LIE n-CENTRALIZERS OF VON NEUMANN ALGEBRAS

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Abstract. Let $\mathcal{U}$ be a von Neumann algebra with a projection $P \in \mathcal{U}$. For any $A_1, A_2, \ldots, A_n \in \mathcal{U}$, define $p_1(A_1) = A_1$, $p_n(A_1, A_2, \ldots, A_n) = [p_{n-1}(A_1, A_2, \ldots, A_{n-1}), A_n]$ for all integers $n \geq 2$, where $[A, B] = AB - BA$ $(A, B \in \mathcal{U})$ denotes the usual Lie product. Assume that $\phi : \mathcal{U} \to \mathcal{U}$ is an additive mapping satisfying

$$\phi(p_n(A_1, A_2, \ldots, A_n)) = p_n(\phi(A_1), A_2, \ldots, A_n) = p_n(A_1, \phi(A_2), \ldots, A_n)$$

for all $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$ In this article, it is shown that the map ϕ is of the form $\phi(A) = WA + \xi(A)$ for all $A \in \mathcal{U}$, where $W \in \mathrm{Z}(\mathcal{U})$, and $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ ($\mathrm{Z}(\mathcal{U})$ is the center of $\mathcal{U}$) is an additive map such that $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. As an application, we characterize generalized Lie n-derivations on arbitrary von Neumann algebras.

1. Introduction and Preliminaries

Let $\mathcal{U}$ be an associative algebra with center $\mathcal{Z}(\mathcal{U})$. A linear mapping $\delta : \mathcal{U} \to \mathcal{U}$ is called a *centralizer* if $\delta(xy) = \delta(x)y = x\delta(y)$ holds for all $x, y \in \mathcal{U}$. For any $x_1, x_2, \ldots, x_n \in \mathcal{U}$, define $p_1(x_1) = x_1$, $p_2(x_1, x_2) = [x_1, x_2]$ and $p_n(x_1, x_2, \ldots, x_n) = [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]$ for all integers $n \geq 2$, where $[x_1, x_2] = x_1x_2 - x_2x_1$ denotes the Lie product of x_1 and x_2 in $\mathcal{U}$. A linear mapping $\delta : \mathcal{U} \to \mathcal{U}$ is said to be a *Lie n-centralizer* $(n \geq 2)$ if $\delta(p_n(x_1, x_2, \ldots, x_n)) = p_n(\delta(x_1), x_2, \ldots, x_n)$ holds for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$. In particular, a Lie 2-centralizer is called a Lie centralizer and a Lie 3-centralizer is said to be a Lie triple centralizer. It is easy to see that every centralizer is a Lie n-centralizer but the converse is not necessarily true. Describing Lie n-centralizers in terms of centralizers is an interesting topics of research. Fošner and Jing in [11] have studied non-additive Lie centralizers on triangular rings, and in [18] non-linear Lie centralizers on generalized matrix algebra have been checked. Jabeen in [16] has described the structure of linear Lie centralizers on a generalized matrix algebra under some conditions. In [9] it has been shown that under some conditions on a unital generalized matrix algebra $\mathcal{U}$, if $\phi : \mathcal{U} \to \mathcal{U}$ is a linear Lie triple centralizer, then $\phi(a) = \lambda a + \xi(a)$ in which $\lambda \in \mathrm{Z}(\mathcal{U})$ and ξ is a linear map from $\mathcal{U}$ into $\mathrm{Z}(\mathcal{U})$ vanishing at every second commutator $[[a, b], c]$ for all$a, b, c \in \mathcal{U}$, where $\mathrm{Z}(\mathcal{U})$ is the center of $\mathcal{U}$. In [21], Lie n-centralizers of generalized matrix algebras have been examined. Fadaee et al. in [3] have studied the characterization of Lie centralizers on non-unital triangular algebras through zero products. In [14], linear Lie centralizers at the zero products on some operator algebras are studied, and in [8], linear Lie centralizers through zero products on a 2-torsion free unital generalized matrix algebra under some mild conditions, are described. In [7], the problem of characterizing linear maps behaving like Lie centralizers at idempotent products on triangular algebras is considered, and in [15], additive Lie centralizers through idempotent-products on a 2-torsion free unital prime ring are determined.

Other important classes of mappings on algebras are derivations, Lie n-derivations and their generalizations, which have been extensively studied. A linear mapping $\delta : \mathcal{U} \to \mathcal{U}$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in \mathcal{U}$. A linear mapping $\delta : \mathcal{U} \to \mathcal{U}$ is said to be a *Lie n-derivation* $(n \geq 2)$ if $\delta(p_n(x_1, x_2, \ldots, x_n)) = p_n(\delta(x_1), x_2, \ldots, x_n) + p_n(x_1, \delta(x_2), \ldots, x_n) + \cdots + p_n(x_1, x_2, \ldots, \delta(x_n))$ holds for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$. Lie n-derivations have been further generalized as follows: Let $\xi : \mathcal{U} \to \mathcal{U}$ be a linear mapping and δ be a Lie derivation on $\mathcal{U}$. Then ξ is called a *generalized Lie n-derivation* associated with the Lie derivation δ if $\xi(p_n(x_1, x_2, \ldots, x_n)) = p_n(\xi(x_1), x_2, \ldots, x_n) + p_n(x_1, \delta(x_2), \ldots, x_n) + \cdots + p_n(x_1, x_2, \ldots, \delta(x_n))$

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holds for all $x_1, x_2, \dots, x_n \in \mathcal{U}$. Note that Lie n-centralizers and Lie n-derivations are special examples of generalized Lie n-derivations. Determining the structure of generalized Lie n-derivations is of special interest. We refer the reader to [1, 2, 4–6, 10, 13, 19, 20] and references therein about characterization of Lie n-derivations and generalized Lie n-derivations. Benkovič [4] recently described the form of generalized Lie n-derivations through the structure of Lie n-centralizers of a unital algebra $\mathcal{U}$ with a nontrivial idempotent and proved that under certain assumptions every generalized Lie n-derivation $G : \mathcal{U} \to \mathcal{U}$ is of the form $G(T) = ZT + D(T)$ for all $T \in \mathcal{U}$, where $Z \in \mathcal{Z}(\mathcal{U})$ and $D : \mathcal{U} \to \mathcal{U}$ is a Lie n-derivation. Motivated by these developments, in the present article, we study Lie n-centralizers on arbitrary von Neumann algebras. It should be noted that most of the previous results about von Neumann algebras are on factor von Neumann algebras or von Neumann algebras without central summands of type I_1, but our results are on a wider class of von Neumann algebras, and some of our results are generalizations of some previous results. It is worth to mention that by using the obtained results, it is possible to characterize generalized Lie n-derivations, generalized Lie derivations, Jordan centralizers and Jordan generalized derivations on von Neumann algebras.

