On a Stationarity Theory for Stochastic Volterra Integral Equations

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September 1, 2025

Abstract

This paper provide a comprehensive analysis of the finite and long time behavior of continuous-time non-Markovian dynamical systems, with a focus on the forward Stochastic Volterra Integral Equations(SVIEs). We investigate the properties of solutions to such equations specifically their stationarity, both over a finite horizon and in the long run. In particular, we demonstrate that such an equation does not exhibit a strong stationary regime unless the kernel is constant or in a degenerate settings. However, we show that it is possible to induce a *fake stationary regime* in the sense that all marginal distributions share the same expectation and variance. This effect is achieved by introducing a deterministic stabilizer ς associated with the kernel. We also look at the L^p -confluence (for $p > 0$) of such process as time goes to infinity(i.e. we investigate if its marginals when starting from various initial values are confluent in L^p as time goes to infinity) and finally the functional weak long-run assymptotics for some classes of diffusion coefficients. Those results are applied to the case of Exponential-Fractional Stochastic Volterra Integral Equations, with an α-gamma fractional integration kernel, where $\alpha \leq 1$ enters the regime of *rough path* whereas $\alpha > 1$ regularizes diffusion paths and invoke *long-term memory*, persistence or long range dependence. With this fake stationary Volterra processes, we introduce a family of stabilized volatility models.

Keywords: Stochastic Volterra Processes, Stochastic Differential Equations, Fourier-Laplace Transforms, Jordan-Cauchy Residue Theorem, Regular Variation, Tauberian Theorems, Limit theorems.

1 Introduction

The theory of stochastic Volterra integral equations (SVIEs) has its origins in the 1980s and has been widely developed since then. These equations which have recently attracted much attention in the mathematical finance community have been introduced mostly with non-singular kernel for modelling in population dynamics, biology and physics [34], in order to generalize modelling to non-Markovian stochastic systems with some memory effect. They were also motivated particularly by the physics of heat transfer [4] and have undergone extensive mathematical study. Early investigations can be traced back to the seminal work of Berger et al. (see [5],[6]) who derived existence and uniqueness results for SVIEs driven by Brownian motion with Lipschitz continuous coefficients. These initial results were subsequently extended in various directions. For instance, [40] generalized the existence and uniqueness results to SVIEs driven by right-continuous semimartingales and smooth kernels. An example of such a kernel is $K(t,s) = (t-s)^{H-\frac{1}{2}}$, where H is known as the Hurst coefficient. Others studies focused on extensions that incorporated anticipative integrands, utilizing Skorokhod integration and Malliavin calculus (This was explored in [38] and [3]). [12] and [14] focused on SVIEs with singular kernels. In a more recent contribution, [49] proved the existence and uniqueness of solutions to SVIEs with singular kernels and non-Lipschitz coefficients, utilizing a condition analogous to that of [50] for stochastic differential equations. Additionally, [51] examined SVIEs in Banach spaces with locally Lipschitz coefficients and singular kernels.

In the late 1990s, attempts were made within the financial community to incorporate long-memory effects into continuous-time stochastic volatility models. This shift was largely motivated by the need to capture persistent dependencies observed in financial markets, particularly through fractional Brownian motion (see [15, 13]). Earlier studies, such as those by Comte and Renault [13], found that $H > 1/2$ was a key parameter in capturing long memory in volatility dynamics. In the early 2000s, research shifted to Volterra equations with singular kernels that blow up as $s \to t$ (i.e., $K(t,s) \to +\infty$ as $s \to t$ or $H < 1/2$), following the empirical observation in [19] that volatility paths exhibit low Hölder regularity ($H \approx 0.1$). As a result, there has been a resurgence of interest in SVIEs within mathematical finance, particularly with the rise of rough volatility models, as highlighted in the work of [16]. These models, which use the above kernel, naturally capture this feature, as their paths have a Hölder continuity exponent H. Singular kernel Volterra equations also arise as limiting dynamics in models of order books via nearly unstable Hawkes processes (see [29, 22]).

In both context, such processes are used to mimic Fractional Brownian motion-driven stochastic differential equations (SDEs). More specifically, within these frameworks, Volterra equations with fractional kernels K provide a more tractable alternative than SDEs involving stochastic integrals with respect to true H-fractional Brownian motions, which would otherwise require the use of “high-order” rough path theory or regularity structures. As the debate on the empirical value of the Hurst index remains controversial in the literature, we note that the setting considered in this paper covers the full range of the Hurst coefficient, namely $H := \alpha - \frac{1}{2} \in (0,1)$.

By considering a deterministic continuous function ϕ, typically normalized such that $\phi(0) = 1$, a rather general form of the stochastic version of the Volterra equation on $[0,T]$ in $\mathbb{R}$ for any $T > 0$ takes the following form:

$$\begin{cases} X_t = X_0\phi(t) + \int_0^t K(t,s)b(s,X_s)\,ds + \int_0^t K(t,s)\sigma(s,X_s)\,dW_s, \quad X_0 \perp\!\!\!\perp W. \\ X_0 : (\Omega,\mathcal{F},\mathbb{P}) \to (\mathbb{R},\mathcal{B}(\mathbb{R})) \text{ is a given initial random variable} \end{cases} \tag{1.1}$$

where $b,\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are Lipstchiz continuous function and $K(t,s)$ a deterministic kernel modeling the memory or hereditary structure of the system. The process $(W_t)_{t\geq 0}$ is an $\mathbb{R}$-valued Brownian motion independent of X_0, both defined on a probability space $(\Omega,\mathcal{A},P)$ and $\mathcal{F}_t \supset \mathcal{F}_{t,X_0,W}$ a filtration satisfying the usual conditions. Such equations (1.1) naturally arise in the modeling of random systems with memory effects and irregular behaviour, including in mathematical finance, physics, and biology.

1.1 Our contribution

In this paper, we investigate a weak form of stationarity for SVIEs with affine drifts and convolutive kernels of the form (1.1). Specifically, our main result follow that of [36] and states that, under a suitable functional equation satisfied by a stabilizing function, the process of the form (1.1) may exhibit a form of *fake stationarity regime*, where the solution has either constant moments up to the order 2 or the same marginal distribution at each time t in the Gaussian case (typically pseudo-Ornstein-Uhlenbeck process, which could be called a fake stationary regime of type II). Moreover, we establish the existence of limiting distributions. Formally, we prove that as $u \to \infty$, the shifted process $(X_t^u)_{t\geq 0}$, defined by $X_t^u := X_{t+u}$, converges in law to a limiting continuous process X^∞. Unlike in [21] (see also [18, 28]), this convergence does not imply that the limiting process is stationary (in the sense that its finite-dimensional distributions are invariant under time shifts). However, we prove that, under *fake stationarity* regime, the limiting process is weak L^2-stationary. Furthermore, since we do not characterize the dynamics of the limiting process, the notion of *fake stationarity* provides a tractable alternative framework for analyzing both short- and long-term behaviors in settings where classical stationarity is either unavailable or analytically intractable.

From an applied perspective, this result may have important implications for volatility models widely used in mathematical finance. In particular, it suggests the possibility of introducing stabilized versions of such models, where the dynamics driving the asset's (typically equity) volatility exhibit constant mean and variance over time. A key advantage of the stabilized formulation lies in its ability to overcome a well-known limitation of classical and rough Heston models [27, 17] driven by mean-reverting CIR or Volterra-CIR dynamics. These models typically display two distinct regimes: a short-maturity regime, where the initial condition (deterministic value at the origin, often the long run mean) is prominent and the variance remains very small, and a long-term regime, which may correspond to the stationary distribution of the process. In contrast, the stabilized model provides a unified and coherent framework that captures both short- and long-maturity behaviors within a single regime, thereby enabling robust and consistent fitting across the full term structure.

1.2 Plan of the paper and Notations

The remainder of the paper is organized as follows: In Section 2, we review key properties of stochastic Volterra equations with convolutive kernels, including results on existence, moment control, and a special focus on processes with affine drift. In this setting, specific analytical tools become available, such as the *resolvent* and the solution of the *Wiener–Hopf equation*. Section 3 investigates the conditions under which SVIEs (1.1) with affine drift admit a weak stationary regime, in the spirit of [36], in a setting where the volatility coefficient is separable in time and state. The time-dependent (deterministic) multiplicative function, referred to as the *stabilizer*, appears in the Brownian convolution and serves to regulate or control the volatility of the process. In the *fake stationarity* regime, this stabilizing function is characterized as the solution to an intrinsic convolution equation involving the derivative of the resolvent associated with the Volterra kernel. Next, in Section 3.3, we provide an example of a *fake stationary regime of order* $p = 2$ when the state-depedndent diffusion coefficient is a trinomial function. It follows in Section 4 the analysis of the confluence and long-run behavior of these time-inhomogenous processes as time tends to infinity. Specifically, we investigate, for such stabilized processes, the functional weak asymptotics of the time-shifted process $(X_{t+s})_{s\geq 0}$ as $t \to +\infty$, which turns out to be a weakly L^2-stationary process. Finally, in Section 5, we apply these results to the case of SVIEs with an α-fractional integration kernel for $\alpha \in \left(1, \frac{3}{2}\right)$ (*long-term memory, persistence or long range dependence*), where the case $\alpha \in \left(\frac{1}{2}, 1\right)$ has been extensively studied in [36, Section 5, Theorem 5.2]. In Section 6, we further extend the application to SVIEs with an α -exponential fractional integration kernel for $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$ involving both the *rough/short memory* and *long-term memory* effects inherent to Volterra equations.

Notations.

• Denote $\mathbb{T} = [0, T] \subset \mathbb{R}_+$, Leb_d the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$, $\mathbb{H} := \mathbb{R}^d$, etc.

• $\mathbb{X} := \mathcal{C}([0,T],\mathbb{H})$(resp. $\mathcal{C}_0([0,T],\mathbb{H})$) denotes the set of continuous functions(resp. null at 0) from $[0,T]$ to $\mathbb{H}$ and $\mathcal{B}or(\mathcal{C}_d)$ denotes the Borel σ-field of $\mathcal{C}_d$ induces by the sup-norm topology.

• For $p \in (0,+\infty)$, $L^p_{\mathbb{H}}(\mathbb{P})$ or simply $L^p(\mathbb{P})$ denote the set of $\mathbb{H}$-valued random vectors X defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ such that $\|X\|_p := (\mathbb{E}[\|X\|^p_{\mathbb{H}}])^{1/p} < +\infty$. For $f : E \to \mathbb{R}$, $\|f\|_{\sup} = \sup_{x\in E} |f(x)|$

• For $f, g \in \mathcal{L}^1_{\mathbb{R}_+,loc}(\mathbb{R}_+, \mathrm{Leb}_1)$, we define their convolution by $f * g(t) = \int_0^t f(t-s)g(s)ds$, $t \geq 0$.

• For $f, g \in \mathcal{L}^2_{\mathbb{R}_+,loc}(\mathbb{R}_+, \mathrm{Leb})$ and W a Brownian motion, we define their stochastic convolution by

$$f \overset{W}{*} g = \int_0^t f(t-s)g(s)dW_s, \quad t \geq 0.$$

• For a random variable/vector/process X, we denote by $L(X)$ or $[X]$ its law or distribution.

• $X \perp\!\!\!\perp Y$ stands for independence of random variables, vectors or processes X and Y.

• $\Gamma(a) = \int_0^{+\infty} u^{a-1}e^{-u}\,du$, $a > 0$, and $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du$, $a, b > 0$. We will extensively use the classical identities: $\Gamma(a+1) = a\,\Gamma(a)$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

2 Background on Stochastic Volterra equations with convolutive kernels

We will assume that, the process $(X_t)_{t\geq 0}$ takes values in $\mathbb{R}$, i.e. $\mathbb{H} = \tilde{\mathbb{H}} = \mathbb{R}$ and $\mathbb{X} := \mathcal{C}([0,T],\mathbb{R})$.

Definition 2.1 (Convolutive kernel and Volterra equations)**.** *A kernel $K : \{(s,t) \in \mathbb{R}^2_+ : 0 \leq s < t\} \to \mathbb{R}_+$ satisfying $\forall\, s, t \geq 0,\ s < t, \quad K(s,t) = K(0, t-s)$ is called a* convolutive kernel*. A Volterra equation based on a convolutive kernel is called a* convolutive Volterra equation*.*

To alleviate notations, we denote from now on $K(t) := K(0,t)$ so that $K(s,t) = K(t-s)$. For convenience we also extend the function $K : \mathbb{R}_+ \to \mathbb{R}_+$ to the whole real line by setting $K(t) = 0$, $t \leq 0$.

2.1 Volterra processes with convolutive kernels

A significant difference between regular diffusion processes and Volterra processes from a technical viewpoint comes from the presence of the kernels which introduces some memory in the dynamics of the process, depriving us of the Markov property and usual tools of stochastic calculus. We are interested in the convolutive stochastic Volterra equation:

$$X_t = X_0\phi(t) + \int_0^t K(t-s)b(s,X_s)ds + \int_0^t K(t-s)\sigma(s,X_s)dW_s, \quad t \geq 0. \tag{2.2}$$

where $b : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are Borel measurable, $K \in \mathcal{L}^2_{loc,\mathbb{R}_+}(\mathrm{Leb}_1)$ is a convolutive kernel and $(W_t)_{t\geq 0}$ is a standard Brownian motion independent from the $\mathbb{R}$-valued random variable X_0 both defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration (satisfying the usual conditions) such that X_0 is $\mathcal{F}_0$-measurable and W is an $(\mathcal{F}_t)$-Brownian motion independent of X_0. $X_0\phi$ is thus a random function evolving deterministically for $t > 0$, i.e. $X_0\phi$ is $\mathcal{F}_0$-measurable.

Assumption 2.2 (On Volterra Equations with convolutive kernels)**.** *Assume that the kernel K satisfies:*

$$\textit{for every } T > 0, \quad (\mathcal{K}^{int}_{T,\beta}) \qquad \exists\, \beta > 1 \quad \textit{such that} \quad K \in L^{2\beta}_{loc}(\mathrm{Leb}_1). \tag{2.3}$$

$$(\mathcal{K}^{cont}_{T,\theta})\ \exists\, \kappa_T < +\infty,\ \exists\, \theta_T > 0,\ \forall\, \delta \in (0,T),\ \sup_{t\in[0,T]} \left[\int_0^t |K\big((s+\delta)\wedge T\big) - K(s))|^2 ds\right]^{\frac{1}{2}} \leq \kappa_T\, \delta^{\theta_T}. \tag{2.4}$$

Assume b and σ satisfy the following Lipschitz-linear growth assumption uniform in time

(i) $\forall t \in [0,T],\ \forall x,y \in \mathbb{R},\ |b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \leq C_{b,\sigma,T}|x-y|$ [1],

(ii) $\sup_{t\in[0,T]} (|b(t,0)| + |\sigma(t,0)|) < +\infty$,

(iii) *Moreover, for some $\delta > 0$, for any $p > 0$ and $T > 0$,*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_0\phi(t)|^p\right) < +\infty, \quad \mathbb{E}|X_0\phi(t') - X_0\phi(t)|^p \leqslant C_{T,p}\left(1 + \mathbb{E}\left[\sup_{t\in[0,T]} |X_0\phi(t)|^p\right]\right)|t'-t|^{\delta p}.$$

Under Assumption (2.2), if $X_0 \in L^p(\mathbb{P})$ for some $p > 0$, then Equation (2.2) admits a unique pathwise continuous solution on $\mathbb{R}_+$ starting from X_0 satisfying (among other properties),

$$\forall T > 0,\ \exists C_{T,p} > 0, \quad \big\| \sup_{t\in[0,T]} |X_t| \big\|_p \leq C_{T,p}\left(1 + \sup_{t\in[0,T]} |\phi(t)| \big\| |X_0| \big\|_p\right). \tag{2.5}$$

This result appears as a generalization of the classical strong existence-uniqueness result of pathwise continuous solutions established in [30, Theorem 1.1] as an improved version of [51, Theorem 3.1 and Theorem 3.3], which holds only when the starting value X_0 has finite polynomial moments of any order (the framework is more general with a function ϕ in front of the starting value).

2.2 Fourier-Laplace transforms and Solvent core of a convolutive kernel

The Laplace transform is a valuable tool, and we provide a brief overview here, as it is particularly effective for addressing the key equation (2.2).

Let us first introduce the Laplace transform of a Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\forall t \geq 0, \quad L_f(t) = \int_0^{+\infty} e^{-tu} f(u)du \in [0,\infty].$$

This Laplace transform is non-increasing and if $L_f(t_0) < +\infty$ for some $t_0 \geq 0$, then $L_f(t) \to 0$ as $t \to +\infty$ by Lebesgue's dominated convergence theorem. One can define the Laplace transform of a Borel function $f : \mathbb{R}_+ \to \mathbb{R}$ on $(0,+\infty)$ as soon as $L_{|f|}(t) < +\infty$ for every $t > 0$ by the above formula. The Laplace transform can be extended to $\mathbb{R}_+$ as an $\mathbb{R}$-valued function if $f \in \mathcal{L}^1_{\mathbb{R}_+}(\mathrm{Leb}_1)$.
Throughout this work, we will adopt the below resolvent definition put forth in [36], which offers a distinct perspective compared to the functional resolvent introduced in [4] and also discussed or presented in works such as [2].

Let K be a convolution kernel satisfying (2.3), (2.4) and $\int_0^t K(u)du > 0$ for every $t > 0$. For every $\lambda \in \mathbb{R}$, the *resolvent or Solvent core* R_λ associated to K and λ is defined as the unique solution – if it exists – to the deterministic Volterra equation

$$\forall t \geq 0, \quad R_\lambda(t) + \lambda \int_0^t K(t-s)R_\lambda(s)ds = 1. \tag{2.6}$$

or, equivalently, written in terms of convolution, $R_\lambda + \lambda K * R_\lambda = 1$. This equation is also known as *resolvent equation* or *renewal equation*. Its solution always satisfies $R_\lambda(0) = 1$ and admits the formal *Neumann series expansion* [2]:

$$R_\lambda = \sum_{k\geq 0} (-1)^k \lambda^k (\mathbf{1} * K^{k*}). \tag{2.7}$$

1. By the Lipschitz-continuity of b and σ in x, uniformly in t, we have $|b(t,x)| \leq K(1 + |b(t,0)| + |x|)$ for some constant K and likewise for σ i.e. b and σ are of linear growth in the sense that there exists a constant $C > 0$ such that:

$$\forall t \in [0,T],\ \forall x \in \mathbb{R},\ |b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|).$$

2. Recall that $K^{1*} = K$ and $K^{k*}(t) = \int_0^t K(t-s) \cdot K^{(k-1)*}(s)\, ds$.

where, K^{k*} denotes the k-th convolution of K or the k-fold $*$ product of k with itself, with the convention in this formula, $K^{0*} = \delta_0$ (Dirac mass at 0). From now on we will assume that the kernel K has a finite Laplace transform $L_K(t) < +\infty$. Note that, as mentioned in [36], if the (non-negative) kernel K satisfies

$$0 \leq K(t) \leq Ce^{bt}t^{a-1} \text{ for some } \; a, \; b, \; C > 0 \in \mathbb{R}_+. \tag{2.8}$$

then, by induction $\mathbf{1} * K^{*n}(t) \leq C^n e^{bt}\frac{\Gamma(a)^n}{\Gamma(an+1)}t^{an}$, so that for such kernels, the above series (2.7) is absolutely converging for every $t > 0$ implying that the function R_λ is well-defined on $(0, +\infty)$.

Remark 1. If K is regular enough (say continuous) the resolvent R_λ is differentiable and one checks that $f_\lambda = -R'_\lambda$ satisfies, for every $t > 0$, $-f_\lambda(t) + \lambda\big(R_\lambda(0)K(t) - K * f_\lambda(t)\big) = 0$ that is f_λ is solution to the equation

$$f_\lambda + \lambda K * f_\lambda = \lambda K. \tag{2.9}$$

2. Taking the Laplace transform from both side of the above equality (2.9), we have that : $L_{f_\lambda}(t)(1 + \lambda L_K(t)) = \lambda L_K(t)$, $t > 0$. Consequently, $L_{f_\lambda}(t) = \frac{\lambda L_K(t)}{1+\lambda L_K(t)}$ so that, for $\lambda \geq 0$, $L_{f_\lambda}(t) \equiv 0$ if and only if $L_K(t) \equiv 0$ i.e. if and only if $K = 0$ by the injectivity of Laplace transform.

3. If $\lim\limits_{t\to+\infty} R_\lambda(t) = 0$ then, one also has that $\int_0^{+\infty} f_\lambda(t)dt = 1 - R_\lambda(+\infty) = 1$.Moreover, if R_λ turns out to be non-increasing, then f_λ is non-negative and satisfies $0 \leq f_\lambda \leq \lambda K$, so that f_λ *is a probability density.*

Example 2.3 (Laplace transform and $\lambda-$ Resolvent associated to the Exponential-fractional Kernel)**.** *The Laplace transform associated to a kernel K always exists and reads, for $t > 0$ $L_K(t) := \int_0^{+\infty} e^{-tu}K(u)du$. When K is the Gamma kernel $K_{b,\alpha,\rho}(t) := be^{-\rho t}\frac{t^{\alpha-1}}{\Gamma(\alpha)} \cdot \mathbf{1}_{(0,\infty)}(t)$, for $b > 0, \alpha > 0$ and $\rho > 0$, then by introducing $v = u(t+\rho)$, we have*

$$L_{K_{b,\alpha,\rho}}(t) = \int_0^\infty be^{-(t+\rho)u}\frac{u^{\alpha-1}}{\Gamma(\alpha)}du = \frac{b(t+\rho)^{-\alpha}}{\Gamma(\alpha)}\int_0^\infty e^{-v}v^{\alpha-1}dv = b(t+\rho)^{-\alpha}.$$

Moreover, one checks that these kernels also satisfy (2.3) *and* (2.4) *for $\alpha > 1/2$ (with $\theta_T = (\alpha - \frac{1}{2}) \wedge 1$) and trivially* (2.8)*. For simplification, assume that $b = 1$. It follows from the easy identity $K_{\alpha,\rho} * K_{\alpha',\rho} = K_{\alpha+\alpha',\rho}$ and the Neumann series expansion provided in equation* (2.7) *that the resolvent reads:*

$$R_{\alpha,\rho,\lambda}(t) = (1 * \delta_0)(t) + \sum_{k\geq 1}(-1)^k\lambda^k(\mathbf{1} * K_{\alpha,\rho}^{(k*)}) = \mathbf{1}_{\mathbb{R}_+}(t) + \sum_{k\geq 1}(-1)^k\lambda^k\int_0^t \frac{e^{-\rho s}s^{k\alpha-1}}{\Gamma(k\alpha)}ds. \tag{2.10}$$

Hence, if $\lambda > 0$, we define the function $f_{\alpha,\rho,\lambda} := -R_{\alpha,\rho,\lambda}$ on $(0, +\infty)$ by:

$$f_{\alpha,\rho,\lambda}(t) = -\frac{d}{dt}R_{\alpha,\rho,\lambda}(t) = -\sum_{k\geq 1}(-1)^k\lambda^k\frac{e^{-\rho t}t^{k\alpha-1}}{\Gamma(k\alpha)} = \lambda e^{-\rho t}t^{\alpha-1}\sum_{k\geq 0}(-1)^k\lambda^k\frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}. \tag{2.11}$$

2.3 Application to the Wiener-Hopf equation

Proposition 2.4 (Wiener-Hopf and Resolvent equations)**.** *Let $g, h : \mathbb{R}_+ \to \mathbb{R}$ be two locally bounded Borel function, let $K \in L^1_{loc}(Leb_{\mathbb{R}_+})$ and let $\lambda \in \mathbb{R}$. Assume that the λ-resolvent R_λ of K is differentiable on $(0, +\infty)$ with a derivative $R'_\lambda \in L^1_{loc}(Leb_{\mathbb{R}_+})$, that both R_λ and R'_λ admit a finite Laplace transform on $\mathbb{R}_+$ and $\lim\limits_{u\to+\infty} e^{-tu}R_\lambda(u) = 0$ for every $t > 0$. Then,*

(a) *The Wiener-Hopf equation $\forall\, t \geq 0, \quad x(t) = g(t) - \lambda\int_0^t K(t-s)x(s)ds$ (also reading $x = g - \lambda K * x$) has a solution given by:*

$$\forall\, t \geq 0, \quad x(t) = g(t) + \int_0^t R'_\lambda(t-s)g(s)ds \quad \textit{or equivalently,} \quad x = g - f_\lambda * g,$$

where $f_\lambda = -R'_\lambda$. This solution is uniquely defined on $\mathbb{R}_+$ up to dt-a.e. equality.

(b) *The integral equation* $\forall\, t \geq 0, \quad x(t) = h(t) - \int_0^t R'_\lambda(t-s)x(s)ds \quad$ *where* $\quad R'_\lambda = -f_\lambda$ *(also reading* $x = h - R'_\lambda * x$*) has a solution given by:*

$\forall\, t \geq 0, \quad x(t) = h(t) + \lambda \int_0^t K(t-s)h(s)ds \quad$ *or equivalently,* $\quad x = h + \lambda K * h.$

This solution is uniquely defined on $\mathbb{R}_+$ *up to* dt*-a.e. equality.*

In Appendix B, we provide a proof of this classical result for the reader's convenience.

3 Investigating stationarity of a scaled stochastic Volterra Integral equation

From now we focus on the special case of a *scaled* stochastic Volterra equation associated to a convolutive kernel $K : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (2.3) and (2.4):

$$X_t = X_0\phi(t) + \int_0^t K(t-s)(\mu(s) - \lambda X_s)ds + \int_0^t K(t-s)\sigma(s, X_s)dW_s, \quad X_0 \perp\!\!\!\perp W. \tag{3.12}$$

where $\lambda > 0$, $\mu : \mathbb{T}_+ \to \mathbb{R}$ is a bounded Borel function (hence having a well-defined finite Laplace transform on $(0, +\infty)$) and $\sigma : \mathbb{T}_+ \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in x, locally uniformly in $t \in \mathbb{T}_+$. Note that the drift $b(t, x) = \mu(t) - \lambda x$ is clearly Lipschitz continuous in x, uniformly in $t \in \mathbb{T}_+$. Then, Equation (3.12) has a unique solution $(X_t)_{t\geq 0}$ adapted to $\mathcal{F}_t^{X_0,W}$, starting from $X_0 \in L^p(\mathbb{P}), p > 0$. This follows by applying the existence Theorem of [51, 30]to each time interval $[0, T]$, $T \in \mathbb{N}$, and gluing the solutions together, utilizing the uniform linear growth of the drift and σ in time.

Note that under our assumptions, if $p > 0$ and $\mathbb{E}[|X_0|^p] < +\infty$, then $\mathbb{E}[\sup_{t\in[0,T]} |X_t|^p] < C_T(1 + \|\phi\|_T^p \mathbb{E}[|X_0|^p]) < +\infty$ for every $T > 0$ (see [30, Theorem 1.1]). Combined with $|\sigma(t, x)| \leq C'_T(1 + |x|)$ for $t \in [0, T]$, this implies $\mathbb{E}[\sup_{t\in[0,T]} |\sigma(t, X_t)|^p] < C'_T(1 + \|\phi\|_T^p \mathbb{E}[|X_0|^p]) < +\infty$ for every $T > 0$, enabling the unrestricted use of both regular and stochastic [3] Fubini's theorems.
Sufficient conditions for interchanging the order of ordinary integration (with respect to a finite measure) and stochastic integration (with respect to a square integrable martingale) are provided in [46, Thm. 1], and further details can be found in [39, Thm. IV.65].(see also [48, Theorem 2.6], [47, Theorem 2.6])

We will always work under the following assumption.

Assumption 3.1 (λ-resolvent R_λ of the kernel)**.** *Throughout the paper, we assume that the* λ*-resolvent* R_λ *of the kernel* K *satisfies the following for every* $\lambda > 0$*:*

$$(\mathcal{K}) \quad \begin{cases} (i) & R_\lambda(t) \text{ is differentiable on } \mathbb{R}^+,\ R_\lambda(0) = 1 \text{ and } \lim_{t\to+\infty} R_\lambda(t) = a \in [0, 1[, \\ (ii) & f_\lambda \in \mathcal{L}^2_{loc}(\mathbb{R}_+, Leb_1), \text{ where we set } \quad f_\lambda := -R'_\lambda \text{ for } t > 0,\ L_{f_\lambda}(t) \neq 0\ dt - a.e., \\ (iii) & \phi \in \mathcal{L}^1_{\mathbb{R}_+}(Leb_1), \text{ is a continuous function satisfying } \lim_{t\to\infty} \phi(t) = \phi_\infty, \text{ with } a\phi_\infty < 1, \\ (iv) & \mu \text{ is a } C^1\text{-function such that } \|\mu\|_{\sup} < \infty \text{ and } \lim_{t\to+\infty} \mu(t) = \mu_\infty \in \mathbb{R}. \end{cases} \tag{3.13}$$

Under assumptions $\mathcal{K}$ (i) and (ii), f_λ is a $(1 - a)$-sum measure, i.e., $\int_0^{+\infty} f_\lambda(s)\, ds = 1 - a$. In fact,
$\int_0^{+\infty} f_\lambda(s)\, ds = [1 - R_\lambda(s)]_{s=0}^{s=+\infty} = -\lim_{s\to+\infty} R_\lambda(s) + R_\lambda(0) = 1 - a$

Lemma 3.1. *Assume that assumption* $(\mathcal{K})$ (ii) *holds, then* $\lim_{t\to+\infty} \int_0^t f_\lambda(t-s)\mu(s)ds = \mu_\infty(1 - a)$ *and* $\lim_{t\to+\infty} \phi(t) - (f_\lambda * \phi)(t) = \phi_\infty\, a$.

For clarity and conciseness, the proof of the above Lemma is postponed to Appendix B, where the main technical results are presented.

