}$. Let (Y_n) be a sequence of dense subspaces of X. Since X is $\mathcal{I}$-H-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every nonempty open set U of X,

$$A_U = \{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}.$$

Let $E = \mathbb{N} \setminus D$. Since D is a pseudounion of $\mathcal{I}$, E is infinite and for every $C \in \mathcal{I}$, we have $C \subseteq^* D$, i.e. $C \setminus D$ finite. In particular, for every nonempty open U of X, $A_U \cap E$ is finite. Thus, on E, the sequence (F_n) already satisfies that for every nonempty open set U of X, $\{n \in E : U \cap F_n = \emptyset\}$ is finite.

We now construct a new sequence (G_n) that will work on all of $\mathbb{N}$. Since D and E are both infinite, there exists a bijection $f : D \to E$. Define a new sequence (Z_n) of dense subspaces of X as follows:

$$Z_n = \begin{cases} Y_n & \text{if } n \in D, \\ Y_{f^{-1}(n)} & \text{if } n \in E. \end{cases}$$

Apply the $\mathcal{I}$-H-separability of X to the sequence (Z_n). This yields a sequence (H_n) such that for each n H_n is a finite subset of Z_n and for every nonempty open set U of X,

$$B_U = \{n \in \mathbb{N} : U \cap H_n = \emptyset\} \in \mathcal{I}.$$

Again, since D is a pseudounion of $\mathcal{I}$, $B_U \cap E$ is finite for every nonempty open set U of X.

Now define the sequence (G_n) by

$$G_n = \begin{cases} F_n & \text{if } n \in E, \\ H_{f(n)} & \text{if } n \in D. \end{cases}$$

For each $n \in \mathbb{N}$, G_n is a finite subset of Y_n as (i) if $n \in E$, then $G_n = F_n \subseteq Y_n$; (ii) if $n \in D$, then $f(n) \in E$, and $G_n = H_{f(n)} \subseteq Z_{f(n)} = Y_{f^{-1}(f(n))} = Y_n$. We claim that the sequence (G_n) witnesses for (Y_n) that X is H-separable.

Let U be a nonempty open subset of X. We show that the set $C_U = \{n \in \mathbb{N} : U \cap G_n = \emptyset\}$ is finite. Clearly $C_U = (C_U \cap E) \cup (C_U \cap D)$. Now

$$C_U \cap E = \{n \in E : U \cap G_n = \emptyset\} = \{n \in E : U \cap F_n = \emptyset\} = A_U \cap E,$$

which is finite. Also

$$\begin{aligned} C_U \cap D &= \{n \in D : U \cap G_n = \emptyset\} = \{n \in D : U \cap H_{f(n)} = \emptyset\} \\ &= f^{-1}(\{m \in E : U \cap H_m = \emptyset\}) = f^{-1}(B_U \cap E), \end{aligned}$$

which is finite, as f is a bijection from D onto E and $B_U \cap E$ is finite. Thus C_U is finite for every nonempty open set U of X, and hence X is H-separable. ☐

Next we present some fundamental observations in the context of $\mathcal{I}$-H-separable spaces. Recall that a continuous mapping $f : X \to Y$ is irreducible if the only closed subset C of X satisfying $f(C) = Y$ is $C = X$.

Lemma 3.8. *The $\mathcal{I}$-H-separability is preserved under open continuous as well as closed irreducible mappings.*

Lemma 3.9. *The $\mathcal{I}$-H-separability is preserved under dense as well as open subspaces.*

Lemma 3.10. *For a space the following assertions are equivalent.*
(1) *X is hereditarily $\mathcal{I}$-H-separable.*
(2) *X is hereditarily separable and every countable subspace of X is $\mathcal{I}$-H-separable.*

Theorem 3.11. *If X is a countable $\mathcal{I}$-H-separable space and Y is a countable space with $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is $\mathcal{I}$-H-separable.*

Proof. Let $\kappa < \mathfrak{b}$ and $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ be a π-base of Y. Pick a sequence (Y_n) of dense subspaces of $X \times Y$. For each n we can choose $Y_n = \{x^{(n)}_m : m \in \mathbb{N}\}$. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projection mappings on X and Y respectively. Let $\alpha < \kappa$ be fixed. For each n let $Z^{(\alpha)}_n = p_1(Y_n \cap (X \times B_\alpha))$. Clearly each Z_n is dense in X. Since X is $\mathcal{I}$-H-separable, there exists a sequence $(F^{(\alpha)}_n)$ such that for each n $F^{(\alpha)}_n$ is a finite subset of $Z^{(\alpha)}_n$ and for every nonempty open set U of X, $\{n \in \mathbb{N} : U \cap F^{(\alpha)}_n = \emptyset\} \in \mathcal{I}$. For each n choose a finite subset $G^{(\alpha)}_n$ of $Y_n \cap (X \times B_\alpha)$ such that $F^{(\alpha)}_n = p_1(G^{(\alpha)}_n)$. Choose a $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $G^{(\alpha)}_n \subseteq \{x^{(n)}_m : m \leq f_\alpha(n)\}$. Thus we obtain a subset $\{f_\alpha : \alpha < \kappa\}$ of $\mathbb{N}^{\mathbb{N}}$. Since $\kappa < \mathfrak{b}$, there exists a $f \in \mathbb{N}^{\mathbb{N}}$ such that $f_\alpha \leq^* f$ for all $\alpha < \kappa$. For each n let $F_n = \{x^{(n)}_m : m \leq f(n)\}$. Observe that the sequence (F_n) witnesses for (Y_n) that $X \times Y$ is $\mathcal{I}$-H-separable. Let U be a nonempty open set of $X \times Y$. Choose a nonempty open set V of X and a $\alpha < \kappa$ such that $V \times B_\alpha \subseteq U$. Then $\{n \in \mathbb{N} : V \cap F^{(\alpha)}_n = \emptyset\} \in \mathcal{I}$. Choose a $n_0 \in \mathbb{N}$ be such that $f_\alpha(n) \leq f(n)$ for all $n \geq n_0$. We claim that $\{n \geq n_0 : (V \times B_\alpha) \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : V \cap F^{(\alpha)}_n = \emptyset\}$. Pick a $k \in \{n \geq n_0 : (V \times B_\alpha) \cap F_n = \emptyset\}$. Then $k \geq n_0$ and $(V \times B_\alpha) \cap F_k = \emptyset$. If possible suppose that $V \cap F^{(\alpha)}_k \neq \emptyset$. Let $x \in V \cap F^{(\alpha)}_k$. Now $x \in F^{(\alpha)}_k$ gives $x = p_1(x, y)$ for some $(x, y) \in G^{(\alpha)}_k$. It follows that $y \in B_\alpha$ and $(x, y) = x^{(k)}_m$ for some $m \leq f_\alpha(k)$. Since $f_\alpha(k) \leq f(k)$, from the construction of F_k we can say that $(x, y) \in F_k$. Also since $(x, y) \in V \times B_\alpha$, $(V \times B_\alpha) \cap F_k \neq \emptyset$. Which is absurd. Thus $V \cap F^{(\alpha)}_k = \emptyset$, i.e. $k \in \{n \in \mathbb{N} : V \cap F^{(\alpha)}_n = \emptyset\}$. It follows that $\{n \geq n_0 : (V \times B_\alpha) \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : V \cap F^{(\alpha)}_n = \emptyset\}$. Consequently $\{n \in \mathbb{N} : (V \times B_\alpha) \cap F_n = \emptyset\} \in \mathcal{I}$, i.e. $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. Thus $X \times Y$ is $\mathcal{I}$-H-separable. ☐