A von Neumann algebra $\mathcal{U}$ is a weakly closed, self-adjoint algebra of operators on a complex Hilbert space $\mathcal{H}$ containing the identity operator I. If $P \in \mathcal{U}$ is idempotent (i. e. $P^2 = P$) and self-adjoint ($P^* = P$), then P is called a *projection*. A projection $P \in \mathcal{U}$ is said to be a *central abelian projection* if $P \in \mathrm{Z}(\mathcal{U})$ and $P\mathcal{U}P$ is abelian. The *central carrier* of $T \in \mathcal{U}$ is the smallest central projection P satisfying $PT = T$, and denoted by $\overline{T}$. It is well known that $\overline{T}$ is the projection whose range is the closed linear span of $\{AT(h) : A \in \mathcal{U}, h \in \mathcal{H}\}$. For each self-adjoint operator $S \in \mathcal{U}$, the *core* of S, denoted by $\underline{S}$, is $\sup\{W \in \mathrm{Z}(\mathcal{U}) : W = W^*, W \leq S\}$. The projection P is a *core-free projection*, if $P \in \mathcal{U}$ is a projection and $\underline{P} = 0$. A routine verifications shows that $\underline{P} = 0$ if and only if $\overline{I - P} = I$. Note that $\mathcal{U}$ is a von Neumann algebra with no central summands of type I_1 if and only if it has a projection P such that $\underline{P} = 0$ and $\overline{P} = I$. If $\mathcal{U}$ is an arbitrary von Neumann algebra, the unit element I of $\mathcal{U}$ is the sum of two orthogonal central projections E_1 and E_2 such that $\mathcal{U} = \mathcal{U}E_1 \oplus \mathcal{U}E_2$, $\mathcal{U}E_1$ is of type I_1 and $\mathcal{U}E_2$ is a von Neumann algebra with no central summands of type I_1. So $\mathcal{U}E_2$ contains a core-free projection with central carrier E_2. We refer the reader to [17] for the theory of von Neumann algebras.

von **Remark 1.1.** Let $\mathcal{U}$ be a von Neumann algebra with no central summands of type I_1, and $P \in \mathcal{U}$ be a projection such that $\underline{P} = 0$ and $\overline{P} = I$. We have $\underline{I - P} = 0$ and $\overline{I - P} = I$.

(i) By [17, Corollary 5.5.7] we have

$$\mathrm{Z}(P\mathcal{U}P) = P\mathrm{Z}(\mathcal{U}) \quad \text{and} \quad \mathrm{Z}((I-P)\mathcal{U}(I-P)) = (I-P)\mathrm{Z}(\mathcal{U}).$$

(ii) It follows from the definition of the central carrier that both *span*$\{AP(h) : A \in \mathcal{U}, h \in \mathcal{H}\}$ and *span*$\{A(I-P)(h) : A \in \mathcal{U}, h \in \mathcal{H}\}$ are dense in $\mathcal{H}$. So $A \in \mathcal{U}$, $A\mathcal{U}P = \{0\}$ implies $A = 0$ and $A\mathcal{U}(I-P) = \{0\}$ implies $A = 0$.

rem2.1 **Remark 1.2.** Let $\mathcal{U}$ be a von Neumann algebra with a projection $P_1 \in \mathcal{U}$, and write $P_2 = I - P_1$. Then, for any $A \in \mathcal{U}$, we have

(i) $p_n(A, P_1, P_1, \dots, P_1) = (-1)^{n-1}P_1AP_2 + P_2AP_1$;
(ii) $p_n(A, P_2, P_2, \dots, P_2) = P_1AP_2 + (-1)^{n-1}P_2AP_1$.

2. Lie n-centralizers of von Neumann algebras

The following theorem is the main result of this article.

main **Theorem 2.1.** *Let $\mathcal{U}$ be a von Neumann algebra with unit element I, and $E_1 + E_2 = I$, where E_1 and E_2 are two orthogonal central projections such that $\mathcal{U}E_1$ is of type I_1 and $\mathcal{U}E_2$ is a von Neumann algebra with no central summands of type I_1. Suppose that $P \in \mathcal{U}E_2$ is a core-free projection with central carrier E_2. Let $\phi : \mathcal{U} \to \mathcal{U}$ be an additive map. Then ϕ satisfies*

$$\phi(p_n(A_1, A_2, \dots, A_n)) = p_n(\phi(A_1), A_2, \dots, A_n) = p_n(A_1, \phi(A_2), \dots, A_n) \quad (\boldsymbol{P})$$

for all $A_1, A_2, \dots, A_n \in \mathcal{U}$ with $A_1A_2 = P$ if and only if $\phi(A) = WA + \xi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map in which $\xi(p_n(A_1, A_2, \dots, A_n)) = 0$ for any $A_1, A_2, \dots, A_n \in \mathcal{U}$ with $A_1A_2 = P$.

The following result is obtained from Theorem 2.1 which is a generalization of the obtained result in [9, Remark 4.4], [8, Corollary 5.2-(iv)] and [12, Corollary 4.3-(ii)] for factor von Neumann algebras.

Corollary 2.2. *Let $\mathcal{U}$ be an arbitrary von Neumann algebra, and $\phi : \mathcal{U} \to \mathcal{U}$ be an additive map. Then ϕ is a Lie n-centralizer if and only if there exist an element $W \in \mathrm{Z}(\mathcal{U})$ and an additive map $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ such that $\phi(A) = WA + \xi(A)$ for any $A \in \mathcal{U}$ and $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$.*

First, we show the main result for von Neumann algebras with no central summands of type I_1 in the following proposition.

Proposition 2.3. *Let $\mathcal{U}$ be a von Neumann algebra with no central summands of type I_1, and P be a core-free projection with central carrier I. Let $\phi : \mathcal{U} \to \mathcal{U}$ be an additive map. Then ϕ satisfies (**P**) if and only if $\phi(A) = WA + \xi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map in which $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1 A_2 = P$.*

Proof. Assume that ϕ satisfies (**P**). Set $P_1 := P$ and $P_2 := I - P_1$. By Remark 1.1 P_2 is also core free and $\overline{P_2} = I$. Set $\mathcal{U}_{ij} = P_i \mathcal{U} P_j$ $(i, j = 1, 2)$, then $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}$. So any element A of $\mathcal{U}$ is of the form $A = A_{11} + A_{12} + A_{21} + A_{22}$ for some $A_{ij} \in \mathcal{U}_{ij}$ $(i, j = 1, 2)$. The continuation of the proof in this case is done through the following lemmas.

Lemma 2.4. $\phi(I), \phi(P_1) \in \mathcal{U}_{11} + \mathcal{U}_{22}$.

Proof. Since $IP_1 = P_1$ and $P_1 P_1 = P_1$, we get

$$\begin{aligned} 0 &= \phi(p_n(I, P_1, \ldots, P_1)) \\ &= p_n(\phi(I), P_1, \ldots, P_1) \\ &= (-1)^{n-1} P_1 \phi(I) P_2 + P_2 \phi(I) P_1 \end{aligned}$$

and

$$\begin{aligned} 0 &= \phi(p_n(P_1, P_1, \ldots, P_1)) \\ &= p_n(\phi(P_1), P_1, \ldots, P_1) \\ &= (-1)^{n-1} P_1 \phi(P_1) P_2 + P_2 \phi(P_1) P_1. \end{aligned}$$

Multiplying the above equations once from left by P_2, and once from right by P_2, we arrive at $P_2\phi(I)P_1 = P_1\phi(I)P_2 = 0$ and $P_2\phi(P_1)P_1 = P_1\phi(P_1)P_2 = 0$, so $\phi(I), \phi(P_1) \in \mathcal{U}_{11} + \mathcal{U}_{22}$. □

Lemma 2.5. $\phi(\mathcal{U}_{ij}) \subseteq \mathcal{U}_{ij}$*, where* $1 \leq i \neq j \leq 2$.

Proof. For any $A_{12} \in \mathcal{U}_{12}$, since $(I + A_{12})P_1 = P_1$, by assumption, we see that

$$\begin{aligned} \phi(A_{12}) &= \phi(p_n(I + A_{12}, P_1, \ldots, P_1)) \\ &= p_n(I + A_{12}, \phi(P_1), P_1, \ldots, P_1) \\ &= p_n(I, \phi(P_1), P_1, \ldots, P_1) + p_n(A_{12}, \phi(P_1), P_1, \ldots, P_1) \\ &= p_n(A_{12}, \phi(P_1), P_1, \ldots, P_1) \\ &= p_{n-1}([A_{12}, \phi(P_1)], P_1, \ldots, P_1) \\ &= (-1)^{n-2} P_1 [A_{12}, \phi(P_1)] P_2 + P_2 [A_{12}, \phi(P_1)] P_1 \\ &= (-1)^{n-2} P_1 [A_{12}, \phi(P_1)] P_2. \end{aligned}$$

Multiplying the above equation once from left and right by P_1, once from left and right by P_2, and once from left by P_2 and from right by P_1, we conclude that $P_1\phi(A_{12})P_1 = 0$, $P_2\phi(A_{12})P_2 = 0$ and $P_2\phi(A_{12})P_1 = 0$. Therefore $\phi(A_{12}) = P_1\phi(A_{12})P_2 \in \mathcal{U}_{12}$.