3. Interchangeability of Lebesgue and stochastic integration.

Proposition 3.2 (Wiener-Hopf transform)**.** *Let $\lambda > 0$ and let $\mu : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function. Assume the kernel K satisfies the above assumptions $(\mathcal{K})$, (2.3) and (2.4) from Assumption 2.2 and its λ-resolvent R_λ is well-defined on $(0, +\infty)$. Then, the solution $(X_t)_{t\geq 0}$ of the Volterra equation* (3.12) *also satisfies:*

$$X_t = X_0\Big(\phi(t) - \int_0^t f_\lambda(t-s)\phi(s)\,ds\Big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\mu(s)\,ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\,dW_s. \quad (3.14)$$

Conversely, any process satisfying (3.14) *also satisfies the original Volterra equation* (3.12)*. Thus, the two formulations are equivalent.*

Proof. Equation (3.12) can be interpreted pathwise as a Wiener-Hopf equation with $x(t) = X_t(\omega)$ and

$$g(t) = X_0(\omega)\phi(t) + (\mu * K)_t + \Big(K \overset{W}{*} \sigma(., X_.(\omega))\Big)_t.$$

This leads to the following expression for X_t:

$$\begin{aligned} X_t &= g(s) + \int_0^t R_\lambda'(t-s)g(s)\,ds = X_0\phi(t) + (\mu * K)_t + \big(K \overset{W}{*} \sigma(., X_.)\big)_t \\ &+ \int_0^t R_\lambda'(t-s)\Big[X_0\phi(s) + (\mu * K)_s + \big(K \overset{W}{*} \sigma(., X_.)\big)_s\Big]ds = X_0\phi(t) + (\mu * K)_t + \Big(K \overset{W}{*} \sigma(., X_.)\Big)_t \\ &+ X_0\int_0^t R_\lambda'(t-s)\phi(s)\,ds + \underbrace{\int_0^t R_\lambda'(t-s)(\mu * K)_s\,ds}_{(a)} + \underbrace{\int_0^t R_\lambda'(t-s)\Big(K \overset{W}{*} \sigma(., X_.)\Big)_s\,ds}_{(b)}. \end{aligned}$$

Using commutativity and associativity (via regular Fubini's theorem) of convolution, we obtain for (a):

$$(a) = -f_\lambda * (\mu * K)_t = -\left((f_\lambda * K) * \mu\right)_t. \quad (3.15)$$

Differentiating Equation (2.6) yields the identity $-f_\lambda * K = \frac{1}{\lambda} f_\lambda - K$, which, upon substitution into (3.15), leads to the following expression in (3.16) for term (a). For term (b), owing to stochastic Fubini's theorem, equation (2.9) provides the below expression in (3.16).

$$(a) = \frac{1}{\lambda}(f_\lambda * \mu)_t - (K * \mu)_t, \quad (b) = \frac{1}{\lambda}\Big(f_\lambda \overset{W}{*} \sigma(\cdot, X_.)\Big)_t - \Big(K \overset{W}{*} \sigma(\cdot, X_.)\Big)_t. \quad (3.16)$$

Substituting (3.16) into (3.12), finally yields

$$X_t = X_0(\phi(t) - (f_\lambda * \phi)_t) + \tfrac{1}{\lambda}(f_\lambda * \mu)_t + \tfrac{1}{\lambda}\Big(f_\lambda \overset{W}{*} \sigma(\cdot, X_.)\Big)_t,$$

The controverse is obtained by solving the corresponding Wiener-Hopf equation. We convolve both sides of Equation (3.14) with the kernel K, using regular and stochastic Fubini's theorem. Details are left to the reader.

Remark 3.3. *1. Notably, in the Markovian case, the Wiener–Hopf equation amounts to applying Itô's lemma to the transformed process $e^{\lambda t}X_t$. In fact, if $K(t) = \mathbf{1}$ in the volterra equation, then $R_\lambda(t) = e^{-\lambda t}$ and $f_\lambda(t) = \lambda e^{-\lambda t}$, so that the above computation corresponds to Itô' s Lemma applied to $e^{\lambda t}X_t$.*

2. Note that if the solution $(X_t)_{t\geq 0}$ is stationary[4]*, and $X_0 \in L^2(\mathbb{P})$, then both the mean and variance of X_t are constant functions of t. Furthermore, the expectations of any function of X_t that grows at most quadratically (see see* (2.5)*) also remain constant. Typically, such is the case of $x \mapsto x$, $x \mapsto x^2$.*

4. In the sense that the shifted processes $(X_{t+u})_{u\geq 0}$ and $(X_u)_{u\geq 0}$ have the same distribution when viewed on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$.

3.1 Towards stationarity of First Moments.

Before investigating the stationary regime of the "scaled" stochastic Volterra equation (3.12), we first determine under which conditions this equation has a constant first moments.

3.1.1 Stationarity of the Mean

We begin by identifying the conditions under which the Volterra Equation (3.12) exhibits a constant mean; that is, when $\mathbb{E}[X_t] = \mathbb{E}[X_0]$ for all $t \geq 0$, assuming that $X_0 \in L^2(\mathbb{P})$. We know that: $\mathbb{E}\left[\left(\int_0^t f_\lambda(t-s)\sigma(s,X_s)dW_s\right)^2\right] = \int_0^t f_\lambda^2(t-s)\mathbb{E}[|\sigma(s,X_s)|^2]\,ds \leq C(1+\|\phi\|_T^2\mathbb{E}[|X_0|^2])\int_0^t f_\lambda^2(u)\,du < +\infty$, which implies $\mathbb{E}\left[\int_0^t f_\lambda(t-s)\sigma(s,X_s)dW_s\right] = 0$. Thus, we have

$$\forall t \geq 0, \quad \mathbb{E}[X_t] = (\phi(t) - (f_\lambda * \phi)_t)\mathbb{E}[X_0] + \frac{1}{\lambda}(f_\lambda * \mu)_t. \tag{3.17}$$

Thus, $\mathbb{E}[X_t]$ is constant if and only if the following condition holds:

$$\forall t \geq 0, \quad \mathbb{E}[X_0]\Big(1 - \phi(t) + (f_\lambda * \phi)_t\Big) = \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\,\mu(s)\,ds. \tag{3.18}$$

Proposition 3.4 (Stationarity of the first moment)**.** *Let $(X_t)_{t\geq 0}$ be a solution to the scaled Volterra equation (3.12) starting from $X_0 \in L^1(\Omega,\mathcal{F},\mathbb{P})$, with $\lambda > 0$ and $\mu_\infty \in \mathbb{R}$. Then the Volterra process $(X_t)_{t\geq 0}$ has constant first moment, if and only if*

$$\mathbb{E}[X_0] = \frac{1-a}{1-a\phi_\infty}\frac{\mu_\infty}{\lambda} := x_\infty \quad \textit{and} \quad \forall\, t \geq 0, \quad \phi(t) = 1 - \lambda\int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda x_\infty} - 1\right)ds. \tag{3.19}$$

so that the equation reads:

$$X_t = X_0 - \frac{1}{\lambda x_\infty}\Big(X_0 - x_\infty\Big)\int_0^t f_\lambda(t-s)\mu(s)ds + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\varsigma(s)\sigma(X_s)dW_s. \tag{3.20}$$

Proof. Case 1 $\mathbb{E}[X_0] = 0$: In this case, equation (3.18) reads: $(f_\lambda * \mu)_t = 0$. By taking the limit in both side and owing to Lemma 3.1, we have $\mu_\infty = 0$. Taking the Laplace transform and owing to assumption $\mathcal{K}(ii)$, we $\mu(t) = 0\ dt - a.e.$ and since μ is C^1 owing to $\mathcal{K}(iv)$, we have $\mu \equiv 0$. In this case, from equation (3.17), we deduce that $\forall t \geq 0, \quad \mathbb{E}[X_t] = \mathbb{E}[X_0] = 0$.

Case 2 $\mathbb{E}[X_0] \neq 0$: In this case, equation (3.18) reads: $\phi(t) - (f_\lambda * \phi)(t) = 1 - \frac{1}{\lambda}(f_\lambda * \frac{\mu}{\mathbb{E}[X_0]})_t$. We may read the above equation as a Wiener-Hopf equation with $x(t) = \phi(t)$ and $h(t) = 1 - \frac{1}{\lambda}(f_\lambda * \frac{\mu}{\mathbb{E}[X_0]})_t$. Then, applying the claim (b) of Proposition 2.4, we get: $\phi(t) = h(t) + \lambda(K * h)_t$. That is:

$$\begin{aligned}\phi(t) &= 1 - (f_\lambda * \frac{\mu}{\lambda\mathbb{E}[X_0]})_t + \lambda(K*1)_t - (K * f_\lambda * \frac{\mu}{\mathbb{E}[X_0]})_t = 1 - \left((f_\lambda + \lambda K * f_\lambda) * \frac{\mu}{\lambda\mathbb{E}[X_0]}\right)_t + \lambda(K*1)_t \\ &\overset{(2.9)}{=} 1 - (K * \frac{\mu}{\mathbb{E}[X_0]})_t + \lambda(K*1)_t = 1 - \lambda\int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda\mathbb{E}[X_0]} - 1\right)ds.\end{aligned}$$

Moreover, by taking the limit in both side of the equality (3.18), we have:

$\mathbb{E}[X_0](1 - a\phi_\infty) := \lim_{t\to+\infty}\mathbb{E}[X_0]\Big(1 - (\phi(t) - (f_\lambda * \phi)_t)\Big) = \lim_{t\to+\infty}\frac{1}{\lambda}(f_\lambda * \mu)_t =: \frac{\mu_\infty}{\lambda}(1-a)$

owing to Lemma 3.1 so that, $\mathbb{E}[X_0] = \frac{1-a}{1-a\phi_\infty}\frac{\mu_\infty}{\lambda} := x_\infty$. Therefore, $\phi(t) = 1 - \lambda\int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda x_\infty} - 1\right)ds$ and $\mathbb{E}[X_0] = x_\infty$. Conversely, as $\phi(t) - (f_\lambda * \phi)(t) = 1 - \frac{1}{\lambda}(f_\lambda * \frac{\mu}{\mathbb{E}[X_0]})_t$, equation (3.17) gives:

$$\forall t \geq 0, \quad \mathbb{E}[X_t] = x_\infty\Big(\phi(t) - (f_\lambda * \phi)_t\Big) + \tfrac{1}{\lambda}(f_\lambda * \mu)_t = x_\infty$$

Thus a necessary and sufficient condition for constant mean is:

$$\mathbb{E}[X_0] = x_\infty,\ \phi(t) - (f_\lambda * \phi)(t) = 1 - \frac{(f_\lambda * \mu)(t)}{\lambda x_\infty} \text{ i.e. } \phi(t) = 1 - \lambda \int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda x_\infty} - 1\right) ds. \quad (3.21)$$

Then Equation (3.14) can be rewritten as (3.20) We can easily check that $\phi(0) = 1$. However, if $\phi(t) \equiv C^{\text{ste}} \equiv 1, (\phi_\infty = 1)$, then by (3.21), we have $\int_0^t K(t-s)\left(\frac{\mu(s)}{\lambda x_\infty} - 1\right) ds \equiv 0\ \forall t \geq 0$, which reduces to the Laplace transform equation $L_K \cdot L_{\frac{\mu(\cdot)}{\lambda x_\infty} - 1} \equiv 0$. Since $L_K(t) > 0\ \forall t \geq 0$ as $K > 0$, we have $L_{\frac{\mu(\cdot)}{\lambda x_\infty} - 1} \equiv 0$ i.e. $\frac{\mu(\cdot)}{\lambda x_\infty} - 1 \equiv 0$ i.e. $\forall t \geq 0, \mu(t) = C^{\text{ste}} = \mu_\infty$. Consequently, the mean is stationary, with the following conditions:

$$\phi(t) = 1 \quad \text{almost surely}, \quad \mu(t) = \mu_\infty \quad \text{almost surely}, \quad \mathbb{E}[X_0] = \tfrac{\mu_\infty}{\lambda}.$$

Conversely, these conditions guarantee that the mean of X_t is constant over time and we recover the case studied in [36]. In the following, we will assume the more general case: (3.21) □

3.1.2 Towards stationarity of the variance

We deduce from the beginning of the section(3) that, the non-negative function defined by

$$t \longmapsto \Xi^2(t) := \mathbb{E}\,\sigma^2(t, X_t), \quad t \geq 0.$$

is locally bounded on $\mathbb{R}_+$ since σ has at most linear growth in space, locally uniformly in $t \geq 0$. To take advantage of this formula, we need to assume that *a priori* $\Xi \in L^2_{loc}(\mathbb{R}_+, \mathrm{Leb}_1)$. First noting that by assuming constant mean as in the above section, i.e. $\forall\, t \geq 0, \mathbb{E}\, X_t = \mathbb{E}\, X_0 = x_\infty$, equation (3.14) reads:

$$X_t - x_\infty = \Big(X_0 - x_\infty\Big)(\phi - f_\lambda * \phi)(t) + \tfrac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s, X_s)\, dW_s.$$

By Itô's isomorphism and Fubini's Theorem

$$\mathrm{Var}\Big(\int_0^t f_\lambda(t-s)\sigma(s, X_s)\, dW_s\Big) = \mathbb{E}\Big[\int_0^t f_\lambda(t-s)\sigma(s, X_s)\, dW_s\Big]^2 = \int_0^t f_\lambda(t-s)^2 \Xi^2(s) ds = (f_\lambda^2 * \Xi^2)(t).$$

Then, it follows from the above equation that: $\forall\, t \geq 0$, by setting $v_0 = \mathrm{Var}(X_0)$, we have:

$$\mathrm{Var}(X_t) = \mathbb{E}\Big[(X_t - x_\infty)^2\Big] = \mathbb{E}\Big[(X_0 - x_\infty)^2\Big](\phi - f_\lambda * \phi)^2(t) + \frac{1}{\lambda^2}\int_0^t f_\lambda(t-s)^2 \Xi^2(s) ds = v_0(\phi - f_\lambda * \phi)^2(t) + \frac{1}{\lambda^2}(f_\lambda^2 * \Xi^2)(t)$$

$$\text{i.e.} \quad \forall\, t \geq 0, \quad \mathrm{Var}(X_t) = v_0(\phi - f_\lambda * \phi)^2(t) + \frac{1}{\lambda^2}(f_\lambda^2 * \Xi^2)(t). \quad (3.22)$$

Examples. ▷ *The case of equation* (3.20) *reads easily owing to* $(\phi - f_\lambda * \phi)(t) = 1 - \frac{(f_\lambda * \mu)_t}{\lambda \mathbb{E}[X_0]} = 1 - \frac{(f_\lambda * \mu)_t}{\lambda x_\infty}$

$$\forall\, t \geq 0, \quad \mathrm{Var}(X_t) = v_0\Big(1 - \tfrac{(f_\lambda * \mu)_t}{\lambda x_\infty}\Big)^2 + \tfrac{1}{\lambda^2}(f_\lambda^2 * \Xi^2)(t).$$

Now, assume a time homogenous or autonomous volatility coefficient, i.e. $\forall\, (t, x) \in \mathbb{T} \times \mathbb{R},\ \sigma(t, x) = \sigma(x)$.

As discussed in Remark 3.3 (2), if the solution $(X_t)_{t \geq 0}$ is stationary and $X_0 \in L^2(\mathbb{P})$ then:

$$\forall\, t \geq 0, \quad \mathbb{E}\, X_t = c^{ste} = x_\infty, \quad \mathrm{Var}(X_t) = c^{ste} = v_0 \geq 0 \quad \text{and} \quad \bar{\sigma}^2(t) := \mathbb{E}\, \sigma^2(X_t) = c^{ste} := \bar{\sigma}^2 \geq 0.$$

so that from equation (3.22) together with the fact that, here $\Xi^2 = \bar{\sigma}^2$, we have:

$$\forall\, t \geq 0, \quad v_0 = \mathrm{Var}(X_t) = v_0(\phi - f_\lambda * \phi)^2(t) + \tfrac{1}{\lambda^2}(f_\lambda^2 * \bar{\sigma}^2)(t) = v_0(\phi - f_\lambda * \phi)^2(t) + \tfrac{\bar{\sigma}^2}{\lambda^2}\int_0^t f_\lambda^2(s) ds$$

or, equivalently

$$\forall\, t \geq 0, \quad v_0\big(1 - (\phi - f_\lambda * \phi)^2(t)\big) = \frac{\bar{\sigma}^2}{\lambda^2}\int_0^t f_\lambda^2(s) ds. \quad (3.23)$$

Consequently,

(*i*) If $\bar{\sigma}^2 = 0$, we get $v_0 = 0$ since $\lim_{t\to+\infty}(\phi - f_\lambda * \phi)(t) = a\phi_\infty < 1 \Rightarrow (\phi - f_\lambda * \phi)(t) \neq 1$ (at least for t large enough). As a consequence, $\mathrm{Var}(X_t) = 0$ for every $t \geq 0$. But, we know that $\mathbb{E}\,X_t = \mathbb{E}\,X_0 = x_\infty$, it follows that $X_t = x_\infty$ $\mathbb{P}$-*a.s.*.

(*ii*) If $\bar{\sigma} > 0$, using equation (3.21) and differentiating this equality implies, owing to $\mathcal{K}$ (*iv*) and Lemma 3.9 (2):

$$\frac{\kappa}{\lambda x_\infty}\left(1 - \frac{(f_\lambda * \mu)_t}{\lambda x_\infty}\right)(\mu(0)f_\lambda(t) + (f_\lambda * \mu')_t) = f_\lambda^2(t) \quad \textit{where} \quad \kappa = 2\frac{\lambda^2 v_0}{\bar{\sigma}^2}.$$

Thus the kernel K must be the function such that its Laplace transform is given (from equation (2.6)) by $L_K(t) = \frac{1}{\lambda}\left(\frac{1}{tL_{R_\lambda}(t)} - 1\right)$ where $f_\lambda := R'_\lambda$ is a solution (if exists any) of the above functional equation. However, in the particular case $\phi \equiv 1$ i.e. $\forall t \geq 0,\ \mu(t) = \mu_\infty$, a.s., as shown in [36], the kernel K is necessary constant, in which case $(X_t)_{t\geq 0}$ is a (Markov) Brownian diffusion process with constant mean and variance, thus *true Volterra equations with non constant kernels are never stationary.*

From now on, we will assume that the volatility coefficient $\sigma(t,x)$ is time-dependent or inhomogenous defined by:

$$\forall\,(t,x) \in \mathbb{T}_+ \times \mathbb{R}, \qquad \sigma(t,x) = \varsigma(t)\sigma(x) \quad \varsigma(t), \sigma(x) > 0.$$

where ς is a (locally) bounded Borel function to be specified later. We assume that the kernel K satisfies equations (2.3) and (2.4) of Assumption 2.2 and σ is *Lipschitz continuous.* As ς is a (locally) bounded Borel function, the scaled Volterra equation (3.12) has a unique $(\mathcal{F}_t^{X_0,W})_{t>0}$-adapted pathwise continuous solution starting from $X_0 \in L^2(\mathbb{P})$ independent of W (still owing to [51, Theorem 3.3.]

Still as a consequence of Remark 3.3 (2), if the solution $(X_t)_{t\geq 0}$ of the Volterra equation (3.12) starting by $X_0 \in L^2(\mathbb{P})$ is stationary, then:

$$\forall\,t \geq 0,\ \mathbb{E}\,X_t = c^{ste} = x_\infty,\ \mathrm{Var}(X_t) = c^{ste} = v_0 \geq 0 \ \text{ and } \ \bar{\sigma}^2(t) := \mathbb{E}\,\sigma^2(X_t) = c^{ste} := \bar{\sigma}_0^2 \geq 0.$$

The theorem below shows what are the consequences of these three constraints in this settings.

Theorem 3.5 (Time-dependent or inhomogenous diffusion coefficient σ)**.** *Let $\sigma(t,x) = \varsigma(t)\sigma(x)$ in the equation* (3.12)*, and assume that $X_0 \in L^2(\mathbb{P})$ with $\mathbb{E}[X_0] = x_\infty$. Suppose the following conditions hold for all $t \geq 0$:*

$$\mathbb{E}[X_t] = x_\infty, \quad Var(X_t) = v_0 \geq 0, \quad \textit{and} \quad \bar{\sigma}^2(t) = \mathbb{E}[X_t] = \bar{\sigma}_0^2 \geq 0.$$

Then, a necessary condition for these relations to be satisfied is that the triplet $(v_0, \bar{\sigma}_0^2, \varsigma(t))$ satisfies the following functional equation *:*

$$(E_{\lambda,c}): \quad \forall\,t \geq 0, \quad c\lambda^2\big(1-(\phi-f_\lambda*\phi)^2(t)\big) = (f_\lambda^2*\varsigma^2)(t) \quad \textit{where} \quad c = \frac{v_0}{\bar{\sigma}_0^2} \quad \textit{and thus} \quad \varsigma = \varsigma_{\lambda,c}. \tag{3.24}$$

Remark With equation (3.21), $(E_{\lambda,c})$ in (3.24) can also be re-written as follows:

$$(E_{\lambda,c}): \quad \forall\,t \geq 0, \quad c\lambda^2\left(1 - \left(1 - \frac{(f_\lambda * \mu)_t}{\lambda x_\infty}\right)^2\right) = (f_\lambda^2 * \varsigma^2)(t) \quad \textit{where} \quad c = \frac{v_0}{\bar{\sigma}_0^2} \quad \textit{and thus} \quad \varsigma = \varsigma_{\lambda,c}.$$

Proof of Theorem 3.5. From Equation (3.22) with $\Xi^2 = \varsigma^2\bar{\sigma}^2$ and the assumption of the theorem :

$$\forall\,t \geq 0, \quad v_0 = \mathrm{Var}(X_t) = v_0(\phi - f_\lambda * \phi)^2(t) + \frac{1}{\lambda^2}(f_\lambda^2 * \bar{\sigma}^2\varsigma^2)(t) = v_0(\phi - f_\lambda * \phi)^2(t) + \frac{\bar{\sigma}^2}{\lambda^2}(f_\lambda^2 * \varsigma^2)(t)$$

or, equivalently

$$\forall\,t \geq 0, \quad v_0\big(1 - (\phi - f_\lambda * \phi)^2(t)\big) = \frac{\bar{\sigma}^2}{\lambda^2}(f_\lambda^2 * \varsigma^2)(t).$$

3.2 Stabilizer and Fake Stationary Regimes

Definition 3.6 (Stationary of Order $p \geq 1$ and Fake stationary regime of type I and II (see. [36]))**.** .

1. *The process $(X_t)_{t\geq 0}$ starting from $X_0 \in L^p(\mathbb{P})$ for $p \geq 1$, exhibit a stationary regime of order p if:*

$$\forall\,t \geq 0, \quad \forall\,k \in \{1,...,p\}, \quad \mathbb{E}\,[X_t^k] = c^{ste} = \mathbb{E}\,[X_0^k].$$

2. The process $(X_t)_{t\geq 0}$ starting from $X_0 \in L^2(\mathbb{P})$, exhibit a fake stationary regime of type I if:
$\forall\, t \geq 0, \quad \mathbb{E}\, X_t = c^{ste} = x_\infty, \quad \mathrm{Var}(X_t) = c^{ste} = v_0 \geq 0$ *and* $\bar{\sigma}^2(t) := \mathbb{E}\,\sigma^2(X_t) = c^{ste} := \bar{\sigma}^2 \geq 0.$

This is equivalent to the definition (1) above, for p=2. In fact,(see proposition 3.1), there is an equivalence between the abose last two equalities, assuming the first one.

3. The process $(X_t)_{t\geq 0}$ starting from X_0 has a fake stationary regime of type II if $(X_t)_{t\geq 0}$ has the same marginal distribution, i.e., $X_t \stackrel{d}{=} X_0$ for every $t \geq 0$. [5]

Definition 3.7. *We will call the stabilizer (or corrector) of the scaled stochastic Volterra equation* (3.12) *a bounded Borel function $\varsigma = \varsigma_{\lambda,c}$, which is a solution(if any) to the functional equation:*

$$(E_{\lambda,c}): \quad \forall\, t \geq 0, \quad c\lambda^2\big(1-(\phi-f_\lambda*\phi)^2(t)\big) = (f_\lambda^2*\varsigma^2)(t) \quad \text{where} \quad c = \frac{v_0}{\bar{\sigma}_0^2} \quad \text{and thus} \quad \varsigma = \varsigma_{\lambda,c}. \tag{3.25}$$

Remark Note that (3.25) has a solution $\varsigma_{\lambda,c}$ for some $c > 0$ if and only if it has a solution $\varsigma_{\lambda,1}$ when $c = 1$, and $\varsigma_{\lambda,c} = \sqrt{c}\varsigma_{\lambda,1}$. Hence, $(E_{\lambda,c})$ can be replaced by $(E_{\lambda,1})$ denoted (E_λ) for simplicity.

Assumption 3.8 (On the stabilizer)**.** *There exists a unique positive bounded Borel solution ς_λ on $]0,+\infty)$ of the equation (E_λ): $\quad \forall\, t > 0, \quad \lambda^2\big(1-(\phi-f_\lambda*\phi)^2(t)\big) = (f_\lambda^2*\varsigma^2)(t)$.*

Lemma 3.9 (On equation *$(E_{\lambda,c})$*: Laplace Transform of $(E_{\lambda,c})$, Uniqueness and Limit of $\varsigma^2_{\lambda,c}$)**.**

1. $$\forall\, t > 0, \quad t\, L_{f_\lambda^2}(t) L_{\varsigma^2}(t) = -2c\lambda^2 L_{(\phi-f_\lambda*\phi)(\phi-f_\lambda*\phi)'}(t). \tag{3.26}$$

2. *$(\phi - f_\lambda * \phi)'(t) = -\frac{1}{\lambda x_\infty}\left(\mu(0)f_\lambda(t) + (f_\lambda * \mu')_t\right)$ so that $(\phi - f_\lambda * \phi)'(t) \stackrel{0}{\sim} -\frac{\mu(0)}{\lambda x_\infty} f_\lambda(t)$.*

3. *c being fixed, the equation $(E_{\lambda,c})$ in* (3.25) *has at most one solution $\varsigma^2_{\lambda,c}$ in $L^1_{loc}(Leb_1)$.*

4. *For fixed c, if $\varsigma^2_{\lambda,c} \in L^1_{loc}(Leb_1)$ is the unique solution of $(E_{\lambda,c})$ in* (3.25) *and $f_\lambda \in L^2(\mathbb{R}_+, Leb_1)$, then*
$$\lim_{t\to+\infty} \varsigma^2_{\lambda,c}(t) = \frac{c\lambda^2(1-a^2\phi^2_\infty)}{\|f_\lambda\|^2_{L^2(Leb_1)}}.$$

Proposition 3.1 (Equivalence)**.** *Let $\lambda > 0$, let $\mu_\infty \in \mathbb{R}$, and let $\sigma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Let $X_0 \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ be such that $\mathbb{E}[X_0] = x_\infty$ and $Var(X_0) = v_0 \geq 0$. Set $\bar{\sigma}_0^2 = \mathbb{E}[\sigma^2(X_0)] > 0$ and set $c = \frac{v_0}{\bar{\sigma}_0^2} \in \mathbb{R}^+$. Assume the kernel K satisfies* (2.3), (2.4), *f_λ^2 has a finite Laplace transform on $(0,+\infty)$, and $(E_{\lambda,c})$ is in force.*

Then, the unique strong solution starting from X_0 of the scaled stochastic Volterra equation (3.12), *where $\varsigma_{\lambda,c}$ is a solution to* (3.25) *has constant mean and satisfies the following equivalence :*
(i) $\forall t \geq 0,\ Var(X_t) = Var(X_0) = v_0$,
(ii) $\forall t \geq 0,\ \mathbb{E}[\sigma^2(X_t)] = \mathbb{E}[\sigma^2(X_0)] = \bar{\sigma}_0^2$.

For clarity and conciseness, the proofs of Lemma 3.9 and Proposition 3.1 are deferred to Appendix B, where the main technical results are presented.

3.3 Examples of fake stationary regimes of type I and II

In this section we specify a family of scaled models where $b(t,x) = \mu(t) - \lambda\, x$ and the diffusion coefficient σ (to be specified later) satisfies the usual conditions (Lipschitz continuous) and is sufficiently regular or smooth, specifically, $\sigma \in C^3(\mathbb{R})$.

5. The distribution of X_0 is not the invariant distribution of the equation since $(X_t)_{t\geq 0}$ is not a stationary process.

Proposition 3.2 (Fake stationary regimes (types I and II) and asymptotics)**.** *Let $X = (X_t)_{t\geq 0}$ be a one-dimensional solution of the stabilized Volterra equation* (3.20) *starting from any random variable X_0 defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\lambda > 0$, $\mu_\infty \in \mathbb{R}$, and a squared diffusion coefficient $\sigma^2 \in C^2_{K,Lip}(\mathbb{R}, \mathbb{R})$, where $\varsigma = \varsigma_{\lambda,c}$, assumed to be the unique continuous solution to Equation* (3.24) *for some $c \in (0, \frac{1}{[\sigma]^2_{Lip}})$ (so that condition $(E_{\lambda,c})$ is satisfied). If $X_0 \in L^2(\mathbb{P})$ is such that $\mathbb{E}[X_0] = x_\infty$, given in* (3.20) *and* $\mathrm{Var}(X_0) = v_0$

1. **Case $\sigma(x) = \sigma$ is constant.** *The solution $(X_t)_{t\geq 0}$ has a constant mean $\frac{\mu_\infty}{\lambda}$ and variance v_0.*
   — *The process exhibits a fake stationary regime of type I i.e.*
   $$\forall t \geq 0, \quad \mathbb{E}[X_t] = x_\infty, \quad Var(X_t) = v_0 = c\sigma^2.$$
   — *Furthermore, if $X_0 \sim \nu^* := \mathcal{N}(x_\infty, v_0)$, this represents a fake stationary regime of type II, since in this case, $X_t \sim X_0$ for all $t \geq 0$. ($(X_t)_{t\geq 0}$ is a Gaussian process with a fake stationary regime of type II. anyway.). ν^* is the 1-marginal distribution.*

2. **Case where σ is not constant.** *Assume that the mean, variance process $(v_t := \mathbb{E}[(X_t - x_\infty)^2])_{t\geq 0}$ and expected squared diffusion process $(\bar{\sigma}^2(t) := \mathbb{E}[\sigma^2(X_t)])_{t\geq 0}$ are constant, i.e.*
   $$\forall t \geq 0, \quad \mathbb{E}[X_t] = x_\infty, \quad \mathrm{Var}(X_t) = v_0 = C^{ste}, \quad \mathbb{E}[\sigma^2(X_t)] = \bar{\sigma}_0^2 = C^{ste}.$$
   Then, a necessary and sufficient condition for this Fake Stationarity Regime of Type I to hold is that there exists a function f not depending on t such that:
   $$\forall u \in [0,1], \quad \forall t \geq 0 \quad \mathbb{E}\left[(X_t - x_\infty)^3 \partial_x^3 \sigma^2(x_\infty + (X_t - x_\infty)u)\right] = f(u). \tag{3.27}$$
   As soon as equation (3.27) *holds, the solution $(X_t)_{t\geq 0}$ to the Volterra equation* (3.12) *starting from X_0 has a fake stationary regime of type I in the sense i.e. for all $t \geq 0$,*
   $$\mathbb{E}[X_t] = x_\infty, \quad Var(X_t) = v_0 = \frac{c(\sigma^2(x_\infty) + r_\infty)}{1 - c\kappa}, \quad \text{and} \quad \mathbb{E}[\sigma^2(X_t)] = \bar{\sigma}_0^2 = \frac{(\sigma^2(x_\infty) + r_\infty)}{1 - c\kappa}. \tag{3.28}$$
   where $\kappa := \frac{1}{2}\partial_x^2 \sigma^2(x_\infty)$ is the curvature of σ^2 and $r_\infty := \int_0^1 \frac{(1-u)^2}{2} f(u)\, du$ provided $\kappa c \neq 1$.

Moreover if $\lim_{t\to+\infty} R_\lambda(t) = 0$ (i.e. $a = 0$) or if $\lim_{t\to+\infty} \phi(t) = 0$ (i.e. $\phi_\infty = 0$), as a consequence of the confluence properties in Proposition4.4, for any starting value $X_0 \in L^2(\mathbb{P})$,
$$\mathbb{E}[X_t] \to x_\infty, \quad \text{and} \quad Var(X_t) \to v_0 \quad \text{as} \quad t \to +\infty.$$

Remark. If $K \equiv 1$, i.e. the solution $(X_t)_{t\geq 0}$ to (3.12) is a (Markov) SDE, and if it admits an invariant distribution (see e.g. [37]) $\nu_\sigma(dx) = \pi_\sigma(x)\, \lambda_1(dx)$, then starting from $X_0 \stackrel{d}{=} \nu_\sigma$ yields a fake stationary regime of type II and, in particular, of type I. In this case, for all $t \geq 0$, equation (3.27) corresponds to the expectation under the invariant distribution, i.e. $\mathbb{E}_{\pi_\sigma}[\cdot]$, and thus the function f does not depend on t.

