A step toward Chen-Lih-Wu conjecture

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Abstract

An equitable k-coloring of a graph is a proper k-coloring where the sizes of any two different color classes differ by at most one. In 1973, Meyer conjectured that every connected graph G has an equitable k-coloring for some $k \leq \Delta(G)$, unless G is a complete graph or an odd cycle. Chen, Lih, and Wu strengthened this in 1994 by conjecturing that for $k \geq 3$, the only connected graphs of maximum degree at most k with no equitable k-coloring are the complete bipartite graph $K_{k,k}$ for odd k and the complete graph K_{k+1}. A more refined conjecture was proposed by Kierstead and Kostochka, relaxing the maximum degree condition to an Ore-type condition. Their conjecture states the following: for $k \geq 3$, if G is an n-vertex graph such that $d(x) + d(y) \leq 2k$ for every edge $xy \in E(G)$, and G admits no equitable k-coloring, then G contains either K_{k+1} or $K_{m,2k-m}$ for some odd m. We prove that for any constant $c > 0$ and all sufficiently large n, the latter two conjectures hold for every $k \geq cn$. Our proof yields an algorithm with polynomial time that decides whether G has an equitable k-coloring, thereby answering a conjecture of Kierstead, Kostochka, Mydlarz, and Szemerédi when $k \geq cn$.

1 Introduction

A *proper k-coloring* of a graph is defined as a vertex coloring from a set of k colors such that no two adjacent vertices share a common color. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors required in a proper coloring. An easy observation deduces that $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. The classical Brooks' theorem states that the chromatic number of a connected graph is at most its maximum degree, unless the graph is a complete graph or an odd cycle.

An *equitable k-coloring* of a graph G is a proper k-coloring where the sizes of any two different color classes differ by at most one. A classical problem in graph coloring theory is to ask whether a graph G has an equitable k-coloring if G has some degree constraints with respect to k. In 1968, Grünbaum [11] conjectured that every graph with maximum degree less than k has an equitable k-coloring. Remarkably, via graph complements, this is equivalent to a conjecture of Erdős from 1964 [8]: every graph on $n = rk$ vertices with minimum degree at least $(1-\frac{1}{r})n$ admits a K_r-factor. In 1970, Hajnal and Szemerédi confirmed the conjecture [12], establishing what is now known as the Hajnal-Szemerédi Theorem. Kierstead and Kostochka [16, 20] found a polynomial time algorithm for such colorings.

Observe that the degree condition in Grünbaum's conjecture is tight, as G admits no equitable k-coloring when it contains K_{k+1} as a subgraph. Perhaps inspired by this and the Brooks' theorem, Meyer [27] further conjectured that every connected graph G has an equitable k-coloring for some

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integer $k \le \Delta(G)$, unless G is a complete graph or an odd cycle. Lih and Wu [25] proved that Meyer's conjecture is true for connected bipartite graphs. In fact, they showed a stronger result that a connected bipartite graph G has an equitable $\Delta(G)$-coloring if G is not the complete bipartite graph $K_{2m+1,2m+1}$ for some $m \ge 0$. This line of research led Chen, Lih, and Wu [5] to propose the following conjecture, which is considered one of the most significant open problems in equitable coloring according to [14].

Conjecture 1.1 (Chen-Lih-Wu Conjecture [5]). *Let G be a connected graph with $\Delta(G) \le k$. Then G has no equitable k-coloring if and only if $G = K_{k+1}$, or $k = 2$ and G is an odd cycle, or k is odd and $G = K_{k,k}$.*

Combined with the Hajnal-Szemerédi Theorem, it suffices to consider the case $k = \Delta(G)$ in Chen-Lih-Wu Conjecture. This conjecture has been verified for several special cases: $k \le 3$ or $k \ge \frac{|V(G)|}{2}$ [5]; $\frac{|V(G)|+1}{3} \le k < \frac{|V(G)|}{2}$ [4]; $k \le 4$ or $k \ge \frac{|V(G)|}{4}$ [18, 19], and some sparse graphs [6, 22, 26, 28, 29, 34, 35, 36]. Moreover, Kierstead and Kostochka [17] proved that there exists a polynomial time algorithm for deciding whether a graph G with $\Delta(G) \le k$ admits an equitable k-coloring.

One may view the Hajnal-Szemerédi Theorem as a Dirac-type result, as it guarantees the existence of a K_r-factor in a graph under the minimum degree condition. About two decades ago, Kostochka and Yu [23] conjectured that every graph G in which $d(x) + d(y) \le 2k$ for every edge $xy \in E(G)$ has an equitable $(k+1)$-coloring. In 2008, Kierstead and Kostochka [15] proved this conjecture with a weaker degree condition.

Theorem 1.2 ([15]). *Every graph satisfying $d(x)+d(y) \le 2k+1$ for every edge xy, has an equitable $(k+1)$-coloring.*

Notice that the degree sum bound in Theorem 1.2 is tight, as the complete graph K_{k+2} admits no proper $(k+1)$-coloring. Moreover, for each odd $m \le k+1$, the graph $K_{m,2k-m+2}$ satisfies $d(x)+d(y) \le 2k+2$ for every edge xy, yet has no equitable $(k+1)$-coloring. In 2008, Kierstead and Kostochka [15] conjectured that the following Ore-type analogue of the Chen-Lih-Wu Conjecture holds.

Conjecture 1.3 ([15]). *Let $k \ge 3$. If G is a graph satisfying $d(x) + d(y) \le 2k$ for every edge xy, and G has no equitable k-coloring, then G contains either K_{k+1} or $K_{m,2k-m}$ for some odd m.*

A conjecture concerning the algorithmic version of the Ore-type condition was proposed by Kierstead, Kostochka, Mydlarz, and Szemerédi [20].

Conjecture 1.4 ([20]). *Let G be a graph with $d(x) + d(y) < 2k$ for all $xy \in E(G)$. Then there exists a polynomial-time algorithm to determine an equitable k-coloring for G.*

Our work answers Conjecture 1.3 and Conjecture 1.4 when $|G|$ is sufficiently large and k is linear with respect to $|G|$.

Theorem 1.5. *For a positive constant c, there exists an integer $n_0 := n_0(c)$ such that the following holds for every $n \ge n_0$. Let k be an integer such that $k \ge cn$, and let G be an n-vertex graph with $d(x) + d(y) \le 2k$ for every edge $xy \in E(G)$. Then either G has an equitable k-coloring, or G contains K_{k+1} or $K_{m,2k-m}$ for some odd integer $m \in [k]$. Moreover, there is an algorithm with polynomial time that decides whether G has an equitable k-coloring.*

By replacing the Ore-type condition in Theorem 1.5 with the maximum degree constraint, we immediately obtain the following corollary. This resolves Conjecture 1.1 for all sufficiently large n when the maximum degree is linear in n.

Corollary 1.6. *For a positive constant c, there exists an integer $n_0 := n_0(c)$ such that the following holds for every $n \ge n_0$. Let k be an integer such that $k \ge cn$ and G be an n-vertex graph with $\Delta(G) \le k$ that admits no equitable k-coloring. Then G contains either K_{k+1} or $K_{k,k}$ (with k odd). If, in addition, G is connected, then G is either K_{k+1} or $K_{k,k}$ (with k odd).*

1.1 Reformulation of Theorem 1.5

We introduce a new approach to prove Theorem 1.5, which differs from standard coloring techniques by treating it as a problem in extremal graph theory. Without loss of generality, we may assume that k divides n. Indeed, if $k \nmid n$, we consider the graph G' obtained by adding a clique on q vertices to G, where q satisfies $k \mid (n+q)$ and $1 \le q \le k-1$. The conclusion for G' immediately implies the result for G since the addition of a complete graph preserves both the degree condition and the nonexistence of equitable colorings.

Let $\overline{G}$ denote the complement graph of G. Then the following are equivalent:

- G admits no equitable k-coloring;
- $\overline{G}$ contains no $K_{\frac{n}{k}}$-factor.

We now define two types of extremal graphs. Let n, r be two nonnegative integers with $r \mid n$, and let s be an odd integer satisfying $1 \le s \le \frac{n}{r}$. We say an n-vertex graph G is *extremal* if one of the following holds.

(EX1) The graph G has an independent set of size $\frac{n}{r}+1$.

(EX2) There exists a partition $A_1 \cup \cdots \cup A_{r-2} \cup B_0 \cup B_1$ of $V(G)$ with $|A_i| = \frac{n}{r}$ for each $i \in [r-2]$ and $|B_0| = s$ such that the edge set of G is

$$E(G) = \bigcup_{i\in[r-2]} \{uv : u \in A_i, v \notin A_i\} \cup \bigcup_{j\in\mathbb{Z}_2} \{uv : u \in B_j, v \notin B_{j+1}\} \text{ (see Figure 1).}$$

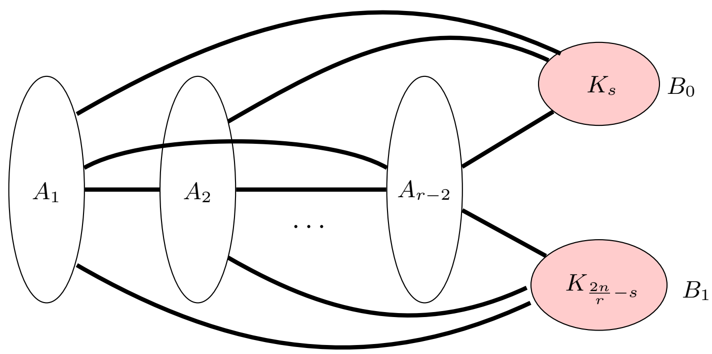


Figure 1: (EX2). Each thick line indicates that all possible edges exist between the two corresponding parts.

Define $\sigma(G) := \min\{d(x)+d(y) : x, y \in V(G) \text{ and } xy \notin E(G)\}$. Hence the condition $d(x)+d(y) \le 2k$ for every edge $xy \in E(G)$ is equivalent to $\sigma(G) \ge 2(n-k)-2$. Having established the above, we now only need to prove the following result, as Theorem 1.5 follows directly by considering the complement graph.

Theorem 1.7. *Let r be a positive integer. There exists an integer $n_0 := n_0(r)$ such that the following holds for every integer $n \ge n_0$ with $r \mid n$. Let G be an n-vertex graph with $\sigma(G) \ge 2(1-\frac{1}{r})n-2$. Then either G admits a K_r-factor or G is extremal. Moreover, there is an algorithm with polynomial time that decides whether G has a K_r-factor.*

1.2 Sketch of the proof of Theorem 1.7

Suppose that $r \ge 3$. Given $\gamma > 0$ and a vertex subset $S \subseteq V(G)$, we say S is a *γ-independent set* of an n-vertex graph G if $|E(G[S])| \le \gamma n^2$. Let G be an n-vertex graph with $\sigma(G) \ge 2(1-\frac{1}{r})n-2$.

We divide the proof of Theorem 1.7 into two parts according to G contains a γ-independent set of size $\frac{n}{r}$ (extremal case) or not (non-extremal case).

Non-extremal case: G contains no γ-independent sets of size $\frac{n}{r}$.

For the non-extremal case, our proof primarily adopts the framework of the absorption method proposed by Rödl, Ruciński and Szemerédi [30], which provides an efficient framework for constructing spanning subgraphs. The absorption method typically decomposes the problem of finding perfect tilings into two steps. First, one constructs a 'small' absorbing set M in the host graph G, which can 'absorb' any small set of 'remaining' vertices. The second step is to find an almost perfect K_r-tiling covering almost all vertices in $G - M$.

Absorbing. We are to find a 'small' K_r-absorbing set M in G that absorbs any 'very small' set of vertices in G. To prove that such a K_r-absorbing set M exists, we refine the widely used method by Rödl, Ruciński, and Szemerédi [30] and Hàn, Person, and Schacht [13]. That is, we show that

$$\textit{almost all } r\text{-subset } T \subseteq V(G) \text{ has } \Omega(n^{r^2}) \ K_r\text{-absorbers for } T.$$

Almost cover. We apply the Regularity Lemma to G and thus obtain a reduced graph R that inherits the Ore's condition of G. Our goal is to find an almost perfect K_r-tiling of R. We begin by greedily selecting a maximal family of vertex-disjoint K_r copies in G. Then, for each $s = r-1, r-2, \dots, 1$ in descending order, we iteratively maximize the number of vertex-disjoint K_s-copies in the remaining graph. This process constructs a maximum $(K_r, K_{r-1}, \dots, K_1)$-factor in R (see Definition 4.11).

Suppose that R contains no almost perfect K_r-tiling. Assume the largest K_r-tiling in R covers exactly $q \le (1-o(1))|R|$ vertices. We then prove that the blow-up graph $R[2^{r-1}]$ admits a maximum $(K_r, K_{r-1}, \dots, K_1)$-factor covering significantly more than $q \cdot 2^{r-1}$ vertices. Thus, crucially, the largest K_r-tiling in $R[2^{r-1}]$ covers a larger proportion of vertices than that in R. By repeating this process, we obtain a blow-up graph R' of R that contains an almost perfect K_r-tiling. Our approach adapts a proof technique of Treglown [33], developed for finding K_r-factors under a degree sequence condition.

Extremal case: G contains a γ-independent set of size $\frac{n}{r}$.

For the extremal case, we will make use of a result concerning K_r-factors in balanced r-partite graphs (Lemma 2.5) as well as the Graph Removal Lemma (Lemma 2.2). Define $S := \{v :\in V(G) : d(x) < (1-\frac{1}{r})n - 1\}$. Clearly, $G[S]$ is a clique by Ore's condition. Since G has a γ-independent set of size $\frac{n}{r}$, one may construct a *good partition* $\mathcal{Q} = \{A_1, A_2, \dots, A_s, B\}$ of $V(G)$ (see Definition 5.2) such that each $|A_i| = \frac{n}{r}$ and $S \subseteq B$, where each A_i is an almost independent set while $G[B]$ has no γ-independent set of size $\frac{n}{r}$.

Under the partition $\mathcal{Q}$, the process of using Lemma 2.5 to construct a K_r-factor in G consists of the following two steps:

Step 1. Cover all non-excellent vertices (i.e., those failing the minimum degree condition) using vertex-disjoint K_r copies. To achieve this, we first identify a small structural 'base' that covers all such vertices. This base is selected with extension properties that guarantee it can be extended into a K_r-tiling while maintaining the original vertex proportion across all parts. Denote the resulting K_r-tiling obtained in this step by $\mathcal{T}$.

Step 2. Find a K_{r-s}-tiling in $G[B \setminus V(\mathcal{T})]$. For $r - s \ge 3$, such a K_{r-s}-tiling can be obtained by using the non-extremal case. Now, suppose that $r - s = 2$ and $G[B \setminus V(\mathcal{T})]$ contains no perfect matching. Assume that G is not extremal. Proposition 2.10 implies that $G[B \setminus V(\mathcal{T})]$ consists of two odd components. By using some structure analysis, we can perform minor adjustments to the bases obtained in Step 1 to construct a new K_r-tiling $\mathcal{T}'$ that covers all non-excellent vertices and $G[B \setminus V(\mathcal{T}')]$ contains a perfect matching.

After the above steps, we contract each K_{r-s} copy in the K_{r-s}-tiling of $G[B \setminus V(\mathcal{T})]$ into a single vertex. This allows us to apply Lemma 2.5 to obtain a K_{s+1}-factor in the remaining graph, which in turn yields a K_r-factor in the graph $G - V(\mathcal{T}')$. Combining this with $\mathcal{T}$ or $\mathcal{T}'$ yields the desired K_r-factor in G.

1.3 Organization

In Section 2, we present some preliminaries and results on matchings in graphs. Section 3 is devoted to the proof of Theorem 1.7, which is divided into two parts: the non-extremal case (Theorem 3.1) and the extremal case (Theorem 3.2). The non-extremal case is proved in Section 4 by using the almost cover lemma (Lemma 4.1, proved in Section 4.2) and the absorbing lemma (Lemma 4.3, proved in Section 4.3). Section 5 deals with the extremal case and provides a proof of Theorem 3.2. The proof consists of two main steps: constructing a good partition of G (Lemma 5.3, proved in Section 6) and covering all non-excellent vertices (Lemma 5.4, proved in Section 7). In the last section, we show that there is an algorithm with polynomial time that decides whether G has a K_r-factor, and some concluding remarks are also given in this section.

2 Notation and Preliminaries

2.1 Notation

Throughout the paper we follow the standard graph-theoretic notation [7]. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. The *order* of a graph G, denoted by $|G|$, is the number of vertices, and its *size*, denoted by $e(G)$, is the number of edges. Let $U \subseteq V(G)$ be a vertex subset. Denote by $G[U]$ the induced subgraph of G on U. The notation $G - U$ is used to denote the induced subgraph after removing U, that is $G - U := G[V(G) \setminus U]$. Given two vertex subsets $A, B \subseteq V(G)$, we use $G[A, B]$ to denote the subgraph of G with vertex set $A \cup B$ and edge set consisting of all edges having one endpoint in A and the other in B. For a family $\mathcal{G}$ of graphs, denote by $|\mathcal{G}|$ the number of graphs in $\mathcal{G}$, and let $V(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} V(G)$.

The *neighborhood* of a vertex v in a graph G, denoted $N_G(v)$, is the set of all vertices adjacent to v. The *degree* of v in G, denoted $d_G(v)$, is the number of vertices in $N_G(v)$; when there is no ambiguity, we abbreviate these as $N(v)$ or $d(v)$. For two vertex subsets $S, A \subseteq V(G)$, denote $N(S) := \bigcap_{v\in S} N(v)$, $N(v, A) := N(v) \cap A$ and $N(S, A) := N(S) \cap A$, and denote by $d(v, A), d(S)$ and $d(S, A)$ the sizes of $N(v, A), N(S)$, and $N(S, A)$, respectively. When A is a subgraph of G, we always abbreviate $V(A)$ as A in the notations above. Let $\delta(G) := \min\{d(v) : v \in V(G)\}$ and $\Delta(G) := \max\{d(v) : v \in V(G)\}$ be the *minimum degree* and *maximum degree* of G respectively.

A *partition* of $V(G)$ is a family $\mathcal{P} = \{A_1, \dots, A_s\}$ of disjoint subsets satisfying $V(G) = \bigcup_{i=1}^{s} A_i$. Given two graphs H and G, an *H-tiling* of G is a collection of vertex-disjoint copies of H in the graph G. A *perfect H-tiling* of G, or an *H-factor*, is an H-tiling that covers all vertices of G. A *matching* in a graph is a set of edges such that no two edges share a common vertex. A *maximum matching* is a matching that contains the largest possible number of edges. We call a matching M of G a *d-matching* if $e(M) = d$.

Consider a graph F and an integer $t \in \mathbb{N}$. Let $F[t]$ denote the graph obtained from F by replacing every vertex of F with an independent set of size t, and connecting two vertices from different sets if and only if their original vertices are adjacent in F. We will refer to $F[t]$ as a *blow-up* copy of F. The *join* of two vertex-disjoint graphs G_1 and G_2, denoted $G_1 \vee G_2$, is formed by taking their disjoint union and then adding every possible edge between vertices of G_1 and vertices of G_2.

Given a set U and an integer k, we write $\binom{U}{k}$ the collection of all k-element subsets of U. For any integers $a \le b$, define $[a, b] := \{i \in \mathbb{N} : a \le i \le b\}$ and $[b] := [1, b]$. When we write $\alpha \ll \beta \ll \gamma$, we always mean that α, β, γ are constants in $(0, 1)$; and $\alpha \ll \beta$ means that there exists $\alpha_0 = \alpha_0(\beta)$ such that the subsequent arguments hold for all $0 < \alpha \le \alpha_0$. Hierarchies of other lengths are defined analogously.

2.2 Preliminaries

In this subsection, we will give some preliminary results that will be used in our proof. The following fact from Ore's condition will be used throughout this paper.

Fact 2.1. *Let t be a positive integer. Assume that G is an n-vertex graph with $\sigma(G) \geq t$. Define $S := \{x \in V(G) : d(x) < \frac{t}{2}\}$. Then $G[S]$ is a complete graph.*

We next recall the definition of γ-independent set. Given an n-vertex graph G and a constant $\gamma > 0$, we define a vertex subset $S \subseteq V(G)$ as a *γ-independent set* if $e(G[S]) \leq \gamma n^2$. Now, we will use the following graph removal lemma to construct a large γ-independent set in graphs.

Lemma 2.2 (Graph Removal Lemma [1])**.** *For any graph H with h vertices and any $\varepsilon > 0$, there exists $\delta > 0$ such that any graph on n vertices which contains at most δn^h copies of H can be made H-free by removing at most εn^2 edges.*

Proposition 2.3. *Let $0 < 1/n \ll \varepsilon \ll \alpha \ll 1/r$ where $r \in \mathbb{N}$ and $r \geq 2$. Suppose that G is an n-vertex graph that contains at most εn^r copies of K_r and*

$$\sigma(G) \geq 2\Big(1 - \frac{1}{r-1} - \alpha\Big)n.$$

Then G admits a $\sqrt{\alpha}$-independent set of size at least $\frac{n}{r-1}$.

Proof. Choose constants satisfying

$$\frac{1}{n} \ll \varepsilon \ll \gamma \ll \alpha \ll \frac{1}{r},$$

let G be a graph as in the assumption. The result is trivial when $r = 2$ as $\varepsilon \ll \alpha$. Thus, one may assume that $r \geq 3$. Since G contains at most εn^r copies of K_r, Lemma 2.2 implies that one can remove at most γn^2 edges from G to obtain a spanning subgraph G' that is K_r-free. Define

$$L := \{v \in V(G) : d_G(v) - d_{G'}(v) > 2\sqrt{\gamma} n\} \text{ and } G'' := G' - L.$$

Clearly, $|L| \leq \sqrt{\gamma} n$. Since $\gamma \ll \alpha$, one has

$$n'' := |G''| \geq (1-\alpha)n, \text{ and } \sigma(G'') \geq 2\Big(1 - \frac{1}{r-1} - 2\alpha\Big)n''.$$

Claim 2.4. *The graph G'' contains an independent set S of size at least $\big(\frac{1}{r-1} - 3r\alpha\big)n$.*

Proof of Claim 2.4. Let $S := \Big\{v \in V(G'') : \ d(v) < (1 - \frac{1}{r-1} - 2\alpha)n''\Big\}$ and $\tilde{G} := G'' - S$. Fact 2.1 indicates that $G''[S]$ is a clique. Notice that G'' is K_r-free, we have $|S| \leq r-1$. Then

$$\delta(\tilde{G}) \geq \Big(1 - \frac{1}{r-1} - 2\alpha\Big)n'' - |S| \geq \Big(1 - \frac{1}{r-1} - 2\alpha\Big)|\tilde{G}| - r. \tag{1}$$

Thus, by Theorem 1.2 (or by Hajnal-Szemerédi Theorem), for each vertex $v \in V(\tilde{G})$, there exists a copy of K_{r-2} containing v in $\tilde{G}$, say K^v. It follows from (1) that

$$\begin{aligned}\Big|\bigcap_{w \in V(K^v)} N_{\tilde{G}}(w)\Big| &\geq \left(1 - \frac{r-2}{r-1} - 2(r-2)\alpha\right)|\tilde{G}| - r(r-2) \\ &\geq \left(\frac{1}{r-1} - 2(r-1)\alpha\right)n \geq \left(\frac{1}{r-1} - 3r\alpha\right)n.\end{aligned}$$

Since G'' is K_r-free, we obtain that $\bigcap_{w \in V(K^v)} N_{G''}(w)$ is an independent set of size at least $\big(\frac{1}{r-1} - 3r\alpha\big)n$ in G'', as desired. $\square$

Let S be an independent set of size at least $\big(\frac{1}{r-1} - 3r\alpha\big)n$ in G''. The construction of G'' ensures that S is a γ-independent set in G. By augmenting S with at most $3r\alpha n$ additional vertices, we obtain a $\sqrt{\alpha}$-independent set in G of size at least $\frac{n}{r-1}$, as required. $\square$

The following theorem provides a degree condition that guarantees the existence of K_r-factors in a balanced r-partite graph, which follows from [9, Lemma 6.3] by using Edmonds algorithm.

