

Scalar molecules $\eta_b B_c^-$ and $\eta_c B_c^+$ with asymmetric quark contents

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(ΩDated: November 6, 2025)

The hadronic scalar molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ with asymmetric quark contents $b\bar{b}\bar{c}$ and $c\bar{c}\bar{b}$ are explored by means of the QCD sum rule method. Their masses and current couplings are calculated using the two-point sum rule approach. The obtained results show that they are strong-interaction unstable particles and transform to ordinary mesons' pairs. The molecule $\mathcal{M}_b$ dissociates through the process $\mathcal{M}_b \rightarrow \eta_b B_c^-$. The decays $\mathcal{M}_c \rightarrow \eta_c B_c^+$ and $J/\psi B_c^{*+}$ are dominant modes for the molecule $\mathcal{M}_c$. The full decay widths of the molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ are estimated using these decay channels, as well as ones generated by the annihilation of $b\bar{b}$ and $c\bar{c}$ quarks in $\mathcal{M}_b$ and $\mathcal{M}_c$, respectively. The QCD three-point sum rule method is employed to find partial widths all of these channels. This approach is required to evaluate the strong couplings at the molecule-meson-meson vertices under consideration. The mass $m = (15728 \pm 90)$ MeV and width $\Gamma[\mathcal{M}_b] = (93 \pm 17)$ MeV of the molecule $\mathcal{M}_b$, and $\tilde{m} = (9712 \pm 72)$ MeV and $\Gamma[\mathcal{M}_c] = (70 \pm 10)$ MeV in the case of $\mathcal{M}_c$ offer valuable guidance for experimental searches at existing facilities.

I. INTRODUCTION

Hadronic four-quark exotic molecular states are already on agenda of high energy physics. Such structures may appear in experiments as a bound and/or resonant states of a pair of ordinary mesons. These molecules are composed of the color-singlet quark-antiquarks, and have internal organizations alternative to those of diquark-antidiquarks: In a diquark-antidiquark picture four-quark mesons are built of colored diquarks and antidiquarks.

Theoretical investigations of hadronic molecules have a rather long history. Thus, existence of the hadronic molecules $c\bar{q}cq$ were supposed in Ref. [1] in light of numerous vector states $J^{PC} = 1^{--}$ observed in e^+e^- annihilation. Analogous ideas were shared by the authors of the publications [2, 3], in which they suggested that four-quark mesons may emerge as bound-resonant states of the D mesons, interacting via conventional light meson exchange mechanism.

The concept of hadronic molecules was later elaborated and advanced in numerous investigations [4–24], in which the authors explored the binding mechanisms of such states, computed their masses, analyzed processes where these particles might be discovered. Needless to say that all available models and methods were applied in these studies to reach reliable conclusions about properties of hadronic molecules.

Another interesting branch of investigations embraces molecules containing only heavy c and b quarks. They may consist of only c (b) quarks, or may be composed of equal number of these quarks. These molecules are hidden charm, bottom, or charm-bottom particles. The

molecules of the first type were examined in Refs. [25–28]. Activity of researches in this field was inspired mainly by observation of new four X structures reported by LHCb-ATLAS-CMS collaborations [29–31]. These structures are presumably scalar resonances made of $c\bar{c}\bar{c}$ quarks. It turns out that some of them may be interpreted as hadronic molecules.

Relevant problems were also addressed in our works [25, 26], in which we considered fully heavy molecules $\eta_c\eta_c$, $\chi_{c0}\chi_{c0}$, and $\chi_{c1}\chi_{c1}$ and computed their masses and decay widths. Our aim was to compare obtained results with measured parameters of different X structures. We argued that the molecule $\eta_c\eta_c$ can be considered as a real candidate to the resonance $X(6200)$, whereas the structure $\chi_{c0}\chi_{c0}$ may be interpreted as $X(6900)$ or one of its components in combination with a scalar diquark-antidiquark state. The mass and width of the molecule $\chi_{c1}\chi_{c1}$ is comparable with those of the structure $X(7300)$, but preferable model for this structure is an admixture of $\chi_{c1}\chi_{c1}$ with sizeable excited diquark-antidiquark component.

There are also various publications devoted to analysis of the molecules with mixed contents [28, 32–36]. The molecules $B_c^{(*)+}B_c^{(*)-}$ were considered in Ref. [32] in the context of the coupled-channel unitary approach. The parameters of the scalar $B_c^+B_c^-$, axial-vector $(B_c^{*+}B_c^- + B_c^+B_c^{*-})/2$ and tensor $B_c^{*+}B_c^{*-}$ mesons were calculated in our articles [34–36]. There, we applied QCD sum rule (SR) method to evaluate masses and full decay widths of these molecules.

Exotic mesons with the asymmetric quark structures $b\bar{b}\bar{c}$ and $c\bar{c}\bar{b}$ also attracted interest of researches. Properties of such diquark-antidiquarks with different spin-parities were investigated in various works (see, the publications [37, 38] and references therein). The hadronic molecules with the same features were considered in Ref. [28].

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In present work, we explore the scalar heavy hadronic molecules $\mathcal{M}_b = \eta_b B_c^-$ and $\mathcal{M}_c = \eta_c B_c^+$ by computing their masses and full decay widths. They have quark contents $bb\bar{b}\bar{c}$ and $cc\bar{c}\bar{b}$, and evidently are molecular analogues of the asymmetric tetraquarks T_b and T_c [38]. Investigations are carried out in the framework of the two-point QCD SR method [39, 40]. Results obtained for the masses of these structures imply that they are strong-interaction unstable particles and convert to a pair of ordinary mesons. The molecule $\mathcal{M}_b$ dissociates to its components $\mathcal{M}_b \rightarrow \eta_b B_c^-$. Apart from this dominant channel, due to annihilation of $b\bar{b}$ quarks, $\mathcal{M}_b$ can transform to pairs of pseudoscalar $B^-\bar{D}^0$, $\bar{B}^0 D^-$, $\bar{B}_s^0 D_s^-$, and vector $B^{*-}\bar{D}^{*0}$, $\bar{B}^{*0} D^{*-}$, $\bar{B}_s^{*0} D_s^{*-}$ mesons. Importance of this mechanism was emphasized in Refs. [41–43] and applied there to diquark-antidiquark mesons.

Dominant channels of the state $\mathcal{M}_c$ are the decays $\mathcal{M}_c \rightarrow \eta_c B_c^+$, $J/\psi B_c^{*+}$, as well as processes $\mathcal{M}_c \rightarrow B^+ D^0$, $B^0 D^+$, $B_s^0 D_s^+$, $B^{*+} D^{*0}$, $B^{*0} D^{*+}$, and $B_s^{*0} D_s^{*+}$. The last six modes are generated because of the $c\bar{c}$ annihilation in $\mathcal{M}_c$.

The widths of the decay channels depend on numerous input parameters of the molecules $\mathcal{M}_b$ and $\mathcal{M}_c$, and of final-state mesons. The masses and couplings of $\mathcal{M}_b$ and $\mathcal{M}_c$ are object of the present studies. The parameters of the conventional mesons are known from experimental measurements or were found using different theoretical methods. Decisive quantities which should be determined are the strong couplings at the, for instance, vertices $\mathcal{M}_b \eta_b B_c^-$, $\mathcal{M}_c \eta_c B_c^+$ and $\mathcal{M}_c J/\psi B_c^{*+}$. They describe the strong interaction of the molecule with ordinary final-state mesons and can be estimated by means of the QCD three-point sum rule method that allows one to evaluate relevant form factors.

This paper is organized in the following way: In Sec. II, we compute the masses and current couplings of the scalar molecules $\mathcal{M}_b$ and $\mathcal{M}_c$. The width of the molecule $\mathcal{M}_b$ is computed in Sec. III. The full width of the structure $\mathcal{M}_c$ saturated by the aforementioned modes is determined in section IV. We make our conclusions in the last part of the article V.

II. MASSES AND CURRENT COUPLINGS OF THE MOLECULES $\mathcal{M}_b$ AND $\mathcal{M}_c$

Here, we consider the masses and current couplings of the molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ in the framework of the two-point QCD sum rule method. To this end, we employ the interpolating currents for the molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ and compute corresponding correlation functions.

Here we give, in details, calculations of $\mathcal{M}_b$ molecule's spectroscopic parameters, but provide only results obtained for the structure $\mathcal{M}_c$. The molecule $\mathcal{M}_b = \eta_b B_c^-$ with quark content $bb\bar{b}\bar{c}$ is interpolated by the current $J(x)$,

$$J(x) = \bar{b}_a(x) i\gamma_5 b_a(x) \bar{c}_b(x) i\gamma_5 b_b(x), \quad (1)$$

where a and b are the color indices.

The scalar molecule $\mathcal{M}_c = \eta_c B_c^+$ has the similar current

$$\tilde{J}(x) = \bar{c}_a(x) i\gamma_5 c_a(x) \bar{b}_b(x) i\gamma_5 b_b(x). \quad (2)$$

A. Parameters of the molecule $\mathcal{M}_b$

To derive the SRs for the mass m and current coupling Λ of $\mathcal{M}_b$, we explore the two-point correlation function

$$\Pi(p) = i \int d^4x e^{ipx} \langle 0 | \mathcal{T} \{ J(x) J^\dagger(0) \} | 0 \rangle, \quad (3)$$

where $\mathcal{T}$ is the time-ordered product of two currents.