Proof. Assume there exists at least a weak solution on the whole non-negative real line of the Stochastic Voltera equation with volatility term $\sigma(t,x) = \varsigma_{\lambda,c}(t)\sigma(x)$ starting from any $X_0 \in L^2(\mathbb{P})$ such that $\mathbb{E}[X_0] = x_\infty$ and $Var[X_0] = v_0$. The first claim (1) is obvious once noted that $(X_t)_{t\geq 0}$ is a Gaussian process (and $[\sigma]_{\mathrm{Lip}} = 0$). The last claim is a straightforward consequence of the confluence property in Proposition 4.4. We know that: $\mathbb{E}\, X_t = \mathbb{E}\, X_0 - \frac{1}{\lambda x_\infty}\Big(\mathbb{E}\, X_0 - x_\infty\Big) \int_0^t f_\lambda(t-s)\mu(s)ds.$

Step 1. *(Conditions for Fake stationary Regime of type I.)* Using the second-Order Taylor Expansion of σ^2 around x_∞ with Integral Remainder, we have:
$$\sigma^2(X_t) = \sigma^2(x_\infty) + \partial_x\sigma^2(x_\infty)Y_t + \frac{\partial_x^2\sigma^2(x_\infty)}{2}Y_t^2 + \int_0^1 \frac{(1-u)^2}{2}\, Y_t^3\, \partial_x^3\sigma^2\left(x_\infty + uY_t\right) du. \tag{3.29}$$
where $Y_t := X_t - x_\infty$ for $\sigma^2 \in C^3(\mathbb{R})$, and the change of variable $u \to u - x_\infty$ in the integral term.

Now, taking the expectation and invoking the standard Fubini lemma, we obtain:
$$\bar{\sigma}(t)^2 := \mathbb{E}[\sigma^2(X_t)] = \sigma^2\left(x_\infty\right) + \kappa\, \mathrm{Var}(X_t) + r_t, \text{ with } r_t = \int_0^1 \frac{(1-u)^2}{2}\, \mathbb{E}\left[Y_t^3\, \partial_x^3\sigma^2\left(x_\infty + uY_t\right)\right] du. \tag{3.30}$$

By the equivalence property, the Fake Stationarity Regime of Type I holds whenever $\bar\sigma(t)$ is constant in which case $\mathrm{Var}(X_t)$ remains constant as well (see Proposition 3.1). It is thus necessary and sufficient that r_t be constant (denoted r_∞) or equivalently, a necessary and sufficient condition is that $\mathbb{E}\left[Y_t^3\partial_x^3\sigma^2(x_\infty+Y_tu)\right]$ is independent of t for any fixed $u\in[0,1]$ i.e. Equation (3.28) holds.

STEP 2. *(Fake stationary Regime of type I.* If $X_0\in L^1(\mathbb{P})$ is such that (3.19) holds, then from equation (3.20) we have constant mean for every $t\geq 0$ i.e. $\mathbb{E}\,X_t=x_\infty$

Assume that the condition $c\kappa\neq 1$ is satisfied and as for the variance, from equation (3.22), we have:

$$\forall\, t\geq 0,\quad \mathrm{Var}(X_t)=\mathrm{Var}(X_0)(\phi-f_\lambda*\phi)^2(t)+\tfrac{1}{\lambda^2}f_\lambda^2*\big(\varsigma^2\,\mathbb{E}\,\sigma^2(X_.)\big)_t$$

Which become, with equation (3.30) in mind:

$$\mathrm{Var}(X_t)=\mathrm{Var}(X_0)(\phi-f_\lambda*\phi)^2(t)+\frac{1}{\lambda^2}(f_\lambda^2*(\varsigma\bar\sigma)^2)_t=\mathrm{Var}(X_0)(\phi-f_\lambda*\phi)^2(t)+\frac{1}{\lambda^2}(f_\lambda^2*(\varsigma^2(\sigma^2(x_\infty)+\kappa\mathrm{Var}(X_.)+r_t)))_t. \tag{3.31}$$

Now, assuming constant variance, $\mathrm{Var}(X_t)=v_0$ for every $t\geq 0$, equation (3.27) holds and equation 3.30 becomes:

$$\bar\sigma(t)^2:=\mathbb{E}[\sigma^2(X_t)]=\sigma^2(x_\infty)+\kappa\mathrm{Var}(X_t)+r_\infty=\sigma^2(x_\infty)+\kappa v_0+r_\infty=:\bar\sigma_0^2.$$

And then, the equation (3.31) above becomes (where in the second line, we take advantage of the identity (3.25) satisfied by $\varsigma=\varsigma_{\lambda,c}$ so that *$(E_{\lambda,c})$* in force),

$$\begin{aligned}\forall\, t\geq 0,\quad v_0=\mathrm{Var}(X_t)&=\mathrm{Var}(X_0)(\phi-f_\lambda*\phi)^2(t)+\frac{\bar\sigma_0^2}{\lambda^2}(f_\lambda^2*\varsigma^2)(t)\\&=v_0(\phi-f_\lambda*\phi)^2(t)+(\sigma^2(x_\infty)+\kappa v_0+r_\infty)c(1-(\phi-f_\lambda*\phi)^2(t))\end{aligned}$$

Which also reads: $v_0(1-(\phi-f_\lambda*\phi)^2(t))=c(\sigma^2(x_\infty)+\kappa v_0+r_\infty)(1-(\phi-f_\lambda*\phi)^2(t)),\ t\geq 0$ i.e. the variance becomes $v_0=\frac{c\left(\sigma^2(x_\infty)+r_\infty\right)}{1-c\kappa}>0$ which is clearly solution to the equation.

Conversely one checks that this constant value for the variance solves the above equation. Let us prove that it is the only one. Assume that there exist two solutions to Equation (3.31) starting from a unique initial value $\mathrm{Var}(X_0)=v_0$, and let $x\in\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ represent the discrepancy over time between those solutions. By the linearity of Equation (3.31), it suffices to show that the equation in $x\in\mathcal{C}(\mathbb{R}_+,\mathbb{R})$

$$x(t)=\frac{\kappa}{\lambda^2}\big(f_\lambda^2*(\varsigma^2.\,x)\big)_t,\quad x(0)=0$$

only has the null function as solution. If x solves the above equation, then

$$|x(t)|\leq\frac{\kappa}{\lambda^2}(f_\lambda^2*\varsigma^2)_t\sup_{0\leq s\leq t}|x(s)|\;=\kappa c|1-(\phi-f_\lambda*\phi)^2(t)|\sup_{0\leq s\leq t}|x(s)|\;\leq c\kappa\sup_{0\leq s\leq t}|x(s)|.$$

where the last inequality comes from $\mathcal{K}(iii)$. If $x\not\equiv 0$, there exist $\varepsilon>0$ such that $\tau_\varepsilon=\inf\{t:|x(t)|>\varepsilon\}<+\infty$. By continuity of x it is clear that $\tau_\varepsilon>0$ and $|x(\tau_\varepsilon)|=\sup_{0:\leq s\leq t}|x(s)|=\varepsilon$ which is impossible since $\kappa c\neq 1$. Consequently $x\equiv 0$. We also have:

$$\bar\sigma_0^2=\sigma^2(x_\infty)+\kappa v_0+r_\infty=\sigma^2(x_\infty)+\kappa\frac{c\left(\sigma^2(x_\infty)+r_\infty\right)}{1-c\kappa}+r_\infty=\frac{\sigma^2(x_\infty)+r_\infty}{1-c\kappa}.$$

Hence $(X_t)_{t\geq 0}$ is a fake stationary regime of type I with the above mean and variance. □

Example 3.10 (Polynomial of degree 2)**.** *Consider as in [36] a squared trinomial diffusion coefficient:*

$$\sigma(x)=\sqrt{\kappa_0+\kappa_1\,x+\kappa_2\,x^2}\quad\textit{with}\quad\kappa_i\geq 0,\ i=0,2,\ \kappa_1^2\leq 4\kappa_2\kappa_0. \tag{3.32}$$

• *The above vol-vol term covers the rough Heston dynamics introduced in [17] (the volatility process $V_t=X_t$ has the vol-vol term as in equation* (3.32) *with $\kappa_0=\kappa_2=a=0$, while the volatility of the traded asset is driven by a* different Brownian motion *).*

• *This type of vol-vol term also appears in the quadratic rough volatility dynamic introduced in [20] ($V_t = \sigma^2(X_t)$). In that model, the asset and its volatility are driven by* the same Brownian motion, *aiming to jointly calibrate the the S&P 500 and VIX smile, accounting for the so-called Zumbach effect, which links the evolution of the asset (here, an index) with its volatility.*

In the next proposition, we assume that when $\kappa_2 = 0$, the associated Volterra Equation has at least a weak solution (see [22]).

Proposition 3.11. *Under the same assumptions as Proposition 3.2, We have the following claims:*

1. *If the diffusion coefficient σ is degenerated in the sense that $\sigma(x_\infty) = 0$, (in particular $\bar\sigma_0^2 = 0$ and $v_0 = 0$) then the solution $X_t = x_\infty$ $\mathbb{P}$-a.s. represents a fake stationary regime (of type I).*
2. *If σ^2 is not constant and not degenerated given by* (3.32) *i.e.* $\sigma^2(x) \in \mathrm{Pol}_2(\mathbb{R})$, *the solution $(X_t)_{t\geq 0}$ to the Volterra equation* (3.12) *has a fake stationary regime of type I, in the sense that*
$$\forall\, t \geq 0, \quad \mathbb{E}[X_t] = x_\infty,\ Var(X_t) = v_0 = \tfrac{c\sigma^2(x_\infty)}{1-c\kappa_2},\ \text{and}\ \mathbb{E}[\sigma^2(X_t)] = \bar\sigma_0^2 = \tfrac{\sigma^2(x_\infty)}{1-c\kappa_2}.$$
Moreover if $a = 0$ of if $\phi_\infty = 0$, for any starting value $X_0 \in L^2(\mathbb{P})$,
$$\mathbb{E}[X_t] \to x_\infty, \quad \text{and} \quad Var(X_t) \to \tfrac{c\sigma^2(x_\infty)}{1-c\kappa_2} \quad \text{as} \quad t \to +\infty.$$

Proof. *(Applicability of equation* (3.27)*).*

1. First, in the degenerate setting $\sigma(x_\infty) = 0$, one has $\bar\sigma_0^2 = \mathbb{E}[\sigma^2(X_0)] = 0$, we get $v_0 = 0$ owing to equation (3.25) since $\lim_{t\to+\infty}(\phi - f_\lambda * \phi)(t) = a\phi_\infty < 1 \Rightarrow \phi(t) - (f_\lambda * \phi)(t) \neq 1$ (at least for t large enough). As a consequence, $\mathrm{Var}(X_t) = 0$ for every $t \geq 0$. But, we know that $\mathbb{E}\, X_t = \mathbb{E}\, X_0 = x_\infty$, it follows that $X_t = x_\infty$ $\mathbb{P}$-*a.s.* and $\forall t \geq 0, \quad \mathcal{L}(Y_t)(dy) = \delta_0(dy)$ so that
$$\forall t \geq 0, \quad \mathbb{E}\left[Y_t^3\, \partial_x^3\sigma^2\left(x_\infty + uY_t\right)\right] = \int_{\mathbb{R}} y^3\, \partial_x^3\sigma^2\left(x_\infty + uy\right)\, \mathcal{L}(Y_t)(dy) = 0 \text{ and } r_\infty = 0.$$
2. Secondly, if $\partial_x^3\sigma^2(v) = 0,\ \forall v \in \mathbb{R}$, then the integral reminder in (3.29) necessarily vanishes. This corresponds to the *trinomial setting*, which has already been extensively studied in [36] and in which case if $\kappa_2 > 0$, $[\sigma]_{\mathrm{Lip}} = \sqrt{\kappa_2}$, the curvature $\kappa = \kappa_2$ and $r_\infty = 0$ (since $f \equiv 0$ in (3.27)). □

Practitioner's corner: 1. The constraint $c \in \left(0, \frac{1}{\kappa}\right)$ implies that we treat c as a free parameter, from which we can deduce v_0 and $\bar\sigma_0^2$. 2. The presence of the *stabilizer* $\varsigma_{\lambda,c}$ allows a better control of the behaviour of the equation since it induces an L^2-confluence and a stability of first two moments if needed. 3. Note that $[\sigma]_{\mathrm{Lip}}^2 = \kappa_2 = \kappa$ so that, in practice, if we rather fix the value of v_0, then $c = \frac{v_0}{\sigma^2(x_\infty)+v_0\kappa}$ so that, σ being $\sqrt{\kappa}$-Lipschitz continuous, one has $c\kappa = \frac{v_0\kappa}{\sigma^2(x_\infty)+v_0\kappa} < 1$ provided $\sigma^2(x_\infty) > 0$ which ensures the L^2-confluence of the paths of the solution (Proposition 4.4 further on).

4 Towards Long run behaviour: asymptotics and confluence

Remark 4.1. *Let $\mu_\infty \in \mathbb{R}$, by assumption* (2.2) *(i), one has for every $x \in \mathbb{R}$,*
$$\sigma^2(x) \leq \Big(\sigma(x_\infty) + [\sigma]_{Lip}|x - x_\infty|\Big)^2 \leq \kappa_0 + \kappa_2|x - x_\infty|^2$$
where $k_0 = k_0(\epsilon) := (1+\epsilon)|\sigma(x_\infty)|^2$ and $k_2 = k_2(\epsilon) := (1 + \frac{1}{\epsilon})[\sigma]_{Lip}^2$, owing to Young's inequality $(a+b)^2 \leq (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})b^2$. Therefore, we can always assume that σ is sublinear *i.e.:*
$$(SL_\sigma) : \exists k_0 = k_0(\epsilon) \in \mathbb{R}_+, k_2 = k_2(\epsilon) \in \mathbb{R}_+ \quad \text{such that} \quad \forall x \in \mathbb{R}, |\sigma(x)|^2 \leq \kappa_0 + \kappa_2|x - x_\infty|^2. \quad (4.33)$$

4.1 Moments control.

Lemma 4.2 (Best constant in a BDG inequality (see Remark 2 in [11]))**.** *Let M be a continuous local martingale null at $t = 0$. Then, for every $p \geq 1$,*
$$\tfrac{1}{\sqrt{p}}\|M_t\|_p \leq 2\|\langle M\rangle_t^{1/2}\|_p.$$

Proposition 4.3 (Moment control)**.** *Assume* (2.2) *(ii) and* $\mathcal{K}$ *(ii) hold. Let* $\sigma(t,x) := \varsigma(t)\sigma(x)$ *where* $\varsigma = \varsigma_{\lambda,c}$ *is a non-negative, continuous and bounded solution to* (3.25) *for some fixed* $\lambda, c > 0$ *(i.e.*(E_λ) *is in force). Let* $(X_t)_{t\geq 0}$ *be the solutions to the Volterra equation* (3.12) *starting from any random variable* X_0*.*

(*a*) First two moments. *Assume* $X_0 \in L^2(\mathbb{P})$ *and* $c \in \big(0, \frac{1}{[\sigma]^2_{Lip}}\big)$*. Then, one has:*

$$\Big|\mathbb{E}\left(X_t\right) - x_\infty\Big| \leq |\phi(t) - (f_\lambda * \phi)(t)|\Big|\mathbb{E}\left(X_0\right) - x_\infty\Big| = \Big|1 - (f_\lambda * \frac{\mu}{\lambda x_\infty})_t\Big|\Big|\mathbb{E}\left(X_0\right) - x_\infty\Big|, \quad t \geq 0$$

$$\sup_{t\geq 0}\Big\||X_t - x_\infty|\Big\|_2 \leq \left[\frac{\sqrt{c}}{1 - [\sigma]_{Lip}\sqrt{c}}|\sigma(x_\infty)|\right] \vee \Big\||X_0 - x_\infty|\Big\|_2 < +\infty.$$

(*b*) L^p-moments. *Let* $p \in (2, +\infty)$*. If* $X_0 \in L^p(\mathbb{P})$ *and* c *is such that* $\rho_p := 4c\,p\,[\sigma]^2_{Lip} < 1$*, then*

$$\sup_{t\geq 0}\Big\||X_t - x_\infty|\Big\|_p \leq \inf_{\epsilon \in (0, \frac{1}{\rho_p} - 1)}\left[\frac{2\sqrt{pc(1+\epsilon)}}{1 - 2[\sigma]_{Lip}\sqrt{pc(1+\epsilon)}}\big|\sigma(x_\infty)\big|^2\right] \vee \left[(1 + 1/\epsilon)^{\frac{1}{2}}\Big\||X_0 - x_\infty|\Big\|_p\right] < +\infty$$

The proof is postponed to Appendix B. It relies on techniques similar to those in [36], which extend the methods developed for the Markovian framework, as discussed extensively in [35, Chapter 7].

4.2 L^p-Confluence or Contraction Properties

Fix $p > 0$. Let $(X_t)_{t\geq 0}$ and $(X'_t)_{t\geq 0}$ be two solutions of the Volterra stochastic equation (3.12) with initial conditions $X_0, X'_0 \in L^p(\mathbb{P})$. According to assumption 2.2 (i), $\exists\, \kappa > 0$ such that for every $t \geq 0$,

$$\Big\||\sigma(X_s) - \sigma(X'_s)|^2\Big\|_{\frac{p}{2}} \leq \kappa\Big\||X_s - X'_s|\Big\|_p^2. \tag{4.34}$$

Proposition 4.4 (L^p-confluence)**.** *Assume assumption* (2.2) *(ii). Assume* $f_\lambda \in L^2(\mathbb{R}_+, Leb_1)$*,* $\sigma(t,x) := \varsigma(t)\sigma(x)$ *where* $\varsigma = \varsigma_{\lambda,c}$ *is a non-negative, continuous and bounded solution to* (3.25) *for some fixed* $\lambda, c > 0$ *(i.e. assumption 3.8 is in force) and* $\sigma : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz continuous function. Let* $p > 0$*, for* $X_0, X'_0 \in L^p(\mathbb{P})$*, we consider the solutions to Volterra equation* (3.12) *denoted* $(X_t)_{t\geq 0}$ *and* $(X'_t)_{t\geq 0}$ *starting from* X_0 *and* X'_0 *respectively. Let* $c \in \big(0, \frac{1}{\kappa}\big)$*, where* κ *is defined in 4.34, set* $\rho_p := c\,(C_p^{\mathrm{BDG}})^2\,\kappa$*. and assume that* $\rho_p < 1 - a^2\phi_\infty^2$*. Then, one has:*

(a) There exists a continuous non-negative function $\varphi_{\infty,p}^{\lambda,c,K,\phi} =: \varphi_{\infty,p} : \mathbb{R}^+ \to [0, \frac{1}{(1-\sqrt{\rho_p})^2}]$*, such that* $\varphi_\infty(0) = \frac{1}{1-\rho_p}$*,* $0 \leq \lim_{t\to\infty}\varphi_{\infty,p}(t) \leq \frac{a^2\phi_\infty^2}{\left(1-\sqrt{\rho_p}\sqrt{1-a^2\phi_\infty^2}\right)^2}$*, only depending on* λ, c, ϕ*, and the kernel* K*, such that :*

$$\forall t \geq 0, \quad \mathbb{E}\left(\Big|X_t - X'_t\Big|^p\right) \leq \varphi_{\infty,p}(t)\mathbb{E}\left(\Big|X_0 - X'_0\Big|^p\right).$$

(b) This result can be written using the p-Wasserstein distance between marginals of X *and* X'*:*

$$\forall t \geq 0, \quad W_p([X_t], [X'_t]) \leq \varphi_{\infty,p}(t)^{1/2}\, W_p([X_0], [X'_0]).$$

(c) In particular, whenever $a = 0$ *or* $\phi_\infty = 0$*, the limit yields* $\varphi_{\infty,p}(t) \to 0$ *and thus the process is contracting in* W_p *as* $t \to \infty$ *i.e more generally finite-dimensional* W_p*-convergence of marginals.*

Proof. *By a Banach fixed point argument on the complete space* $(C_b([0,\infty), \mathbb{R}), \|\cdot\|_\infty)$*.*
Fix $p > 0$. Let $(X_t)_{t\geq 0}$ and $(X'_t)_{t\geq 0}$ be two solutions of the same SVIE with initial conditions $X_0, X'_0 \in L^p(\mathbb{P})$. Set $\Delta_t = X_t - X'_t \in L^p(\mathbb{P})$ for every $t \geq 0$. one writes owing to equation 3.14:

$$X_t - X'_t = \big(\phi(t) - (f_\lambda * \phi)(t)\big)\big(X_0 - X'_0\big) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\varsigma(s)\Big(\sigma(X_s) - \sigma(X'_s)\Big)dW_s$$

Let $\bar\delta_t = \Big\| |\Delta_t| \Big\|_p$ for convenience. Under our assumptions, $t \mapsto \bar\delta_t$ is continuous (see [30]). Let $C_p^{\mathrm{BDG}} > 0$ denote a BDG constant in L^p. Set $\rho_p := c\,(C_p^{\mathrm{BDG}})^2\,\kappa$. Owing to the triangle inequality and applying the *BDG* inequality to the (a priori) local martingale $M_u = \int_0^u f_\lambda(t-s)\varsigma(s)\sigma(X_s)dW_s$, $0 \le s \le t$, (see [41, Proposition 4.3]) follow by the generalized Minkowski inequality, we get:

$$\begin{aligned}\Big\| |X_t - X'_t| \Big\|_p &\le \Big\| |X_0 - X'_0| \Big\|_p \Big|\phi(t) - (f_\lambda * \phi)(t)\Big| + \frac{C_p^{BDG}}{\lambda} \Big\| \Big(f_\lambda^2 \overset{dt}{*} \varsigma^2(\cdot)|\sigma(X_s) - \sigma(X'_s)|^2 \Big)_t \Big\|_{\frac{p}{2}}^{\frac{1}{2}} \\ &\le \Big\| |X_0 - X'_0| \Big\|_p \Big|\phi(t) - (f_\lambda * \phi)(t)\Big| + \frac{C_p^{BDG}}{\lambda} \Big(\int_0^t f_\lambda^2(t-s)\varsigma^2(s) \big\| |\sigma(X_s) - \sigma(X'_s)|^2 \big\|_{\frac{p}{2}} \Big)^{\frac{1}{2}}\end{aligned}$$

Fix $\epsilon > 0$, using the elementary inequality $(a+b)^2 \le (1+1/\epsilon)a^2 + (1+\epsilon)b^2$ for $\epsilon \in (0, 1/\rho_p - 1)$ i.e. $\beta := \rho_p(1+\varepsilon) \;<\; 1$, it follows owing to equation (4.34) that:

$$\Big\| |X_t - X'_t| \Big\|_p^2 \le \Big\| |X_0 - X'_0| \Big\|_p^2 \Big|\phi(t) - (f_\lambda * \phi)(t)\Big|^2 (1+1/\epsilon) + \frac{\rho_p}{c\lambda^2}(1+\epsilon) \int_0^t f_\lambda^2(t-s)\varsigma^2(s) \Big\| |X_s - X'_s|^2 \Big\|_p ds$$

which entails:

$$\Big\| |\Delta_t| \Big\|_p^2 \le \Big\| |\Delta_0| \Big\|_p^2 \Big|\phi(t) - (f_\lambda * \phi)(t)\Big|^2 \left(1 + \frac{1}{\epsilon}\right) + (1+\epsilon)\,\frac{\rho_p}{c\lambda^2} \int_0^t f_\lambda^2(t-s)\varsigma^2(s) \Big\| |\Delta_s| \Big\|_p^2 ds. \tag{4.35}$$

i.e. we obtain, for all $t \ge 0$,

$$\bar\delta_t^2 \le \bar\delta_0^2 \left|\phi(t) - (f_\lambda * \phi)(t)\right|^2 \left(1 + \frac{1}{\varepsilon}\right) + \rho_p(1+\varepsilon)\frac{1}{\lambda^2 c} \int_0^t f_\lambda^2(t-s)\varsigma^2(s)\,\bar\delta_s^2\,ds. \tag{4.36}$$

Step 1. *Non-expansivity via a deterministic stopping-time argument:* For the fixed $\varepsilon > 0$ such that $\beta := \rho_p(1+\varepsilon) < 1$, let $\eta > 0$ such that $1 + \frac{1}{\epsilon} < \rho_p(1+\epsilon)(1+\eta)^2$ and define the stopping time

$$\tau_\eta := \inf\{t \ge 0 : \bar\delta_t > (1+\eta)\bar\delta_0\}$$

(with the convention $\inf \varnothing = +\infty$). If $\tau_\eta < \infty$, then for $s \le \tau_\eta$ we have $\bar\delta_s \le (1+\eta)\bar\delta_0$ and by continuity $\bar\delta_{\tau_\eta} = (1+\eta)\bar\delta_0$. Evaluating (4.36) at $t = \tau_\eta$ and bounding $\bar\delta_s^2 \le (1+\eta)^2\bar\delta_0^2$ in the integral combined with the identity $f_\lambda^2 * \varsigma^2 = c\lambda^2(1 - (\phi - f_\lambda * \phi)^2)$ yields:

$$\begin{aligned}\bar\delta_{\tau_\eta}^2 &\le \bar\delta_0^2 \left[(\phi - f_\lambda * \phi)^2(\tau_\eta)(1 + \frac{1}{\varepsilon}) + (1 - (\phi - f_\lambda * \phi)^2(\tau_\eta))\rho_p(1+\varepsilon)(1+\eta)^2 \right] \\ &\le \delta_0^2 \left[(\phi - f_\lambda * \phi)^2(\tau_\eta) \left(1 + \frac{1}{\varepsilon} - \rho_p(1+\varepsilon)(1+\eta)^2\right) + \rho_p(1+\varepsilon)(1+\eta)^2 \right] \\ &< \rho_p(1+\varepsilon)(1+\eta)^2\bar\delta_0^2 < (1+\eta)^2\bar\delta_0^2.\end{aligned}$$

which leads to a contradiction. Whence $\tau_\eta = +\infty$ i.e., $\bar\delta_s \le (1+\eta)\bar\delta_0$ for all $s \ge 0$. This holds for every $\eta > 0$, implying the non-expansivity bound $\bar\delta_t \le \bar\delta_0$ for all $t \ge 0$ when letting $\eta \downarrow 0$.

Step 2. *Iteration and the Volterra map:* Substituting this (i.e. $\bar\delta_t \le \bar\delta_0$) into (4.35) combined with the stabilizer identity $f_\lambda^2 * \varsigma^2 = c\lambda^2(1 - (\phi - f_\lambda * \phi)^2)$ gives, for all $t > 0$,

$$\bar\delta_t^2 \le \bar\delta_0^2\,\varphi_{1,p}^\varepsilon(t), \quad \text{where} \quad \varphi_{1,p}^\varepsilon(t) := \Big(1 + \frac{1}{\varepsilon}\Big)(\phi - f_\lambda * \phi)^2(t) + \rho_p(1+\varepsilon)(1 - (\phi - f_\lambda * \phi)^2(t)).$$

Note that $\varphi_{1,p}^\varepsilon(t) = \rho_p(1+\varepsilon) + (\phi - f_\lambda * \phi)^2(t)(1 + \frac{1}{\varepsilon} - \rho_p(1+\varepsilon))$ satisfies:

$$\varphi_{1,p}^\varepsilon(0) = 1 + \frac{1}{\varepsilon}, \quad \varphi_1 \text{ is continuous, } M_1 := \|\varphi_{1,p}^\varepsilon\|_\infty \le 1 + \frac{1}{\epsilon} + \rho_p(1+\varepsilon)$$

Substituting this upper bound for δ_t^2 (i.e. $\bar{\delta}_t^2 \le \bar{\delta}_0^2\varphi_{1,p}^\varepsilon(t)$) into (4.35) yields

$$\bar{\delta}_t^2 \le \bar{\bar{\delta}}_0^2\varphi_{2,p}^\varepsilon(t), \quad \text{where} \quad \varphi_{2,p}^\varepsilon(t) := \Big(1+\frac{1}{\varepsilon}\Big)(\phi - f_\lambda * \phi)^2(t) + \rho_p(1+\varepsilon)\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\varphi_{1,p}^\varepsilon(s)\frac{ds}{\lambda^2 c}.$$

and inductively for $k \ge 2$

$$\bar{\delta}_t^2 \le \bar{\bar{\delta}}_0^2\varphi_{k,p}^\varepsilon(t), \quad \text{where} \quad \varphi_{k,p}^\varepsilon(t) := (\phi - f_\lambda * \phi)^2(t)\Big(1+\frac{1}{\varepsilon}\Big) + \rho_p(1+\varepsilon)\frac{1}{\lambda^2 c}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\,\varphi_{k-1,p}^\varepsilon(s)\,ds. \tag{4.37}$$

To obtain a uniform sup-bound, put $M_k := \|\varphi_{k,p}^\varepsilon\|_\infty$. From (4.37) and $(\phi - f_\lambda * \phi)(t) \le 1$, $1 - (\phi - f_\lambda * \phi)(t) \le 1$, we get $M_k \le \Big(1+\frac{1}{\varepsilon}\Big) + \beta M_{k-1}$. Iterating yields (since $\beta < 1$):

$M_k \le \Big(1+\frac{1}{\varepsilon}\Big)\sum_{j=1}^{k-1}\beta^j + \beta^{k-1}M_1 \le \max(M_1, \frac{1+\frac{1}{\varepsilon}}{1-\beta}) \le \max(1+\frac{1}{\varepsilon}+\beta, \frac{1+\frac{1}{\varepsilon}}{1-\beta}) = \frac{1+\frac{1}{\varepsilon}}{1-\beta}$. Thus for every $k \ge 1$ and $t \ge 0$ one has the uniform bound

$$0 \le \varphi_{k,p}^\varepsilon(t) \le \frac{1+1/\varepsilon}{1-\rho_p(1+\varepsilon)} := M_*^\epsilon. \tag{4.38}$$

Step 3. *Define the operator $\mathcal{T} : C_b([0,\infty)) \to C_b([0,\infty))$:*

$$(\mathcal{T}\psi)(t) := (\phi - f_\lambda * \phi)^2(t)\Big(1+\frac{1}{\varepsilon}\Big) + \rho_p(1+\varepsilon)\frac{1}{\lambda^2 c}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\,\psi(s)\,ds. \tag{4.39}$$

and, for $k \ge 2$, set $\varphi_{k,p}^\varepsilon = \mathcal{T}\varphi_{k-1,p}^\varepsilon$. The operator $\mathcal{T}$ is linear in its last term and for any $\psi_1, \psi_2 \in C_b$,

$$\|\mathcal{T}\psi_1 - \mathcal{T}\psi_2\|_\infty \le \rho_p(1+\varepsilon)\cdot\sup_{t\ge 0}\frac{1}{\lambda^2 c}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\,ds \;\cdot\; \|\psi_1-\psi_2\|_\infty = \rho_p(1+\varepsilon)\|\psi_1-\psi_2\|_\infty$$

because the convolution integral equals $1 - (\phi - f_\lambda * \phi)(t) \le 1$. By assumption $\rho_p(1+\varepsilon) < 1$, so $\mathcal{T}$ is a strict contraction in $\|\cdot\|_\infty$ with Lipschitz constant $\rho_p(1+\varepsilon) < 1$ on the complete or Banach space $C_b([0,\infty))$ with the sup norm. The Banach fixed point theorem therefore provides a unique fixed point $\varphi_{\infty,p}^\varepsilon \in C_b([0,\infty))$ and, moreover, $\varphi_{k,p}^\varepsilon = \mathcal{T}^{k-1}\varphi_{1,p}^\varepsilon \xrightarrow[k\to\infty]{\|\cdot\|_\infty} \varphi_{\infty,p}^\varepsilon$.