Lemma 2.5 ([9]). *For any $r, n \in \mathbb{N}$, let G be an r-partite graph on vertex classes $V_1, \dots, V_r$ where $|V_i| = n$ for each $1 \leq i \leq r$. If*

$$\min_{i\in[r]}\min_{v\in V_i}\min_{j\in[r]\setminus\{i\}} |N(v) \cap V_j| \geq \Big(1 - \frac{1}{2r}\Big)n,$$

then in time $O(rn^4)$ we can find a K_r-factor of G.

We will use the following version of Chernoff's bound.

Lemma 2.6. *Let X be a random variable with binomial or hypergeometric distribution, and let $0 < \varepsilon < \frac{3}{2}$. Then*

$$\mathbb{P}[|X - \mathbb{E}(X)| \geq \varepsilon\mathbb{E}(X)] \leq 2e^{-\frac{\varepsilon^2}{3}\mathbb{E}(X)}.$$

2.3 Matchings in graphs

In this subsection, we introduce some preliminaries that are related to matchings in graphs.

Lemma 2.7 ([32]). *Let $d, n \in \mathbb{N}$. Suppose that G is a graph on $n \geq 2d$ vertices with $\delta(G) \geq d$. Let $X \subseteq V(G)$ be a subset of size d. Then G contains a d-matching that covers all vertices in X.*

Let M be a maximum matching of a graph G and v be a vertex in $V(G) \setminus V(M)$, define

$$SN(v) := \{z \in V(M) : wz \in E(M) \text{ for some } w \in N(v)\}.$$

The following fact follows immediately from the definition of a maximum matching; the proof is omitted.

Fact 2.8. *Let M be a maximum matching of a graph G and x, y be two distinct vertices in $V(G) \setminus V(M)$. Then $E(G[SN(x), SN(y)]) = \emptyset$.*

Lemma 2.9. *Let $d \geq 1$ and $n \geq 3d + 2$. Assume that G is an n-vertex graph with $d \leq \delta(G) \leq \Delta(G) \leq n - 2d - 1$. Then G admits a matching of size $d + 1$.*

Proof. Choose a maximum matching, say M, inside G. Suppose that $e(M) \leq d+1$. Together with Lemma 2.7, we obtain $e(M) = d$. Let x_1, x_2, x_3 be three distinct vertices in $V(G) \setminus V(M)$. Clearly, $N(x_i) \subseteq V(M)$ for each $i \in [3]$. We are to prove that $N(x_1) = N(x_2) = N(x_3)$.

For each $i \in [3]$, assume that there are a_i edges in M such that both endpoints of them are adjacent to x_i. If follows from Fact 2.8 that $E(SN(x_i), SN(x_j)) = \emptyset$ for $1 \leq i < j \leq 3$. Therefore, $d(x_i) \leq 2a_i + d - (a_1 + a_2 + a_3)$ for each $i \in [3]$. Together with $\delta(G) \geq d$, one has

$$3d \leq d(x_1) + d(x_2) + d(x_3) \leq 3d - (a_1 + a_2 + a_3).$$

Hence $a_1 = a_2 = a_3 = 0$ and $d(x_1) = d(x_2) = d(x_3) = d$. Consequently, every vertex in $V(G) \setminus V(M)$ is adjacent to exactly one vertex in each edge of M. Applying Fact 2.8 again yields that every two distinct vertices in $V(G) \setminus V(M)$ share the same neighbors in $V(M)$. Hence each vertex $v \in N(x_1)$ satisfies $d(v) \geq n - 2d$, which contradicts the maximum degree condition of G. $\square$

The main goal of this subsection is to prove the following proposition.

Proposition 2.10. *Let n be a positive even integer and γ be a constant such that $0 < 1/n \ll \gamma \ll 1$. Suppose that G is a graph on n vertices so that*

$$\sigma(G) \geq n - \gamma n. \tag{2}$$

Then one of the following holds:

(i) *G admits a perfect matching;*

(ii) *G contains a 2γ-independent set of size at least $\frac{n}{2}$;*

(iii) *G consists of two odd components C_1 and C_2. Moreover, if $|C_i| \le \frac{1-\gamma}{2}n$ for some $i \in [2]$, then C_i is a clique.*

Proof. Suppose that G has no perfect matching. Let M be a maximum matching in G. Then, $|V(M)| \le n-2$ and there are two distinct vertices $x, y \in V(G) \setminus V(M)$. The maximality of M implies that $N(x) \cup N(y) \subseteq V(M)$, and so $SN(x) \cup SN(y) \subseteq V(M)$. Since $xy \notin E(G)$, by (2) we have

$$d(x) + d(y) = |SN(x)| + |SN(y)| \ge n - \gamma n. \tag{3}$$

Define $SN(x, y) := SN(x) \cap SN(y)$. Fact 2.8 implies that $SN(x, y)$ is an independent set of G and

$$d(x) + d(y) \le 2e(M) \le n - 2. \tag{4}$$

We divide our proof into two cases.

Case 1. $SN(x, y) \neq \emptyset$.

In this case, we will show that (ii) holds. Without loss of generality, assume that $d(x) \le d(y)$. Then (4) implies $d(x) < \frac{n}{2}$. Choose $z \in SN(x, y)$ and assume $zz' \in E(M)$. Hence $z' \in N(x) \cap N(y)$ and $N(z) \subseteq V(M)$. Thus, (2) implies that

$$d(z) \ge n - \gamma n - d(x) > \frac{n}{2} - \gamma n.$$

Notice that $N(z) \cap (SN(x) \cup SN(y)) = \emptyset$. Otherwise, there exists a matching of size $|E(M)| + 1$, a contradiction. Thus, $|SN(x) \cup SN(y)| \le n - d(z) < \frac{n}{2} + \gamma n$. Together with (3), one has

$$|SN(x) \cap SN(y)| = |SN(x)| + |SN(y)| - |SN(x) \cup SN(y)| \ge \frac{n}{2} - 2\gamma n.$$

Recall that $SN(x, y)$ is an independent set in G. By adding at most $2\gamma n$ arbitrary vertices to $SN(x, y)$, we obtain a 2γ-independent set of size at least $\frac{n}{2}$ in G, as desired.

Case 2. $SN(x, y) = \emptyset$.

In this case, one may assume that $SN(x, y) = \emptyset$ for every maximum matching M of G and any two distinct vertices $x, y \in V(G) \setminus V(M)$. Suppose that $|V(G) \setminus V(M)| > 2$. Then there is a vertex $z \in V(G) \setminus (V(M) \cup \{x, y\})$. It follows from (2) that

$$d(x) + d(y) + d(z) \ge \frac{3}{2}(n - \gamma n) > |V(M)|.$$

Recall that $N(x) \cup N(y) \cup N(z) \subseteq V(M)$. Therefore, there are two vertices $u, v \in \{x, y, z\}$ such that $N(u) \cap N(v) \neq \emptyset$, thus $SN(u, v) \neq \emptyset$, a contradiction. Hence, $|V(G) \setminus V(M)| = 2$. We will show that (iii) holds.

Since $SN(x, y) = \emptyset$ and $E(G[SN(x), SN(y)]) = \emptyset$, there are two vertex-disjoint subgraphs $G_1 := G[\{x\} \cup SN(x)]$ and $G_2 := G[\{y\} \cup SN(y)]$ in G such that $E(G[V(G_1), V(G_2)]) = \emptyset$. Define $R := V(G) \setminus (V(G_1) \cup V(G_2))$. If $R = \emptyset$, then the two endpoints of each edge in M are adjacent to the same vertex in $\{x, y\}$. It follows that G consists of two connected odd components G_1 and G_2, as desired. Suppose that $R \neq \emptyset$. In order to prove (iii) holds, it is necessary to assign vertices in R to G_1 or G_2. Choose a vertex $v \in R$ with $vv' \in E(M)$. Hence $xv', yv' \notin E(G)$. Together with $xy \notin E(G)$ and (2), we have

$$d(x) + d(y) + d(v') = |SN(x)| + |SN(y)| + |N(v')| \ge \frac{3}{2}(n - \gamma n).$$

Recall that $SN(x, y) = \emptyset$. Hence,

$$\text{either } SN(x) \cap N(v') \neq \emptyset \text{ or } SN(y) \cap N(v') \neq \emptyset. \tag{5}$$

• Assume $v' \notin R$. Then $v' \in SN(x) \cup SN(y)$. Without loss of generality, assume that $v' \in SN(x)$, that is, $xv \in E(G)$ and $yv \notin E(G)$. Suppose that $N(v) \cap SN(y) \neq \emptyset$. Then choose $u \in N(v) \cap SN(y)$ with $uu' \in E(M)$. Hence $M' := (M \setminus \{vv', uu'\}) \cup \{vu, u'y\}$ is also a maximum matching of G. Notice that $v \in N(x) \cap N(v')$. Therefore, under the matching M' we have $SN(x, v') \neq \emptyset$, contradicting our initial assumption in Case 2. Thus, $N(v) \cap SN(y) = \emptyset$. Together with $vy \notin E(G)$, we know $N(v) \cap V(G_2) = \emptyset$. Thus, one may add such a vertex v to $V(G_1)$.

• Assume $v' \in R$. Then $xv, yv \notin E(G)$. By a similar argument as (5), we obtain that

$$\text{either } SN(x) \cap N(v) \neq \emptyset \text{ or } SN(y) \cap N(v) \neq \emptyset. \tag{6}$$

We claim that none of the following holds:

(1) $SN(x) \cap N(v) \neq \emptyset$ and $SN(y) \cap N(v') \neq \emptyset$;

(2) $SN(x) \cap N(v') \neq \emptyset$ and $SN(y) \cap N(v) \neq \emptyset$.

Otherwise, without loss of generality, suppose the former case happens. Let $a \in SN(x) \cap N(v)$ and $b \in SN(y) \cap N(v')$, and assume $aa', bb' \in E(M)$. Then $M' := (M \setminus \{aa', bb', vv'\}) \cup \{xa', av, v'b, b'y\}$ is a perfect matching of G, a contradiction. Together with (5) and (6), we obtain that

$$\text{either } SN(x) \cap (N(v) \cup N(v')) = \emptyset \text{ or } SN(y) \cap (N(v) \cup N(v')) = \emptyset. \tag{7}$$

Together with $xv, xv', yv, yv' \notin E(G)$, we have either $(N(v) \cup N(v')) \cap V(G_1) = \emptyset$ or $(N(v) \cup N(v')) \cap V(G_2) = \emptyset$. Therefore, v and v' can be assigned to G_1 or G_2 based on where their neighbors are located, which preserves the absence of edges between G_1 or G_2.

By sequentially applying the above operation to each vertex in R, we can assign them to either G_1 or G_2 accordingly. In order to preserve the disconnectivity between G_1 and G_2, we only need to show that in the above process, for each edge in $G[R]$, its two endpoints are assigned to the same part. Let $uv \in E(G[R])$ with $uu', vv' \in E(M)$. We proceed our proof by considering the following three subcases.

Subcase 2.1. $u', v' \notin R$.

We claim that there exists a vertex $z \in \{x, y\}$ such that $u', v' \in SN(z)$. Otherwise, suppose that $u' \in SN(x)$ and $v' \in SN(y)$. By the argument in the above, we obtain that $N(u) \cap SN(y) = \emptyset$ and $N(v) \cap SN(x) = \emptyset$. Without loss of generality, assume that $d(x) \leq d(y)$. Then $d(x) \leq \frac{n}{2}$. It follows from $v \in R$ that $xv' \notin E(G)$. Together with (2) one has $d(v') \geq \frac{(1-2\gamma)n}{2}$. By $v' \in SN(y)$ and Fact 2.8, we obtain $N(v') \cap SN(x) = \emptyset$. Thus, $N(v') \subseteq SN(y) \cup R$. Notice that $|R| \leq \gamma n$. Hence, there exists a vertex $a \in SN(y) \cap N(v')$. Therefore, $M' := (M \setminus \{aa', uu', vv'\}) \cup \{uv, v'a, a'y\}$ is also a maximum matching in G. But $SN(x, u') \neq \emptyset$, a contradiction. This implies that there exists a vertex $z \in \{x, y\}$ such that $u', v' \in SN(z)$. Therefore, u and v are assigned to the same part by using the above argument.

Subcase 2.2. $u' \in R$, $v' \notin R$.

By a similar discussion as Subcase 2.1, we can show that u and v are assigned to the same part.

Subcase 2.3. $u', v' \in R$.

We claim that there exists a vertex $z \in \{x, y\}$ such that

$$SN(z) \cap (N(v) \cup N(v')) = \emptyset \text{ and } SN(z) \cap (N(u) \cup N(u')) = \emptyset.$$

Otherwise, based on (7), one may assume that $SN(x) \cap (N(v) \cup N(v')) = \emptyset$ and $SN(y) \cap (N(u) \cup N(u')) = \emptyset$. Furthermore, $SN(y) \cap (N(v) \cup N(v')) \neq \emptyset$ and $SN(x) \cap (N(u) \cup N(u')) \neq \emptyset$. By

$uv \in E(G)$ and the maximality of M, one has either $SN(y) \cap N(v') = \emptyset$ or $SN(x) \cap N(u') = \emptyset$. Assume, without loss of generality, that $SN(y) \cap N(v') = \emptyset$. Together with $SN(x) \cap N(v') = \emptyset$, one has $N(v') \subseteq R$. Notice that $xv', yv' \notin E(G)$. By (2), one has

$$d(x), d(y) \geq n - 2\gamma n,$$

which contradicts (4).

With the above discussion, by assigning vertices in R to G_1 or G_2 accordingly, we know that G can be decomposed into two connected components, say C_1 and C_2, each of which has odd order. Furthermore, by Ore's condition, if $|C_i| \leq \frac{1-\gamma}{2}n$ for some $i \in [2]$, then C_i is a clique, as desired. $\square$

3 Proof of Theorem 1.7

In this section, we will give a quick proof of Theorem 1.7. The proof is divided into two cases, depending on whether the graph contains a γ-independent set of size n/r (the extremal case) or not (the non-extremal case).

Theorem 3.1 (Non-extremal case)**.** *Let $r \geq 3$ be an integer. For any $0 < \alpha \ll \gamma \ll \frac{1}{r}$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ with $r \mid n$. Assume that G is a graph on n vertices satisfying $\sigma(G) \geq 2(1 - \frac{1}{r} - \alpha)n$. If G contains no γ-independent set of size $\frac{n}{r}$, then G has a K_r-factor.*

Theorem 3.2 (Extremal case)**.** *Let $r \geq 3$ be an integer. There exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for each $n \geq n_0$ with $r \mid n$. Suppose that G is a graph on n vertices with $\sigma(G) \geq 2(1 - \frac{1}{r})n - 2$. If G contains a γ-independent set of size $\frac{n}{r}$, then*

(i) *either G has a K_r-factor, or*

(ii) *G is an extremal graph, i.e., (EX1) or (EX2).*

We now prove the first part of Theorem 1.7; the algorithmic part will be presented in the final section.

Proof of Theorem 1.7. The case $r = 1$ is trivial. If $r \geq 3$, then by combining Theorem 3.1 and Theorem 3.2, we obtain Theorem 1.7 immediately. In the following, we consider the case that $r = 2$ and $n \geq 4$ is even.

Suppose that G is an n-vertex graph with $\sigma(G) \geq n - 2$. Let $P := u_1u_2 \ldots u_t$ be a longest path in G. Hence $N(u_1) \cup N(u_t) \subseteq V(P)$. If $t = n$, then G admits a K_2-factor, as desired. In the following, we assume that $t \leq n - 1$.

We first assume that $G[V(P)]$ contains a Hamilton cycle. Then any vertex in P is not adjacent to any vertex outside P. Hence for each $i \in [t]$ and each $w \in V(G) \setminus V(P)$, we have

$$n - 2 \leq d(u_i) + d(w) \leq (t-1) + (n-t-1) = n - 2.$$

Thus, $d(u_i) = t - 1$ for each $i \in [t]$ and $d(w) = n - t - 1$ for each $w \in V(G) \setminus V(P)$. It follows that $G = K_t \cup K_{n-t}$. Therefore, G has a K_2-factor if t is even, and G satisfies (EX2) otherwise, as desired.

Next, we assume that $G[V(P)]$ contains no Hamilton cycles. Then $u_1u_t \notin E(G)$. Let

$$A = \{j \in [t-2] : u_1u_{j+1} \in E(G)\} \text{ and } B = \{j \in [2, t-1] : u_tu_j \in E(G)\}.$$

Obviously, $A \cap B = \emptyset$ as $G[V(P)]$ contains no Hamilton cycles. Together with Ore's condition, one has

$$n - 2 \leq d(u_1) + d(u_t) = |A| + |B| = |A \cup B| \leq t - 1 \leq n - 2.$$

Thus, $t = n - 1$ and $A \cup B = [t-1]$. Without loss of generality, assume that $|A| \leq |B|$. Hence $|A| \leq \frac{n}{2} - 1 \leq |B|$. Define

$$C = \{j \in [t] : u_tu_{j-1} \in E(G)\}.$$

Then $C \subseteq [3,t]$ and $|C| = |B|$. Moreover,

$$t \in C \setminus A, \text{ and } 1 \in A \setminus C.$$

For each $j \in A$, there is a path $u_j u_{j-1} \dots u_1 u_{j+1} \dots u_t$ with endpoints u_j and u_t; for each $j \in C$, there is a path $u_1 u_2 \dots u_{j-1} u_t u_{t-1} \dots u_j$ with endpoints u_1 and u_j. Let x be the unique vertex in $V(G) \setminus V(P)$. Recall that the longest path in G has order $n-1$. Hence $xu_j \notin E(G)$ for every $j \in A \cup C$. In particular, $xu_1 \notin E(G)$. The Ore's condition implies that

$$n-2 \leq d(u_1) + d(x) \leq |A| + (n-1-|A \cup C|) \leq n-1-|C \setminus A| \leq n-2.$$

It follows that $d(x) = n-1-|A \cup C|$ and $C \setminus A = \{t\}$. Together with the fact that $1 \in A \setminus C$, we have $C \setminus \{t\} \subseteq A \setminus \{1\}$. Therefore,

$$n-2 = |A| + |B| = |A| + |C| = |A| + |C \setminus \{t\}| + 1 \leq |A| + |A \setminus \{1\}| + 1 = 2|A| \leq n-2.$$

It follows that $C \setminus \{t\} = A \setminus \{1\}$ and $|A| = |B| = |C| = \frac{n}{2} - 1$.

Choose $j \in A \cap C$. Then $j \in [3, n-3]$ and $j-1 \in B$. Together with $A \cap B = \emptyset$, one has $j-1 \notin A$. Since $|A \cap C| = \frac{n}{2} - 2$, we obtain $A \cap C = \{3, 5, \dots, n-3\}$. Therefore, $N(u_1) = N(u_{n-1}) = \{u_2, u_4, \dots, u_{n-2}\}$. For each even integer $i \in [n-1]$, define $P^i := u_1 u_2 \dots u_i u_{n-1} u_{n-2} \dots u_{i+1}$ to be a new path with endpoints u_1 and u_{i+1}. By a similar discussion as above, we obtain that for each even integer $i \in [n-1]$,

$$N(u_1) = N(u_{i+1}) = \{u_2, u_4, \dots, u_{n-2}\} \text{ and } xu_{i+1} \notin E(G).$$

It follows that $\{x, u_1, u_3, \dots, u_{n-1}\}$ forms an independent set of G with size $\frac{n}{2} + 1$, as desired. $\square$

4 Non-extremal case

For the non-extremal case, our proof primarily adopts the framework of the absorption method proposed by Rödl, Ruciński and Szemerédi [30]. The absorption method typically decomposes the problem of finding perfect tilings into two steps: almost cover and the absorption of the remaining vertices.

Lemma 4.1 (Almost cover). *Let $0 < \frac{1}{n} \ll \mu \ll \alpha \ll \gamma \ll \frac{1}{r}$ and $r \geq 3$. Assume that G is a graph on n vertices such that $\sigma(G) \geq 2(1 - \frac{1}{r} - \alpha)n$. If G does not contain any γ-independent set of size $\frac{n}{r}$, then G has a K_r-tiling that covers all but at most μn vertices.*

We need the following notation of absorbers and absorbing sets.

Definition 4.2 (Absorbing set, abosrber). Let G be an n-vertex graph and $r \in \mathbb{N}$.

- A subset $M \subseteq V(G)$ is an *ε-absorbing set* in G for some constant ε if for any subset $U \subseteq V(G) \setminus M$ of size at most εn and $|M \cup U| \in r\mathbb{N}$, $G[M \cup U]$ contains a K_r-factor.
- For an r-set $Q \subseteq V(G)$, a subset $S \subseteq V(G)$ is called a *K_r-absorber* for Q if $|S| = r^2$ and both $G[S]$ and $G[S \cup Q]$ contain K_r-factors.

For example, in Figure 2, $\{v_1, \dots, v_9\}$ is a K_3-absorber for $\{u_1, u_2, u_3\}$.

Lemma 4.3 (Absorbing lemma). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \alpha \ll \xi \ll \gamma \ll \frac{1}{r}$ and $r \geq 3$. Assume that G is a graph on n vertices such that $\sigma(G) \geq 2(1 - \frac{1}{r} - \alpha)n$. If G does not contain any γ-independent set of size $\frac{n}{r}$, then G contains an ε-absorbing set M of size at most $2\xi n$.*

Using Lemmas 4.1 and 4.3, whose proofs are deferred to later subsections, we now prove Theorem 3.1.

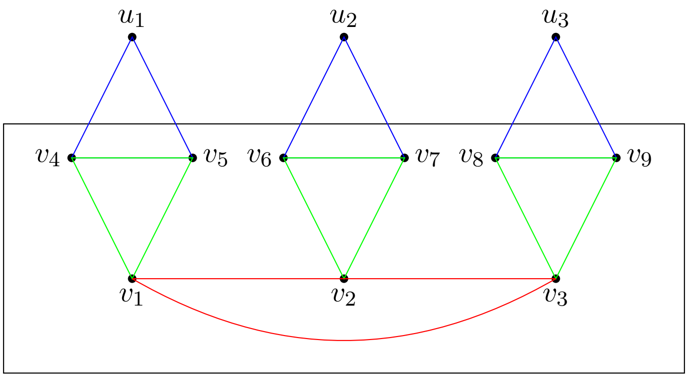


Figure 2: A K_3-absorber for $\{u_1, u_2, u_3\}$.