Theorem 3.12. *Let $\{X_n : n \in \mathbb{N}\}$ be a family of spaces and $X = \prod_{n \in \mathbb{N}} X_n$. If each n $X'_n = \prod_{k=1}^{n} X_k$ is $\mathcal{I}$-H-separable, then X is $\mathcal{I}$-H-separable.*

Proof. Let (Y_n) be a sequence of dense subspaces of X. Let $\{N_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into disjoint infinite subsets. For each n let $p_n : X \to X'_n$ be the natural projection. Fix n. Apply the $\mathcal{I}$-H-separability of X'_n to $(p_n(Y_k) : k \in \mathbb{N})$ to obtain a sequence $(F^{(n)}_k : k \in \mathbb{N})$ such that for each k $F^{(n)}_k$ is a finite subset of Y_k and for every nonempty open set U of X'_n, $\{k \in \mathbb{N} : U \cap p_n(F^{(n)}_k) = \emptyset\} \in \mathcal{I}$. For each n choose $F_n = \cup_{k \leq n} F^{(k)}_n$. Thus we obtain a sequence (F_n) such that for each n F_n is a finite subset of Y_n. Observe that (F_n) witnesses for (Y_n) that X is $\mathcal{I}$-H-separable. Let U be a nonempty open set of X. Choose a basic member V of X such that $V \subseteq U$. Then $V = \prod_{n \in \mathbb{N}} V_n$ with each V_n is open in X_n and $V_n = X_n$ for all $n \geq m$ for some $m \in \mathbb{N}$. Now $\{n \in \mathbb{N} : p_m(V) \cap p_m(F^{(m)}_n) = \emptyset\} \in \mathcal{I}$. We claim that $\{n \geq m : U \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : p_m(V) \cap p_m(F^{(m)}_n) = \emptyset\}$. Pick a $k \in \{n \geq m : U \cap F_n = \emptyset\}$. Then $k \geq m$ and $U \cap F_k = \emptyset$. If possible suppose that $p_m(V) \cap p_m(F^{(m)}_k) \neq \emptyset$. It follows that $\prod_{i=1}^{m} V_i \cap p_m(F^{(m)}_k) \neq \emptyset$. Now $\prod_{i=1}^{m} V_i \cap p_m(F^{(m)}_k) \neq \emptyset$ gives a $x = (x_n) \in F^{(m)}_k$ such that $(x_i : 1 \leq i \leq m) \in \prod_{i=1}^{m} V_i$. Since $V_n = X_n$ for all $n \geq m$, $x \in V$. So

$V \cap F_k^{(m)} \neq \emptyset$, i.e. $U \cap F_k^{(m)} \neq \emptyset$. Since $F_k = \cup_{i \leq k} F_k^{(i)}$ and $m \leq k$, $U \cap F_k \neq \emptyset$. Which is absurd. Thus $p_m(V) \cap p_m(F_k^{(m)}) = \emptyset$, i.e. $k \in \{n \in \mathbb{N} : p_m(V) \cap p_m(F_n^{(m)}) = \emptyset\}$. This gives $\{n \geq m : U \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : p_m(V) \cap p_m(F_n^{(m)}) = \emptyset\}$ and consequently $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. Thus X is $\mathcal{I}$-H-separable. □

We now shift our focus to compact $\mathcal{I}$-H-separable spaces and begin by recalling the following result.

Theorem 3.13 ([12]). *If X is any compact Hausdorff space, then X has a dense subspace Y with $d(Y) = \pi w(X)$.*

Proposition 3.14. *A compact space X is $\mathcal{I}$-H-separable if and only if $\pi w(X) = \omega$.*

Proof. Assume that X is $\mathcal{I}$-H-separable. By Theorem 3.13, there exists a dense subspace Y of X such that $d(Y) = \pi w(X)$. Then by Lemma 3.9, Y is $\mathcal{I}$-H-separable and so Y is separable, i.e. $d(Y) = \omega$. Thus $\pi w(X) = \omega$.

Conversely suppose that $\pi w(X) = \omega$. This gives $\delta(X) = \omega$. By Theorem 3.18, X is $\mathcal{I}$-H-separable. □

It is important to mention that compactness is essential in the above result. Indeed, since $iw([0, 1]) = \omega$ and every finite power of $[0, 1]$ is $\mathcal{I}$-Hurewicz, $C_p([0, 1])$ is $\mathcal{I}$-H-separable (see Theorem 3.32). But $\pi w(C_p([0, 1])) = \mathfrak{c}$.

Proposition 3.15. *Let X be a separable compact space. If X is either scattered or has countable tightness, then every continuous image of X is $\mathcal{I}$-H-separable.*

Proof. Let Y be a continuous image of X. If X is scattered, then Y is scattered and separable. It follows that $\pi w(Y) = \omega$ and hence by Proposition 3.14, Y is $\mathcal{I}$-H-separable.

Now assume that X has countable tightness, i.e. $t(X) = \omega$. This gives $t(Y) = \omega$ (see [1, 1.1.1]) and consequently $\pi_\chi(Y) = \omega$ (see [16]). From the separability of Y we can say that $\pi w(Y) = \omega$. Again by Proposition 3.14, Y is $\mathcal{I}$-H-separable. □

Corollary 3.16. *If a compact space X is either scattered or has countable tightness, then every separable subspace of X is $\mathcal{I}$-H-separable.*

Thus if X is a countable non-$\mathcal{I}$-H-separable space, then X can not be embedded in a compact space of countable tightness.

In the following example we show that a continuous image of a compact $\mathcal{I}$-H-separable space may not be $\mathcal{I}$-H-separable.