In particular the convergence is uniform on $[0,\infty)$ i.e. $\varphi_{k,p}^\varepsilon = \mathcal{T}^{k-1}\varphi_{1,p}^\varepsilon$ converges uniformly (on $[0,\infty)$) to $\varphi_{\infty,p}^\varepsilon$. For every $t \ge 0$ the L^p-norm satisfies $\bar{\delta}_t^2 \le \bar{\delta}_0^2\,\varphi_{\infty,p}^\varepsilon(t)$.

Step 4. *Limit equation and ε-dependent asymptotic bound:* Passing to the limit in (4.37) yields that $\varphi_{\infty,p}^\varepsilon$ satisfies the Volterra or functional fixed-point equation

$$\varphi_{\infty,p}^\varepsilon(t) = (\phi - f_\lambda * \phi)^2(t)\Big(1+\frac{1}{\varepsilon}\Big) + \rho_p(1+\varepsilon)\frac{1}{\lambda^2 c}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\,\varphi_{\infty,p}^\varepsilon(s)\,ds. \tag{4.40}$$

By the uniform bound in equation (4.38), $\varphi_{\infty,p}^\varepsilon$ is bounded and nonnegative on $[0,\infty)$ i.e. $\forall t \ge 0, \quad 0 \le \varphi_{\infty,p}^\varepsilon(t) \le \frac{1+1/\varepsilon}{1-\rho_p(1+\varepsilon)}$. Taking $\liminf_{t\to+\infty}, \limsup_{t\to\infty}$ in (4.40) and using $(\phi - f_\lambda * \phi)^2(t) \to a^2$ and the stabilizer identity we obtain $\underline{\ell}_{\infty,p}^\varepsilon, \ell_{\infty,p}^\varepsilon := \liminf_{t\to+\infty}\varphi_{\infty,p}^\varepsilon(t), \limsup_{t\to+\infty}\varphi_{\infty,p}^\varepsilon(t) \in [0, M_*^\epsilon]$. Now, $\underline{\ell}_{\infty,p}^\varepsilon, \ell_{\infty,p}^\varepsilon \in [0, M_*^\epsilon]$ implies that for any $\eta > 0$, there exists $t_\eta \in \mathbb{R}^+$ such that for $t \ge t_\eta$, $\underline{\ell}_{\infty,p}^\varepsilon - \eta \le \varphi_{\infty,p}^\varepsilon(t) \le \ell_{\infty,p}^\varepsilon + \eta$. Then, we obtain on the first hand,

$$\begin{aligned}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\varphi_{\infty,p}^\varepsilon(s)\frac{ds}{\lambda^2 c} &\le \frac{1}{c\lambda^2}\int_{t_\eta}^t f_\lambda^2(t-s)\varsigma^2(s)(\ell_{\infty,p}^\varepsilon + \eta)\,ds + \frac{1}{c\lambda^2}\int_0^{t_\eta} f_\lambda^2(t-s)\varsigma^2(s)\varphi_{\infty,p}^\varepsilon(s)\,ds \\ &\le \frac{1}{c\lambda^2}\int_{t_\eta}^t f_\lambda^2(t-s)\varsigma^2(s)(\ell_{\infty,p}^\varepsilon + \eta)\,ds + \frac{1}{c\lambda^2}M_*^\epsilon\int_{t-t_\eta}^t f_\lambda^2(u)\varsigma^2(t-s)\,du.\end{aligned}$$

where the second term on the right-hand side of the last inequality follows from the fact that $\varphi^{\varepsilon}_{\infty,p}(t-u) \le M^{\epsilon}_{*}$ for all $u \le t \le t_\eta$ and vanishes as t goes to infinity.

Since $f_\lambda \in L^2(\mathrm{Leb}_1)$ and $\lim_{t\to+\infty}(\phi - f_\lambda * \phi)^2(t) = a^2\phi_\infty^2$ both owing to Assumption 3.1, we conclude from equation (4.40) and the identity satisfied by ς:

$$\ell^{\varepsilon}_{\infty,p} =: \limsup_{t\to+\infty} \varphi^{\varepsilon}_{\infty,p}(t) \le (1+\frac{1}{\varepsilon})\, a^2\phi^2_\infty + \rho_p(1+\varepsilon)(\ell^{\varepsilon}_{\infty,p}+\eta)(1-a^2\phi^2_\infty) \overset{\eta\to0}{\Longrightarrow} \ell^{\varepsilon}_{\infty,p} \le \frac{(1+\frac{1}{\varepsilon})\, a^2\phi^2_\infty}{1-\rho_p(1+\varepsilon)(1-a^2\phi^2_\infty)}$$

On the other hand, we also have:

$$\int_0^t f^2_\lambda(t-s)\varsigma^2(s)\varphi^{\varepsilon}_{\infty,p}(s)\frac{ds}{\lambda^2 c} \ge \frac{1}{c\lambda^2}\int_{t_\eta}^t f^2_\lambda(t-s)\varsigma^2(s)(\underline{\ell}^{\varepsilon}_{\infty,p}-\eta)\, ds + \int_{t-t_\eta}^t f^2_\lambda(u)\,\varsigma^2(t-u)\varphi^{\varepsilon}_{\infty,p}(t-u)\,\frac{du}{c\lambda^2}$$
$$\ge \frac{1}{c\lambda^2}\int_{t_\eta}^t f^2_\lambda(t-s)\varsigma^2(s)(\underline{\ell}^{\varepsilon}_{\infty,p}-\eta)\, ds.$$

Therefore, still with the fact that $f_\lambda \in L^2(\mathrm{Leb}_1)$ and $\lim_{t\to+\infty}(\phi - f_\lambda * \phi)^2(t) = a^2$, we obtain from equation (4.40) and the identity satisfied by ς:

$$\underline{\ell}^{\varepsilon}_{\infty,p} =: \liminf_{t\to+\infty} \varphi^{\varepsilon}_{\infty,p}(t) \ge (1+\frac{1}{\varepsilon})\, a^2\phi^2_\infty + \rho_p(1+\varepsilon)(\underline{\ell}^{\varepsilon}_{\infty,p}-\eta)(1-a^2\phi^2_\infty) \overset{\eta\to0}{\Longrightarrow} \underline{\ell}^{\varepsilon}_{\infty,p} \ge \frac{(1+\frac{1}{\varepsilon})\, a^2\phi^2_\infty}{1-\rho_p(1+\varepsilon)(1-a^2\phi^2_\infty)}.$$

Consequently, $\underline{\ell}^{\varepsilon}_{\infty,p} = \ell^{\varepsilon}_{\infty,p} = \frac{(1+\frac{1}{\varepsilon})\, a^2\phi^2_\infty}{1-\rho_p(1+\varepsilon)(1-a^2\phi^2_\infty)} := L(\epsilon)$. The minimizer of $L(\epsilon)$ in $(0, 1/\rho_p - 1)$ is $\varepsilon_* = \frac{1}{\sqrt{\rho_p(1-a^2\phi^2_\infty)}} - 1$, which is admissible iff $\rho_p < 1 - a^2\phi^2_\infty$. In that admissible case one obtains the optimal asymptotic value: $\inf_{\varepsilon\in(0,1/\rho_p-1)} L(\epsilon) = \ell^{\varepsilon_*}_{\infty,p} = \frac{a^2\phi^2_\infty}{\left(1-\sqrt{\rho_p}\sqrt{1-a^2\phi^2_\infty}\right)^2}$.

Step 5. *Passage to the ε-Free Control:* Finally, optimizing $\varphi^{\varepsilon}_{\infty,p}$ over admissible ε gives the ε-free control i.e. passing to the infimum over admissible ε gives the claimed ε-free control with $\varphi_{\infty,p}(t)$. $\bar\delta^2_t \le \bar\delta^2_0\, \varphi_{\infty,p}(t), \qquad \varphi_{\infty,p}(t) := \inf_{\varepsilon\in(0,1/\rho_p-1)} \varphi^{\varepsilon}_{\infty,p}(t).$ Now, note that

$$\forall t \ge 0, \quad 0 \le \varphi_{\infty,p}(t) := \inf_{\varepsilon\in(0,1/\rho_p-1)} \varphi^{\varepsilon}_{\infty,p}(t) \le \inf_{\varepsilon\in(0,1/\rho_p-1)} \frac{1+1/\varepsilon}{1-\rho_p(1+\varepsilon)} = M_*^{\epsilon=\frac{1}{\sqrt{\rho_p}}-1} = \frac{1}{(1-\sqrt{\rho_p})^2}.$$

Moreover, $\limsup_{t\to\infty}\varphi_{\infty,p}(t) = \limsup_{t\to\infty}\inf_{\varepsilon\in(0,1/\rho_p-1)}\varphi^{\varepsilon}_{\infty,p}(t) \le \limsup_{t\to\infty}\varphi^{\varepsilon_*}_{\infty,p}(t) = \ell^{\varepsilon_*}_{\infty,p}$ so that $\lim_{t\to\infty}\varphi_{\infty,p}(t) \le \frac{a^2\phi^2_\infty}{\left(1-\sqrt{\rho_p}\sqrt{1-a^2\phi^2_\infty}\right)^2}$, with equality if the uniform convergence $\sup_{\varepsilon\in(0,1/\rho_p-1)}\left|\varphi^{\varepsilon}_{\infty,p}(t) - \ell(\varepsilon)\right| \xrightarrow[t\to\infty]{} 0$ holds. Hence $\|X_t - X'_t\|_p \le \varphi_{\infty,p}(t)^{1/2}\,\|X_0 - X'_0\|_p$ for every $t \ge 0$, and therefore, by coupling, for the p-Wasserstein distance between marginals,$W_p([X_t],[X'_t]) \le \varphi_{\infty,p}(t)^{1/2}\,W_p([X_0],[X'_0])$. In particular, if $a = 0$ the asymptotic bound above yields $\varphi_{\infty,p}(t) \to 0$ and thus the process is contracting in W_p as $t\to\infty$. This completes the proof. □

Remark. 1. The function $\varphi_{\infty,p}$ quantifies the time decay of the L^p discrepancy between two solutions of the SVIE with different initial values. If ς is bounded (i.e. $\|\varsigma^2\|_\infty < +\infty$) and both $\kappa < \frac{\lambda^2}{(C^{\mathrm{BDG}}_p)^2\,(1+\varepsilon_*)\,\|\varsigma^2\|_\infty\int_0^{+\infty} f^2_\lambda(u)\,du}$ and $(\phi - f_\lambda * \phi) \in L^2(\mathrm{Leb}_1)$, then one derives from equation (4.40) and using Fubini-Tonelli's theorem that:

$$\int_0^{+\infty}\varphi_{\infty,p}(s)\,ds \le \int_0^{+\infty}\varphi^{\varepsilon_*}_{\infty,p}(s)\,ds \le \frac{\lambda^2\left(1+\frac{1}{\varepsilon_*}\right)}{\lambda^2 - (C^{\mathrm{BDG}}_p)^2\,\kappa(1+\varepsilon_*)\|\varsigma^2\|_\infty\|f_\lambda\|^2_{L^2(\mathrm{Leb}_1)}}\int_0^{+\infty}(\phi - f_\lambda*\phi)^2(t)\,dt < \infty.$$

2. L^2*-confluence:* Under the assumption of Proposition 4.4 with $c \in \left(0, \frac{1}{\kappa}\right)$, $\rho := c\kappa$ and $X_0, X'_0 \in L^2(\mathbb{P})$. By [21, Proposition 5.3] (which use Itô's Isometry and the first Dini Lemma), one has that the solutions to Volterra equation (3.12) denoted $(X_t)_{t\ge0}$ and $(X'_t)_{t\ge0}$ starting from X_0 and X'_0 respectively satisfies:

(a) There exists a continuous non-negative function $\varphi_\infty^{\lambda,c,K,\phi} =: \varphi_\infty : \mathbb{R}^+ \to [0,1]$, such that $\varphi_\infty(0) = 1$, $\lim_{t\to+\infty} \varphi_\infty(t) = \frac{a^2\phi_\infty^2}{1-\rho(1-a^2\phi_\infty^2)}$, only depending on λ, c, ϕ, and K, such that :

$$\forall t \geq 0,\ \mathbb{E}\left(\left|X_t - X'_t\right|\right)^2 \leq \varphi_\infty(t)\mathbb{E}\left(\left|X_0 - X'_0\right|\right)^2 \quad W_2([X'_t],[X_t]) \leq \varphi_\infty(t)^{1/2} W_2([X'_0],[X_0]).$$

(b) In particular, if $a = 0$ or $\phi_\infty = 0$ and X has a fake stationary regime of type I, $\mathbb{E}X'_t \to x_\infty$, $\mathrm{Var}(X'_t) \to v_0$ as $t \to +\infty$. And more generally finite-dimensional W_2-convergence. Thus, the process X' mixes: as time increases, the random variable X'_t progressively forgets its initial mean and variance and converges to those of the limiting fake stationary regime. While, if X has a fake stationary regime of type II, its marginal distribution is unique.

4.3 Asymptotics: Long run functional weak behaviour:

In the following, $\overset{(C)}{\to}$ stands for functional weak convergence on $C(\mathbb{R}_+, \mathbb{R})$ equipped with the topology of uniform convergence on compact sets. To establish relative compactness in (b) of the below theorem, in terms of functional $\mathcal{W}_2$-compactness (quadratic Wasserstein distance), we require that $\|\sup_{t\geq 0}|X_t|\|_p < +\infty$ for some $p > 2$.

Assumption 4.5 (Integrability and Uniform Hölder continuity)**.** *Let $\lambda, c > 0$, and assume the kernel K and its λ-resolvent R_λ satisfy*

$$\int_0^{+\infty} f_\lambda^{2\beta}(u)\,du < +\infty \quad \text{for some } \beta \geq 1, \quad \text{so that} \quad f_\lambda \in \mathcal{L}^2(Leb_1),$$

and there exists $\vartheta \in (0,1]$, and a real constant $C < +\infty$ such that[6]

$$\max_{i=1,2}\left[\int_0^{+\infty} |f_\lambda(u+\bar\delta) - f_\lambda(u)|^i\,du\right]^{\frac{1}{i}} \leq C\bar\delta^\vartheta.$$

Theorem 4.6. *Let $\lambda, c > 0$, let $\mu_\infty \in \mathbb{R}$, and let $\mu : \mathbb{R} \to \mathbb{R}$ a bounded bornel funtion, $\sigma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function satisfying equation (4.33) (SL_σ). Assume Assumption 4.5 and Assumption 3.8 on Equation (E_λ) are in force. Let $(X_t)_{t\geq 0}$ be the solution to the scaled Stochastic Volterra Integral Equation (3.12) starting from $X_0 \in L^p(\mathbb{P})$ for some suitable p.*

(a) C-tightness of time-shifted processes. *Assume*

$$X_0 \in L^p(\mathbb{P}) \quad \text{and} \quad \begin{cases} p = 2 \quad \text{and} \quad c < \frac{1}{\kappa_2} \quad \text{if} \quad (\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta}) > \frac{1}{2}, \\ p > \frac{1}{\delta\wedge\vartheta\wedge\frac{\beta-1}{2\beta}} \quad \text{and} \quad c < \frac{1}{(C_p^{BDG})^2\kappa_2} \quad \text{if} \quad (\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta}) \leq \frac{1}{2}. \end{cases} \tag{4.41}$$

Then, the family of shifted processes $(X_{t+u})_{u\geq 0}$ is C-tight, uniformly integrable, and square uniformly integrable for $p > 2$ as $t \to +\infty$. For any limiting distribution P on $\Omega_0 := C(\mathbb{R}_+, \mathbb{R})$, the canonical process $Y_t(\omega) = \omega(t)$ has a $\left(\delta \wedge \vartheta \wedge \frac{\beta-1}{2\beta} - \frac{1}{p} - \eta\right)$-Hölder pathwise continuous P-modification for sufficiently small $\eta > 0$.

That is, there exists a process X^∞ with continuous sample paths such that

$$(X_{t+u})_{t\geq 0} \Rightarrow (X_t^\infty)_{t\geq 0} \quad \text{weakly in } C(\mathbb{R}_+;\mathbb{R}) \text{ as } u \to \infty.$$

Any limiting process X^∞ satisfies $\forall t \geq 0$, $X_t^\infty \in L^p(\mathbb{P})$ for each $p \geq 2$ and its first moment is given by $\mathbb{E}[X_t^\infty] = a\phi_\infty\mathbb{E}[X_0] + (1-a)\frac{\mu_\infty}{\lambda}$.

Moreover, if $a = 0$, the shifted processes of two solutions $(X_t)_{t\geq 0}$ and $(X'_t)_{t\geq 0}$ are L^2-confluent, i.e. there exists a non-increasing function $\bar\varphi_\infty : \mathbb{R}_+ \to [0,1]$ with $\lim_{t\to+\infty}\bar\varphi_\infty(t) = 0$, and

$$W_2\left([(X_{t+t_1},\ldots,X_{t+t_N})], [(X'_{t+t_1},\ldots,X'_{t+t_N})]\right) \to 0 \quad \text{as} \quad t \to +\infty.$$

6. Uniform Hölder continuity or Hölder regularity with exponent ϑ for the function $f_{\alpha,\lambda}$, ensuring controlled behavior as t and $t+\delta$ become arbitrarily close.

Hence, the functional weak limiting distributions of $[X_{t+\cdot}]$ and $[X'_{t+\cdot}]$ coincide, meaning that if $[X_{t_n+\cdot}] \overset{(C)}{\to} P$ for some subsequence $t_n \to +\infty$, then $[X'_{t_n+\cdot}] \overset{(C)_w}{\to} P$ and vice versa.

(b) Functional weak long-run behavior. *Assume furthermore that the solution $(X_t)_{t\geq 0}$ of the volterra equation (3.12) has a fake stationary regime of type I, starting from a random variable $X_0 \in L^2(P)$ with mean $\frac{\mu_\infty}{\lambda}$ and variance v_0.Then for any limiting distribution X^∞, $\mathbb{E}[X_t^\infty] = \frac{\mu_\infty}{\lambda}$ while its autocovariance function is, for $t_1, t_2 \geq 0$, $t_1 \leq t_2$, given by $Cov(X_{t_1}^\infty, X_{t_2}^\infty) = C_{f_\lambda}(t_1, t_2)$*

$$Cov(X_{t+t_1}, X_{t+t_2}) \overset{t\to+\infty}{\to} C_{f_\lambda}(t_1, t_2) := a^2\phi_\infty^2 \mathrm{Var}(X_0) + \frac{(1-a^2\phi_\infty^2)v_0}{\int_0^{+\infty} f_\lambda^2(s)ds} \int_0^{+\infty} f_\lambda(t_2 - t_1 + u) f_\lambda(u) du. \quad (4.42)$$

Thus, under any limiting distribution P, the canonical process Y is a (weak) L^2-stationary process[7] *with mean x_∞ and covariance function $\bar{C}_{f_\lambda}(s, t)$, for $s, t \geq 0$.*

(c) Stationary Gaussian Case. *If $\sigma(x) = \sigma > 0$ is constant and X_0 has a Gaussian distribution, (say $X_0 \sim \mathcal{N}(x_\infty, v_0)$), then $(X_t)_{t\geq 0}$ satisfies $X_{t+\cdot} \xrightarrow{(C)} \mathcal{GP}(f_\lambda)$ as $t \to +\infty$, where $\mathcal{GP}(f_\lambda)$ is the stationary Gaussian process with mean x_∞ and covariance function $C_{f_\lambda}(\cdot)$.*

Remark: Be aware that at this stage, we do not have uniqueness of the limit distributions since they are not characterized by their mean and covariance functions, except in Gaussian setting.

5 Applications to Fractional Stochastic Volterra Integral equations

Let consider the below **Fractional integration kernel** where $\alpha = H + \frac{1}{2}$, with H denoting the Hurst coefficient:

$$K(t) = K_\alpha(t) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} \mathbf{1}_{\mathbb{R}_+}(t), \quad \alpha > 0. \quad (5.43)$$

This family of kernels corresponds to the fractional integrations of order $\alpha > 0$ and satisfy (2.8), (2.3) and (2.4) for $\alpha > 1/2$ (with $\theta_T = (\alpha - \frac{1}{2}) \wedge 1$, see [42, 30]) among many others. It follows from the easy identity $K_\alpha * K_{\alpha'} = K_{\alpha+\alpha'}$, that $R_{\alpha,\lambda}(t) = \sum_{k\geq 0}(-1)^k \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k+1)} = E_\alpha(-\lambda t^\alpha) = e_\alpha(\lambda^{1/\alpha} t)$ $t \geq 0$, where E_α denotes the standard Mittag-Leffler function and e_α, the alternate Mittag-Leffler function.

$$E_\alpha(t) = \sum_{k\geq 0} \frac{t^k}{\Gamma(\alpha k + 1)}, \; t \in \mathbb{R} \quad \textit{and} \quad e_\alpha(t) := E_\alpha(-t^\alpha) = \sum_{k\geq 0} (-1)^k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad t \geq 0. \quad (5.44)$$

In section 5 of [36] (see [25] further on), the author demonstrated that for such kernels K_α, with $\frac{1}{2} < \alpha < 1$ (" rough models"), E_α is increasing and differentiable on the real line with $\lim_{t\to+\infty} E_\alpha(t) = +\infty$ and $E_\alpha(0) = 1$. In particular, E_α is an homeomorphism from $(-\infty, 0]$ to $(0, 1]$. Consequently, the resolvent $R_{\alpha,\lambda}$ satisfies its established monotonicity condition $(\mathcal{K})$ for all $\lambda > 0$. Moreover, it was shown that if $\lambda > 0$, the function $f_{\alpha,\lambda} := -R_{\alpha,\lambda}$ exists and is defined on $(0, +\infty)$ by:
$f_{\alpha,\lambda}(t) = -R'_{\alpha,\lambda}(t) = \alpha\lambda t^{\alpha-1} E'_\alpha(-\lambda t^\alpha) = \lambda t^{\alpha-1} \sum_{k\geq 0} (-1)^k \lambda^k \frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}$ so that for $\alpha \in (\frac{1}{2}, 1)$, $f_{\alpha,\lambda}$ is a probability densitycalled *Mittag-Leffler density* and is square-integrable with respect to the Lebesgue measure on $\mathbb{R}_+$. Consequently, the results established in [36], particularly in Sections 2, 3, and 4, apply to the case $\sigma(t, x) = \sigma(t)$ (Gaussian setting) and $\sigma(t, x) = \varsigma(t)\sigma(x)$.

Note that, in this paper, *our Assumption $(\mathcal{K})$ is a slightly relaxed version of that of [36].* The purpose of this part is to extend these results to the general case where $\alpha \in \mathbb{R}_+^*$. We show that for

7. Weak-stationarity in the sense of constant mean, variance and stable autocovariance function (see for example [31]) in constrat to Strong-stationarity where all finite-dimensional distributions are invariant under time shifts.

$0 < \alpha < 2$, the resolvent $R_{\alpha,\lambda}$ of K_α satisfy our relaxed monotonicity assumption $(\mathcal{K})$ for all $\lambda > 0$, and that $f_{\alpha,\lambda} := -R_{\alpha,\lambda}$ exists and is square-integrable with respect to the Lebesgue measure on $\mathbb{R}_+$, both for $\frac{1}{2} < \alpha < 1$ ("rough models") and $\frac{1}{2} < \alpha < \frac{3}{2}$ ("long memory volatility models"). As a result, the findings from Sections 3 and 4, will be applicable in the cases where $\sigma(t,x) = \sigma(x)$ (Gaussian setting) and $\sigma(t,x) = \varsigma(t)\sigma(x)$. To this end, by a scaling property, it is enough to study $R_{\alpha,1}$ ($\lambda = 1$) given by its expansion $E_\alpha(-t^\alpha)$ where E_α in the literature is known as *Mittag-Leffler function*.

We will leverage the conducive class of completely monotone functions. Let us recall that a function $\varphi : (0,+\infty) \to [0,+\infty)$ is called a *completely monotone* (CM) function if it is non-negative, C^∞ (i.e. it is infinitely differentiable on $(0,+\infty)$), and satisfies $(-1)^n\varphi^{(n)}(t) \geq 0$ for all $n \in \mathbb{N}$ and $t > 0$.

Crucially, "Bernstein-Widder theorem" [44, Theorem 1.4] (see also [7]) provides a necessary and sufficient condition a function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ to be CM. More specifically, φ is CM if it is a (real valued) Laplace transform of a unique non-negative measure μ on $[0,\infty)$. Futhermore, a result by Pollard [45] state that a CM function can be obtained by composing a CM function with a Bernstein function [8]

5.1 α-fractional kernels with $\alpha > 0$

The Mittag-Leffler function $E_\alpha(z)$, with $\alpha > 0$, generalizes the exponential function (attained with $\alpha = 1$). It is defined by a power series, which converges on the entire complex plane. In particular, we are interested in the alternate Mittag-Leffler function reading:

$$e_\alpha(t) := E_\alpha(-t^\alpha) = \sum_{k\geq 0}(-1)^k\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}, \quad t \geq 0, \quad E_\alpha(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0, \quad z \in \mathbb{C}.$$

In the limiting cases $\alpha = 1$ and $\alpha = 2$, $e_\alpha(t)$ are elementary functions, namely $e_1(t) = e^{-t}$ and $e_2(t) = \cos t$. Integral representations of the Mittag-Leffler function E_α were first established in [1], followed by further results in [33], where they were connected by the Laplace transform. For instance, (see (F.12) in [33]), the Laplace transform of $E_\alpha(-at^\alpha)$, with $a \in \mathbb{C}$, is given by: $L_{E_\alpha(-at^\alpha)}(z) = \frac{z^{\alpha-1}}{z^\alpha + a}, \quad z \in \mathbb{C},\ \Re(z) > |a|^{1/\alpha}, \quad \alpha > 0$. From this, we can deduce the Laplace transform of e_α, which is given by: $L_{e_\alpha}(z) = \frac{z^{\alpha-1}}{z^\alpha+1}, \quad z \in \mathbb{C},\ \Re(z) > 1, \quad \alpha > 0$. Here, we define $z^\alpha := |z|^\alpha e^{i\alpha\arg(z)}$, where $-\pi < \arg(z) < \pi$, that is in the complex z-plane cut along the negative real axis

5.1.1 α-fractional kernels for $\alpha \in \mathbb{R}_+^*$

Proposition 5.1. *The followings hold for the the alternate Mittag-Leffler function for any $t \geq 0$:*

1. *If $\alpha \in \mathbb{R}_+^* \setminus \mathbb{N}^*$, $e_\alpha(t) = F_\alpha(t) + G_\alpha(t)$ where $F_\alpha(t) := \int_0^{+\infty} e^{-tu}H_\alpha(u)\,du$ with $\forall\, u \in \mathbb{R}_+$, $H_\alpha(u) = \frac{\sin(\alpha\pi)}{\pi}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\pi\alpha)+1}$ and $G_\alpha(t) := \frac{2}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\exp\left[t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right]\cos\left[t\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right]$*
2. *If $\alpha \in \mathbb{N}^*$, $e_\alpha(t) = G_\alpha(t) = \frac{2}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\exp\left[t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right]\cos\left[t\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right]$*

The result or representation of the above proposition 5.1 is an extension of the case $\alpha \in (0,2)$ studied in [23] to the general case $\alpha \in \mathbb{R} \setminus \mathbb{N}$. The second claim is straigthforward as the function H_α vanishes identically if α is an integer.

Proof. Based on the inverse Laplace transform (Bromwich-Mellin formula [9]), we have the below representation as a Laplace inverse integral: For γ larger than the real parts of all poles of the integrand,

$$e_\alpha(t) = \frac{1}{2\pi i}\int_{Br(\gamma,\infty)} e^{zt}\frac{z^{\alpha-1}}{z^\alpha+1}\,dz, = \frac{1}{2i\pi}\int_{z=\gamma-i\cdot\infty}^{z=\gamma+i\cdot\infty} e^{zt}\frac{z^{\alpha-1}}{z^\alpha+1}dz = \frac{1}{2\pi i}\lim_{R\to+\infty}\int_{Br(\gamma,R)} e^{zt}\frac{z^{\alpha-1}}{z^\alpha+1}\,dz. \tag{5.45}$$

8. A function $\psi : \mathbb{R}_+ \to \mathbb{R}$ is called a Bernstein function if it is of class $\mathcal{C}^\infty$, is non-negative, and its derivative is CM.
9. on the Bromwich path, i.e., a line $\mathrm{Re}\{z\} = a$ with $a \geq 1$, and $\mathrm{Im}\{z\}$ running from $-\infty$ to $+\infty$.

Let $J_\alpha(t,\cdot) : z \to e^{zt}\frac{z^{\alpha-1}}{z^\alpha+1}$ be the integrand of the above representaion. The relevant poles of $J_\alpha(t,\cdot)$ or rather $\frac{z^{\alpha-1}}{z^\alpha+1}$ is the set $\mathbb{S} := \{z_n = \exp\left(i\frac{(2n+1)\pi}{\alpha}\right), n = 0, \cdots, \lfloor \alpha - 1 \rfloor\}$. $J_\alpha(t,\cdot)$ is thus holomorphic/analytic on $\mathbb{C} \setminus \mathbb{S}$. And since 0 is a brand-point of the integrand $J_\alpha(t,\cdot)$, we consider $\Gamma_{\gamma,\delta,R}$ the Jordan contour (see Figure 1) defined as the union of the below-represented several distinct paths:

$$\Gamma_{\gamma,\delta,R} = Br(\gamma, R) \cup C^+ \cup C_R^+ \cup (-H(\delta, \tfrac{1}{R})) \cup C_R^- \cup C^-,$$

— $H(\delta, \frac{1}{R})$ is the *Hankel Contour* given by $H(\delta, \frac{1}{R}) := [-R + i\delta, -c + i\delta] \cup C_{\frac{1}{R}} \cup [-R - i\delta, -c - i\delta]$, where $C_{\frac{1}{R}}$ is the small circular arc $|s| = \frac{1}{R}$.

— $Br(\gamma, R)$, the *truncated Bromwich Path* i.e. $Br(\gamma, R) := [\gamma - iR, \gamma + iR]$, where $\gamma \geq 1$ and $\text{Re}\{z\} = \gamma$, with $\text{Im}\{z\} \in [-R, R]$.

— $C^+ := [\gamma + iR, iR]$ and $C^- := [-iR, \gamma - iR]$.

— C_R^+ and C_R^- denote the upper and lower semi-circular arcs, respectively, of a circle of radius R; C_R^+ runs from iR to $-R+i\delta$, and C_R^- from $-R - i\delta$ to $-iR$.

