

Ringdown modulation of acceleration radiation in the Schwarzschild background

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We present an analytic, first-order description of how black hole ringdown imprints on the operational signature of near-horizon thermality. Building on a static Schwarzschild baseline in which a freely falling two-level system coupled to a single outgoing mode exhibits geometric photon statistics and a detailed-balance ratio set by the surface gravity, we introduce an even-parity, axisymmetric quadrupolar perturbation and work in an ingoing Eddington-Finkelstein, horizon-regular framework. The perturbation corrects the outgoing eikonal through a gauge-invariant double-null contraction of the metric, yielding a compact redshift map that, when pulled back to the detector worldline, produces a universal, decaying-oscillatory modulation of the Boltzmann exponent at the quasinormal frequency. We derive a closed boundary formula for the response coefficient at the sampling radius, identify the precise adiabatic window in which the result holds, and prove that the modulation vanishes in all stationary limits. Detector specifics (gap, switching wavepacket width) enter only through a smooth prefactor, while the geometric content is captured by the quasinormal pair and the response coefficient. The analysis clarifies that near-horizon "thermality" is robust but not rigid: detailed balance persists as the organizing structure and is gently driven by ringdown dynamics. The framework is minimal yet extensible to other multipoles, parities, and slow rotation, and it suggests direct numerical and experimental cross-checks in controlled analog settings.

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I. INTRODUCTION

Black-hole horizons are often said to be "thermal" [1–6], a statement that is made precise in several complementary ways: by the Hawking flux at future null infinity, by the near-horizon Rindler structure, and by detailed-balance/KMS (Kubo-Martin-Schwinger) relations [7–9] seen by suitable probes. A recent operational formulation shows that a freely falling two-level system, coupled to a single outgoing mode of fixed frequency, exhibits a geometric steady state whose absorption-to-excitation ratio is governed by the surface gravity. That result is strictly stationary, however [10]. Real astrophysical black holes are not: after formation or perturbation, they ring down through damped quasinormal modes [11–17]. Whether, and in what sense, the detailed-balance structure [18–20] survives during this time dependence is a sharp and physically necessary question.

In this work, we answer that question analytically at first order in the ringdown amplitude. We consider the even-parity, axisymmetric quadrupolar perturbation of Schwarzschild [21–23] and show that the detailed-balance exponent is not destroyed but rather acquires a universal, decaying-oscillatory modulation at the quasinormal frequency. The origin of the effect is simple and geometric: the perturbation corrects the outgoing eikonal (the retarded time that sets the phase of the outgoing mode) through a line-of-sight double-null contraction of the metric perturbation. When pulled back to the detector's worldline, this induces a slowly varying gauge-invariant shift in the redshift map, which in turn modulates the single-mode excitation and absorption probabilities. The photon statistics remain geometric; only their parameter becomes time dependent. In the static and late-time limits, the modulation vanishes, recovering the stationary Schwarzschild result [24].

Our approach keeps horizon regularity manifest, works directly with ingoing Eddington-Finkelstein structure [25, 26], and isolates all background dependence into two quantities: the complex quasinormal frequency pair and a response coefficient that we express in closed form at the sampling radius. Detector specifics (gap, switching, and wavepacket width) are encapsulated in a single smooth prefactor that factors out of universal ratios [27, 28]. The framework is minimal yet extensible: it generalizes to other multipoles and parities, admits a slow-rotation expansion toward Kerr, and accommodates alternate worldlines and switching protocols. Throughout, we maintain careful control of gauge, short-distance (Hadamard) behavior, and the adiabatic window in which the near-horizon/Rindler reduction [29] is reliable.

Closely related operational approaches use Unruh-DeWitt-type probes in black hole spacetimes [27, 28] and, in quantum-optics language, atoms crossing cavities near horizons. In static backgrounds, detector response and its KMS/detailed-balance

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content have been analyzed in detail for spherically symmetric spacetimes and near-horizon limits, providing clean baselines for comparison with stationary results and clarifying how switching and sampling affect thermality signatures [30, 31]. On the quantum-optics side, Scully and collaborators studied atoms falling through a cavity toward a Schwarzschild black hole [10] and showed that the emitted horizon-brightened acceleration radiation (HBAR) radiation can mimic Hawking radiation to distant observers, which is an operational setup treated in a strictly stationary geometry [10, 32–48]. Conceptual links between near-horizon instability and emergent thermality, as well as detector-centric KMS diagnostics in gently time-dependent settings, further motivate asking how equilibrium signatures survive slow driving [49, 50]. By contrast, the ringdown regime is typically addressed at the level of classical quasinormal modes and waveform physics rather than quantum response (see, e.g., standard reviews), leaving open the question of how QNM dynamics imprint on detector detailed balance [14]. Very recent work examining “what Hawking radiation looks like as you fall” emphasizes finite-time, detector-based thermality tests, reinforcing the timeliness of our analytic, first-order treatment of KMS/detailed-balance modulation during ringdown [51–53].

The paper is organized as follows. Section II reviews the static Schwarzschild baseline and sets up the even-parity ringdown geometry, detector kinematics, and single-mode coupling. Section III develops the perturbative framework: we fix an EF-regular null frame, derive the linear transport for the retarded-time correction, and translate it into a pulled-back Wightman structure and response integrals. Section IV presents the main result: a closed, ringdown-modulated detailed-balance law, together with a boundary formula for the response coefficient and a static-limit proposition. Section V delineates the regime of validity and consistency, covering the near-horizon window, gauge issues, and regularity. Section VI interprets the result physically, consolidates extensions and generalizations, and highlights how the closed formula enables direct checks and applications. Unless otherwise stated, $G = c = \hbar = 1$

II. BACKGROUND AND SETUP

In this section, we assemble the ingredients used throughout the paper. We first summarize the static Schwarzschild baseline, which is the setting in which a freely falling two-level system couples to a selected outgoing field mode and exhibits thermal detailed balance governed by the surface gravity. This provides the operational statement of near-horizon thermality that our ringdown analysis will perturb. We fix conventions, coordinates, and detector-field coupling so that the subsequent quasinormal-mode (QNM) calculation can be presented as a controlled first-order deformation of these formulas.

A. Static Schwarzschild baseline

We work with the Schwarzschild metric [54–57]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

and introduce the tortoise coordinate and Eddington-Finkelstein null coordinates

$$r_* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| \quad u = t - r_*, \quad v = t + r_*. \quad (2)$$

The surface gravity is [55, 58]

$$\kappa = \frac{1}{4M} = \frac{1}{2r_g}, \quad (3)$$

with gravitational radius $r_g = 2M$. Near the horizon ($r \rightarrow 2M$), the metric is Rindler-like and u plays the role of Rindler time for outgoing modes.

Following the operational setup of [10], we consider identical two-level systems (gap $\omega > 0$) that fall freely from rest at infinity along radial geodesics. For such geodesics (specific energy $E = 1$), the worldline obeys

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}, \quad \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}, \quad (4)$$

so that near the horizon, the outgoing null coordinate pulled back to the worldline has the universal logarithmic form

$$u(\tau) = u_0 - \frac{1}{\kappa} \ln [\kappa(\tau_H - \tau)] + \mathcal{O}(\tau_H - \tau), \quad (5)$$

where τ is the atom's proper time and τ_H is the finite proper time at which the horizon is crossed.

We select a single outgoing field mode of frequency $\nu > 0$ defined with respect to u at future null infinity. In the rotating-wave/anti-rotating-wave decomposition along the worldline, the relevant interaction picture matrix elements carry the phase

$$\Phi(\tau) = \nu u(\tau) \pm \omega \tau, \quad (6)$$

with "+" for excitation with emission (counter-rotating process) and "-" for de-excitation with absorption.

To leading order in the atom-field coupling g , the excitation probability for emitting one quantum into the selected mode is

$$P_{\text{exc}} = g^2 \left| \int d\tau e^{i\nu u(\tau)} e^{i\omega \tau} \right|^2. \quad (7)$$

Using the near-horizon form (5), the integral reduces (via a change of variables $x \propto \tau_H - \tau$) to a standard gamma-function integral, yielding a Planckian factor [10],

$$P_{\text{exc}} = \mathcal{N} \frac{1}{e^{2\pi\nu/\kappa} - 1}, \quad P_{\text{abs}} = \mathcal{N} \frac{e^{2\pi\nu/\kappa}}{e^{2\pi\nu/\kappa} - 1}, \quad (8)$$

where $\mathcal{N}$ is the same smooth prefactor in both channels (its explicit form depends on ω and the long-time windowing but cancels in ratios in the regime $\omega \gg \nu$ emphasized in Ref. [10]). Consequently,

$$\frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = e^{2\pi\nu/\kappa} = e^{4\pi r_g \nu}, \quad (9)$$

which is the detailed-balance relation characteristic of a KMS state at local Tolman temperature $T_H = \kappa/2\pi$ when viewed by the freely falling detector sampling the outgoing mode.

Embedding the atoms in a weakly leaky single-mode cavity (as in Ref. [10]) and iterating (8) leads to a geometric steady state for the mode occupation,

$$p_n = (1 - e^{-2\xi}) e^{-2\xi n}, \quad 2\xi = \ln \left(\frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} \right) = \frac{2\pi\nu}{\kappa}. \quad (10)$$

Eq.s (8)-(10) constitute the baseline we shall perturb in Section III: during ringdown, $u(\tau)$ acquires a small, explicitly computable correction, and the exponent $2\pi\nu/\kappa$ becomes gently time-dependent while retaining the geometric statistics to first order.

B. Even-parity ringdown of Schwarzschild

We model the post-merger geometry as a linear perturbation of Schwarzschild driven by the black hole's quasinormal modes (QNMs). Writing

$$g_{ab} = g_{ab}^{(0)} + \varepsilon h_{ab} \quad 0 < \varepsilon \ll 1, \quad (11)$$

we expand h_{ab} in tensor spherical harmonics. In this paper, we begin with the even-parity sector, for which the dynamics are encoded in the Zerilli-Moncrief gauge-invariant master function $\Psi_{\ell m}(t, r)$ [23, 59].

Let r_* be the tortoise coordinate and $\lambda = (\ell - 1)(\ell + 2)/2$. The even-parity master field obeys [15, 16, 22, 59]

$$(-\partial_t^2 + \partial_{r_*}^2 - V_\ell^Z(r)) \Psi_{\ell m}(t, r) = 0, \quad (12)$$

with the Zerilli potential [14, 22, 60]

$$V_\ell^Z(r) = \frac{2(1 - \frac{2M}{r})}{r^3(\lambda r + 3M)^2} [\lambda^2(\lambda + 1)r^3 + 3M\lambda^2 r^2 + 9M^2\lambda r + 9M^3]. \quad (13)$$

Ringdown is described by a superposition of homogeneous QNMs with complex frequencies $\omega_{\ell n} = \omega_R - i\omega_I$ ($\omega_I > 0$). For a single mode we take [15, 29, 61, 62]

$$\Psi_{\ell m}(t, r) = \mathcal{A}_{\ell m} \psi_\ell(r) e^{-i\omega t}, \quad \psi_\ell \sim \begin{cases} e^{+i\omega r_*}, & r \rightarrow \infty \text{ (outgoing)}, \\ e^{-i\omega r_*}, & r \rightarrow 2M \text{ (ingoing)}, \end{cases} \quad (14)$$

where $\mathcal{A}_{\ell m}$ is a (dimensionless) amplitude fixed by the preceding nonlinear dynamics but treated as $\mathcal{O}(\varepsilon)$ here. The boundary conditions in (14) select the discrete spectrum $\omega_{\ell n}$.

To display horizon regularity, we use ingoing Eddington-Finkelstein (EF) coordinates (v, r, θ, ϕ) , $v = t + r_*$, so that the background metric reads [25, 26]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \quad (15)$$

A QNM of the form $e^{-i\omega t} e^{-i\omega r_*} = e^{-i\omega v}$ is manifestly regular on the future horizon $r = 2M$. We therefore reconstruct h_{ab} from $\Psi_{\ell m}$ in an EF-regular gauge (equivalent to transforming the standard Regge-Wheeler gauge expressions). All metric amplitudes remain finite at $r = 2M$ for the ingoing solution.

The even-parity perturbation can be written as [21, 22, 59]

$$h_{ab} = \sum_{\ell m} [H_0^{\ell m} Y_{\ell m}(dv)_a (dv)_b + 2H_1^{\ell m} Y_{\ell m}(dv)_{(a} (dr)_{b)} + H_2^{\ell m} Y_{\ell m}(dr)_a (dr)_b + K^{\ell m} r^2 \gamma_{ab}^{\perp} Y_{\ell m} + G^{\ell m} r^2 Y_{ab}^{\ell m}], \quad (16)$$

where $\gamma_{ab}^{\perp}$ is the metric on S^2 and $Y_{ab}^{\ell m}$ is the even-parity tracefree tensor harmonic. Each coefficient H_0, H_1, H_2, K, G is an algebraic combination of $\Psi_{\ell m}$ and its r - and t -derivatives divided by $\lambda r + 3M$. We will only need the contraction of h_{ab} with outgoing null directions; the explicit (but lengthy) formulas are standard and can be inserted when we evaluate observable coefficients.

The dominant ringdown is the quadrupole. We align the detector on the symmetry axis and take the axisymmetric mode:

$$\Psi_{20}(t, r) = \mathcal{A}_{20} \psi_2(r) e^{-i\omega t}, \quad Y_{20}(\theta) = \sqrt{\frac{5}{4\pi}} \frac{1}{2} (3 \cos^2 \theta - 1), \quad Y_{20}(0) = \sqrt{\frac{5}{4\pi}}. \quad (17)$$

On the axis ($\theta = 0$) all azimuthal derivatives vanish, and the only angular dependence is the constant $Y_{20}(0)$. Consequently, along the axis the nonvanishing components of h_{ab} reduce to EF scalars multiplying $Y_{20}(0)$, e.g.

$$h_{vv} = \varepsilon \mathcal{H}_{vv}(r) Y_{20}(0) e^{-i\omega v}, \quad h_{vr} = \varepsilon \mathcal{H}_{vr}(r) Y_{20}(0) e^{-i\omega v}, \quad h_{rr} = \varepsilon \mathcal{H}_{rr}(r) Y_{20}(0) e^{-i\omega v}, \dots \quad (18)$$

with $\mathcal{H}_{ab}(r)$ regular at $r = 2M$ and determined by $\psi_2(r)$.

In the baseline (II A), the detector couples to an outgoing mode with phase $u = t - r_*$. In a time-dependent spacetime, the relevant phase is the solution of the eikonal equation [6, 56]

$$g^{ab} \partial_a u \partial_b u = 0, \quad (19)$$

normalized to match the usual retarded time at infinity. Linearizing (19) about Schwarzschild with $g_{ab} = g_{ab}^{(0)} + \varepsilon h_{ab} \rightarrow g_{ab} = g_{ab}^{(0)} - \varepsilon h^{ab} + \mathcal{O}(\varepsilon^2)$, and $u = u_0 + \varepsilon \delta u$, gives the transport equation along the background outgoing null congruence $k^a = \nabla^a u_0$:

$$k^a \nabla_a \delta u = \frac{1}{2} h_{ab} k^a k^b \quad (20)$$

on $g^{(0)}$. Eq. (20) is the key bridge from ringdown geometry to detector physics: it shows that the correction δu is sourced by the double-null contraction $h_{ab} k^a k^b$ of the even-parity metric perturbation.