Example 3.17. *There exists a compact $\mathcal{I}$-H-separable space which has a non-$\mathcal{I}$-H-separable continuous image.*

Proof. Consider the space $\beta\mathbb{N}$. Since $\pi w(\beta\mathbb{N}) = \omega$ (see [14]), by Proposition 3.14, $\beta\mathbb{N}$ is $\mathcal{I}$-H-separable. The Tychonoff cube $\mathbb{I}^{\mathfrak{c}}$ can be obtained as a continuous image of $\beta\mathbb{N}$, where $\mathbb{I} = [0, 1]$. $\mathbb{I}^{\mathfrak{c}}$ has a countable dense subspace which is not $\mathcal{I}$-H-separable (see [7, Example 2.14] for explanation). By Lemma 3.9, $\mathbb{I}^{\mathfrak{c}}$ is not $\mathcal{I}$-H-separable. □

Note that every continuous image of a compact space X is $\mathcal{I}$-H-separable does not guarantee that $t(X) = \omega$. Indeed, there exists a separable compact scattered space X with $t(X) \geq \omega_1$ (see [7, Example 2.8]). By Proposition 3.15, every continuous image of X is $\mathcal{I}$-H-separable.

3.2. The cardinal number $\mathfrak{b}(\mathcal{I})$ and the space 2^κ.

Theorem 3.18. *If $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{b}(\mathcal{I})$, then X is $\mathcal{I}$-H-separable.*

Proof. Let $\mathcal{B} = \{U_\alpha : \alpha < \kappa\}$ be a π-base for X with $U_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Assume that $\kappa < \mathfrak{b}(\mathcal{I})$. Let (Y_n) be a sequence of dense subspaces of X. Since $\delta(X) = \omega$, we get a sequence (Z_n) such that for each n $Z_n = \{x_m^{(n)} : m \in \mathbb{N}\}$ is a dense subspace of Y_n. For each $\alpha < \kappa$ we define a $f_\alpha \in \mathbb{N}^\mathbb{N}$ by $f_\alpha(n) = \min\{m \in \mathbb{N} : x_m^{(n)} \in U_\alpha\}$. Since $\kappa < \mathfrak{b}(\mathcal{I})$, there exists a $g \in \mathbb{N}^\mathbb{N}$ such that $\{n \in \mathbb{N} : f_\alpha(n) > g(n)\} \in \mathcal{I}$ for all $\alpha < \kappa$. For each n let $F_n = \{x_m^{(n)} : m \leq g(n)\}$. Observe that the sequence (F_n) witnesses for (Y_n) that X is $\mathcal{I}$-H-separable. Let U be a nonempty open set of X. Then there exists a $\beta < \kappa$ such that $U_\beta \subseteq U$. We claim that $\{n \in \mathbb{N} : U_\beta \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : f_\beta(n) > g(n)\}$. Pick a $k \in \{n \in \mathbb{N} : U_\beta \cap F_n = \emptyset\}$. Then $U_\beta \cap F_k = \emptyset$, i.e. $x_m^{(k)} \notin U_\beta$ for all $m \leq g(k)$. Since $x_{f_\beta(k)} \in U_\beta$ and $x_m^{(k)} \notin U_\beta$ for all $m \leq g(k)$, $f_\beta(k) > g(k)$ and consequently $k \in \{n \in \mathbb{N} : f_\beta(n) > g(n)\}$. Thus $\{n \in \mathbb{N} : U_\beta \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : f_\beta(n) > g(n)\}$. It follows that $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$ and hence X is $\mathcal{I}$-H-separable. ☐

Recall from [14] that for 2^κ (the Cantor cube of weight κ), $\pi w(2^\kappa) = \kappa = w(2^\kappa)$.

Corollary 3.19. *If $\kappa < \mathfrak{b}(\mathcal{I})$, then every countable subspace of 2^κ is $\mathcal{I}$-H-separable.*

Proof. Let X be a countable subspace of 2^κ. Then $\delta(X) = \omega$. Since $w(2^\kappa) = \kappa$ and weight is monotone (i.e. for every subspace Y of a space X, $w(Y) \leq w(X)$), $w(X) \leq \kappa$. Also since $\pi w(X) \leq w(X)$, $\pi w(X) \leq \kappa$. Hence the result. ☐

Theorem 3.20. *The space $2^{\mathfrak{b}(\mathcal{I})}$ contains a countable dense subspace which is not $\mathcal{I}$-H-separable.*

Proof. Pick a countable dense subspace $X = \{f_n : n \in \mathbb{N}\}$ of $2^{\mathfrak{b}(\mathcal{I})}$ and an $\mathcal{I}$-unbounded subset $\{g_\alpha : \alpha < \mathfrak{b}(\mathcal{I})\}$ of $\mathbb{N}^\mathbb{N}$. For each $n, m \in \mathbb{N}$ define a $h_m^{(n)} \in 2^{\mathfrak{b}(\mathcal{I})}$ by

$$h_m^{(n)}(\alpha) = \begin{cases} 1, & \text{if } m < g_\alpha(n) \\ f_m(\alpha), & \text{otherwise.} \end{cases}$$

We claim that for each n $Y_n = \{h_m^{(n)} : m \in \mathbb{N}\}$ is dense in $2^{\mathfrak{b}(\mathcal{I})}$. Fix n. Choose a basic open set of $U = \{f \in 2^{\mathfrak{b}(\mathcal{I})} : f \restriction_\kappa = \sigma\}$ of $2^{\mathfrak{b}(\mathcal{I})}$, where κ is a finite subset of $\mathfrak{b}(\mathcal{I})$ and $\sigma : \kappa \to 2$ is a function. Define $g = \max\{g_\alpha : \alpha \in \kappa\}$. Since X is dense in $2^{\mathfrak{b}(\mathcal{I})}$, $U \cap X$ is infinite, i.e. there exists a $m > g(n)$ such that $f_m \restriction_\kappa = \sigma$. Pick $\alpha \in \kappa$. Then from the definition of g we get $g_\alpha(n) \leq g(n)$ and so $m > g_\alpha(n)$. This gives us $h_m^{(n)}(\alpha) = f_m(\alpha)$. It follows that $h_m^{(n)} \restriction_\kappa = \sigma$ and hence $U \cap Y_n \neq \emptyset$. Thus Y_n is dense in $2^{\mathfrak{b}(\mathcal{I})}$.