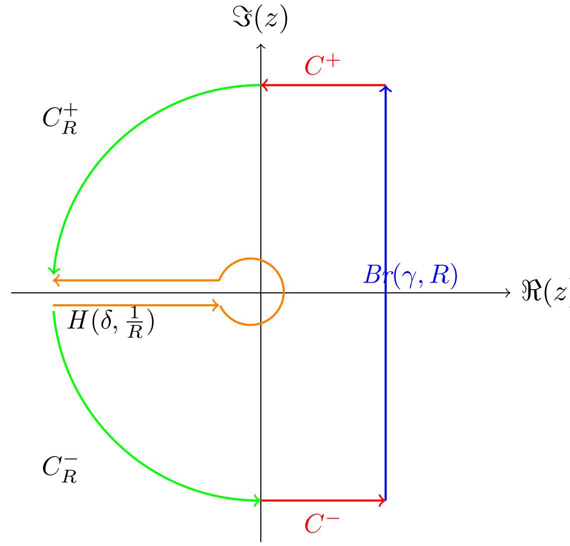

Figure 1 – Jordan contour $\Gamma_{\gamma,\delta,R}$.

For small values of δ, large values of R, and $\gamma \geq 1$, the Jordan contour $\Gamma_{\gamma,\delta,R}$ encloses all poles of $J_\alpha(t,\cdot)$. Therefore, by the *Jordan-Cauchy Residue Theorem*, we have:

$$\sum_{z\in\mathbb{C}\setminus\{-1\}:z^\alpha=-1} \text{Res}(J_\alpha(t,\cdot), z) = \frac{1}{2\pi i}\oint_{\Gamma_{\gamma,\delta,R}} J_\alpha(t,z)\,dz = \frac{1}{2\pi i}\int_{Br(\gamma,R)} J_\alpha(t,z)\,dz + \frac{1}{2\pi i}\int_{C^+} J_\alpha(t,z)\,dz$$
$$+ \frac{1}{2\pi i}\int_{C_R^+} J_\alpha(t,z)\,dz - \frac{1}{2\pi i}\int_{H(\delta,\frac{1}{R})} J_\alpha(t,z)\,dz + \frac{1}{2\pi i}\int_{C_R^-} J_\alpha(t,z)\,dz + \frac{1}{2\pi i}\int_{C^-} J_\alpha(t,z)\,dz.$$

Taking the limit as $R \to \infty$ and $\delta \to 0$, we may decompose (5.45) as follows

$$e_\alpha(t) := \frac{1}{2\pi i}\lim_{R\to+\infty}\int_{Br(\gamma,R)} J_\alpha(t,z)\,dz = \sum_{z\in\mathbb{C}\setminus\{-1\}:z^\alpha=-1} \text{Res}(J_\alpha(t,\cdot), z) + \frac{1}{2\pi i}\lim_{R\to+\infty}\lim_{\delta\to0}\int_{H(\delta,\frac{1}{R})} J_\alpha(t,z)\,dz$$
$$- \frac{1}{2\pi i}\left(\lim_{R\to+\infty}\int_{C^+} J_\alpha(t,z)\,dz + \lim_{R\to+\infty}\int_{C_R^+} J_\alpha(t,z)\,dz + \lim_{R\to+\infty}\int_{C_R^-} J_\alpha(t,z)\,dz + \lim_{R\to+\infty}\int_{C^-} J_\alpha(t,z)\,dz\right).$$

We now examine these six terms. The contribution from the Hankel path is given by $\frac{1}{2\pi i}\int_{H(\delta,\frac{1}{R})} J_\alpha(t,z)\,dz$, whose limit coincides with the usual contour representation of the Mittag-Leffler function for $\alpha \in (0, 1)$.

$$\frac{1}{2\pi i}\int_{H(\delta,\frac{1}{R})} e^{zt}\frac{z^{\alpha-1}}{z^\alpha+1}\,dz \overset{R\to+\infty,\delta\to0}{=} \int_0^{+\infty} e^{-tu} H_\alpha(u)\,du = L_{H_\alpha}(t) =: F_\alpha(t). \tag{5.46}$$

where a synthetic formula was found for H_α in [33] (see (F.22) p.31, see also [32] in the case $0 < \alpha < 1$).

$$\forall\, u\in\mathbb{R}_+,\; H_\alpha(u) = -\frac{1}{2\pi}\cdot 2\,\Im\text{m}\Big(\frac{z^{\alpha-1}}{z^\alpha+1}\Big)_{|z=ue^{i\pi}} = \frac{\sin(\alpha\pi)}{\pi}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\pi\alpha)+1} \tag{5.47}$$

Note that this representation of F_α in term of the Laplace transform of a non-negative Lebesgue integrable function (see Equation (5.46) above) was first established in [1].

Also note that the function H_α vanishes identically if α is an integer. The limit of the other integrals vanishes. In fact: $\left|\int_{C_R^+} J_\alpha(t,z)\,\mathrm{d}z\right| \leq \int_{\frac{\pi}{2}}^{\pi} R|J_\alpha(t,e^{i\theta})|\,\mathrm{d}\theta$ and $|J_\alpha(t,e^{i\theta})| \leq \frac{R^{\alpha-1}}{R^\alpha-1}e^{tR\cos(\theta)} \leq \frac{R^{\alpha-1}}{R^\alpha-1}e^{tR(-\frac{2}{\pi}\theta+1)}$ where in the last inequality, we used the trick $\cos(\theta) \leq -\frac{2}{\pi}\theta+1 \quad \forall\theta\in[\frac{\pi}{2},\pi]$. Consequently, $\left|\int_{C_R^+} J_\alpha(t,z)\,\mathrm{d}z\right| \leq \frac{R^{\alpha-1}}{R^\alpha-1}\times\frac{\pi}{-2tR}\left[e^{tR(-\frac{2}{\pi}\theta+1)}\right]_{\theta=\frac{\pi}{2}}^{\theta=\pi} = \frac{\pi R^\alpha}{2t(R^{\alpha+1}-R)}(1-e^{-tR}) \overset{R\to\infty}{\longrightarrow} 0$ Likewise $\lim_{R\to+\infty}\left|\int_{C_R^-} J_\alpha(t,z)\,\mathrm{d}z\right| = 0$. Moreover: $\int_{C^+} J_\alpha(t,z)\,dz = \int_{\gamma+iR}^{iR} J_\alpha(t,z)\,dz = \int_\gamma^0 J_\alpha(t,x+iR)\,dx$. Now, observe that: $|J_\alpha(t,x+iR)| \leq \frac{(x^2+R^2)^{\frac{\alpha-1}{2}}}{(x^2+R^2)^{\frac{\alpha}{2}}-1}e^{tx} \leq e^{tx}\frac{(\gamma^2+R^2)^{\frac{\alpha-1}{2}}}{R^\alpha-1}$. As a consequence, $\left|\int_{C^+} J_\alpha(t,z)\,\mathrm{d}z\right| \leq \frac{(\gamma^2+R^2)^{\frac{\alpha-1}{2}}}{R^\alpha-1}\int_\gamma^0 e^{tx}\,\mathrm{d}x \overset{R\to\infty}{\longrightarrow} 0$. Likewise for $\lim_{R\to+\infty}\left|\int_{C^-} J_\alpha(t,z)\,\mathrm{d}z\right| = 0$. Finally,

$$G_\alpha(t) := \sum_{z\in\mathbb{C}\backslash\{-1\}:z^\alpha=-1} \mathrm{Res}(J_\alpha(t,\cdot),z) = \sum_{z_n\in\mathbb{S}} \mathrm{Res}(J_\alpha(t,\cdot),z_n) = \sum_{n=0}^{\lfloor\alpha-1\rfloor} e^{z_nt}\,\mathrm{Res}\left[\frac{z^{\alpha-1}}{z^\alpha+1}\right]_{z_n} = \frac{1}{\alpha}\sum_{n=0}^{\lfloor\alpha-1\rfloor} e^{z_nt},$$

Note that, $e^{z_nt}+e^{\bar{z}_nt} = e^{\mathrm{Re}\{z_n\}t}\left(e^{\mathrm{Im}\{z_n\}t}+e^{-\mathrm{Im}\{z_n\}t}\right) = 2e^{\mathrm{Re}\{z_n\}t}\cos\left(\mathrm{Im}\{z_n\}t\right)$ and $\sum_{z_n\in\mathbb{S}}\mathrm{Res}(J_\alpha(t,\cdot),z_n) = \frac{1}{\alpha}\sum_{n=0}^{\lfloor\alpha-1\rfloor} e^{z_nt} = \frac{1}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\left(e^{z_nt}+e^{\bar{z}_nt}\right)$. As a consequence,

$$G_\alpha(t) := \sum_{z\in\mathbb{C}\backslash\{-1\}:z^\alpha=-1} \mathrm{Res}(J_\alpha(t,\cdot),z) = \frac{1}{\alpha}\sum_{n=0}^{\lfloor\alpha-1\rfloor} e^{z_nt} = \frac{2}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\exp\left[t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right]\cos\left[t\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right]$$

Remark: 1. For $0<\alpha<1$, there are no relevant poles since $|\arg(z_k)|>\pi$, so $G_\alpha(t)\equiv 0$, and we obtain $e_\alpha(t) = F_\alpha(t)$, for $0<\alpha<1$. For $1<\alpha<2$, there are exactly two relevant poles, $z_0 = \exp(i\pi/\alpha)$ and $z_{-1} = \exp(-i\pi/\alpha) = \bar{z}_0$, located in the left half-plane. In this case, we have $G_\alpha(t) = \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\cos\left(t\sin\left(\frac{\pi}{\alpha}\right)\right)$ and $e_\alpha(t) = \int_0^{+\infty} e^{-tu}H_\alpha(u)\,du + \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\cos\left(t\sin\left(\frac{\pi}{\alpha}\right)\right)$. It is clear that the function $e_\alpha(t)$ oscillates in an evanescent manner to 0 as $t\to+\infty$. We note that this function exhibits oscillations with circular frequency and an exponentially decaying amplitude (see Figure 3). Note that, the above expression of e_α is the same for $2<\alpha<3$ with the only difference that the two poles are now located in the right haft plane, and so providing amplified oscillations.
2. In the case $2<\alpha<+\infty$, however, certains poles are located in the right half plane, so providing amplified oscillations. This common instability for $\alpha>2$ is the reason why we will limit ourselves to consider α in the range $0<\alpha<2$ as highlighted by the below proposition.

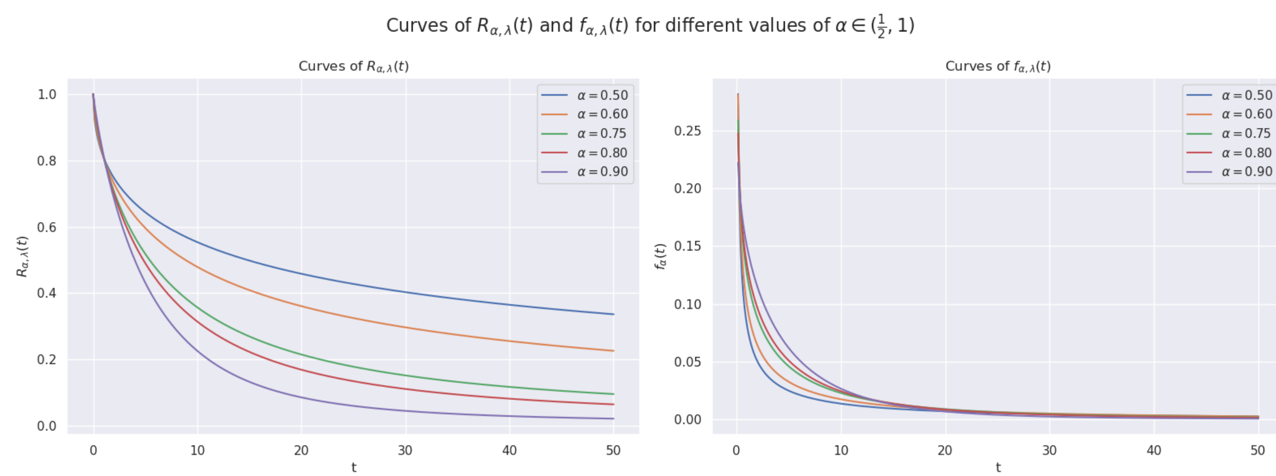

Figure 2 – Curves of $R_{\alpha,\lambda}(t)$ and $f_{\alpha,\lambda}(t)$ for different values of $\alpha\in[\frac{1}{2},1)$

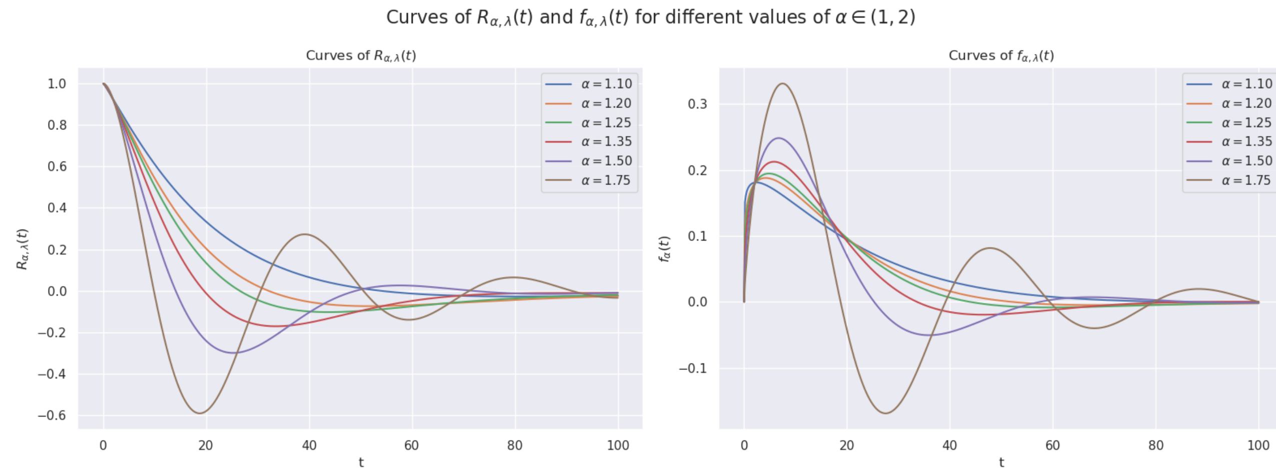

Figure 3 – Curves of $R_{\alpha,\lambda}(t)$ and $f_{\alpha,\lambda}(t)$ for different values of $\alpha \in (1,2)$

Proposition 5.1. *Let $\lambda > 0$ and let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$.*
(a) The function $(-1)^{\lfloor \alpha \rfloor} F_\alpha$ is completely monotonic(thus convex), hence infinitely differentiable on $\mathbb{R}_+^*$.
(b) The λ-resolvent $R_{\alpha,\lambda}$ satisfies $R_{\alpha,1} = e_\alpha$ and $R_{\alpha,\lambda} = R_{\alpha,1}(\lambda^{1/\alpha}\cdot)$. The function $R_{\alpha,\lambda}$ is infinitely differentiable i.e. $\mathcal{C}^\infty$ on $(0,+\infty)$. Moreover $R_{\alpha,\lambda}(0) = 1$, $R_{\alpha,\lambda} \in \mathcal{L}^r(\mathrm{Leb}_1)$ for every $r > \frac{1}{\alpha}$ and $\alpha \leq 2$. $f_{\alpha,\lambda}(t) := -R'_{\alpha,\lambda}(t)$ is infinitely differentiable and satisfy : $\forall\, t > 0$, $f_{\alpha,\lambda}(t) =$

$$\lambda^{\frac{1}{\alpha}} \left(\int_0^{+\infty} e^{-\lambda^{\frac{1}{\alpha}} tu} u H_\alpha(u) du - \frac{2}{\alpha} \sum_{n=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \exp\left[t\lambda^{\frac{1}{\alpha}} \cos\left(\frac{(2n+1)\pi}{\alpha} \right) \right] \cos\left[t\lambda^{\frac{1}{\alpha}} \sin\left(\frac{(2n+1)\pi}{\alpha} \right) - \frac{(2n+1)\pi}{\alpha} \right] \right)$$

so that, $R_{\alpha,\lambda}$ converges to $a \in [0,1)$ and $f_{\alpha,\lambda} \in \mathcal{L}^{2\beta}(\mathrm{Leb}_1)$ for every $\beta > 0$ provided $\alpha \in (0,2)$.
(c) if $\alpha \geq 2$ the $\mathcal{L}^2(\mathbb{R}_+)$-$\vartheta$-Hölder continuity of $f_{\alpha,\lambda}$ as stated in Assumption 4.5 does not holds.

Furthermore, the function $R_{\alpha,\lambda}$ satisfies the assumptions $\mathcal{K}$ (i), specifically, that $R_{\alpha,\lambda}$ converges to 0, along with the function $f_{\alpha,\lambda}$ satisfying the assumption 4.5[10] *(for the weak functional behavior property) if and only if $\alpha \in (0,2)$.*

Proposition 5.2 (α-fractional kernels $1 < \alpha < 2$)**.** *Let $\lambda > 0$ and let $\alpha \in (1,2)$.*
(a) The λ-resolvent $R_{\alpha,\lambda}$ satisfies $R_{\alpha,1} = e_\alpha$ and $R_{\alpha,\lambda} = R_{\alpha,1}(\lambda^{1/\alpha}\cdot)$. The function e_α and thus $R_{\alpha,\lambda}$ are infinitely differentiable i.e. $\mathcal{C}^\infty$ on $(0,+\infty)$) with:

$$\forall k \in \mathbb{N}, \quad e_\alpha^{(k)}(t) = F_\alpha^{(k)}(t) + G_\alpha^{(k)}(t) \quad \textit{where} \quad F_\alpha^{(k)}(t) = \int_0^{+\infty} e^{-tu} H_\alpha^{(k)}(u)\, du \tag{5.48}$$

$$H_\alpha^{(k)}(u) := (-1)^k \frac{\sin(\alpha\pi)}{\pi} \frac{u^{\alpha-1+k}}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1} \;\textit{and}\; G_\alpha^{(k)}(t) = \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \cos\left[t\sin\left(\frac{\pi}{\alpha}\right) - \frac{k\pi}{\alpha} \right]. \tag{5.49}$$

Moreover $R_{\alpha,\lambda}(0) = 1$, $R_{\alpha,\lambda}(t) \leq 1 \quad \forall t \geq 0$, $R_{\alpha,\lambda}$ converges to 0. $R_{\alpha,\lambda} \in \mathcal{L}^r(\mathrm{Leb}_1)$ for every $r > \frac{1}{\alpha}$ and $f_{\alpha,\lambda} := -R'_{\alpha,\lambda}$ is infinitely differentiable , converges to 0 and satisfy:

$$\forall\, t > 0, \quad f_{\alpha,\lambda}(t) := -R'_{\alpha,\lambda}(t) = \lambda^{\frac{1}{\alpha}} \left(\int_0^{+\infty} e^{-\lambda^{\frac{1}{\alpha}} tu} u H_\alpha(u) du - \frac{2}{\alpha} e^{t\lambda^{\frac{1}{\alpha}} \cos\left(\frac{\pi}{\alpha}\right)} \cos\left[t\lambda^{\frac{1}{\alpha}} \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha} \right] \right).$$

10. Uniform Hölder continuity or Hölder regularity with exponent ϑ for the function $f_{\alpha,\lambda}$, ensuring controlled behavior as t and $t+\delta$ become arbitrarily close.

(b) *Moreover, if* $\alpha \in (1,2)$, $f_{\alpha,\lambda} \in \mathcal{L}^{2\beta}(\mathrm{Leb}_1)$ *for every* $\beta > 0$ *and for* $i \in \{1,2\}$, *for every* $\vartheta \in \big(0, \alpha - \frac{\mathbf{1}_{i=2}}{i}\big)$, *there exists a real constant* $C_{\vartheta,\lambda} > 0$ *such that*

$$\forall\, \delta > 0, \quad \Big[\int_0^{+\infty} \big(f_{\alpha,\lambda}(t+\delta) - f_{\alpha,\lambda}(t)\big)^i\, dt\Big]^{1/i} \le C_{\vartheta,\lambda}\delta^{\vartheta}.$$

For clarity and conciseness, the proofs of Propositions 5.1 and 5.2 are postponed to Appendix B.

Theorem 5.3. *Let* $\alpha \in \big(1, \frac{3}{2}\big)$ *(and more generally* $\alpha \in \big(\frac{1}{2}, \frac{3}{2}\big)$*), let* $K(t) = K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ *the fractional kernel, let* $\sigma(t,x) := \varsigma(t)\sigma(x)$ *with* σ *a Lipschitz continuous function given by equation* (3.32), *thus satisfying a relation of the type* (4.33) (SL_σ) *with* $\kappa := \kappa_2 > 0$, *let* $c \in \Big(0, \frac{1}{\kappa_2}\Big)$ *with* $\varsigma = \varsigma_{\lambda,c}$, $\lambda > 0$ *and let* $X_0 \in L^2(\mathbb{P})$ *such that* $\mathbb{E}[X_0] = x_\infty$ *and* $Var(X_0) = v_0 = \frac{c\sigma^2(x_\infty)}{1-c\kappa_2}$. *Then,*

1. *For fractional kernels* K_α *with* $1 < \alpha < 2$, *the solution* $(X_t)_{t\ge0}$ *to the Volterra equation* (3.12) *starting from* X_0 *has a fake stationary regime of type I in the sense that:*
$$\forall\, t \ge 0, \qquad \mathbb{E}\, X_t = x_\infty, \quad \mathrm{Var}(X_t) = v_0 = \frac{c\sigma^2(x_\infty)}{1-c\kappa_2} \;\; \text{and} \quad \mathbb{E}\,\sigma^2(X_t) = \bar\sigma_0^2 = \frac{\sigma^2(x_\infty)}{1-c\kappa_2}.$$
2. *If* $a = 0$ *or* $\phi_\infty = 0$, $\forall X_0' \in L^2(\mathbb{P})$, *a solution to* (3.12) *starting from* X_0' *satisfies* $\|X_t' - X_t\|_2 \underset{t\to\infty}{\to} 0$
3. *The family of shifted processes* $X_{t+\cdot}, t \ge 0$, *is* C*-tight as* $t \to +\infty$ *and its (functional) limiting distributions are all* L^2*-stationary processes with covariance function* C_∞ *given by* (4.42).

Proof. (1), (2) are consequences of Proposition 3.11. If $0 < \vartheta < \alpha - \frac{1}{2}$ and $\beta > 1$, Theorem 4.6 applies.

5.2 The function $\varsigma^2_{\alpha,\lambda,c}$ solution of the stabilizer equation when $\alpha \in (0,2)$

In this section we want to compute $\varsigma_{\lambda,c}$ as a power series in $t^{k\alpha}$. To this end we rely on the Laplace version of the equation *$(E_{\lambda,c})$* in (3.25) satisfied by $\varsigma^2_{\lambda,c}$: $c\lambda^2\big(1 - (\phi - f_\lambda * \phi)^2(t)\big) = (f_\lambda^2 * \varsigma^2)(t)\ \forall\, t \ge 0$, for which the laplace transform is given by equation (3.26) in Lemma 3.9:

$$\forall\, t > 0, \quad t\, L_{f_\lambda^2}(t).L_{\varsigma^2}(t) = -2\, c\lambda^2 L_{(\phi - f_\lambda * \phi)(\phi - f_\lambda*\phi)'}(t).$$

Given the kernel $K_\alpha(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}$ and the expansion of the resolvents $R_{\alpha,\lambda}$ and it derivative $-f_{\alpha,\lambda}$, $\forall t \ge 0$,

$$R_{\alpha,\lambda}(t) = \sum_{k\ge0} (-1)^k \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k+1)} = E_\alpha(-\lambda t^\alpha),\ f_{\alpha,\lambda}(t) = \alpha\lambda t^{\alpha-1}E'_\alpha(-\lambda t^\alpha) = \lambda t^{\alpha-1}\sum_{k\ge0}(-1)^k\frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha(k+1))}. \quad (5.50)$$

Since $\phi(t) - (f_\lambda * \phi)(t) = 1 - \frac{(f_\lambda*\mu)_t}{\lambda x_\infty}$ we have $\phi(t) - (f_\lambda * \phi)(t) \overset{0}{\sim} 1$ and by Lemma 3.9 (2) $(\phi(t) - (f_\lambda * \phi)(t))' \overset{0}{\sim} -\frac{\mu(0)}{\lambda x_\infty} f_\lambda(t)$, so that: $(\phi - f_\lambda * \phi)(\phi - f_\lambda * \phi)'(t) \overset{0}{\sim} -\frac{\mu(0)}{\lambda x_\infty}\frac{\lambda t^{\alpha-1}}{\Gamma(\alpha)}$ and $f_\lambda^2(t) \overset{0}{\sim} \frac{\lambda^2 t^{2(\alpha-1)}}{\Gamma(\alpha)^2}$.
It follows that – at least heuristically ([11]) –

$$L_{(\phi - f_\lambda*\phi)(\phi - f_\lambda*\phi)'}(t) \overset{+\infty}{\sim} -\lambda\frac{\mu(0)}{\lambda x_\infty}t^{-\alpha} \quad \text{and} \quad L_{f_\lambda^2}(t) \overset{+\infty}{\sim} \frac{\lambda^2\Gamma(2\alpha-1)t^{-(2\alpha-1)}}{\Gamma(\alpha)^2}.$$

This implies that $L_{\varsigma^2}(t) \overset{+\infty}{\sim} 2\lambda\, c\frac{\mu(0)}{\lambda x_\infty}\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)}t^{-(2-\alpha)}$ owing to Equation (3.26). This in turn suggests that

$$\varsigma^2(t) \overset{0}{\sim} \frac{2\lambda c\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\lambda x_\infty}t^{1-\alpha} \quad \text{so that} \begin{cases} (i) & \varsigma(0) = 0 \text{ if } \alpha < 1, \\ (ii) & \lim_{t\to0^+}\varsigma(t) = +\infty \text{ if } \alpha > 1 \text{ provided } \frac{\mu(0)}{\lambda x_\infty} > 0. \end{cases} \quad (5.51)$$

This suggests to search $\varsigma^2(t)$ of the form (*Power Series Ansatz*):

$$\varsigma^2(t) = \varsigma^2_{\alpha,\lambda,c}(t) := 2\,\lambda\, c\, t^{1-\alpha}\sum_{k\ge0}(-1)^k c_k\lambda^k t^{\alpha k} \quad \text{with} \quad c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\lambda x_\infty}. \quad (5.52)$$

11. We use here heuristically a dual version of the Hardy-Littlewood Tauberian theorem for Laplace transform, namely $\varsigma^2(t) \overset{0}{\sim} Ct^\gamma$, $\gamma > -1$, iff $L_{\varsigma^2}(t) \overset{+\infty}{\sim} Ct^{-(\gamma+1)}\Gamma(\gamma+1)$. We refer to [8, 26] for a general theory of regular variation.

Remark: 1. At this point, it is crucial to emphasize that, for a fixed value of α, all functions $\varsigma^2_{\alpha,\lambda,c}$ from equation (6.58) are derived or generated from a common function, defined as

$$\varsigma^2_{\alpha,\lambda,c}(t) = c\lambda^{2-\frac{1}{\alpha}}\varsigma^2_{\alpha}\left(\lambda^{\frac{1}{\alpha}}t\right) \quad \text{with} \quad \varsigma^2_{\alpha}(t) := 2\,t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k t^{\alpha k}. \tag{5.53}$$

where the coefficients c_k depend on α. Thus, for simplicity in what follows, we will assume $c = \lambda = 1$.

2. For the computation of the function $\varsigma^2_{\alpha,\lambda,c}$, we need to establish a recurrence formula satisfied by the coefficients c_k, which involves knowing the form of the function ϕ or more specificcally, the mean-reverting function μ. In practice, since this function is usually taken to be constant equal to μ_0, we are going in the next subsection to compute and study the function $\varsigma^2_{\alpha,\lambda,c}$ when $\mu(t) = \mu_0$ a.e. and $\alpha \in (1, \frac{3}{2})$ bearing in mind that, the case when $\alpha \in (\frac{1}{2}, 1)$ have been intensively study in [36].

5.2.1 Existence and computation of the function $\varsigma^2_{\alpha,\lambda,c}$ solution of the stabilizer equation when $\alpha \in (1, \frac{3}{2})$

The recurrence formula satisfied by the coefficients c_k, which make possible the computation of the functions $\varsigma_{\alpha,\lambda,c}$ are established in the same manner as in [36]. We consider the case where $\mu(t) = \mu_0$ a.e., so that $\mu_\infty = \mu(0) = \mu_0$, and assume $\phi \equiv 1$ as in the previous subsection. We then have the following proposition, whose proof is postponed to Appendix B.

Proposition 5.2 (Existence of the function $\varsigma^2_{\alpha,\lambda,c}$ for $\alpha \in (1,2)$)**.** *Let $\alpha \in (1,2)$:*

1. $\lim_{t\to 0}\varsigma^2_{\alpha,\lambda,c} = +\infty$, *and* $\lim_{t\to+\infty}\varsigma^2_{\alpha,\lambda,c}(t) = \frac{c\lambda^2}{\|f_{\alpha,\lambda}\|^2_{L^2(Leb_1)}}$.
2. $\varsigma^2_{\alpha,\lambda,c}(t) = c\lambda^{2-\frac{1}{\alpha}}\varsigma^2_\alpha(\lambda^{\frac{1}{\alpha}}t)$ *where* $\varsigma^2_\alpha(t) := 2\,t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k t^{\alpha k}$ *and the coefficients* $(c_k)_{k\geq 0}$ *are defined as follows:* $c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}$ *and for every* $k \geq 1$,
$$c_k = \frac{\Gamma(\alpha)^2\Gamma(\alpha(k+1))}{\Gamma(2\alpha-1)\Gamma(\alpha k+2-\alpha)}\left[(a*b)_k - \alpha(k+1)\sum_{\ell=1}^{k} B\big(\alpha(\ell+2)-1, \alpha(k-\ell-1)+2\big)(b^{*2})_\ell c_{k-\ell}\right]. \tag{5.54}$$
where for two sequences of real numbers $(u_k)_{k\geq 0}$ *and* $(v_k)_{k\geq 0}$, *the Cauchy product is defined as* $(u*v)_k = \sum_{\ell=0}^{k} u_\ell v_{k-\ell}$ *and* $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$ *denoting the beta function.*
3. *The convergence radius* $\rho_\alpha = \left(\liminf_k \left(|c_k|^{1/k}\right)\right)^{\frac{-1}{\alpha}}$ *of the power series* $\sum_{k\geq 0} c_k t^{\alpha k}$, *defined by the coefficients* c_k, *is infinite. Specifically, there exist constants* $K \geq 1$ *and* $A \geq 2^{\alpha+2}$ *such that for all* $k \geq 0$, *the following inequality holds:* $|c_k| \leq \frac{KA^k}{\Gamma(\alpha(k-1)+2)}$. *As a consequence, the expansion in equation 5.53 converges for all* $t \in \mathbb{R}^+$, *and in fact, for all* $t \in \mathbb{R}$.