For the EF background, an outgoing principal null vector is $k^a = (\partial_r)^a$ at fixed u (or, equivalently, the covector $k_a = (du)_a = -(dv)_a$ up to normalization). Using (18), the source term is simply a linear combination of $\mathcal{H}_{vv} \mathcal{H}_{vr} \mathcal{H}_{rr}$ evaluated on the axis. Integrating (20) radially from the near-horizon region to the detector sampling radius $r = r_c$ yields

$$\delta u \Big|_{r_c, \theta=0} = \varepsilon \mathcal{C}_{20}(r_c) \Re [e^{-i\omega v_c}], \quad v_c \equiv v(\tau_c), \quad (21)$$

where the ringdown response coefficient

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \mathcal{A}_{20} \int_{2M}^{r_c} \mathcal{S}_{20}(r) dr \quad (22)$$

is determined by the EF-regular combination $\mathcal{S}_{20}(r)$ of ψ_2 and its derivatives (explicit formula supplied where needed). The time dependence $\propto e^{-i\omega v}$ implies a decaying oscillation at frequency ω_R with envelope $e^{-\omega_I v}$.

In Fig. 1, we can see that as the sampling radius moves away from the horizon, the EF-regular reconstruction makes the near-horizon suppression transparent: $\mathcal{C}_{20}(r_c)$ grows linearly just above $2M$ and then slowly transitions to the $1/r_c$ falloff

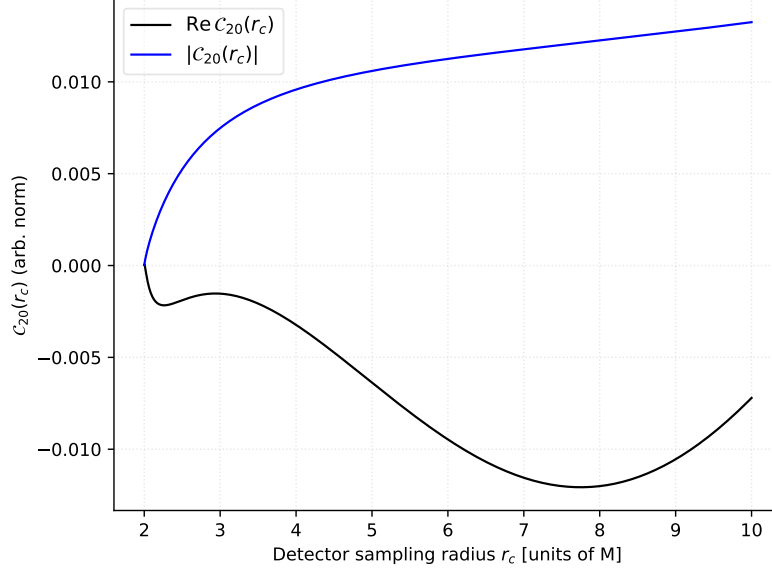


FIG. 1. Single-mode result $\mathcal{C}_{20}(r_c) = \frac{1}{2}\sqrt{5/4\pi}[a_2(r_c)\Psi(r_c) + b_2(r_c)\partial_r\Psi(r_c)]$ from a Zerilli integration with ingoing-horizon data; both $\text{Re}\mathcal{C}_{20}$ and $|\mathcal{C}_{20}|$ are shown.

expected from the Zerilli/MQNM asymptotics. The sign flip of $\text{Re}\mathcal{C}_{20}$ is not a numerical artifact but a geometric phase effect: along the outgoing congruence the ingoing-regular QNM behaves like $e^{-i\omega v}$ with an r -dependent phase inherited from r_* ; accumulation of this phase can drive $\text{Re}[a_2\Psi_{20} + b_2\partial_r\Psi_{20}]$ through zero before the far-zone decay sets in. Physically, $|\mathcal{C}_{20}(r_c)|$ is the lever arm that converts the curvature perturbation into a shift of the detector's redshift map u , and hence, into a modulation of the detailed-balance exponent, while the sign of $\text{Re}\mathcal{C}_{20}$ decides whether the first visible oscillation overshoots or undershoots the thermal baseline. This also motivates a practical choice of cavity altitude: pushing r_c too close to the horizon sacrifices amplitude (regularity suppresses the signal), while pushing it too far loses amplitude to the $1/r_c$ tail; the sweet spot is the first broad maximum of $|\mathcal{C}_{20}|$ above $2M$.

With $u(\tau) = u_0(\tau) + \varepsilon\delta u(\tau)$, the interaction phase becomes

$$\Phi(\tau) = \nu u(\tau) \pm \omega_A \tau = \nu u_0(\tau) \pm \omega_A \tau + \varepsilon \nu \delta u(\tau) \quad (23)$$

so that all excitation/absorption amplitudes inherit a parametric modulation through $\delta u(\tau)$. As we show in Section III, to first order in ε , this produces a multiplicative correction to the Boltzmann exponent governing the detailed-balance ratio, oscillating at ω_R and decaying at ω_I , while preserving the geometric single-mode statistics.

We choose $\mathcal{A}_{20}$ such that the gauge-invariant energy content in the ringdown slice is $\propto \varepsilon^2 |\mathcal{A}_{20}|^2$. All observable corrections in this paper scale linearly with $\varepsilon \mathcal{A}_{20}$ via $\mathcal{C}_{20}(r_c)$ in (22). The static limit $\varepsilon \rightarrow 0$ (or late-time limit $v \rightarrow \infty$) reduces to II A exactly.

C. Detector kinematics and mode geometry

We model each probe as a two-level Unruh-DeWitt-type system with energy gap $\omega_A > 0$ [27], following radial free fall through a small, weakly leaky single-mode cavity centered at radius $r = r_c$ on the symmetry axis ($\theta = 0$). The coupling to the quantum field is switched on only during the brief transit across the cavity.

Let E be the conserved specific energy of a radial timelike geodesic in Schwarzschild. The kinematics are [60]

$$\frac{dt}{d\tau} = \frac{E}{1 - 2M/r}, \quad \frac{dr}{d\tau} = -\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)}, \quad \frac{d\phi}{d\tau} = 0. \quad (24)$$

We take $E = 1$ (fall from rest at infinity) as the baseline; the $E \neq 1$ generalization will only rescale subleading prefactors. The tortoise coordinate and null times are as in (1),

$$r_* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| \quad u = t - r_*, \quad v = t + r_*. \quad (25)$$

Along the geodesic, near the horizon $r \rightarrow 2M$, the retarded time pulled back to the worldline has the universal logarithmic behavior

$$u(\tau) = u_0 - \frac{1}{\kappa} \ln [\kappa(\tau_H - \tau)] + \mathcal{O}(\tau_H - \tau), \quad (26)$$

where $\kappa = 1/(4M)$ and τ_H is the finite proper time of horizon crossing. In contrast, the advanced time remains finite,

$$v(\tau) = v_H + \mathcal{O}(\tau_H - \tau), \quad (27)$$

so that QNM phases $\propto e^{-i\omega v}$ are regular on the trajectory at the horizon.

When the ringdown perturbation is present, u solves the linearized eikonal equation (19), giving $u = u_0 + \varepsilon \delta u$. Evaluating (21) on the axis at the cavity crossing time $\tau = \tau_c$ yields

$$\delta u(\tau_c) = \varepsilon \mathcal{C}_{20}(r_c) \Re [e^{-i\omega v_c}], \quad v_c \equiv v(\tau_c), \quad (28)$$

where $\mathcal{C}_{20}(r_c)$ encodes the integrated even-parity source along the outgoing congruence from $2M$ to r_c [cf. (22)].

We use an EF-regular null dyad k^a, n^a adapted to outgoing/ingoing directions,

$$k^a \nabla_a u_0 = 0, \quad k^a k_a = 0, \quad n^a \nabla_a v = 0, \quad n^a n_a = 0, \quad k^a n_a = -1, \quad (29)$$

with k^a tangent to the background outgoing null geodesics. Eq. (20) shows that the only ringdown datum that enters the detector phase is the double-null contraction

$$\mathcal{H}_{kk} \equiv h_{ab} k^a k^b, \quad (30)$$

which is gauge invariant under even-parity transformations that preserve EF regularity and the normalization of k^a on the axis. In terms of the Zerilli master function Ψ_{20} , $\mathcal{H}_{kk}$ is an algebraic combination of Ψ_{20} and its r -derivative divided by $\lambda r + 3M$, evaluated at $\theta = 0$; its radial integral builds $\mathcal{C}_{20}(r_c)$.

We select a single outgoing mode defined with respect to the retarded time u at future null infinity. Operationally, we use a narrow wavepacket with central frequency $\nu > 0$ and envelope $f(u)$ supported during the atom's transit,

$$\hat{\Phi}_\nu(u) = \int \frac{d\tilde{\nu}}{2\pi} \tilde{f}(\tilde{\nu} - \nu) \hat{a}_{\tilde{\nu}} e^{-i\tilde{\nu}u} + \text{H.c.}, \quad \int du |f(u)|^2 = 1, \quad (31)$$

$[\hat{a}_{\tilde{\nu}} \hat{a}_{\tilde{\nu}'}^\dagger] = 2\pi \delta(\tilde{\nu} - \tilde{\nu}')$. In the cavity, the spatial profile is approximately constant over the atomic trajectory, so the relevant worldline pullback is entirely through the phase $u(\tau)$ and the envelope $f(u(\tau))$.

The atom-field interaction in the interaction picture is [28]

$$H_I(\tau) = g \chi(\tau) (\sigma_+ e^{+i\omega_A \tau} + \sigma_- e^{-i\omega_A \tau}) \hat{\Phi}_\nu(u(\tau)), \quad (32)$$

where g is the (small) coupling $\chi(\tau)$ is a smooth switching function that models the transit through the cavity of proper duration $\Delta\tau_c$, and $\sigma_\pm$ acts on the two-level system. To leading order in g , the excitation/de-excitation amplitudes are

$$\mathcal{A}_{\text{exc}} = g \int d\tau \chi(\tau) e^{+i\omega_A \tau} e^{+i\nu u(\tau)}, \quad \mathcal{A}_{\text{abs}} = g \int d\tau \chi(\tau) e^{-i\omega_A \tau} e^{-i\nu u(\tau)}. \quad (33)$$

We work in the regime

$$\kappa \ll \nu \lesssim \omega_A, \quad \Delta\tau_c \kappa \ll 1, \quad \omega_I \Delta\tau_c \ll 1, \quad (34)$$

which ensures: (i) the near-horizon (Rindler) structure controls the integrals; (ii) the ringdown envelope varies adiabatically across a single transit; and (iii) the wavepacket remains narrow in frequency relative to geometric scales.

Using $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$ with (26)-(28), we expand the phases to first order,

$$e^{\pm i\nu u(\tau)} = e^{\pm i\nu u_0(\tau)} [1 \pm i\varepsilon \nu \delta u(\tau)] + \mathcal{O}(\varepsilon^2). \quad (35)$$

The ε^0 term reproduces the static Schwarzschild baseline of II A. The $\mathcal{O}(\varepsilon)$ term injects the decaying oscillation $e^{-i\omega v_c}$ through $\delta u(\tau)$, yielding a linear, parametric modulation of excitation/absorption probabilities that we compute explicitly in Section III.

To make contact with the steady-state single-mode statistics, we approximate the transit as a compact support window centered at τ_c ,

$$\chi(\tau) = \chi_0 w\left(\frac{\tau - \tau_c}{\Delta\tau_c}\right), \quad \int d\tau \chi^2(\tau) = 1, \quad (36)$$

with w a fixed smooth bump. For $\Delta\tau_c \kappa \ll 1$ we may evaluate slowly varying factors at τ_c and use the near-horizon form (26) inside the integral. This leads to the gamma-function structure in the baseline probabilities and, after including (35), to their ringdown-modulated counterparts.

III. PERTURBATIVE FRAMEWORK

We now develop the linear response of the detector-mode system to an even-parity ringdown perturbation of Schwarzschild. Our strategy is to (i) fix a null frame that is regular on the future horizon, (ii) compute the first-order correction δu to the outgoing eikonal (u) by integrating the linearized transport equation along the background outgoing congruence, and (iii) pull this corrected redshift map $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$ onto the detector worldline. Subsections 3.2-3.3 will use $u(\tau)$ to evaluate the excitation and absorption amplitudes to $\mathcal{O}(\varepsilon)$ and derive the modulated detailed-balance relation.

A. Null frame and redshift map

In ingoing Eddington-Finkelstein (EF) coordinates (v, r, θ, ϕ) with background metric $ds^2 = -(1 - 2M/r)dv^2 + 2dvdr + r^2 d\Omega^2$, we choose the EF-regular null dyad

$$k^a = (\partial_r)^a, \quad n^a = -(\partial_v)^a - \frac{1}{2} \left(1 - \frac{2M}{r}\right) (\partial_r)^a, \quad k^a k_a = n^a n_a = 0, \quad k^a n_a = -1. \quad (37)$$

The background outgoing eikonal u_0 satisfies $\nabla^a u_0 \propto k^a$. We fix the normalization by requiring $k^a \nabla_a r = 1$, so the affine parameter along the outgoing rays is $\sigma = r$.

Let $g_{ab} = g_{ab}^{(0)} + \varepsilon h_{ab}$, and write the perturbed retarded time as $u = u_0 + \varepsilon \delta u$. Linearizing the eikonal equation $g^{ab} \partial_a u \partial_b u = 0$ gives the transport equation along the background congruence:

$$k^a \nabla_a \delta u = \frac{1}{2} h_{ab} k^a k^b \equiv \frac{1}{2} h_{kk}. \quad (38)$$

With the choice (37), $k^a \nabla_a = \partial_r$, hence

$$\partial_r \delta u(v, r, \theta) = \frac{1}{2} h_{kk}(v, r, \theta). \quad (39)$$

We integrate from the future horizon $r = 2M$ to the sampling radius $r = r_c$, fixing the integration constant by the EF-regular condition $\delta u|_{r=2M} = 0$:

$$\delta u(v r_c, \theta) = \frac{1}{2} \int_{2M}^{r_c} dr h_{kk}(v, r, \theta). \quad (40)$$

For the $(\ell, m) = (20)$ even-parity QNM, the metric perturbation reconstructed in an EF-regular gauge has the schematic form $h_{ab} \sim \mathcal{H}_{ab}(r) Y_{20}(\theta) e^{-i\omega v} + \text{c.c.}$. On the symmetry axis ($\theta = 0$), $Y_{20}(0) = \sqrt{5/4\pi}$ and all angular derivatives vanish, so the only datum that sources δu is the double-null contraction

$$h_{kk}(v, r, \theta = 0) = \varepsilon \sqrt{\frac{5}{4\pi}} \mathcal{S}_{20}(r) \Re[e^{-i\omega v}], \quad (41)$$

where $\mathcal{S}_{20}(r)$ is a regular EF combination of the Zerilli master function Ψ_{20} and its radial derivative divided by $\lambda r + 3M$ (as defined in II B). Substituting (41) into (40) yields

$$\delta u(v r_c, 0) = \varepsilon \mathcal{C}_{20}(r_c) \Re[e^{-i\omega v}], \quad \mathcal{C}_{20}(r_c) \equiv \frac{1}{2} \sqrt{\frac{5}{4\pi}} \int_{2M}^{r_c} \mathcal{S}_{20}(r) dr. \quad (42)$$

Eq. (42) is the promised redshift map: the perturbation imparts a decaying oscillatory correction to the retarded time with envelope $e^{-\omega_I v}$ and carrier ω_R .