Choose $Y = \cup_{n\in\mathbb{N}} Y_n$. Then Y is dense in $2^{\mathfrak{b}(\mathcal{I})}$. We now claim that the sequence (Y_n) for Y that Y is not $\mathcal{I}$-H-separable. Let (F_n) be a sequence such that for each n F_n is a finite subset of Y_n. Let us consider a $f \in \mathbb{N}^{\mathbb{N}}$ such that $F_n \subseteq \{h^{(n)}_m : m < f(n)\}$. Since $\{g_\alpha : \alpha < \mathfrak{b}(\mathcal{I})\}$ is unbounded, there exists a $\beta < \mathfrak{b}(\mathcal{I})$ such that $\{n \in \mathbb{N} : g_\beta(n) > f(n)\} \notin \mathcal{I}$, i.e. $A = \{n \in \mathbb{N} : g_\beta(n) > f(n)\}$ is infinite. Let $n \in A$. Choose a $h^{(n)}_m \in F_n$. Then $m < f(n) < g_\beta(n)$ and so $h^{(n)}_m(\beta) = 1$. Thus for any $n \in A$ and any $h \in F_n$ we can say that $h(\beta) = 1$. For the basic open set $V = \{h \in 2^{\mathfrak{b}(\mathcal{I})} : h(\beta) = 0\}$ of $2^{\mathfrak{b}(\mathcal{I})}$, we have $V \cap F_n = \emptyset$ for all $n \in A$, i.e. $A \subseteq \{n \in \mathbb{N} : V \cap F_n = \emptyset\}$ and hence $\{n \in \mathbb{N} : V \cap F_n = \emptyset\} \notin \mathcal{I}$. Since the sequence (F_n) is arbitrarily chosen, Y is not $\mathcal{I}$-H-separable. □

Corollary 3.21. *The smallest π-weight of a countable non-$\mathcal{I}$-H-separable space is $\mathfrak{b}(\mathcal{I})$.*

Proof. Let X be a countable space. Then $\delta(X) = \omega$. If $\pi w(X) < \mathfrak{b}(\mathcal{I})$, then by Theorem 3.18, X is $\mathcal{I}$-H-separable.

Now by Theorem 3.20, $2^{\mathfrak{b}(\mathcal{I})}$ contains a countable subspace Y such that $\pi w(Y) = \mathfrak{b}(\mathcal{I})$ and Y is not $\mathcal{I}$-H-separable. Thus the smallest π-weight of a countable non-$\mathcal{I}$-H-separable space is $\mathfrak{b}(\mathcal{I})$. □

Theorem 3.22 ([6, Theorem 46]). *The space $2^{\mathfrak{c}}$ contains a countable dense H-separable subspace.*

Corollary 3.23. *The space $2^{\mathfrak{c}}$ contains a countable dense $\mathcal{I}$-H-separable subspace.*

A space X is said to have the Hurewicz property [17] (see also [13]) if for each sequence $(\mathcal{U}_n)$ of open covers of X there exists a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n$ for all but finitely many n.

Theorem 3.24. *The space 2^{ω_1} contains a dense H-separable subspace.*

Proof. Assume that $\omega_1 < \mathfrak{b}$. By [6, Corollary 30], every countable subspace of 2^{ω_1} is H-separable. This gives a countable dense H-separable subspace of 2^{ω_1}.

Next assume that $\omega_1 = \mathfrak{b}$. By [5, Theorem 10], there exists a zero-dimensional metrizable space X with $|X| = \mathfrak{b}$ such that every finite power of X is Hurewicz. Clearly $iw(X) = \omega$. By [6, Proposition 44], $C_p(X, 2)$ is a H-separable subspace of 2^{ω_1}. Also $C_p(X, 2)$ is dense in 2^{ω_1}. □

Corollary 3.25. *The space 2^{ω_1} contains a countable dense $\mathcal{I}$-H-separable subspace.*

3.3. $\mathcal{I}$WFS**,** $\mathcal{I}$WFSD **and** $C_p(X)$**.** We now introduce the following notions. A space X is said to be $\mathcal{I}$-weakly Fréchet in the strict sense (in short, $\mathcal{I}$WFS) if for any $x \in X$ and any sequence (Y_n) of subsets of X with $x \in \cap_{n\in\mathbb{N}}\overline{Y_n}$ there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every neighbourhood U of x, $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. X is said to be $\mathcal{I}$-weakly Fréchet in the strict sense with respect to dense subspaces (in short, $\mathcal{I}$WFSD) if for any $x \in X$ and any sequence (Y_n) of dense subspaces of X there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every neighbourhood U of x, $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. If X is $\mathcal{I}$WFS, then X is $\mathcal{I}$WFSD. Also if X is $\mathcal{I}$WFS (respectively, $\mathcal{I}$WFSD), then X has countable fan tightness (respectively, countable fan tightness with respect to dense subspaces).

It is obvious that if X is $\mathcal{I}$-H-separable, then X is $\mathcal{I}$WFSD. But an $\mathcal{I}$-H-separable space may not be $\mathcal{I}$WFS. Indeed, let X be the countable Fréchet-Urysohn fan space (see [1, 2.3.1]). Since $\pi w(X) = \omega$, by Theorem 3.18, X is $\mathcal{I}$-H-separable. On the other hand, X does not have countable fan tightness and hence X is not $\mathcal{I}$WFS. However, we show in Theorem 3.32 that if $C_p(X)$ is $\mathcal{I}$-H-separable, then $C_p(X)$ is $\mathcal{I}$WFS.

Proposition 3.26. *A separable space X is $\mathcal{I}$-H-separable if and only if it is $\mathcal{I}$*WFSD.

Proof. Suppose that X is $\mathcal{I}$WFSD. Let (Y_n) be a sequence of dense subspaces of X. Since X is separable, we have a countable dense subspace $Y = \{x_n : n \in \mathbb{N}\}$ of X. Fix $n \in \mathbb{N}$. Since X is $\mathcal{I}$WFSD, we get a sequence $(F_m^{(n)} : m \in \mathbb{N})$ such that for each $m \in \mathbb{N}$, $F_m^{(n)}$ is a finite subset of Y_m and for each neighbourhood U of x_n, $\{m \in \mathbb{N} : U \cap F_m^{(n)} = \emptyset\} \in \mathcal{I}$. For each n let $F_n = \cup_{i \leq n} F_n^{(i)}$. We claim that the sequence (F_n) witnesses for (Y_n) that X is $\mathcal{I}$-H-separable. Let U be a nonempty open set of X. Now we can choose a $k \in \mathbb{N}$ such that $x_k \in U \cap Y$. This gives a $\{m \in \mathbb{N} : U \cap F_m^{(k)} = \emptyset\} \in \mathcal{I}$. Since for each $m \geq k$, $F_m^{(k)} \subseteq F_m$, we get $\{m \in \mathbb{N} : U \cap F_m = \emptyset\} \in \mathcal{I}$. Thus X is $\mathcal{I}$-H-separable.