Proof of Theorem 3.1. Let $0 < \frac{1}{n} \ll \mu \ll \varepsilon \ll \alpha \ll \xi \ll \gamma \ll \frac{1}{r}$. Assume that G is a graph on n vertices and $\sigma(G) \geq 2(1 - \frac{1}{r} - \alpha)n$. By Lemma 4.3, G contains an ε-absorbing set M such that $|M| \leq 2\xi n$. Let $G' := G - M$ and $n' := |G'| \geq (1 - 2\xi)n$. Therefore,

$$\sigma(G') \geq 2\Big(1 - \frac{1}{r} - \alpha\Big)n - 4\xi n \geq 2\Big(1 - \frac{1}{r} - 4\xi\Big)n'.$$

Applying Lemma 4.1 on G' yields that G' has a K_r-tiling, say $\mathcal{K}$, that covers all but at most $\mu n'$ vertices. Denote by U the set of vertices in G' that are not covered by $\mathcal{K}$. Together with the absorbing property of M, one has $G[M \cup U]$ contains a K_r-factor $\mathcal{K}'$. Hence, $\mathcal{K} \cup \mathcal{K}'$ is a K_r-factor of G, as desired. □

4.1 Regularity

We begin by defining regularity for graphs.

Definition 4.4. Given a graph G and a pair (X, Y) of vertex-disjoint subsets in $V(G)$, the *density* of (X, Y) is defined as

$$d(X, Y) = \frac{e(G[X, Y])}{|X||Y|}.$$

Given a constant $\varepsilon > 0$, we say that (X, Y) is an *ε-regular pair* in G if for all $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have

$$\big|d(X', Y') - d(X, Y)\big| \leq \varepsilon.$$

Moreover, a pair (X, Y) is called *(ε, d)-regular* for some $d > 0$ if (X, Y) is ε-regular and $d(X, Y) \geq d$.

Lemma 4.5 (Slicing lemma [21])**.** *Let $0 < \varepsilon < \alpha$ and $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$. Let (X, Y) be an ε-regular pair of density d. Suppose $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \alpha|X|$ and $|Y'| \geq \alpha|Y|$. Then (X', Y') is an ε'-regular pair with density d' where $|d' - d| < \varepsilon$.*

We next state the degree form of Szemerédi's Regularity Lemma, which can be easily derived from the classical version [31].

Lemma 4.6 (Degree form of the Regularity Lemma [1, 31])**.** *For every $\varepsilon > 0$, there exist integers M and n_0 such that the following holds for any $d \in [0, 1]$ and any positive integer $n \geq n_0$. Let G be a graph on n vertices. Then we can find in time $O(n^{2.376})$ a partition $\mathcal{P} := \{V_0, V_1, \ldots, V_k\}$ of $V(G)$ and a spanning subgraph $G' \subseteq G$ with the following properties:*

(1) $\frac{1}{\varepsilon} \leq k \leq M$;

(2) $|V_0| \leq \varepsilon n$ and $|V_1| = |V_2| = \cdots = |V_k| =: m$;

(3) $d_{G'}(v) > d_G(v) - (d + 2\varepsilon)n$ for every $v \in V(G)$;

(4) $e(G'[V_i]) = 0$ for each $i \in [k]$;

(5) for $1 \le i < j \le k$, each pair (V_i, V_j) is ε-regular in G' with density either 0 or at least d.

The sets V_i are called *clusters* and V_0 is the *exceptional set*. A widely used auxiliary graph associated with the regular partition is the *reduced graph* defined as follows.

Definition 4.7 (Reduced graph)**.** Let G be an n-vertex graph. Given parameters $\varepsilon > 0$ and $d \in [0, 1]$, let G' be a spanning subgraph of G and $\mathcal{P} = \{V_0, V_1, \ldots, V_k\}$ be a partition of $V(G)$, as given in Lemma 4.6. We define the *reduced graph* $R := R(\varepsilon, d)$ for $\mathcal{P}$ as follows: R is a graph on vertex set $\{V_1, \ldots, V_k\}$ in which V_i and V_j are adjacent if and only if (V_i, V_j) has density at least d in G'.

The next lemma states that clusters inherit the Ore-type condition in the reduced graph.

Lemma 4.8 ([24])**.** *Given a constant c, let G be a graph on n vertices such that $\sigma(G) \ge cn$. Suppose we have applied Lemma 4.6 with parameters ε and d to G. Let R be the corresponding reduced graph. Then $\sigma(R) \ge (c - 2d - 4\varepsilon)|R|$.*

The following result will be used to convert an almost perfect K_r-tiling in a reduced graph into an almost perfect K_r-tiling in the original graph G.

Lemma 4.9 ([2])**.** *Let $\varepsilon, d > 0$ and $m, r \in \mathbb{N}$ such that $0 < 1/m \ll \varepsilon \ll d \ll 1/r$. Let H be a graph obtained from K_r by replacing every vertex with m independent vertices and every edge with an ε^2-regular pair of density at least d. Then H admits a K_r-tiling that covers all but at most εmr vertices.*

Our next lemma shows that the reduced graph inherits the property of having no large almost independent set.

Lemma 4.10. *Let $r, n, t \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon \ll d \ll \gamma \ll 1/r$, and let G be an n-vertex graph containing no γ-independent set of size n/r. Suppose we have applied Lemma 4.6 with parameters ε and d to G, yielding a reduced graph R for a partition $\mathcal{P}$ of $V(G)$. Then, $R[t]$ can be viewed as a subgraph of some reduced graph $R' := R(2\varepsilon t, d/2)$ for a refinement partition of $\mathcal{P}$. Moreover, $R[t]$ contains no $\gamma/2$-independent set of size tk/r.*

Proof. Let $\mathcal{P} := \{V_0, V_1, \ldots, V_k\}$ be a partition of $V(G)$ obtained in Lemma 4.6. For each $i \in [k]$, partition V_i into $V_{i,0} \cup V_{i,1} \cup \cdots \cup V_{i,t}$, where $m' := |V_{i,j}| = \lfloor \frac{m}{t} \rfloor \ge \frac{m}{2t}$ for each $1 \le j \le t$. Since $mk \ge (1 - \varepsilon)n$, one has

$$m' \cdot |R[t]| = \lfloor \frac{m}{t} \rfloor \cdot kt \ge mk - kt \ge (1 - 2\varepsilon)n.$$

Lemma 4.5 implies that if $(V_{i_1}, V_{i_2})_{G'}$ is ε-regular with density at least d, then $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon t$-regular with density at least $d - \varepsilon \ge d/2$ for all $1 \le j_1, j_2 \le t$. In particular, we can label the vertex set of $R[t]$ so that $V(R[t]) = \{V_{i,j} : 1 \le i \le k, 1 \le j \le t\}$, where $V_{i_1,j_1}V_{i_2,j_2} \in E(R[t])$ implies that $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon t$-regular with density at least $d - \varepsilon \ge d/2$. That is, $R[t]$ can be viewed as a subgraph of some reduced graph $R' := R(2\varepsilon t, d/2)$ for the partition $\{(\bigcup_{i\in[k]} V_{i,0} \cup V_0), V_{1,1}, \ldots, V_{1,t}, \ldots, V_{k,1}, \ldots, V_{k,t}\}$.

Next, we prove that $R[t]$ contains no $\gamma/2$-independent set of size tk/r. Suppose that there exists a set $S := \{V_{i_1,j_1}, \ldots, V_{i_{tk/r},j_{tk/r}}\} \subseteq V(R[t])$ such that $|E((R[t])[S])| \le \frac{\gamma}{2}(tk)^2$. Therefore,

$$|V_{i_1,j_1} \cup \cdots \cup V_{i_{tk/r},j_{tk/r}}| \ge \frac{(1 - 2\varepsilon)n}{r} \text{ and } |E(G[V_{i_1,j_1} \cup \cdots \cup V_{i_{tk/r},j_{tk/r}}])| \le \frac{\gamma}{2}(tk)^2 m'^2 \le \frac{\gamma}{2}n^2.$$

By augmenting $V_{i_1,j_1} \cup \cdots \cup V_{i_{tk/r},j_{tk/r}}$ with at most $2\varepsilon n/r$ additional vertices, we obtain a γ-independent set in G of size at least $\frac{n}{r}$, a contradiction. $\square$

4.2 Proof of Lemma 4.1

We first prove the almost cover lemma.

Definition 4.11. A *maximum* $(K_r, K_{r-1}, \dots, K_1)$*-factor* of a graph G is a collection of vertex-disjoint subgraphs $H_1, \dots, H_k$ of G such that

- each H_i is isomorphic to some K_s where $1 \le s \le r$;
- the vertex sets of these subgraphs form a partition $V(G) = U_r \cup U_{r-1} \cup \cdots \cup U_1$, where U_s consists of the vertices covered by K_s components;
- for each $s \in \{r, r-1, \dots, 1\}$, the number of K_s copies in $G\left[V(G) \setminus \left(\bigcup_{i=s+1}^{r} U_i\right)\right]$ is maximized.

Proof of Lemma 4.1. Given parameters satisfying

$$0 < \frac{1}{n} \ll \varepsilon \ll d \ll \mu \ll \alpha \ll \gamma \ll \frac{1}{r},$$

assume that G is an n-vertex graph with $\sigma(G) \ge 2(1 - 1/r - \alpha)n$ and G contains no γ-independent set of size n/r. By applying Lemma 4.6 on G with parameters ε and d, we obtain a partition $\mathcal{P} := \{V_0, V_1, \dots, V_k\}$ with $k \ge \frac{1}{\varepsilon}$ and a spanning subgraph $G' \subseteq G$ with properties (1)-(5) as stated. Let R be the reduced graph corresponding to $\mathcal{P}$. Based on Lemma 4.8, we have $\sigma(R) \ge 2(1 - \frac{1}{r} - 2\alpha)k$. Define $S := \{v \in V(R) : d_R(v) < (1 - \frac{1}{r} - 2\alpha)k\}$.

Choose a maximum $(K_r, K_{r-1}, \dots, K_1)$-factor of R and assume $V(R) = U_r \cup U_{r-1} \cup \cdots \cup U_1$, where U_s consists of the vertices covered by K_s components for each $s \in [r]$. Hence $|(V(G) \setminus U_r) \cap S| \le r - 1$. Therefore, there are at most $r-1$ components K_t with $t < r$ that intersect with S. In the following, if K_t is a clique in the maximum $(K_r, K_{r-1}, \dots, K_1)$-factor of R, then we simply say $K_t \in U_t$ for each $t \in [r]$.

Claim 4.12. $|U_r| \ge (1 - \alpha r^2)k$.

Proof of Claim 4.12. Choose a minimum constant ζ satisfying $\zeta \ge r\alpha$ and $r|(1+\zeta)k$, let $R^* := R \vee K_{\zeta k}$ and $k^* := |V(R^*)| = (1+\zeta)k$. Notice that if $xy \notin E(R^*)$, then $xy \notin E(R)$. Hence, for each edge $xy \notin E(R^*)$, we have

$$d_{R^*}(x) + d_{R^*}(y) = d_R(x) + d_R(y) + 2\zeta k \ge 2\Big(1 - \frac{1}{r} - \alpha + \zeta\Big)k \ge 2\Big(1 - \frac{1}{r}\Big)k^*,$$

where the last inequality holds as $\zeta \ge r\alpha$ and $r \ge 3$. Together with Theorem 1.2, we obtain that R^* contains a K_r-factor. It follows from $\zeta k \le r\alpha k + r$ that $|U_r| \ge (1 - \alpha r^2)k$, as desired. □

In the following claim, we consider K_r-tilings in the blow-up graph of R.

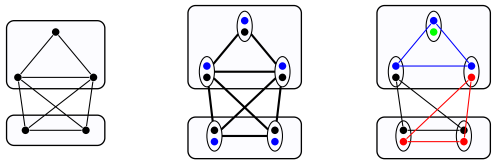
Claim 4.13. *There exists an integer z such that the blow-up graph $R[2^{z(r-1)}]$, has a K_r-tiling covering at least $(1 - \mu/2)|R[2^{z(r-1)}]|$ vertices.*

Proof of Claim 4.13. If $|V(R) \setminus U_r| \le \frac{\mu}{2}k$, then we are done. So, we assume that $|U_r| := q$ with $(1 - \alpha r^2)k \le q \le (1 - \mu/2)k$. Then there are at least $\frac{\mu}{2}k - (r-1)^2$ vertices that are in $K_t \in U_t$ with $t < r$ and $V(K_t) \cap S = \emptyset$.

We construct an auxiliary bipartite graph $H = (A, B)$, where each $K_r \in U_r$ corresponds to a vertex v_{K_r} in A and each $K_t \in U_t$ with $t < r$ and $V(K_t) \cap S = \emptyset$ corresponds to a vertex v_{K_t} in B; $v_{K_r}v_{K_t} \in E(H)$ if and only if $|N_R(K_t) \cap V(K_r)| \ge r - t + 1$. Choose a maximum matching M in H. Hence, for each $v_{K_t} \in B \setminus V(M)$, we have $|N_R(K_t) \cap V(K_r)| \le r - t$ for each $v_{K_r} \in A \setminus V(M)$. Furthermore, Claim 4.12 implies that $|E(M)| \le \alpha r^2 k$. Next, we will show that each $K_t \in U_t$ with $t < r$ and $V(K_t) \cap S = \emptyset$ can be augmented to a copy of K_{t+1} in the blow-up graph $R[2]$.

Fact 4.14. *If $v_{K_r}v_{K_t} \in E(M)$, then $(K_r \cup K_t)[2]$ contains $2K_r \cup K_{t+1} \cup K_{t-1}$ as a subgraph.*

Proof of Fact 4.14. Notice that $|N_R(K_t) \cap V(K_r)| \geq r - t + 1$. Let $V(K_r) = \{u_1, \ldots, u_r\}$, $V(K_t) = \{w_1, \ldots, w_t\}$, and assume $\{u_1, \ldots, u_{r-t+1}\} \subseteq N_R(K_t) \cap V(K_r)$. Consider the blow-up graph $(K_r \cup K_t)[2]$ with vertex set $\{u^1, u^2 : u \in V(K_r \cup K_t)\}$. It is straightforward to verify that $(K_r \cup K_t)[2]$ contains four disjoint subgraphs induced by the following vertex sets: $\{u_1^1, \ldots, u_r^1\}$, $\{u_1^2, \ldots, u_{r-t}^2, w_1^1, \ldots, w_t^1\}$, $\{w_1^2, \ldots, w_t^2, u_{r-t+1}^2\}$, and $\{u_{r-t+2}^2, \ldots, u_r^2\}$. That is, $(K_r \cup K_t)[2]$ contains $2K_r \cup K_{t+1} \cup K_{t-1}$ as a subgraph. (see Figure 3 for $r = 3$ and $t = 2$). $\square$



$K_3 \cup K_2$ $\qquad\qquad\qquad\qquad$ $3K_3 \cup K_1$

Figure 3: $|N_R(K_t) \cap V(K_r)| \geq r - t + 1$ for $r = 3$ and $t = 2$.

Choose a clique $K_t \in U_t$ with $v_{K_t} \in B \setminus V(M)$, say K. Denote

$$X := \{v_{K_r} \in A \setminus V(M) : |N_R(K) \cap V(K_r)| = r - t\} \text{ and } Y := A \setminus (V(M) \cup X),$$

assume $x := |X|$ and $y := |Y|$. It follows from Claim 4.12 that $x + y \geq (\frac{1}{r} - \alpha r - \alpha r^2)k$. Moreover,

$$tx + (t+1)y \leq |V(R) \setminus N_R(K)| = k - |N_R(K)| \leq t\Big(\frac{1}{r} + 2\alpha\Big)k,$$

the last inequality follows by $V(K) \cap S = \emptyset$. Therefore, $y \leq t(2\alpha + \alpha r + \alpha r^2)k$ and $x \geq (\frac{1}{r} - 2\alpha r^3)k$.

If $|N_R(w) \cap V(K_r)| = r$ for some $w \in V(K)$ and $v_{K_r} \in X$, then a similar proof to Fact 4.14 shows that $(K_r \cup K)[2]$ contains $2K_r \cup K_{t+1} \cup K_{t-1}$ as a subgraph. Now, we assume that $|N_R(w) \cap V(K_r)| \leq r - 1$ for each $w \in V(K)$ and $v_{K_r} \in X$. Choose $w \in V(K)$, denote

$$\begin{aligned} X^w &:= \{v_{K_r} \in A \setminus V(M) : |N_R(w) \cap V(K_r)| = r - 1\} \text{ and} \\ Y^w &:= \{v_{K_r} \in A \setminus V(M) : |N_R(w) \cap V(K_r)| < r - 1\}, \end{aligned}$$

assume $x^w := |X^w|$ and $y^w := |Y^w|$. Similarly, we have

$$x^w + y^w \geq \Big(\frac{1}{r} - \alpha r - \alpha r^2\Big)k \text{ and } x^w + 2y^w \leq \Big(\frac{1}{r} + 2\alpha\Big)k.$$

Hence $y^w \leq (2\alpha + 2\alpha r + 2\alpha r^2)k$ and $x^w \geq (\frac{1}{r} - 4\alpha r^2)k$. Note that $X \cup (\bigcup_{u \in V(K)} X^u) \subseteq A \setminus V(M)$ and $|A| \leq \frac{|U_r|}{r} < \frac{k}{r}$. Thus,

$$\ell := \Big|X \cap (\bigcap_{u \in V(K)} X^u)\Big| \geq \Big(\frac{1}{r} - 7\alpha r^3\Big)k.$$

For each $i \in [\ell]$, choose a vertex w_i from each copy of K_r, say K^i, with $v_{K^i} \in X \cap (\bigcap_{u \in V(K)} X^u)$ such that $w_i w \notin E(R)$. It follows from Lemma 4.10 that R contains no $\gamma/2$-independent set of size k/r. Thus, there is an edge, say $w_1 w_2$, in $R[\{w_1, \ldots, w_\ell\}]$. We claim that w_1, w_2 are adjacent to all vertices in $V(K) \setminus \{w\}$. Otherwise, suppose that $w_1 w' \notin E(R)$ for some $w' \in V(K) \setminus \{w\}$. By the choice of $X \cap (\bigcap_{u \in V(K)} X^u)$, we know that at most t edges are missing between $V(K)$ and $V(K^1)$ in R. However, at least two vertices in $V(K)$ are not adjacent to the vertex w_1. Hence, at most $t - 1$ vertices in K^1 has non-neighbors in K. It follows that $|N_R(K) \cap V(K^1)| \geq r - t + 1$, a contradiction. Therefore, $(K^1 \cup K^2 \cup K)[2]$ contains $4K_r \cup K_{t+1} \cup K_{t-1}$, as shown by an argument similar to Fact 4.14 (see Figure 4 for $r = 3$ and $t = 2$).

By the above discussion, for each $K_t \in U_t$ with $t < r$ and $V(K_t) \cap S = \emptyset$, we conclude that

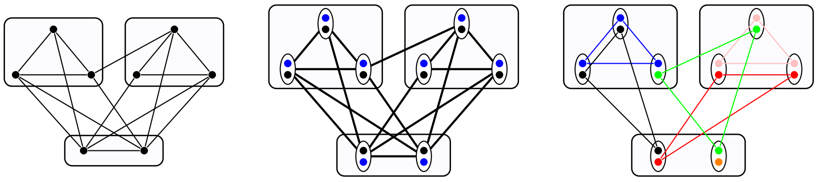


Figure 4: $|N_R(K_t) \cap V(K_r)| \leq r - t$ for $r = 3$ and $t = 2$.

- either there exists a copy of $K_r \in U_r$ such that $(K_r \cup K_t)[2]$ contains $2K_r \cup K_{t+1} \cup K_{t-1}$,
- or there are two copies of $K_r \in U_r$ such that $(2K_r \cup K_t)[2]$ contains $4K_r \cup K_{t+1} \cup K_{t-1}$.

Moreover, we can choose the K_r copies for different K_t so that they are pairwise disjoint. Let q be the number of vertices covered by U_r. Assume $|U_j| = \max\{|U_i| : i \in [r-1]\}$. Then $|U_j| \geq \frac{k-q}{2(r-1)}$. By applying the augmentation process to the cliques in U_j, we obtain a maximum $(K_r, K_{r-1}, \ldots, K_1)$-factor of $R[2]$ such that copies of K_i cover at least $2|U_i|$ vertices for all $i \in [r] \setminus \{j, j+1\}$, and copies of K_{j+1} cover at least $2|U_{j+1}| + |U_j|$ vertices. (Here, $2|U_{j+1}|$ and $|U_j|$ come from the original cliques in U_{j+1} and the augmentation of cliques in U_j, respectively). By iterating this augmentation–first on the new K_{j+1} copies, then on the resulting K_{j+2} copies, and so forth, up to the K_{r-1} copies, we ultimately construct a K_r-tiling within $R[2^{r-j}]$ of order at least

$$2^{r-j}|U_r| + 2^{r-j-1}|U_{r-1}| + \cdots + |U_j| \geq 2^{r-j}q + |U_j|.$$

Therefore, there exists a K_r-tiling in $R[2^{r-1}]$ covering at least

$$2^{r-1}q + \frac{|U_j|}{2^{r-1}} \cdot 2^{r-1} \geq q2^{r-1} + \frac{\mu k}{4(r-1)2^{r-1}} \cdot 2^{r-1}$$

vertices, as $k - q \geq \mu k/2$. Thus, an additional η-proportion of the vertices in $R[2^{r-1}]$ are covered, where $\eta = \frac{\mu}{4(r-1)2^{r-1}}$.

By Claim 4.12 and Lemma 4.10, we repeat this argument until the blow-up graph of R has a K_r-tiling covering all but at most $\mu/2$-proportion of its vertices. Clearly, this process will terminate within $z := \frac{1}{\eta}(\alpha r^2 - \mu/2)$ iterations. □

Denote $R' := R[2^{z(r-1)}]$ and $z' := 2^{z(r-1)}$. For each $1 \leq i \leq k$, partition V_i into $V_{i,0} \cup V_{i,1} \cup \cdots \cup V_{i,z'}$, where $m' := |V_{i,j}| = \lfloor \frac{m}{z'} \rfloor \geq \frac{m}{2z'}$ for each $1 \leq j \leq z'$. By Lemma 4.10, we can label the vertex set of R' so that $V(R') = \{V_{i,j} : 1 \leq i \leq k, 1 \leq j \leq z'\}$, where $V_{i_1,j_1}V_{i_2,j_2} \in E(R')$ implies that $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon z'$-regular with density at least $d - \varepsilon \geq d/2$.