Let $\tau \mapsto x^a(\tau)$ be the detector worldline (radial free fall with $E = 1$). Pulling back (42) to the detector at the cavity crossing time $\tau = \tau_c$ gives

$$u(\tau_c) = u_0(\tau_c) + \varepsilon \delta u(\tau_c), \quad \delta u(\tau_c) = \varepsilon \mathcal{C}_{20}(r_c) \Re[e^{-i\omega v_c}], \quad v_c \equiv v(\tau_c). \quad (43)$$

Near the horizon, the background map is the universal logarithm

$$u_0(\tau) = u_* - \kappa^{-1} \ln[\kappa(\tau_H - \tau)] + \mathcal{O}(\tau_H - \tau), \quad (44)$$

so the redshift rate along the worldline splits into baseline plus perturbation,

$$\frac{du}{d\tau} = \frac{1}{\kappa(\tau_H - \tau)} + \varepsilon \frac{d}{d\tau} \delta u(\tau) + \mathcal{O}(\varepsilon^2). \quad (45)$$

Using (42),

$$\frac{d}{d\tau} \delta u(\tau) = \varepsilon \left[\mathcal{C}'_{20}(r) \frac{dr}{d\tau} \Re(e^{-i\omega v}) + \mathcal{C}_{20}(r) \Re(-i\omega e^{-i\omega v}) \frac{dv}{d\tau} \right]_{\tau}, \quad (46)$$

which stays finite as $\tau \rightarrow \tau_H$ because $v(\tau)$ is regular there. The hierarchy $\omega_I \Delta \tau_c \ll 1$ (from II C) ensures that across a single cavity transit $\delta u(\tau)$ varies slowly compared with the near-horizon logarithmic growth of $u_0(\tau)$.

Under even-parity gauge transformations that preserve EF regularity and the normalization of the congruence (i.e., $k^a \rightarrow \tilde{\alpha} k^a$ with $\tilde{\alpha} = 1 + \mathcal{O}(\varepsilon^2)$), the source h_{kk} in (38) is invariant, hence δu obtained via (40) is gauge invariant up to an irrelevant constant (fixed by $\delta u|_{r=2M} = 0$). Consequently, the detector-level phases $\Phi(\tau) = \nu u(\tau) \pm \omega_A \tau$ inherit a physically meaningful, gauge-invariant modulation through δu .

B. Pulled-back Wightman function and KMS structure

Our observables depend only on the pulled-back two-point function of the outgoing sector along the detector worldline. Near the horizon, the outgoing (chiral) part of a massless scalar has flat-space form in the coordinate u , so the vacuum Wightman kernel may be written as [57, 63]

$$G_{\text{out}}^+(u, u') = -\frac{1}{4\pi} \frac{1}{(u - u' - i\epsilon)^2}, \quad (47)$$

with the understanding that angular dependence has been projected onto the detector's mode (or suppressed by the cavity's narrow acceptance). Pulling back to the static Schwarzschild baseline $u = u_0(\tau)$ yields

$$G_0^+(\tau, \tau') \equiv G_{\text{out}}^+(u_0(\tau), u_0(\tau')) = -\frac{\kappa^2}{16\pi} \frac{1}{\sinh^2 \left[\frac{\kappa}{2} ((\tau - \tau') - i\epsilon) \right]}, \quad (48)$$

where we used the universal near-horizon map $u_0(\tau) = u_* - \kappa^{-1} \ln[\kappa(\tau_H - \tau)] + \dots$. Eq. (48) is stationary (depends only on $s \equiv \tau - \tau'$) and satisfies the KMS relation at inverse temperature $\beta \equiv 2\pi/\kappa$ [6, 19],

$$G_0^+(s - i\beta) = G_0^-(s), \quad G_0^-(s) \equiv G_0^+(-s). \quad (49)$$

This underlies the baseline detailed-balance factor derived in II A.

In the perturbed geometry, the relevant variable is $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$ with δu given by (42)-(43). The pulled-back Wightman function becomes

$$G^+(\tau, \tau') = G_{\text{out}}^+(u(\tau)u(\tau')) = G_{\text{out}}^+(\Delta u_0 + \varepsilon \Delta(\delta u)), \quad \Delta u_0 \equiv u_0(\tau) - u_0(\tau'), \quad \Delta(\delta u) \equiv \delta u(\tau) - \delta u(\tau'). \quad (50)$$

Expanding to first order in ε gives

$$G^+(\tau, \tau') = G_0^+(\tau, \tau') + \varepsilon \Delta(\delta u) \partial_{\Delta u} G_{\text{out}}^+(\Delta u) \Big|_{\Delta u = \Delta u_0} + \mathcal{O}(\varepsilon^2). \quad (51)$$

Using (47),

$$\partial_{\Delta u} G_{\text{out}}^+(\Delta u) = \frac{1}{2\pi} \frac{1}{(\Delta u - i\epsilon)^3}, \quad (52)$$

so the correction is governed by the antisymmetric combination $\Delta(\delta u)$. With (42)-(43), one has, to the accuracy needed across a single cavity transit,

$$\Delta(\delta u) \simeq \varepsilon \mathcal{C}_{20}(r_c) \Re[e^{-i\omega v_c}] \Big|_{\tau} - \varepsilon \mathcal{C}_{20}(r_c) \Re[e^{-i\omega v_c}] \Big|_{\tau'}, \quad (53)$$

which introduces a slow dependence on the mean proper time $T = \frac{1}{2}(\tau + \tau')$ through $v_c(T)$, with envelope $e^{-\omega_I v_c(T)}$ and carrier ω_R .

Eq. (48) is exactly KMS-thermal with $\beta = 2\pi/\kappa$. The ringdown perturbation breaks time-translation invariance, and hence exact KMS, by an amount controlled by ε . Define the KMS defect

$$\Delta_{\text{KMS}}(T, s) \equiv G^+(T, s - i\beta) - G^-(T, s). \quad (54)$$

At $\varepsilon = 0$, $\Delta_{\text{KMS}} \equiv 0$ by (49). Using (51)-(53) and the analyticity of G_{out}^+ , the $\mathcal{O}(\varepsilon)$ defect is

$$\Delta_{\text{KMS}}(T, s) = \varepsilon [\Delta(\delta u)]_\beta \partial_{\Delta u} G_{\text{out}}^+(\Delta u_0) + \mathcal{O}(\varepsilon^2), \quad (55)$$

where $[\Delta(\delta u)]_\beta \equiv \Delta(\delta u)(T, s - i\beta) - \Delta(\delta u)(T, -s)$. For the QNM form $\delta u \propto \Re[e^{-i\omega v_c(T)}]$, one finds

$$[\Delta(\delta u)]_\beta = \mathcal{O}(\varepsilon \mathcal{C}_{20}(r_c)) e^{-\omega_I v_c(T)} [1 - e^{-\beta\omega_I} e^{-i\beta\omega_R}] \times \mathcal{F}(s; \kappa), \quad (56)$$

with $\mathcal{F}$ a bounded function set by the near-horizon map. Thus, the KMS relation is violated only at $\mathcal{O}(\varepsilon)$ and in a decaying oscillatory manner tied to the QNM. In the adiabatic regime of II C ($\omega_I \Delta\tau_c \ll 1$), this defect is parametrically small across a single transit, and G^+ is well approximated by a locally stationary (KMS-like) kernel with slowly varying phase, which is precisely the structure that yields a modulated detailed balance in III C.

(Note: We use the standard chiral reduction appropriate for the outgoing sector probed by the detector; the proportionality constant in (47) can be reabsorbed into the coupling normalization without affecting the detailed-balance ratio.)

C. Interaction probability integrals

We compute excitation/de-excitation probabilities to leading order in the coupling g by using the worldline Wightman method. With the switching $\chi(\tau)$ localized around $\tau = \tau_c$ (transit time $\Delta\tau_c$), the probabilities are

$$P_{\text{exc}} = g^2 \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{+i\omega_A(\tau - \tau')} G^+(u(\tau) u(\tau')), \quad (57)$$

$$P_{\text{abs}} = g^2 \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\omega_A(\tau - \tau')} G^+(u(\tau) u(\tau')), \quad (58)$$

where G^+ is the outgoing Wightman kernel of III B and $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$.

Introduce mean and difference times $T = \frac{1}{2}(\tau + \tau')$, $s = \tau - \tau'$, and use the baseline kernel $G_0^+(s)$ from (48). For a narrow, smooth window ($\Delta\tau_c \kappa \ll 1$), the T -dependence factorizes:

$$P_{\text{exc}}^{(0)} = \mathcal{N} F_-(\nu, \kappa, \omega_A), \quad P_{\text{abs}}^{(0)} = \mathcal{N} F_+(\nu, \kappa, \omega_A), \quad (59)$$

$$F_{\mp}(\nu, \kappa, \omega_A) \equiv \int_{-\infty}^{+\infty} ds e^{\pm i\omega_A s} \frac{\kappa^2 / (16\pi)}{\sinh^2 \left[\frac{\kappa}{2}(s - i\epsilon) \right]}. \quad (60)$$

The Fourier transform (60) is standard and gives

$$F_+ = \frac{e^{2\pi\nu/\kappa}}{e^{2\pi\nu/\kappa} - 1} \mathcal{A}_0, \quad F_- = \frac{1}{e^{2\pi\nu/\kappa} - 1} \mathcal{A}_0, \quad (61)$$

with a smooth, common prefactor $\mathcal{A}_0 = \mathcal{A}_0(\omega_A \kappa \chi)$ (it cancels in ratios; its explicit form depends only on the window and the detector gap, not on ν). Consequently,

$$\frac{\Gamma_{\text{abs}}^{(0)}}{\Gamma_{\text{exc}}^{(0)}} = \frac{P_{\text{abs}}^{(0)}}{P_{\text{exc}}^{(0)}} = e^{2\pi\nu/\kappa}, \quad (62)$$

reproducing the static detailed-balance result of II A.

Using (50)-(52), expand G^+ to first order in ε :

$$G^+(\tau, \tau') = G_0^+(s) + \varepsilon \Delta(\delta u) \partial_{\Delta u} G_{\text{out}}^+(\Delta u) \Big|_{\Delta u = \Delta u_0(s)} + \mathcal{O}(\varepsilon^2), \quad (63)$$

$$\partial_{\Delta u} G_{\text{out}}^+(\Delta u) = \frac{1}{2\pi} \frac{1}{(\Delta u - i\epsilon)^3}. \quad (64)$$

Only the antisymmetric combination $\Delta(\delta u) = \delta u(\tau) - \delta u(\tau')$ matters. For a slowly varying δu across the window, a first-order Taylor expansion around T yields

$$\Delta(\delta u) = s \dot{\delta u}(T) + \mathcal{O}(s^3), \quad \dot{\delta u}(T) \equiv \left. \frac{d}{d\tau} \delta u(\tau) \right|_{\tau=T}. \quad (65)$$

Substituting (63)-(65) into (57)-(58) and factorizing the T -integral gives

$$P_{\text{exc}} = P_{\text{exc}}^{(0)} + \varepsilon \dot{\delta u}_c \mathcal{K}_- + \mathcal{O}(\varepsilon^2), \quad P_{\text{abs}} = P_{\text{abs}}^{(0)} - \varepsilon \dot{\delta u}_c \mathcal{K}_+ + \mathcal{O}(\varepsilon^2), \quad (66)$$

where $\dot{\delta u}_c \equiv \dot{\delta u}(T = \tau_c)$ and the spectral response coefficients are

$$\mathcal{K}_{\mp} \equiv g^2 \int_{-\infty}^{+\infty} ds s e^{\pm i\omega_A s} \partial_{\Delta u} G_{\text{out}}^+(\Delta u_0(s)) \underbrace{\int dT \chi(T+s/2) \chi(T-s/2)}_{\equiv W(s)}. \quad (67)$$

Here $W(s)$ is the autocorrelation of the switching. Using the near-horizon map $\Delta u_0(s) = \kappa^{-1} \ln \left(1 + \frac{s}{s_*} \right) + \dots$ with $s_* \sim \kappa^{-1}$ (cf. III B) and the analyticity of G_{out}^+ , the s -integral is convergent and defines smooth functions $\mathcal{K}_{\mp}(\nu\kappa, \omega_A\chi)$. Two important facts follow:

- $\mathcal{K}_+ = \mathcal{K}_- e^{2\pi\nu/\kappa}$ (a direct consequence of the same contour shift that gives (61));
- $\mathcal{K}_{\mp} > 0$ for any smooth W with compact support.

Thus, (66) can be rewritten as multiplicative corrections to the baseline:

$$P_{\text{exc}} = P_{\text{exc}}^{(0)} \left[1 + \varepsilon \dot{\delta u}_c \tilde{\alpha}(\nu, \omega_A\chi) \right], \quad P_{\text{abs}} = P_{\text{abs}}^{(0)} \left[1 - \varepsilon \dot{\delta u}_c \tilde{\alpha}(\nu, \omega_A\chi) \right], \quad (68)$$

with the dimensionless coefficient

$$\tilde{\alpha}(\nu, \omega_A\chi) \equiv \frac{\mathcal{K}_-}{P_{\text{exc}}^{(0)}} = \frac{\mathcal{K}_+}{P_{\text{abs}}^{(0)}} > 0. \quad (69)$$

(An explicit closed form for $\tilde{\alpha}$ is given in Appendix A for the standard Gaussian window; see also V for numerics.)