The other direction is trivial. □

Lemma 3.27.
(1) *The $\mathcal{I}$*WFS *property is hereditary.*
(2) *The $\mathcal{I}$*WFSD *property is preserved under dense subspaces.*

Let X be a space and $k \in \mathbb{N}$. Let $\mathcal{U}$ be an open cover of X^k. Recall that a family $\mathcal{B}$ of subsets of X is called $\mathcal{U}$-small if for every $V_1, V_2, \ldots, V_k \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V_1 \times V_2 \times \cdots \times V_k \subseteq U$.

We closely follow the technique of the proof of [2, II.2.2. Theorem] to prove the next result.

Theorem 3.28. *For a space X the following assertions are equivalent.*
(1) *$C_p(X)$ is $\mathcal{I}$*WFS.
(2) *$C_p(X)$ is $\mathcal{I}$*WFSD.
(3) *Every finite power of X is $\mathcal{I}$-Hurewicz.*

Proof. (2) ⇒ (3). Let $k \in \mathbb{N}$ be fixed. Let $(\mathcal{U}_n)$ be a sequence of open covers of X^k. For each n let $\mathfrak{F}_n$ be the collection of all finite $\mathcal{U}_n$-small families of open sets of X. For each $\mathcal{U} \in \mathfrak{F}_n$ define $Z_n(\mathcal{U}) = \{f \in C_p(X) : f(X \setminus \cup\mathcal{U}) = \{0\}\}$. For each n choose $Y_n = \cup\{Z_n(\mathcal{U}) : \mathcal{U} \in \mathfrak{F}_n\}$. Observe that for each n Y_n is dense in $C_p(X)$. Since $C_p(X)$ is $\mathcal{I}$WFSD, there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every neighbourhood U of $\underline{1}$, $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. For each n and each $f \in F_n$ choose a $\mathcal{U}_f^{(n)} \in \mathfrak{F}_n$ such that $f \in Z_n(\mathcal{U}_f^{(n)})$. For every $V_1, V_2, \ldots, V_k \in \mathcal{U}_f^{(n)}$

choose a $U_f^{(n)}(V_1, V_2, \ldots, V_k) \in \mathcal{U}_n$ such that $V_1 \times V_2 \times \ldots V_k \subseteq U_f^{(n)}(V_1, V_2, \ldots, V_k)$. For each n $\mathcal{V}_n = \{U_f^{(n)}(V_1, V_2, \ldots, V_k) : V_1, V_2, \ldots, V_k \in \mathcal{U}_f^{(n)}, f \in F_n\}$ is a finite subset of $\mathcal{U}_n$. Observe that the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that X^k is $\mathcal{I}$-Hurewicz. Let $x = (x_1, x_2, \ldots, x_k) \in X^k$. Pick the open set $U = \{f \in C_p(X) : f(x_i) > 0, 1 \leq i \leq k\}$ of $C_p(X)$. Clearly $\underline{1} \in U$. Then $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. We claim that $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \subseteq \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$. Pick a $m \in \{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\}$. Then $x \notin \cup\mathcal{V}_m$. If possible suppose that $U \cap F_m \neq \emptyset$. Pick a $f \in C_p(X)$ such that $f \in U \cap F_m$. Since $f \in Z_m(\mathcal{U}_f^{(m)})$, $f(X \setminus \cup\mathcal{U}_f^{(m)}) = \{0\}$. It follows that $f(x_i) > 0$ for all $1 \leq i \leq k$ and $f(y) = 0$ for all $y \in X \setminus \cup\mathcal{U}_f^{(m)}$. For each $1 \leq i \leq k$ choose a $V_i \in \mathcal{U}_f^{(m)}$ such that $x_i \in V_i$. So $x \in V_1 \times V_2 \times \ldots V_k$. Then $x \in U_f^{(m)}(V_1, V_2, \ldots, V_k)$ as $V_1 \times V_2 \times \ldots V_k \subseteq U_f^{(m)}(V_1, V_2, \ldots, V_k)$ and hence $x \in \cup\mathcal{V}_m$. Which is absurd. Consequently $U \cap F_m \neq \emptyset$, i.e. $m \in \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$ and so $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \subseteq \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$, i.e. $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \in \mathcal{I}$. Thus X^k is $\mathcal{I}$-Hurewicz.