By Claim 4.13, R' has a K_r-tiling $\mathcal{M}$ that contains at least $(1-\mu/2)|R'|$ vertices. Consider any copy of K_r, say K in $\mathcal{M}$ and let $V(K) = \{V_{i_1,j_1}, V_{i_2,j_2}, \ldots, V_{i_r,j_r}\}$. Set $V' := V_{i_1,j_1} \cup V_{i_2,j_2} \cup \cdots \cup V_{i_r,j_r}$. Note that $0 < 1/m' \ll 2\varepsilon z' \ll d/2 \ll \gamma \ll 1/r$. Applying Lemma 4.9 yields that $G'[V']$ contains a K_r-tiling covering all but at most $\sqrt{2\varepsilon z'}m'r$ vertices. By considering each copy of K_r in $\mathcal{M}$ we conclude that $G' \subseteq G$ contains a K_r-tiling covering at least

$$(1 - \sqrt{2\varepsilon z'})m'r \times (1 - \mu/2)|R'|/r \geq (1 - \mu)n$$

vertices, as desired. □

4.3 Proof of Lemma 4.3

In this subsection, we will give the proof of Lemma 4.3.

Proof of Lemma 4.3. Given parameters satisfying

$$0<\frac{1}{n}\ll\varepsilon\ll\alpha\ll\xi\ll\gamma\ll\frac{1}{r},$$

assume that G is an n-vertex graph with $\sigma(G)\geq 2(1-\frac{1}{r}-\alpha)n$ and G contains no γ-independent set of size $\frac{n}{r}$. Denote $S:=\{v\in V(G):d(v)<(1-\frac{1}{r}-\alpha)n\}$. Clearly, $G[S]$ forms a clique. We aim to establish that almost all r-sets in $V(G)$ have $\Omega(n^{r^2})$ K_r-absorbers. To this end, we proceed by considering the following two cases.

Case 1. $|S|\leq\xi n$.

Let $\{v_1,\dots,v_r\}$ be a set in $V(G)\setminus S$. For any set $W\subseteq V(G)\setminus S$ with size at most $r-1$, we have $|N(W)\setminus S|\geq\big(1-\frac{|W|}{r}-|W|\alpha-\xi\big)n$. Hence, there are at least

$$(n-\xi n-r)\Big(\big(1-\frac{1}{r}-\alpha-\xi\big)n-r-1\Big)\cdots\Big(\big(1-\frac{r-1}{r}-(r-1)\alpha-\xi\big)n-2r+1\Big)\geq\Big(\frac{n}{2r}\Big)^r$$

r-sets, say $\{u_1,u_2,\dots,u_r\}$, in $V(G)\setminus(S\cup\{v_1,v_2,\dots,v_r\})$ such that each spans a copy of K_r in G. For every pair (u_i,v_i) with $i\in[r]$, the number of $(r-1)$-sets $X\subseteq V(G)\setminus S$ for which both $G[X\cup\{u_i\}]$ and $G[X\cup\{v_i\}]$ are copies of K_r is at least

$$\Big(\big(1-\frac{2}{r}-2\alpha-\xi\big)n-2r\Big)\cdots\Big(\big(1-\frac{r-2}{r}-(r-2)\alpha-\xi\big)n-3r\Big)\cdot\frac{\gamma}{2}n^2\geq\Big(\frac{n}{2r}\Big)^{r-3}\frac{\gamma}{2}n^2.$$

Here, the last two vertices have at least $\frac{\gamma}{2}n^2$ choices since G contains no γ-independent set of size $\frac{n}{r}$. It follows from $|S|\leq\xi n$ that all but at most ξn^r r-sets have at least $\Omega(n^{r^2})$ K_r-absorbers.

Case 2. $|S|>\xi n$.

Let $\{v_1,\dots,v_r\}$ be a set in $V(G)$. Clearly, there are at least $\binom{|S|-r}{r}$ r-sets, say $\{u_1,\dots,u_r\}$, in S that span copies of K_r in $G[S]-\{v_1,v_2,\dots,v_r\}$. Fix an $i\in[r]$, we consider the pair (u_i,v_i). If $|N(v_i)\cap S|\geq\frac{\xi}{4}n$, then the number of $(r-1)$-sets X such that both $G[X\cup\{u_i\}]$ and $G[X\cup\{v_i\}]$ are copies of K_r is at least $c\xi^{r-1}n^{r-1}$ for some constant c. If $|N(v_i)\cap S|\leq\frac{\xi}{4}n$, then there are at least $\frac{3\xi}{4}n$ choices for $u_i\in S\setminus N(v_i)$. By Ore's condition, we have $d(v_i)+d(u_i)\geq 2(1-\frac{1}{r}-\alpha)n$. Hence

$$|(N(v_i)\cap N(u_i))\setminus S|\geq|N(v_i)\cap N(u_i)|-\frac{\xi}{4}n\geq\frac{r-2}{r}n-\frac{\xi}{3}n.$$

Let

$$W^j:=\big((N(v_i)\cap N(u_i))\setminus S\big)\cap\big(\bigcap_{\ell\in[j-1]}N(w^\ell)\big)\text{ for each }j\in[r-2],$$
$$\text{and }w^j\in W^j\text{ for each }j\in[r-3].$$

Similarly, for each $j\in[2,r-3]$, one has $|W^j|\geq\frac{r-2-j+1}{r}n-\frac{\xi}{3-j/r}n$. Thus, w^j is well-defined for each $j\in[r-2]$. Together with G contains no γ-independent set of size $\frac{n}{r}$, there are at least $\frac{\gamma}{2}n^2$ edges in $G[W^{r-2}]$. Thus, the number of $(r-1)$-sets $X\subseteq(N(v_i)\cap N(u_i))\setminus S$ that forms a clique is at least $\frac{\gamma}{2\cdot r^{r-3}}n^{r-1}$. Therefore, for each r-set in $V(G)$, there are at least $\Omega(n^{r^2})$ K_r-absorbers for it.

Let $\mathcal{Q}$ denote the set of r-set in $V(G)$ each of which has at least $\Omega(n^{r^2})$ K_r-absorbers. Then $\mathcal{Q}=\binom{V(G)}{r}$ if $|S|>\xi n$ and $|\mathcal{Q}|\geq(1-o(1))n^r$ otherwise. Given an r-set $Q\subseteq\mathcal{Q}$, let $L_Q\subseteq\binom{V(G)}{r^2}$ denote the family of all K_r-absorbers for Q. Thus, $|L_Q|\geq\sqrt{\varepsilon}n^{r^2}$ for each $Q\in\mathcal{Q}$. Let F be the family obtained by selecting each of the $\binom{n}{r^2}$ elements of $\binom{V(G)}{r^2}$ independently with probability $p:=\frac{\sqrt{\varepsilon}}{n^{r^2-1}}$. Then

$$\mathbb{E}(|F|)=p\binom{n}{r^2}<\frac{\sqrt{\varepsilon}}{r^2!}n,\text{ and }\mathbb{E}(|L_Q\cap F|)\geq p\sqrt{\varepsilon}n^{r^2}=\varepsilon n\text{ for each }Q\in\mathcal{Q}.$$

By Lemma 2.6, with high probability we have

$$|F| \leq 2\mathbb{E}(|F|) < \frac{2\sqrt{\varepsilon}}{r^2!}n, \tag{8}$$

and

$$|L_Q \cap F| \geq \frac{1}{2}\mathbb{E}(|L_Q \cap F|) \geq \frac{\varepsilon}{2}n \text{ for each } Q \in \mathcal{Q}. \tag{9}$$

Let Y be the number of intersecting pairs of members in F. Then

$$\mathbb{E}(Y) \leq p^2\binom{n}{r^2}r^2\binom{n}{r^2-1} \leq \frac{\varepsilon n}{(r^2-1)!(r^2-1)!}.$$

By Markov's bound, the probability that $Y \leq \frac{\varepsilon n}{(r^2-1)!(r^2-1)!}$ is at least $\frac{1}{2}$. Therefore, we can find a family F of r^2-sets satisfying (8) and (9), and having at most $\frac{\varepsilon n}{(r^2-1)!(r^2-1)!}$ intersecting pairs. Removing one set from each of the intersecting pairs in F, we get a family F' such that $|F'| \leq |F| < \frac{2\sqrt{\varepsilon}}{r^2!}n$ and for all $Q \in \mathcal{Q}$ we have

$$|L_Q \cap F'| \geq \frac{\varepsilon}{2}n - \frac{\varepsilon n}{(r^2-1)!(r^2-1)!} > \frac{\varepsilon}{r}n. \tag{10}$$

Assume that $|S| \leq \xi n$ and $X \subseteq V(G) \setminus S$ is a set with size at most $2\xi n$. By a similar discussion as Case 2, we can prove that for any vertex $w \in S$, there exist $\Omega(n^{r-1})$ copies of K_r in $G - X$ with $V(K_r) \cap S = \{w\}$. Thus, for any set $T \subseteq S$ with size at most $r-1$, we can find a set $T' \subseteq V(G) \setminus (S \cup X)$ with size at most $(r-1)^2$ such that $G[T \cup T']$ contains a perfect K_r-tiling.

Let M' denote the disjoint union of the sets in F'. Define $M := M'$ if $|S| \geq \xi n$ and $M := M' \cup S \cup S'$ otherwise, where $|S'| \leq (r-1)^2$ and $S' \subseteq V(G) \setminus (S \cup M')$ such that $G[(M' \setminus S) \cup S']$ contains a perfect K_r-tiling. Then $|M| = |F'|r^2 + \xi n + (r-1)^2 \leq 2\xi n$. Since F' consists of disjoint K_r-absorbers and each K_r-absorber is covered by a perfect K_r-tiling, $G[M]$ contains a perfect K_r-tiling. Now, let $W \subseteq V(G) \setminus M$ be a set of at most εn vertices such that $|W| = r\ell$ for some $\ell \in \mathbb{N}$. We arbitrarily partition W into r-sets $Q_1, \ldots, Q_\ell$. Notice that $Q_i \in \mathcal{Q}$ for each $i \in [\ell]$ as $S \subseteq M$ if $|S| \leq \xi n$. Together with (10), we are able to find a different K_r-absorber from $L_{Q_i} \cap F'$ for each Q_i with $i \in [\ell]$. Therefore, $G[M \cup W]$ contains a K_r-factor, as desired. □

5 Extremal case

We begin by introducing some necessary parameters and definitions central to our proof. Let r, s, n be three positive integers satisfying

$$r \geq 3,\ 1 \leq s \leq r \text{ and } r \mid n. \tag{11}$$

Define constants $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_r, \gamma_{r+1}$ satisfying

$$0 < \frac{1}{n} \ll \gamma \ll \gamma_1 \ll \cdots \ll \gamma_r \ll \gamma_{r+1} = \frac{1}{r}. \tag{12}$$

When the constant s is fixed, we define

$$\gamma_s \ll \alpha \ll \beta' \ll \beta \ll \gamma_{s+1}. \tag{13}$$

Let G be an n-vertex graph with $r \mid n$. We say a vertex partition $\mathcal{P} := \{A_1, \ldots, A_s, B\}$ is an (r,s)-*partition* of G if each A_i with $i \in [s]$ has size exactly $\frac{n}{r}$. Given a constant $\delta > 0$, we define the following three types of vertices with respect to an (r,s)-partition $\mathcal{P}$ of G.

- For a vertex $x \in A_i$ with $i \in [s]$, we say x is (δ, A_i)-*bad* (or simply (δ, i)-bad) if $d(x, A_i) \geq \delta n$. Let $V_b(\mathcal{P}, \delta, i)$ be the set of (δ, i)-bad vertices.

- For a vertex $x \in V(G) \setminus A_i$ with $i \in [s]$, we say that x is (δ, A_i)-*exceptional* (or simply (δ, i)-exceptional) if $d(x, A_i) \leq \delta n$. Let $V_{ex}(\mathcal{P}, \delta, i)$ be the set of (δ, i)-exceptional vertices.

- For a vertex $x \in V(G) \setminus A_i$ with $i \in [s]$, we say that x is (δ, A_i)-*excellent* (or simply (δ, i)-excellent) if $d(x, A_i) \geq |A_i| - \delta n$. Let $V_e(\mathcal{P}, \delta, i)$ be the set of (δ, i)-excellent vertices, and let $V_{ne}(\mathcal{P}, \delta, i) := V(G) \setminus (A_i \cup V_e(\mathcal{P}, \delta, i))$.

- A vertex $x \in V(G) \setminus B$ is called (δ, B)-*excellent* if $d(x, B) \geq |B| - \delta n$. Denote by $V_e(\mathcal{P}, \delta, B)$ the set of all such (δ, B)-excellent vertices, and let $V_{ne}(\mathcal{P}, \delta, B) := V(G) \setminus (B \cup V_e(\mathcal{P}, \delta, B))$.

When the partition $\mathcal{P}$ is clear from context and $\bullet \in \{b, ex, e, ne\}$, we abbreviate $V_\bullet(\mathcal{P}, \delta, i)$ to $V_\bullet(\delta, i)$ when no ambiguity arises. Define $S := \{x \in V(G) : d(x) < (1 - \frac{1}{r})n - 1\}$, and denote

$$V_\bullet^S(\beta, i) := V_\bullet(\beta, i) \cap S \text{ and } V_\bullet^L(\beta, i) := V_\bullet(\beta, i) \setminus S.$$

By the Pigeonhole Principle, we obtain the following fact, which will be used frequently in the subsequent proof.

Fact 5.1. *Let G be an n-vertex graph with an (r, s)-partition $\mathcal{P} = \{A_1, \ldots, A_s, B\}$. If $d(v) > (1 - \frac{2}{r} + 2\delta)n$ for some vertex $v \in V(G)$, then there exists at most one $i \in [s]$ such that $d(v, A_i) \leq \delta n$.*

We divide the proof of Theorem 3.2 into two parts. In Section 5.1, we show that for any n-vertex graph G satisfying $\sigma(G) \geq 2\left(1 - \frac{1}{r}\right)n - 2$, the vertex set of G admits a "good partition" (see Lemma 5.3). Then, in Section 5.2, under such a partition, we construct a K_r-tiling that covers all non-excellent vertices (see Lemma 5.4). Furthermore, the remaining graph still preserves an (r, s)-partition. By using those two lemmas, we provide a proof of Theorem 3.2 in Section 5.3. The proofs of Lemma 5.3 and Lemma 5.4 will be given in Section 6 and Section 7, respectively.

5.1 Partition of graphs

We aim to establish a partition of the graph G that exhibits certain desirable properties. For this, we introduce the following definition.

Definition 5.2 (Good partition)**.** Let G be an n-vertex graph. Given constants $0 \leq \delta \ll \zeta \ll 1/r$ and a vertex subset $S \subseteq V(G)$, an (r, s)-partition $\mathcal{Q} = \{A_1, \ldots, A_s, B\}$ with $S \subseteq B$ is called $(r, s; \delta, \zeta; S)$-*good* if there exist constants α, β', β with $\delta \ll \alpha \ll \beta' \ll \beta \ll \zeta$ such that the following conditions hold.

(A1) For each $i \in [s]$, A_i is a $\sqrt{\alpha}$-independent set of size $\frac{n}{r}$ in G.

(A2) If $|B| \geq \frac{2n}{r}$, then $G[B]$ does not contain any $\frac{\zeta^2}{4}$-independent set of size $\frac{n}{r}$.

(A3) For each $i \in [s]$, $|V_b(2\beta, i)| \leq \sqrt{\alpha}n$ and $|V_{ne}(2\beta', i)| \leq \sqrt{\alpha}n$.

(A4) For each $i \in [s]$, either $V_b(2\beta, i) = \emptyset$ or $V_{ex}^L(\beta/2, i) = \emptyset$. Moreover, $G[V_{ex}^L(\beta/2, i), A_i]$ contains a matching M_i covering $V_{ex}^L(\beta/2, i)$ such that $V(M_i) \cap V(M_j) = \emptyset$ for $i \neq j$ (here $M_i = \emptyset$ if $V_{ex}^L(\beta/2, i) = \emptyset$).

Lemma 5.3. *Given the parameters defined in (11)-(13), let G be an n-vertex graph with $\sigma(G) \geq 2\big(1 - \frac{1}{r}\big)n - 2$ and suppose G has a γ-independent set of size $\frac{n}{r}$. Then either G is the extremal graph (EX1), or there exists an $(r, s; \gamma_s, \gamma_{s+1}; S)$-good partition for some $s \in [r]$, where $S := \big\{x \in V(G) : d(x) < (1 - \frac{1}{r})n - 1\big\}$.*

5.2 Covering non-excellent vertices

Note that Lemma 5.3 yields an $(r, s; \gamma_s, \gamma_{s+1}; S)$-good partition of G. Under such a partition, there will be some vertices that disrupt the uniformity–specifically, the non-excellent vertices. To address this issue, we introduce the following lemma to cover such vertices.

Lemma 5.4. *Given the parameters defined in (11)-(13), let G be an n-vertex graph with $\sigma(G) \geq 2\left(1 - \frac{1}{r}\right)n - 2$ and suppose G has a γ-independent set of size $\frac{n}{r}$. Let $\mathcal{Q}$ be an $(r, s; \gamma_s, \gamma_{s+1}; S)$-good partition of G for some $s \in [r]$, where $S := \{x \in V(G) : d(x) < (1 - \frac{1}{r})n - 1\}$. Then one of the following holds.*

(B1) *G is the extremal graph (EX2).*

(B2) *There exists a K_r-tiling $\mathcal{T}$ with at most $5r\alpha^{1/4}n$ vertices such that*

- *$\bigcup_{i\in[s]} V_{ne}(2\beta', i) \cup V_{ne}(2\beta', B) \subseteq V(\mathcal{T})$;*
- *the partition $\mathcal{Q}' = \{A'_1, \ldots, A'_s, B'\}$, which is obtained from $\mathcal{Q}$ by deleting all vertices in $\mathcal{T}$, is an (r, s)-partition of $G - V(\mathcal{T})$;*
- *and there exists a K_{r-s}-factor in $G[B']$.*

5.3 Proof of Theorem 3.2

Based on Lemma 5.3 and Lemma 5.4, we give the proof of Theorem 3.2.

Proof of Theorem 3.2. Given the parameters defined in (11)-(13), let G be an n-vertex graph with $\sigma(G) \geq 2(1-\frac{1}{r})n-2$. Assume that G has a γ-independent set of size $\frac{n}{r}$. By Lemma 5.3, either G has an independent set of size $\frac{n}{r} + 1$ or G has an $(r, s; \gamma_s, \gamma_{s+1}; S)$-good partition $\mathcal{Q}$ for some $s \in [r]$, where $S := \{x \in V(G) : d(x) < (1-\frac{1}{r})n-1\}$. For the former case, G is the extremal graph (EX1). For the latter case, together with Lemma 5.4, we conclude that either G is the extremal graph (EX2) or (B2) holds.

Assume that there exists a K_r-tiling $\mathcal{T}$ satisfying (B2). Let $\mathcal{Q}' = \{A'_1, \ldots, A'_s, B'\}$ be the (r, s)-partition obtained from $\mathcal{Q}$. Then for each $D' \in \mathcal{Q}'$ and each $v \notin D'$ we have

$$d(v, D') \geq |D'| - 2\beta' n - 5r\alpha^{1/5}n \geq |D'| - \beta n. \tag{14}$$

Notice that there exist $t := \frac{|B'|}{r-s}$ vertex-disjoint copies of K_{r-s}, say $K^1, \ldots, K^t$, in $G[B']$. Contracting each K^i (where $i \in [t]$) into a vertex w_i yields a new vertex set B^* of size t. We are to construct an auxiliary graph G^* with $V(G^*) = \left(\bigcup_{i\in[s]} A'_i\right) \cup B^*$, and $xy \in E(G^*)$ if and only if

- $x \in A'_i$, $y \in A'_j$, and $xy \in E(G)$ for $i, j \in [s]$ with $i \neq j$; or
- $x = w_i \in B^*$, $y \in A'_j$, and $y \in N(V(K^i))$ for $j \in [s]$ and $i \in [t]$.

Thus, $\mathcal{Q}^* = \{A'_1, \ldots, A'_s, B^*\}$ is a $(s + 1, s)$-partition of G^*. Then (14) implies that

$$d_{G^*}(v, D^*) \geq |D^*| - (r - s)\beta n \geq \left(1 - \frac{1}{2(s + 1)}\right)|D^*| \text{ for each } D^* \in \mathcal{Q}^* \text{ and each } v \notin D^*.$$

By Lemma 2.5, we conclude that G^* admits a K_{s+1}-factor. Consequently, $G - V(\mathcal{T})$ admits a K_r-factor. Combining this with $\mathcal{T}$ yields the desired K_r-factor in G. □

6 Proof of Lemma 5.3

In this section, we prove Lemma 5.3 by constructing the required good partition of G.

Proof of Lemma 5.3. Given the parameters defined in (11)-(13), let G be an n-vertex graph with $\sigma(G) \geq 2\left(1-\frac{1}{r}\right)n-2$ and G has a γ-independent set of size $\frac{n}{r}$. Denote $S := \{x \in V(G) : d(x) < (1-\frac{1}{r})n-1\}$. We first show that G admits an (r,s)-partition for some $s \in [r]$.

Claim 6.1. *There exists an integer $s \in [r]$ such that G has an (r,s)-partition $\mathcal{P} := \{A_1,\ldots,A_s,B\}$ with $S \subseteq B$. Moreover, A_i is a γ_i-independent set in G for each $i \in [s]$ and $G[B \setminus S]$ contains no γ_{s+1}-independent set of size $\frac{n}{r}$.*

Proof of Claim 6.1. It follows from Fact 2.1 that $G[S]$ is a clique. Let A be a γ-independent set of size $\frac{n}{r}$ in G. Then,

$$\binom{|A\cap S|}{2} = e(G[A\cap S]) \leq e(G[A]) \leq \gamma n^2.$$

Thus, $|A \cap S| \leq 2\sqrt{\gamma}n$. Let A' be a subset of $V(G) \setminus (A \cup S)$ with size exactly $|A \cap S|$, and let $A_1 = A' \cup (A \setminus S)$. Since $\gamma \ll \gamma_1$, we have

$$e(G[A_1]) \leq e(G[A]) + \frac{n}{r}|A'| \leq (\gamma + 2\sqrt{\gamma})n^2 < \gamma_1 n^2,$$

that is, A_1 is a γ_1-independent set of size $\frac{n}{r}$ in $G-S$.