Remark. The equation in (5.54), which provides the coefficients of the expansion for $\varsigma^2_{\alpha,\lambda,c}$ when $\alpha \in (1, \frac{3}{2})$, closely resembles that obtained for $\alpha \in (\frac{1}{2}, 1)$ in [36], although the properties of the two functions differ significantly. By the scaling property (5.53), we may assume now that $c = \lambda = 1$.

Proposition 5.3 (Existence of $\varsigma_{\alpha,\lambda,c}$ i.e. positivity computation of the function $\varsigma^2_{\alpha,\lambda,c}$ solution of the stabilizer equation for $\alpha \in (1, \frac{3}{2})$)**.** *Let $\alpha \in (1, \frac{3}{2})$ and consider the volterra equation of the first kind,*

$$\kappa\left(1 - R^2_\alpha(t)\right) = (f^2_\alpha * g_\alpha)(t), \quad \forall t \geq 0, \quad \kappa > 0. \tag{5.55}$$

with $R_\alpha : \mathbb{R}^+ \to \mathbb{R}$, $f_\alpha := -R'_\alpha$ *satisfy* $R_\alpha(0) = 1$, $\lim_{t\to+\infty} R_\alpha(t) = 0$, *and* $f_\alpha(0) = 0$, $\lim_{t\to+\infty} f_\alpha(t) = 0$

(a) Then equation (6.59) has at most one solution in $L^1_{loc}(Leb_1)$ *that converges to a finite limit.*

(b) If the equation (6.59) has a continuous solution g_α defined on $I \subseteq (0, +\infty)$, then $g_\alpha \geq 0$ on $I \subseteq \mathbb{R}^+$, so that the function $\sqrt{g_\alpha}$ is well-defined on $I \subseteq \mathbb{R}_+$.

Proof. The argument is similar to that of Proposition 6.1 and is therefore left to the reader.

5.3 Numerical illustration of Fake Stationarity for Fractional SVIE with $\alpha \in (\frac{1}{2}, \frac{3}{2})$

In this section we specify a family of scaled volterra equations where $b(x) = \mu_0 - \lambda x$ for $\lambda > 0$ and a diffusion coefficient σ to be specified later. Let c be such that $c[\sigma]^2_{Lip} < 1$. For the numerical illustrations, we consider the case $\phi(t) = C^{\text{ste}} = \phi(0) = 1$ almost surely, in which case the equation with constant mean reads :

$$X_t = \frac{\mu_0}{\lambda} + \Big(X_0 - \frac{\mu_0}{\lambda}\Big)R_\lambda(t) + \frac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma(s)\sigma(X_s)dW_s. \tag{5.56}$$

The reader is invited to take a look to the Appendix A for the semi-integrated Euler scheme introduce in this setting for the above equation and to the captions of the differents figures for the numerical values of the parameters of the Stochastic Volterra equation.

5.3.1 A numerical illustration of Fake Stationarity in SVIE with α-Fractional Kernels for $\alpha \in (1, \frac{3}{2})$ and (stabilized) quadratic Diffusion coefficient

We consider an α-fractional kernel for $\alpha \in (1, \frac{3}{2})$ ("Long Memory") and a squared trinomial diffusion coefficient of the form 3.32, $\sigma(x) = \sqrt{\kappa_0 + \kappa_1\,(x - \frac{\mu_0}{\lambda}) + \kappa_2\,(x - \frac{\mu_0}{\lambda})^2}$, $\kappa_i \geq 0,\ i = 0, 2,\ \kappa_1^2 \leq 4\kappa_2\kappa_0$.

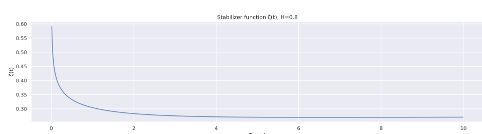

Figure 4 – Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 10 for a value of the Hurst esponent $H = 0.8$, $\lambda = 0.2$, c = 0.3.

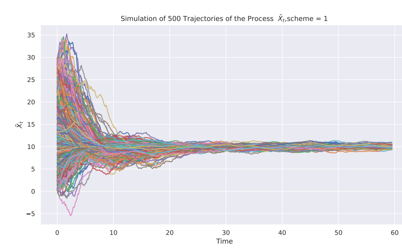

Figure 5 – Confluence from a [0,30]-Uniform Distribution, T=60, $H = 0.8$, $\lambda = 0.2$, c = 0.36.

Figure 5 shows L^2-confluence of the the process's marginals for different initial values as time increases.

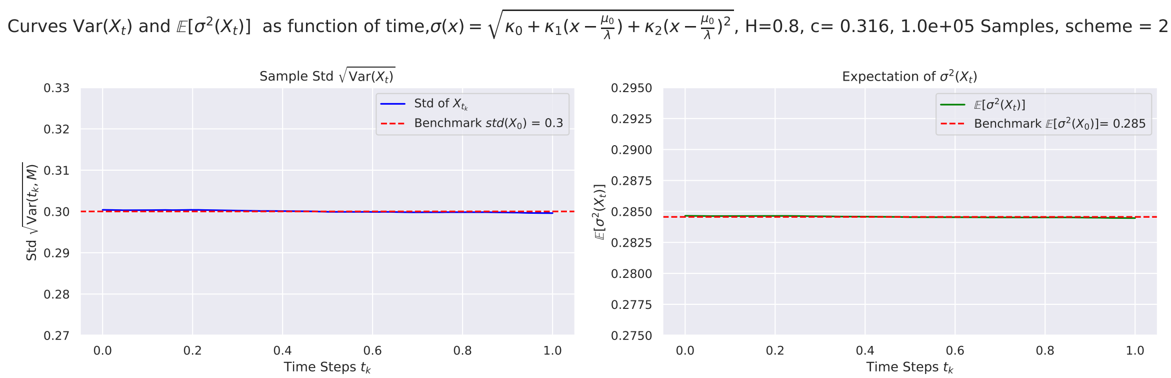

Figure 6 – Graph of $t_k \mapsto \text{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over $[0, T]$, $T = 1$, $H = 0.8$, $\mu_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, and $\text{StdDev}(X_0) = 0.3$. Number of steps: $n = 800$, Simulation size: $M = 100000$.

5.3.2 A numerical illustration of the degenerate case of Fake Stationarity in SVIE with α-Fractional Kernels for $\alpha \in (\frac{1}{2}, \frac{3}{2})$ and a (stabilized) tanh Diffusion coefficient

In this section we specify a family of scaled models where $b(x) = \mu_0 - \lambda\, x$ and $\sigma(x) = \sqrt{\frac{\tanh(x - \frac{\mu_0}{\lambda})}{2}}$, $\lambda > 0$.

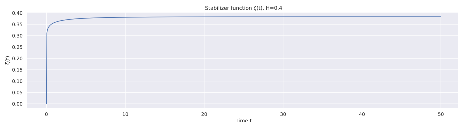

Figure 7 – Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 50 for a value of the Hurst esponent $H = 0.4$, $\lambda = 0.2$, $c = 0.36$.

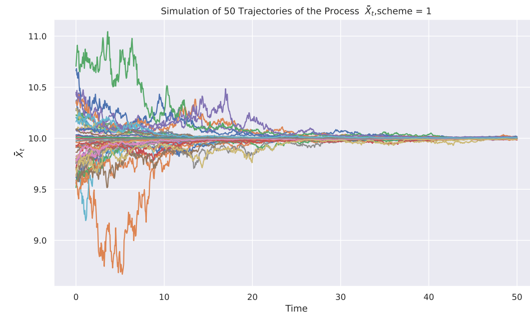

Figure 8 – Confluent trajectories in the degenerate case, $T = 50$, $H = 0.4$, $\lambda = 0.2$, $c = 0.36$.

6 Applications to Exponential-Fractional Stochastic Volterra Equations

Let consider the below *Gamma Fractional integration kernel* or *Exponential-Fractional integration kernel* defined in Example 2.3, where $\alpha = H + \frac{1}{2}$, with H denoting the Hurst coefficient:

$$K(t) = K_{\alpha,\rho}(t) = e^{-\rho t}\frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}_+}(t), \quad \text{with} \quad \alpha, \rho > 0.$$

The purpose of this part is to extend the results of the preceeding section to the general case of a gamma fractional integration kernel where $\alpha \in (\frac{1}{2}, \frac{3}{2})$. Note that, this is a generalization of the exponential kernel and the fractional integration kernel. The gamma kernel is often adopted in the Quadratic Rough Heston model (see, e.g., [9]) due to its numerical convenience, flexibility, and the availability of a closed-form expression for its resolvent of the second kind. We show that for such kernels $K_{\alpha,\rho}$, the resolvent $R_{\alpha,\rho,\lambda}$ satisfy our standing assumption $(\mathcal{K})$ for all $\lambda > 0$, and that $f_{\alpha,\rho,\lambda} := -R_{\alpha,\rho,\lambda}$ exists and is square-integrable with respect to the Lebesgue measure on $\mathbb{R}_+$, both for $\frac{1}{2} < \alpha < 1$ ("rough models") and $\frac{1}{2} < \alpha < \frac{3}{2}$ ("long memory volatility models"). As a result, the findings from Sections 3, 3.3 and 4, will be applicable in the cases where $\sigma(t,x) = \sigma(x)$ (Gaussian setting) and $\sigma(t,x) = \varsigma(t)\sigma(x)$.

6.1 $\alpha-$ Exponential Fractional kernels $\frac{1}{2} < \alpha < \frac{3}{2}$

By definition, $\mathcal{L}[R_{\alpha,\rho,\lambda}](s) = \frac{1}{s(1+\mathcal{L}[K_{\alpha,\rho}](s))} = \frac{1}{s(1+\lambda(s+\rho)^{-\alpha})}$ (owing to Example 2.3) so that, by the Tauberian Final Value Theorem [12] : $a := \lim_{t\to\infty} R_{\alpha,\rho,\lambda}(t) = \lim_{s\to 0} s\mathcal{L}[R_{\alpha,\rho,\lambda}](s) = \frac{1}{1+\lambda\rho^{-\alpha}} \in [0,1)$. If $\lambda > 0$, we define the function $f_{\alpha,\rho,\lambda} := -R_{\alpha,\rho,\lambda}$ on $(0,+\infty)$ (see (2.11) in Example2.3) by noticing that :

$$\mathcal{L}[f_{\alpha,\rho,\lambda}](s) = \mathcal{L}[-R'_{\alpha,\rho,\lambda}](s) = -s\mathcal{L}[R_{\alpha,\rho,\lambda}](s) + R_{\alpha,\rho,\lambda}(0) = \frac{-s}{s(1+\lambda(s+\rho)^{-\alpha})} + 1 = \frac{\lambda}{\lambda + (s+\rho)^{\alpha}} = \mathcal{L}[e^{-\rho\cdot} f_{\alpha,\lambda}](s)$$

i.e. by injectivity of the Laplace transform, $f_{\alpha,\rho,\lambda}(t) = e^{-\rho t} f_{\alpha,\lambda}(t) = \alpha\lambda e^{-\rho t} t^{\alpha-1} E'_\alpha(-\lambda t^\alpha)$. Likewise, using Tauberian Final Value Theorem, $\lim_{t\to\infty} f_{\alpha,\rho,\lambda}(t) = \lim_{s\to 0} s\mathcal{L}[-R'_{\alpha,\rho,\lambda}](s)$, that is

$$\lim_{t\to\infty} f_{\alpha,\rho,\lambda}(t) = -\lim_{s\to 0} s\left(s\mathcal{L}[R_{\alpha,\rho,\lambda}](s) - R_{\alpha,\rho,\lambda}(0)\right) = -\lim_{s\to 0} \frac{s}{(1+\lambda(s+\rho)^{-\alpha})} - s = 0$$

12. $f : [0,\infty) \to \mathbb{C}$ continuous, $\lim_{t\to\infty} f(t) = f_\infty$, the Laplace transform $L_f(s)$ exists for $s > 0$ and $\lim_{s\to 0^+} sL_f(s) = f_\infty$.

Remark: Note that we recover the exponential kernel if $\alpha = \rho = 1$. In fact, if $K(t) = e^{-t}\mathbf{1}_{\mathbb{R}_+}(t)$, $R_{1,1,\lambda}$ reads:

$$R_{1,1,\lambda}(t) = \mathbf{1}_{\mathbb{R}_+}(t) + \sum_{k\geq 1}(-1)^k\lambda^k \int_0^t \frac{e^{-s}s^{k-1}}{\Gamma(k)}\, ds = \mathbf{1}_{\mathbb{R}_+}(t) + \int_0^t e^{-s}\sum_{k\geq 1}(-1)^k\lambda^k \frac{s^{k-1}}{k!}\, ds = 1 - \lambda\int_0^t e^{-(\lambda+1)s}\, ds$$

So that we recover the resolvent of the exponential kernel given in [36]:

$$K(t) = e^{-t}, \quad \text{which are } R_\lambda(t) = \begin{cases} t+1 & \text{if } \lambda = -1 \\ \frac{1+\lambda e^{-(\lambda+1)t}}{\lambda+1} & \text{if } \lambda \neq -1 \end{cases}$$

Proposition 6.1. *Let $\lambda > 0$ and let $\alpha \in (0, 2)$. (a) The λ-resolvent $R_{\alpha,\rho,\lambda}$ is infinitely differentiable i.e. $\mathcal{C}^\infty$ on $(0, +\infty)$ and completely monotonic if $\alpha < 1$. Moreover $R_{\alpha,\rho,\lambda}(0) = 1$, $R_{\alpha,\rho,\lambda}$ converges to $a := \frac{1}{1+\lambda\rho^{-\alpha}} \in [0, 1[$. $R_{\alpha,\rho,\lambda} \in \mathcal{L}^r(\mathrm{Leb}_1)$ for every $r > \frac{1}{\alpha}$.*
(b) $f_{\alpha,\rho,\lambda} := -R'_{\alpha,\rho,\lambda}$ is infinitely differentiable, converges to 0, and satisfy :

$$\forall\, t > 0, \quad f_{\alpha,\rho,\lambda}(t) := e^{-\rho t} f_{\alpha,\lambda}(t) = \lambda^{\frac{1}{\alpha}} \int_0^{+\infty} e^{-(\rho+\lambda^{\frac{1}{\alpha}})tu} u H_\alpha(u)du - \frac{2}{\alpha} e^{t(\lambda^{\frac{1}{\alpha}}\cos(\frac{\pi}{\alpha})-\rho)} \cos\left[t\lambda^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right].$$

If $\alpha < 1$, $f_{\alpha,\rho,\lambda}$ is a completely monotonic function (hence convex), decreasing to 0 while $1 - R_{\alpha,\rho,\lambda}$ is a Bernstein function.
(c) If $\alpha \in (\frac{1}{2}, \frac{3}{2})$, $f_{\alpha,\rho,\lambda}$ is $\mathcal{L}^{2\beta}$-integrable $\forall \beta \in \left(0, \frac{1}{2(1-\alpha)}\right)$ if $\alpha < 1$ and for every β if $\alpha > 1$.
Moreover, for $i \in \{1, 2\}$ and for every $\vartheta \in \left(0, \alpha - \frac{\mathbf{1}_{i=2}}{i}\right)$, there exists a real constant $C_{\vartheta,\rho,\lambda} > 0$ such that

$$\forall\, \delta > 0, \quad \left[\int_0^{+\infty}\left(f_{\alpha,\rho,\lambda}(t+\delta) - f_{\alpha,\rho,\lambda}(t)\right)^i\right]^{1/i} \leq C_{\vartheta,\rho,\lambda}\delta^\vartheta.$$

For clarity and conciseness, the proof is postponed to Appendix B.

Theorem 6.2. *Let $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $\rho > 0$, let $K(t) = K_{\alpha,\rho}(t) = e^{-\rho t}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ the Gamma fractional kernel, let $\sigma(t, x) := \varsigma(t)\sigma(x)$ with σ be a Lipschitz continuous function given by (3.32), thus satisfying a relation of the type (4.33) (SL_σ) with $\kappa := \kappa_2 > 0$, let $c \in \left(0, \frac{1}{\kappa_2}\right)$ with $\varsigma = \varsigma_{\lambda,c}$, $\lambda > 0$ and let $X_0 \in L^2(\mathbb{P})$ such that $\mathbb{E}[X_0] = x_\infty$ and $Var(X_0) = v_0 = \frac{c\sigma^2(x_\infty)}{1-c\kappa_2}$. Then,*

1. *For exponential-fractional kernels $K_{\alpha,\rho}$ with $\frac{1}{2} < \alpha < \frac{3}{2}$, the solution $(X_t)_{t\geq 0}$ to the Volterra equation (3.12) starting from X_0 has a fake stationary regime of type I in the sense that:* $\forall\, t \geq 0, \quad \mathbb{E}\, X_t = x_\infty, \quad \mathrm{Var}(X_t) = v_0 = \frac{c\sigma^2(x_\infty)}{1-c\kappa_2}$ *and* $\mathbb{E}\,\sigma^2(X_t) = \bar{\sigma}_0^2 = \frac{\sigma^2(x_\infty)}{1-c\kappa_2}$.
2. *If $\phi_\infty = 0$, for every $X'_0 \in L^2(\mathbb{P})$, a solution to (3.12) starting from X'_0 satisfies $\|X'_t - X_t\|_2 \underset{t\to\infty}{\to} 0$.*
3. *The family of shifted processes $X_{t+\cdot}, t \geq 0$, is C-tight as $t \to +\infty$ and its (functional) limiting distributions are all L^2-stationary processes with covariance function C_∞ given by (4.42).*

Proof. The (1) is a consequence of Proposition 3.11. If $0 < \vartheta < \alpha - \frac{1}{2}$ and $\beta > 1$, Theorem 4.6 applies.

6.2 Existence of $\varsigma_{\alpha,\rho,\lambda,c}$ i.e. positivity computation of the function $\varsigma^2_{\alpha,\rho,\lambda,c}$ solution of the stabilizer equation when $\alpha \in (\frac{1}{2}, \frac{3}{2})$

In this section we want to compute $\varsigma_{\lambda,c}$. To this end we rely on the Laplace version of the equation *($E_{\lambda,c}$)* in (3.25) satisfied by $\varsigma^2_{\lambda,c}$, namely $\forall\, t \geq 0, \quad c\lambda^2\big(1 - (\phi - f_\lambda * \phi)^2(t)\big) = (f_\lambda^2 * \varsigma^2)(t)$, for which the laplace transform is given by equation (3.26) in Lemma 3.9:

$$\forall\, t > 0, \quad t\, L_{f_\lambda^2}(t).L_{\varsigma^2}(t) = -2\, c\lambda^2 L_{(\phi - f_\lambda*\phi)(\phi - f_\lambda*\phi)'}(t).$$

Given the form of the kernel $K_{\alpha,\rho}(u) = e^{-\rho u}\frac{u^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\mathbb{R}}(u), \quad \alpha, \rho > 0$ and the expansion of the resolvents $R_{\alpha,\lambda}$ and it derivative $-f_{\alpha,\lambda}$, in Example 2.3

$$R_{\alpha,\rho,\lambda}(t) = 1 + \sum_{k\geq 1}(-1)^k\lambda^k\int_0^t \frac{e^{-\rho s}s^{k\alpha-1}}{\Gamma(k\alpha)}\,ds, \quad f_{\alpha,\rho,\lambda}(t) = e^{-\rho t}f_{\alpha,\lambda}(t) = \lambda e^{-\rho t}t^{\alpha-1}\sum_{k\geq 0}(-1)^k\lambda^k\frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}$$

Since $\phi(t) - (f_\lambda * \phi)(t) = 1 - \frac{(f_\lambda * \mu)_t}{\lambda x_\infty}$ we have $\phi(t) - (f_\lambda * \phi)(t) \overset{0}{\sim} 1$ and by Lemma 3.9 (2) $(\phi(t) - (f_\lambda * \phi)(t))' \overset{0}{\sim} -\frac{\mu(0)}{\lambda x_\infty}f_\lambda(t)$, so that:

$$e^{2\rho t}(\phi - f_\lambda * \phi)(\phi - f_\lambda * \phi)'(t) \overset{0}{\sim} -\frac{\mu(0)}{\lambda x_\infty}\frac{\lambda t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{and} \quad e^{2\rho t}f_\lambda^2(t) \overset{0}{\sim} \frac{\lambda^2 t^{2(\alpha-1)}}{\Gamma(\alpha)^2}$$

It follows that (heuristically) $L_{e^{2\rho\cdot}(\phi - f_\lambda*\phi)(\phi-f_\lambda*\phi)'}(t) \overset{+\infty}{\sim} -\lambda\frac{\mu(0)}{\lambda x_\infty}t^{-\alpha}$ and $L_{e^{2\rho\cdot}f_\lambda^2}(t) \overset{+\infty}{\sim} \frac{\lambda^2\Gamma(2\alpha-1)t^{-(2\alpha-1)}}{\Gamma(\alpha)^2}$. So, roughly, this implies that

$$L_{e^{2\rho\cdot}\varsigma^2}(t) = L_{\varsigma^2}(t-2\rho) = \frac{-2c\lambda^2 L_{(\phi-f_\lambda*\phi)(\phi-f_\lambda*\phi)'}(t-2\rho)}{(t-2\rho)L_{e^{2\rho\cdot}f_\lambda^2}(t-2\rho)} \overset{+\infty}{\sim} 2\lambda\, c\frac{\mu(0)}{\lambda x_\infty}\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)}t^{-(2-\alpha)}$$

owing to Equation (3.26) evaluated at $(t-2\rho)$. This in turn suggests that

$$\varsigma^2(t) \overset{0}{\sim} \frac{2\lambda c\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\lambda x_\infty}e^{-2\rho t}t^{1-\alpha} \quad \text{so that} \begin{cases} (i) & \varsigma(0) = 0 \text{ if } \alpha < 1, \\ (ii) & \lim_{t\to 0^+}\varsigma(t) = +\infty \text{ if } \alpha > 1 \text{ provided } \frac{\mu(0)}{\lambda x_\infty} > 0. \end{cases} \tag{6.57}$$

This suggests to search $\varsigma^2(t)$ of the form (*Exponential Power Series Ansatz*):

$$\varsigma^2(t) = \varsigma^2_{\alpha,\rho,\lambda,c}(t) := 2\,\lambda\, c\, e^{-2\rho t}t^{1-\alpha}\sum_{k\geq 0}(-1)^k c_k\lambda^k t^{\alpha k}, \text{ with } c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}\frac{\mu(0)}{\lambda x_\infty}. \tag{6.58}$$

so that, there exists η small enough such that $\forall t \in (0,\eta), \quad \varsigma^2_{\alpha,\rho,\lambda,c}(t) \approx e^{-2\rho t}\varsigma^2_{\alpha,\lambda,c}(t)$.

Remark: 1. For the computation of the function $\varsigma^2_{\alpha,\lambda,c}$, establishing a recurrence formula satisfied by the coefficients c_k turns out to be quite challenging. We rather solve the functional equation numerically. This involves knowing the *form of the mean-reverting function* μ. In practice, since this function is usually taken to be constant equal to μ_0, we are study $\varsigma^2_{\alpha,\rho,\lambda,c}$ when $\mu(t) = \mu_0$ a.e. and $\alpha \in (\frac{1}{2}, \frac{3}{2})$.
2. With that in mind, on a time grid $t_k = k\frac{T}{n}, k = 0, ..., n.$, we use the discretization
$\forall\, k \geq 1, \quad c\lambda^2\big(1 - R^2_{\alpha,\rho,\lambda,c}(t_k)\big) = (f^2_{\alpha,\rho,\lambda,c} * \varsigma^2_{\alpha,\rho,\lambda,c})(t_k) = \sum_{j=0}^{k-1} f^2_{\alpha,\rho,\lambda,c}(t_k - t_j)\varsigma^2_{\alpha,\rho,\lambda,c}(t_{j+1})(t_{j+1} - t_j)$.
which we can solve step by step (Lower-Triangular system) to recover the values $\varsigma^2_{\alpha,\rho,\lambda,c}(t_k)$.

From now on, we consider the case $\mu(t) = \mu_0$ a.e., such that $\mu_\infty = \mu(0) = \mu_0$ and $\phi \equiv 1$.

Proposition 6.1 (Existence and Properties of the function $\varsigma^2_{\alpha,\rho,\lambda,c}$ for $\alpha \in (\frac{1}{2}, \frac{3}{2})$)**.** *Let $\alpha \in (\frac{1}{2}, \frac{3}{2})$:*

1. *In reference to the remark on the stabilizer, consider the following equation for $c, \lambda > 0$:*

$$c\lambda^2\left(1 - R^2_{\alpha,\rho,\lambda}(t)\right) = (f^2_{\alpha,\rho,\lambda} * g_{\alpha,\rho,\lambda})(t), \quad \forall t \geq 0. \tag{6.59}$$

where $R_{\alpha,\rho,\lambda} : \mathbb{R}^+ \to \mathbb{R}$ and $f_{\alpha,\rho,\lambda} := -R'_{\alpha,\rho,\lambda}$ satisfy $R_{\alpha,\rho,\lambda}(0) = 1$, $\lim_{t\to+\infty}R_{\alpha,\rho,\lambda}(t) = a$, and $\lim_{t\to+\infty}f_{\alpha,\rho,\lambda}(t) = 0$.

   (a) *Then equation (6.59) has at most one solution in $L^1_{loc}(Leb_1)$ that converges to a finite limit.*

   (b) *If the equation (6.59) has a continuous solution $g_{\alpha,\rho,\lambda}$ defined on $I \subseteq (0,+\infty)$, then $g_{\alpha,\rho,\lambda} \geq 0$ on $I \subseteq \mathbb{R}^+$, so that the function $\sqrt{g_{\alpha,\rho,\lambda}}$ is well-defined on $I \subseteq \mathbb{R}_+$. If $\alpha < 1$, $g_{\alpha,\rho,\lambda}$ is concave, non-decreasing and non-negative on $[0,+\infty)$. In particular, we have $\forall t > 0 \quad g_{\alpha,\rho,\lambda}(t) > 0$.*

2. *The stabilizer $\varsigma^2_{\alpha,\rho,\lambda,c}$ exists as a non-negative function on $I \subseteq (0,+\infty)$ and*

$$\lim_{t\to 0}\varsigma_{\alpha,\rho,\lambda,c} = \begin{cases} 0 \text{ if } \alpha \leq 1, \\ +\infty \text{ if } \alpha > 1. \end{cases} \quad \text{and} \quad \lim_{t\to+\infty}\varsigma_{\alpha,\rho,\lambda,c}(t) = \frac{\sqrt{c(1-a^2\phi^2_\infty)\lambda}}{\|f_{\alpha,\rho,\lambda}\|_{L^2(Leb_1)}}, \quad a = \frac{1}{1+\lambda\rho^{-\alpha}}.$$

Proof. Claim 1(a) comes from Lemma 3.9 (3). Claim (2) follows from 1(b), equation (6.57) and Lemma 3.9 (4). The proof of 1(b) is postponed to Appendix B.

6.3 Numerical illustration of Fake Stationarity for α-Gamma Fractional SVIE With (Stabilized) Quadratic Diffusion Coefficient and $\alpha \in (\frac{1}{2}, \frac{3}{2})$

In this section we specify a family of scaled volterra equations where $b(x) = \mu_0 - \lambda\, x$ for some $\lambda > 0$ and a diffusion coefficient σ given by (6.60). Let c such that $c[\sigma]^2_{Lip} < 1$. For the numerical illustrations, we consider the case $\phi(t) = C^{\text{ste}} = \phi(0) = 1$ almost surely, in which case the equation with constant mean reads :

$$X_t = \tfrac{\mu_0}{\lambda} + \Big(X_0 - \tfrac{\mu_0}{\lambda}\Big) R_\lambda(t) + \tfrac{1}{\lambda}\int_0^t f_{\alpha,\lambda}(t-s)\varsigma(s)\sigma(X_s)dW_s.$$

The reader is invited to take a look to the Appendix A for the semi-integrated Euler scheme introduce in this setting for the above equation and to the captions of the differents figures for the numerical values of the parameters of the Stochastic Volterra equation. We consider an α-Gamma Fractional kernel for $\alpha \in (\frac{1}{2}, \frac{3}{2}) \subset (0, 2)$ ("Rough and Long Memory models ") and a squared trinomial diffusion coefficient σ of the form 3.32 and given by:

$$\sigma(x) = \sqrt{\kappa_0 + \kappa_1\,(x - \frac{\mu_0}{\lambda}) + \kappa_2\,(x - \frac{\mu_0}{\lambda})^2} \quad \text{with} \quad \kappa_i \geq 0,\ i = 0, 2,\ \kappa_1^2 \leq 4\kappa_2\kappa_0. \tag{6.60}$$

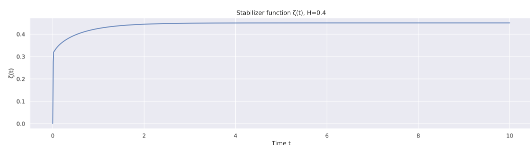

Figure 9 – Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 10 for a value of the Hurst esponent $H = 0.4$, $\lambda = 0.2$, $\rho = 1.2$, c = 0.36.

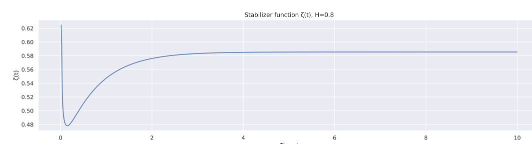

Figure 10 – Graph of the stabilizer $t \to \varsigma_{\alpha,\lambda,c}(t)$ over time interval [0, T], T = 10 for a value of the Hurst esponent $H = 0.8$, $\lambda = 0.2$, $\rho = 1.2$, c = 0.36.

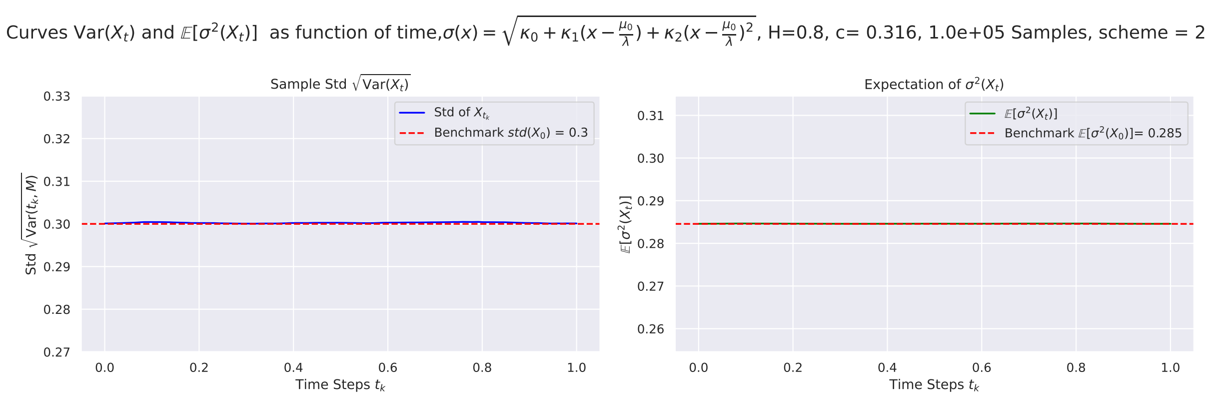

Figure 11 – Graph of $t_k \mapsto \text{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.8$, $\mu_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, $\rho = 1.2$, and $\text{StdDev}(X_0) = 0.3$. Number of steps: $n = 800$, Simulation size: $M = 100000$.