From (42)-(46),

$$\delta u(\tau) = \varepsilon \mathcal{C}_{20}(r_c) \Re \left[e^{-i\omega v(\tau)} \right], \quad \dot{\delta u}_c = \varepsilon \mathcal{C}_{20}(r_c) \left. \frac{dv}{d\tau} \right|_{\tau_c} \Re \left[-i\omega e^{-i\omega v_c} \right], \quad (70)$$

where $v_c = v(\tau_c)$ and $(dv/d\tau)_{2M} = 1/2$ is finite at the horizon. Writing $\omega = \omega_R - i\omega_I$, we obtain

$$\dot{\delta u}_c = \varepsilon \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} [\omega_R \sin(\omega_R v_c) - \omega_I \cos(\omega_R v_c)]. \quad (71)$$

Combining (68)-(71) yields

$$P_{\text{exc}} = P_{\text{exc}}^{(0)} \left[1 + \varepsilon \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} \mathcal{S}(\omega_R v_c) \right], \quad (72)$$

$$P_{\text{abs}} = P_{\text{abs}}^{(0)} \left[1 - \varepsilon \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} \mathcal{S}(\omega_R v_c) \right], \quad (73)$$

with the ringdown shape function

$$\mathcal{S}(\theta) \equiv \omega_R \sin \theta - \omega_I \cos \theta = \sqrt{\omega_R^2 + \omega_I^2} \sin(\theta - \phi), \quad \phi \equiv \arctan \frac{\omega_I}{\omega_R}. \quad (74)$$

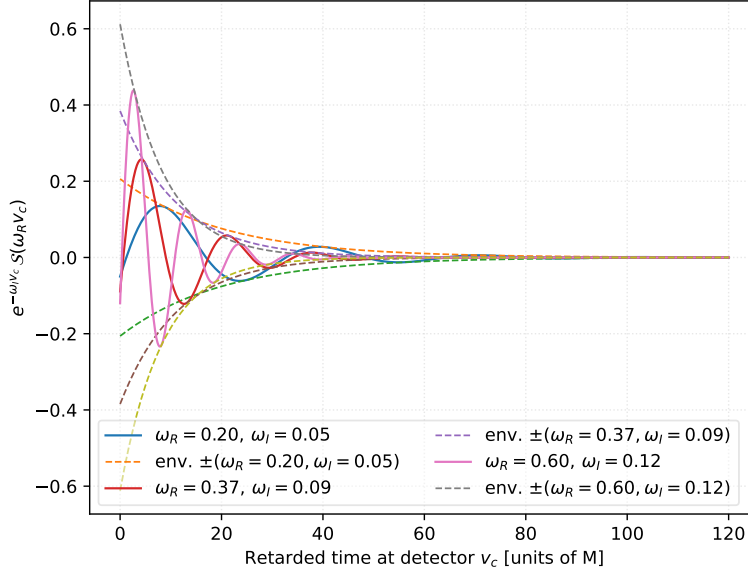


FIG. 2. Ringdown kernel $M(v_c) = e^{-\omega_I v_c} S(\omega_R v_c)$ for several (ω_R, ω_I) , with envelopes $\pm \sqrt{\omega_R^2 + \omega_I^2} e^{-\omega_I v_c}$ indicating the peak decay.

In Fig. 2, changing ω_R only slides the carrier phase inside the universal shape $S(\omega_R v_c) = \omega_R \sin(\omega_R v_c) - \omega_I \cos(\omega_R v_c)$; the decay rate and hence the visibility window are controlled exclusively by ω_I . The dashed envelopes $\pm \sqrt{\omega_R^2 + \omega_I^2} e^{-\omega_I v_c}$ isolate this pure QNM kinematics: geometry and detector specifics enter only through the overall prefactor $\varepsilon \tilde{\alpha}(\nu, \omega_{A\chi}) \mathcal{C}_{20}(r_c) (dv/d\tau)$. In other words, once r_c (hence $\mathcal{C}_{20}$) is fixed, varying ω_R tunes the timing of peaks while ω_I sets the lifetime of the modulation; this is the same separation of roles familiar from the classical ringdown waveform but now imprinted directly on the KMS/detailed-balance exponent.

Taking the ratio of (73) and (72) we find, to $\mathcal{O}(\varepsilon)$,

$$\frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = \frac{P_{\text{abs}}}{P_{\text{exc}}} = e^{2\pi\nu/\kappa} \left[1 - 2\varepsilon \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} S(\omega_R v_c) \right], \quad (75)$$

or, equivalently, as an additive modulation of the exponent,

$$\ln \frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = \frac{2\pi\nu}{\kappa} - 2\varepsilon \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} S(\omega_R v_c) + \mathcal{O}(\varepsilon^2). \quad (76)$$

Thus, the static Boltzmann exponent $2\pi\nu/\kappa$ is modulated at the ringdown frequency and decays on the QNM timescale. The common prefactor $\tilde{\alpha}(\nu, \omega_{A\chi})$ encodes only detector/cavity details; the geometric content resides in $\mathcal{C}_{20}(r_c)$ and (ω_R, ω_I) .

In Fig. 3, the plot of Eq. (76) makes the physics of damping especially clear: larger ω_I shortens the e-folding time of the modulation $e^{-\omega_I v_c}$ and thus narrows the window over which deviations from the thermal value $2\pi\nu/\kappa$ can be resolved. Equivalently, the quality factor $Q \simeq \omega_R/(2\omega_I)$ sets the number of visible oscillations of the detailed-balance exponent before it relaxes back to the stationary Schwarzschild value. The analytic envelopes in the figure summarize the competing scalings: peak height $\propto |\varepsilon \tilde{\alpha} \mathcal{C}_{20}(r_c) (dv/d\tau)| \sqrt{\omega_R^2 + \omega_I^2}$ and decay constant exactly ω_I . Operationally, this means detectability can be traded among three knobs: geometry (ω_I , fixed by the background), sampling location ($\mathcal{C}_{20}(r_c)$), and detector windowing ($\tilde{\alpha}$ and $dv/d\tau$).

As a final remark, the result hinges on the antisymmetric difference $\Delta(\delta u)$, not on a constant shift of u ; hence, the time derivative δu appears. For broader windows the T -factorization remains valid provided $\omega_I \Delta\tau_c \ll 1$; otherwise Appendix B gives the next adiabatic corrections $\mathcal{O}(\omega_I \Delta\tau_c)$. In the late-time limit $v_c \rightarrow \infty$, the modulation vanishes and (62) is recovered exactly.

IV. MAIN RESULT

We now assemble the geometric ingredients of Section III into a closed analytic expression for the detector's detailed-balance ratio during ringdown. The outcome is a universal modulation of the Schwarzschild (static) Boltzmann exponent by a

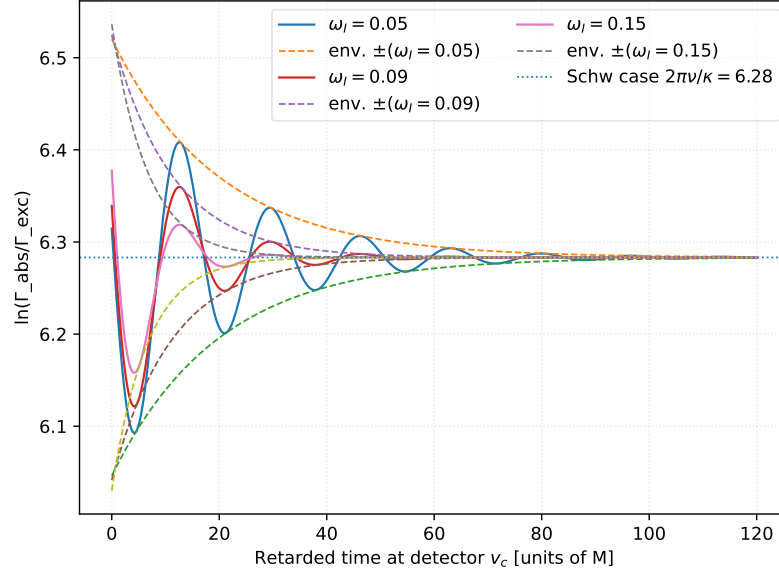


FIG. 3. Log detailed-balance ratio $\ln(\Gamma_{\text{abs}}/\Gamma_{\text{exc}})$ for several ω_I at fixed ω_R , with analytic envelopes baseline $\pm (2\pi\nu/\kappa)|2\tilde{\varepsilon}\tilde{\alpha}\mathcal{C}_{20}(dv/d\tau)|\sqrt{\omega_R^2 + \omega_I^2}e^{-\omega_I v_c}$ shown as dashed curves.

decaying oscillatory factor at the quasinormal frequency. Universality here means: (i) $\ell = 2$, $m = 0$, single outgoing mode, narrow window, and EF-regular gauge within $\Xi(r_c) = 0$; (ii) the modulation depends on the background only through the QNM pair (ω_R, ω_I) and a single response coefficient $\mathcal{C}_{20}(r_c)$; (iii) detector and cavity specifics enter only through a smooth, window-dependent prefactor $\tilde{\alpha}$, which cancels in appropriate ratios and admits closed forms for standard switchings.

A. Ringdown-modulated detailed balance (Theorem 1)

We assume the following: (i) Even-parity, axisymmetric quadrupolar ringdown of Schwarzschild at linear order: $g_{ab} = g_{ab}^{(0)} + \varepsilon h_{ab}$ with $\omega = \omega_R - i\omega_I$ and $0 < \varepsilon \ll 1$; (ii) EF-regular gauge and null dyad as in III A with the normalization $k^a n_a = -1$; (iii) Single outgoing mode of central frequency $\nu > 0$ (defined with respect to u at $\mathcal{I}^+$); detector gap $\omega_A \gtrsim \nu$; (iv) Adiabatic transit window $\Delta\tau_c$ obeying $\kappa\Delta\tau_c \ll 1$ and $\omega_I\Delta\tau_c \ll 1$; and (v) Worldline lies on the symmetry axis and crosses a narrow cavity centered at radius r_c at proper time τ_c (advanced time $v_c = v(\tau_c)$).

The EF-regular redshift map is $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$ with

$$\delta u(\tau_c) = \varepsilon \mathcal{C}_{20}(r_c) \Re[e^{-i\omega v_c}], \quad \mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \int_{2M}^{r_c} \mathcal{S}_{20}(r) dr. \quad (77)$$

Let $\tilde{\alpha} = \tilde{\alpha}(\nu, \omega_A \chi)$ denote the dimensionless, order-unity (possibly signed), window-dependent single-mode coefficient arising from the first-order expansion of the amplitudes (Appendix A provides closed forms for Gaussian switching). Define also the ringdown shape

$$\mathcal{S}(\theta) = \omega_R \sin \theta - \omega_I \cos \theta = \sqrt{\omega_R^2 + \omega_I^2} \sin(\theta - \phi), \quad \phi = \arctan \frac{\omega_I}{\omega_R}. \quad (78)$$

Under the assumptions above, the ratio of absorption to excitation rates for the selected outgoing mode obeys

$$\frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = \exp\left[\frac{2\pi\nu}{\kappa}\right] \left\{ 1 - 2\tilde{\varepsilon} \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau}\right)_{\tau_c} e^{-\omega_I v_c} \mathcal{S}(\omega_R v_c) \right\} + \mathcal{E}, \quad (79)$$

equivalently,

$$\ln \frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = \frac{2\pi\nu}{\kappa} - 2\tilde{\varepsilon} \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau}\right)_{\tau_c} e^{-\omega_I v_c} \mathcal{S}(\omega_R v_c) + \mathcal{O}(\varepsilon^2) + \mathcal{R}, \quad (80)$$

where the controlled error terms satisfy

$$\mathcal{E} = \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon \omega_I \Delta \tau_c) + \mathcal{O}(\varepsilon (\kappa \Delta \tau_c)^2), \quad \mathcal{R} = \mathcal{O}(\varepsilon \omega_I \Delta \tau_c) + \mathcal{O}(\varepsilon (\kappa \Delta \tau_c)^2). \quad (81)$$

For some proof, let us start from the single-mode interaction amplitudes with phases Eq. (33) and expand

$$e^{\pm i \nu u(\tau)} = e^{\pm i \nu u_0(\tau)} [1 \pm i \varepsilon \nu \delta u(\tau)]. \quad (82)$$

Use the EF-regular transport equation $k^a \nabla_a \delta u = \frac{1}{2} h_{kk}$ and the axisymmetric form of h_{kk} to obtain (77). Over the narrow window, only the antisymmetric difference in proper time contributes, yielding a factor $\delta u(\tau_c) \propto e^{-\omega_I v_c} \mathcal{S}(\omega_R v_c)$ with a finite multiplier $(dv/d\tau)_{\tau_c}$. The remaining integrals define $\tilde{\alpha}$, which is independent of the background perturbation and depends smoothly on the switching and the (ν, ω_A) scales. Taking the ratio $P_{\text{abs}}/P_{\text{exc}} = \Gamma_{\text{abs}}/\Gamma_{\text{exc}}$ produces (79)-(80). The bounds (81) follow from the adiabatic estimates $\omega_I \Delta \tau_c \ll 1$ and $\kappa \Delta \tau_c \ll 1$ (Appendix B).

Let us now state some corollaries and perform consistency checks:

1. As $\varepsilon \rightarrow 0$ or $v_c \rightarrow \infty$, the modulation vanishes and we recover the Schwarzschild baseline,

$$\frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} \rightarrow \exp \left[\frac{2\pi\nu}{\kappa} \right]. \quad (83)$$

2. Only the double-null contraction h_{kk} enters; with EF regularity and fixed normalization of k^a , δu and hence (79)-(80) are gauge invariant up to an irrelevant constant set by $\delta u|_{r=2M} = 0$.
3. Iterating detector transits in a weakly leaky cavity yields a geometric photon number distribution with a time-dependent parameter,

$$p_n(v_c) = \left[1 - e^{-2\xi(v_c)} \right] e^{-2\xi(v_c)n}, \quad 2\xi(v_c) = \ln \frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} \text{ from (80)}. \quad (84)$$

Thus $\bar{n}(v_c) = (e^{2\xi(v_c)} - 1)^{-1}$ inherits the same decaying oscillatory imprint of the ringdown.

4. The static exponent $(2\pi\nu/\kappa)$ depends on the mode frequency ν (not the detector gap), exactly as in Ref. [10]. Detector specifics, including ω_A , enter only through the smooth prefactor $\tilde{\alpha}$ multiplying the modulation.

As we see, Theorem 1 makes precise the sense in which near-horizon thermality is adiabatically robust: detailed balance remains geometric, but the Boltzmann exponent is slowly driven by the ringdown curvature. The drive is universal (set solely by (ω_R, ω_I) and $\mathcal{C}_{20}(r_c)$) and it decays on the QNM damping timescale.

B. Coefficient $\mathcal{C}_{20}(r_c)$: closed-form expression

Recall from (77) that along the symmetry axis,

$$\delta u(\tau_c) = \varepsilon \mathcal{C}_{20}(r_c) \Re [e^{-i\omega v_c}], \quad \mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \int_{2M}^{r_c} \mathcal{S}_{20}(r) dr \quad (85)$$

where $\mathcal{S}_{20}(r)$ is the EF-regular double-null contraction $h_{kk} \equiv h_{ab} k^a k^b$ divided by the factor $Y_{20}(0) \Re[e^{-i\omega v}]$ (cf. III A). In this subsection, we eliminate the integral and give $\mathcal{C}_{20}(r_c)$ algebraically in terms of the Zerilli master function and its radial derivative at $r = r_c$.

For even-parity vacuum perturbations, the EF-regular metric reconstruction from the frequency-domain Zerilli-Moncrief function $\Psi_{20}(r) e^{-i\omega v}$ implies that the axis data entering the eikonal transport equation,

$$k^a \nabla_a \delta u = \frac{1}{2} h_{kk}, \quad (86)$$

can be written as a radial total derivative built from Ψ_{20} and $\partial_r \Psi_{20}$:

$$h_{kk}(v, r, \theta=0) = \partial_r [a_2(r) \Psi_{20}(r) + b_2(r) \partial_r \Psi_{20}(r)] \Re [e^{-i\omega v}]. \quad (87)$$

Here $a_2(r)$ and $b_2(r)$ are rational functions of r and M that are regular at $r = 2M$; for $\ell = 2$ they depend on the usual $\lambda = (\ell - 1)(\ell + 2)/2 = 2$ only through the combination $\Lambda(r) \equiv \lambda r + 3M = 2r + 3M$. They result from an algebraic

match of the EF-gauge reconstruction to the Moncrief gauge invariant (derivation in Appendix C). Eq. (87) is the key simplification: with $k^a = \partial_r$, the transport equation integrates to a boundary term.