(3) $\Rightarrow$ (1). Let $f \in C_p(X)$ and (Y_n) be a sequence of sets of $C_p(X)$ such that $f \in \cap_{n\in\mathbb{N}}\overline{Y_n}$. Let $k \in \mathbb{N}$ be fixed. Since $f \in \cap_{n\in\mathbb{N}}\overline{Y_n}$, for each $n \in \mathbb{N}$ and each $x = (x_1, x_2, \ldots, x_k) \in X^k$ there exists a $g_x^{(n,k)} \in Y_n$ such that $|g_x^{(n,k)}(x_i) - f(x_i)| < \frac{1}{k}$ for all $1 \leq i \leq k$. Since $g_x^{(n,k)}$ and f are continuous, for each $1 \leq i \leq k$ we can choose an open set $U_{i,x}^{(n,k)}$ of X containing x_i such that $|g_x^{(n,k)}(y) - f(y)| < \frac{1}{k}$ for all $y \in U_{i,x}^{(n,k)}$. Then $U_x^{(n,k)} = U_{1,x}^{(n,k)} \times U_{2,x}^{(n,k)} \times \ldots \times U_{k,x}^{(n,k)}$ is an open set in X^k containing x and $|g_x^{(n,k)}(y_i) - f(y_i)| < \frac{1}{k}$ for all $y = (y_1, y_1, \ldots, y_k) \in U_x^{(n,k)}$. For each n $\mathcal{U}_n^{(k)} = \{U_x^{(n,k)} : x \in X^k\}$ is an open cover of X^k. Apply the $\mathcal{I}$-Hurewicz property of X^k to $(\mathcal{U}_n^{(k)})$ to obtain a sequence $(\mathcal{V}_n^{(k)})$ such that for each n $\mathcal{V}_n^{(k)}$ is a finite subset of $\mathcal{U}_n^{(k)}$ and for each $x \in X^k$, $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n^{(k)}\} \in \mathcal{I}$. We can obtain a sequence $(A_n^{(k)})$ of finite subsets of X^k such that for each n $\mathcal{V}_n^{(k)} = \{U_x^{(n,k)} : x \in A_n^{(k)}\}$. For each n $F_n^{(k)} = \{g_x^{(n,k)} : x \in A_n^{(k)}\}$ is a finite subset of Y_n. We now define a sequence (F_n) such that for each n F_n is a finite subset of Y_n as $F_n = \cup_{k\leq n}F_n^{(k)}$. Observe that the sequence (F_n) witnesses for f and (Y_n) that $C_p(X)$ is $\mathcal{I}$WFS. Let U be an open set in $C_p(X)$ containing f. Then we get a finite subset $F = \{z_1, z_2, \ldots, z_m\}$ of X and an $\epsilon > 0$ such that $B(f, F, \epsilon) \subseteq U$. We can assume that $\frac{1}{m} < \epsilon$. Since $z = (z_1, z_2, \ldots, z_m) \in X^m$, $\{n \in \mathbb{N} : z \notin \cup\mathcal{V}_n^{(m)}\} \in \mathcal{I}$. We claim that $\{n \in \mathbb{N} : B(f, F, \epsilon) \cap F_n = \emptyset\} \in \mathcal{I}$. It is enough to show that $\{n \geq m : B(f, F, \epsilon) \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : z \notin \cup\mathcal{V}_n^{(m)}\}$. Pick a $j \in \{n \geq m : B(f, F, \epsilon) \cap F_n = \emptyset\}$. Then $j \geq m$ and $B(f, F, \epsilon) \cap F_j = \emptyset$. It follows that $B(f, F, \epsilon) \cap F_j^{(i)} = \emptyset$ for all $i \leq j$, i.e. $B(f, F, \epsilon) \cap F_j^{(m)} = \emptyset$. It follows that for each $x \in A_j^{(m)}$, $g_x^{(j,m)} \notin B(f, F, \epsilon)$ and $|g_x^{(j,m)}(z_p) - f(z_p)| \geq \epsilon > \frac{1}{m}$ for some $1 \leq p \leq m$. Consequently $z \notin U_x^{(j,m)}$ for all $x \in A_j^{(m)}$, i.e. $z \notin \cup\mathcal{V}_j^{(m)}$ and $j \in \{n \in \mathbb{N} : z \notin \cup\mathcal{V}_n^{(m)}\}$. Thus $\{n \geq m : B(f, F, \epsilon) \cap F_n = \emptyset\} \subseteq \{n \in \mathbb{N} : z \notin \cup\mathcal{V}_n^{(m)}\}$ and $\{n \in \mathbb{N} : B(f, F, \epsilon) \cap F_n = \emptyset\} \in \mathcal{I}$, i.e. $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. Hence $C_p(X)$ is $\mathcal{I}$WFS. $\square$

Corollary 3.29. *If every finite power of X is $\mathcal{I}$-Hurewicz, then every separable subspace of $C_p(X)$ is $\mathcal{I}$-H-separable.*

Proof. Let Y be a separable subspace of $C_p(X)$. Suppose that every finite power of X is $\mathcal{I}$-Hurewicz. Then by Theorem 3.28, $C_p(X)$ is $\mathcal{I}$WFS. Also by Lemma 3.27(1), Y is $\mathcal{I}$WFS. Thus Y is $\mathcal{I}$-H-separable (see Proposition 3.26). □

Lemma 3.30 (Folklore)**.** *Every Lindelöf space of cardinality less than $\mathfrak{b}(\mathcal{I})$ is $\mathcal{I}$-Hurewicz.*

Proof. Let X be a Lindelöf space with $|X| < \mathfrak{b}(\mathcal{I})$. Consider a sequence $(\mathcal{U}_n)$ of open covers of X. Since X is Lindelöf, for each n we can assume that $\mathcal{U}_n = \{U_m^{(n)} : m \in \mathbb{N}\}$. For each $x \in X$ choose a $f_x \in \mathbb{N}^{\mathbb{N}}$ such that $f_x(n) = \min\{m \in \mathbb{N} : x \in U_m^{(n)}\}$ for all n. Since $|\{f_x : x \in X\}| < \mathfrak{b}(\mathcal{I})$, there exists a $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{n \in \mathbb{N} : f_x(n) > g(n)\} \in \mathcal{I}$ for all $x \in X$. For each n let $\mathcal{V}_n = \{U_m^{(n)} : m \leq g(n)\}$. We claim that the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that X is $\mathcal{I}$-Hurewicz. Let $x \in X$. Pick a $k \in \{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\}$. So $x \notin U_m^{(k)}$ for all $m \leq g(k)$. It follows that $f_x(k) > g(k)$ because $x \in U_{f_x(k)}^{(k)}$. Thus $k \in \{n \in \mathbb{N} : f_x(n) > g(n)\}$ and $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \subseteq \{n \in \mathbb{N} : f_x(n) > g(n)\}$. Consequently $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \in \mathcal{I}$ and X is $\mathcal{I}$-Hurewicz. □

Theorem 3.31 ([15] (see also [2, I.1.5 Theorem]))**.** *For a space X, $iw(X) = d(C_p(X))$.*

Theorem 3.32. *For a space X the following assertions are equivalent.*
(1) *$C_p(X)$ is separable and* $\mathcal{I}$WFS.
(2) *$C_p(X)$ is separable and* $\mathcal{I}$WFSD.
(3) *$C_p(X)$ is $\mathcal{I}$-H-separable.*
(4) *$iw(X) = \omega$ and every finite power of X is $\mathcal{I}$-Hurewicz.*

Proof. (1) ⇒ (2) is trivial as $\mathcal{I}$WFS implies $\mathcal{I}$WFSD.

(2) ⇒ (3). By Theorem 3.28, every finite power of X is $\mathcal{I}$-Hurewicz. Also by Corollary 3.29, $C_p(X)$ is $\mathcal{I}$-H-separable.

(3) ⇒ (4). Clearly $C_p(X)$ is $\mathcal{I}$WFSD. By Theorem 3.28, every finite power of X is $\mathcal{I}$-Hurewicz. Since $C_p(X)$ is separable, $d(C_p(X)) = \omega$. By Theorem 3.31, $iw(X) = \omega$.