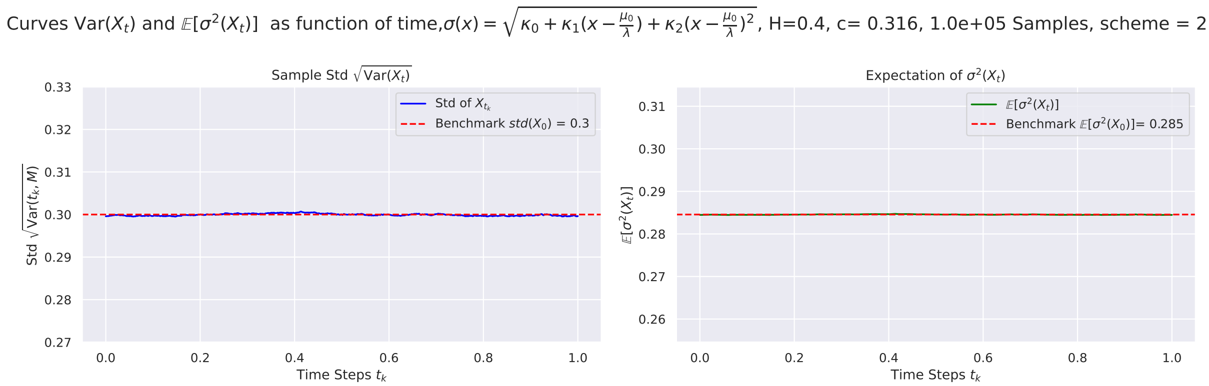

Figure 12 – Graph of $t_k \mapsto \mathrm{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.4$, $\mu_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, $\rho = 1.2$, and $\mathrm{StdDev}(X_0) = 0.3$. Number of steps: $n = 800$, Simulation size: $M = 100000$.

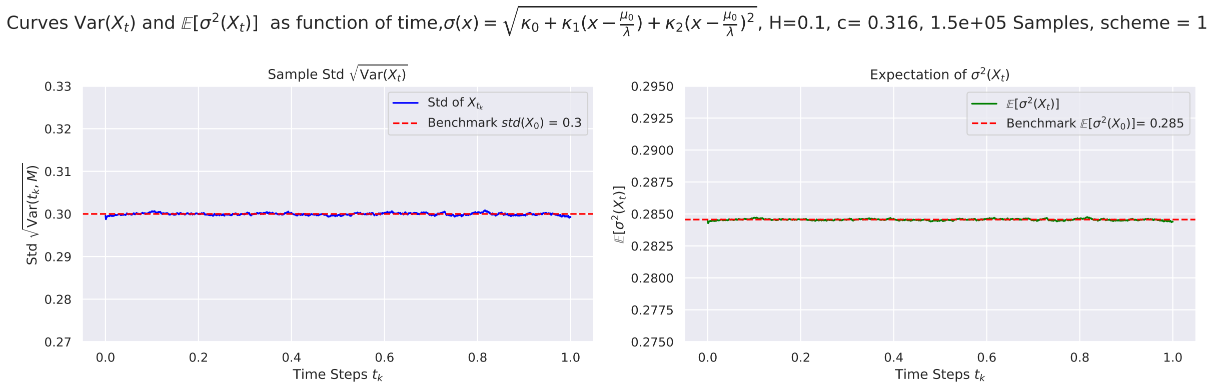

Figure 13 – Graph of $t_k \mapsto \mathrm{StdDev}(t_k, M)$ and $t_k \mapsto \mathbb{E}[\sigma^2(X_{t_k}, M)]$ over the time interval $[0, T]$, $T = 1$, $H = 0.1$, $\mu_0 = 2$, $\lambda = 0.2$, $v_0 = 0.09$, $\rho = 1.2$, and $\mathrm{StdDev}(X_0) = 0.3$. Number of steps: $n = 800$, Simulation size: $M = 150000$.

Acknowledgement: The authors thank J-F. Chassagneux and M. Rosenbaum for insightful discussions, help and comments.

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A About the Simulation of the semi-integrated scheme for stochastic Volterra integral Equations (SVIE)

We aim at providing numerical approximation for the equation:

$$X_t = \underbrace{X_0(\phi(t) - \int_0^t f_\lambda(t-s)\phi(s)\,ds) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\theta(s)ds}_{=:g(t)} + \underbrace{\frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s,X_s)dW_s}_{=:(b)}. \tag{A.61}$$

We want to provide a more generalized scheme for equations of the type:

$$X_t = g(t) + \int_0^t f(t,s)\,\sigma(s,X_s)dW_s.$$

Where g(t) can be computed using quadrature formulae on different intervals(Gauss-Legendre, Gauss-Laguerre etc.) We introduce the following Euler-Maruyama scheme for the above equation:

$$\overline{X}_{t_k} = g(t_k) + \sum_{\ell=1}^{k}\int_{t_{\ell-1}}^{t_\ell} f(t_k,s)\,\sigma(t_{\ell-1},\overline{X}_{t_{\ell-1}})dW_s = g(t_k) + \sum_{\ell=1}^{k}\sigma(t_{\ell-1},\overline{X}_{t_{\ell-1}})I_k^{n,l} \tag{A.62}$$

where $I_k^{n,l} = \int_{t_{\ell-1}}^{t_\ell} f(t_k,s)dW_s$ on the time grid $t_k = t_k^n = \frac{kT}{n}, k = 0,\ldots,n$. Due to the lack of Markovianity, $\bar{X}_{t_k^n}$ is generally not a function of $(\bar{X}_{t_{k-1}^n}, W_{t_k^n} - W_{t_{k-1}^n})$. However, it can be computed uniquely from $(\bar{X}_0^h,\ldots,\bar{X}_{t_{k-1}^n})$ and the Gaussian vector $\left(\int_{t_{\ell-1}}^{t_\ell} f(t_k^n,s)dW_s\right)_{\ell=1,\ldots,k}$, ensuring that the Euler-Maruyama scheme is well-defined by induction. The exact simulation of the Euler-Maruyama scheme (A.62) involves simulating the independent random vectors: $\left(\int_{t_\ell^n}^{t_{\ell+1}^n} K_2(t_k^n,s)\,dW_s\right)_{\ell\le k\le n}$, $\ell = 1,\ldots,n$. **Practitioner's corner:**. We aim at providing all the $I_k^{n,l}$ at once.

$$\begin{array}{c|c}
 & \begin{array}{ccccc} I_1^{n,\ell} & I_2^{n,\ell} & & \cdots & I_n^{n,\ell} \end{array} \\
\begin{array}{c} G_1^{n,\ell} \\ G_2^{n,\ell} \\ G_3^{n,\ell} \\ \vdots \\ G_n^{n,\ell} \end{array} &
\begin{bmatrix}
\Delta W_{t_\ell} & \int_{t_{\ell-1}}^{t_\ell} f(t_1,u)f(t_2,u)\,dW_u & \int_{t_{\ell-1}}^{t_\ell} f(t_1,u)f(t_3,u)\,dW_u & \cdots & \int_{t_{\ell-1}}^{t_\ell} f(t_1,u)f(t_n,u)\,dW_u \\
0 & \Delta W_{t_\ell} & \int_{t_{\ell-1}}^{t_\ell} f(t_2,u)f(t_3,u)\,dW_u & \cdots & \int_{t_{\ell-1}}^{t_\ell} f(t_2,u)f(t_n,u)\,dW_u \\
0 & 0 & \Delta W_{t_\ell} & \cdots & \int_{t_{\ell-1}}^{t_\ell} f(t_3,u)f(t_n,u)\,dW_u \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Delta W_{t_\ell}
\end{bmatrix}
\end{array}$$

We will rather consider and simulate the n independent Gaussian vectors:

$$G^{n,\ell} = \left(\Delta W_{t_l}, \left[\int_{t_\ell^n}^{t_{\ell+1}^n} f(t_k^n,s)\,dW_s\right]_{k=\ell,\ldots,n}\right) = \left(\Delta W_{t_l}, \left[I_k^{n,l}\right]_{k=\ell,\ldots,n}\right), \quad \ell = 1,\ldots,n.$$

Remark: Note that we consider the Brownian increment in the above vector because, in applications to volatility model dynamics, the dynamics of the traded asset and its volatility process can be jointly driven by the same Brownian motion (see for exemple the quadratic rough volatility dynamic introduced in [20]). This approach takes into account, among other factors, the so-called Zumbach effect, which links the evolution of the asset or an index with its volatility.

The covariance matrix of $\left(\left[I_k^{n,l}\right]_{k=\ell,\ldots,n}\right)$ is symmetric and $(n-\ell+1)\times(n-\ell+1)$, given by:

$$\Sigma^{n,\ell} = \left[Cov(I_i^{n,l}, I_j^{n,l})\right]_{\ell\le i,j\le n} = \left[\int_{t_{\ell-1}}^{t_\ell} f(t_i,u)f(t_j,u)du\right]_{\ell\le i,j\le n} = \left[\int_0^{T/n} f(t_i^n, t_\ell^n+u)f(t_j^n, t_\ell^n+u)\,du\right]_{\ell\le i,j\le n}.$$

The covariance matrix of $G^{n,\ell}$ will be symmetric and $(n-\ell+2)\times(n-\ell+2)$ $C := C^{n+1,1}$, given by:

$$C = \begin{pmatrix} \frac{T}{n} & {}^tC^{0,1} \\ C^{0,1} & \Sigma^{n,1} \end{pmatrix},\; C^{0,\ell} = \left[Cov(\Delta W_{t_l}, I_i^{n,l})\right]_{\ell\le i\le n} = \left[\int_0^{T/n} f(t_i^n, t_\ell^n+u)\,du\right]_{\ell\le k,k'\le n} \quad \ell=1,\ldots,n,$$

At this stage, we can compute any fixed sub-matrix of C by a cubature formula (such as Trapezoid, Midpoint, Simpson, higher-order Newton-Cote, or Gauss-Legendre integration formulas) and then perform a numerically stable extended Cholesky transform. This results in the decomposition:

$$[C_{ij}]_{1\le i,j\le n+1} = T^{(n)}D^{(n)}T^{(n)*} \quad T^{(n)} \text{ lower triangular.}$$

$T^{(n)}$ is a lower triangular matrix with diagonal entries $T_{ii}^{(n)}=1$, and $D^{(n)}$ is a diagonal matrix with non-negative entries. Then, taking advantage of the telescopic feature and the structure of this Cholesky transform one has:

$$[C_{ij}]_{1\le i,j\le n+1-\ell} = [T_{ij}^{(n)}]_{1\le i,j\le n+1-\ell}[D_{ij}^{(n)}]_{1\le i,j\le n+1-\ell}[T_{ij}^{(n)}]^*_{1\le i,j\le n+1-\ell},\; \ell=1,\ldots,n.$$

Finally, for each $\ell=1,\ldots,n$, we have: $(G^{n+1,\ell})_{\ell=1,\ldots,n} \overset{d}{=} (\tilde{T}^{(n+1-\ell)}Z^{(\ell)})_{\ell=1,\ldots,n}$, where

$$Z^{(\ell)} \sim \mathcal{N}(0, I_{n-\ell+2}) \quad \text{and} \quad \tilde{T}^{(n+1-\ell)} = [T_{ij}^{(n)}]_{1\le i,j\le n+1-\ell}[\sqrt{D_{ij}^{(n)}}]_{1\le i,j\le n+1-\ell}.$$

Remark: This Cholesky matrix is usually quite sparse (*when H is small* in the case of fractional kernel for example) since, all entries beyond the fourth column are numerically 0 (in fact smaller than 10^{-4}). This is due to the fact that such singular kernels have essentially no memory for small H. This feature quickly disappears when running the procedure with $H>1/2$.

Application in the Fake Stationary case with $\phi(t)=1\ \forall t\ge 0$. (A.61) can be re-written as follow:

$$X_t = \frac{\mu_0}{\lambda} + (X_0 - \frac{\mu_0}{\lambda})R_\lambda(t) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\sigma(s,X_s)dW_s. \tag{A.63}$$

knowing that $\mu(s)=\mu_0$ and noting that $\int_0^t f_\lambda(s)ds = 1-R_\lambda(t)$. Here $f(t,s)=\frac{1}{\lambda}f_\lambda(t-s)$. The Euler-Maruyama scheme (A.62) on the time grid $t_k=t_k^n=\frac{kT}{n}, k=0,\ldots,n$, write recursively:

$$\overline{X}_{t_k} = \frac{\mu_0}{\lambda} + \big(X_0 - \frac{\mu_0}{\lambda}\big)R_\lambda(t_k) + \sum_{\ell=1}^k \int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k-s)\,\frac{\varsigma(t_\ell)}{\lambda}\sigma(\overline{X}_{t_{\ell-1}})dW_s = g(t_k) + \sum_{\ell=1}^k \frac{\varsigma(t_\ell)}{\lambda}\sigma(\overline{X}_{t_{\ell-1}})I_k^{n,l} \tag{A.64}$$

where the integrals $\left(I_k^{n,l} = \int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k-s)dW_s\right)_k$ can be simulated on the discrete grid $(t_k^n)_{0\le k\le n}$ by generating an independent sequence of gaussian vectors $G^{n,l}, l=1\cdots n$ using the Cholesky decomposition of the covariance matrix C of these vectors which read (setting $u=\frac{T}{n}(\ell-v)$, $v\in[0,1]$):

$$\begin{aligned}\Sigma^{n,\ell} &= \left[Cov(I_k^{n,l}, I_{k'}^{n,l})\right]_{\ell\le k,k'\le n} = \left[\int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k-u)f_\lambda(t_{k'}-u)du\right]_{\ell\le k,k'\le n},\; \ell=1,\ldots,n,\\ &= \left(\frac{T}{n}\right)\left[\int_0^1 f_\lambda(\frac{T}{n}(k-l+v))f_\lambda(\frac{T}{n}(k'-l+v))dv\right]_{\ell\le k,k'\le n} = \left(\frac{T}{n}\right)\left[\Omega^n_{k-\ell,k'-\ell}\right]_{\ell\le k,k'\le n},\end{aligned}$$

where the symmetric matrix Ω^n is defined by $\Omega^n := \Big[\int_0^1 f_\lambda(\frac{T}{n}(i+v))f_\lambda(\frac{T}{n}(j+v))dv\Big]_{i,j\geq 0}$ and

$$\begin{aligned}\Sigma^{0,\ell} &= \Big[\mathrm{Cov}\left(\Delta_{t_\ell},\, I_k^{n,\ell}\right)\Big]_{\ell\leq k,k'\leq n} = \left[\int_{t_{\ell-1}}^{t_\ell} f_\lambda(t_k-u)\,\mathrm{d}u\right]_{\ell\leq k,k'\leq n}\\ &= [R_\lambda(t_k-t_\ell)-R_\lambda(t_k-t_{\ell-1})]_{\ell\leq k,k'\leq n} = \left(\frac{T}{n}\right)\left[\int_0^1 f_\lambda\left(\frac{T}{n}(k-\ell+v)\right)\mathrm{d}v\right]_{\ell\leq k,k'\leq n}\\ &= \left(\frac{T}{n}\right)\left[\Omega^0_{k-\ell,k'-\ell}\right]_{\ell\leq k,k'\leq n},\quad \ell=1,\ldots,n.\quad \text{so that}\quad C:=C^{n+1}=\left(\frac{T}{n}\right)\begin{pmatrix}1 & \Omega^0\\ \Omega^0 & \Omega^n\end{pmatrix}.\end{aligned}$$

Remark 1. If the fonction f_λ is a monone (case where we replace it mutantis mutandis by the fractional integration kernel $K(u)=K_{1,\alpha,0}(u)=\frac{u^{\alpha-1}}{\Gamma(\alpha)}, u\in[0,T]$, where $\alpha\in(1/2,1)$), we will have the fact that C^n is a certain power factor of $\left(\frac{T}{n}\right)$, say $\Psi\left(\frac{T}{n}\right)^\beta$, times an infinite symmetric matrix $(\bar\Gamma)$ (not depending on n anyway) defined by $\bar\Gamma := \Big[\int_0^1\big((i+v)(j+v)\big)^{(\alpha-1)}dv\Big]_{i,j\geq 0}$. In this case, the diagonal entries of $\bar\Gamma$ have closed form formular and the matrices of interest $[C_{ij}]_{0\leq,i,j\leq n-1}$, $n\geq 1$ are telescopic sub-matrices of $\bar\Gamma$ times the factor $\Psi\left(\frac{T}{n}\right)^\beta$.

2. For comprehensive results concerning convergence rates and *a priori* error estimates related to the approximation of the stochastic Volterra process (A.63) by the semi-integrated Euler–Maruyama scheme (A.64), as well as its continuous-time (or "genuine") extension, the reader is referred to [30].

B Supplementary material and Proofs.

Proof of Proposition 2.4: Convoluting $x(t)+\lambda\int_0^t K(t-s)x(s)ds$ with R'_λ together with the fact that $\lambda K * R'_\lambda = -\lambda K - R'_\lambda$(see equation 2.9), we obtain:

$$\begin{aligned}\int_0^t g(s)R'_\lambda(t-s)ds &= \int_0^t x(s)R'_\lambda(t-s)ds + \lambda\int_0^t\int_0^s K(s-u)x(u)duR'_\lambda(t-s)ds\\ &= \int_0^t x(s)R'_\lambda(t-s)ds + \lambda\int_0^t\int_0^{t-u} K(t-u-s)R'_\lambda(s)dsx(u)du\\ &= \int_0^t x(s)R'_\lambda(t-s)ds + \int_0^t\big(-\lambda K(t-u)-R'_\lambda(t-u)\big)x(u)du = -\lambda\int_0^t K(t-s)x(s)ds.\end{aligned}$$

Inserting this in the Wiener-Hopf equation gives the results. For the second claim, we can use the Laplace transform in the integral equation and deduced that:

$$L_x(t) = \frac{L_h(t)}{1+L_{R'_\lambda}(t)} = L_h(t)\big(1+\lambda L_K(t)\big) = L_{h+\lambda K*h}(t)$$

where the penultimate equality comes from applying the Laplace transform to $R'_\lambda * K = -K - \frac{R'_\lambda}{\lambda}$ (see Equation (2.9)). We then conclude by the injectivity of the Laplace transform.

Proof of Lemma 3.1. For our convenience, we will consider two cases:
Case 1 *(f_λ is a probability density).* If $f_\lambda > 0$ on $(0,+\infty)$ (i.e. R_λ decreases), then the function f_λ is a probability density. Upon replacing μ with $\frac{\mu}{\mu_\infty}$, we can assume that $\mu(t)$ tends to 1 as t becomes large.

$$\begin{aligned}\int_0^t f_\lambda(t-s)\mu(s)ds - \mu_\infty(1-a) &= \int_0^t f_\lambda(t-s)\mu(s)ds - \mu_\infty\int_0^{+\infty} f_\lambda(s)\,ds\\ &= \int_0^t f_\lambda(t-s)(\mu(s)-\mu_\infty)\,ds - \mu_\infty\int_t^{+\infty} f_\lambda(s)\,ds\end{aligned}$$

so that, by the triangle inequality, we have:

$$\left|\int_0^t f_\lambda(t-s)\mu(s)ds - \mu_\infty(1-a)\right| \le \int_0^t f_\lambda(t-s)\left|\mu(s)-\mu_\infty\right| ds + \mu_\infty \int_t^{+\infty} f_\lambda(s)\, ds$$

First note that we can split the first integral as follows:

$$\begin{aligned}\int_0^t f_\lambda(t-s)\left|\mu(s)-\mu_\infty\right| ds &= \int_0^{t-A_\epsilon} f_\lambda(t-s)\left|\mu(s)-\mu_\infty\right| ds + \int_{t-A_\epsilon}^t f_\lambda(t-s)\left|\mu(s)-\mu_\infty\right| ds \\ &= \int_{A_\epsilon}^t f_\lambda(s)\left|\mu(t-s)-\mu_\infty\right| ds + \int_0^{A_\epsilon} f_\lambda(s)\left|\mu(t-s)-\mu_\infty\right| ds.\end{aligned}$$

where A_ϵ is chosen such that for all $s \ge A_\epsilon$, we have $|\mu(s)-\mu_\infty| \le \epsilon$. Moreover $\forall s \in]0, A_\epsilon[, \quad t-s \ge t-A_\epsilon \ge A_\epsilon$ for t large enough, (say $t \ge 2A_\epsilon$), and hence, this implies that $|\mu(t-s)-\mu_\infty| \le \epsilon, \quad \forall s \in]0, A_\epsilon[$.

We thus have: $\int_0^t f_\lambda(t-s)\left|\mu(s)-\mu_\infty\right| ds \le \|\mu-\mu_\infty\|_{\sup} \int_{A_\epsilon}^t f_\lambda(s)\, ds + \epsilon \int_0^{A_\epsilon} f_\lambda(s)\, ds$. And hence,

$$\begin{aligned}\lim_{t\to\infty}\left|\int_0^t f_\lambda(t-s)\mu(s)ds - \mu_\infty(1-a)\right| &\le \|\mu-\mu_\infty\|_\infty \lim_{t\to\infty}\int_{A_\epsilon}^t f_\lambda(s)\, ds + \epsilon\int_0^{A_\epsilon} f_\lambda(s)\, ds \\ &\quad + \mu_\infty \lim_{t\to\infty}\int_t^{+\infty} f_\lambda(s)\, ds \le \epsilon(1-a) \le \epsilon, \quad \text{since} \int_0^\infty f_\lambda(s)\, ds = 1-a.\end{aligned}$$

Case 2 *(f_λ is just a 1-sum measure).* If $\int_0^{+\infty} f_\lambda(s)\, ds = 1$, a more rigorous Approach to prove the above Lemma make used of Laplace Transforms - and Tauberian Final Value Theorem. Let's assume the L^1-integrability of f_λ, i.e., $\int_0^\infty |f_\lambda(s)|ds < \infty$ so that $L_{|f_\lambda|}(t) < +\infty$ for every $t > 0$: f_λ has subsequently a finite Laplace transform defined (at least) on $\mathbb{R}^+$.

Since $\lim_{t\to+\infty}\mu(t) = \mu_\infty$, we have by Tauberian Final Value Theorem $\lim_{z\to 0} zL_\mu(z) = \mu_\infty$. As the Laplace transform of the convolution is the product of the Laplace transforms, we have:

$$\mathcal{L}\left(\int_0^t f_\lambda(t-s)\mu(s)ds\right)(z) = L_{f_\lambda}(z)L_\mu(z)$$

Therefore, by Tauberian Final Value Theorem if $\lim_{t\to+\infty}\int_0^t f_\lambda(t-s)\mu(s)ds$ exists, then

$$\lim_{t\to+\infty}\int_0^t f_\lambda(t-s)\mu(s)ds = \lim_{z\to 0} z\mathcal{L}\left(\int_0^t f_\lambda(t-s)\mu(s)ds\right)(z) = \lim_{z\to 0} zL_\mu(z)L_{f_\lambda}(z) = \mu_\infty \lim_{z\to 0} L_{f_\lambda}(z)$$

However, by our assumption $\int_0^\infty f_\lambda(s)ds = 1-a$, we have $\lim_{z\to 0} L_{f_\lambda}(z) = \lim_{z\to 0}\int_0^\infty e^{-zs} f_\lambda(s)ds = L_{f_\lambda}(0) = \int_0^\infty f_\lambda(s)ds = 1-a$.
Consequently, we have $\lim_{t\to+\infty}\phi(t) - (f_\lambda * \phi)(t) = \phi_\infty - \phi_\infty(1-a) = \phi_\infty\, a$. This completes the proof.

Proof of Lemma 3.9. Step 1. The equation 3.25 can be expressed in terms of the Laplace transform as follows: $c\lambda^2 L_{1-(\phi-f_\lambda*\phi)^2} = L_{f_\lambda^2} L_{\varsigma^2}$.

In order to get rid of the Laplace transform of $1-(\phi-f_\lambda*\phi)^2$, we apply integration by parts using $(\phi-f_\lambda*\phi)$ as the integrator, treating it as a single function:

$$\begin{aligned}L_{1-(\phi-f_\lambda*\phi)^2}(t) &= L_1(t) - L_{(\phi-f_\lambda*\phi)^2}(t) \\ &= \frac{1}{t} - \left(\frac{(\phi-f_\lambda*\phi)^2(0) - \lim_{u\to+\infty} e^{-tu}(\phi-f_\lambda*\phi)^2(u)}{t} + \frac{2}{t}L_{(\phi-f_\lambda*\phi)(\phi-f_\lambda*\phi)'}(t)\right) \\ &= -\frac{2}{t}L_{(\phi-f_\lambda*\phi)(\phi-f_\lambda*\phi)'}(t).\end{aligned}$$

Thus, the Laplace counterpart of equation (3.25) simplifies to equation (3.26).

Step 2. The second assertion is straightforward, noting that $\phi(t) - (f_\lambda*\phi)(t) = 1 - \frac{(f_\lambda*\mu)_t}{\lambda x_\infty}$, and applying the Leibniz rule for differentiating an integral along with the fact that the space $(L^1(\mathbb{R}), +, \cdot, *)$

is a commutative algebra:

$$\frac{d}{dt}\left(\int_0^t \mu(t-s)f_\lambda(s)\,ds\right) = \mu(t-t)f_\lambda(t) + \int_0^t \frac{\partial}{\partial t}\left(\mu(t-s)f_\lambda(s)\right)\,ds = \mu(0)f_\lambda(t) + \int_0^t \mu'(t-s)f_\lambda(s)\,ds.$$

One recognises hereinabove the equation given in the Lemma.

Step 3. If $\varsigma^1_{\lambda,c}$ and $\varsigma^2_{\lambda,c}$ are two solutions of the equation *($E_{\lambda,c}$)* in (3.25), then $f^2_\lambda * \delta\varsigma_{\lambda,c} = 0$ in $L^1_{\text{loc}}(\mathbb{R}^+)$ with $\delta\varsigma_{\lambda,c} = \varsigma^1_{\lambda,c} - \varsigma^2_{\lambda,c}$. As $L_{f^2_\lambda}(t) > 0$ for $t > 0$ (by Assumption $(\mathcal{K})$(ii)), then $\delta g = 0$, which implies $\delta\varsigma_{\lambda,c} = 0$ in $L^1_{\text{loc}}(\mathbb{R}^+)$. Thus, the solution $\varsigma^2_{\lambda,c}$ of Equation (3.25) if any, is unique.

We would also note that, c being fixed, $L_{f^2_\lambda}(t) > 0$ for $t > 0$ (by Assumption $(\mathcal{K})$(ii)). Then $L_{\varsigma^2_{\lambda,c}}$ is uniquely determined by (3.26), which in turn implies the uniqueness of $\varsigma^2_{\lambda,c}$ (at least dt-a.e.).

Step 4. $\varsigma^2_{\lambda,c}$ is non-negative and has a limit $l_\infty \in (0,+\infty]$ as $t \to +\infty$. If $l_\infty = +\infty$, then for every $A > 0$, there exists t_A such that for $t \geq t_A$, $\varsigma^2_{\lambda,c}(t) \geq A$. Hence
$(f^2_\lambda * \varsigma^2_{\lambda,c})(t) = \int_0^{t_A} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + \int_{t_A}^t f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds \geq \int_0^{t_A} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + A\int_{t_A}^t f^2_\lambda(t-s)\,ds$
i.e.

$$(f^2_\lambda * \varsigma^2_{\lambda,c})(t) \geq \int_0^{t_A} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + A\int_0^{t-t_A} f^2_\lambda(s)\,ds.$$

Consequently, as $(f^2_\lambda * \varsigma^2_{\lambda,c})(t) = c\lambda^2(1-(\phi - f_\lambda * \phi)^2(t)) \to c\lambda^2(1-a^2\phi^2_\infty)$ as $t \to +\infty$ owing to Lemma 3.1, we have: $c\lambda^2(1-a^2\phi^2_\infty) = \lim_{t\to+\infty}(f^2_\lambda * \varsigma^2_{\lambda,c})(t) \geq A\int_0^{+\infty} f^2_\lambda(u)\,du$. As $f_\lambda \in L^2(\mathbb{R}_+, \text{Leb}_1)$, this yields a contradiction by letting $A \to \infty$. Hence, $l_\infty < +\infty$. Now, still by the same arguments, $\lim_{t\to+\infty}\varsigma^2_{\lambda,c}(t) = l_\infty \in (0,+\infty[\implies \forall\eta > 0, \exists t_\eta \in \mathbb{R}^+$ such that $\forall t > t_\eta \quad l_\infty - \eta \leq \varsigma^2_{\lambda,c}(t) \leq l_\infty + \eta$

On the first hand, we have:

$$(f^2_\lambda * \varsigma^2_{\lambda,c})(t) \geq \int_0^{t_\eta} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + (l_\infty - \eta)\int_{t_\eta}^t f^2_\lambda(t-s)\,ds = \int_0^{t_\eta} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + (l_\infty - \eta)\int_0^{t-t_\eta} f^2_\lambda(s)\,ds.$$

Hence, we obtain:
$c\lambda^2(1-a^2\phi^2_\infty) = \lim_{t\to+\infty}(f^2_\lambda * \varsigma^2_{\lambda,c})(t) \geq (l_\infty - \eta)\int_0^{+\infty} f^2_\lambda(u)\,du \implies l_\infty \leq \frac{c\lambda^2(1-a^2\phi^2_\infty)}{\int_0^{+\infty} f^2_\lambda(s)\,ds}$ *by letting* $\eta \to 0$.

On the other hand, we also have:

$$(f^2_\lambda * \varsigma^2_{\lambda,c})(t) \leq \int_0^{t_\eta} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + (l_\infty + \eta)\int_{t_\eta}^t f^2_\lambda(t-s)\,ds = \int_0^{t_\eta} f^2_\lambda(t-s)\varsigma^2_{\lambda,c}(s)\,ds + (l_\infty + \eta)\int_0^{t-t_\eta} f^2_\lambda(s)\,ds.$$

Therefore, we obtain:
$c\lambda^2(1-a^2\phi^2_\infty) = \lim_{t\to+\infty}(f^2_\lambda * \varsigma^2_{\lambda,c})(t) \leq (l_\infty + \eta)\int_0^{+\infty} f^2_\lambda(u)\,du \implies l_\infty \geq \frac{c\lambda^2(1-a^2\phi^2_\infty)}{\int_0^{+\infty} f^2_\lambda(s)\,ds}$ *as* $\eta \to 0$.
This completes the proof and we are done. □

Proof of Proposition 3.1. We adapt the proof of the corresponding Proposition in [36] in order to prove the equivalence of the two statements or claims above.

$\boxed{(i) \Rightarrow (ii)}$ Assume $\text{Var}(X_t) = \text{Var}(X_0) = v_0$ for every $t \geq 0$. If $v_0 = 0$, then $X_t = x_\infty$ *a.s.* for every $t \geq 0$. Consequently $\mathbb{E}\,\sigma^2(X_t) = \mathbb{E}\,\sigma^2\left(\frac{\mu_\infty}{\lambda}\right)$ which is constant over time anyway. Assume that $v_0 > 0$. Then, since the constant c is finite, it follows that $\mathbb{E}[\sigma^2(X_0)] > 0$. Define the function

$$g(t) = \varsigma^2_{\lambda,c}\left(\frac{\mathbb{E}[\sigma^2(X_t)]}{\mathbb{E}[\sigma^2(X_0)]} - 1\right).$$

We can check using equation (3.22) and $(E_{\lambda,c})$ that this function satisfies the convolution equation $f^2_\lambda * g = 0$ with the initial condition $g(0) = 0$. Furthermore, under the assumption that σ has linear growth, the expectation $\mathbb{E}[\sigma^2(X_t)]$ remains bounded due to the boundedness of $\mathbb{E}[X_t^2]$. As a consequence, the function g, along with its positive and negative parts g^+ and g^-, admits a Laplace transform.