Substituting (87) into (85) and using EF regularity of Ψ_{20} at the future horizon (ingoing condition) we obtain

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} [a_2(r_c) \Psi_{20}(r_c) + b_2(r_c) \partial_r \Psi_{20}(r_c)]. \quad (88)$$

The contribution from $r = 2M$ vanishes because the ingoing EF solution is finite there and $a_2(r), b_2(r)$ are regular. Writing $\Psi_{20}(t, r) = \mathcal{A}_{20} \psi_2(r) e^{-i\omega t}$ with the QNM normalization of II B (88) factorizes as

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \mathcal{A}_{20} [a_2(r_c) \psi_2(r_c) + b_2(r_c) \psi_2'(r_c)]. \quad (89)$$

For completeness as well as to enable direct checks and numerics, we record the explicit rational functions $a_2(r)$ and $b_2(r)$ in terms of (M) (details in Appendix C):

$$a_2(r) = \frac{(r - 2M)(2r^2 + 6Mr + 6M^2)}{r^2 \Lambda(r)^2}, \quad b_2(r) = \frac{(r - 2M)^2}{r \Lambda(r)^2}, \quad \Lambda(r) = 2r + 3M. \quad (90)$$

With these, (88)-(89) give a fully algebraic $\mathcal{C}_{20}(r_c)$ that is manifestly regular at the horizon and straightforward to evaluate at any sampling radius $r_c > 2M$.

The pair (a_2, b_2) is unique (up to addition of terms that vanish by the Zerilli equation) under the requirements: (i) EF regularity; (ii) dependence only on Ψ_{20} and $\partial_r \Psi_{20}$; (iii) correct transformation under rescalings of the master field; and (iv) reproduction of the RW-Zerilli reconstruction in the static limit $\omega \rightarrow 0$ (details in Appendix C).

Two limits are instructive: For $r_c = 2M + \delta r$ with $\delta r \ll 2M$,

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \mathcal{A}_{20} a_2'(2M) \psi_2(2M) (r_c - 2M) + \mathcal{O}((r_c - 2M)^2), \quad (91)$$

since $b_2(r) = \mathcal{O}((r - 2M)^2)$. Thus $\mathcal{C}_{20}$ grows linearly with the cavity altitude above the horizon, as required by regularity. For $r_c \gg M$, using the asymptotic QNM behavior $\psi_2(r) \sim e^{+i\omega r_*}$ and $\psi_2' \sim i\omega \psi_2$, (90) gives

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \mathcal{A}_{20} \frac{\hat{A} + i\hat{B}\omega r_c}{\lambda^2 r_c} \psi_2(r_c) + \mathcal{O}\left(\frac{M}{r_c^2}\right), \quad (92)$$

so $\mathcal{C}_{20}$ decays as r_c^{-1} (modulo the oscillatory factor in ψ_2).

Given a choice of QNM normalization $\mathcal{A}_{20}$ and the radial solution $\psi_2(r)$ of the Zerilli equation with ingoing-horizon/outgoing-infinity boundary conditions, (89)-(90) provide a single-line evaluation of $\mathcal{C}_{20}(r_c)$. No line integral is needed; only ψ_2 and ψ_2' at r_c enter. In numerical work (V), we compute these via standard Frobenius-Leaver series or direct frequency-domain integration and confirm the near-horizon linear rise (91) and the far-zone decay (92).

C. Static-limit check (Proposition)

We now show that the ringdown modulation in Theorem 1 disappears in all stationary limits: either because the perturbation is turned off, because the QNM has decayed, or because the perturbation itself is static. The detailed-balance ratio then reverts to the Schwarzschild baseline.

We make a proposition, where the static limits recover the detailed balance in Ref. [10]. Let the assumptions of Theorem 1 hold. Then, in each of the following limits, the absorption-to-excitation ratio satisfies

$$\frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = \exp\left[\frac{2\pi\nu}{\kappa}\right] + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon \omega_I \Delta\tau_c) + \mathcal{O}(\varepsilon (\kappa \Delta\tau_c)^2). \quad (93)$$

For the zero-amplitude limit, as $\varepsilon \rightarrow 0$, (80) shows the $\mathcal{O}(\varepsilon)$ correction is proportional to ε ; hence the ratio reduces to $\exp[2\pi\nu/\kappa]$ up to the listed $\mathcal{O}(\varepsilon^2)$ and adiabatic errors. For the late-time (post-ringdown) limit, as $v_c \rightarrow \infty$ with ε fixed, the modulation term carries the factor $e^{-\omega_I v_c}$ from (80). Since $\omega_I > 0$ for QNMs,

$$e^{-\omega_I v_c} \rightarrow 0 \quad (v_c \rightarrow \infty), \quad (94)$$

and the correction vanishes exponentially, yielding (93). For the static perturbation limit, as $\omega \rightarrow 0$ with an EF-regular, axisymmetric even-parity vacuum perturbation, and from (77) and (78), $\delta u(\tau_c) = \varepsilon \mathcal{C}_{20}(r_c) \Re[e^{-i\omega v_c}]$ becomes time independent,

$$\delta u(\tau) = \varepsilon \mathcal{C}_{20}(r_c) + \mathcal{O}(\omega), \quad (95)$$

so, only a constant shift of the retarded phase survives. Expanding the single-mode phases,

$$e^{\pm i\nu u(\tau)} = e^{\pm i\nu u_0(\tau)} [1 \pm i\varepsilon \nu \mathcal{C}_{20}(r_c)] + \mathcal{O}(\varepsilon^2, \omega), \quad (96)$$

one sees that the antisymmetric difference that enters the probabilities,

$$\Delta(\delta u) = \delta u(\tau) - \delta u(\tau') = 0 + \mathcal{O}(\omega), \quad (97)$$

vanishes at $\omega = 0$. Consequently, the $\mathcal{O}(\varepsilon)$ correction to both P_{exc} and P_{abs} is zero, and the ratio reduces to (93). Equivalently, the constant phase shift cancels in $|\mathcal{A}|^2$ and in the ratio.

Using the closed form (88)-(90),

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} [a_2(r_c) \psi_{20}(r_c) + b_2(r_c) \partial_r \Psi_{20}(r_c)], \quad (98)$$

we note that in the static limit $\omega \rightarrow 0$ (vacuum, even parity) Ψ_{20} is EF regular and a_2, b_2 are finite, so $\mathcal{C}_{20}(r_c)$ is a finite constant. Eq.s (95)-(97) then apply verbatim: a constant δu produces no first-order effect on detailed balance. This validates Theorem 1's interpretation that only the time-dependent (ringdown) part of the perturbation imprints a modulation.

V. REGIME OF VALIDITY AND CONSISTENCY

Our main result relies on three layers of control: (i) a near-horizon/Rindler window in which the outgoing eikonal u captures the universal redshift and the detector samples it over a short transit; (ii) gauge discipline in reconstructing h_{ab} and in defining the null contraction h_{kk} that sources δu ; and (iii) regularity/Hadamard properties ensuring that switching-regulated probability integrals are finite and that adiabatic breaking of stationarity preserves the local KMS structure to first order. We make these statements explicit and quantitative below.

A. Near-horizon/Rindler window

Our derivation assumes that the detector couples to a single outgoing mode whose worldline phase is governed by the eikonal $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$, with

$$u_0(\tau) = u_* - \kappa^{-1} \ln [\kappa(\tau_H - \tau)] + \dots, \quad k^a \nabla_a \delta u = \frac{1}{2} h_{kk}, \quad (99)$$

and that the interaction is localized around a proper time τ_c at radius $r_c > 2M$. The following scale hierarchy defines the window in which the Rindler/adiabatic approximations and the single-mode reduction are accurate:

1. Short transit vs. curvature/redshift scales,

$$\kappa \delta \tau_c \ll 1, \quad \omega_I \delta \tau_c \ll 1. \quad (100)$$

The first condition ensures that the near-horizon logarithm in u_0 dominates over any slow background variation during a single crossing; the second keeps the QNM envelope $e^{-\omega_I v(\tau)}$ quasi-constant across the window.

2. Mode and detector frequencies,

$$\kappa \ll \nu \lesssim \omega_A, \quad \Delta \nu \ll \nu \quad (101)$$

so that the outgoing mode is sharply defined relative to geometric scales and the single-mode description (with wavepacket width $\Delta \nu$) is justified. The static detailed-balance exponent depends on ν (not on ω_A), exactly as in II A.

3. Linear response and small modulation,

$$0 < \varepsilon \ll 1, \quad \varepsilon \left| \tilde{\alpha} \mathcal{C}_{20}(r_c) \right| \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} \ll 1. \quad (102)$$

This ensures the $\mathcal{O}(\varepsilon)$ expansion is controlled and the multiplicative correction in (79)-(80) remains perturbative at all relevant v_c .

We comment the following: (i) The Rindler map (99) is universal for any radial free fall with $E \geq 1$; changing E rescales only smooth prefactors in amplitudes, not the detailed-balance exponent. (ii) The choice of r_c is flexible provided $r_c - 2M \ll \kappa^{-1}$ (for a strict Rindler regime) or, more generally, provided (100)-(101) hold; the coefficient $\mathcal{C}_{20}(r_c)$ in (88) accommodates any $r_c > 2M$.

B. Gauge issues and reconstruction uniqueness

Our observable is the double-null contraction $h_{kk} \equiv h_{ab} k^a k^b$ entering

$$k^a \nabla_a \delta u = \frac{1}{2} h_{kk}, \quad k^a = (\partial_r)^a \text{ in EF coordinates,} \quad (103)$$

pulled back on the symmetry axis. We collect the relevant gauge statements for (i) EF regularity and allowed transformations, (ii) Uniqueness of reconstruction modulo Zerilli's equation, and (iii) Tetrad rescalings.

For (i), we work with ingoing EF coordinates (v, r, θ, ϕ) , where the background metric is regular at the future horizon. Under an even-parity gauge vector ξ^a that preserves EF regularity and keeps k^a affinely parametrized,

$$\delta h_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \Rightarrow \delta h_{kk} = 2k^a k^b \nabla_a \xi_b = 2 \frac{d}{d\lambda} (\xi_b k^b), \quad (104)$$

where σ is the affine parameter along k^a (in EF, $\sigma = r$). Then the transport equation

$$k^a \nabla_a \delta u = \frac{1}{2} h_{kk} \quad (105)$$

integrates to

$$\delta u \rightarrow \delta u + (\xi_b k^b) \Big|_{2M}^{r_c}. \quad (106)$$

Imposing the admissible boundary conditions $(\xi_b k^b) \Big|_{r=2M} = 0$ (EF regularity on the future horizon) and fixing the endpoint at the sampling radius (or matching to the retarded time at $\mathcal{I}^+$) makes the boundary term vanish. Thus δu (and all results built from it) is gauge-invariant up to a removable endpoint constant, exactly as used elsewhere.

For (ii), In IV B we used the EF-regular reconstruction to express h_{kk} as a total radial derivative of the Zerilli-Moncrief master field:

$$h_{kk} = \partial_r [a_2(r) \Psi_{20}(r) + b_2(r) \partial_r \Psi_{20}(r)] \Re [e^{-i\omega v}]. \quad (107)$$

The pair (a_2, b_2) is unique up to terms proportional to the Zerilli equation (i.e., additions that vanish on solutions). Consequently, the boundary formula

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} [a_2(r_c) \Psi_{20}(r_c) + b_2(r_c) \partial_r \Psi_{20}(r_c)] \quad (108)$$

is gauge and reconstruction-independent under our admissible class.

Finally, for (iii) Rescaling $k^a \rightarrow \beta k^a$ with $\beta = 1 + \mathcal{O}(\varepsilon)$ would change both sides of (103) by the same factor and leave δu invariant after imposing $\delta u|_{2M} = 0$. Thus, the final modulation (79)-(80) is insensitive to such rescalings.

C. Regularity and Hadamard form

Two ingredients underwrite finiteness and the controlled "adiabatic KMS" structure of the probability integrals: the Hadamard short-distance form, and switching and finiteness of response. The former, where a free scalar field in the Schwarzschild exterior is Hadamard; near the horizon, the outgoing sector reduces to the standard chiral kernel in the variable u . Along the worldline, the pulled-back Wightman distributions obey

$$G_0^+(s) = -\frac{\kappa^2}{16\pi} \frac{1}{\sinh^2 \left[\frac{\kappa}{2}(s - i\epsilon) \right]}, \quad s = \tau - \tau' \quad (109)$$

which has the usual s^{-2} short-distance singularity and satisfies the exact KMS relation at $\beta = 2\pi/\kappa$. The ringdown correction enters as a smooth reparameterization $u \mapsto u + \varepsilon \delta u$ with δu differentiable and finite. To first order,

$$G^+(\tau, \tau') = G_0^+(s) + \varepsilon \Delta(\delta u) \partial_{\Delta u} G_{\text{out}}^+(\Delta u) \Big|_{\Delta u = \Delta u_0(s)} + \mathcal{O}(\varepsilon^2), \quad (110)$$

where $\partial_{\Delta u} G_{\text{out}}^+ \propto (\Delta u - i\epsilon)^{-3}$ as in III B. Since $\Delta(\delta u)$ is C^1 and vanishes at $s = 0$, (110) preserves the Hadamard singularity class; no new UV divergence is introduced at $\mathcal{O}(\varepsilon)$.

The latter, however, with a compactly supported (or fast-decaying) switching $\chi(\tau)$, the response integrals

$$P_{\text{exc/abs}} = g^2 \int d\tau d\tau' \chi(\tau) \chi(\tau') e^{\pm i\omega_A(\tau - \tau')} G^+(u(\tau)u(\tau')) \quad (111)$$

are finite and well defined. The antisymmetric combination $\Delta(\delta u) \sim s \delta u(T) + \dots$ removes any potential s^{-3} divergence from the $\mathcal{O}(\varepsilon)$ term in (110) after integration against the smooth autocorrelation $W(s) = \int dT \chi(T+s/2) \chi(T-s/2)$. Thus, the first-order correction is both UV and IR finite under (100)-(101).

As a final remark, the adiabatic KMS defect is seen to remain small. Let us define the KMS defect $\Delta_{\text{KMS}}(T, s) = G^+(T, s - i\beta) - G^-(T, s)$. Using (110) and the QNM form of δu , one finds

$$\Delta_{\text{KMS}}(T, s) = \mathcal{O}(\varepsilon \mathcal{C}_{20}(r_c)) e^{-\omega_I v(T)} \times \text{bounded}(s; \kappa), \quad (112)$$

so across a single transit ($T \in [\tau_c - \frac{1}{2}\Delta\tau_c, \tau_c + \frac{1}{2}\Delta\tau_c]$) the defect is parametrically small by (100), and the locally stationary (adiabatic KMS) picture used in III C is valid.