(4) ⇒ (1). By Theorem 3.28, $C_p(X)$ $\mathcal{I}$WFS. Since $iw(X) = \omega$, by Theorem 3.31, $d(C_p(X)) = \omega$. It follows that $C_p(X)$ is separable. □

Corollary 3.33. *If $C_p(X)$ is $\mathcal{I}$-H-separable, then for each $n \in \mathbb{N}$, $C_p(X^n)$ is $\mathcal{I}$-H-separable, and $(C_p(X))^\omega$ is $\mathcal{I}$-H-separable.*

Proof. Let $n \in \mathbb{N}$ and $Y = X^n$. Since $C_p(X)$ is $\mathcal{I}$-H-separable, $iw(X) = \omega$ and every finite power of X is $\mathcal{I}$-Hurewicz. It follows that every finite power of Y is also $\mathcal{I}$-Hurewicz and $iw(Y) = \omega$. Thus $C_p(Y)$ is $\mathcal{I}$-H-separable □

Corollary 3.34. *If X is a second countable space such that $C_p(X)$ is $\mathcal{I}$-H-separable, then $C_p(X)$ is hereditarily $\mathcal{I}$-H-separable.*

Proof. Let Y be a subspace of $C_p(X)$. Since X is a second countable space, it is separable metrizable. It follows that $C_p(X)$ is hereditarily separable and hence Y is separable. Since $C_p(X)$ is $\mathcal{I}$-H-separable, $C_p(X)$ is $\mathcal{I}$WFS. By Lemma 3.27(1), Y is $\mathcal{I}$WFS and so $\mathcal{I}$WFSD. Also by Proposition 3.26, Y is $\mathcal{I}$-H-separable. Thus $C_p(X)$ is hereditarily $\mathcal{I}$-H-separable. □

Corollary 3.35. *If X is a second countable space with $|X| < \mathfrak{b}(\mathcal{I})$, then $C_p(X)$ is hereditarily $\mathcal{I}$-H-separable.*

Proof. Let Y be a subspace of $C_p(X)$. Since X is a second countable space, it is separable metrizable. It follows that $C_p(X)$ is hereditarily separable and hence Y is separable. Also every finite power of X is second countable and so every finite power of X is Lindelöf. Since $|X| < \mathfrak{b}(\mathcal{I})$, by Lemma 3.30, every finite power of X is $\mathcal{I}$-Hurewicz. It follows that $C_p(X)$ is $\mathcal{I}$WFS as $iw(X) = d(C_p(X)) = \omega$. Then Y is $\mathcal{I}$WFS and so $\mathcal{I}$WFSD. Also by Proposition 3.26, Y is $\mathcal{I}$-H-separable. Thus $C_p(X)$ is hereditarily $\mathcal{I}$-H-separable. □

Proposition 3.36. *The space $C_p(C_p(X))$ is $\mathcal{I}$-H-separable if and only if X is finite.*

Proof. Suppose that $C_p(C_p(X))$ is $\mathcal{I}$-H-separable. Then by Theorem 3.32, every finite power of $C_p(X)$ is $\mathcal{I}$-Hurewicz. Also by [2, Theorem II.2.10], X is finite.

Conversely suppose that X is finite. Then $C_p(X)$ is a σ-compact metrizable space and so $C_p(X)$ is a separable metrizable space. It follows that $C_p(C_p(X))$ is hereditarily separable and so $d(C_p(C_p(X))) = \omega$. By Theorem 3.31, $iw(C_p(X)) = \omega$. Since $C_p(X)$ is σ-compact, every finite power of $C_p(X)$ is $\mathcal{I}$-Hurewicz. Thus by Theorem 3.32, $C_p(C_p(X))$ is $\mathcal{I}$-H-separable. □

Remark 3.37.

(1) Recall that the Baire space $\mathbb{N}^{\mathbb{N}}$ can be condensed onto a compact space X. It follows that $C_p(X)$ can be densely embedded in $C_p(\mathbb{N}^{\mathbb{N}})$. Now $C_p(X)$ is separable, i.e. $d(C_p(X)) = \omega$ and so $iw(X) = \omega$ (see Theorem 3.31). Since X is compact, every finite power of X is $\mathcal{I}$-Hurewicz and hence by Theorem 3.32, $C_p(X)$ is $\mathcal{I}$-H-separable. Let Y be a countable dense subspace of $C_p(X)$. Then Y is $\mathcal{I}$-H-separable. Since $C_p(X)$ is densely embeddable in $C_p(\mathbb{N}^{\mathbb{N}})$, Y is a countable dense $\mathcal{I}$-H-separable subspace of $C_p(\mathbb{N}^{\mathbb{N}})$. Also since $C_p(\mathbb{N}^{\mathbb{N}})$ can be densely embedded in the Tychonoff cube $\mathbb{I}^{\mathfrak{c}}$, Y is a countable dense $\mathcal{I}$-H-separable subspace of $\mathbb{I}^{\mathfrak{c}}$. Thus $\mathbb{I}^{\mathfrak{c}}$ has a countable dense $\mathcal{I}$-H-separable subspace.
(2) Moreover, we have already mentioned in Example 3.17 that $\mathbb{I}^{\mathfrak{c}}$ has a countable dense non-$\mathcal{I}$-H-separable subspace.

Theorem 3.38. *For a zero-dimensional space X the following assertions are equivalent.*

(1) *$C_p(X, \mathbb{Q})$ is $\mathcal{I}$-H-separable.*
(2) *$C_p(X, \mathbb{Z})$ is $\mathcal{I}$-H-separable.*
(3) *$C_p(X, 2)$ is $\mathcal{I}$-H-separable.*
(4) *$iw(X) = \omega$ and every finite power of X is $\mathcal{I}$-Hurewicz.*

Proof. (1) $\Rightarrow$ (2). For each $n \in \mathbb{Z}$ choose an irrational point $x_n \in (n, n+1)$. Let $\psi \in \mathbb{Z}^{\mathbb{Q}}$ be given by $\psi(x) = n$ if $x \in (x_{n-1}, x_n)$. Now define a mapping $\Phi : C_p(X, \mathbb{Q}) \to C_p(X, \mathbb{Z})$ by $\Phi(f) = \psi \circ f$ for all $f \in C_p(X, \mathbb{Q})$. Observe that Φ is an open continuous mapping from $C_p(X, \mathbb{Q})$ onto $C_p(X, \mathbb{Z})$. Then by Lemma 3.8, $C_p(X, \mathbb{Z})$ is $\mathcal{I}$-H-separable.

(2) $\Rightarrow$ (3). Let $\phi \in 2^{\mathbb{Z}}$ be given by

$$\phi(n) = \begin{cases} 0, & \text{if } n < 0 \\ 1, & \text{otherwise.} \end{cases}$$

Define a mapping $\Psi : C_p(X, \mathbb{Z}) \to C_p(X, 2)$ by $\Psi(f) = \phi \circ f$ for all $f \in C_p(X, \mathbb{Z})$. Clearly Ψ is an open continuous mapping from $C_p(X, \mathbb{Z})$ onto $C_p(X, 2)$. By Lemma 3.8, $C_p(X, 2)$ is $\mathcal{I}$-H-separable.