For any $A_{21} \in \mathcal{U}_{21}$, since $P_1(I + A_{21}) = P_1$, we have

$$\begin{aligned} \phi(A_{21}) &= \phi(p_n(P_1, I + A_{21}, P_1, \ldots, P_1)) \\ &= p_n(\phi(P_1), I + A_{21}, P_1, \ldots, P_1) \\ &= p_n(\phi(P_1), I, P_1, \ldots, P_1) + p_n(\phi(P_1), A_{21}, P_1, \ldots, P_2) \end{aligned}$$

$$
\begin{aligned}
&= p_n(\phi(P_1), A_{21}, P_1, \dots, P_1) \\
&= p_{n-1}([\phi(P_1), A_{21}], P_1, \dots, P_1) \\
&= (-1)^{n-2} P_1[\phi(P_1), A_{21}]P_2 + P_2[\phi(P_1), A_{21}]P_1 \\
&= P_2[\phi(P_1), A_{21}]P_1.
\end{aligned}
$$

Multiplying the above equation once from left and right by P_1, once from left and right by P_2, and once from left by P_2 and from right by P_1, and we arrive at $\phi(A_{21}) \in \mathcal{U}_{21}$. $\square$

13 **Lemma 2.6.** *$\phi(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{22}$, for $i \in \{1, 2\}$.*

Proof. For any invertible $A_{11} \in \mathcal{U}_{11}$, since $A_{11}A_{11}^{-1} = P_1$, we have

$$
\begin{aligned}
0 &= \phi(p_n(A_{11}, A_{11}^{-1}, P_1, \dots, P_1)) \\
&= p_n(\phi(A_{11}), A_{11}^{-1}, P_1, \dots, P_1) \\
&= p_{n-1}([\phi(A_{11}), A_{11}^{-1}], P_1, \dots, P_1) \\
&= (-1)^{n-2} P_1[\phi(A_{11}), A_{11}^{-1}]P_2 + P_2[\phi(A_{11}), A_{11}^{-1}]P_1 \\
&= (-1)^{n-1} A_{11}^{-1}\phi(A_{11})P_2 + P_2\phi(A_{11})A_{11}^{-1}.
\end{aligned}
$$

Write $\phi(A_{11}) = \sum_{i,j=1}^{2} T_{ij}$. It follows from above equation that $T_{12} = T_{21} = 0$. Consequently, $\phi(A_{11}) \in \mathcal{U}_{11} + \mathcal{U}_{22}$, for any invertible $A_{11} \in \mathcal{U}_{11}$. For any $A_{11} \in \mathcal{U}_{11}$, we may find a sufficiently big number m such that $mP_1 - A_{11}$ is invertible. Thus, by the above and Lemma 2.4, we have

$$\phi(A_{11}) = m\phi(P_1) - \phi(mP_1 - A_{11}) \in \mathcal{U}_{11} + \mathcal{U}_{22}.$$

For any $B_{22} \in \mathcal{U}_{22}$, write $\phi(B_{22}) = \sum_{i,j=1}^{2} S_{ij}$. By the equation $(P_1 + B_{22})P_1 = P_1$ and Lemma 2.4, we have

$$
\begin{aligned}
0 &= \phi(p_n(P_1 + B_{22}, P_1, \dots, P_1)) \\
&= p_n(\phi(P_1 + B_{22}), P_1, \dots, P_1) \\
&= p_n(\phi(P_1), P_1, \dots, P_1) + p_n(\phi(B_{22}), P_1, \dots, P_1) \\
&= p_n(\phi(B_{22}), P_1, \dots, P_1) \\
&= (-1)^{n-1} P_1\phi(B_{22})P_2 + P_2\phi(B_{22})P_1.
\end{aligned}
$$

It follows that $S_{12} = S_{21} = 0$. Therefore $\phi(B_{22}) \in \mathcal{U}_{11} + \mathcal{U}_{22}$.

$\square$

14 **Lemma 2.7.** *For $i = 1, 2$, there exists an additive map $h_i : \mathcal{U}_{ii} \to \mathrm{Z}(\mathcal{U})$ such that $P_j\phi(A_{ii})P_j = h_i(A_{ii})P_j$ for any $A_{ii} \in \mathcal{U}_{ii}$, where $1 \le i \ne j \le 2$.*

Proof. For any invertible element $A_{11} \in \mathcal{U}_{11}$, and each $B_{22} \in \mathcal{U}_{22}$ we have $A_{11}(A_{11}^{-1} + B_{22}) = P_1$, and $(A_{11} + B_{22})A_{11}^{-1} = P_1$. First, suppose that $n \ge 3$. For each $C_{ij} \in \mathcal{U}_{ij}$, where $(1 \le i \ne j \le 2)$, we get

$$
\begin{aligned}
0 &= \phi(p_n(A_{11}, A_{11}^{-1} + B_{22}, C_{21}, P_1, \dots, P_1)) \\
&= p_n(\phi(A_{11}), A_{11}^{-1} + B_{22}, C_{21}, P_1, \dots, P_1) \\
&= p_n(\phi(A_{11}), A_{11}^{-1}, C_{21}, P_1, \dots, P_1) + p_n(\phi(A_{11}), B_{22}, C_{21}, P_1, \dots, P_1) \\
&= \phi(p_n(A_{11}, A_{11}^{-1}, C_{21}, P_1, \dots, P_1)) + p_n(\phi(A_{11}), B_{22}, C_{21}, P_1, \dots, P_1) \\
&= p_n(\phi(A_{11}), B_{22}, C_{21}, P_1, \dots, P_1) \\
&= p_{n-2}([[\phi(A_{11}), B_{22}], C_{21}], P_1, \dots, P_1) \\
&= (-1)^{n-3} P_1[[\phi(A_{11}), B_{22}], C_{21}]P_2 + P_2[[\phi(A_{11}), B_{22}], C_{21}]P_1 \\
&= P_2[[\phi(A_{11}), B_{22}], C_{21}]P_1
\end{aligned}
$$

and

$$0 = \phi(p_n(A_{11}^{-1} + B_{22}, A_{11}, C_{12}, P_1, \dots, P_1))$$

$$
\begin{aligned}
&= p_n(\phi(A_{11}^{-1}+B_{22}), A_{11}, C_{12}, P_1, \ldots, P_1) \\
&= p_n(\phi(A_{11}^{-1}), A_{11}, C_{12}, P_1, \ldots, P_1) + p_n(\phi(B_{22}), A_{11}, C_{12}, P_1, \ldots, P_1) \\
&= \phi(p_n(A_{11}^{-1}, A_{11}, C_{12}, P_1, \ldots, P_1)) + p_n(\phi(B_{22}), A_{11}, C_{12}, P_1, \ldots, P_1) \\
&= p_n(\phi(B_{22}), A_{11}, C_{12}, P_1, \ldots, P_1) \\
&= p_{n-2}([[\phi(B_{22}), A_{11}], C_{12}], P_1, \ldots, P_1) \\
&= (-1)^{n-3} P_1[[\phi(B_{22}), A_{11}], C_{12}]P_2 + P_2[[\phi(B_{22}), A_{11}], C_{12}]P_1 \\
&= (-1)^{n-3} P_1[[\phi(B_{22}), A_{11}], C_{12}]P_2.
\end{aligned}
$$