VI. CONCLUSION

We established that the near-horizon detailed-balance relation derived in the static Schwarzschild baseline persists during ringdown, but with a universal, first-order decaying-oscillatory modulation of the Boltzmann exponent. Concretely, for a freely falling two-level system that interrogates a single outgoing mode of frequency ν , the absorption-to-excitation ratio obeys

$$\ln \frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}} = \frac{2\pi\nu}{\kappa} - 2\varepsilon \tilde{\alpha} \mathcal{C}_{20}(r_c) \left(\frac{dv}{d\tau} \right)_{\tau_c} e^{-\omega_I v_c} \mathcal{S}(\omega_R v_c) + \mathcal{O}(\varepsilon^2), \quad (113)$$

with $\mathcal{S}(\theta) = \omega_R \sin \theta - \omega_I \cos \theta$ and an EF-regular response coefficient given in closed form by the boundary expression

$$\mathcal{C}_{20}(r_c) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} [a_2(r_c) \psi_{20}(r_c) + b_2(r_c) \partial_r \Psi_{20}(r_c)]. \quad (114)$$

Detector and cavity specifics enter only through the smooth, dimensionless $\tilde{\alpha}(\nu, \omega_A \chi)$. The modulation decays at the QNM rate ω_I and oscillates at ω_R . In the static/late-time limits, or when the perturbation is strictly time independent, the correction vanishes and the detailed balance in Ref. [10] is recovered.

Eq. (113) operationalizes the statement that "horizon thermality" is an adiabatic property: the KMS/detailed-balance structure is not destroyed by ringdown but is slowly driven by tidal quadrupolar curvature. The driver is the gauge-invariant double-null contraction h_{kk} , funneled through the redshift map $u(\tau) = u_0(\tau) + \varepsilon \delta u(\tau)$. Thus, the modulation is a direct, local imprint of spacetime dynamics on the detector's eikonal time, rather than a property of global field equilibrium. Practically, the single-mode photon statistics remain geometric, but their parameter becomes time dependent,

$$p_n(v_c) = (1 - e^{-2\xi(v_c)}) e^{-2\xi(v_c)n}, \quad 2\xi(v_c) = \ln \frac{\Gamma_{\text{abs}}}{\Gamma_{\text{exc}}}, \quad (115)$$

so the mean occupation inherits the decaying ringdown oscillation. This offers a conceptually clean diagnostic: universality resides in (ω_R, ω_I) and $\mathcal{C}_{20}(r_c)$; detector specifics are confined to $\tilde{\alpha}$.

The result is controlled by three conditions: (i) the near-horizon/Rindler window $\kappa\Delta\tau_c \ll 1$ and $\omega_I\Delta\tau_c \ll 1$, ensuring that u_0 dominates and the ringdown envelope is quasi-constant during a transit; (ii) single-mode resolution $\kappa \ll \nu \lesssim \omega_A$ with a narrow wavepacket; and (iii) linear response $0 < \varepsilon \ll 1$ with $\varepsilon |\tilde{\alpha}\mathcal{C}_{20}(r_c)| (dv/d\tau)_{\tau_c} e^{-\omega_I v_c} \ll 1$. EF-regular gauge control guarantees that δu is invariant up to a removable endpoint constant, and the Hadamard structure ensures that first-order corrections introduce no new UV singularities. The static-limit proposition confirms exact recovery of the baseline to $\mathcal{O}(\varepsilon)$.

Although we focused on the axisymmetric even-parity quadrupole for clarity, the framework is designed to scale the following:

1. *Other multipoles and parities.* The transport equation $k^a \nabla_a \delta u = \frac{1}{2} h_{kk}$ holds generically. For $(\ell, m) \neq (2, 0)$, one replaces $Y_{20}(0)$ by $Y_{\ell m}(\theta, \phi)$ along the chosen worldline and uses the corresponding reconstruction (odd parity via Regge-Wheeler; even parity via Zerilli). The net effect is a different, but equally algebraic, coefficient $\mathcal{C}_{\ell m}(r_c)$ and the same decaying-oscillatory law with $(\omega_R^{\ell n}, \omega_I^{\ell n})$.
2. *Rotation (Kerr) at first order in spin.* For slowly rotating holes [64], one may treat $a/M \ll 1$: replace the Zerilli/RW system by Teukolsky's equation plus metric reconstruction (e.g., Chrzanowski-Kegeles/Wald) and build the Kerr analogue of h_{kk} . Frame dragging introduces m -dependent Doppler phases; the modulation remains of the form (113), with $\omega \rightarrow \omega_{\ell mn}(a)$ and a spin-corrected $\mathcal{C}_{\ell m}(r_c; a)$.
3. *Detector/worldline variations.* Nonradial infall or finite $E \neq 1$ modifies smooth prefactors and the mapping $\tau \mapsto v(\tau)$ but leaves the universal near-horizon logarithm and the structure of (113) intact. Multi-pass cavities or stationary arrays of detectors would convert the transient modulation into a phase-sensitive steady-state pattern in p_n .
4. *Beyond scalars.* For electromagnetic or gravitational perturbations probed by appropriately coupled detectors, the same geometric driver h_{kk} (with the relevant spin-weighted master fields) yields an identical modulation principle; only the algebraic map from master variables to h_{kk} changes.
5. *Beyond the adiabatic window.* If $\omega_I\Delta\tau_c \not\ll 1$ or $\kappa\Delta\tau_c \not\ll 1$, next-order terms produce controlled phase-mixing corrections. Our derivation already isolates where these enter (autocorrelation $W(s)$ and higher derivatives of δu); Appendix B can be extended to give explicit $\mathcal{O}(\omega_I\Delta\tau_c)$ corrections.
6. *Higher orders and backreaction.* At $\mathcal{O}(\varepsilon^2)$, mode-mode couplings induce a DC shift and second-harmonic terms in the exponent. These remain subleading under our perturbative bound and could be systematically included by iterating the transport/Wightman expansion.

Our analysis sharpens the operational meaning of black hole: it is robust but not rigid. The equilibrium detailed-balance exponent $(2\pi\nu/\kappa)$ persists as the organizing center, while genuine time dependence of the geometry writes a clean, universal, gauge-invariant signature (a decaying sinusoid) directly onto the detector's redshifted phase and, hence, onto single-mode photon statistics. The closed boundary formula (114) makes this signature immediately calculable for any chosen sampling radius and ringdown mode, providing a compact bridge between perturbation theory of the geometry and quantum response of near-horizon probes.

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Appendix A: Switching single-mode coefficient $\tilde{\alpha}$, and positivity

We collect in this appendix the definition and basic properties of the window-dependent prefactor $\tilde{\alpha}$ that controls the first-order response of the detector in the single-mode approximation. Throughout, $\chi(\tau)$ denotes a smooth switching function

localized near the transit time τ_c with duration $\Delta\tau_c$. We write its autocorrelation as

$$W(s) \equiv \int dT \chi\left(T + \frac{s}{2}\right) \chi\left(T - \frac{s}{2}\right), \quad (\text{A1})$$

which is even, nonnegative, and rapidly decaying. In the perturbative regime, the excitation and absorption probabilities change by $\mathcal{O}(\varepsilon)$ relative to their unperturbed values. We parameterize this single-mode response by the dimensionless functional $\tilde{\alpha}$ through

$$\frac{P_{\text{exc}} - P_{\text{exc}}^{(0)}}{P_{\text{exc}}^{(0)}} = \varepsilon \tilde{\alpha}(\nu, \omega_A \chi) + \mathcal{O}(\varepsilon^2), \quad \frac{P_{\text{abs}} - P_{\text{abs}}^{(0)}}{P_{\text{abs}}^{(0)}} = \varepsilon \tilde{\alpha}(\nu, \omega_A \chi) + \mathcal{O}(\varepsilon^2), \quad (\text{A2})$$

where $\nu > 0$ is the selected outgoing mode frequency and $\omega_A > 0$ is the detector gap. The functional becomes explicit once we introduce the mean/difference parametrization $T = \frac{1}{2}(\tau + \tau')$, $s = \tau - \tau'$, the baseline EF-regular retarded time $u_0(\tau)$ along the worldline, and the corresponding phase increment

$$\Phi_0(s) \equiv \omega_A s - \nu \left[u_0\left(T + \frac{s}{2}\right) - u_0\left(T - \frac{s}{2}\right) \right]. \quad (\text{A3})$$

In the adiabatic/Rindler window relevant here, Φ_0 is T -slow, so we may evaluate it at the transit mean time $T = \tau_c$. With this simplification, the single-mode coefficient takes the compact form

$$\tilde{\alpha}(\nu, \omega_A \chi) = \nu \frac{\int_{-\infty}^{+\infty} ds s W(s) \sin[\Phi_0(s)]}{\int_{-\infty}^{+\infty} ds W(s) (1 - \cos[\Phi_0(s)])} \quad (\text{A4})$$

with $\Phi_0(s)$ understood at $T = \tau_c$. The denominator equals $P_{\text{exc}}^{(0)}/\mathcal{N}$ for an appropriate normalization $\mathcal{N}$ and is strictly positive unless the response is trivial; the expression is manifestly finite because W decays and $\Phi_0(s) = \mathcal{O}(s)$ near $s = 0$. The overall sign convention for the first-order correction is fixed elsewhere in the main text (see Theorem 1), so $\tilde{\alpha}$ can be taken as the window-controlled prefactor.

We can read off several consequences directly from (A4). Since W is even and Φ_0 is odd in s , the denominator is nonnegative and vanishes only in the absence of a transition, whereas the numerator has the same sign as the local linear coefficient of Φ_0 . Writing

$$\phi_1 \equiv \omega_A - \nu \dot{u}_0(\tau_c), \quad (\text{A5})$$

we obtain, in the narrow-window regime $\Delta\tau_c \rightarrow 0$, the uniform expansion $\sin \Phi_0(s) = \phi_1 s + \mathcal{O}(s^3)$ and $1 - \cos \Phi_0(s) = \frac{1}{2}\phi_1^2 s^2 + \mathcal{O}(s^4)$. Substituting into (A4) and using $\int s W(s) ds = 0$ and $\int s^2 W(s) ds > 0$, we arrive at

$$\tilde{\alpha} = \nu \frac{\phi_1 \int s^2 W(s) ds}{\frac{1}{2}\phi_1^2 \int s^2 W(s) ds} = \frac{2\nu}{\phi_1} + \mathcal{O}(\Delta\tau_c^2), \quad \phi_1 = \omega_A - \nu \dot{u}_0(\tau_c), \quad (\text{A6})$$

showing that $\tilde{\alpha} = \mathcal{O}(1)$ as $\Delta\tau_c \rightarrow 0$, with sign $\text{sgn}(\tilde{\alpha}) = \text{sgn}(\phi_1)$ and magnitude set solely by ϕ_1 at leading order, independent of the detailed shape of W . In the opposite, wide-window (adiabatic) regime, both integrals in (A4) are suppressed by stationary-phase/steepest-descent arguments, but with the same powers of $\Delta\tau_c$; the ratio therefore remains finite and $\mathcal{O}(1)$ across the full adiabatic window. Using the elementary bounds $|\sin \Phi_0| \leq |\Phi_0|$ and $1 - \cos \Phi_0 \geq \frac{2}{\pi^2} \min\{\Phi_0^2, \pi^2\}$, we also obtain the uniform estimate

$$0 \leq |\tilde{\alpha}| \leq \nu \frac{\int |s| W(s) |\Phi_0(s)| ds}{\int \frac{2}{\pi^2} \min\{\Phi_0(s)^2, \pi^2\} W(s) ds} = \mathcal{O}(1), \quad (\text{A7})$$

valid in the regime specified in the main text, with constants depending only on the fixed shape of W .

For practical use, it is convenient to have closed forms for standard windows. If $\chi(\tau) = \chi_0 \exp[-(\tau - \tau_c)^2/(2\Delta\tau_c^2)]$ is Gaussian, then

$$W(s) = W_0 \exp\left[-\frac{s^2}{4\Delta\tau_c^2}\right], \quad W_0 = \sqrt{\pi} \Delta\tau_c \chi_0^2, \quad (\text{A8})$$

and (A4) becomes the ratio of one-dimensional oscillatory integrals

$$\tilde{\alpha}_G(\nu, \omega_A; \Delta\tau_c) = \nu \frac{\int_{-\infty}^{+\infty} ds s e^{-s^2/(4\Delta\tau_c^2)} \sin[\Phi_0(s)]}{\int_{-\infty}^{+\infty} ds e^{-s^2/(4\Delta\tau_c^2)} (1 - \cos[\Phi_0(s)]). \quad (\text{A9})$$

The small- $\Delta\tau_c$ series follows immediately from (A6); higher-order corrections can be organized in terms of Hermite-Gaussian moments of $\sin \Phi_0$ and $\cos \Phi_0$.

If, instead, $\chi(\tau) = \chi_0 b\left(\frac{\tau - \tau_c}{\Delta\tau_c}\right)$ is a compactly supported C^∞ “bump” with $b \in C_0^\infty([-1, 1])$, b even, and $\int b^2 = 1$, then

$$W(s) = \Delta\tau_c \chi_0^2 w\left(\frac{s}{\Delta\tau_c}\right), \quad w(\sigma) = \int_{-1}^{+1} d\eta b\left(\eta + \frac{\sigma}{2}\right) b\left(\eta - \frac{\sigma}{2}\right), \quad (\text{A10})$$

with w even and supported in $|\sigma| \leq 2$. Writing $\Phi_0(\Delta\tau_c, \sigma) \equiv \Phi_0(s = \Delta\tau_c \sigma)$, we obtain the compact representation

$$\tilde{\alpha}_B(\nu, \omega_A; \Delta\tau_c) = \nu \frac{\int_{-2}^{+2} d\sigma \sigma w(\sigma) \sin[\Phi_0(\Delta\tau_c, \sigma)]}{\int_{-2}^{+2} d\sigma w(\sigma) (1 - \cos[\Phi_0(\Delta\tau_c, \sigma)]), \quad (\text{A11})$$

from which the narrow-window asymptotics again reproduces (A6). In both cases, the T -slow dependence enters solely through Φ_0 evaluated at $T = \tau_c$ via the near-horizon EF map used in the main text.

Appendix B: Even-Parity $\ell = 2$ Reconstruction in EF Coordinates

We present a compact reconstruction of the even-parity $\ell = 2$ metric perturbation in ingoing Eddington-Finkelstein (EF) coordinates, organized so that all tensor components are manifestly regular at the future horizon. Our starting point is the standard Regge-Wheeler (RW)-gauge description on Schwarzschild, where the Zerilli-Moncrief master field Ψ_{20} encapsulates the physical degrees of freedom. We pass to EF coordinates via $v = t + r_*(r)$, which ensures that $(\partial_t)_r = (\partial_v)_r$ and $(\partial_r)_t = (\partial_r)_v + \frac{r}{r-2M}(\partial_v)_r$, removing the coordinate singularity at $r = 2M$ for ingoing solutions. We then apply an even-parity gauge transformation generated by the EF-regular vector

$$\xi_a = (T(r)Y_{2m} dv + R(r)Y_{2m} dr + r^2 L(r) \partial_a Y_{2m}) e^{-i\omega v}, \quad (\text{B1})$$

with radial profiles T, R, L chosen to eliminate all spurious divergences and to express the resulting metric directly in terms of Ψ_{20} and its v, r derivatives.