In the sum rule approach this correlator has to be presented in two forms. First, it should be expressed using the physical parameters m and Λ of the molecule $\mathcal{M}_b$. The correlator $\Pi^{\text{Phys}}(p)$ obtained by this way is, shortly, the physical side of the required SRs. To find it, we take into account that $\Pi^{\text{Phys}}(p)$ is given by the formula

$$\Pi^{\text{Phys}}(p) = \frac{\langle 0 | J | \mathcal{M}_b \rangle \langle \mathcal{M}_b | J^\dagger | 0 \rangle}{m^2 - p^2} + \dots, \quad (4)$$

and contains the contribution of the ground-state particle, as well as those of the higher resonances and continuum states: The latter are shown in Eq. (4) by the dots.

We rewrite $\Pi^{\text{Phys}}(p)$ using the matrix element

$$\langle 0 | J | \mathcal{M}_b \rangle = \Lambda, \quad (5)$$

and get

$$\Pi^{\text{Phys}}(p) = \frac{\Lambda^2}{m^2 - p^2} + \dots. \quad (6)$$

The term $\Lambda^2/(m^2 - p^2)$ is the invariant amplitude $\Pi^{\text{Phys}}(p^2)$ required for following analysis.

Second, $\Pi(p)$ is calculated in the operator product expansion (OPE) by employing heavy quark propagators. The result of these computations

$$\begin{aligned} \Pi^{\text{OPE}}(p) = i \int d^4x e^{ipx} \text{Tr} \left\{ \left[\gamma_5 S_b^{aa'}(x) \gamma_5 S_b^{a'a}(-x) \right] \right. \\ \times \text{Tr} \left[\gamma_5 S_b^{bb'}(x) \gamma_5 S_c^{b'b}(-x) \right] - \text{Tr} \left[\gamma_5 S_b^{ab'}(x) \gamma_5 S_c^{b'b}(-x) \right] \\ \left. \times \gamma_5 S_b^{ba'}(x) \gamma_5 S_b^{a'a}(-x) \right\}, \end{aligned} \quad (7)$$

is the QCD side $\Pi^{\text{OPE}}(p)$ of the sum rules, where $S_{b(c)}^{ab}(x)$ are the propagators of b and c quarks [44].

The function $\Pi^{\text{OPE}}(p)$ has also the simple Lorentz structure: We label as $\Pi^{\text{OPE}}(p^2)$ the corresponding invariant amplitude. By equating two formulas for the amplitudes and applying the assumption about the hadron-quark duality, and performing some manipulations, we

get the SRs for m and Λ (for further details see, for example, Ref. [38])

$$m^2 = \frac{\Pi'(M^2, s_0)}{\Pi(M^2, s_0)}, \quad (8)$$

and

$$\Lambda^2 = e^{m^2/M^2} \Pi(M^2, s_0). \quad (9)$$

In Eq. (8), we employ $\Pi'(M^2, s_0) = d\Pi(M^2, s_0)/d(-1/M^2)$. Here, $\Pi(M^2, s_0)$ is the amplitude $\Pi^{\text{OPE}}(p^2)$ after the Borel transformation and continuum subtraction procedures. The Borel transformation is necessary to suppress contribution of higher resonances and continuum states. The continuum subtraction allows us to remove the suppressed terms from the QCD side of the relevant equality. As a result, $\Pi(M^2, s_0)$ acquires a dependence on the Borel M^2 and continuum subtraction s_0 parameters, and has the form

$$\Pi(M^2, s_0) = \int_{(3m_b+m_c)^2}^{s_0} ds \rho^{\text{OPE}}(s) e^{-s/M^2} + \Pi(M^2). \quad (10)$$

The spectral density $\rho^{\text{OPE}}(s)$ is found as an imaginary part of the function $\Pi^{\text{OPE}}(p^2)$. Because in the current paper we consider only the perturbative and dimension-four nonperturbative contributions $\sim \langle \alpha_s G^2/\pi \rangle$ to $\Pi^{\text{OPE}}(p^2)$, $\rho^{\text{OPE}}(s)$ contains terms $\rho^{\text{pert.}}(s)$ and $\rho^{\text{Dim4}}(s)$. The nonperturbative function $\Pi(M^2)$ is calculated directly from the correlator $\Pi^{\text{OPE}}(p)$ and embrace effects of terms which are not included into the spectral density.

To carry out the numerical calculations, we have to fix the parameters in Eqs. (8) and (9). The b and c quarks' masses and gluon condensate $\langle \alpha_s G^2/\pi \rangle$ are universal quantities. In the current article, we employ

$$\begin{aligned} m_c &= (1.2730 \pm 0.0046) \text{ GeV}, \\ m_b &= (4.183 \pm 0.007) \text{ GeV}, \\ \langle \alpha_s G^2/\pi \rangle &= (0.012 \pm 0.004) \text{ GeV}^4. \end{aligned} \quad (11)$$

Quark masses m_c and m_b are calculated in the $\overline{\text{MS}}$ scheme [45]. The condensate $\langle \alpha_s G^2/\pi \rangle$ was estimated in Refs. [39, 40] from studies of different processes.

The parameters M^2 and s_0 depend on a analyzing problem and have to satisfy standard restrictions of SR analyses. In the SR method the pole contribution (PC) should dominate in obtained quantities, therefore, in computations we require fulfilment $\text{PC} \geq 0.5$. Convergence of OPE is another condition for reliable SR studies. In our case, the correlation function contains only dimension-4 term $\Pi^{\text{Dim4}}(M^2, s_0)$. Then, the constraint $|\Pi^{\text{Dim4}}(M^2, s_0)| \leq 0.05|\Pi(M^2, s_0)|$ is enough to ensure convergence of OPE. Last but not least is stability of final results upon variations of M^2 and s_0 .

Numerical analysis is carried out over a broad range of the parameters M^2 and s_0 . Collected results permits us to limit the working regions for M^2 and s_0 , where all

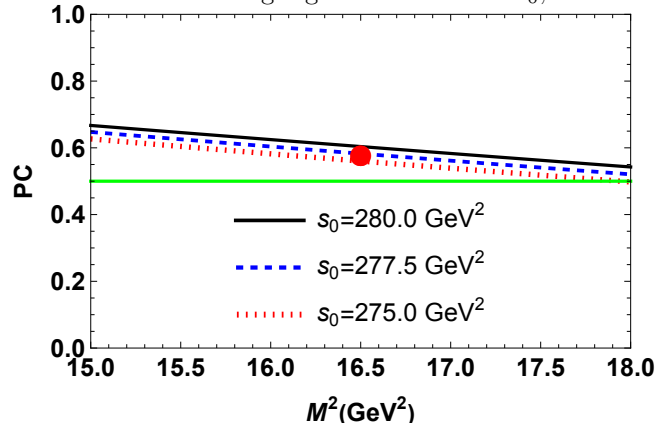


FIG. 1: Pole contribution PC as a function of M^2 at some s_0 . The circle labels the point $M^2 = 16.5 \text{ GeV}^2$ and $s_0 = 277.5 \text{ GeV}^2$.

standard conditions are satisfied. We conclude that the intervals

$$M^2 \in [15, 18] \text{ GeV}^2, \quad s_0 \in [275, 280] \text{ GeV}^2, \quad (12)$$

meet all these conditions. In fact, at maximal and minimal M^2 the pole contribution averaged over s_0 is $\text{PC} \approx 0.52$ and $\text{PC} \approx 0.65$. At $M^2 = 15 \text{ GeV}^2$ the nonperturbative contribution constitutes approximately 1.5% of the full result. The PC as a function of the Borel parameter is presented in Fig. 1, where all lines overshoot the border $\text{PC} = 0.5$.

We calculate m and Λ as their mean values in the windows Eq. (12) and get

$$\begin{aligned} m &= (15728 \pm 90) \text{ MeV}, \\ \Lambda &= (3.09 \pm 0.32) \text{ GeV}^5. \end{aligned} \quad (13)$$

The predictions in Eq. (13) amount to SR results at $M^2 = 16.5 \text{ GeV}^2$ and $s_0 = 277.5 \text{ GeV}^2$, where $\text{PC} \approx 0.58$, which guaranties the prevalence of PC in extracted quantities. The ambiguities in Eq. (13) are formed due to choices of M^2 and s_0 : Uncertainties connected with errors in quark masses and gluon condensate are negligible.

The errors in Eq. (13) amount to $\pm 0.6\%$ of the mass m , which proves the stability of this result. Uncertainties of Λ are larger and equal to $\pm 10\%$ remaining nevertheless inside borders reasonable for the SR analysis. In Fig. 2, we plot dependence of m on the parameters M^2 and s_0 .