Considering above equations, and using Lemma 2.6, we arrive at

$$(P_2\phi(A_{11})P_2B_{22} - B_{22}P_2\phi(A_{11})P_2)CP_1 = 0$$

and

$$(P_1\phi(B_{22})P_1A_{11} - A_{11}P_1\phi(B_{22})P_1)CP_2 = 0$$

for any $C \in \mathcal{U}$ and any invertible element $A_{11} \in \mathcal{U}_{11}$. For any $A_{11} \in \mathcal{U}_{11}$, we may find a sufficiently big number m such that $mP_1 - A_{11}$ is invertible. So,

$$(P_2\phi(A_{11})P_2B_{22} - B_{22}P_2\phi(A_{11})P_2)CP_1 = 0$$

and

$$(P_1\phi(B_{22})P_1A_{11} - A_{11}P_1\phi(B_{22})P_1)CP_2 = 0$$

for all $C \in \mathcal{U}$, $A_{11} \in \mathcal{U}_{11}$, and $B_{22} \in \mathcal{U}_{22}$. From Remark 1.1, we conclude that $P_2\phi(A_{11})P_2 \in \mathrm{Z}(\mathcal{U}_{22})$ and $P_1\phi(B_{22})P_1 \in \mathrm{Z}(\mathcal{U}_{11})$, and by fact that $\mathrm{Z}(\mathcal{U}_{22}) = \mathrm{Z}(\mathcal{U})P_2$ and $\mathrm{Z}(\mathcal{U}_{11}) = \mathrm{Z}(\mathcal{U})P_1$ we have $P_2\phi(A_{11})P_2 \in \mathrm{Z}(\mathcal{U})P_2$ and $P_1\phi(B_{22})P_1 \in \mathrm{Z}(\mathcal{U})P_1$. Therefore, for any $A_{11} \in \mathcal{U}_{11}$ and $B_{22} \in \mathcal{U}_{22}$, there are $Z_1, Z_2 \in \mathrm{Z}(\mathcal{U})$ such that $P_2\phi(A_{11})P_2 = Z_1P_2$ and $P_1\phi(B_{22})P_1 = Z_2P_1$.

Now, suppose that $n = 2$. Then

$$
\begin{aligned}
0 &= \phi([p_n(A_{11}, A_{11}^{-1}+B_{22}]) \\
&= [\phi(A_{11}), A_{11}^{-1}+B_{22}] \\
&= [\phi(A_{11}), A_{11}^{-1}] + [\phi(A_{11}), B_{22}] \\
&= \phi([p_n(A_{11}, A_{11}^{-1}]) + [\phi(A_{11}), B_{22}] \\
&= [\phi(A_{11}), B_{22}] = [P_2\phi(A_{11})P_2, B_{22}]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \phi([p_n(A_{11}^{-1}+B_{22}]) \\
&= [\phi(A_{11}^{-1}+B_{22}), A_{11}] \\
&= [\phi(A_{11}^{-1}), A_{11}] + [\phi(B_{22}), A_{11}] \\
&= \phi([p_n(A_{11}^{-1}, A_{11}]) + [\phi(B_{22}), A_{11}] \\
&= [\phi(B_{22}), A_{11}] = [P_1\phi(B_{22})P_1, A_{11}].
\end{aligned}
$$

Again, by Remark 1.1, we conclude that $P_2\phi(A_{11})P_2 = Z_1P_2$ and $P_1\phi(B_{22})P_1 = Z_2P_1$ for some $Z_1, Z_2 \in \mathrm{Z}(\mathcal{U})$.

So we can define the maps $h_1 : \mathcal{U}_{11} \to \mathrm{Z}(\mathcal{U})$ by $h_1(A_{11}) = Z_1$ for any $A_{11} \in \mathcal{U}_{11}$ and $h_2 : \mathcal{U}_{22} \to \mathrm{Z}(\mathcal{U})$ by $h_2(B_{22}) = Z_2$ for any $B_{22} \in \mathcal{U}_{22}$. Suppose that $h_1(A_{11}) = Z_1 \in \mathrm{Z}(\mathcal{U})$ and $h_1(A_{11}) = Z_1' \in \mathrm{Z}(\mathcal{U})$. Then we have $\phi(A_{11}) - Z_1 \in \mathcal{U}_{11}$ and $\phi(A_{11}) - Z_1' \in \mathcal{U}_{11}$. It follows that $Z_1' - Z_1 = (\phi(A_{11}) - Z_1) - (\phi(A_{11}) - Z_1') \in \mathcal{U}_{11} \cap \mathrm{Z}(\mathcal{U}) = \{0\}$. So $Z_1 = Z_1'$. In a similar way it is proved that Z_2 is unique. By the uniqueness of Z_1 and Z_2 the maps h_1 and h_2 are well-defined. Moreover, from the uniqueness of Z_1 and Z_2 and additivity of ϕ it follows that h_1 and h_2 are additive. Also,

$$P_2\phi(A_{11})P_2 = h_1(A_{11})P_2 \quad \text{and} \quad P_1\phi(B_{22})P_1 = h_2(B_{22})P_1.$$

□

Now, for any $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathcal{U}$, we define two additive maps $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ and $\psi : \mathcal{U} \to \mathcal{U}$ by

$$\xi(A) = h_1(A_{11}) + h_2(A_{22}) \quad \text{and} \quad \psi(A) = \phi(A) - \xi(A).$$

By Lemmas 1-3, it is clear that $\psi(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii}$ for $i = 1, 2$, and $\psi(\mathcal{U}_{ij}) = \phi(\mathcal{U}_{ij}) \subseteq \mathcal{U}_{ij}$ for $1 \leq i \neq j \leq 2$.

15 **Lemma 2.8.** *There is an element $W \in \mathrm{Z}(\mathcal{U})$ such that $\psi(A) = WA$ for all $A \in \mathcal{U}$.*

Proof. We divide the proof into the following steps.

Step 1. The following statements hold:

(i) $\psi(A_{ii}B_{ij}) = \psi(A_{ii})B_{ij} = A_{ii}\psi(B_{ij})$ for all $A_{ii} \in \mathcal{U}_{ii}$ and $B_{ij} \in \mathcal{U}_{ij}$, where $1 \leq i \neq j \leq 2$;

(ii) $\psi(B_{ij}A_{jj}) = \psi(B_{ij})A_{jj} = B_{ij}\psi(A_{jj})$ for all $B_{ij} \in \mathcal{U}_{ij}$ and $A_{jj} \in \mathcal{U}_{jj}$, where $1 \leq i \neq j \leq 2$