Next, we iteratively select all possible sets A_i satisfying $A_i \cap S = \emptyset$, $|A_i| = \frac{n}{r}$ and A_i is a γ_i-independent set in $G - \bigcup_{j\in[i-1]} A_j$, until no such sets remain. Suppose the process terminates after s steps, resulting in sets $A_1, A_2, \ldots, A_s$. We denote the remaining vertices by B. Clearly, $G[B \setminus S]$ contains no γ_{s+1}-independent set of size $\frac{n}{r}$, as desired. □

Let $\mathcal{P}_0 := \{A_1^0,\ldots,A_s^0,B^0\}$ be an (r,s)-partition of G obtained in Claim 6.1. Next, we estimate the number of bad vertices and non-excellent vertices under the partition $\mathcal{P}_0$. For each $i \in [s]$, since A_i^0 is a γ_i-independent set in G and $\gamma_i \ll \alpha \ll \beta$, we have

$$|V_b(\mathcal{P}_0,\beta,i)| \leq \frac{2e(G[A_i^0])}{\beta n} \leq \frac{2\gamma_i n^2}{\beta n} \leq \alpha n.$$

Observe that at least $\binom{|A_i^0|}{2} - \gamma_i n^2$ edges are missing in $G[A_i^0]$. By Ore's condition, the number of edges in $G[A_i^0, V(G)\setminus A_i^0]$ is at least

$$\sum_{x\in A_i^0} d(x) - 2e(G[A_i^0]) \geq \frac{1}{|A_i^0|}\cdot e\left(\overline{G[A_i^0]}\right)\cdot\sigma(G) - 2e\left(G[A_i^0]\right) \geq |A_i^0||V(G)\setminus A_i^0| - 4\gamma_i n^2.$$

Since $\gamma_i \ll \alpha \ll \beta' \ll \beta$, for each $i \in [s]$, we have that

$$|V_{ex}(\mathcal{P}_0,\beta,i)| \leq |V_{ne}(\mathcal{P}_0,\beta',i)| \leq \frac{4\gamma_i n^2}{\beta' n} \leq \alpha n.$$

In the following part, we are to construct an $(r,s;\gamma_s,\gamma_{s+1};S)$-good partition by utilizing the following operations. Recall that $\mathcal{P}_0 = \{A_1^0,\ldots,A_s^0,B^0\}$. For each $k \in [s]$, we recursively construct an (r,s)-partition $\mathcal{P}_k = \{A_1^k,\ldots,A_s^k,B^k\}$ of G from the previous partition $\mathcal{P}_{k-1} = \{A_1^{k-1},\ldots,A_s^{k-1},B^{k-1}\}$. Let

$$X_k := V_{ex}^L(\mathcal{P}_{k-1},\beta+(k-1)\alpha,k),\ Y_k := V_b(\mathcal{P}_{k-1},\beta+(k-1)\alpha,k),\ \text{and } H_k := G[(A_k^{k-1}\cup X_k)\setminus Y_k].$$

Denote $t_k := \min\{|X_k|,|Y_k|\}$. Assume $\{x_1^k,\ldots,x_{t_k}^k\} \subseteq X_k$ and $\{y_1^k,\ldots,y_{t_k}^k\} \subseteq Y_k$. We now perform the following steps on $\mathcal{P}_{k-1}$ to construct the partition $\mathcal{P}_k$.

- For each pair of vertices x_j^k, y_j^k with $j \in [t_k]$, we move x_j^k to A_k^{k-1} and reassign y_j^k to the original part that x_j^k belongs to (see Figure 5).
- If $|X_k| > |Y_k|$, then we will find a matching M_k of size $c_k := |X_k| - |Y_k|$ in H_k. Otherwise, there exists an independent set of size $\frac{n}{r} + 1$.
- A suitable exchange of at most c_k vertices between X_k and A_k^{k-1} (denote the resulting set by A_k^k) ensures that each edge in M_k has exactly one endpoint in A_k^k, which yields the partition $\mathcal{P}_k = \{A_1^k, \ldots, A_s^k, B^k\}$.

The details of the last two steps and the verification that $\mathcal{P}_s$ satisfies Definition 5.2 will be provided in the subsequent claim.

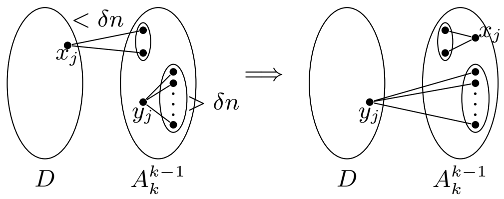


Figure 5: Vertices x_j, y_j exchanged in the k-th step, where $\delta = \beta + (k-1)\alpha$ and $D \in \mathcal{P}_{k-1} \setminus A_k^{k-1}$.

Claim 6.2. *Either G contains an independent set of size $\frac{n}{r}+1$, or for each $k \in [s]$, there exists an (r,s)-partition $\mathcal{P}_k := \{A_1^k, \ldots, A_s^k, B^k\}$ satisfying the following.*

(C1) *For each $i \in [s]$, A_i^k is a $(2^k-1)\alpha$-independent set of size $\frac{n}{r}$ in G.*

(C2) *If $|B^k| \geq \frac{2n}{r}$, then $G[B^k \setminus S]$ contains no $(\gamma_{s+1} - (2^k-1)\alpha)$-independent set of size $\frac{n}{r}$.*

(C3) *For each $i \in [s]$, we have*

$$|V_b(\mathcal{P}_k, \beta + (2^k-1)\alpha, i)| \leq 2^k \alpha n \text{ and } |V_{ne}(\mathcal{P}_k, \beta' + (2^k-1)\alpha, i)| \leq 2^k \alpha n.$$

(C4) *For each $i \in [k]$, there exists a matching M_i of size c_i in H_i such that each edge in M_i has exactly one endpoint in A_i^k, and $V(M_i) \cap V(M_j) = \emptyset$ for all $i \neq j$. If $M_i \neq \emptyset$, then there are no $(\beta + (2^k-1)\alpha, i)$-bad vertices in A_i^k.*

(C5) *For each $i \in [k]$, there are no $(\beta - (2^k-1)\alpha, i)$-exceptional vertices in $V(G) \setminus (V(H_i) \cup S)$.*

Proof of Claim 6.2. We prove this claim by induction on k. We first consider the case that $k=1$. Observe that for each $i \in [s]$, the number of vertices exchanged from $\mathcal{P}_0$ to $\mathcal{P}_1$ within A_i^1 (resp. B^1) is bounded by $|V_{ex}^L(\mathcal{P}_0, \beta, 1)| \leq \alpha n$. Thus, under the partition $\mathcal{P}_1$, for each $i \in [s]$, we have

$$e(G[A_i^1]) \leq \gamma_i n^2 + \alpha n \cdot \frac{n}{r} \leq \alpha n^2,$$

and for any set $B' \subseteq B^1 \setminus S$ of size $\frac{n}{r}$, we have

$$e(G[B']) \geq \gamma_{s+1} n^2 - \alpha n \cdot \frac{n}{r} \geq (\gamma_{s+1} - \alpha) n^2.$$

Therefore, A_i^1 is an α-independent set of size $\frac{n}{r}$ for each $i \in [s]$, and $G[B^1 \setminus S]$ contains no $(\gamma_{s+1} - \alpha)$-independent set of size $\frac{n}{r}$. Hence, (C1) and (C2) hold.

For each vertex $v \in V(G)$, it is straightforward to check that for each $i \in [s]$,

$$d(v, A_i^0) - \alpha n \leq d(v, A_i^1) \leq d(v, A_i^0) + \alpha n. \tag{15}$$

Therefore, if v is not moved in the first step and $v \notin V_{ne}(\mathcal{P}_0, \beta', i) \cup V_b(\mathcal{P}_0, \beta, i)$, then $v \notin V_{ne}(\mathcal{P}_1, \beta' + \alpha, i) \cup V_b(\mathcal{P}_1, \beta + \alpha, i)$. Thus, for each $i \in [s]$ we have

$$|V_{ne}(\mathcal{P}_1, \beta' + \alpha, i)| \leq |V_{ne}(\mathcal{P}_0, \beta', i)| + \alpha n \leq 2\alpha n \text{ and } |V_b(\mathcal{P}_1, \beta + \alpha, i)| \leq |V_b(\mathcal{P}_0, \beta, i)| + \alpha n \leq 2\alpha n.$$

Hence (C3) holds.

Note that $|V(H_1)| = \frac{n}{r} + c_1$. If $c_1 = 1$, then either H_1 has a matching M_1 of size c_1 or $V(H_1)$ is an independent set of size $\frac{n}{r} + 1$. If $c_1 \geq 2$, then since $d_G(v) \geq (1 - \frac{1}{r})n - 1$ for each $v \in V(H_1)$, we have $\delta(G[H_1]) \geq c_1 - 1$. Recall that $H_1 = G[(A_1^0 \cup X_1) \setminus Y_1]$, (15) implies that for each $v \in V(H_1)$ we have

$$d_{H_1}(v) \leq \beta n + |V_{ex}^L(\mathcal{P}_0, \beta, 1)| \leq (\beta + \alpha)n,$$

that is, $\Delta(H_1) \leq (\beta + \alpha)n$. Combining with Lemma 2.9, one has that H_1 contains a matching M_1 of size c_1. Notice that the number of matching edges of M_1 within $A_1^0 \cup \{x_1^1, \dots, x_{t_1}^1\}$ is the same as the number of matching edges of M_1 within $X_1 \setminus \{x_1^1, \dots, x_{t_1}^1\}$. Consequently, we can perform vertex exchanges in $V(H_1)$ to ensure that under the partition $\mathcal{P}_1$, every $e \in E(M_1)$ has exactly one endpoint in A_1^1 (see Figure 6). This can be achieved by exchanging at most c_1 vertices. Furthermore, if $M \neq \emptyset$, then we have $A_1^1 \subseteq V(H_1)$, that is, $V_b(\mathcal{P}_1, \beta + \alpha, 1) = \emptyset$. Thus, (C4) holds.

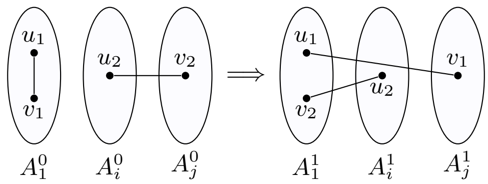


Figure 6: Here, $u_1v_1, u_2v_2 \in E(M_1)$ with $u_1, v_1 \in A_1^0 \cup \{x_1^1, \dots, x_{t_1}^1\}$, $u_2, v_2 \in X_1 \setminus \{x_1^1, \dots, x_{t_1}^1\}$. Exchange v_1 and v_2 to guarantee that every edge in M_1 has exactly one endpoint in A_1^1.

Let v be a vertex in $V(G) \setminus (V(H_1) \cup S)$. If $d(v, A_1^0) \geq \beta n$, then $d(v, A_1^1) \geq (\beta - \alpha)n$, that is, there are no $(\beta - \alpha, i)$-exceptional vertices in $V(G) \setminus V(G) \setminus (V(H_i) \cup S)$ for partition $\mathcal{P}_1$. Thus, (C5) holds. Hence, (C1)-(C5) hold when $k = 1$. Next, we assume that (C1)-(C5) hold for $k - 1$, and we will prove that $\mathcal{P}_k$ satisfies all of them.

By induction, for each $i \in [s]$, we know that A_i^{k-1} is a $(2^{k-1} - 1)\alpha$-independent set and

$$\left|V_{ex}(\mathcal{P}_{k-1}, \beta, i)\right| \leq \left|V_{ne}^L(\mathcal{P}_{k-1}, \beta + (2^{k-1} - 1)\alpha, i)\right| \leq 2^{k-1}\alpha n.$$

Hence, at most $2^{k-1}\alpha n$ vertices are exchanged in the k-th step. Thus, we have

$$d(x, A_i^{k-1}) - 2^{k-1}\alpha n \leq d(x, A_i^k) \leq d(x, A_i^{k-1}) + 2^{k-1}\alpha n. \tag{16}$$

By an argument analogous to that for $\mathcal{P}_1$, (C1)-(C3) also hold for $\mathcal{P}_k$. Hence, it remains to prove (C4) and (C5).

For (C4), the same method as for $k = 1$ yields an independent set of size $\frac{n}{r} + 1$ or a matching M_k of size c_k in H_k such that $V_b(\mathcal{P}_k, \beta + (2^k - 1)\alpha, k) = \emptyset$ whenever $M_k \neq \emptyset$. By induction, for each $i \in [k-1]$ with $M_i \neq \emptyset$, there are no $(\beta + (2^{k-1} - 1)\alpha, i)$-bad vertices in A_i^{k-1}. Given that $d(v) \geq \left(1 - \frac{1}{r}\right)n - 1$ for every $v \in V(G) \setminus S$, and by Fact 5.1, it follows that no vertex in A_i is exchanged in the k-th step. Therefore, $V_b\big(\mathcal{P}_k, \beta + (2^k - 1)\alpha, i\big) = \emptyset$, and every edge in M_i has exactly one endpoint in A_i^k. Moreover, by induction, the inequality (16) implies that

$$d(x_i, A_i^k) \leq (\beta + (2^{k-1} - 1)\alpha)n + 2^{k-1}\alpha n \leq (\beta + (2^k - 1)\alpha)n$$

for each $i \in [k-1]$ and each $x_i \in V(H_i)$. For H_k with any $x_k \in V(H_k)$, we have

$$d(x_k, A_k^k) \leq (\beta + (k-1)\alpha)n + 2^{k-1}\alpha n \leq (\beta + (2^k - 1)\alpha)n.$$

Combining with Fact 5.1, since $d(v) \geq (1-\frac{1}{r})n-1$ for any $v \in V(G) \setminus S$, we have that $V(H_i) \cap V(H_j) = \emptyset$ for all $i \neq j \in [k]$, which implies $V(M_i) \cap V(M_j) = \emptyset$ for all $i \neq j \in [k]$. Thus, (C4) holds.

By a similar argument, we have that there are no $(\beta-(2^k-1)\alpha, k)$-exceptional vertices in $V(G) \setminus (V(H_k) \cup S)$. For each $i \in [k-1]$, let $v \in V(G) \setminus (V(H_i) \cup S)$. By induction, we have $d(v, A_i^{k-1}) \geq (\beta-(2^{k-1}-1)\alpha)n$. Thus, (16) implies that

$$d(v, A_i^k) \geq (\beta-(2^{k-1}-1)\alpha)n - 2^{k-1}\alpha n \geq (\beta-(2^k-1)\alpha)n.$$

Hence, (C5) holds. $\square$

By Claim 6.2, either G has an independent set of size $\frac{n}{r}+1$, or it admits an (r,s)-good partition. In the former case, the proof is complete. We may thus assume the latter, and let $\mathcal{Q} := \{A_1, \ldots, A_s, B\}$ be the partition $\mathcal{P}_s$ obtained by Claim 6.2. We will show that $\mathcal{Q}$ is the desired good partition. Clearly, (A1) follows by (C1) and the fact that $(2^s-1)\alpha \leq \sqrt{\alpha}$. By (C4), there exist s vertex-disjoint matchings $M_1, \ldots, M_s$. Thus (C4) and (C5) implies (A4).

It is routine to check that for two constants μ_1, μ_2 with $\mu_1 \leq \mu_2$, for each $i \in [s]$ we have

$$|V_{ne}(\mu_1, i)| \geq |V_{ne}(\mu_2, i)|,\ |V_b(\mu_1, i)| \geq |V_b(\mu_2, i)| \text{ and } |V_{ex}(\mu_2, i)| \geq |V_{ex}(\mu_1, i)|.$$

Hence, (A3) holds by (C3).

Finally, we prove (A2) by considering the following two cases: $|B \setminus S| \geq \frac{n}{r}$ and $|B \setminus S| < \frac{n}{r}$. We start with $|B \setminus S| \geq \frac{n}{r}$. Let $B' \subseteq B$ be an arbitrary subset of size $\frac{n}{r}$, and let $C \subseteq B \setminus (S \cup B')$ be a set with size $|B' \cap S|$. If $|B' \cap S| \geq \gamma_{s+1} n$, then

$$e(G[B']) \geq e(G[B' \cap S]) \geq \frac{\gamma_{s+1}^2}{4} n^2.$$

If $|B' \cap S| < \gamma_{s+1} n$, then

$$e(G[B']) - e(G[(B' \setminus S) \cup C]) \geq e(G[B' \setminus S, B' \cap S]) - e(G[B' \setminus S, C]) \geq -|B' \setminus S||C| \geq -\frac{n}{r} \cdot \gamma_{s+1} n.$$

Therefore, by (C2) we have

$$e(G[B']) \geq e(G[(B' \setminus S) \cup C]) - \frac{n}{r} \cdot \gamma_{s+1} n \geq (\gamma_{s+1} - (2^s-1)\alpha) n^2 - \frac{n}{r} \cdot \gamma_{s+1} n \geq \frac{\gamma_{s+1}^2}{4} n^2.$$

That is, B' is not a $\frac{\gamma_{s+1}^2}{4}$-independent set in G.

Now, we consider that $|B \setminus S| < \frac{n}{r}$. It is easy to see that if $|B \setminus S| \leq \frac{n}{2r}$, then (A2) holds immediately. Hence we may assume that $|B \setminus S| > \frac{n}{2r}$ in the following. Since $|B| \geq \frac{2n}{r}$, we obtain $|S| > \frac{n}{r}$. Let v be a vertex in S. If v is $(2\beta', i)$-excellent for all $i \in [s]$, then $d(v, B \setminus S) < 2r\beta' n$, as $d(v) < (1-\frac{1}{r})n - 2$. It follows that

$$e(G[B \setminus S, S]) \leq d(v, B \setminus S) \cdot |S| + |B \setminus S| \cdot \Big| \bigcup_{i \in [s]} V_{ne}(2\beta', i) \Big| \leq 2r\beta' n \cdot |S| + \frac{n}{r} \cdot r\sqrt{\alpha} n = 2r\beta' n \cdot |S| + \sqrt{\alpha} n^2.$$

Thus, we obtain that

$$\begin{aligned} 2e(G[B \setminus S]) &\geq \Big((1-\frac{1}{r})n - 1 - \frac{r-2}{r} n \Big) |B \setminus S| - e(G[S, B \setminus S]) \\ &\geq \Big(\frac{n}{r} - 1 \Big) \frac{n}{2r} - 2r\beta' n \Big(\frac{n}{r} + \beta n + 1 \Big) - \sqrt{\alpha} n^2 \\ &\geq \frac{\gamma_{s+1}^2}{2} n^2. \end{aligned}$$

Therefore, any subset of B with size $\frac{n}{r}$ is not a $\frac{\gamma_{s+1}^2}{4}$-independent set in G. Hence (A2) holds. $\square$

7 Proof of Lemma 5.4

With a good partition of G provided by Lemma 5.3, our task in this section is to construct a K_r-tiling that covers all non-excellent vertices under the partition. Throughout, we assume the following hypothesis:

(†) Given the parameters defined in (11)-(13), let G be an n-vertex graph with $\sigma(G) \geq 2\left(1 - \frac{1}{r}\right) n - 2$ and suppose G has a γ-independent set of size $\frac{n}{r}$. Let $\mathcal{Q} := \{A_1, \dots, A_s, B\}$ be an $(r, s; \gamma_s, \gamma_{s+1}; S)$-good partition of G for some $s \in [r]$, where $S := \{x \in V(G) : d(x) < (1 - \frac{1}{r})n - 1\}$.

We first define some necessary notation and provide several properties of the $(r, s; \gamma_s, \gamma_{s+1}; S)$-good partition $\mathcal{Q}$. Define

$$V_{ex}(\beta) := \bigcup_{i=1}^{s} V_{ex}(\beta, i),\ \ V_{ex}^S(\beta) := V_{ex}(\beta) \cap S \text{ and } V_{ex}^L(\beta) := V_{ex}(\beta) \setminus V_{ex}^S(\beta);$$
$$V_e(2\beta') := \bigcap_{D \in \mathcal{Q}} (V_e(2\beta', D) \cup D),\ \ V_e^S(2\beta') := V_e(2\beta') \cap S \text{ and } V_e^L(2\beta') := V_e(2\beta') \setminus V_e^S(2\beta').$$

Let $V_{ne}(2\beta') := V(G) \setminus V_e(2\beta')$. We claim that

$$|V_{ne}(2\beta', B)| \leq \alpha^{1/4} n.$$

Otherwise, suppose that $|V_{ne}(2\beta', B)| > \alpha^{1/4} n$. By the Pigeonhole Principle, there exists some $i \in [s]$ such that there are at least $\frac{\alpha^{1/4} n}{s}$ vertices $v \in A_i$ satisfying $d(v, B) < |B| - 2\beta' n$. Since $d(v) \geq (1 - \frac{1}{r})n - 1$ for each $v \in A_i$, we obtain that

$$|E(G[A_i])| \geq \frac{1}{2} \cdot \frac{\alpha^{1/4} n}{s} \cdot (2\beta' n - 1) \geq \sqrt{\alpha} n^2,$$

which contradicts (A1) of Definition 5.2. Combining with (A3), we get

$$|V_{ne}(2\beta')| \leq 2\alpha^{1/4} n. \tag{17}$$

To prove Lemma 5.4, we employ a critical structure, called a *base*, that covers all non-excellent vertices and whose extension property ensures the existence of a vertex-disjoint K_r-tiling.

Definition 7.1 (Base)**.** Given integers n, r, s with $n \gg r \geq 3$ and a positive constant ξ, let G be an n-vertex graph and $\mathcal{P}$ be an (r, s)-partition of G.

- We say a clique C is a ξ-*base* in G if for each $D \in \mathcal{P}$ we have
  (D1) $|V(C) \cap D| \leq \frac{r|D|}{n}$,
  (D2) $|N(V(C)) \cap D| \geq |D| - (|V(C) \cap D| + 1) \cdot \frac{n}{r} + \xi n$.
- We say a pair of vertex-disjoint cliques C_0 and C_1 is a $(2, \xi)$-*base* in G if there exist two different parts $D_0, D_1 \in \mathcal{P}$ satisfying the following conditions.
  (D3) For each $i \in \mathbb{Z}_2$, we have $|V(C_i) \cap D_i| = \frac{r|D_i|}{n} + 1$ and $|V(C_i) \cap D| \leq \frac{r|D|}{n}$ for each $D \in \mathcal{P} \setminus \{D_i\}$.
  (D4) For each $i \in \mathbb{Z}_2$, (D2) holds for all $D \in \mathcal{P} \setminus \{D_{i+1}\}$, and
  $$|N(V(C_i)) \cap D_{i+1}| \geq |D_{i+1}| - (|V(C) \cap D_{i+1}| + 2) \cdot \frac{n}{r} + \xi n.$$

A collection $\mathcal{H}$ of vertex-disjoint subgraphs of G is called a *ξ-base set* if for each $H \in \mathcal{H}$, there exists a parameter $\tau \geq \xi$ for which H is either a τ-base or a $(2,\tau)$-base. Here, (D2) and (D4) provide the fundamental criterion for extending a base structure to a copy of K_r, while (D1) and (D3) ensure that after extending the base set to a K_r-tiling, the partition $\mathcal{P}$ retains its (r,s)-partition property in the remaining graph.

In the following, we split the proof of Lemma 5.4 into two parts. In Section 7.1, we construct a $\beta/4$-base set that covers all vertices in $V_{ne}(2\beta')$. In Section 7.2, we extend the base set to a K_r-tiling satisfying (B2).