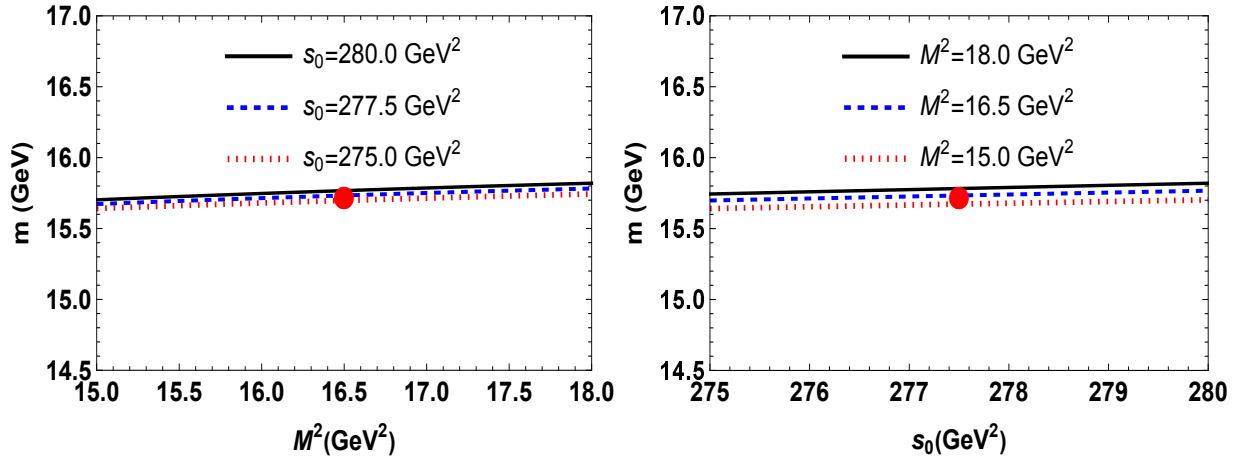


FIG. 2: Dependence of the mass m on the parameters M^2 (left panel), and s_0 (right panel).

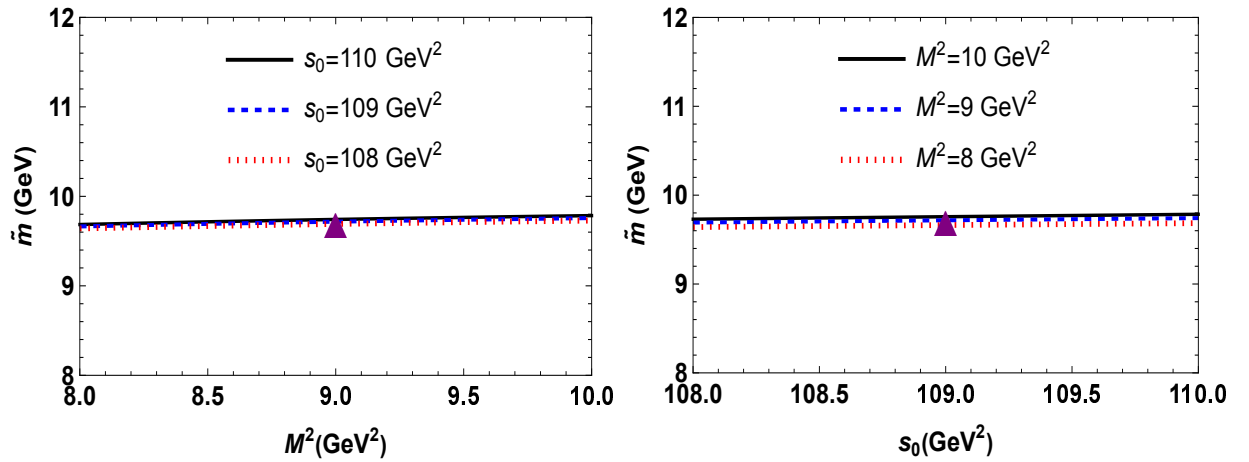


FIG. 3: Mass $\tilde{m}$ as a function on the parameters M^2 (left panel), and s_0 (right panel). The purple triangle shows point $M^2 = 9$ GeV² and $s_0 = 109$ GeV².

B. Mass and current coupling of the molecule $\mathcal{M}_c$

The correlators $\tilde{\Pi}^{\text{Phys}}(p)$ and $\tilde{\Pi}^{\text{OPE}}(p)$, and SRs for parameters $\tilde{m}$ and $\tilde{\Lambda}$ of the molecule $\mathcal{M}_c = \eta_c B_c^+$ do not differ considerably from those of $\mathcal{M}_b$. Therefore, it is enough to present windows for M^2 and s_0 . Numerical calculations demonstrate that

$$M^2 \in [8, 10] \text{ GeV}^2, \quad s_0 \in [108, 110] \text{ GeV}^2, \quad (14)$$

satisfy all restrictions. Indeed, at maximal $M^2 = 10$ GeV² the pole contribution is $\text{PC} \approx 0.50$, while at $M^2 = 8$ GeV² it amounts to $\text{PC} \approx 0.75$. The nonperturbative contribution at $M^2 = 8$ GeV² constitutes 2% of the full result.

The mass $\tilde{m}$ and current coupling $\tilde{\Lambda}$ of the molecule

$\mathcal{M}_c$ are

$$\begin{aligned} \tilde{m} &= (9712 \pm 72) \text{ MeV}, \\ \tilde{\Lambda} &= (5.11 \pm 0.48) \times 10^{-1} \text{ GeV}^5. \end{aligned} \quad (15)$$

These predictions effectively amount to the sum rule results at $M^2 = 9$ GeV² and $s_0 = 109$ GeV², where $\text{PC} \approx 0.62$. The mass $\tilde{m}$ as a function of the Borel and continuum subtraction parameters M^2 and s_0 is depicted in Fig. 3.

III. FULL DECAY WIDTH OF $\mathcal{M}_b$

In this section we calculate the full decay width of the hadronic molecule $\mathcal{M}_b$. Information on the mass of $\mathcal{M}_b$ permits one to find its decay channels. The process $\mathcal{M}_b \rightarrow \eta_b B_c^-$ is kinematically allowed decay channel of

$\mathcal{M}_b$. In fact, the masses $m_{\eta_b} = (9398.7 \pm 2.0)$ MeV and $m_{B_c} = (6274.47 \pm 0.27 \pm 0.17)$ MeV [45] of the final-state mesons establish the threshold 15673 MeV which is less than m . This is the dominant mode of $\mathcal{M}_b$, because all of its valence quarks appear in the final-state particles.

Another decay channels of the molecule $\mathcal{M}_b$ are ones generated by annihilation of $b\bar{b}$ quarks in $\mathcal{M}_b$ to $q\bar{q}$ and $s\bar{s}$ pairs. Then initial b and $\bar{c}$ quarks from $\mathcal{M}_b$ and light quarks form pairs of BD mesons with appropriate quantum numbers and charges. Because in the SR method the vacuum expectation value $\langle\bar{b}b\rangle$ of b quarks is replaced by the gluon condensate $\langle\alpha_s G^2/\pi\rangle$, these processes are sub-leading modes of the molecule $\mathcal{M}_b$. Nevertheless, total contribution of such channel to the full decay width of $\mathcal{M}_b$ may be sizeable. Here, we are going to take into account decays to mesons $B-\bar{D}^0$, $\bar{B}^0 D^-$, $\bar{B}_s^0 D_s^-$, $B^{*-}\bar{D}^{*0}$, $\bar{B}^{*0} D^{*-}$, and $\bar{B}_s^{*0} D_s^{*-}$.

A. Process $\mathcal{M}_b \rightarrow \eta_b B_c^-$

The width of the process $\mathcal{M}_b(p) \rightarrow \eta_b(p') B_c^-(q)$ besides the known parameters depends on the strong coupling g at the vertex $\mathcal{M}_b \eta_b B_c^-$. In its turn, g can be computed at the mass shell $q^2 = m_{B_c}^2$ using the form factor $g(q^2)$. To evaluate $g(q^2)$ we analyze the following three-point correlation function

$$\Pi(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J^{\eta_b}(y) \times J^{B_c^-}(0) J^\dagger(x) \} | 0 \rangle, \quad (16)$$

with $J^{\eta_b}(x)$ and $J^{B_c^-}(x)$ being the currents which interpolate the pseudoscalar mesons η_b and B_c^- , and have the forms

$$J^{\eta_b}(x) = \bar{b}_i(x) i \gamma_5 b_i(x), \quad J^{B_c^-}(x) = \bar{c}_j(x) i \gamma_5 b_j(x). \quad (17)$$

Here, i and j are the color indices. The four-momentum p of the molecule $\mathcal{M}_b$ is connected by the equality $p = p' + q$ to momenta of mesons.

It is known that the correlator Eq. (16) expressed using parameters of particles $\mathcal{M}_b$, η_b and B_c^- is the phenomenological side of SR $\Pi^{\text{Phys}}(p, p')$. To find $\Pi^{\text{Phys}}(p, p')$, we insert into Eq. (16) full system of intermediate states for the particles $\mathcal{M}_b$, η_b and B_c^- and carry out four-integrals over x and y . Having dissected the contribution of the ground-state particles and using a naive factorization approximation, we obtain

$$\begin{aligned} \Pi^{\text{Phys}}(p, p') &= \frac{\langle 0 | J^{\eta_b} | \eta_b(p') \rangle \langle 0 | J^{B_c^-} | B_c^-(q) \rangle}{p'^2 - m_{\eta_b}^2} \frac{q^2 - m_{B_c}^2}{q^2 - m_{B_c}^2} \\ &\times \langle \eta_b(p') B_c^-(q) | \mathcal{M}_b(p) \rangle \frac{\langle \mathcal{M}_b(p) | J^\dagger | 0 \rangle}{p^2 - m^2} \\ &+ \dots \end{aligned} \quad (18)$$

The ellipses above denote effects of excited and continuum states.