For any invertible element $A_{11} \in \mathcal{U}_{11}$ and any $B_{12} \in \mathcal{U}_{12}$, since $(A_{11}^{-1} + B_{12})A_{11} = P_1$, by Lemma 2.5, we have

$$\begin{aligned}
(-1)^{n-1}\psi(A_{11}B_{12}) &= \phi((-1)^{n-1}A_{11}B_{12}) \\
&= \phi(p_n(A_{11}^{-1} + B_{12}, A_{11}, P_1, \ldots, P_1)) \\
&= p_n(\phi(A_{11}^{-1} + B_{12}), A_{11}, P_1, \ldots, P_1) \\
&= p_n(\phi(A_{11}^{-1}) + \phi(B_{12}), A_{11}, P_1, \ldots, P_1) \\
&= p_n(\phi(A_{11}^{-1}), A_{11}, P_1, \ldots, P_1) + p_n(\phi(B_{12}), A_{11}, P_1, \ldots, P_1) \\
&= \phi(p_n(A_{11}^{-1}, A_{11}, P_1, \ldots, P_1)) + p_n(\phi(B_{12}), A_{11}, P_1, \ldots, P_1) \\
&= p_n(\phi(B_{12}), A_{11}, P_1, \ldots, P_1) \\
&= p_{n-1}([\phi(B_{12}), A_{11}], P_1, \ldots, P_1) \\
&= (-1)^{n-2}P_1[\phi(B_{12}), A_{11}]P_2 + P_2[\phi(B_{12}), A_{11}]P_1 \\
&= (-1)^{n-2}P_1[\phi(B_{12}), A_{11}]P_2 \\
&= (-1)^{n-1}A_{11}\phi(B_{12}) \\
&= (-1)^{n-1}A_{11}\psi(B_{12})
\end{aligned}$$

and

$$\begin{aligned}
(-1)^{n-1}\psi(A_{11}B_{12}) &= \phi((-1)^{n-1}A_{11}B_{12}) \\
&= \phi(p_n(A_{11}^{-1} + B_{12}, A_{11}, P_1, \ldots, P_1)) \\
&= p_n(A_{11}^{-1} + B_{12}, \phi(A_{11}), P_1, \ldots, P_1) \\
&= p_n(\phi(A_{11}^{-1}, \phi(A_{11}), P_1, \ldots, P_1) + p_n(B_{12}, \phi(A_{11}), P_1, \ldots, P_1) \\
&= \phi(p_n(A_{11}^{-1}, A_{11}, P_1, \ldots, P_1)) + p_n(B_{12}, \phi(A_{11}), P_1, \ldots, P_1) \\
&= p_n(B_{12}, \phi(A_{11}), P_1, \ldots, P_1) \\
&= p_{n-1}([B_{12}, \phi(A_{11})], P_1, \ldots, P_1) \\
&= (-1)^{n-2}P_1[B_{12}, \phi(A_{11})]P_2 + P_2[B_{12}, \phi(A_{11})]P_1 \\
&= (-1)^{n-2}P_1[B_{12}, \phi(A_{11})]P_2 \\
&= (-1)^{n-1}\phi(A_{11})B_{12} \\
&= (-1)^{n-1}\psi(A_{11})B_{12}.
\end{aligned}$$

For any $A_{11} \in \mathcal{U}_{11}$, there exists an integer m such that $mP_1 - A_{11}$ is invertible. Note that mP_1 is also invertible. By above results we have $\psi(mP_1B_{12}) = mP_1\psi(B_{12}) = m\psi(P_1)B_{12}$ and $\psi((mP_1 - A_{11})B_{12}) = (mP_1 - A_{11})\psi(B_{12}) = \psi(mP_1 - A_{11})B_{12}$. Thus, $\psi(A_{11}B_{12}) = A_{11}\psi(B_{12}) = \psi(A_{11})B_{12}$, for any $A_{11} \in \mathcal{U}_{11}$ and any $B_{22} \in \mathcal{U}_{22}$.

For any invertible element $A_{11} \in \mathcal{U}_{11}$ and any $B_{21} \in \mathcal{U}_{21}$, since $A_{11}(A_{11}^{-1} + B_{21}) = P_1$ and $p_n(A_{11}, A_{11}^{-1} + B_{21}, P_1, \ldots, P_1) = -B_{21}A_{11}$, and with the similar arguments as above, it can be checked that

$$\psi(B_{21}A_{11}) = B_{21}\psi(A_{11}) = \psi(B_{21})A_{11}$$

for any $A_{11} \in \mathcal{U}_{11}$ and $B_{21} \in \mathcal{U}_{21}$. For any $A_{22} \in \mathcal{U}_{22}$ and $B_{21} \in \mathcal{U}_{21}$, we have $(P_1 + A_{22} - A_{22}B_{21})(P_1 + B_{21}) = P_1$. By properties of ψ and ξ, we see that

$$\begin{aligned}
-\psi(B_{21}) = -\phi(B_{21})&\\
&= \phi(p_n(P_1 + A_{22} - A_{22}B_{21}, P_1 + B_{21}, P_1, \ldots, P_1))\\
&= p_n(\phi(P_1) + \phi(A_{22}) - \phi(A_{22}B_{21}), P_1 + B_{21}, P_1, \ldots, P_1)\\
&= p_n(\phi(P_1), P_1, P_1, \ldots, P_1) + p_n(\phi(P_1), B_{21}, P_1, \ldots, P_1)\\
&\quad + p_n(\phi(A_{22}), P_1, P_1, \ldots, P_1) + p_n(\phi(A_{22}), B_{21}, P_1, \ldots, P_1)\\
&\quad - p_n(\phi(A_{22}B_{21}), P_1, P_1, \ldots, P_1) - p_n(\phi(A_{22}B_{21}), B_{21}, P_1, \ldots, P_1)\\
&= p_n(\phi(P_1), B_{21}, P_1, \ldots, P_1) + p_n(\phi(A_{22}), B_{21}, P_1, \ldots, P_1)\\
&\quad - p_n(\phi(A_{22}B_{21}), P_1, P_1, \ldots, P_1)\\
&= p_n(\psi(P_1) + \xi(P_1), B_{21}, P_1, \ldots, P_1) + p_n(\psi(A_{22}) + \xi(A_{22}), B_{21}, P_1, \ldots, P_1)\\
&\quad - p_n(\psi(A_{22}B_{21}), P_1, P_1, \ldots, P_1)\\
&= p_n(\psi(P_1), B_{21}, P_1, \ldots, P_1) + p_n(\psi(A_{22}), B_{21}, P_1, \ldots, P_1)\\
&\quad - p_n(\psi(A_{22}B_{21}), P_1, P_1, \ldots, P_1)\\
&= -B_{21}\psi(P_1) + \psi(A_{22})B_{21} - \psi(A_{22}B_{21}).
\end{aligned}$$

Hence, we have

$$\psi(A_{22}B_{21}) = \psi(A_{22})B_{21}$$

for all $A_{22} \in \mathcal{U}_{22}$ and $B_{21} \in \mathcal{U}_{21}$, because $\psi(B_{21}) = \psi(B_{21}P_1) = B_{21}\psi(P_1)$. Also,

$$\psi(A_{22}B_{21}) = \psi(A_{22}B_{21}P_1) = A_{22}B_{21}\psi(P_1) = A_{22}\psi(B_{21}),$$

for any $A_{22} \in \mathcal{U}_{22}$ and $B_{21} \in \mathcal{U}_{21}$.

For any $A_{22} \in \mathcal{U}_{22}$ and $B_{12} \in \mathcal{U}_{12}$, since $(P_1 + B_{12})(P_1 + A_{22} - B_{12}A_{22}) = P_1$ and $p_n(P_1 + B_{12}, P_1 + A_{22} - B_{12}A_{22}, P_2, \ldots, P_2) = -B_{12}$, and with the similar arguments as above, it can be checked that

$$\psi(B_{12}A_{22}) = B_{12}\psi(A_{22}) = \psi(B_{12})A_{22}.$$

Step 2. $\psi(A_{ii}B_{ii}) = \psi(A_{ii})B_{ii} = A_{ii}\psi(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathcal{U}_{ii}$, where $i \in \{1, 2\}$.
For any $A_{ii}, B_{ii} \in \mathcal{U}_{ii}$ and any $S_{ij} \in \mathcal{U}_{ij}$ $(1 \leq i \neq j \leq 2)$, by Step 1, we have

$$\psi(A_{ii}B_{ii}S_{ij}) = \psi(A_{ii}B_{ii})S_{ij},$$

and on other hand

$$\psi(A_{ii}B_{ii}S_{ij}) = A_{ii}\psi(B_{ii}S_{ij}) = A_{ii}\psi(B_{ii})S_{ij},$$