7.1 Covering vertices in $V_{ne}(2\beta')$

Denote $|V_{ex}^S(\beta/2,i)| = s_i$ for each $i \in [s]$. Let $f(x,y)$ be a function of x and y such that

$$f(x,y) = \begin{cases} 1 & x = 1, \\ \lceil x/(y+1) \rceil + 1 & x \geq 2. \end{cases}$$

Notice that $V_{ne}(2\beta')$ can be partitioned as follows:

$$V_{ne}(2\beta') = V_{ex}(\beta/2) \cup (V_{ne}(2\beta') \setminus V_{ex}(\beta/2)).$$

The subsequent lemma constructs a base set that covers all vertices in $V_{ex}(\beta/2)$.

Lemma 7.2. *Suppose that (†) holds. Then there exists a $\beta/4$-base set $\mathcal{F}$ in G that covers all vertices in $V_{ex}(\beta/2)$.*

Moreover, for any $i \in [s]$ with $s_i > 0$, there exist families $\mathcal{F}_i$ and $\mathcal{F}_i'$ of vertex-disjoint subgraphs, with $V_{ex}^S(\beta/2,i) \subseteq V(\mathcal{F}_i)$, $|\mathcal{F}_i| = \lceil s_i/(r-s+1) \rceil$ and $|\mathcal{F}_i'| = f(s_i, r-s)$, such that each $F \in \mathcal{F}_i$ and $F' \in \mathcal{F}_i'$ satisfy that $F \cup F'$ is a $(2,\beta/4)$-base in G with $D_0 = B$ and $D_1 = A_i$.

Proof. Since $\mathcal{Q} := \{A_1, \ldots, A_s, B\}$ is an $(r,s;\gamma_s,\gamma_{s+1};S)$-good partition of G, (A1)-(A4) in Definition 5.2 hold. Let M_i be the matching given in Definition 5.2 (A4). Recall that $V_{ex}(\beta/2) = V_{ex}^L(\beta/2) \cup V_{ex}^S(\beta/2)$. We will construct two vertex-disjoint $\beta/4$-base sets $\mathcal{F}^L$ and $\mathcal{F}^S$ covering $V_{ex}^L(\beta/2)$ and $V_{ex}^S(\beta/2)$, respectively. The subsequent claim considers $\mathcal{F}^L$.

Claim 7.3. *Let M_i with $i \in [s]$ be the matching given in Definition 5.2 (A4). Define $\mathcal{F}^L := \bigcup_{i\in[s]} E(M_i)$. Then $\mathcal{F}^L$ is a $\beta/4$-base set that covers $V_{ex}^L(\beta/2)$.*

Proof of Claim 7.3. It is clear that $\mathcal{F}^L$ covers all vertices in $V_{ex}^L(\beta/2)$ and any two elements of $\mathcal{F}^L$ are vertex-disjoint. Let uv be an arbitrary edge in $\mathcal{F}^L$. It suffices to show that uv is a $\beta/4$-base in G. Without loss of generality, assume that $u \in V_{ex}^L(\beta/2,i)$ for some $i \in [s]$. Then, $d(u,A_i) \leq \beta n/2$ and $d(v,A_i) \leq 2\beta n$.

By Definition 5.2 (A4), (D1) holds obviously. As $u, v \notin S$, one has

$$d(u, V(G) \setminus A_i) + d(v, V(G) \setminus A_i) \geq 2\Big(1 - \frac{1}{r}\Big)n - 3\beta n.$$

Hence $|N(v,D) \cap N(u,D)| \geq |D| - 3\beta n$ for any $D \in \mathcal{Q} \setminus \{A_i\}$. Thus, (D2) holds when $\xi = \beta/4$. □

Next, we consider vertices in $V_{ex}^S(\beta/2)$, and begin by estimating the degrees of those in $V_{ne}(2\beta', i)$.

Claim 7.4. *If $u \in V_{ne}(2\beta', i)$ for some $i \in [s]$, then $d(u) \geq \left(1 - \frac{1}{r}\right) n - 2\alpha^{1/4} n$.*

Proof of Claim 7.4. Since A_i is a $\sqrt{\alpha}$-independent set, one has

$$\Big|\Big\{v \in A_i : d(v) \leq \Big(1 - \frac{1}{r}\Big)n + \alpha^{1/4}n\Big\}\Big| \geq \Big|\Big\{v \in A_i : d(v,A_i) \leq \alpha^{1/4}n\Big\}\Big| \geq \frac{n}{r} - 2\alpha^{1/4}n.$$

Therefore, for any $u \in V_{ne}(2\beta', i)$, there exists a vertex $v \in A_i \setminus N(u)$ such that $d(v) \leq (1 - \frac{1}{r})n + \alpha^{1/4}n$. By Ore's condition, for any $u \in V_{ex}(\beta/2, i)$, one has

$$d(u) \geq 2\Big(1 - \frac{1}{r}\Big)n - 2 - \Big(1 - \frac{1}{r}\Big)n - \alpha^{1/4}n \geq \Big(1 - \frac{1}{r}\Big)n - 2\alpha^{1/4}n,$$

as desired. □

By Fact 5.1 and Claim 7.4, one has

$$V_{ex}^S(\beta/2,i)\cap V_{ex}^S(\beta/2,j)=\emptyset \text{ for any distinct } i,j\in[s].$$

We now construct the $\beta/4$-base set $\mathcal{F}^S$, in which each base consists of two components: one for covering the vertices in $V_{ex}^S(\beta/2)$, while the other ensures that (D4) is satisfied. The following claim describes the construction of the former one.

Claim 7.5. *For each $i\in[s]$, there exists a family $\mathcal{F}_i$ of $\lceil s_i/(r-s+1)\rceil$ vertex-disjoint K_{r-s+1} copies such that $V_{ex}^S(\beta/2,i)\subseteq V(\mathcal{F}_i)\subseteq (B\cap V_e(2\beta'))\cup V_{ex}^S(\beta/2,i)$. Furthermore, the families $\{\mathcal{F}_i : i\in[s]\}$ are pairwise vertex-disjoint.*

Proof of Claim 7.5. Let $\mathcal{F}_i=\emptyset$ if $s_i=0$ and $i\in[s]$. Next, fix an $i\in[s]$ with $s_i>0$, suppose the desired $\mathcal{F}_k$ have been constructed for all $k<i$. Note that $G[V_{ex}^S(\beta/2,i)]$ is a clique. Consequently, there are $\lfloor s_i/(r-s+1)\rfloor$ vertex-disjoint K_{r-s+1} copies in $G[V_{ex}^S(\beta/2,i)]$. Let T_i denote the set of remaining vertices in $V_{ex}^S(\beta/2,i)$ and denote $t_i:=|T_i|$, where $0\le t_i\le r-s$. If $t_i=0$, then we get the desired family $\mathcal{F}_i$. Next, we consider $t_i>0$. Define $B_{T_i}:=\big(N(T_i,B)\cap V_e(2\beta')\big)\setminus V(\bigcup_{k<i}\mathcal{F}_k)$. We will find a $K_{r-s+1}\subseteq G[B_{T_i}]$ that covers T_i (see Figure 7).

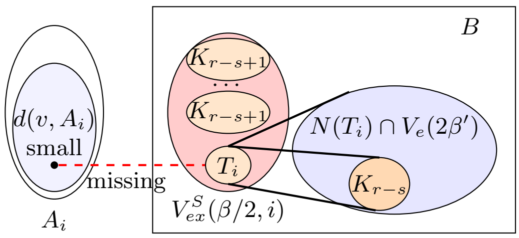


Figure 7: Process of covering $V_{ex}^S(\beta/2,i)$.

By Claim 7.4, one has $d(u,B)\ge |B|-\beta n/2-2\alpha^{1/4}n$ for each $u\in T_i$. Thus,

$$|N(T_i,B)|\ge |B|-t_i\cdot\frac{\beta n}{2}-t_i\cdot 2\alpha^{1/4}n\ge |B|-r\cdot\beta n. \tag{18}$$

By Definition 5.2 (A3), we get that $|B\cap V_e(2\beta')|\ge |B|-r\sqrt{\alpha}n$. Then

$$|B_{T_i}|\ge |B|-r\beta n-r\sqrt{\alpha}n-\sum_{k<i}\left\lceil\frac{s_k}{r-s+1}\right\rceil\cdot(r-s+1)\ge |B|-(r+1)\beta n.$$

It follows from Ore's condition that $\sigma(G[B])\ge 2\big(1-\frac{1}{r-s}\big)|B|-2$. Thus,

$$\begin{aligned}\sigma(G[B_{T_i}])&\ge\sigma(G[B])-2(r+1)\beta n\ge 2\Big(1-\frac{1}{r-s}\Big)|B|-(2r+3)\beta n\\&\ge 2\Big(1-\frac{1}{r-s-1}\Big)|B_{T_i}|.\end{aligned}$$

Note that $G[B_{T_i}]$ contains no $\frac{\gamma_{s+1}^2}{4}$-independent set of size $\frac{n}{r}$, and hence no $\frac{\gamma_{s+1}^2}{10}$-independent set of size $\frac{|B_{T_i}|}{r-s}$. By Proposition 2.3, we conclude that $G[B_{T_i}]$ contains K_{r-s} as a subgraph. By selecting $r-s+1-t_i$ vertices from this K_{r-s} and combining them with T_i, we obtain a K_{r-s+1} that covers T_i. Together with the existing $\lfloor s_i/(r-s+1)\rfloor$ copies, it forms the desired family $\mathcal{F}_i$. $\square$

Let $\mathcal{F}_i$ be the family obtained in Claim 7.5 for each $i\in[s]$. For all distinct $i,j\in[s]$ and each $K_{r-s+1}\in\mathcal{F}_i$, since $V(\mathcal{F}_i)\subseteq(B\cap V_e(2\beta'))\cup V_{ex}^S(\beta/2,i)$, together with Claim 7.4 we have

$$|N(V(K_{r-s+1}),A_j)|\ge |A_j|-(r-s+1)\cdot\max\left\{\frac{\beta}{2}n+2\alpha^{1/4}n,2\beta'n\right\}\ge(1-r^2\beta)\frac{n}{r}. \tag{19}$$

Next, we are to construct the second component of each base in $\mathcal{F}^S$.

Claim 7.6. *For each $i \in [s]$ with $s_i > 0$, there exists a family $\mathcal{F}_i'$ of vertex-disjoint subgraphs satisfying*

- $|\mathcal{F}_i'| = f(s_i, r-s)$;
- $|V(F') \cap A_i| = 2$ *for each* $F' \in \mathcal{F}_i'$;
- *for any pair* $F \in \mathcal{F}_i$ *and* $F' \in \mathcal{F}_i'$, *their union* $F \cup F'$ *forms a* $(2, \beta/4)$*-base in* G.

Proof of Claim 7.6. Fix an $i \in [s]$ with $s_i > 0$, suppose the desired $\mathcal{F}_k'$ have been constructed for all $k < i$. We split our proof into the following two cases.

Case 1. $|V_b(2\beta, i)| \geq f(s_i, r-s)$.

In this case, we construct $\mathcal{F}_i'$ by utilizing vertices in $V_b(2\beta, i)$. Observe that $V_b(2\beta, i)$ contains the following two types of vertices:

Type 1 Vertices that are $(\beta/2, j)$-exceptional for some $j \in [s] \setminus \{i\}$;

Type 2 Vertices that are not $(\beta/2, j)$-exceptional for any $j \in [s]$.

Let x be a vertex in $V_b(2\beta, i)$ that is of **Type 1**, that is, $x \in V_{ex}(\beta/2, j)$ for some $j \in [s] \setminus \{i\}$. By Definition 5.2 (A4), there exists an edge $xy \in E(M_j)$ with $y \in A_j$ and $d(x, A_j), d(y, A_j) \leq 2\beta n$. Together with $x, y \notin S$, we obtain $d(\{x,y\}, D) \geq |D| - 4\beta n - 2$ for each $D \in \mathcal{Q} \setminus \{A_j\}$. Recall that (17) implies that at most $4r\alpha^{1/4}n$ vertices have been used in all previously constructed bases, including those in $\mathcal{F}_k'$ with $k < s$. Together with $|V_b(2\beta, i)| \leq \sqrt{\alpha}n$, there are at least

$$d(\{x,y\}, A_i) - \big|V_{ne}(2\beta') \cup V_b(2\beta, i)\big| - 4r\alpha^{1/4}n \geq \frac{n}{r} - 5\beta n$$

vertices in $\big(V_e(2\beta') \setminus V_b(2\beta, i)\big) \cap N(\{x,y\}, A_i)$ that remain unused by any previous base. Choose z to be one such vertex. Then for any $D \in \mathcal{Q} \setminus \{A_i, A_j, B\}$ we have

$$d(\{x,y,z\}, D) \geq |D| - 4\beta n - 2 - 2\beta' n \geq |D| - 5\beta n, \text{ and } d(\{x,y,z\}, B) \geq |B| - 4\beta n - 2 - \big(\frac{n}{r} + 1\big).$$

Together with (19), the union of $G[\{x,y,z\}]$ and any $F \in \mathcal{F}_i$ satisfy (D3)-(D4), and thus forms a $(2, \beta/4)$-base in G.

Next, suppose $x \in V_b(2\beta, i)$ is of **Type 2**. Then, $d(x, A_j) \geq \beta n/2$ for all $j \in [s]$ and $d(x, B) \geq |B| - n/r - 1$. Since $|V_{ne}(2\beta')| \leq 2\alpha^{1/4}n$ by (17), there are at least

$$d(x, A_i) - \big|V_{ne}(2\beta') \cup V_b(2\beta, i)\big| - 4\alpha^{1/4}n \geq \frac{\beta n}{3}$$

vertices in $\big(V_e(2\beta') \setminus V_b(2\beta, i)\big) \cap N(x, A_i)$ that are not used in any previous bases. Choose z to be one such vertex. Then for any $D \in \mathcal{Q} \setminus \{A_i, B\}$ we have

$$d(\{x,z\}, D) \geq d(x, D) - \big(|D| - d(z, D)\big) \geq \frac{\beta n}{2} - 2\beta' n \geq \frac{\beta n}{3}, \text{ and } d(\{x,z\}, B) \geq |B| - \frac{n}{r} - 3\beta' n.$$

Combining with (19), the union of $G[\{x,z\}]$ and any $F \in \mathcal{F}_i$ satisfy (D3)-(D4), and thus forms a $(2, \beta/4)$-base in G.

As $\alpha \ll \beta$, we can construct a family $\mathcal{F}_i'$ of $f(s_i, r-s)$ vertex-disjoint subgraphs such that for every $F' \in \mathcal{F}_i'$ and every $F \in \mathcal{F}_i$, the union $F \cup F'$ forms a $(2, \beta/4)$-base in G.

Case 2. $|V_b(2\beta, i)| < f(s_i, r-s)$.

We claim that $G[A_i \setminus (V_b(2\beta, i) \cup V(M_i))]$ contains a matching M_i^* of size $f(s_i, r-s) - |V_b(2\beta, i)|$. If so, then for any $uv \in E(M_i^*)$ and $D \in \mathcal{Q} \setminus \{A_i\}$ we have $d(\{u,v\}, D) \geq |D| - 4\beta n - 2$. Therefore, (19) implies that the union of any edge in M_i^* and any $F \in \mathcal{F}_i$ satisfy (D3)-(D4), and thus forms a $(2, \beta/4)$-base in G. In the following, we will prove the existence of M_i^*.

Notice that $e(G[V_{ex}(\beta/2,i), A_i]) \le \frac{\beta}{2}n \cdot |V_{ex}(\beta/2,i)|$. Therefore,

$$\Big|\{v \in A_i : d(v, V_{ex}(\beta/2,i)) \ge \frac{1}{9}|V_{ex}(\beta/2,i)|\}\Big| \le \frac{|V_{ex}(\beta/2,i)|\beta n}{2} \cdot \frac{1}{\frac{1}{9}|V_{ex}(\beta/2,i)|} \le 5\beta n.$$

Furthermore, by (A4) one has

$$\Big|\{v \in A_i : d(v, V(M_i) \cap A_i) \ge \frac{8}{9}|E(M_i)|\}\Big| \le \frac{e(G[V(M_i) \cap A_i, A_i])}{\frac{8}{9}|E(M_i)|} \le \frac{2\beta n|E(M_i)|}{\frac{8}{9}|E(M_i)|} \le 4\beta n.$$

Hence, there exists a set $A'_i \subseteq A_i \setminus (V_b(2\beta,i) \cup V(M_i) \cup N(V^S_{ex}(\beta/2,i)))$ of size at least $\frac{n}{2r}$ such that each vertex $v \in A'_i$ satisfying

$$d(v, V_{ex}(\beta/2,i)) < \frac{|V_{ex}(\beta/2,i)|}{9}, \text{ and } d(v, V(M_i) \cap A_i) < \frac{8}{9}|E(M_i)|.$$

Choose $v \in A'_i$. Since $v \notin N(V^S_{ex}(\beta/2,i))$, i.e., there exists a vertex $u \in S$ such that $uv \notin E(G)$, Ore's condition implies that $d(v) \ge (1-\frac{1}{r})n$. Thus,

$$d(v, A_i) \ge \Big(1-\frac{1}{r}\Big)n - \big|V(G) \setminus (A_i \cup V_{ex}(\beta/2,i))\big| - d(v, V_{ex}(\beta/2,i)) \ge \Big\lceil \frac{8|V_{ex}(\beta/2,i)|}{9} \Big\rceil.$$

Therefore,

$$\begin{aligned} d\big(v, A_i \setminus (V_b(2\beta,i) \cup V(M_i))\big) &\ge \big(d(v,A_i) - d(v, V(M_i))\big) - |V_b(2\beta,i)| \ge \Big\lceil \frac{8s_i}{9} \Big\rceil - |V_b(2\beta,i)| \\ &\ge f(s_i, r-s) - |V_b(2\beta,i)|. \end{aligned}$$

Together with $|A'_i| \gg 2s_i$, there exists a matching M^*_i of size $f(s_i, r-s) - |V_b(2\beta,i)|$ inside $G[A'_i, A_i \setminus (V_b(2\beta,i) \cup V(M_i))]$.

Note that all other existing bases can be chosen to be vertex-disjoint from all M^*_i. Together with the argument in Case 1, there exists a family of $|V_b(2\beta,i)|$ vertex-disjoint subgraphs that covers $V_b(2\beta,i)$. Combining those with M^* gives a family $\mathcal{F}'_i$ of size $f(s_i, r-s)$ such that for any $F' \in \mathcal{F}'_i$ and $F \in \mathcal{F}_i$, the union $F \cup F'$ forms a $(2,\beta/4)$-base in G. □

By the above argument, we have $\lceil s_i/(r-s+1)\rceil = |\mathcal{F}_i| \le |\mathcal{F}'_i| = f(s_i, r-s)$. Notice that when $s_i \ge 2$ and $r-s \ge 2$, it holds that $|\mathcal{F}'_i| = |\mathcal{F}_i| + 1$. Let $\mathcal{F}''_i$ be a subfamily of $\mathcal{F}'_i$ with size exactly $|\mathcal{F}_i|$. Thus, by combining $\mathcal{F}_i$ and $\mathcal{F}''_i$, we obtain a $\beta/4$-base set $\mathcal{F}^S_i$ of size $\lceil s_i/(r-s+1)\rceil$. Define $\mathcal{F}^S := \bigcup_{i\in[s]} \mathcal{F}^S_i$. Clearly, $\mathcal{F}^S$ is a $(2,\beta/4)$-base set that covers $V^S_{ex}(\beta/2)$.

Let $\mathcal{F}^L_*$ be a maximal subset of $\mathcal{F}^L$ where every graph is vertex-disjoint from all graphs in $\mathcal{F}^S$. Hence, $\mathcal{F} := \mathcal{F}^L_* \cup \mathcal{F}^S$ is the desired $\beta/4$-base set. □

Lemma 7.2 guarantees the existence of a $\beta/4$-base set $\mathcal{F}$ covering $V_{ex}(\beta/2)$. However, vertices in $V_{ne}(2\beta') \setminus V_{ex}(\beta/2)$ may still fail to satisfy the conditions of Lemma 2.5. We next construct a base set covering $V_{ne}(2\beta') \setminus V_{ex}(\beta/2)$.

Lemma 7.7. *Suppose that (†) holds, and let $U \subseteq V(G)$ be a vertex subset with $|U| \le 4r\alpha^{1/4}n$. Then there exists a $\beta/4$-base set that covers $V_{ne}(2\beta') \setminus V_{ex}(\beta/2)$ and avoids U. Moreover, every vertex in this base set that does not belong to B has degree at most $(1-2\beta')n$.*

Proof. Choose $v \in V_{ne}(2\beta') \setminus V_{ex}(\beta/2)$. If $s = r$, then $\delta(G) \ge (1-\frac{1}{r})n - 1$ and one may assume $v \in A_i$ for some $i \in [r]$. Thus, $d(v, A_j) \ge \beta n/2$ for every $j \in [n] \setminus \{i\}$. Therefore, the vertex v itself is a $\beta/4$-base under the partition $\mathcal{Q}$. In what follows, we only consider the case that $s < r$.

Recall that

$$v \in V_{ne}(2\beta') = V(G) \setminus V_e(2\beta') \text{ and } V_e(2\beta') = \bigcap_{D\in\mathcal{Q}} (V_e(2\beta', D) \cup D).$$

Hence there exists a set $D \in \mathcal{Q}$ such that $v \in V(G) \setminus (V_e(2\beta', D) \cup D)$. We split the proof into the following two cases.

Case 1. $v \notin B$.

Since $v \notin B$, without loss of generality, assume that $v \in A_i$ for some $i \in [s]$. Let D be a set in $\mathcal{Q}$ such that $v \in V(G) \setminus (V_e(2\beta', D) \cup D)$. Since $v \notin V_e(2\beta', D)$, we obtain

$$d(v, A_i) \geq \Big(1 - \frac{1}{r}\Big)n - 1 - |V(G) \setminus (A_i \cup D)| - d(v, D) \geq |D| - 1 - (|D| - 2\beta' n) = 2\beta' n - 1.$$

As $|V_{ne}(2\beta')| \leq 2\alpha^{1/4}n$ and $|U| \leq 4r\alpha^{1/4}n$, we obtain

$$\big|(N(v, A_i) \cap V_e(2\beta')) \setminus U\big| = d(v, A_i) - |V_{ne}(2\beta')| - |U| \geq \beta' n.$$

Since $G[A_i]$ is $\sqrt{\alpha}$-independent, there are at most $2r\sqrt{\alpha}n (\ll \beta' n)$ vertices $u \in A_i$ such that $d(u) \geq n - 2\beta' n$. Choose $w \in (N(v, A_i) \cap V_e(2\beta')) \setminus U$ such that $d(w) \leq n - 2\beta' n$. Since $v \notin V_{ex}(\beta/2)$ and $w \in V_e(2\beta')$, for each $D' \in \mathcal{Q} \setminus \{A_i, B\}$, we deduce

$$|N(\{v, w\}, D')| \geq \frac{\beta n}{2} - 2\beta' n \geq \frac{\beta n}{4} \text{ and } |N(\{v, w\}, B)| \geq |B| - \frac{n}{r} - 3\beta' n. \tag{20}$$

Let $B' := B \setminus (V_{ne}(2\beta') \cup U)$. Then

$$|B'| \geq \frac{r-s}{r}n - |V_{ne}(2\beta')| - |U| \geq \frac{r-s}{r}n - 5r\alpha^{1/4}n.$$

Based on $\sigma(G) \geq 2(1 - \frac{1}{r})n - 2$, we get

$$\sigma(G[B']) \geq \Big(1 - \frac{1}{r-s} - \alpha^{1/5}\Big)|B'|.$$

Since $G[B]$ contains no $\gamma_{s+1}^2/4$-independent set of size $\frac{n}{r}$, Proposition 2.3 implies that there exists a copy of K_{r-s+1} in $G[B']$. It is straightforward to check that such a K_{r-s+1} together with the edge vw satisfy (D3) and (D4) with $\xi = \beta/4$, and thus forms a $(2, \beta/4)$-base in G, as desired.