By applying to Eq. (18) the matrix elements of the mesons η_b and B_c^-

$$\begin{aligned} \langle 0 | J^{\eta_b} | \eta_b(p') \rangle &= \frac{f_{\eta_b} m_{\eta_b}^2}{2m_b}, \\ \langle 0 | J^{B_c^-} | B_c^-(q) \rangle &= \frac{f_{B_c} m_{B_c}^2}{m_b + m_c}, \end{aligned} \quad (19)$$

one can simplify Π^{Phys} . Above, f_{η_b} and f_{B_c} are the decay constants of the corresponding mesons. We have to introduce also a formula for the vertex $\langle \eta_b(p') B_c^-(q) | \mathcal{M}_b(p) \rangle$. It has a simple form

$$\langle \eta_b(p') B_c^-(q) | \mathcal{M}_b(p) \rangle = g(q^2) p \cdot p'. \quad (20)$$

As a result, we get

$$\begin{aligned} \Pi^{\text{Phys}}(p, p') &= g(q^2) \frac{\Lambda f_{\eta_b} m_{\eta_b}^2 f_{B_c} m_{B_c}^2}{2m_b(m_b + m_c)(p^2 - m^2)} \\ &\times \frac{1}{(p'^2 - m_{\eta_b}^2)(q^2 - m_{B_c}^2)} \frac{m^2 + m_{\eta_b}^2 - q^2}{2} + \dots \end{aligned} \quad (21)$$

This is the invariant amplitude $\Pi^{\text{Phys}}(p^2, p'^2, q^2)$ which will be used to obtain SR for $g(q^2)$.

The correlator $\Pi(p, p')$ computed in terms of quark propagators reads

$$\begin{aligned} \Pi^{\text{OPE}}(p, p') &= \int d^4x d^4y e^{ip'y} e^{-ipx} \{ \text{Tr} [\gamma_5 S_b^{ia}(y-x) \\ &\times \gamma_5 S_b^{ai}(x-y)] \text{Tr} [\gamma_5 S_c^{jb}(-x) \gamma_5 S_c^{bj}(x)] \\ &- \text{Tr} [\gamma_5 S_b^{ib}(y-x) \gamma_5 S_c^{bj}(x) \gamma_5 S_b^{ja}(-x) \gamma_5 S_b^{ai}(x-y)] \}. \end{aligned} \quad (22)$$

The correlator $\Pi^{\text{OPE}}(p, p')$ has a simple Lorentz ~ 1 organization as well, and is equal to the amplitude $\Pi^{\text{OPE}}(p^2, p'^2, q^2)$. In the present work, this amplitude is calculated by taking into account Dim4 terms $\sim \langle \alpha_s G^2/\pi \rangle$.

Having equated $\Pi^{\text{Phys}}(p^2, p'^2, q^2)$ and $\Pi^{\text{OPE}}(p^2, p'^2, q^2)$, performed the double Borel transformations over the variables $-p^2$, $-p'^2$ and under the quark-hadron duality assumption subtracted contributions of excited and continuum states from the QCD side of this equality, we derive the sum rule for $g(q^2)$

$$\begin{aligned} g(q^2) &= \frac{4m_b(m_b + m_c)(q^2 - m_{B_c}^2)}{\Lambda f_{\eta_b} m_{\eta_b}^2 f_{B_c} m_{B_c}^2 (m^2 + m_{\eta_b}^2 - q^2)} \\ &\times e^{m^2/M_1^2} e^{m_{\eta_b}^2/M_2^2} \Pi(\mathbf{M}^2, \mathbf{s}_0, q^2). \end{aligned} \quad (23)$$

Here $\Pi(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is given by the expression

$$\begin{aligned} \Pi(\mathbf{M}^2, \mathbf{s}_0, q^2) &= \int_{(3m_b+m_c)^2}^{s_0} \int_{4m_b^2}^{s_0'} ds ds' e^{-s/M_1^2} \\ &\times e^{-s'/M_2^2} \rho(s, s', q^2), \end{aligned} \quad (24)$$

Mesons	mass (MeV)	DC (MeV)
η_b	9398.7 ± 2.0	724
$B_c^\pm$	$6274.47 \pm 0.27 \pm 0.17$	371 ± 37
$B_c^{*\pm}$	6338	471
η_c	2984.1 ± 0.4	421 ± 35
J/ψ	3096.900 ± 0.006	411 ± 7
$\overline{D}^0$	1864.84 ± 0.05	211.9 ± 1.1
$D^\pm$	1869.66 ± 0.05	211.9 ± 1.1
$D_s^\pm$	1968.35 ± 0.07	249.9 ± 0.5
$\overline{D}^{*0}$	2006.85 ± 0.05	252.2 ± 22.66
$D^{*\pm}$	2010.26 ± 0.05	252.2 ± 22.66
$D_s^{*\pm}$	2112.2 ± 0.4	268.8 ± 6.5
$\overline{B}^0$	5279.72 ± 0.08	206
$B^\pm$	5279.41 ± 0.07	206
$\overline{B}_s^0$	5366.93 ± 0.10	234
$\overline{B}^{*0}, B^{*\pm}$	5324.75 ± 0.20	210 ± 6
$\overline{B}_s^{*0}$	5415.4 ± 1.4	221

TABLE I: Masses and decay constants (DC) of the mesons that appear in decays of the hadronic molecules $\mathcal{M}_b$ and $\mathcal{M}_c$.

where the spectral density $\rho(s, s', q^2)$ amounts to the imaginary part of $\Pi^{\text{OPE}}(s, s', q^2)$.

The correlator $\Pi(\mathbf{M}^2, \mathbf{s}_0, q^2)$ depends on the parameters $\mathbf{M}^2 = (M_1^2, M_2^2)$ and $\mathbf{s}_0 = (s_0, s'_0)$ where the pairs (M_1^2, s_0) and (M_2^2, s'_0) are related to $\mathcal{M}_b$ and η_b channels. Restrictions imposed on $\mathbf{M}^2$ and $\mathbf{s}_0$ are standard in SR calculations and have been detailed above (see, Sec. II). Our analysis demonstrates that Eq. (12) for the parameters (M_1^2, s_0) and

$$M_2^2 \in [9, 11] \text{ GeV}^2, \quad s'_0 \in [95, 99] \text{ GeV}^2. \quad (25)$$

for (M_2^2, s'_0) meet these requirements. The mass and decay constant of the mesons η_b and B_c^- necessary for numerical computations, as well as parameters of particles that emerge while studying other decays are collected in Table I. The masses of the mesons are borrowed from Ref. [45]. The parameters of the B_c^* meson are model-dependent predictions from Refs. [46, 47]. Other decay constants were extracted from experimental measurements or computed using various theoretical methods (see, for instance, Refs. [48–50]).

The SR method leads to credible results in the Euclidean region $q^2 < 0$. At the same time, $g(q^2)$ becomes equal to g at the mass shell $q^2 = m_{B_c}^2$. For this reason, we use the function $g(Q^2)$ with $Q^2 = -q^2$ and utilize it in following analysis. The SR predictions for $g(Q^2)$ are shown in Fig. 4, where Q^2 changes within borders $Q^2 = 2 - 30 \text{ GeV}^2$.

To extract g at the mass shell $q^2 = -Q^2 = m_{B_c}^2$, we employ the extrapolating function $\mathcal{G}(Q^2, m^2)$ which at $Q^2 > 0$ coincides with SR data, but can also be applied in the domain $Q^2 < 0$. This function has the analytical

form

$$\mathcal{G}_i(Q^2, m^2) = \mathcal{G}_i^0 \exp \left[c_i^1 \frac{Q^2}{m^2} + c_i^2 \left(\frac{Q^2}{m^2} \right)^2 \right], \quad (26)$$

where $\mathcal{G}_i^0$, c_i^1 , and c_i^2 are constants obtained from comparison with SR data. Then, it is not difficult to find

$$\mathcal{G}^0 = 0.81 \text{ GeV}^{-1}, c^1 = 10.99, \text{ and } c^2 = -3.46. \quad (27)$$

In Fig. 4 we plot $\mathcal{G}(Q^2, m^2)$ as well: Nice agreement of $\mathcal{G}(Q^2, m^2)$ and SR data is evident. Then, for g we obtain

$$g \equiv \mathcal{G}(-m_{B_c}^2, m^2) = (1.3 \pm 0.2) \times 10^{-1} \text{ GeV}^{-1}. \quad (28)$$

The width of the decay $\mathcal{M}_b \rightarrow \eta_b B_c^-$ is given by the formula

$$\Gamma[\mathcal{M}_b \rightarrow \eta_b B_c^-] = g^2 \frac{m_{\eta_b}^2 \lambda}{8\pi} \left(1 + \frac{\lambda^2}{m_{\eta_b}^2} \right), \quad (29)$$

where $\lambda = \lambda(m, m_{\eta_b}, m_{B_c})$ and

$$\lambda(a, b, c) = \frac{\sqrt{a^4 + b^4 + c^4 - 2(a^2 b^2 + a^2 c^2 + b^2 c^2)}}{2a}. \quad (30)$$

Then, we obtain

$$\Gamma[\mathcal{M}_b \rightarrow \eta_b B_c^-] = (37.8 \pm 15.4) \text{ MeV}. \quad (31)$$

The error above is generated by the ambiguities of the coupling g and the masses of the particles $\mathcal{M}_b$ (upper limit), m_{η_b} and m_{B_c} .