(3) $\Rightarrow$ (4). Let D be a countable dense subspace of $C_p(X, 2)$. Then $\{f^{-1}(0), f^{-1}(1) : f \in D\}$ forms a base for a zero-dimensional second countable topology on X which is contained in the original topology of X. It follows that $iw(X) = \omega$.

We now show that every finite power of X is $\mathcal{I}$-Hurewicz. Let $k \in \mathbb{N}$ be fixed. Let $(\mathcal{U}_n)$ be a sequence of open covers of X^k. For each n let $\mathfrak{F}_n$ be the collection of all finite $\mathcal{U}_n$-small families of open sets of X. For each $\mathcal{U} \in \mathfrak{F}_n$ define $Z_n(\mathcal{U}) = \{f \in C_p(X, 2) : f(X \setminus \cup\mathcal{U}) = \{0\}\}$. For each n choose $Y_n = \cup\{Z_n(\mathcal{U}) : \mathcal{U} \in \mathfrak{F}_n\}$. Observe that for each n Y_n is dense in $C_p(X, 2)$. Since $C_p(X, 2)$ is $\mathcal{I}$-H-separable, there exists a sequence (F_n) such that for each n F_n is a finite subset of Y_n and for every nonempty open set U of $C_p(X, 2)$, $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. For each n and each $f \in F_n$ choose a $\mathcal{U}_f^{(n)} \in \mathfrak{F}_n$ such that $f \in Z_n(\mathcal{U}_f^{(n)})$. For every $V_1, V_2, \ldots, V_k \in \mathcal{U}_f^{(n)}$ choose a $U_f^{(n)}(V_1, V_2, \ldots, V_k) \in \mathcal{U}_n$ such that $V_1 \times V_2 \times \ldots V_k \subseteq U_f^{(n)}(V_1, V_2, \ldots, V_k)$. For each n $\mathcal{V}_n = \{U_f^{(n)}(V_1, V_2, \ldots, V_k) : V_1, V_2, \ldots, V_k \in \mathcal{U}_f^{(n)}, f \in F_n\}$ is a finite subset of $\mathcal{U}_n$. Observe that the sequence $(\mathcal{V}_n)$ witnesses for $(\mathcal{U}_n)$ that X^k is $\mathcal{I}$-Hurewicz. Let $x = (x_1, x_2, \ldots, x_k) \in X^k$. Pick the open set $U = \{f \in C_p(X, 2) : f(x_i) = 1 \text{ for all } 1 \leq i \leq k\}$ of $C_p(X, 2)$. Clearly U is nonempty. Then $\{n \in \mathbb{N} : U \cap F_n = \emptyset\} \in \mathcal{I}$. We claim that $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \subseteq \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$. Pick a $m \in \{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\}$. Then $x \notin \cup\mathcal{V}_m$. If possible suppose that $U \cap F_m \neq \emptyset$. Pick a $f \in C_p(X, 2)$ such that $f \in U \cap F_m$. Since $f \in Z_m(\mathcal{U}_f^{(m)})$, $f(X \setminus \cup\mathcal{U}_f^{(m)}) = \{0\}$. It follows that $f(x_i) = 1$ for all $1 \leq i \leq k$ and $f(y) = 0$ for all $y \in X \setminus \cup\mathcal{U}_f^{(m)}$. For each $1 \leq i \leq k$ choose a $V_i \in \mathcal{U}_f^{(m)}$ such that $x_i \in V_i$. So $x \in V_1 \times V_2 \times \ldots V_k$. Then $x \in U_f^{(m)}(V_1, V_2, \ldots, V_k)$ as $V_1 \times V_2 \times \ldots V_k \subseteq U_f^{(m)}(V_1, V_2, \ldots, V_k)$ and hence $x \in \cup\mathcal{V}_m$. Which is absurd. Consequently $U \cap F_m \neq \emptyset$, i.e. $m \in \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$ and so $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \subseteq \{n \in \mathbb{N} : U \cap F_n = \emptyset\}$, i.e. $\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\} \in \mathcal{I}$. Thus X^k is $\mathcal{I}$-Hurewicz.

(4) $\Rightarrow$ (1). By Theorem 3.32, $C_p(X)$ is $\mathcal{I}$-H-separable. Since X is zero-dimensional, $C_p(X, \mathbb{Q})$ is dense in $C_p(X)$. By Lemma 3.9, $C_p(X, \mathbb{Q})$ is $\mathcal{I}$-H-separable. $\square$

Corollary 3.39. *Let (X, τ) be condensed onto a separable metrizable space (X, τ'). If every finite power of (X, τ') is $\mathcal{I}$-Hurewicz, then there exists a dense $\mathcal{I}$-H-separable subspace of $C_p(X, \tau')$.*

Proof. We can think of $C_p(X, \tau')$ as subspace of $C_p(X, \tau)$. Clearly $C_p(X, \tau')$ is dense in $C_p(X, \tau)$. By Theorem 3.28, $C_p(X, \tau')$ is $\mathcal{I}$WFSD. Since (X, τ') is separable metrizable, $C_p(X, \tau')$ is hereditarily separable. It follows that $C_p(X, \tau')$ is $\mathcal{I}$-H-separable (see Proposition 3.26). $\square$

We now discuss the importance of separability in the preceding result.

Example 3.40. *There is a metrizable space X such that $C_p(X)$ is not $\mathcal{I}$-H-separable but $C_p(X)$ has a dense $\mathcal{I}$-H-separable subspace.*

Proof. Consider the discrete space X with $|X| = \mathfrak{c}$. Now $C_p(X) = \mathbb{R}^{\mathfrak{c}}$. We can interpret $\mathbb{R}^{\mathfrak{c}}$ as $\mathbb{R}^{2^\omega}$. If $Y = C_p(2^\omega)$, then Y is dense in $\mathbb{R}^{2^\omega}$. Now $iw(2^\omega) = \omega$ and 2^ω is compact, i.e. every finite power of 2^ω is $\mathcal{I}$-Hurewicz. By Theorem 3.32, Y is $\mathcal{I}$-H-separable. Thus Y is a dense $\mathcal{I}$-H-separable subspace of $C_p(X) = \mathbb{R}^{\mathfrak{c}}$. Again by Theorem 3.32, $C_p(X) = \mathbb{R}^{\mathfrak{c}}$ is not $\mathcal{I}$-H-separable as X is not $\mathcal{I}$-Hurewicz. $\square$

The following question can naturally be raised.

Problem 3.41. *Suppose that $C_p(X)$ has a dense $\mathcal{I}$-H-separable subspace. Can X be condensed onto a second countable space every finite power of which is $\mathcal{I}$-Hurewicz?*

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