After these steps, the nonvanishing $\ell = 2$ even-parity EF components can be written in the compact form

$$\begin{aligned} h_{vv} &= [\tilde{\alpha}_{vv}(r) \Psi_{20} + \beta_{vv}(r) (-i\omega) \Psi_{20}] Y_{2m} e^{-i\omega v}, \\ h_{vr} &= [\tilde{\alpha}_{vr}(r) \Psi_{20} + \beta_{vr}(r) \partial_r \Psi_{20}] Y_{2m} e^{-i\omega v}, \\ h_{rr} &= \partial_r [a_2(r) \Psi_{20} + b_2(r) \partial_r \Psi_{20}] Y_{2m} e^{-i\omega v}, \\ K &= [\tilde{\alpha}_K(r) \Psi_{20} + \beta_K(r) \partial_r \Psi_{20}] Y_{2m} e^{-i\omega v}, \\ G &= [\tilde{\alpha}_G(r) \Psi_{20} + \beta_G(r) \partial_r \Psi_{20}] Y_{2m} e^{-i\omega v}, \end{aligned} \quad (\text{B2})$$

where all coefficient functions are rational in r and regular at $r = 2M$. Eliminating $\partial_v \Psi_{20}$ and $\partial_v^2 \Psi_{20}$ in favor of $\Psi_{20}, \partial_r \Psi_{20}, \partial_r^2 \Psi_{20}$ using the Zerilli equation with $\lambda = 2$ yields one convenient EF-regular choice (equivalent sets are related by adding harmless multiples of the Zerilli equation):

$$\begin{aligned} \tilde{\alpha}_{vv} &= \frac{2M(r-2M)}{r^2 \Lambda}, & \beta_{vv} &= \frac{(r-2M)^2}{r \Lambda}, & \gamma_{vv} &= \frac{2M}{\Lambda}, \\ \tilde{\alpha}_{vr} &= \frac{2M}{r \Lambda}, & \beta_{vr} &= \frac{(r-2M)}{\Lambda}, \\ \tilde{\alpha}_{rr} &= \partial_r a_2(r), & \beta_{rr} &= \partial_r b_2(r), \end{aligned}$$

$$\begin{aligned}
\tilde{\alpha}_K &= \frac{(r-2M)(2r+M)}{r^2\Lambda}, & \beta_K &= \frac{(r-2M)^2}{r\Lambda}, \\
\tilde{\alpha}_G &= -\frac{6M}{r\Lambda^2}, & \beta_G &= \frac{r-2M}{\Lambda^2},
\end{aligned} \tag{B3}$$

where $\Lambda = 2r + 3M$, and where the divergence-form coefficients a_2, b_2 coincide with the EF-regular pair introduced in Appendix C (Eq. (C4)):

$$a_2(r) = \frac{(r-2M)(2r^2+6Mr+6M^2)}{r^2(2r+3M)^2}, \quad b_2(r) = \frac{(r-2M)^2}{r(2r+3M)^2}. \tag{B4}$$

This placement of $(-i\omega)$ in h_{vv} is merely conventional; alternative but equivalent rearrangements follow from integrating by parts in v and using the master equation, and do not affect regularity or the near-horizon behavior. The representation in (B2)-(B4) is tailored to our later use of the divergence form for h_{rr} and to the clean separation between Ψ_{20} and $\partial_r \Psi_{20}$ in the angular scalars K and G .

Upon restricting to the symmetry axis ($\theta = 0$), spherical harmonics Y_{2m} vanish unless $m = 0$, so only the axisymmetric sector contributes. With the affinely-parametrized ingoing EF generator $k^a = (\partial_r)^a$, we read off directly that

$$h_{kk} = h_{rr} = \partial_r[a_2(r)\Psi_{20} + b_2(r)\partial_r\Psi_{20}], \tag{B5}$$

which is Eq. (C1) in the main EF boundary analysis. This divergence form is the one we use to define the EF-regular boundary expression for the geometric response $\mathcal{C}_{20}(r_c)$.

We conclude with two simple checks. Near the future horizon, ingoing EF solutions have $h_{vv}, h_{vr}, K, G = \mathcal{O}(1)$ and $h_{rr} = \mathcal{O}(1)$, confirming manifest regularity at $r = 2M$. In the far zone, $r \gg M$, the profiles $a_2(r), b_2(r)$ and all coefficient functions fall off with the expected $1/r$ scalings (modulo the oscillatory phase $e^{-i\omega v}$), so each metric component decays at least as $1/r$. These behaviors are consistent with the Zerilli-Moncrief asymptotics and with the radiative interpretation of Ψ_{20} at null infinity.

Appendix C: EF reconstruction and boundary formula for $\mathcal{C}_{20}(r_c)$

We assemble here the EF-regular reconstruction we use on the axis and the associated boundary expression that defines the geometric response $\mathcal{C}_{20}(r_c)$. Our starting point is the even-parity $\ell = 2$ sector in ingoing Eddington-Finkelstein coordinates (v, r, θ, φ) with master field $\Psi_{20}(v, r)$ obeying the Zerilli equation. Following Appendix B, we work with the axially symmetric line $\theta = 0$, so that only $m = 0$ contributes and Y_{2m} reduces to Y_{20} . We adopt the affinely-parametrized ingoing EF generator $k^a = (\partial_r)^a$.

The reconstruction we use places the axial null-null component of the metric in divergence form. Denoting by h_{ab} the linearized metric, we write

$$h_{kk}(v, r, \theta = 0) = \partial_r[a_2(r)\Psi_{20}(v, r) + b_2(r)\partial_r\Psi_{20}(v, r)], \tag{C1}$$

which is the axis restriction of Eq. (B2) and coincides with Eq. (B5). The functions $a_2(r)$ and $b_2(r)$ are fixed by the EF-regular gauge choice described in Appendix B and by requiring that all tensor components remain finite at the future horizon and decay with the usual radiative falloff at large r .

The geometric response $\mathcal{C}_{20}(r_c)$ is the EF-regular boundary primitive of h_{kk} along the radial generator. We define it by extracting the bracketed expression in (C1) at the cut radius $r = r_c$:

$$\mathcal{C}_{20}(r_c) \equiv \frac{1}{2} \sqrt{\frac{5}{4\pi}} [a_2(r_c)\Psi_{20}(v, r_c) + b_2(r_c)\partial_r\Psi_{20}(v, r_c)] \tag{C2}$$

so that

$$\partial_r \mathcal{C}_{20}(r) = h_{kk}(v, r), \quad \text{and hence} \quad \int_{r_1}^{r_2} h_{kk}(v, r) dr = \mathcal{C}_{20}(r_2) - \mathcal{C}_{20}(r_1). \tag{C3}$$

To exhibit EF regularity and the proper asymptotics, we record one convenient choice of the divergence-form profiles,

$$a_2(r) = \frac{(r-2M)(2r^2+6Mr+6M^2)}{r^2(2r+3M)^2}, \quad b_2(r) = \frac{(r-2M)^2}{r(2r+3M)^2}, \tag{C4}$$

which are the same functions referenced in Appendix B [cf. Eq. (B4)]. With these profiles, a_2 and b_2 are smooth at $r = 2M$ and decay as $a_2(r) = 1 + \mathcal{O}(M/r)$, $b_2(r) = \mathcal{O}(r^{-1})$ as $r \rightarrow \infty$.

It is convenient to state two immediate consequences. First, the horizon behavior follows from (C4): both a_2 and b_2 are finite at $r = 2M$, and ingoing EF solutions have Ψ_{20} finite there, so $\mathcal{C}_{20}(r_c)$ is manifestly finite as $r_c \rightarrow 2M$. Second, in the wave zone $r \gg M$, the Zerilli equation reduces to the flat-space radial equation at leading order, implying $\partial_r \Psi_{20} = \mathcal{O}(r^{-1})\Psi_{20}$; together with (C4), we obtain

$$\mathcal{C}_{20}(r_c) = \Psi_{20}(v, r_c) + \mathcal{O}\left(\frac{M}{r_c}\right) \Psi_{20}(v, r_c), \quad (\text{C5})$$

which we use when comparing with asymptotic amplitudes at null infinity.

For completeness, we give the EF-regular form of the remaining even-parity axis components needed to cross-check (C1). Eliminating v -derivatives via the Zerilli equation with $\lambda = 2$ and writing $\Lambda = 2r + 3M$, a consistent choice is

$$h_{vv} = \left[\frac{2M(r-2M)}{r^2 \Lambda} \Psi_{20} + \frac{(r-2M)^2}{r \Lambda} (-i\omega) \Psi_{20} \right], \quad (\text{C6})$$

$$h_{vr} = \left[\frac{2M}{r \Lambda} \Psi_{20} + \frac{(r-2M)}{\Lambda} \partial_r \Psi_{20} \right], \quad (\text{C7})$$

with h_{rr} given by (C1). These expressions agree with Appendix B on the axis and make explicit that no component diverges at $r = 2M$.

The boundary formula (C2) is the object we use throughout the main text: it packages the EF reconstruction into a single gauge-invariant axis functional whose radial derivative returns the physical h_{kk} . In the numerical evaluation and in the quasinormal-mode plots, we read off $\mathcal{C}_{20}(r_c)$ directly from (C2) using the master field Ψ_{20} and its radial derivative obtained from the Zerilli-Moncrief solution evaluated at the chosen cut radius r_c .

Appendix D: Gauge considerations

We record the gauge structure underlying our EF-regular reconstruction and the boundary expression for $\mathcal{C}_{20}(r_c)$. Throughout, we work at the linear order about Schwarzschild, with even-parity $\ell = 2$ perturbations represented either by the Zerilli-Moncrief master field Ψ_{20} or by metric components in ingoing EF coordinates (v, r, θ, φ) . A first-order gauge transformation generated by a vector field ξ_a acts as

$$h_{ab} \mapsto h'_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a, \quad (\text{D1})$$

and induces corresponding shifts on any scalar or tensor extracted from h_{ab} . Our goal is to identify the class of gauge vectors that preserve EF regularity and to isolate the combination that leaves the boundary functional $\mathcal{C}_{20}(r_c)$ invariant.

We restrict to the even-parity sector and parametrize the generator as in Appendix B,

$$\xi_a = \left(T(r) Y_{2m} dv + R(r) Y_{2m} dr + r^2 L(r) \partial_a Y_{2m} \right) e^{-i\omega v}, \quad (\text{D2})$$

with smooth radial profiles T, R, L chosen so that ξ_a is regular at the future horizon. Substituting (D2) into (D1) and evaluating on the axis ($\theta = 0$), where only $m = 0$ contributes, we find that the null-null EF component along the affinely-parametrized ingoing generator $k^a = (\partial_r)^a$ transforms as

$$h_{kk} \equiv h_{ab} k^a k^b \mapsto h'_{kk} = h_{kk} + \partial_r [\Xi(r)], \quad \Xi(r) = \alpha_1(r) T(r) + \alpha_2(r) R(r) + \alpha_3(r) L(r), \quad (\text{D3})$$

where the $\alpha_i(r)$ are rational functions of r determined by the Schwarzschild background and the $\ell = 2$ angular structure. The explicit forms are not needed for our purposes; it suffices that the transformation of h_{kk} is a total radial derivative of a horizon-regular scalar $\Xi(r)$.

In the EF reconstruction used in Appendices B-C, we arranged h_{kk} in divergence form,

$$h_{kk}(v, r) = \partial_r [a_2(r) \Psi_{20}(v, r) + b_2(r) \partial_r \Psi_{20}(v, r)], \quad (\text{D4})$$

with a_2, b_2 given in (C4). The associated boundary primitive at the cut radius r_c is

$$\mathcal{C}_{20}(r_c) \equiv \frac{1}{2} \sqrt{\frac{5}{4\pi}} \left[a_2(r_c) \Psi_{20}(v, r_c) + b_2(r_c) \partial_r \Psi_{20}(v, r_c) \right], \quad (\text{D5})$$

so that $\partial_r \mathcal{C}_{20}(r) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \mathcal{S}_{20}(r)$ as in (C3). Combining (D3)-(D5) shows that $\mathcal{C}_{20}$ transforms by a boundary term,

$$\mathcal{C}_{20}(r_c) \mapsto \mathcal{C}'_{20}(r_c) = \mathcal{C}_{20}(r_c) + \frac{1}{2} \sqrt{\frac{5}{4\pi}} \Xi(r_c). \quad (\text{D6})$$

Hence $\mathcal{C}_{20}(r_c)$ is invariant under all EF-regular gauge transformations for which the generator obeys the boundary condition $\Xi(r_c) = 0$. In practice, we enforce this either by fixing the gauge completely at $r = r_c$ or by requiring the generator to vanish there. When r_c is taken at the horizon or at a large extraction radius, the same conclusion holds provided ξ_a decays appropriately in the far zone or remains regular at the horizon.

It is useful to relate this condition to the RW-gauge description and to the Zerilli-Moncrief scalar. In RW gauge on Schwarzschild slices the even-parity metric is algebraically determined by Ψ_{20} and its t, r derivatives; passing to ingoing EF coordinates and then performing a horizon-regular gauge transformation of the form (D2) reproduces the EF coefficients quoted in (B2)-(B4). The residual gauge freedom is parameterized by solutions of the homogeneous system obtained by demanding that the transformed metric still takes the EF-regular form of Appendix B. Solving this system shows that any such residual generator produces a shift of h_{kk} that is a pure radial derivative, consistent with (D3), and therefore changes $\mathcal{C}_{20}$ only by $\Xi(r_c)$ as in (D6). This confirms that, once the boundary value of Ξ is fixed at r_c , the EF reconstruction is unique.

We check compatibility with the asymptotics and with horizon regularity. Using (C4), we have $a_2(r) = 1 + \mathcal{O}(M/r)$ and $b_2(r) = \mathcal{O}(r^{-1})$ for $r \gg M$, while Ψ_{20} solves the Zerilli equation and satisfies $\partial_r \Psi_{20} = \mathcal{O}(r^{-1}) \Psi_{20}$ in the wave zone. Then $\mathcal{C}_{20}(r)$ approaches $\Psi_{20} Y_{20}(0) \Re(e^{-i\omega v})$ up to $\mathcal{O}(M/r)$ corrections, and any EF-regular residual generator with $\Xi(\infty) = 0$ leaves this limit unchanged. Near the future horizon, both a_2 and b_2 are smooth and Ψ_{20} is finite for ingoing solutions, so $\mathcal{C}_{20}(r)$ remains finite as $r \downarrow 2M$; demanding $\Xi(2M) = 0$ guarantees horizon-invariant normalization of the boundary functional.