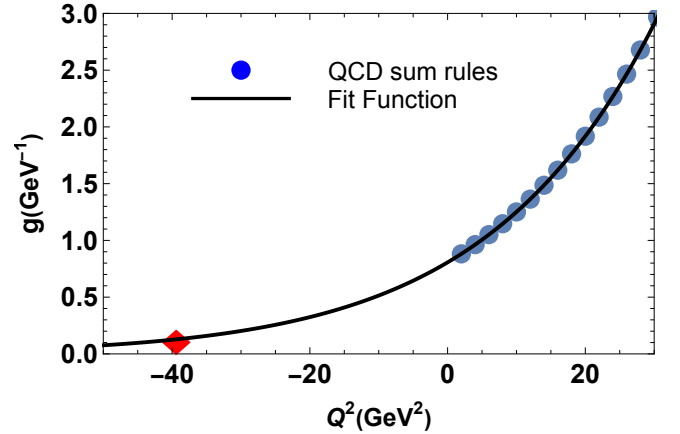


FIG. 4: The sum rule's data and extrapolating function $\mathcal{G}(Q^2, m^2)$. The diamond is placed at $Q^2 = -m_{B_c}^2$.

B. Decays of $\mathcal{M}_b$ triggered by $\bar{b}b$ annihilation

As it has been explained, annihilation of $\bar{b}b$ quarks gives rise to numerous decay channels of the molecule $\mathcal{M}_b$. The processes $\mathcal{M}_b \rightarrow B^- \overline{D}^0, \overline{B}^0 D^-, \overline{B}_s^0 D_s^-$,

and vector $B^{*-}\bar{D}^{*0}$, $\bar{B}^{*0}D^{*-}$, $\bar{B}_s^{*0}D_s^{*-}$ are among these modes. Let us first consider the decays to pairs of pseudoscalar mesons. In our present studies we adopt the approximations $m_u = m_d = 0$ and $m_s = (93.5 \pm 0.8)$ MeV. The correlation functions for the decays $\mathcal{M}_b \rightarrow B^-\bar{D}^0$ and $\mathcal{M}_b \rightarrow \bar{B}^0D^-$ contain u and d quark propagators which are the same in this approximation. The partial widths of these processes may differ from each other due to parameters of the particles involved into decays. We use the same decay constants for the neutral and charged mesons, therefore their masses are only possible sources of potential variations. From Table I it is seen that differences between the masses of the mesons B^- and $\bar{B}^0$, as well as $\bar{D}^0$ and D^- ones are very small. For this reason, we calculate the partial width $\Gamma[\mathcal{M}_b \rightarrow B^-\bar{D}^0]$ of the decay $\mathcal{M}_b \rightarrow B^-\bar{D}^0$, and employ an approximate relation $\Gamma[\mathcal{M}_b \rightarrow \bar{B}^0D^-] \approx \Gamma[\mathcal{M}_b \rightarrow B^-\bar{D}^0]$. The similar arguments are valid in the case of the decays to vector mesons as well.

Let us consider the decay $\mathcal{M}_b \rightarrow B^-\bar{D}^0$ of the molecule $\mathcal{M}_b$ in a detailed form. Our aim is to extract the strong coupling g at the vertex $\mathcal{M}_b B^-\bar{D}^0$. To this end, we investigate the three-point correlator

$$\Pi_1(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J^{B^-}(y) \times J^{\bar{D}^0}(0) J^\dagger(x) \} | 0 \rangle, \quad (32)$$

where $J^{B^-}(x)$ and $J^{\bar{D}^0}(x)$ are currents for the mesons B^- and $\bar{D}^0$. They have the following forms

$$J^{B^-}(x) = \bar{u}_i(x) i\gamma_5 b_i(x), \quad J^{\bar{D}^0}(x) = \bar{c}_j(x) i\gamma_5 u_j(x). \quad (33)$$

The matrix elements of these mesons employed to calculate the physical side of the sum rule for the relevant form factor $g_1(q^2)$ are

$$\begin{aligned} \langle 0 | J^{B^-} | B^-(p') \rangle &= \frac{f_B m_B^2}{m_b}, \\ \langle 0 | J^{\bar{D}^0} | \bar{D}^0(q) \rangle &= \frac{f_D m_{\bar{D}^0}^2}{m_c}. \end{aligned} \quad (34)$$

In formulas above m_B , $m_{\bar{D}^0}$ and f_B , f_D are the masses and decay constants of these particles. The vertex $\langle B^-(p') \bar{D}^0(q) | \mathcal{M}_b(p) \rangle$ and correlator $\Pi_1^{\text{Phys}}(p, p')$ are similar to those obtained in the previous subsection.

The QCD side of SR for the form factor $g_1(q^2)$ is given by the expression

$$\begin{aligned} \Pi_1^{\text{OPE}}(p, p') &= \frac{\langle \bar{b}b \rangle}{3} \int d^4x d^4y e^{ip'y} e^{-ipx} \text{Tr} [\gamma_5 S_b^{ia}(y-x) \\ &\times S_c^{aj}(x) \gamma_5 S_u^{ji}(-y)]. \end{aligned} \quad (35)$$

To continue calculations, we utilize the relation

$$\langle \bar{b}b \rangle = -\frac{1}{12m_b} \langle \frac{\alpha_s G^2}{\pi} \rangle \quad (36)$$

extracted in Ref. [39] using the sum rule method.

The form factor $g_1(Q^2)$ is computed in the region $Q^2 = 2 - 20$ GeV². In numerical calculations for parameters (M_1^2, s_0) we have used Eq. (12), whereas (M_2^2, s'_0) have been chosen in the following intervals

$$M_2^2 \in [5.5, 6.5] \text{ GeV}^2, \quad s'_0 \in [33.5, 34.5] \text{ GeV}^2. \quad (37)$$

Predictions obtained for $g_1(Q^2)$ are displayed in Fig. 5. The extrapolating function $\mathcal{G}_1(Q^2, m^2)$ is fixed by the constants $\mathcal{G}_1^0 = 0.026$ GeV⁻¹, $c_1^1 = 4.88$, and $c_1^2 = -6.70$. Then the coupling g_1 can be extracted at the point $Q^2 = -m_{\bar{D}^0}^2$ and is equal to

$$g_1 \equiv \mathcal{G}_1(-m_{\bar{D}^0}^2, m^2) = (2.42 \pm 0.39) \times 10^{-2} \text{ GeV}^{-1}. \quad (38)$$

This leads to the following results for width of the decay $\mathcal{M}_b \rightarrow B^-\bar{D}^0$

$$\Gamma[\mathcal{M}_b \rightarrow B^-\bar{D}^0] = (11.9 \pm 2.8) \text{ MeV}. \quad (39)$$

Note that uncertainties in the width is total errors connected by uncertainties both in g_1 and the masses $\mathcal{M}_b$, m_B and $m_{\bar{D}^0}$. The decay $\mathcal{M}_b \rightarrow \bar{B}_s^0 D_s^-$ is investigated

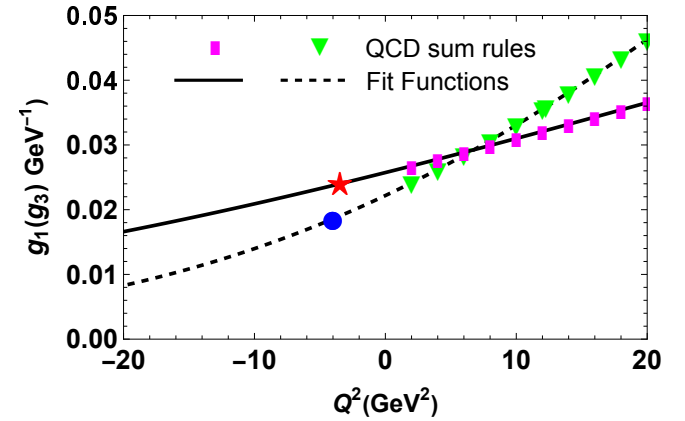


FIG. 5: The QCD data for the form factors $g_1(Q^2)$ and $g_3(Q^2)$ and fit functions $\mathcal{G}_1(Q^2, m^2)$ (solid line), $\mathcal{G}_3(Q^2, m^2)$ (dashed line). The red star and blue circle show positions of the points $Q^2 = -m_{\bar{D}^0}^2$ and $Q^2 = -m_{D_s^*}^2$, respectively.

by the same manner. Our results for the strong coupling g_2 and partial width of this process read:

$$g_2 \equiv \mathcal{G}_2(-m_{D_s^*}^2, m^2) = (1.84 \pm 0.32) \text{ GeV}^{-1}, \quad (40)$$

and

$$\Gamma[\mathcal{M}_b \rightarrow \bar{B}_s^0 D_s^-] = (6.8 \pm 1.8) \text{ MeV}. \quad (41)$$

It is worth noting that the coupling g_2 has been found using the fit function $\mathcal{G}_2(Q^2, m^2)$ with parameters $\mathcal{G}_2^0 = 0.02$ GeV⁻¹, $c_2^1 = 4.74$, and $c_2^2 = -6.42$.