Case 2. $v \in B$.

In this case, there exists an $i \in [s]$ such that $v \in V(G) \setminus (V_e(2\beta', A_i) \cup A_i)$. It follows from Claim 7.4 that

$$d(v, B) \geq \frac{r-s-1}{r}n + 2(\beta' - \alpha^{1/4})n.$$

Let $B' := N(v, B) \setminus (V_{ne}(2\beta') \cup U)$. It is straightforward to check that $|B'| > \frac{r-s-1}{r}n$ and $\sigma(G[B']) \geq 2(1 - \frac{1}{r-s-1})|B'| - 2$. Recall that $G[B]$ contains no $\gamma_{s+1}^2/4$-independent set of size $\frac{n}{r}$. Thus, Proposition 2.3 implies that $G[B']$ contains a copy of K_{r-s-1}. Therefore, there exists a copy of K_{r-s} containing v, which forms a $\beta/4$-base under the partition $\mathcal{Q}$, as desired.

As $\alpha \ll \beta'$ and $|V_{ne}(2\beta')| \leq 2\alpha^{1/4}n$, by applying the above argument iteratively, we deduce that for each vertex $v \in V_{ne}(2\beta')$, there is a base covering v that avoids all vertices used in any previously constructed bases. $\square$

7.2 Proof of Lemma 5.4

In this subsection, we will give the proof of Lemma 5.4. First, we show that the bases constructed above can be extended to some vertex-disjoint K_r and the rest partition still has some good properties.

Lemma 7.8. *Suppose that (†) holds. Let H be a $\beta/5$-base or $(2, \beta/5)$-base in G, and W be a vertex subset satisfying $|W| \leq 5r\alpha^{1/4}n$. Then, there exists a K_r-tiling T_H in $G - W$ that covers $V(H)$, consisting of at most two vertex-disjoint cliques and satisfying*

$$|T_H \cap V(D)| = \frac{kr|D|}{n} \textit{ for every } D \in \mathcal{Q}.$$

Proof. We first consider a $\beta/5$-base $H \in \mathcal{H}$. Let $t_H := r - s - |V(H) \cap B|$. By (D2) one has $|N(V(H), B)| \geq (t_H - 1)\frac{n}{r} + \frac{\beta n}{5}$. Let $G' := G[(N(V(H), B) \setminus W) \cap V_e(2\beta')]$. Clearly,

$$|V(G')| \geq (t_H - 1)\frac{n}{r} + \frac{\beta n}{5} - 5r\alpha^{1/4}n - 2\alpha^{1/4}n \geq (t_h - 1)\frac{n}{r} + \frac{\beta n}{8}.$$

Since $\sigma(G) \geq 2(1 - \frac{1}{r})n - 2$, one has

$$\sigma(G') \geq 2|V(G')| - \frac{2n}{r} - 2 \geq 2\Big(1 - \frac{1}{t_H - 1}\Big)|V(G')|.$$

By (A2), G' contains no $\frac{\gamma_{s+1}^2}{4}$-independent set of size $\frac{n}{r}$. Proposition 2.3 implies that there exists a K_{t_H} in G'. Define H' as the join of this K_{t_H} and H. Since H is a $\beta/4$-base, for every $A_i \in \mathcal{Q}$ with $|A_i \cap V(H)| = 0$ we have

$$|(N(H', A_i) \setminus W) \cap V_e(2\beta')| \geq \frac{\beta n}{5} - r \cdot 2\beta' n - 5r\alpha^{1/4}n - 2\alpha^{1/4}n \geq \frac{\beta n}{8}.$$

This enables us to iteratively select vertices $x_i \in \big(N(H', A_i) \cap V_e(2\beta')\big) \setminus W$ for each $i \in [s]$ with $|A_i \cap V(H)| = 0$. Therefore, $G[V(H') \cup \{x_i : |A_i \cap V(H)| = 0 \text{ and } i \in [s]\}]$ is a clique K_r, as desired.

Next, we assume that H is a $(2, \beta/5)$-base in $\mathcal{H}$. Without loss of generality, assume that C_0 and C_1 are two vertex-disjoint cliques in H and $D_0, D_1 \in \mathcal{Q}$ are two parts satisfying $|N(C_i) \cap D_i| = \frac{r|D_i|}{n} + 1$. For C_0, we will find a vertex set $T_1 \subseteq D_1 \setminus W$ of size $t_1 := \frac{r|D_1|}{n} - |V(C_0) \cap D_1| - 1$ such that $G[V(C_0) \cup T_1]$ is a clique. If $t_1 \leq 0$, then it is trivial. Thus, we only need to consider $D_1 = B$ and $r - s \geq 2$. Let $G' := G[(N(C_0, B) \cap V_e^S(2\beta')) \setminus W]$. By applying a similar argument as the case that H is a $\beta/5$-base, we conclude that G' contains K_{t_1} as a subgraph and there exists a copy of K_r covering $K_{t_1} \cup C_0$ that avoids W and all used vertices.

By the same argument, there exists a copy of K_r covering C_1 that avoids both W and all previously used vertices. Moreover, the union of these two K_r copies uses two distinct vertices from each $A_i \setminus W$ for each $i \in [s]$ and $2(r - s)$ vertices from $B \setminus W$, as required. $\square$

Now, we are ready to prove Lemma 5.4.

Proof of Lemma 5.4. Recall that $|V_{ne}(2\beta')| \leq 2\alpha^{1/4}n$. By Lemma 7.2 and Lemma 7.7, we obtain a $\beta/4$-base set $\mathcal{H}$ covering $V_{ne}(2\beta')$ with at most $12\alpha^{1/4}n$ vertices. Let $W := V(\mathcal{H})$. We then proceed iteratively: at each step, we apply Lemma 7.8 to extend a base to a K_r-tiling, and then update W by adding the vertices of this new K_r-tiling. Therefore, we obtain a K_r-tiling $\mathcal{T}$ of order at most $4r\alpha^{1/4}n$ that covers $V_{\mathrm{ne}}(2\beta')$. Let $G' := G - V(\mathcal{T})$, $A_i' := A_i \setminus V(\mathcal{T})$ for each $i \in [s]$, and $B' := B \setminus V(\mathcal{T})$. By Lemma 7.8, one has that $\{A_1', \ldots, A_s', B'\}$ is an (r, s)-partition of G' and $(r - s) \mid |B'|$. Furthermore,

$$\sigma(G') \geq 2\Big(1 - \frac{1}{r}\Big)n - 8r\alpha^{1/4}n - 2 \text{ and } \sigma(G[B']) \geq 2\Big(1 - \frac{1}{r - s} - \alpha^{1/5}\Big)|B'|. \tag{21}$$

Observe that $G[B']$ contains no $\frac{\gamma_{s+1}^2}{8}$-independent set of size $\frac{n}{r}$, and hence no $\frac{\gamma_{s+1}^2}{10}$-independent set of size $\frac{|B'|}{r-s}$. Clearly, (B2) is trivial for $r - s = 1$. If $r - s \geq 3$, then by Theorem 3.1 we obtain that there exists a K_{r-s}-tiling in $G[B']$, i.e., (B2) holds. In what follows, it suffices to consider $r - s = 2$. We will show that either G is the extremal graph (EX2), or $G[B']$ contains a perfect matching, or by modifying some bases in $\mathcal{H}$ we can get a new K_r-tiling $\mathcal{T}'$ such that $G[B \setminus V(\mathcal{T}')]$ admits a perfect matching.

If $G[B']$ contains a perfect matching, then we are done. Suppose that $G[B']$ contains no perfect matching. By Proposition 2.10, we know that $G[B']$ consists of two odd components, say F_0 and F_1.

Suppose that $V_{ne}(2\beta') = \emptyset$. Then $\mathcal{H} = \emptyset$ and $B' = B$. Without loss of generality, assume that $|F_0| \leq |F_1|$. By Ore's condition, we conclude that $N(z) = V(G) \setminus (V(F_{i+1}) \cup \{z\})$ for each $z \in$

$V(F_i)$ with $i \in \mathbb{Z}_2$. Hence F_0, F_1 are two cliques with odd orders. If $G[A_i] = \emptyset$ for all $i \in [r-2]$, then G is the extremal graph described in (EX2). Suppose that there exists an edge $uv \in E(G[A_i])$ for some $i \in [r-2]$. Then there exists a vertex $w \in V(F_0)$ such that $G[\{u, v, w\}]$ forms a K_3, say K^1. Since F_1 is a clique and $|F_1| \geq |B'|/2$, there exists a K_3, say K^2, in F_1. Since $V_{ne}(2\beta') = \emptyset$, it is easy to check that $K^1 \cup K^2$ is a $(2, \beta/4)$-base in G. By applying Lemma 7.8 with $W = \emptyset$, we obtain a K_r-tiling $\mathcal{T}$ of G such that $|V(F_i) \setminus V(\mathcal{T})|$ is even for every $i \in \mathbb{Z}_2$. Applying Proposition 2.10 again yields that $G[B \setminus V(\mathcal{T})]$ contains a perfect matching, as desired. Thus, overall, it suffices to consider the case where

$$r - s = 2,\ G[B'] = F_0 \cup F_1 \text{ with } |F_0| \text{ and } |F_1| \text{ both odd, and } V_{ne}(2\beta') \neq \emptyset.$$

Our goal is to modify the K_r-tiling $\mathcal{T}$ into a new K_r-tiling $\mathcal{T}'$ that satisfies (B2). We begin with an analysis of the properties of F_0 and F_1. Since S is a clique in G, at most one of the components F_i with $i \in \mathbb{Z}_2$ satisfies $V(F_i) \cap S \neq \emptyset$. Without loss of generality, assume that $V(F_1) \cap S = \emptyset$. Therefore, for a vertex $v \in V(F_1)$, one has

$$\Big(1 - \frac{1}{r}\Big)n - 1 \leq d(v) \leq n - |B'| + d(v, F_1),$$

which implies

$$|F_1| \geq d(v, F_1) \geq \frac{n}{r} - 1 - 4r\alpha^{1/4}n \geq \frac{n}{r} - \alpha^{1/5}n.$$

Furthermore, if $|F_1| \geq \frac{n}{r} + 1$, then $d(u) \leq (1 - \frac{1}{r})n - 2$ for every $u \in V(F_0)$. That is, $V(F_0) \subseteq S$. Similarly, if $V(F_0) \setminus S \neq \emptyset$, then $|F_0| \geq \frac{n}{r} - \alpha^{1/5}n$. In particular, if $|F_0| \leq \frac{n}{r} - 3\beta n$, then by Ore's condition one has

$$G[V_{ex}^L(\beta/2) \cup (V(G) \setminus (V_b(2\beta) \cup B)), V(F_0)] \text{ is a complete bipartite graph.} \tag{22}$$

For any $\tilde{\mathcal{H}} \subseteq \mathcal{H}$, let $\tilde{\mathcal{T}} \subseteq \mathcal{T}$ be the set of K_r copies obtained by extending the bases in $\widetilde{\mathcal{H}}$. Let $X_{\tilde{\mathcal{H}}} := \big(B \cap V(\tilde{\mathcal{T}})\big) \setminus V_{ex}^S(\beta/2)$. It follows from (17) that $|X_{\tilde{\mathcal{H}}}| \leq 8\alpha^{1/4}n$. If $|F_1| \geq \frac{3n}{2r}$, then define

$$F_0'(\tilde{H}) := G\big[V(F_0) \cup (X_{\tilde{\mathcal{H}}} \cap S)\big] \text{ and } F_1'(\tilde{H}) := G\big[(V(F_1) \cup X_{\tilde{\mathcal{H}}}) \setminus S\big]; \tag{23}$$

if $|F_1| < \frac{3n}{2r}$, then define

$$F_1'(\tilde{H}) := G\big[V(F_1) \cup \{x \in X_{\tilde{\mathcal{H}}} : d(x, F_1) \geq \frac{2}{3}|F_1|\}\big] \text{ and } F_0'(\tilde{H}) := G\big[(V(F_0) \cup X_{\tilde{\mathcal{H}}}) \setminus V(F_1')\big]. \tag{24}$$

For brevity, when $\widetilde{H}$ is clear from context, we write F_0' and F_1' for $F_0'(\widetilde{H})$ and $F_1'(\widetilde{H})$, respectively. Note that if $V_{ex}^S(\beta/2) \cap V(\tilde{\mathcal{H}}) = \emptyset$, then

$$|V(F_0 \cup F_1)| \text{ and } |V(F_0' \cup F_1')| \text{ are even.} \tag{25}$$

The following claim demonstrates that both F_0' and F_1' remain connected even after the deletion of any small subset of vertices.

Claim 7.9. *For any $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ and any $X' \subseteq V(F_0' \cup F_1')$ of size at most $10\alpha^{1/4}n$, we have that $F_0'' := G[V(F_0') \setminus X']$ and $F_1'' := G[V(F_1') \setminus X']$ are connected.*

Proof of Claim 7.9. We first consider that $|F_1| \geq \frac{3n}{2r}$. Clearly, F_0' is a clique, and thus F_0'' is connected. For two vertices $x \in V(F_0)$ and $y \in V(F_1)$, we have

$$2\Big(1 - \frac{1}{r}\Big)n - 2 \leq d(x) + d(y) \leq 2(n - |B'|) + |F_0| - 1 + d(y, F_1).$$

Thus,

$$d(y, F_1'') \geq d(y, F_1') - |X'| \geq d(y, F_1) - |X'| \geq |F_1| - 1 - |B \setminus B'| - |X'| \geq |F_1| - 20\alpha^{1/4}n. \tag{26}$$

For any $y' \in V(F_1') \setminus V(F_1)$, we have

$$d(y', F_1'') \geq d(y', F_1') - |X'| \geq d(y', B) - (|F_0| + |B \setminus B'| + |X'|) \geq \frac{n}{r} - 1 - \left(\frac{n}{2r} + 20\alpha^{1/4}\right) \geq \frac{n}{3r}.$$

Therefore, for any two distinct vertices $u, v \in V(F_1'')$, we obtain $N_{F_1''}(u) \cap N_{F_1''}(v) \neq \emptyset$ unless $u, v \in V(F_1'') \setminus V(F_1)$. Consequently, F_1'' is connected.

Now, we assume $|F_1| < \frac{3n}{2r}$. For any $y' \in V(F_1'') \setminus V(F_1)$, by the construction of F_1' we have $d(y', F_1'') \geq \frac{2}{3}|F_1| - |X'| \geq \frac{1}{2}|F_1''| + 1$. Together with (26) and the classical Dirac's Theorem, we conclude that F_1'' is connected.

We now show that F_0'' is connected. Similar to (26) we obtain $d(y, F_0'') \geq |F_0| - 20\alpha^{1/4} \geq \frac{4}{5}|F_0''|$ for any $y \in V(F_0 \cap F_0'')$. Let $y' \in V(F_0'') \setminus V(F_0)$. If there is a vertex $x \in V(F_0) \setminus N_G(y')$, then by Ore's condition we have

$$2\left(1 - \frac{1}{r}\right)n - 2 \leq d(x) + d(y') \leq 2(n - |B'|) + |F_0| - 1 + \frac{2}{3}|F_1| + d(y', F_0).$$

Thus,

$$d(y', F_0'') \geq d(y', F_0) - |X'| \geq \frac{1}{3}|F_1| - 20\alpha^{1/4}n \geq \frac{n}{4r} > \frac{1}{5}|F_0''|,$$

the last two inequalities hold by $|F_1| \geq \frac{n}{r} - \alpha^{1/5}n$. Therefore, for any two distinct vertices $u, v \in V(F_0'')$, we obtain $N_{F_0''}(u) \cap N_{F_0''}(v) \neq \emptyset$ unless $u, v \in V(F_0'') \setminus V(F_0)$. It follows that F_0'' is connected. $\square$

To allow for minor modifications, we introduce a claim that modifies a given base into a desired form.

Claim 7.10. *For any subset $\tilde{\mathcal{H}} \subseteq \mathcal{H}$, let F_0' and F_1' be given by (23)-(24). Let K be a clique satisfying (D3) and (D4) with $\xi = \beta/4$, $C_0 = K$, $D_0 = A_i$ for some $i \in [r-2]$, and $D_1 = B$. Suppose that $V_{ex}^S(\beta/2) \cap V(K) = \emptyset$ and $d(K, F_j') \geq \frac{n}{3r}$ for some $j \in \mathbb{Z}_2$. If $|F_0'| \geq 2$ or $d(x) \leq n - 2\beta' n$ for every $x \in V(K)$, then there exists a $(2, \beta/5)$-base H' such that $V(K) \subseteq V(H')$, $|B \cap V(H')| = 4$ and $|V(F_\ell') \setminus V(H')|$ is even for every $\ell \in \mathbb{Z}_2$.*

Proof of Claim 7.10. We first consider $|F_0'| \geq 2$. Since $d(K, F_j') \geq \frac{n}{3r}$ for some $j \in \mathbb{Z}_2$, there exists a vertex $w \in V(F_j') \subseteq V_e(2\beta')$ such that $G[V(K) \cup \{w\}]$ forms a clique K_{k+1}. Let such a K_{k+1} be C_0. By Ore's condition, for each $x \in V(F_1')$ one has

$$d(x, F_1' - w) \geq |F_1'| - |B \setminus B'| - 2 \geq \frac{1}{2}|F_1'| + 1.$$

Thus, Mantel's Theorem implies that $F_1' - w$ contains a K_3. Similarly, one may obtain that F_0 contains a K_3 if $|F_0'| \geq \frac{n}{7r}$, and F_0' is a clique if $2 \leq |F_0'| < \frac{n}{7r}$. Consequently, there exists a K_3 in $F_{j+\ell}' - V(K) \cup \{w\}$, where $\ell \equiv |F_j'| \pmod 2$. Define C_1 to be such a K_3. It is easy to check that $H' := C_0 \cup C_1$ satisfy (D3)-(D4) with $D_0 := A_i$, $D_1 := B$, and $\xi := \beta/5$. Together with (25), H' is a $(2, \beta/5)$-base such that $|B \cap V(H')| = 4$ and $|V(F_i') \setminus V(H')|$ is even for every $i \in \mathbb{Z}_2$.

Now, we consider $d(x) \leq n - 2\beta' n$ for every $x \in V(K)$ and $|F_0'| = 1$. Let w be the unique vertex in F_0'. Then $d(w) \leq (1 - \frac{2}{r})n + 8\alpha^{1/4}n$. Since $\beta' \gg \alpha$, Ore's condition implies that $G[V(K) \cup \{w\}]$ is a clique, say C_0. Recall that F_1' contains a K_3, say C_1. Thus, $H' := C_0 \cup C_1$ is the desired $(2, \beta/5)$-base H'. $\square$

Based on the construction of bases in Lemma 7.2 and Lemma 7.7, there are three possible types for each base $H \in \mathcal{H}$: (1) $V(H) \cap V_{ne}(2\beta') \subseteq V_{ne}(2\beta') \setminus V_{ex}(\beta/2)$ (in Lemma 7.7); (2) $V(H) \cap V_{ne}(2\beta') \subseteq V_{ex}^L(\beta/2)$ (in Claim 7.3); (3) $V(H) \cap V_{ex}^S(\beta/2) \neq \emptyset$ (in Claims 7.5 and 7.6). We then modify $\mathcal{T}$ into a K_r-tiling $\mathcal{T}'$ based on the types of bases in $\mathcal{H}$.

Case 1. There exists a base $H \in \mathcal{H}$ satisfying $V(H) \cap V_{ne}(2\beta') \subseteq V_{ne}(2\beta') \setminus V_{ex}(\beta/2)$.

By the proof of Lemma 7.7, one has

- $V(H) \cap V_{ne}(2\beta')$ contains exactly one vertex, say v;

- if $v \notin B$, then $v \in A_i$ for some $i \in [r-2]$ and H is a $(2, \beta/4)$-base. More precisely, H has two components C_0 and C_1 satisfying (D3)-(D4), where C_0 is an edge vw in $G[A_i]$ with $d(w) \leq n - 2\beta' n$ and C_1 is a K_3 copy in $G[B \cap V_e(2\beta')]$;

- if $v \in B$, then H is an edge vw in $G[B]$, which forms a $\beta/4$-base.

Suppose that F_0' and F_1' are defined as in (23) and (24) by letting $\tilde{\mathcal{H}} := \{H\}$. We proceed by considering whether v belongs to B or not.

We first assume $v \notin B$. Recall that H is extended to two vertex-disjoint copies of K_r, denoted K^1 and K^2, in $\mathcal{T}$ by Lemma 7.8. By (20) one has $d(\{v, w\}, F_j') \geq \frac{n}{3r}$ for some $j \in \mathbb{Z}_2$. Claim 7.10 implies that there exists a $(2, \beta/4)$-base H' such that $\{v, w\} \subseteq V(H')$, $|V(H') \cap B| = 4$ and $|V(F_\ell') \setminus V(H')|$ is even for every $\ell \in \mathbb{Z}_2$. Applying Lemma 7.8 to H' with $W := V(\mathcal{T}) \setminus V(K^1 \cup K^2)$, one may extend it to two vertex-disjoint copies of K_r that avoids W, denoted $\tilde{K}^1$ and $\tilde{K}^2$. Let $\mathcal{T}' := \mathcal{T} \setminus \{K^1, K^2\} \cup \{\tilde{K}^1, \tilde{K}^2\}$. Clearly, $\mathcal{T}'$ is a K_r-tiling covering $V_{ne}(2\beta')$. For each $\ell \in \mathbb{Z}_2$, Claim 7.9 ensures that $G[V(F_\ell') \setminus V(\tilde{K}^1 \cup \tilde{K}^2)]$ is a connected graph with even order. Combined with Proposition 2.10, we know that $G[B \setminus V(\mathcal{T}')]$ contains a perfect matching.