Comparing the above two equations, we see that $\psi(A_{ii}B_{ii})S_{ij} = A_{ii}\psi(B_{ii})S_{ij}$ holds for all $S_{ij} \in \mathcal{U}_{ij}$. From Remark 1.1, it follows that $\psi(A_{ii}B_{ii}) = A_{ii}\psi(B_{ii})$ for any $A_{ii}, B_{ii} \in \mathcal{U}_{ii}$, where $i = 1, 2$. Also, for any $A_{ii}, B_{ii} \in \mathcal{U}_{ii}$ and any $S_{ji} \in \mathcal{U}_{ji}$ $(1 \leq i \neq j \leq 2)$, by Step 1, we get

$$\psi(S_{ji}A_{ii}B_{ii}) = S_{ji}\psi(A_{ii}B_{ii}),$$

and on other hand

$$\psi(S_{ji}A_{ii}B_{ii}) = \psi(S_{ji}A_{ii})B_{ii} = S_{ji}\psi(A_{ii})B_{ii},$$

Comparing the above two equations and by Remark 1.1, we see that $\psi(A_{ii}B_{ii}) = \psi(A_{ii})B_{ii}$ for any $A_{ii}, B_{ii} \in \mathcal{U}_{ii}$, where $i \in \{1, 2\}$.

Step 3. $\psi(A_{ij}B_{ji}) = \psi(A_{ij})B_{ji} = A_{ij}\psi(B_{ji})$ for all $A_{ij} \in \mathcal{U}_{ij}$ and $B_{ji} \in \mathcal{U}_{ji}$, where $1 \leq i \neq j \leq 2$.
Assume that $A_{ij} \in \mathcal{U}_{ij}$ and $B_{ji} \in \mathcal{U}_{ji}$, $1 \leq i \neq j \leq 2$. It follows from Steps 1 and 2 that

$$\psi(A_{ij}B_{ji}) = \psi(P_iA_{ij}B_{ji}) = \psi(P_i)A_{ij}B_{ji} = \psi(A_{ij})B_{ji},$$

and

$$\psi(A_{ij}B_{ji}) = \psi(A_{ij}B_{ji}P_i) = A_{ij}B_{ji}\psi(P_i) = A_{ij}\psi(B_{ji}).$$

Step 4. The desired result in Lemma 2.8 is valid.

From Steps 1-3 and the fact that each $\mathcal{U}_{ij}$ is a invariant subspace for ψ, it follows that

$$\psi(AB) = A\psi(B) = \psi(A)B$$

for all $A, B \in \mathcal{U}$. Set $W := \psi(I)$. So

$$\psi(A) = \psi(AI) = A\psi(I) = AW \quad \text{and} \quad \psi(A) = \psi(IA) = \psi(I)A = WA$$

for all $A, B \in \mathcal{U}$, and $W \in Z(\mathcal{U})$. $\square$

16 **Lemma 2.9.** *$\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for all $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$.*

Proof. For any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$, by Lemma 2.8 we have

$$\begin{aligned}
\xi(p_n(A_1, A_2, \ldots, A_n)) &= \phi(p_n(A_1, A_2, \ldots, A_n)) - \psi(p_n(A_1, A_2, \ldots, A_n)) \\
&= p_n(\phi(A_1), A_2, \ldots, A_n) - \psi(p_n(A_1, A_2, \ldots, A_n)) \\
&= p_n(\psi(A_1) + \xi(A_1), A_2, \ldots, A_n) - \psi(p_n(A_1, A_2, \ldots, A_n)) \\
&= p_n(\psi(A_1), A_2, \ldots, A_n) - \psi(p_n(A_1, A_2, \ldots, A_n)) \\
&= Wp_n(A_1, A_2, \ldots, A_n) - Wp_n(A_1, A_2, \ldots, A_n) \\
&= 0,
\end{aligned}$$

since $W \in Z(\mathcal{U})$. $\square$

Now, by the definition of ψ and Lemma 2.8, we get $\phi(A) = WA + \xi(A)$ for any $A \in \mathcal{U}$, where $W \in Z(\mathcal{U})$. From Lemma 2.9, it follows that $\xi : \mathcal{U} \to Z(\mathcal{U})$ is an additive mapping in which $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. So the desired result is valid.

The converse is clear. $\square$

Now we are ready to present the proof of the main theorem.

Proof of Theorem 2.1. Let ϕ satisfies (**P**). Suppose that $A, B, Y \in \mathcal{U}$ such that $BYE_2 = P$. Put $X := AE_1 + BE_2$. Since $\mathcal{U}E_1 \subseteq Z(\mathcal{U})$, it follows that $[X, YE_2] = [BE_2, YE_2] = [B, Y]E_2$. Moreover, we have $XYE_2 = BYE_2 = P$. Hence, for any $C_3, C_4, \ldots, C_n \in \mathcal{U}$, we get $\phi(p_n(X, YE_2, C_3, C_4, \ldots, C_n)) = p_n(\phi(X), YE_2, C_3, C_4, \ldots, C_n) = p_n(X, \phi(YE_2), C_3, C_4, \ldots, C_n)$. So

$$\begin{aligned}
\phi(p_n(BE_2, YE_2, C_3, C_4, \ldots, C_n)) &= \phi(p_n(X, YE_2, C_3, C_4, \ldots, C_n)) \\
&= p_n(\phi(X), YE_2, C_3, C_4, \ldots, C_n) \\
&= p_n(\phi(AE_1), YE_2, C_3, C_4, \ldots, C_n) + p_n(\phi(BE_2), YE_2, C_3, C_4, \ldots, C_n).
\end{aligned}$$

Multiplying the right side of the above identity by E_2, we obtain

$$\phi(p_n(BE_2, YE_2, C_3, C_4, \ldots, C_n))E_2 = p_n(\phi(AE_1), YE_2, C_3, C_4, \ldots, C_n) + p_n(\phi(BE_2), YE_2, C_3, C_4, \ldots, C_n). \tag{2.1}$$ v1

By setting $A = 0$ in (2.1) we see that

$$\phi(p_n(BE_2, YE_2, C_3, C_4, \ldots, C_n))E_2 = p_n(\phi(BE_2), YE_2, C_3, C_4, \ldots, C_n). \tag{2.2}$$ v2

for all $B, Y \in \mathcal{U}E_2$ with $BYE_2 = P$. Also, we have

$$\phi(p_n(BE_2, YE_2, C_3, C_4, \ldots, C_n))E_2 = p_n(AE_1, \phi(YE_2), C_3, C_4, \ldots, C_n) + p_n(BE_2, \phi(YE_2), C_3, C_4, \ldots, C_n).$$

So by the fact that $\mathcal{U}E_1 \subseteq Z(\mathcal{U})$ we arrive at

$$\phi(p_n(BE_2, YE_2, C_3, C_4, \ldots, C_n))E_2 = p_n(BE_2, \phi(YE_2), C_3, C_4, \ldots, C_n). \tag{2.3}$$ v3

for all $B, Y \in \mathcal{U}E_2$ with $BYE_2 = P$. Equations (2.2) and (2.3) show that the additive mapping $\varphi : \mathcal{U}E_2 \to \mathcal{U}E_2$ defined by $\varphi(AE_2) = \phi(AE_2)E_2$, on $\mathcal{U}E_2$ satisfies (**P**). By our assumption $\mathcal{U}E_2$ is a von Neumann algebra with no central summands of type I_1, and $P \in \mathcal{U}E_2$ is a projection such that $\underline{P} = 0$ and $\overline{P} = E_2$.