The next channels of the hadronic molecule $\mathcal{M}_b$ are decays to the vector mesons' pairs $B^{*-}\bar{D}^{*0}$, $\bar{B}^{*0}D^{*-}$,

$\overline{B}_s^{*0} D_s^{*-}$. As a sample, we analyze the mode $\mathcal{M}_b \rightarrow B^{*-} \overline{D}^{*0}$ and write down formulas for this decay. The correlator to be analyzed in this case is

$$\Pi_{\mu\nu}(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J_\mu^{B^*}(y) \times J_\nu^{\overline{D}^{*0}}(0) J^\dagger(x) \} | 0 \rangle. \quad (42)$$

Here, $J_\mu^{B^*}(x)$ and $J_\nu^{\overline{D}^{*0}}(x)$ are currents which interpolate the vector particles B^{*-} and $\overline{D}^{*0}$

$$J_\mu^{B^*}(x) = \overline{u}_i(x) \gamma_\mu b_i(x), \quad J_\nu^{\overline{D}^{*0}}(x) = \overline{c}_j(x) \gamma_\nu u_j(x). \quad (43)$$

To derive the physical side of the SR for the form factor $g_3(q^2)$ describing the strong interactions of particles at the vertex $\mathcal{M}_b B^{*-} \overline{D}^{*0}$ we use the expression

$$\begin{aligned} \Pi_{\mu\nu}^{\text{Phys}}(p, p') &= \frac{\langle 0 | J_\mu^{B^*} | B^{*-}(p', \varepsilon_1) \rangle \langle 0 | J_\nu^{\overline{D}^{*0}} | \overline{D}^{*0}(q, \varepsilon_2) \rangle}{p'^2 - m_{B^*}^2} \frac{\langle \mathcal{M}_b(p) | J^\dagger | 0 \rangle}{q^2 - m_{D^*}^2} \\ &\times \langle B^{*-}(p', \varepsilon_1) \overline{D}^{*0}(q, \varepsilon_2) | \mathcal{M}_b(p) \rangle \frac{\langle \mathcal{M}_b(p) | J^\dagger | 0 \rangle}{p^2 - m^2} \\ &+ \dots \end{aligned} \quad (44)$$

In Eq. (44) m_{B^*} and m_{D^*} are the masses of the final-state mesons, whereas ε_1 and ε_2 are their polarization vectors.

The correlation function $\Pi_{\mu\nu}^{\text{Phys}}(p, p')$ can be rewritten in the following form

$$\begin{aligned} \Pi_{\mu\nu}^{\text{Phys}}(p, p') &= g_3(q^2) \frac{\Lambda f_{B^*} m_{B^*} f_{D^*} m_{D^*}}{(p^2 - m^2)(p'^2 - m_{B^*}^2)} \\ &\times \frac{1}{q^2 - m_{D^*}^2} \left[\frac{m^2 - m_{B^*}^2 - q^2}{2} g_{\mu\nu} - p'_\nu q_\mu \right] \\ &+ \dots \end{aligned} \quad (45)$$

This expression has been obtained by applying the matrix elements

$$\begin{aligned} \langle 0 | J_\mu^{B^*} | B^{*-}(p', \varepsilon_1) \rangle &= f_{B^*} m_{B^*} \varepsilon_{1\mu}, \\ \langle 0 | J_\nu^{\overline{D}^{*0}} | \overline{D}^{*0}(q, \varepsilon_2) \rangle &= f_{D^*} m_{D^*} \varepsilon_{2\nu}, \\ \langle B^{*-}(p', \varepsilon_1) \overline{D}^{*0}(q, \varepsilon_2) | \mathcal{M}_b(p) \rangle &= g_3(q^2) \\ &\times [q \cdot p' \varepsilon_1^* \cdot \varepsilon_2^* - q \cdot \varepsilon_1^* p' \cdot \varepsilon_2^*]. \end{aligned} \quad (46)$$

The QCD side of the SR is equal to

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p, p') &= \frac{\langle \overline{b}b \rangle}{3} \int d^4x d^4y e^{ip'y} e^{-ipx} \text{Tr} [\gamma_\mu S_b^{ia}(y-x) \\ &\times S_c^{aj}(x) \gamma_\nu S_u^{ji}(-y)]. \end{aligned} \quad (47)$$

To find SR for the form factor $g_3(q^2)$ we utilize amplitudes which correspond to terms $\sim g_{\mu\nu}$ both in $\Pi_{\mu\nu}^{\text{Phys}}(p, p')$ and $\Pi_{\mu\nu}^{\text{OPE}}(p, p')$. As a result, we get

$$\begin{aligned} g_3(q^2) &= \frac{2(q^2 - m_{D^*}^2)}{\Lambda f_{B^*} m_{B^*} f_{D^*} m_{D^*} (m^2 - m_{B^*}^2 - q^2)} \\ &\times e^{m^2/M_1^2} e^{m_{B^*}^2/M_2^2} \Pi_3(\mathbf{M}^2, \mathbf{s}_0, q^2), \end{aligned} \quad (48)$$

where $\Pi_3(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is transformed amplitude $\Pi_3^{\text{OPE}}(s, s', q^2)$ from $\Pi_{\mu\nu}^{\text{OPE}}(p, p')$.

Numerical analysis is carried out by employing parameters of the particles $\mathcal{M}_b$, B^{*-} , and $\overline{D}^{*0}$ and

$$M_2^2 \in [5.5, 6.5] \text{ GeV}^2, \quad s'_0 \in [34, 35] \text{ GeV}^2. \quad (49)$$

The parameters of the extrapolating function are $\mathcal{G}_3^0 = 0.022 \text{ GeV}^{-1}$, $c_3^1 = 10.65$, and $c_3^2 = -19.06$. The coupling g_3 amounts to

$$g_3 = (1.86 \pm 0.35) \times 10^{-2} \text{ GeV}^{-1}. \quad (50)$$

Results obtained for $g_3(Q^2)$ and fit function $\mathcal{G}_3(Q^2, m^2)$ are shown in Fig. 5.

We calculate the width of this decay by means of the formula

$$\Gamma[\mathcal{M}_b \rightarrow B^{*-} \overline{D}^{*0}] = g_3^2 \frac{\lambda_3}{4\pi} \left(\lambda_3^2 + \frac{3m_{B^*}^2 m_{D^*}^2}{2m^2} \right), \quad (51)$$

where $\lambda_3 = \lambda(m, m_{B^*}, m_{D^*})$. This expression leads to the prediction

$$\Gamma[\mathcal{M}_b \rightarrow B^{*-} \overline{D}^{*0}] = (8.8 \pm 2.4) \text{ MeV}. \quad (52)$$

The widths of the decays $\mathcal{M}_b \rightarrow \overline{B}^{*0} D^{*-}$ and $\mathcal{M}_b \rightarrow B^{*-} \overline{D}^{*0}$ are equal to each other provided one neglects differences in masses of the involved conventional mesons. Therefore, we employ

$$\Gamma[\mathcal{M}_b \rightarrow \overline{B}^{*0} D^{*-}] \approx \Gamma[\mathcal{M}_b \rightarrow B^{*-} \overline{D}^{*0}]. \quad (53)$$

The process $\mathcal{M}_b \rightarrow \overline{B}_s^{*0} D_s^{*-}$ is studied by the similar manner. The coupling g_4 is equal to

$$g_4 = (1.72 \pm 0.31) \times 10^{-2} \text{ GeV}^{-1}, \quad (54)$$

extracted the parameters

$$M_2^2 \in [6, 7] \text{ GeV}^2, \quad s'_0 \in [35, 36] \text{ GeV}^2. \quad (55)$$

For the partial width of this mode, we find

$$\Gamma[\mathcal{M}_b \rightarrow \overline{B}_s^{*0} D_s^{*-}] = (7.3 \pm 1.9) \text{ MeV}. \quad (56)$$

By taking into account all these decay channels, and results for their partial widths it is not difficult to estimate the full decay width of the hadronic molecule:

$$\Gamma[\mathcal{M}_b] = (93 \pm 17) \text{ MeV}. \quad (57)$$

IV. WIDTH OF THE MOLECULE $\mathcal{M}_c$

Here, we evaluate the width of the molecule $\mathcal{M}_c$ by studying the decays $\mathcal{M}_c \rightarrow \eta_c B_c^+$ and $\mathcal{M}_c \rightarrow J/\psi B_c^{*+}$ which are dominant modes $\mathcal{M}_c$. It is clear that both these modes are permitted channels for $\mathcal{M}_c$. Indeed, the

mass $\tilde{m} = 9712$ MeV of $\mathcal{M}_c$ exceeds thresholds for these processes which amount to 9259 MeV and 9435 MeV.