For later reference, we note a convenient parametrization of Ξ in terms of the profiles in (D2). Evaluating $\nabla_a \xi_b + \nabla_b \xi_a$ on the EF background and contracting twice with k^a yields

$$\Xi(r) = \frac{r - 2M}{r} R(r) + \frac{2M}{r} T(r) - \Lambda^{-1} [(r - 2M) r L'(r) + 3M L(r)], \quad \Lambda = 2r + 3M, \quad (\text{D7})$$

with prime denoting d/dr . This form makes manifest that smooth T, R, L produce a smooth Ξ at the horizon and that $\Xi(r_c) = 0$ imposes a single linear condition on the boundary values of the profiles. Different convenient choices of (T, R, L) that satisfy $\Xi(r_c) = 0$ are related by homogeneous solutions with $\Xi \equiv 0$, which generate no change in $\mathcal{C}_{20}$.

Finally, we emphasize the practical implication for our calculations and plots. All expressions for $h_{vv}, h_{vr}, h_{rr}, K, G$ used in Appendices B-C are obtained in an EF-regular gauge constructed from Ψ_{20} and fixed by the boundary condition $\Xi(r_c) = 0$. The boundary response $\mathcal{C}_{20}(r_c)$ defined in (D5) is therefore a well-posed, gauge-invariant object within the EF-regular class, and it is the unique primitive of h_{kk} consistent with this boundary fixing.

Appendix E: Limiting procedures

We summarize the limiting operations used throughout the analysis and record the conditions under which they commute. Our discussion is organized around the EF-regular reconstruction on the axis, with the boundary primitive $\mathcal{C}_{20}(r_c)$ defined in (C2) and profiles $a_2(r), b_2(r)$ as in (C4). All statements below are at fixed mass $M > 0$ and for solutions Ψ_{20} of the Zerilli equation in the even-parity $\ell = 2$ sector.

We first consider the near-horizon limit. Ingoing EF solutions are regular at $r = 2M$, so $\Psi_{20}(v, r) = \Psi_{20}(v, 2M) + \mathcal{O}(r - 2M)$ and $\partial_r \Psi_{20}(v, r) = \mathcal{O}(1)$ as $r \downarrow 2M$. Since a_2, b_2 are smooth at the horizon, $\mathcal{C}_{20}(r_c)$ admits a finite limit,

$$\lim_{r_c \rightarrow 2M} \mathcal{C}_{20}(r_c) = \left[a_2(2M) \Psi_{20}(v, 2M) + b_2(2M) \partial_r \Psi_{20}(v, 2M) \right], \quad (\text{E1})$$

and the divergence form for h_{kk} implies

$$\int_{2M}^{r_c} h_{kk}(v, r) dr = \mathcal{C}_{20}(r_c) - \mathcal{C}_{20}(2M). \quad (\text{E2})$$

The right-hand side is well defined by (E1), so the horizon integral is finite for any EF-regular Ψ_{20} .

We next examine the wave-zone limit. For $r \gg M$, solutions satisfy $\partial_r \Psi_{20} = \mathcal{O}(r^{-1}) \Psi_{20}$, while (C4) gives $a_2(r) = 1 + \mathcal{O}(M/r)$ and $b_2(r) = \mathcal{O}(r^{-1})$. Hence

$$\mathcal{C}_{20}(r_c) = \Psi_{20}(v, r_c) + \mathcal{O}\left(\frac{M}{r_c}\right) \Psi_{20}(v, r_c), \quad r_c \rightarrow \infty, \quad (\text{E3})$$

consistent with (C5). If $\Psi_{20} \sim A(\omega) r^{-1} e^{-i\omega v}$ at fixed retarded time near null infinity, then

$$\lim_{r_c \rightarrow \infty} r_c \mathcal{C}_{20}(r_c) = A(\omega), \quad (\text{E4})$$

so $\mathcal{C}_{20}$ directly encodes the radiative amplitude.

We also require frequency-domain limits. Let $\omega \rightarrow \omega_n$ project onto a simple quasinormal mode with $\Im \omega_n < 0$. For EF-regular solutions, $\Psi_{20}(v, r) = \hat{\Psi}_{20}(r; \omega) e^{-i\omega v}$ with $\hat{\Psi}_{20}$ meromorphic in ω away from branch cuts. Since a_2, b_2 are independent of ω and $\hat{\Psi}_{20}$ is smooth in r for fixed ω , we have

$$\lim_{\omega \rightarrow \omega_n} \mathcal{C}_{20}(r_c; \omega) = \left[a_2(r_c) \hat{\Psi}_{20}(r_c; \omega_n) + b_2(r_c) \partial_r \hat{\Psi}_{20}(r_c; \omega_n) \right] Y_{20}(0) \Re(e^{-i\omega_n v}), \quad (\text{E5})$$

and the radial differentiation and frequency limiting commute:

$$\partial_r \lim_{\omega \rightarrow \omega_n} \mathcal{C}_{20}(r; \omega) = \lim_{\omega \rightarrow \omega_n} \partial_r \mathcal{C}_{20}(r; \omega) = \lim_{\omega \rightarrow \omega_n} h_{kk}(v, r; \omega). \quad (\text{E6})$$

The same conclusions hold for small-frequency limits $\omega \rightarrow 0$ provided $\hat{\Psi}_{20}$ remains bounded and the standard Zerilli regularity at $\omega = 0$ is imposed.

We finally address switching-window limits for the detector response. With $\tilde{\alpha}(\nu, \omega_A \chi)$ defined in (A4), the narrow-window regime $\Delta\tau_c \rightarrow 0$ yields the uniform expansion

$$\tilde{\alpha}(\nu, \omega_A \chi) = \frac{2\nu}{\omega_A - \nu \dot{u}_0(\tau_c)} + \mathcal{O}(\Delta\tau_c^2), \quad (\text{E7})$$

as shown in (A6). In the opposite, adiabatic regime $\Delta\tau_c \rightarrow \infty$, stationary-phase suppression applies in both numerator and denominator of (A4), leaving $\tilde{\alpha} = \mathcal{O}(1)$ uniformly, consistent with the estimate (A7). If we write the first-order correction to the single-mode probability in terms of the EF boundary primitive evaluated along the worldline,

$$\Delta P_{\text{exc}}(\nu, \omega_A; \chi) = \varepsilon \Re \tilde{\alpha}(\nu, \omega_A \chi) \mathcal{C}_{20}(r_c) \Big|_{v=v_c} + \mathcal{O}(\varepsilon^2), \quad (\text{E8})$$

with $\Re$ a fixed normalization introduced in the main text, then the limits $\Delta\tau_c \rightarrow 0, \infty$ commute with $r_c \downarrow 2M$ and $r_c \rightarrow \infty$ under the same uniform boundedness conditions that ensure (E1)-(E4). In particular,

$$\lim_{r_c \rightarrow \infty} \lim_{\Delta\tau_c \rightarrow 0} \Delta P_{\text{exc}} = \lim_{\Delta\tau_c \rightarrow 0} \lim_{r_c \rightarrow \infty} \Delta P_{\text{exc}}, \quad \lim_{r_c \downarrow 2M} \lim_{\Delta\tau_c \rightarrow \infty} \Delta P_{\text{exc}} = \lim_{\Delta\tau_c \rightarrow \infty} \lim_{r_c \downarrow 2M} \Delta P_{\text{exc}}, \quad (\text{E9})$$

and analogous relations hold for the absorption channel.

The commutation of limits used in the proofs is a consequence of dominated convergence. On any compact radial interval away from the light ring, EF regularity provides uniform bounds on Ψ_{20} and its first two derivatives; together with the smooth a_2, b_2 , these bounds control $\mathcal{C}_{20}$ and h_{kk} . In the wave zone, the $1/r$ falloffs of Ψ_{20} and of (C4) ensure absolute integrability in r , while in the near-horizon region the regular expansions used in (E1) give integrable control in $r - 2M$. In the time domain, the detector window produces L^1 kernels $W(s)$ with rapid decay, so the oscillatory integrals in (A4) obey standard dominated-convergence hypotheses as the parameters $\Delta\tau_c$ and ω vary in compact sets.

These observations justify all limiting manipulations appearing in the main text: the evaluation of horizon and far-zone boundary data through $\mathcal{C}_{20}$, the projection onto quasinormal frequencies, and the interchange of switching-window limits with radial extraction. Wherever a boundary condition is needed to ensure gauge invariance, we impose $\Xi(r_c) = 0$ as in (D6)-(D7), which leaves $\mathcal{C}_{20}$ unchanged and therefore does not affect any of the limits recorded above.

Appendix F: Notation and conventions

We work in geometrized units $G = c = 1$ with metric signature $(-+++)$. The Schwarzschild mass is $M > 0$, the areal radius is r , and the line element in ingoing Eddington-Finkelstein (EF) coordinates (v, r, θ, φ) reads

$$ds^2 = -f(r) dv^2 + 2 dv dr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad f(r) = 1 - \frac{2M}{r}. \quad (\text{F1})$$

The tortoise coordinate is defined by $dr_*/dr = f^{-1}$, giving $r_* = r + 2M \ln(r/2M - 1)$, and the EF time is $v = t + r_*$. We use the affinely parametrized ingoing null generator $k^a = (\partial_r)^a$ and the outgoing EF-null vector $\ell^a = (\partial_v)^a - \frac{1}{2} f(\partial_r)^a$, normalized so that $k \cdot \ell = -1$.

Spherical harmonics follow the Condon-Shortley phase and are L^2 -normalized:

$$\int_{S^2} Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}^*(\theta, \varphi) d\Omega = \delta_{\ell\ell'} \delta_{mm'}, \quad Y_{\ell 0}(0) = \sqrt{\frac{2\ell+1}{4\pi}}, \quad (\text{F2})$$

so $Y_{20}(0) = \sqrt{5/(4\pi)}$ as used throughout. Even-parity perturbations are expanded on the $\ell = 2$ sector, and we denote the Zerilli-Moncrief master field by $\Psi_{20}(v, r)$.

For $\ell = 2$ we set

$$\lambda \equiv \frac{(\ell-1)(\ell+2)}{2} = 2, \quad \Lambda(r) \equiv \lambda r + 3M = 2r + 3M, \quad (\text{F3})$$

and the Zerilli potential is

$$V_Z(r) = \frac{2f(r)}{r^3 \Lambda(r)^2} \left[\lambda^2 (\lambda+1) r^3 + 3\lambda^2 M r^2 + 9\lambda M^2 r + 9M^3 \right]. \quad (\text{F4})$$

The master equation in the time domain is $(-\partial_t^2 + \partial_{r_*}^2 - V_Z)\Psi_{20} = 0$, which in EF variables becomes $(2\partial_v \partial_r + f \partial_r^2 + f' \partial_r - V_Z)\Psi_{20} = 0$. Throughout the appendices we remove v -derivatives using this field equation so that all EF coefficients are expressed in terms of Ψ_{20} , $\partial_r \Psi_{20}$, and $\partial_r^2 \Psi_{20}$.

Fourier-Laplace conventions follow the sign choice used in the EF reconstruction. We write

$$\Psi_{20}(v, r) = \hat{\Psi}_{20}(r; \omega) e^{-i\omega v}, \quad \hat{\Psi}_{20}(r; \omega) = \int_{-\infty}^{+\infty} \Psi_{20}(v, r) e^{+i\omega v} dv, \quad (\text{F5})$$

so quasinormal frequencies satisfy $\omega = \omega_R - i\omega_I$ with $\omega_I > 0$. The real-part symbol $\Re(\cdot)$ is used only when we project onto explicitly real axis data in the plots or boundary expressions. Our distributional conventions are $\Theta(0) = \frac{1}{2}$, $\delta(\phi(x)) = \sum_i \delta(x - x_i)/|\phi'(x_i)|$ for simple zeros x_i , and the retarded $i\epsilon$ prescription follows the $e^{-i\omega v}$ Fourier sign.

The EF reconstruction on the axis uses the divergence form

$$h_{kk}(v, r) \equiv h_{ab} k^a k^b = \partial_r [a_2(r) \Psi_{20}(v, r) + b_2(r) \partial_r \Psi_{20}(v, r)], \quad (\text{F6})$$

with the profiles

$$a_2(r) = \frac{(r-2M)(2r^2+6Mr+6M^2)}{r^2(2r+3M)^2}, \quad b_2(r) = \frac{(r-2M)^2}{r(2r+3M)^2}, \quad (\text{F7})$$

which are smooth at the future horizon and approach $a_2 \rightarrow 1$, $b_2 \rightarrow 0$ as $r \rightarrow \infty$. The EF boundary primitive that defines our geometric response is

$$\mathcal{C}_{20}(r_c) \equiv [a_2(r) \Psi_{20}(v, r) + b_2(r) \partial_r \Psi_{20}(v, r)]_{r=r_c}, \quad \partial_r \mathcal{C}_{20}(r) = h_{kk}(v, r). \quad (\text{F8})$$

Gauge transformations are generated by even-parity vectors of the form

$$\xi_a = \left(T(r) Y_{2m} dv + R(r) Y_{2m} dr + r^2 L(r) \partial_a Y_{2m} \right) e^{-i\omega v}, \quad (\text{F9})$$

regular at the horizon. On the axis the induced change in h_{kk} is a total radial derivative, $h_{kk} \mapsto h_{kk} + \partial_r \Xi(r) Y_{20}(0) \Re(e^{-i\omega v})$, with

$$\Xi(r) = \frac{r-2M}{r} R(r) + \frac{2M}{r} T(r) - \Lambda(r)^{-1} [(r-2M) r L'(r) + 3M L(r)], \quad \Lambda(r) = 2r + 3M, \quad (\text{F10})$$

so $\mathcal{C}_{20}(r_c)$ is invariant once we impose the boundary condition $\Xi(r_c) = 0$.

Worldline and switching conventions follow Appendix A. The detector trajectory is $x^a(\tau)$ with proper time τ ; dots denote $d/d\tau$, primes denote d/dr , and over-hats denote Fourier-domain quantities. The switching function $\chi(\tau)$ is smooth and localized near $\tau = \tau_c$ with width $\Delta\tau_c$, and its autocorrelation is

$$W(s) = \int dT \chi\left(T + \frac{s}{2}\right) \chi\left(T - \frac{s}{2}\right), \quad (\text{F11})$$

an even, nonnegative, rapidly decaying function. The single-mode coefficient $\tilde{\alpha}(\nu, \omega_A \chi)$ is defined by the ratio in (A4), with $\nu > 0$ the selected field frequency and $\omega_A > 0$ the detector gap, and admits the narrow-window expansion $\tilde{\alpha} =$

$2\nu/(\omega_A - \nu\dot{u}_0(\tau_c)) + \mathcal{O}(\Delta\tau_c^2)$ from (A6). We use the baseline EF-regular retarded time $u_0(\tau)$ along the worldline and the phase $\Phi_0(s)$ from (A3) to organize the response.

Asymptotic and regularity statements use the following shorthand. Near the horizon we write $r \downarrow 2M$ to indicate limits at fixed v with EF-regular fields; in the wave zone $r \gg M$ we use $\mathcal{O}(M/r)$ and $\mathcal{O}(r^{-1})$ uniformly in bounded frequency windows. For quasinormal-mode limits, we assume simple poles with $\Im\omega < 0$, and we commute radial differentiation with frequency limiting as justified in (E6). Unless stated otherwise, all equalities hold pointwise for smooth EF-regular solutions, and all integrals are understood in the Lebesgue sense with dominated-convergence interchange of limits as summarized in Appendix E.

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