So by Proposition 2.3, there are $W_1 \in Z(\mathcal{U}E_2) \subseteq Z(\mathcal{U})$ and an additive mapping $\xi_1 : \mathcal{U}E_2 \to Z(\mathcal{U}E_2) \subseteq Z(\mathcal{U})$ such that

$$\phi(AE_2)E_2 = \varphi(AE_2) = W_1 AE_2 + \xi_1(AE_2) \tag{2.4}$$

for all $A \in \mathcal{U}$ and $\xi_1(p_n(AE_2, BE_2, C_3E_2, \ldots, C_nE_2)) = 0$ for all $A, B, C_3, \ldots, C_n \in \mathcal{U}$ with $ABE_2 = P$. Assume that for $Y \in \mathcal{U}$ the element YE_2 is invertible in $\mathcal{U}E_2$ (i.e., $YE_2 \in Inv(\mathcal{U}E_2)$). Taking $B := P(YE_2)^{-1}$ and $X = AE_1 + BE_2$ for $A \in \mathcal{U}$. So $BYE_2 = P$, and from (2.1) and (2.2), for any $C_3, \ldots, C_n \in \mathcal{U}$ it follows that $p_n(\phi(AE_1)E_2, YE_2, C_3, \ldots, C_n) = 0$. Since each element of $\mathcal{U}E_2$ is a sum of two invertible elements of $\mathcal{U}E_2$, it results that $p_n(\phi(AE_1)E_2, YE_2, C_3, \ldots, C_n) = 0$ for all $A, Y, C_3, \ldots, C_n \in \mathcal{U}$. However, by Kleinecke-Shirokov Theorem and the fact that the spectral radius is submultiplicative on commuting elements, it results that $[\phi(AE_1)E_2, YE_2] = 0$ for all $A, Y \in \mathcal{U}$. So

$$\phi(AE_1)E_2 \in Z(\mathcal{U}E_2) \subseteq Z(\mathcal{U})$$

for all $A \in \mathcal{U}$. Also

$$\varphi(A)E_1 \in \mathcal{U}E_1 \subseteq Z(\mathcal{U})$$

for all $A \in \mathcal{U}$. Now by (2.4) we have

$$\begin{aligned}\phi(A) &= \phi(A)E_1 + \phi(AE_1)E_2 + \phi(AE_2)E_2\\ &= \phi(A)E_1 + \phi(AE_1)E_2 + W_1AE_2 + \xi_1(AE_2)\\ &= WA + \xi(A),\end{aligned}$$

for all $A \in \mathcal{U}$, where $W := W_1E_2 \in Z(\mathcal{U})$ and $\xi : \mathcal{U} \to \mathcal{U}$ is an additive map defined by $\xi(A) = \phi(A)E_1 + \phi(AE_1)E_2 + \xi_1(AE_2)$. By above all three summands lie in $Z(\mathcal{U})$, thus ξ maps $\mathcal{U}$ into $Z(\mathcal{U})$. Finally according to these results for $A, B, C_3, \ldots, C_n \in \mathcal{U}$ where $AB = P$ we have

$$\begin{aligned}\xi(p_n(A, B, C_3, \ldots, C_n)) &= \phi(p_n(A, B, C_3, \ldots, C_n)) - Wp_n(A, B, C_3, \ldots, C_n)\\ &= p_n(\phi(A), B, C_3, \ldots, C_n) - p_n(WA, B, C_3, \ldots, C_n)\\ &= p_n(\xi(A), B, C_3, \ldots, C_n) = 0.\end{aligned}$$

The converse is clear. □

Proof of Corollary 2.2. Let ϕ be a Lie n-centralizer. If $\mathcal{U}$ is an abelian von Neumann algebra, then ϕ maps $\mathcal{U}$ into $Z(\mathcal{U}) = \mathcal{U}$. Also, from the fact that ϕ is a Lie n-centralizer, it follows that $\phi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$. So in this case the result is valid. Now let's assume that $\mathcal{U}$ is non-abelian. In this case the unit element I of $\mathcal{U}$ is the sum of two orthogonal central projections E_1 and E_2 such that $\mathcal{U} = \mathcal{U}E_1 \oplus \mathcal{U}E_2$, $\mathcal{U}E_1$ is of type I_1 and $\mathcal{U}E_2$ is a von Neumann algebra with no central summands of type I_1. So there exist a core-free projection $P \in \mathcal{U}E_2$ with central carrier E_2. Because ϕ is a Lie n-centralizer, it satisfies the condition (**P**) on $\mathcal{U}$. Hence by Theorem 2.1, $\phi(A) = WA + \xi(A)$ for all $A \in \mathcal{U}$, where $W \in Z(\mathcal{U})$, $\xi : \mathcal{U} \to Z(\mathcal{U})$ is an additive map. It is sufficient to prove that for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ we have $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$. This part can be proved in similar manner with the proof of Theorem 2.1.

The converse is clear. □

3. Applications

In this section, we present some applications of the results obtained in the previous section. As an application of Theorem 2.1, we characterize the generalized Lie n-derivations of arbitrary von Neumann algebras. First, we characterize generalized Lie n-derivations of von Neumann algebras with no central summands of type I_1, which is a partial generalization of [2, Theorem 3.11]

Theorem 3.1. *Let $\mathcal{U}$ be a von Neumann algebra with no central summands of type I_1, and P be a core-free projection with central carrier I. Let $\mathcal{L}, G_{\mathcal{L}} : \mathcal{U} \to \mathcal{U}$ be additive maps satisfying*

$$\mathcal{L}(p_n(A_1, A_2, \ldots, A_n)) = \sum_{i=1}^{n} p_n(A_1, A_2, \ldots, A_{i-1}, \mathcal{L}(A_i), A_{i+1}, \ldots, A_n) \tag{3.1}$$

and

$$\begin{aligned} G_{\mathcal{L}}(p_n(A_1, A_2, \ldots, A_n)) &= p_n(G_{\mathcal{L}}(A_1), A_2, \ldots, A_n) + \sum_{i=2}^{n} p_n(A_1, A_2, \ldots, A_{i-1}, \mathcal{L}(A_i), A_{i+1}, \ldots, A_n) \\ &= p_n(\mathcal{L}(A_1), A_2, \ldots, A_n) + p_n(A_1, G_{\mathcal{L}}(A_2), \ldots, A_n) \\ &\quad + \sum_{i=3}^{n} p_n(A_1, A_2, \ldots, A_{i-1}, \mathcal{L}(A_i), A_{i+1}, \ldots, A_n) \qquad (3.2) \end{aligned}$$

for all $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. Then $G_{\mathcal{L}}$ is of the form $G_{\mathcal{L}}(A) = WA + \delta(A) + \chi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\delta : \mathcal{U} \to \mathcal{U}$ is an additive derivation and $\chi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map satisfying $\chi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$.