Investigation of the decay $\mathcal{M}_c \rightarrow \eta_c B_c^+$ does not differ considerably from analysis performed in the previous section. Here, we should calculate the form factor $\tilde{g}_1(q^2)$ and find the strong coupling $\tilde{g}_1$ at the vertex $\mathcal{M}_c \eta_c B_c^+$. We start to consider the correlator

$$\begin{aligned} \tilde{\Pi}(p, p') &= i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J^{B_c^+}(y) \\ &\quad \times J^{\eta_c}(0) \tilde{J}^\dagger(x) \} | 0 \rangle, \end{aligned} \quad (58)$$

with $J^{B_c^+}(x)$ and $J^{\eta_c}(x)$ being the interpolating currents for the mesons B_c^+ and η_c , respectively

$$J^{B_c^+}(x) = \bar{b}_i(x) i\gamma_5 c_i(x), \quad J^{\eta_c}(x) = \bar{c}_j(x) i\gamma_5 c_j(x). \quad (59)$$

We compute the physical side of SR using the following matrix elements

$$\begin{aligned} \langle 0 | J^{\eta_c} | \eta_c(q) \rangle &= \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c}, \\ \langle 0 | J^{B_c^+} | B_c^+(p') \rangle &= \frac{f_{B_c} m_{B_c}^2}{m_b + m_c}, \end{aligned} \quad (60)$$

and

$$\langle \eta_c(q) B_c^+(p') | \mathcal{M}_c(p) \rangle = \tilde{g}_1(q^2) p \cdot p'. \quad (61)$$

In the formulas above, the mass and decay constant of the pseudoscalar meson η_c are denoted as m_{η_c} and f_{η_c} , respectively.

The phenomenological and QCD components of this SR have analytical forms presented in Sec. III with evident replacements. As a result, the SR for $\tilde{g}_1(q^2)$ reads

$$\begin{aligned} \tilde{g}_1(q^2) &= \frac{4m_c(m_b + m_c)(q^2 - m_{\eta_c}^2)}{\Lambda f_{\eta_c} m_{\eta_c}^2 f_{B_c} m_{B_c}^2 (m^2 + m_{B_c}^2 - q^2)} \\ &\quad \times e^{m^2/M_1^2} e^{m_{B_c}^2/M_2^2} \tilde{\Pi}_1(\mathbf{M}^2, \mathbf{s}_0, q^2). \end{aligned} \quad (62)$$

The correlation function $\tilde{\Pi}_1(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is determined by the expression

$$\begin{aligned} \tilde{\Pi}(\mathbf{M}^2, \mathbf{s}_0, q^2) &= \int_{(m_b+3m_c)^2}^{s_0} \int_{(m_b+m_c)^2}^{s'_0} ds ds' e^{-s/M_1^2} \\ &\quad \times e^{-s'/M_2^2} \tilde{\rho}(s, s', q^2). \end{aligned} \quad (63)$$

In calculations the parameters (M_1^2, s_0) in the channel of the molecule $\mathcal{M}_c$ are chosen as in Eq. (14). The intervals for (M_2^2, s'_0) in the B_c^+ channel are chosen as

$$M_2^2 \in [6.5, 7.5] \text{ GeV}^2, \quad s'_0 \in [45, 47] \text{ GeV}^2. \quad (64)$$

The function $\tilde{g}_1(Q^2)$ is calculated at $Q^2 = 2 - 20 \text{ MeV}^2$. The extrapolating function $\tilde{\mathcal{G}}_1(Q^2, \tilde{m}^2)$ has the form Eq. (26) with m^2 substituted by $\tilde{m}^2$. The function $\tilde{\mathcal{G}}_1$ has the parameters $\tilde{\mathcal{G}}_1^0 = 0.132 \text{ GeV}^{-1}$, $\tilde{c}_1^1 = 3.148$, and $\tilde{c}_1^2 = -2.152$.

The coupling $\tilde{g}_1$ extracted at the mass shell $q^2 = m_{\eta_c}^2$ amounts to

$$\tilde{g}_1 \equiv \tilde{\mathcal{G}}_1(-m_{\eta_c}^2, \tilde{m}^2) = (9.63 \pm 1.86) \times 10^{-2} \text{ GeV}^{-1}. \quad (65)$$

We evaluate the partial width of this channel by employing the expression

$$\Gamma[\mathcal{M}_c \rightarrow \eta_c B_c^+] = \tilde{g}_1^2 \frac{m_{B_c}^2 \tilde{\lambda}_1}{8\pi} \left(1 + \frac{\tilde{\lambda}_1^2}{m_{B_c}^2} \right), \quad (66)$$

where $\tilde{\lambda}_1$ is $\lambda(\tilde{m}, m_{B_c}, m_{\eta_c})$. Our prediction is

$$\Gamma[\mathcal{M}_c \rightarrow \eta_c B_c^+] = (21.0 \pm 6.0) \text{ MeV}. \quad (67)$$

The second dominant channel of the molecule $\mathcal{M}_c$ is the decay to particles J/ψ and B_c^{*+} . To find the coupling $\tilde{g}_2$ at the vertex $\mathcal{M}_c J/\psi B_c^{*+}$, one should compute the relevant form factor $\tilde{g}_2(q^2)$, which can be obtained from the sum rule for this function. To this end, we consider the correlator

$$\begin{aligned} \tilde{\Pi}_{\mu\nu}(p, p') &= i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J_\mu^{B_c^*}(y) \\ &\quad \times J_\nu^{J/\psi}(0) \tilde{J}^\dagger(x) \} | 0 \rangle, \end{aligned} \quad (68)$$

with $J_\mu^{B_c^*}(x)$ and $J_\nu^{J/\psi}(x)$ being the currents that interpolate vector mesons B_c^{*+} and J/ψ , respectively

$$J_\mu^{B_c^*}(x) = \bar{b}_i(x) \gamma_\mu c_i(x), \quad J_\nu^{J/\psi}(x) = \bar{c}_j(x) \gamma_\nu c_j(x). \quad (69)$$

The phenomenological side of SR $\Pi_{\mu\nu}^{\text{Phys}}(p, p')$ is given by the standard expression

$$\begin{aligned} \Pi_{\mu\nu}^{\text{Phys}}(p, p') &= \frac{\langle 0 | J_\mu^{B_c^*} | B_c^{*+}(p', \epsilon_1) \rangle \langle 0 | J_\nu^{J/\psi} | J/\psi(q, \epsilon_2) \rangle}{p'^2 - m_{B_c^*}^2} \frac{q^2 - m_{J/\psi}^2}{q^2 - m_{J/\psi}^2} \\ &\quad \times \langle B_c^{*+}(p', \epsilon_1) J/\psi(q, \epsilon_2) | \mathcal{M}_c(p) \rangle \frac{\langle \mathcal{M}_c(p) | \tilde{J}^\dagger | 0 \rangle}{p^2 - \tilde{m}^2} + \dots \end{aligned} \quad (70)$$

Here, $m_{J/\psi}$ and $m_{B_c^*}$ are the masses of the mesons, and ϵ_1, ϵ_2 the polarization vectors of these particles.

The correlator $\Pi_{\mu\nu}^{\text{Phys}}$ can be rewritten by using the matrix elements

$$\begin{aligned} \langle 0 | J_\mu^{B_c^*} | B_c^{*+}(p') \rangle &= f_{B_c^*} m_{B_c^*} \epsilon_{1\mu}, \\ \langle 0 | J_\nu^{J/\psi} | J/\psi(q) \rangle &= f_{J/\psi} m_{J/\psi} \epsilon_{2\nu}, \end{aligned} \quad (71)$$

and

$$\begin{aligned} \langle B_c^{*+}(p', \epsilon_1) J/\psi(q, \epsilon_2) | \mathcal{M}_c(p) \rangle &= \tilde{g}_2(q^2) \\ &\quad \times [q \cdot p' \epsilon_1^* \cdot \epsilon_2^* - q \cdot \epsilon_1^* p' \cdot \epsilon_2^*]. \end{aligned} \quad (72)$$

In Eq. (72) $f_{J/\psi}$ and $f_{B_c^*}$ are the decay constants of J/ψ and B_c^{*+} , respectively.