Next, assume that $v \in B$. Then, $d(v, A_i) \leq |A_i| - 2\beta' n$ for some $i \in [r-s]$. Recall that H is extended to a copy of K_r, say K, in $\mathcal{T}$. Suppose that $d(v, F_j) = 0$ for some $j \in \mathbb{Z}_2$ and let x be a vertex in F_j. Then

$$
\begin{aligned}
2\Big(1 - \frac{1}{r}\Big)n - 2 \leq d(v) + d(x) &\leq 2(n - |B'|) - |A_i| + d(v, A_i) + d(v, F_{j+1}) + d(x, F_j) \\
&\leq 2n - |B'| - 2 - 2\beta' n < 2n - |B| - 2 = 2\Big(1 - \frac{1}{r}\Big)n - 2,
\end{aligned}
$$

a contradiction. Hence, $d(v, F_j) > 0$ for each $j \in \mathbb{Z}_2$. Thus, there exists a $\beta/4$-base H' such that $|V(H') \cap B| = 2$ and $|V(F_i') \setminus V(H')|$ is even for every $i \in \mathbb{Z}_2$. By Lemma 7.8, Claim 7.9 and Proposition 2.10, there exists a K_r-tiling $\mathcal{T}'$ such that $G[B \setminus V(\mathcal{T}')]$ admits a perfect matching.

Case 2. There exists a base $H \in \mathcal{H}$ satisfying $V(H) \cap V_{ne}(2\beta') \subseteq V_{ex}^L(\beta/2)$.

By the construction of Claim 7.3 in Lemma 7.2, we know that H is an edge uv with $u \in A_i \cap V_{ex}^L(\beta/2, j)$ and $v \in A_j \cap V_e(2\beta')$ for some distinct $i, j \in [s]$, which forms a $\beta/4$-base. Suppose that F_0' and F_1' are defined as in (23)-(24) by letting $\tilde{\mathcal{H}} := \{H\}$. It is routine to check that there exists a vertex $v' \in (A_i \cap V_e(2\beta')) \setminus (V(\mathcal{T}) \cup V_b(2\beta'))$ such that $G[\{u, v, v'\}]$ forms a K_3. Note that $d(u), d(v), d(v') \leq n - 2\beta' n$ and $N(\{u, v, v'\}, F_j) \geq \frac{n}{3r}$ for some $j \in \mathbb{Z}_2$. Applying Claim 7.10, Lemma 7.8, Claim 7.9 and Proposition 2.10 again yield a K_r-tiling $\mathcal{T}'$ such that $G[B \setminus V(\mathcal{T}')]$ admits a perfect matching.

Case 3. There exists a base $H \in \mathcal{H}$ satisfying $V(H) \cap V_{ex}^S(\beta/2) \neq \emptyset$.

In this case, one may assume that no base in $\mathcal{H}$ satisfies Case 1 or Case 2. Thus, $V_{ne}(2\beta') = V_{ex}(\beta/2)$ and $|V_{ex}^S(\beta/2)| \geq 1$. Denote $s_i := |V_{ex}^S(\beta/2, i)|$ for each $i \in [r-2]$. By Claims 7.5 and 7.6 in Lemma 7.2, there exist two families $\mathcal{F}_i$ and $\mathcal{F}_i'$ of subgraphs of G for each $i \in [r-2]$ with $s_i > 0$, where

- $\mathcal{F}_i$ consists of $\lceil s_i/3 \rceil$ vertex-disjoint K_3 copies such that $V_{ex}^S(\beta/2, i) \subseteq V(\mathcal{F}_i) \subseteq (B \cap V_e(2\beta')) \cup V_{ex}^S(\beta/2, i)$,

- $\mathcal{F}_i'$ consists of $f(s_i, 2)$ vertex-disjoint subgraphs each of which has exactly two vertices in A_i, and for any pair $F \in \mathcal{F}_i$ and $F' \in \mathcal{F}_i'$, their union $F \cup F'$ forms a $(2, \beta/4)$-base in G.

Moreover, each graph $F' \in \mathcal{F}_i'$ is one of the following three types:

- F' is a triangle xyz, where $x \in V_b(2\beta, i) \cap V_{ex}^L(\beta/2, j)$, $y \in V_e(2\beta') \cap A_j$, $z \in \big(V_e(2\beta') \setminus V_b(2\beta, i)\big) \cap N(\{x, y\}, A_i)$;

- F' is an edge xz, where $x \in V_b(2\beta, i) \setminus V_{ex}^L(\beta/2)$ and $z \in \big(V_e(2\beta') \setminus V_b(2\beta, i)\big) \cap N(x, A_i)$;
- F' is an edge in a matching of $G[A_i \setminus (V_b(2\beta, i) \cup V(M_i))]$.

Let $\mathcal{F}_i = \mathcal{F}'_i = \emptyset$ for each $i \in [r-2]$ with $s_i = 0$. Define F'_0 and F'_1 as in (23) and (24) with $\tilde{\mathcal{H}} := \mathcal{H}$. Clearly, $V(F'_0 \cup F'_1) \subseteq V_e(2\beta')$.

Assume that there is an edge $uv \in E(G[V(F'_0), V(F'_1)])$. By the construction of $\mathcal{F}_i$ in Claim 7.5, one may avoid u, v in $\bigcup_{i\in[r-2]} \mathcal{F}_i$. Based on Lemma 7.8 with $\{u, v\} \subseteq W$, there exists a K_r-tiling $\mathcal{T}'$ that covers all bases in $\mathcal{H}$ and avoids $\{u, v\}$. By Claim 7.9, the graph $F''_j := G[V(F'_j) \setminus V(\mathcal{T}')]$ is connected for each $j \in \mathbb{Z}_2$. As $uv \in E(G[V(F''_0), V(F''_1)])$, the graph $G[V(F''_0 \cup F''_1)]$ is connected. Hence, Proposition 2.10 guarantees the existence of a perfect matching in $G[V(F''_0 \cup F''_1)]$. Thus, in what follows, it suffices to consider

$$E(G[V(F'_0), V(F'_1)]) = \emptyset.$$

We first assume that there exists an integer $k \in [r-2]$ such that $s_k \geq 3$. Thus, $|\mathcal{F}'_k| \geq |\mathcal{F}_k| + 1$. Choose $Q \in \mathcal{F}'_k$. Then $|V(Q) \cap A_k| = 2$. Observe that $\mathcal{T}$ can be constructed with $V(Q) \cap V(\mathcal{T}) = \emptyset$. Assume that $|F_0| = 1$, and let w be the unique vertex in F_0. Clearly, $w \in S$. Note that there exists a $(2, \beta/4)$-base $H := H_1 \cup H_2 \in \mathcal{H}$ with $|V(H_1) \cap V_{ex}^S(\beta/2, k)| = 3$ and $|V(H_2) \cap A_i| = 2$. Assume that H is extended to two copies of K_r by Lemma 7.8, denoted K^1 and K^2, in $\mathcal{T}$ with $H_2 \subseteq K^2$. Let $\{x, y, z\} \subseteq V(H)$ be the three vertices in $V(H_1) \cap V_{ex}^S(\beta/2, k)$. In fact, one may assume that $|V(K^2) \cap V(F'_1)| = 1$. Otherwise, by replacing the vertex in $V(K^2) \cap V(F'_0)$ with a vertex in $N(H_2, F_1) \cap V_e(2\beta')$, we get a new tiling $\mathcal{T}'$ such that $G[B \setminus V(\mathcal{T}')]$ contains a perfect matching. Since $z \in V_{ex}^S(\beta/2, i)$, Claim 7.4 implies that there exist $z_1, z_2 \in V(F'_1)$ such that $G[\{z, z_1, z_2\}]$ is a K_3. Thus, there exist two vertex-disjoint $(2, \beta/4)$-bases $H' := G[\{w, x, y\}] \cup H_2$ and $H'' := G[\{z, z_1, z_2\}] \cup Q$ in G. By applying Lemma 7.8, Claim 7.9 and Proposition 2.10, we obtain a K_r-tiling $\mathcal{T}'$ covering $V_{ex}(\beta/2)$ such that $G[B \setminus V(\mathcal{T}')]$ admits a perfect matching. For the case $|F_0| \geq 2$, reapplying these results together with Claim 7.10 yields a K_r-tiling $\mathcal{T}'$ with the same property.

Next, we assume that there exists an integer $k \in [r-2]$ such that $|V_{ex}^S(\beta/2, k)| = 2$. Let $\{u, v\} = V_{ex}^S(\beta/2, k)$. It follows from (18) that $N(\{u, v\}, F'_i) \neq \emptyset$ for each $i \in \mathbb{Z}_2$. Hence, there exists a vertex $w \in V(F'_j)$ for some $j \in \mathbb{Z}_2$ satisfying the following:

- $H := G[\{u, v, w\}] \cup F'$ forms a $(2, \beta/4)$-base, where $F' \in \mathcal{F}'_k$.
- Assume $K^1 \cup K^2$ are two vertex-disjoint K_r copies covering H and avoiding $V(\mathcal{T})$ (via Lemma 7.8). Define $\mathcal{T}'$ as the K_r-tiling obtained from $\mathcal{T} \cup \{K^1, K^2\}$ by deleting the K_r copies that covered the base containing $\{u, v\}$ in $\mathcal{T}$. Then $|V(F'_\ell) \setminus V(\mathcal{T})|$ is even for each $\ell \in \mathbb{Z}_2$.

Together with Claim 7.9, and Proposition 2.10, there exists a perfect matching in $G[B \setminus V(\mathcal{T}')]$. Thus, in what follows, we consider that

$$|V_{ex}^S(\beta/2, k)| \leq 1 \text{ for each } k \in [r-2]. \tag{27}$$

Suppose that there exists a subgraph $L \in \bigcup_{k\in[r-2]} \mathcal{F}'_k$ such that $N(L, F'_i) \neq \emptyset$ for all $i \in \mathbb{Z}_2$. By an argument similar to the case that $|V_{ex}^S(\beta/2, k)| = 2$ for some $k \in [s]$, we conclude that there exists a K_r-tiling $\mathcal{T}'$ covering $V_{ex}(\beta/2)$ for which $G[B \setminus V(\mathcal{T}')]$ contains a perfect matching. In the following, assume that

$$\text{for any } L \in \bigcup_{k\in[r-2]} \mathcal{F}'_k, \text{ we have } N(L, F'_i) = \emptyset \text{ for some } i \in \mathbb{Z}_2. \tag{28}$$

Fix a $k \in [s]$ with $s_k = 1$, let $u_k v_k \in E(G[A_k]) \setminus E(G[A_k \cap V_b(2\beta, k)])$. Since $V_{ne}(2\beta') = V_{ex}(\beta/2)$, one has $A_k \setminus V_b(2\beta, k) \subseteq V_e(2\beta')$. Without loss of generality, assume that $u_k \in V_e(2\beta') \setminus V_b(2\beta)$.

Suppose that $v_k \notin V_b(2\beta, k)$. Then $N(\{u_k, v_k\}, B) \geq \frac{2n}{r} - 4\beta n$ and u_kv_k forms a $\beta/4$-base in G. It follows from (28) that $|F_0| \leq 5\beta n$. However, (22) implies that $G[\{u_k, v_k\}, V(F_0)]$ is a complete bipartite graph, which also contradicts (28). Thus, $v_k \in V_b(2\beta, k)$. Similarly, we can show that $v_k \notin V_{ex}^L(\beta/2)$, that is, $v_k \in V_b(2\beta, k) \setminus V_{ex}^L(\beta/2)$. Therefore, each graph in $\mathcal{F}_k'$ is exactly an edge. Moreover,

$$E(G[A_k]) \setminus E(G[A_k \cap V_b(2\beta, k)]) = E(G[V_e(2\beta') \setminus V_b(2\beta, k), V_b(2\beta, k) \setminus V_{ex}^L(\beta/2)]).$$

Next, we show that $E(G[A_k]) \setminus E(G[A_k \cap V_b(2\beta)])$ contains no edge that is vertex-disjoint from u_kv_k. Assume for contradiction that it contains two independent edges, say u_kv_k and $u'v'$. Assume that $u' \in V_e(2\beta') \setminus V_b(2\beta, k)$ and $v' \in V_b(2\beta, k) \setminus V_{ex}^L(\beta/2)$. If $|F_0| \geq 2$, then applying Claim 7.10 with $K = u'v'$, one may obtain a $(2, \beta/5)$-base H' such that $\{u, v\} \subseteq V(H')$, $|B \cap V(H')| = 4$ and $|V(F_i) \setminus V(H')|$ is even for every $i \in \mathbb{Z}_2$. Thus, Lemma 7.8 and Proposition 2.10 guarantee the existence of a new K_r-tiling $\mathcal{T}'$ such that $G[B \setminus V(\mathcal{T}')]$ has a perfect matching. Now, we assume $|F_0| = 1$ and let w be the unique vertex in F_0. Let $M^* := \left(\bigcup_{j\in[r-2]} \mathcal{F}_j'\right) \cup \{u'v'\}$. Clearly, M^* is a matching of size $|V_{ex}^S(\beta/2)| + 1$. Since $E(G[V(F_0'), V(F_1')]) = \emptyset$, every vertex $u \in V(F_1')$ satisfies $d(u) \leq n - |F_0'| - 1$. By Ore's condition, one has

$$d(w) \geq n - |F_1'| - |V_{ex}^S(\beta/2)| - 1.$$

Therefore, there exists an edge $xy \in E(M^*)$ such that $\{x, y\} \subseteq N(w)$. Notice that $d(\{x, y\}, B) \geq \frac{n}{r} - 2\beta n$ and $|V(F_0') \cup V_{ex}^S(\beta/2)| < 2\beta n$. Hence, $N(\{x, y\}, F_1') > 0$, which contradicts (28). Thus, we have that $G[A_k]$ is a star with center v_k together with some isolated vertices. Let K^k be the K_r copy in $\mathcal{T}$ that covers v_k.

Let w_k denote the unique vertex in $V_{ex}^S(\beta/2, k)$. Suppose that there exists a vertex $x_k \in A_k \setminus V_b(2\beta, k)$ such that $w_kx_k \in E(G)$. Since $d(x_k) \geq (1-1/r)n - 1$, all but at most two vertices in B are adjacent to x_k. By Claim 7.4, (22) and the fact that $w_k \in V_{ex}^S(\beta/2, k)$, it is easy to check that $N(\{w_k, x_k\}, F_i') \neq \emptyset$ for each $i \in \mathbb{Z}_2$. Assume that $V(K^k) \cap V(F_j') \neq \emptyset$ for some $j \in \mathbb{Z}_2$. Choose $r_k \in N(\{w_k, x_k\}, F_{j+1}')$ such that $G[\{w_k, x_k, r_k\}]$ is a $\beta/4$-base. By replacing the $(2, \beta/4)$-base covering w_k with $G[\{w_k, x_k, r_k\}]$ in $\mathcal{H}$, together with Lemma 7.8, Claim 7.9, and Proposition 2.10, we obtain a K_r-tiling $\mathcal{T}'$ covering $V_{ex}(\beta/2)$ such that $G[B \setminus V(\mathcal{T}')]$ contains a perfect matching. In what follows, we assume that

$$w_kx \notin E(G) \text{ for any } x \in A_k \setminus V_b(2\beta, k).$$

Thus, $d(x) \leq (1 - \frac{1}{r})n$ for every $x \in A_k \setminus V_b(2\beta)$. Together with $w_k \in S$ and Ore's condition, one has $d(x) = (1 - \frac{1}{r})n$ for every $x \in A_k \setminus V_b(2\beta)$, and $d(w_k) = (1 - \frac{1}{r})n - 2$. Thus,

$$G[A_k] \text{ is a star with center } v_k \text{ for all } k \in [r-2] \text{ with } s_k = 1.$$

A similar argument shows that for all $k \in [r-2]$ with $s_k = 0$, we have $G[A_k] = \emptyset$, whence Ore's condition implies that $G[A_k, V_{ex}^S(\beta/2)]$ is complete.

The above argument implies that for every $x \in A_i \setminus \{v_i\}$ and each $i \in [r-2]$, one has

$$N(x) = \begin{cases} \big(V(G) \setminus (A_i \cup \{w_i\})\big) \cup \{v_i\} & \text{if } s_i = 1, \\ V(G) \setminus A_i & \text{if } s_i = 0. \end{cases} \tag{29}$$

Moreover, $|V_b(2\beta)| = |V_{ex}^S(\beta/2)| =: q$, and by (28) one has $N(v_i, F_j') = \emptyset$ for some $j \in \mathbb{Z}_2$ and all $v_i \in V_b(2\beta)$. Let

$$Y_0 := \{y \in V_b(2\beta) : N(y, F_0') = \emptyset\} \text{ and } Y_1 := \{y \in V_b(2\beta) : N(y, F_1') = \emptyset\}.$$

Choose $z_0 \in V(F_0')$ and $z_1 \in V(F_1')$. By Ore's condition, one has

$$2\Big(1 - \frac{1}{r}\Big)n - 2 \leq d(z_0) + d(z_1) \leq 2\Big(1 - \frac{2}{r}\Big)n - (|Y_0| + |Y_1|) + 2q + |F_0'| + |F_1'| - 2 = 2\Big(1 - \frac{1}{r}\Big)n - 2.$$

Thus, $d(z_0, V^S_{ex}(\beta/2)) = d(z_1, V^S_{ex}(\beta/2)) = q$. Hence $G[V^S_{ex}(\beta/2), V(F_0' \cup F_1')]$ is a complete bipartite graph.

Define $\hat{G} := G[V^S_{ex}(\beta/2), V_b(2\beta)]$. Recall that $d(w_i) = (1-\frac{1}{r})n - 2$ for each $w_i \in V^S_{ex}(\beta/2)$. Together with (29), one has $d(w_i, V_b(2\beta)) = q - 2$ for each $w_i \in V^S_{ex}(\beta/2)$, this implies $e(\hat{G}) = q(q-2)$. Thus, there exists a vertex $v_t \in V_b(2\beta)$ such that $d(v_t, V^S_{ex}(\beta/2)) \le q-2$. Without loss of generality, assume that $v_t \in Y_0$. Choose $y \in V(F_0')$. By Ore's condition one has

$$\begin{aligned}2\Big(1-\frac{1}{r}\Big)n - 2 \le d(v_t) + d(y) \le & \Big(\Big(1-\frac{2}{r}\Big)n - |V_b(2\beta)| + d(v_t, V_b(2\beta)) + d(v_t, V^S_{ex}(\beta/2)) + |F_1'|\Big) \\ & + \Big(\Big(1-\frac{2}{r}\Big)n - |Y_0| + |F_0'| - 1 + |V^S_{ex}(\beta/2)|\Big) \\ \le & 2\Big(1-\frac{1}{r}\Big)n - 3 + d(v_t, V_b(2\beta)) - |Y_2|.\end{aligned}$$

It follows that $d(v_t, V_b(2\beta)) \ge |Y_0| + 1$. That is, there exists a vertex $v_\ell \in Y_1 \cap A_\ell$ for some $\ell \in [r-2]$ such that $v_t v_\ell \in E(G)$.

Obviously, there exists a copy of K_r, say K, containing v_t, v_ℓ such that $|V(K) \cap A_j| = 1$ for each $j \in [r-2] \setminus \{t, \ell\}$ and $|V(K) \cap A_j| = 2$ for each $j \in \{t, \ell\}$. Recall that $G[V^S_{ex}(\beta/2), V(F_0' \cup F_1')]$ is a complete bipartite graph. Hence, there are two vertices $z, z' \in V(F_i)$ for some $i \in \mathbb{Z}_2$ such that $G[\{w_t, w_\ell, z, z'\}]$ forms a clique K_4. Moreover, there is a K_r copy, say K', containing $\{w_t, w_\ell, z, z'\}$ such that $|V(K') \cap A_j| = 1$ for each $j \in [r-2] \setminus \{t, \ell\}$. Let $\mathcal{T}' := (\mathcal{T} \setminus \{K_r \in \mathcal{T} : V(K_r) \cap \{v_t, v_\ell, w_t, w_\ell\} \ne \emptyset\}) \cup \{K, K'\}$. Since $E(G[F_0', F_1']) = \emptyset$, every K_r in $\mathcal{T}$ covering some $w_i \in V^S_{ex}(\beta/2)$ must have two vertices within either F_0' or F_1'. It follows that $G[B \setminus V(\mathcal{T}')]$ consists of two components with even order. Therefore, Proposition 2.10 implies that there exists a perfect matching in $G[B \setminus V(\mathcal{T}')]$.

By the above argument, we obtain that when $r - s = 2$, either G is the extremal graph in (EX2), or there exists a K_r-tiling $\mathcal{T}'$ satisfying (B2), as desired. □

8 Concluding remarks

8.1 Algorithmic aspects of the proof

We point out those parts of the proof where translation into an algorithm is nontrivial.

Non-extremal case: To construct the desired absorbing set, we first note that for almost all r-set in G, $\Omega(n^{r^2})$ absorbers can be found in $O(n^{r^2})$ time. In our proof, the existence of the absorbing set is established by probabilistic arguments, which naturally lead to a randomized construction. However, we can directly apply the result of Garbe and Mycroft [10, Proposition 4.7], which provides an $O(n^{4r^2})$-time deterministic algorithm via the conditional expectation method.

For the almost cover, the algorithm of Alon et al. [1] yields a regular partition that satisfies the Regularity lemma in $O(n^{2.376})$ time. As the order of the reduced graph is bounded, Claims 4.12 and 4.13 take constant time. The blow-up process via Lemmas 4.5 and 4.9 takes $O(n^{r+1})$ time. Thus, for the non-extremal case, the overall complexity of constructing a K_r-tiling is $n^{O_r(1)}$.

Extremal case: For the extremal case, an (r, s)-partition as in Claim 6.1 can be found in $O(n^{2.376})$ time using Algorithm 2 of Gan, Han, and Hu [9]. The remaining steps are completed by a greedy algorithm, yielding an overall time complexity of $n^{O_r(1)}$.

When r is a constant, Theorem 1.7 can be achieved in polynomial time. Therefore, Conjecture 1.4 holds for $k \ge cn$, where $n := |G|$ is sufficiently large and $c > 0$ is any fixed constant.

8.2 Further research

In this paper, we prove that the Chen-Lih-Wu conjecture holds when n is sufficiently large and $k \ge cn$ for any positive constant c. In fact, we establish a stronger result under the Ore-type condition. Our proof necessitates the condition $k = \Omega(n)$ due to the following requirements: in the non-extremal case, $n \geq r^{2^{c_0 r}}$ for Lemma 4.1 and Lemma 4.3; in the extremal case, $n \geq r^{2^{c_1 r}}$ due to the parameter relation $\gamma_i \leq \gamma_{i+1}^5$ from (11)-(13). Given that $n \leq rk$, these bounds are satisfied for sufficiently large n if $k = \Omega(n)$, thereby establishing Conjectures 1.1 and 1.3 under this condition. A natural and interesting direction for future work is to consider the case when $k = o(n)$.

Our paper focuses on the problem of equitable coloring under Ore-type conditions. Another natural direction for future research is to extend this work to degree sequence conditions. Inspired by the degree sequence version of the Hajnal-Szemerédi conjecture proposed by Balogh, Kostochka, and Treglown [3], we formulate a corresponding conjecture for equitable coloring, thereby introducing a new perspective rooted in degree sequences.

Conjecture 8.1. *Let $n, k \in \mathbb{N}$ such that $k \leq n$. Suppose that G is an n-vertex graph with degree sequence $d_1 \geq \cdots \geq d_n$ such that*

- $d_i < 2k - i$ *for all* $i < k$*;*
- $d_{k+1} < k$.

Then G contains an equitable k-coloring.

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