Proof. Let $\phi = G_{\mathcal{L}} - \mathcal{L}$. Then, ϕ satisfies (**P**). Hence, it follows from Proposition 2.3 that ϕ is of the form $\phi(A) = WA + \xi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map satisfying $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. Therefore, $G_{\mathcal{L}}(A) = WA + \xi(A) + \mathcal{L}(A)$ for all $A \in \mathcal{U}$. By [2, Theorem 3.11], $\mathcal{L}(A) = \delta(A) + \gamma(A)$, where $\delta : \mathcal{U} \to \mathcal{U}$ is an additive derivation and $\gamma : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map satisfying $\gamma(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. Suppose that $\chi = \xi + \gamma$. Then $\chi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map satisfying $\chi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$ and $G_{\mathcal{L}}(A) = WA + \delta(A) + \chi(A)$ $(A \in \mathcal{U})$, as desired. □

The following corollary is an easy consequence of the above theorem.

cor1 main2 **Corollary 3.2.** *Let $\mathcal{U}$ be a von Neumann algebra with no central summands of type I_1, and $G_{\mathcal{L}} : \mathcal{U} \to \mathcal{U}$ be a generalized Lie n-derivation (with associated Lie n-derivation $\mathcal{L}$). Then $G_{\mathcal{L}}$ is of the form $G_{\mathcal{L}}(A) = WA + \delta(A) + \chi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\delta : \mathcal{U} \to \mathcal{U}$ is an additive derivation and $\chi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map satisfying $\chi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$.*

Now, let us state and prove Theorem 3.1 for an arbitrary von Neumann algebra.

main3 **Theorem 3.3.** *Let $\mathcal{U}$ be a von Neumann algebra with unit element I, and $E_1 + E_2 = I$, where E_1 and E_2 are two orthogonal central projections such that $\mathcal{U}E_1$ is of type I_1 and $\mathcal{U}E_2$ is a von Neumann algebra with no central summands of type I_1. Suppose that $P \in \mathcal{U}E_2$ is a core-free projection with central carrier E_2. Let $\mathcal{L}, G_{\mathcal{L}} : \mathcal{U} \to \mathcal{U}$ be additive maps satifying (3.1) and (3.2). Then $G_{\mathcal{L}}$ is of the form $G_{\mathcal{L}}(A) = WA + \chi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\chi : \mathcal{U} \to \mathcal{U}$ is an additive map satisfying (3.1) for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$.*

Proof. Let $\phi = G_{\mathcal{L}} - \mathcal{L}$. Then, ϕ satisfies (**P**). Hence, it follows from Theorem 2.1 that ϕ is of the form $\phi(A) = WA + \xi(A)$ $(A \in \mathcal{U})$, where $W \in \mathrm{Z}(\mathcal{U})$, $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ is an additive map satisfying $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. Therefore, $G_{\mathcal{L}}(A) = WA + \xi(A) + \mathcal{L}(A)$ for all $A \in \mathcal{U}$. Set $\chi = \xi + \mathcal{L}$. Then it is easy to see that $\chi : \mathcal{U} \to \mathcal{U}$ is an additive map satisfying (3.1) for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. Consequently, $G_{\mathcal{L}}(A) = WA + \chi(A)$ $(A \in \mathcal{U})$, as desired. □

cor1 main3 **Corollary 3.4.** *Let $\mathcal{U}$ be an arbitrary von Neumann algebra,and $G_{\mathcal{L}} : \mathcal{U} \to \mathcal{U}$ be a generalized Lie n-derivation (with associated Lie n-derivation $\mathcal{L}$). Then there exist an element $W \in \mathrm{Z}(\mathcal{U})$ and a Lie n-derivation $\chi : \mathcal{U} \to \mathcal{U}$ such that $\phi(A) = WA + \chi(A)$ for any $A \in \mathcal{U}$.*

Proof. Let $G_{\mathcal{L}} : \mathcal{U} \to \mathcal{U}$ be a generalized Lie n-derivation with associated Lie n-derivation $\mathcal{L}$. Then $\phi = G_{\mathcal{L}} - \mathcal{L}$ is a Lie n-centralizer on $\mathcal{U}$. Hence, it follows from Corollary 3.2 that there exist an element $W \in \mathrm{Z}(\mathcal{U})$ and an additive map $\xi : \mathcal{U} \to \mathrm{Z}(\mathcal{U})$ such that $\phi(A) = WA + \xi(A)$ for any $A \in \mathcal{U}$ and $\xi(p_n(A_1, A_2, \ldots, A_n)) = 0$ for any $A_1, A_2, \ldots, A_n \in \mathcal{U}$. Therefore, $G_{\mathcal{L}}(A) = WA + \xi(A) + \mathcal{L}(A)$ for all $A \in \mathcal{U}$. Set $\chi = \xi + \mathcal{L}$. Then it is easy to see that $\chi : \mathcal{U} \to \mathcal{U}$ is a Lie n-derivation. Consequently, $G_{\mathcal{L}}(A) = WA + \chi(A)$ $(A \in \mathcal{U})$, as desired. □

Declarations.

- All authors contributed to the study conception and design and approved the final manuscript.

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References

ash 1. M. Ashraf and M.A. Ansari, *Multiplicative generalized Lie n-derivations of unital rings with idempotents*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 116, 92 (2022), https://doi.org/10.1007/s13398-022-01233-5 2

AA21 2. M. Ashraf and A. Jabeen, *Characterization of Lie type derivation on von Neumann algebra with local actions*, Bull. Korean Math. Soc. 58 (5) (2021), 1193–1208. 2, 9, 10

beh 3. R. Behfar and H. Ghahramani, *Lie maps on triangular algebras without assuming unity*, Mediterranean J. Math. 18 (2021), 1–28. 1

B18 4. D. Benkovič, *Generalized Lie derivations of unital algebras with idempotents.* Oper. Matrices **12**(2), 357–367 (2018) 2

B19 5. D. Benkovič, *Generalized Lie n-derivations of triangular algebras.* Comm. Algebra **47**(12), 5294–5302 (2019) 2

be12 6. D. Benkovič and D. Eremita, *Multiplicative Lie n-derivations of triangular rings.* Linear Algebra Appl. **436**, 4223–4240 (2012) 2

fad2 7. B. Fadaee, A. Fošner and H. Ghahramani, *Centralizers of Lie Structure of triangular algebras*, Results Math, 77 (2022), https://doi.org/10.1007/s00025-022-01756-8. 1

fad3 8. B. Fadaee and H. Ghahramani, *Lie centralizers at the zero products on generalized matrix algebras*, J. Algebra Appl. 21 (2022), https://doi.org/10.1142/S0219498822501651. 1, 3

fad4 9. B. Fadaee, H. Ghahramani and W. Jing, *Lie triple centralizers on generalized matrix algebras*, Quaestiones Math. 46 (2023), 281–300. 1, 3

FengQi 10. X. Feng and X. Qi, *Nonlinear generalized Lie n-derivations on von Neumann algebras*, Bull. Iranian Math. Soc. 45 (2019), 569–581. 2

fo 11. A. Fošner and W. Jing, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory, 4 (2019), 342–350. 1

fos2 12. A. Fošner, H. Ghahramani and F. Wei, *Lie centralizers and generalized Lie derivations at zero products*, Rocky Mountain J. Math, to appear. 3

fwx13 13. A. Fošner, F. Wei and Z.-K., Xiao, *Nonlinear Lie-type derivations of von Neumann algebras and related topics.* Colloq. Math. **132**, 53–71 (2013) 2

gh3 14. H. Ghahramani and W. Jing, *Lie centralizers at zero products on a class of operator algebras*, Ann. Funct. Anal. 12 (2021), 1–12. 1

goo 15. Sh. Goodarzi and A. Khotanloo, *Lie Centralizers and generalized Lie derivations on prime rings by local actions*, Commun. Algebra, (2023), https://doi.org/10.1080/00927872.2023.2228429. 1

jab 16. A. Jabeen, Lie (Jordan) centralizers on generalized matrix algebras, Commun. Algebra, (2020), https://doi.org/10.1080/00927872.2020.1797759. 1

ka 17. R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras*, Vol. I, Academic Press, New York, 1983, Vol. II, Academic Press, New York, 1986. 2

liu2 18. L. Liu, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear and Multilinear Algebra, (2020), https://doi.org/10.1080/03081087.2020.1810605. 1

W14 19. Y. Wang, *Lie n-derivations of unital algebras with idempotents.* Linear Algebra Appl. **458**, 512–525 (2014) 2

WW13 20. Y. Wang, and Y. Wang, *Multiplicative Lie n-derivations of generalized matrix algebras.* Linear Algebra Appl. **438**, 2599–2616 (2013) 2

yu 21. H. Yuan and Z. Liu, *Lie n-centralizers of generalized matrix algebras*, AIMS Math. 8 (2023), 14609–14622. 1

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