Then, for $\tilde{\Pi}_{\mu\nu}^{\text{Phys}}(p, p')$ we get

$$\tilde{\Pi}_{\mu\nu}^{\text{Phys}}(p, p') = \frac{\tilde{g}_2(q^2)\tilde{\Lambda}f_{B_c^*}m_{B_c^*}f_{J/\psi}m_{J/\psi}}{(p^2 - \tilde{m}^2)(p'^2 - m_{B_c^*}^2)(q^2 - m_{J/\psi}^2)} \times \left[\frac{(m^2 - m_{B_c^*}^2 - q^2)}{2}g_{\mu\nu} - q_\mu p'_\nu + \dots \right]. \quad (73)$$

The correlation function $\tilde{\Pi}_{\mu\nu}(p, p')$ expressed using quark propagators becomes equal to

$$\tilde{\Pi}_{\mu\nu}^{\text{OPE}}(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \text{Tr} [\gamma_\mu S_c^{ja}(-x) \times \gamma_5 S_c^{ai}(x-y) \gamma_\nu S_c^{ib}(y-x) \gamma_5 S_b^{bj}(x)]. \quad (74)$$

To derive SR for the form factor $\tilde{g}_2(q^2)$, we utilize the invariant amplitudes which correspond to terms proportional to $g_{\mu\nu}$ in Eqs. (73) and (74). Then, we find for $\tilde{g}_2(q^2)$

$$\tilde{g}_2(q^2) = \frac{2(q^2 - m_{J/\psi}^2)}{\tilde{\Lambda}f_{B_c^*}m_{B_c^*}f_{J/\psi}m_{J/\psi}(m^2 - m_{B_c^*}^2 - q^2)} \times e^{m^2/M_1^2} e^{m_{B_c^*}^2/M_2^2} \tilde{\Pi}_2(\mathbf{M}^2, \mathbf{s}_0, q^2). \quad (75)$$

Operations to find the coupling $\tilde{g}_2$ have been explained above so we give final results without details. Note that the function $\tilde{g}_2(Q^2)$ is calculated for $Q^2 = 2 - 30 \text{ MeV}^2$. In the $\mathcal{M}_c$ channel parameters (M_1^2, s_0) are chosen as in Eq. (14). In the B_c^{*+} channel, we have varied (M_2^2, s'_0) inside windows

$$M_2^2 \in [6.5, 7.5] \text{ GeV}^2, \quad s'_0 \in [50, 51] \text{ GeV}^2. \quad (76)$$

The function $\tilde{g}_2(Q^2, \tilde{m}^2)$ is fixed by constants: $\tilde{g}_2^0 = 0.40 \text{ GeV}^{-1}$, $\tilde{c}_2^1 = 6.88$, and $\tilde{c}_2^2 = -5.65$. Then, the coupling $\tilde{g}_2$ is equal to

$$\tilde{g}_2 \equiv \tilde{g}_2(-m_{J/\psi}^2, \tilde{m}^2) = (1.9 \pm 0.4) \times 10^{-1} \text{ GeV}^{-1}. \quad (77)$$

The width of the decay $\mathcal{M}_c \rightarrow J/\psi B_c^{*+}$ is obtained using the formula

$$\Gamma[\mathcal{M}_c \rightarrow J/\psi B_c^{*+}] = g_2^2 \frac{\lambda_2}{4\pi} \left(\lambda_2^2 + \frac{3m_{B_c^*}^2 m_{J/\psi}^2}{2\tilde{m}^2} \right), \quad (78)$$

where λ_2 is $\lambda(\tilde{m}, m_{B_c^*}, m_{J/\psi})$. We find

$$\Gamma[\mathcal{M}_c \rightarrow J/\psi B_c^{*+}] = (22.1 \pm 7.5) \text{ MeV}. \quad (79)$$

We have explored also six decay channels $\mathcal{M}_c \rightarrow B^+ D^0$, $B^0 D^+$, $B_s^0 D_s^+$, $B^{*+} D^{*0}$, $B^{*0} D^{*+}$, and $B_s^{*0} D_s^{*+}$ triggered by annihilation of $c\bar{c}$ quarks. We have benefited from the facts $\Gamma[\mathcal{M}_c \rightarrow B^+ D^0] \approx \Gamma[\mathcal{M}_c \rightarrow B^0 D^+]$ and $\Gamma[\mathcal{M}_c \rightarrow B^{*+} D^{*0}] \approx \Gamma[\mathcal{M}_c \rightarrow B^{*0} D^{*+}]$. Final information on remaining four channels are presented in Table II.

The full width of the molecule $\mathcal{M}_c$ saturated by these decay channels is

$$\Gamma[\mathcal{M}_c] = (70 \pm 10) \text{ MeV}. \quad (80)$$

i	Channels	$\tilde{g}_i \text{ (GeV}^{-1}) \times 10^2$	$\Gamma_i \text{ (MeV)}$
1	$B^+ D^0$	3.2 ± 0.6	4.8 ± 1.3
2	$B_s^0 D_s^+$	2.9 ± 0.5	3.7 ± 0.9
3	$B^{*+} D^{*0}$	4.3 ± 0.7	4.8 ± 1.2
4	$B_s^{*0} D_s^{*+}$	4.1 ± 0.6	4.0 ± 0.9

TABLE II: Decay channels of the molecule $\mathcal{M}_c$ due to $c\bar{c}$ annihilation, corresponding strong couplings $\tilde{g}_i$ and widths Γ_i .

V. CONCLUSIONS

Investigations carried out in the present work is a new step towards understanding of the internal structure and properties of the potential all heavy four-quark mesons. We have considered the scalar structures $b\bar{b}c\bar{c}$ and $c\bar{c}b\bar{b}$ organized as hadronic molecules $\mathcal{M}_b = \eta_b B_c^-$ and $\mathcal{M}_c = \eta_c B_c^+$. We have calculated their masses and evaluated decay widths by analyzed the dominant and some of subleading decay channels.

The masses and current couplings of these molecules have been calculated by means of QCD two-point sum rule method. Predictions $m = (15728 \pm 90) \text{ MeV}$ and $\tilde{m} = (9712 \pm 72) \text{ MeV}$ obtained for the masses of $\mathcal{M}_b$ and $\mathcal{M}_c$ have allowed us to determine their possible decay channels. In our studies we have distinguished the dominant and subleading decay mechanisms of these particles. The dominant mechanism is one in which all constituent quarks participate in producing of ordinary final-state mesons. For molecule $\mathcal{M}_b$ breakdown to η_b and B_c^- mesons is the dominant process. The dominant channels of $\mathcal{M}_c$ are the processes $\mathcal{M}_c \rightarrow \eta_c B_c^+$ and $\mathcal{M}_c \rightarrow J/\psi B_c^{*+}$. In the last decay $\mathcal{M}_c$ falls to vector partners of the constituent mesons.

Another mechanism of decays is generated by annihilation of constituent $b\bar{b}$ or $c\bar{c}$ quarks inside of the molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ and producing $B_{(s)}^{(*)} D_{(s)}^{(*)}$ pairs with appropriate charges and spin-parities. This mechanism has been included into the SR framework after replacing in the correlation functions the vacuum expectation values $m_b \langle \bar{b}b \rangle$ and $m_c \langle \bar{c}c \rangle$ by a term $\sim \langle \alpha_s G^2 / \pi \rangle$. It is worth emphasizing that relations between the heavy quark and gluon condensates were extracted within the SR method and are approximate expressions.

All decay channels considered in this work have been explored using the three-point SR approach. This approach have permitted us to estimate the strong couplings g_i and $\tilde{g}_i$ at the vertices $\mathcal{M}_b M_1 M_2$ and $\mathcal{M}_c M_1 M_2$, where M_1 and M_2 are the final-state mesons. Our predictions $\Gamma[\mathcal{M}_b] = (93 \pm 17) \text{ MeV}$ and $\Gamma[\mathcal{M}_c] = (70 \pm 10) \text{ MeV}$ for the widths of the molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ mean that they may be interpreted as relatively broad structures. Note that numerous subleading processes form sizeable parts of these parameters.

As it has been emphasized in Sec. I that the exotic scalar mesons T_b and T_c with the same contents but

diquark-antidiquark structures were explored in our work [38]. It is interesting to compare parameters of these states with ones obtained in the present article. It is easy to see, that the molecules are heavier than their diquark-antidiquark counterparts. But relevant mass gaps are small and within errors of calculations one may state that the molecule and diquark-antidiquark exotic mesons have approximately the similar masses. The molecules are relatively broad structures than diquark-antidiquark T_b and T_c states. But here one should take into account that widths of T_b and T_c tetraquarks were estimated by analyzing only their dominant decay channels.

The hadronic molecules composed of four b and c quarks in various combinations were studied in Ref. [28]. There, authors used the local gauge formalism to investigate the meson-meson interactions in such systems. In the scalar sector of this model, the molecular states rest above the relevant two-meson thresholds. Our findings for the scalar molecules $\mathcal{M}_b$ and $\mathcal{M}_c$ are qualitatively consistent with this conclusion of Ref. [28]. In this ar-

ticle the authors gave also information on parameter-dependent masses of axial-vector molecules ΥB_c^- , $\eta_b B_c^{*-}$, and ΥB_c^{*-} which lie below the corresponding two-meson thresholds. In other words, these molecules can not dissociate to their ingredients, and in this sense, are stable structures. Of course, this does not mean that they are stable against the strong decays through annihilation mechanisms, which may lead to considerably broad structures even in these cases. Predictions of Ref. [28] are interesting for understanding of the internal organizations and binding mechanisms of the fully heavy hadronic molecules, but need to be confirmed using alternative approaches including the sum rule method. This problem is beyond the scope of the present article, but eventually may be addressed in our future works.

The studies carried out in the present paper provide valuable information on parameters of hadronic molecules built of heavy quarks and may be useful for experimental analysis of such systems.

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