Since the Laplace transform of f_λ^2, denoted $L_{f_\lambda^2}$, is not identically zero and is strictly positive on $(0, +\infty)$, we obtain:

$$L_{f_\lambda^2} \cdot L_{g^+} = L_{f_\lambda^2} \cdot L_{g^-}.$$

This implies $L_{g^+} = L_{g^-}$, hence $g^+ = g^-$, and consequently $g = 0$. $\boxed{(ii) \Rightarrow (i)}$ First, we have that $\bar{\sigma}_0^2 = \bar{\sigma}_t^2 = \mathbb{E}\,\sigma^2(X_t)$, $t \geq 0$, so that it follows from Equation (3.22) and $(E_{\lambda,c})$.

$$\begin{aligned}\mathrm{Var}(X_t) &= \mathrm{Var}(X_0)(\phi - f_\lambda * \phi)^2(t) + \frac{\bar{\sigma}_0^2}{\lambda^2}(f_\lambda^2 * \varsigma_{\lambda,c}^2)(t) = \mathrm{Var}(X_0)(\phi - f_\lambda * \phi)^2(t) + \frac{v_0}{c\lambda^2}(f_\lambda^2 * \varsigma_{\lambda,c}^2)(t)\\ &\overset{(E_{\lambda,c})}{=} v_0(\phi - f_\lambda * \phi)^2(t) + v_0\big(1 - (\phi - f_\lambda * \phi)^2(t)\big) = v_0.\end{aligned}$$

Proof of Proposition 4.3. Using equation (3.2) and owing to (3.21), we have: $\forall\, t \geq 0$,

$$\begin{aligned}X_t - x_\infty &= \Big(X_0 - x_\infty\Big)(\phi - f_\lambda * \phi)(t) + x_\infty\Big((\phi - f_\lambda * \phi)(t) - 1\Big) + \frac{1}{\lambda}(f_\lambda * \mu)_t + \frac{1}{\lambda}\Big(f_\lambda \overset{W}{*} \varsigma(\cdot)\sigma(X_\cdot)\Big)_t\\ &= \Big(X_0 - x_\infty\Big)(\phi - f_\lambda * \phi)(t) + \frac{1}{\lambda}\int_0^t f_\lambda(t-s)\varsigma(s)\sigma(X_s)\,dW_s.\end{aligned}$$

so that in particular, $\Big|\mathbb{E}\left(X_t\right) - x_\infty\Big| \leq |\phi(t) - (f_\lambda * \phi)(t)|\Big|\mathbb{E}\left(X_0\right) - x_\infty\Big| = \Big|1 - (f_\lambda * \frac{\mu}{\lambda x_\infty})_t\Big|\Big|\mathbb{E}\left(X_0\right) - x_\infty\Big|$.
(a) Using elementary computations and Itô's Isometry show that for every $t \geq 0$

$$\mathbb{E}\left(\Big|X_t - x_\infty\Big|\right)^2 \leq \mathbb{E}\left(\Big|(X_0 - x_\infty)(\phi(t) - (f_\lambda * \phi)(t))\Big|^2\right) + \frac{1}{\lambda^2}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\mathbb{E}\left(\sigma^2(X_s)\right)ds.$$

Set $\rho := c\,[\sigma]_{\mathrm{Lip}}^2 \in (0,1)$ and let ϵ in Remark 4.1 be equal to $\epsilon = \frac{\rho}{\eta}$ where $\eta \in (0, 1-\rho)$ is a free parameter such that $\rho + \eta \in (0,1)$. From equation (4.33), The real constants κ_i, $i = 0, 2$ depending on η and given by $k_0 = k_0(\eta) := (1 + \frac{\rho}{\eta})|\sigma(x_\infty)|^2$ and $k_2 = k_2(\eta) := (1 + \frac{\eta}{\rho})[\sigma]_{\mathrm{Lip}}^2$ so that $c\,\kappa_2 = \rho + \eta < 1$.

Next, we have using equations (4.33) and (3.25) ($f_\lambda^2 * \varsigma^2 = c\lambda^2(1 - (\phi - f_\lambda * \phi)^2)$):

$$\begin{aligned}\mathbb{E}\left(|X_t - x_\infty|\right)^2 &\leq \mathbb{E}\left(|X_0 - x_\infty|\right)^2(\phi - f_\lambda * \phi)^2(t) + \kappa_0 c\big(1 - (\phi - f_\lambda * \phi)^2(t)\big)\\ &\quad + \frac{\kappa_2}{\lambda^2}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\mathbb{E}\left(|X_s - x_\infty|^2\right)ds.\end{aligned}$$

Now let $A > \bar{A}_\eta := \frac{\kappa_0 c}{1 - \kappa_2 c} \vee \mathbb{E}\left(|X_0 - x_\infty|\right)^2$, $\delta > 0$ and $t_\delta = \inf\left\{t \geq 0 : \mathbb{E}\left(|X_t - x_\infty|\right)^2 \geq A + \delta\right\}$. As $t \mapsto \mathbb{E}\left(|X_t - x_\infty|\right)^2$ is continuous and $A > \mathbb{E}\left(|X_0 - x_\infty|\right)^2$ it follows from the above inequality and the identity satisfied by ς [13] that, if $t_\delta < +\infty$, then $\mathbb{E}\left(|X_s - x_\infty|\right)^2 < A + \delta \quad \forall s \leq t_\delta$ and we have: $A + \delta = \mathbb{E}\left(|X_{t_\delta} - \frac{\mu_\infty}{\lambda}|\right)^2 < A(\phi - f_\lambda * \phi)^2(t_\delta) + \big(\kappa_0 c + \kappa_2 c(A + \delta)\big)\big(1 - (\phi - f_\lambda * \phi)^2(t_\delta)\big)$. Now, as $\kappa_0 c + \kappa_2 c A < A$ by construction of A, we have:

$$A + \delta = \mathbb{E}\left(|X_{t_\delta} - x_\infty|\right)^2 < A(\phi - f_\lambda * \phi)^2(t_\delta) + A(1 - (\phi - f_\lambda * \phi)^2(t_\delta)) + \kappa_2 c\delta\big((1 - (\phi - f_\lambda * \phi)^2(t_\delta)\big) < A + \delta.$$

As c is so that $c\kappa_2 < 1$. This yields a contradiction. Consequently, $t_\delta = +\infty$ which implies that $\mathbb{E}\left(|X_t - x_\infty|\right)^2 \leq A + \delta$ for every $t \geq 0$. Letting $\delta \to 0$ and $A \to \bar{A}_\eta$ successively, yields

13. $f_\lambda^2 * \varsigma^2 = c\lambda^2(1 - (\phi - f_\lambda * \phi)^2)$

$$\sup_{t\geq 0}\mathbb{E}\left(|X_t-x_\infty|\right)^2 \leq \bar{A}_\eta = \frac{c(1+\frac{\rho}{\eta})|\sigma(x_\infty)|^2}{1-(1+\frac{\eta}{\rho})c[\sigma]^2_{\rm Lip}} = c|\sigma(x_\infty)|^2\frac{\eta+\rho}{\eta(1-\rho-\eta)}.$$

A straightforward computation shows that $\eta\mapsto \bar{A}_\eta$ attains its minimum on $(0,\,1-\rho)$ at $\eta=\sqrt{\rho}-\rho$. This minimum is given by $\frac{c}{(1-\sqrt{\rho})^2}|\sigma(x_\infty)|^2$ which completes the proof of the stated result.

(b) Let $p\geq 2$. Set $\rho_p := c\,(C_p^{BDG})^2\,[\sigma]^2_{\rm Lip}\in(0,\,1)$. Owing to the triangle inequality and applying the BDG inequality to the (a priori) local martingale $M_u=\int_0^u f_\lambda(t-s)\varsigma(s)\sigma(X_s)dW_s$, $0\leq s\leq t$, (see [41, Proposition 4.3]) follow by the generalized Minkowski inequality, we get:

$$\begin{aligned}\Big\||X_t-x_\infty|\Big\|_p &\leq \Big\||X_0-x_\infty|\Big\|_p\Big|\phi(t)-(f_\lambda*\phi)(t)\Big| + \frac{C_p^{BDG}}{\lambda}\Big\|\Big(f_\lambda^2\overset{dt}{*}\varsigma^2(\cdot)|\sigma(X_\cdot)|^2\Big)_t\Big\|^{\frac12}_{\frac{p}{2}}\\ &\leq \Big\||X_0-x_\infty|\Big\|_p\Big|\phi(t)-(f_\lambda*\phi)(t)\Big| + \frac{C_p^{BDG}}{\lambda}\Big(\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\Big\||\sigma(X_s)|^2\Big\|_{\frac{p}{2}}\Big)^{\frac12}.\end{aligned}$$

Owing to the elementary inequality $(a+b)^2\leq(1+\frac{1}{\epsilon})a^2+(1+\epsilon)b^2$ $\forall\epsilon\in(0,1/\rho_p-1)$, it follows that:

$$\Big\||X_t-x_\infty|\Big\|^2_p \leq \Big\||X_0-x_\infty|\Big\|^2_p\Big|\phi(t)-(f_\lambda*\phi)(t)\Big|^2(1+1/\epsilon)+\frac{(C_p^{BDG})^2}{\lambda^2}(1+\epsilon)\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\||\sigma(X_s)|^2\|_{\frac{p}{2}}ds$$

Likewise, set $\widetilde{\rho}_p := c\,(C_p^{BDG})^2\,[\sigma]^2_{\rm Lip}(1+\varepsilon)=\rho_p(1+\varepsilon)\in(0,\,1)$ and let ϵ in Remark 4.1 be equal to $\frac{\widetilde{\rho}_p}{\eta}$ where $\eta\in(0,\,1-\widetilde{\rho}_p)$ is a free parameter such that $\widetilde{\rho}_p+\eta\in(0,\,1)$. From equation (4.33), The real constants κ_i, $i=0,2$ depending on η are given by $k_0=k_0(\eta):=(1+\frac{\widetilde{\rho}_p}{\eta})|\sigma(x_\infty)|^2$ and $k_2=k_2(\eta):=(1+\frac{\eta}{\widetilde{\rho}_p})[\sigma]^2_{\rm Lip}$ so that $c\,(C_p^{BDG})^2\,(1+\varepsilon)\,\kappa_2=\widetilde{\rho}_p+\eta<1$. As $\frac{p}{2}\geq 1$, according to the remark 4.1, $\Big\||\sigma(X_s)|^2\Big\|_{\frac{p}{2}}\leq\kappa_0+\kappa_2\Big\||X_s-x_\infty|\Big\|^2_p$ which entails, combined with the identity $f_\lambda^2*\varsigma^2=c\lambda^2(1-(\phi-f_\lambda*\phi)^2)$, that, for every $t\geq 0$,

$$\begin{aligned}\Big\||X_t-x_\infty|\Big\|^2_p &\leq \Big\||X_0-x_\infty|\Big\|^2_p\Big|\phi(t)-(f_\lambda*\phi)(t)\Big|^2(1+\frac{1}{\epsilon})\\ &\quad+(C_p^{BDG})^2(1+\epsilon)\Big(\kappa_0 c\big(1-(\phi-f_\lambda*\phi)^2(t)\big)+\frac{\kappa_2}{\lambda^2}\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\Big\||X_s-x_\infty|\Big\|^2_p ds\Big).\end{aligned}\tag{B.65}$$

Now let $A>\bar{A}_{\eta,\epsilon}:=\frac{\kappa_0 c(C_p^{BDG})^2(1+\epsilon)}{1-\kappa_2 c(C_p^{BDG})^2(1+\epsilon)}\vee\Big[(1+1/\epsilon)\Big\||X_0-x_\infty|\Big\|^2_p\Big]$, $\delta>0$ and $t_\delta=\inf\Big\{t\geq 0:\Big\||X_t-x_\infty|\Big\|^2_p\geq A+\delta\Big\}$. If $t_\delta<+\infty$, then, on the one hand, it follows from the continuity of $t\mapsto\Big\||X_t-x_\infty|\Big\|^2_p$ that $A+\delta=\Big\||X_{t_\delta}-x_\infty|\Big\|^2_p$ and, on the other hand, from Equation (3.25) satisfied by ς, that $\int_0^t f_\lambda^2(t-s)\varsigma^2(s)\Big\||X_s-x_\infty|\Big\|^2_p ds\leq Ac\lambda^2(1-(\phi-f_\lambda*\phi)^2(t))$. Moreover, since $A>\|X_0-x_\infty\|^2_p(1+\frac{1}{\epsilon})$, we deduce from (B.65) the inequalities:

$$\begin{aligned}A+\delta=\Big\||X_{t_\delta}-x_\infty|\Big\|^2_p &< A(\phi-f_\lambda*\phi)^2(t_\delta)+(C_p^{BDG})^2(1+\epsilon)\big(\kappa_0 c+\kappa_2 c(A+\delta)\big)\big(1-(\phi-f_\lambda*\phi)^2(t_\delta)\big)\\ &< A(\phi-f_\lambda*\phi)^2(t_\delta)+A(1-(\phi-f_\lambda*\phi)^2(t_\delta))+(C_p^{BDG})^2(1+\epsilon)c\kappa_2\delta\big((1-(\phi-f_\lambda*\phi)^2(t_\delta)\big)\\ &< A+\delta\big((1-(\phi-f_\lambda*\phi)^2(t_\delta)\big)<A+\delta.\end{aligned}$$

Here, the second inequality uses the bound $(C_p^{\rm BDG})^2(1+\epsilon)\,c\,(\kappa_0+\kappa_2 A)<A$ which holds by the very definition of A, while the penultimate inequality follows from the assumption that $(C_p^{BDG})^2(1+\epsilon)c\kappa_2<1$.

This yields a contradiction. Consequently, $t_\delta = +\infty$ which implies that $\Big\| |X_{t_\delta} - x_\infty| \Big\|_p^2 \leq A + \delta$ for every $t \geq 0$. Letting $\delta \to 0$ and $A \to \bar{A}_{\eta,\epsilon}$ successively, yields

$$\sup_{t\geq 0} \Big\| |X_t - x_\infty| \Big\|_p \leq \bar{A}_{\eta,\epsilon}^{\frac{1}{2}} = \left(\frac{\widetilde{\rho}_p}{[\sigma]_{\mathrm{Lip}}^2} |\sigma(x_\infty)|^2 \frac{\eta + \widetilde{\rho}_p}{\eta(1-\widetilde{\rho}_p - \eta)} \right)^{\frac{1}{2}} < +\infty.$$

A straightforward computation shows that the mapping $\eta \mapsto \bar{A}_{\eta,\eta}$ attains its minimum on the interval $(0,\, 1-\widetilde{\rho}_p)$ at $\eta = \sqrt{\widetilde{\rho}_p} - \widetilde{\rho}_p$, this minimum being $\frac{\widetilde{\rho}_p}{[\sigma]_{\mathrm{Lip}}^2 (1-\sqrt{\widetilde{\rho}_p})^2} \big|\sigma(x_\infty)\big|^2 = \frac{c(C_p^{BDG})^2(1+\epsilon)}{(1-[\sigma]_{\mathrm{Lip}} \sqrt{c(C_p^{BDG})^2(1+\epsilon)})^2} \big|\sigma(x_\infty)\big|^2$, which completes the proof. The stated results follows by setting $C_p^{BDG} = 2\sqrt{p}$ owing to Lemma 4.2.□

Proof of Theorem 4.6. It follows from (4.41) that either $p = 2$ and $c < \frac{1}{\kappa_2}$, or $p > 2$ and $c < \frac{1}{(C_p^{BDG})^2 \kappa_2}$. Hence, Proposition 4.3 implies that $\sup_{t\geq 0} \Big\| |X_t - x_\infty| \Big\|_p < +\infty$. As a consequence of σ having at most affine growth, we derive that $\sup_{t\geq 0} \Big\| |\sigma(X_t)| \Big\|_p < +\infty$.

Step 1. *(Kolmogorov criterion).* Now, we can establish C-tightness by the Kolmogorov criterion. Let p be given by (4.41). One writes for $s,\, t \geq 0$ with $s \leq t$ and owing to equation 3.14:

$$X_t - X_s = \big((\phi - f_\lambda * \phi)(t) - (\phi - f_\lambda * \phi)(s)\big) X_0 + \frac{1}{\lambda}\Big(J(t) - J(s) + I(t) - I(s)\Big).$$

Where we set: $J(t) := \int_0^t f_\lambda(t-u)\varsigma(u)\sigma(X_u)dW_u$ and $I(t) = \int_0^t f_\lambda(t-u)\mu(u)\, du$. On the first hand,

$$\begin{aligned}\Big|(f_{\alpha,\lambda} * \phi)(t) - (f_{\alpha,\lambda} * \phi)(s)\Big| &= \Big| \int_0^s [f_{\alpha,\lambda}(t-u) - f_{\alpha,\lambda}(s-u)]\, \phi(u)\, du + \int_s^t f_{\alpha,\lambda}(t-u)\phi(u)\, du \Big| \\ &\leq \sup_{u\geq 0} |\phi(u)| \left(\int_0^s |\, (f_{\alpha,\lambda}(t-u) - f_{\alpha,\lambda}(s-u))\, |du + \int_s^t |f_{\alpha,\lambda}(t-u)|du \right).\end{aligned}$$

Consequently, we obtain the following bound:

$$\begin{aligned}&\Big\| |\big((\phi - f_\lambda * \phi)(t) - (\phi - f_\lambda * \phi)(s)\big) X_0| \Big\|_p = \Big\| |X_0| \Big\|_p \Big(|(f_{\alpha,\lambda} * \phi)(s) - (f_{\alpha,\lambda} * \phi)(t)| + |\phi(t) - \phi(s)| \Big) \\ &\leq \Big\| |X_0| \Big\|_p \|\phi\|_\infty \left(C\, |t-s|^\vartheta + \left(\int_0^{+\infty} f_{\alpha,\lambda}^{2\beta}(u) du \right)^{\frac{1}{2\beta}} |t-s|^{1-\frac{1}{2\beta}} \right) + C_p' \left(1 + \|\phi\|_\infty \, \||X_0|\|_p\right) |t-s|^\delta \\ &\leq C_{p,X_0,\phi,\beta,f_\lambda} |t-s|^{\vartheta(1-\frac{1}{2\beta})\wedge\delta}.\end{aligned}$$

where the penultimate inequality come from assumption 2.2 (iii). Next, by using generalized Minkowski inequalities, one gets similarly:

$$\begin{aligned}\Big\| |I(t) - I(s)| \Big\|_p &\leq \Big\| |\int_s^t f_\lambda(t-u)\mu(u)\, du| \Big\|_p + \Big\| |\int_0^s (f_\lambda(t-u) - f_\lambda(s-u))\, \mu(u)\, du| \Big\|_p \\ &\leq \sup_{u\geq 0} |\mu(u)| \times \left(\int_s^t |f_\lambda(t-u)|\, du + \int_0^s |\, (f_\lambda(t-u) - f_\lambda(s-u))\, |\, du \right) \\ &\leq \|\mu\|_\infty \times \left(\Big(\int_0^{+\infty} f_\lambda^{2\beta}(u) du \Big)^{\frac{1}{2\beta}} (t-s)^{1-\frac{1}{2\beta}} + \int_0^s |f_\lambda(t-u) - f_\lambda(s-u)|\, du \right) \\ &\leq C_{p,\mu,f_\lambda} |t-s|^{(\vartheta\wedge(1-\frac{1}{2\beta}))}.\end{aligned}$$

On the other hand, combining the L^p-BDG and the generalized Minkowski inequality, one derives from (4.41) that,

$$
\begin{aligned}
&\Big\||J(t)-J(s)|\Big\|_p \le \Big\||\int_s^t f_\lambda(t-u)\varsigma(u)\sigma(X_u)\mathrm{d}W_u|\Big\|_p + \Big\||\int_0^s \left(f_\lambda(t-u)-f_\lambda(s-u)\right)\varsigma(u)\sigma(X_u)dW_u|\Big\|_p \\
&\le C_p\|\varsigma^2\|_\infty\left[\Big(\int_s^t f^2_{\alpha,\lambda}(t-u)\big\||\sigma(X_u)|\big\|_p^2 du\Big)^{\frac{1}{2}} + \Big(\int_0^s \left(f_{\alpha,\lambda}(t-u)-f_{\alpha,\lambda}(s-u)\right)^2\big\||\sigma(X_u)|\big\|_p^2 du\Big)^{\frac{1}{2}}\right] \\
&\le C_p\|\varsigma^2\|_\infty \sup_{u\ge 0}\Big\||\sigma(X_u)|\Big\|_p\left[\Big(\int_0^{+\infty} f^{2\beta}_{\alpha,\lambda}(u)du\Big)^{\frac{1}{2\beta}}|t-s|^{\frac{\beta-1}{2\beta}} + \left(\int_0^{+\infty}\left(f_{\alpha,\lambda}(t-s+u)-f_{\alpha,\lambda}(u)\right)^2 du\right)^{\frac{1}{2}}\right] \\
&\le C_{p,\varsigma,f_\lambda}\cdot\left(1+\|\phi\|_T\Big\||X_0|\Big\|_p\right)|t-s|^{\vartheta\wedge\frac{\beta-1}{2\beta}} := C_{p,T,\sigma,\varsigma,f_\lambda,X_0}\cdot|t-s|^{\vartheta\wedge\frac{\beta-1}{2\beta}} \quad \text{where} \quad C_p \equiv C_p^{BDG}.
\end{aligned}
$$

Finally, putting all these estimates together, since $\frac{\beta-1}{2\beta} \le 1-\frac{1}{2\beta}$ we have the existence of a real constant $C_{p,X_0,\phi,\beta,\lambda,f_\lambda} > 0$ such that:

$$\mathbb{E}\left(|X_t - X_s|\right)^p \le C_{p,X_0,\phi,\beta,\lambda,f_\lambda}|t-s|^{p(\delta\wedge\vartheta\wedge\frac{\beta-1}{2\beta})}$$

Define for $u \ge 0$ the process X^u by $X^u_t = X_{t+u}$, where $t \ge 0$. Then X^u has continuous sample paths and satisfies $\quad \sup_{u\ge 0}\mathbb{E}[|X^u_t - X^u_s|^p] \le C(p)|t-s|^{p(\delta\wedge\vartheta\wedge\frac{\beta-1}{2\beta})}$ for $0 \le t-s \le 1$.

As $p(\delta\wedge\vartheta\wedge\frac{\beta-1}{2\beta}) > 1$ according to equation (4.41), it follows from Kolmogorov's C-tightness criterion (see [41, Theorem 2.1, p. 26, 3rd edition] [14] or [43, Lemma 44.4, Section IV.44, p.100]), that the family of shifted processes $X_{t+\cdot}$, $t \ge 0$, is C-tight i.e. $(X^u)_{u\ge 0}$ is tight on $C(\mathbb{R}_+;\mathbb{R})$ (hence the existence of a weak continuous accumulation point thanks to Prokhorov's theorem) with limiting distributions P under which the canonical process has the announced Hölder pathwise regularity. Therefore, we conclude that along a sequence $u_k \uparrow \infty$, the process X^{u_k} converges in law to some continuous process X^∞.

An application of Fatou's lemma shows that any limiting process (resp. the limit distribution) has a finite moment of any order, i.e., $\quad \forall t > 0, \quad \mathbb{E}[|X^\infty_t|^p] \le \sup_{u\ge 0}\mathbb{E}[|X_u|^p] < \infty$.

For the first moment formula, we note using equation (3.17) and Lemma 3.1 that

$$\mathbb{E}[X_t] \longrightarrow a\phi_\infty\mathbb{E}[X_0] + (1-a)\frac{\mu_\infty}{\lambda} \quad \text{as } t\to\infty.$$

Since $\sup_{t\ge 0}\mathbb{E}[|X_t|^2] < \infty$, we easily conclude that $\quad \lim_{t\to\infty}\mathbb{E}[X_t] = \mathbb{E}[X^\infty_t]$.

Step 3. *(b) Asymptotic weak stationarity.* Now let us consider the asymptotic covariance between X_{t+t_1} and X_{t+t_2}, $0 < t_1 < t_2$ when X_t starts for X_0 with mean $\frac{\mu_\infty}{\lambda}$, variance v_0 and $\bar\sigma^2 = \mathbb{E}\,\sigma(X_t)^2$, $t \ge 0$ constant over time. Using $\mathrm{Cov}(aU+b, cV+d) = ac\,\mathrm{Cov}(U,V)$ and equation (3.14), we have:

$$
\begin{aligned}
\mathrm{Cov}(X_{t+t_1}, X_{t+t_2}) &= \mathrm{Var}(X_0)\left((\phi - f_\lambda*\phi)(t+t_1)\right)\left((\phi - f_\lambda*\phi)(t+t_2)\right) \\
&\qquad + \frac{1}{\lambda^2}\mathbb{E}\left[\int_0^{t+t_1} f_\lambda(t+t_2-s)f_\lambda(t+t_1-s)\varsigma^2(s)\sigma^2(X_s)ds\right] \\
&= \mathrm{Var}(X_0)\left((\phi - f_\lambda*\phi)(t+t_1)\right)\left((\phi - f_\lambda*\phi)(t+t_2)\right) + \frac{\bar\sigma^2}{\lambda^2}\int_0^{t+t_1} f_\lambda(t_2-t_1+u)f_\lambda(u)\varsigma^2(t+t_1-u)du.
\end{aligned}
$$

As $f_\lambda(t_2-t_1+\cdot)f_\lambda \in \mathcal{L}^2(\mathrm{Leb}_1)$ since $f_\lambda \in \mathcal{L}^2(\mathrm{Leb}_1)$, $\mathbf{1}_{\{0\le u\le t+t_1\}}\varsigma^2(t+t_1-u) \to \frac{c\lambda^2(1-a^2\phi^2_\infty)}{\int_0^{+\infty} f^2_\lambda(s)ds}$ for every $u \in \mathbb{R}_+$ as $t \to +\infty$ (owing to Lemma 3.9) and $\lim_{t\to+\infty}(\phi - f_\lambda*\phi)(t) = a\phi_\infty$, $v_0 = c\bar\sigma^2$, we have:

$$\mathrm{Cov}(X_{t+t_1}, X_{t+t_2}) \overset{t\to+\infty}{\longrightarrow} a^2\phi^2_\infty\mathrm{Var}(X_0) + \frac{c\bar\sigma^2(1-a^2\phi^2_\infty)}{\int_0^{+\infty} f^2_\lambda(s)ds}\int_0^{+\infty} f_\lambda(t_2-t_1+u)f_\lambda(u)du =: C_{f_\lambda}(t_1,t_2).$$

The confluence result follows from the Remark (2) in Proposition 4.4 with $\bar\varphi_\infty(t) = \sup_{u\ge t}\varphi_\infty(u)$. Let X and X' be two solutions of Equation (3.14) starting from X_0 and X'_0 respectively, both square integrable.

14. If a process X taking values in a Polish space (S,ρ) satisfies $\mathbb{E}[\rho(X_s,X_t)^\alpha] \le c|s-t|^{\beta+d}$ for some constants $\alpha,\beta,c > 0$ and all $s,t \in \mathbb{R}$, then X admits a continuous modification whose paths are Hölder continuous of any order $\gamma \in (0, \frac{\beta}{\alpha})$.

Using the Remark (2) in Proposition 4.4, we derive that for every $0 \le t_1 < t_2 < \cdots < t_N < +\infty$
$$\mathcal{W}_2\big([(X_{t+t_1}, \cdots, X_{t+t_N})], [(X'_{t+t_1}, \cdots, X'_{t+t_N})]\big) \to 0 \text{ as } t \to +\infty.$$

As a consequence, the weak limiting distributions of $[X_{t+\cdot}]$ and $[X'_{t+\cdot}]$ are the same in the sense that, if $[X_{t_n+\cdot}] \xrightarrow{(C)} P$ for some subsequence $t_n \to +\infty$ (where P is a probability measure on $C(\mathbb{R}_+, \mathbb{R})$ equipped with the Borel σ-field induced by the sup-norm topology), then $[X'_{t_n+\cdot}] \xrightarrow{(C)_w} P$ and conversely.

Step 4. *(c) Stationary Gausian case.* This result stems first from the fact that $(X_t)_{t\ge 0}$ is a Gaussian process, implying that its limiting distributions in the functional weak sense are also Gaussian. Secondly, a Gaussian process is completely characterized by its mean and covariance functions.
In fact, when $\sigma(x) = \sigma > 0 \quad \forall x \in \mathbb{R}$ and X_0 follows a Gaussian distribution, the process X is Gaussian, which implies (at least for finite-dimensional weak convergence, i.e., weak convergence of all marginals of any order) that,$(X_{t+\cdot}) \xrightarrow{(C)} \mathcal{GP}(f_\lambda) \quad$ as $\quad t \to +\infty$, where $\mathcal{GP}(f_\lambda)$ is a Gaussian process with mean x_∞ and covariance function given above. □

Lemma B.1 (Expansions)**.** *We have the following inequalities:*

1. $0 \le 1 - e^{-v} \le (1 - e^{-v})^\vartheta \le v^\vartheta$*, for every* $v \ge 0$*, and* $\vartheta \in (0, 1]$.
2. $\sin(v) \le v^\vartheta$*, for every* $v \ge 0$*, and* $\vartheta \in (0, 1]$.

Proof. The claim (1) is straightforward since $\vartheta \in (0, 1)$, while for the second claim, we have:
— if $0 \le v \le 1$, then $\sin(v) \le v \le v^\vartheta$, for every $\vartheta \in (0, 1]$.
— if $v \ge 1$, then $v^\vartheta \ge 1 \ge \sin(v)$, for every $\vartheta \in (0, 1]$.

Proof of Proposition 5.1: Step 1. As $\forall \alpha \in \mathbb{R} \setminus \mathbb{N}$, $(-1)^{\lfloor \alpha \rfloor} \sin(\alpha\pi) > 0$, we have the inequality:

$$u^{2\alpha} + 2u^\alpha \cos(\pi\alpha) + 1 \ge 1 - \cos^2(\alpha\pi) = \sin^2(\alpha\pi) > 0 \quad (or \quad \ge (u^\alpha - 1)^2 > 0). \tag{B.66}$$

i.e., $(-1)^{\lfloor \alpha \rfloor} H_\alpha(u)$ is non-negative for all u in the integral 5.46. Therefore, $(-1)^{\lfloor \alpha \rfloor} F_\alpha(t)$ is the Laplace transform of a non-negative Lebesgue integrable function $(-1)^{\lfloor \alpha \rfloor} H_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, and, by the "Bernstein theorem", $(-1)^{\lfloor \alpha \rfloor} F_\alpha(t)$ is *completely monotone* (CM) in the real line,in the sense that $(-1)^n(-1)^{\lfloor \alpha \rfloor} F_\alpha^{(n)}(t) \ge 0$ at every order $n \ge 0$. However, the CM property of $(-1)^{\lfloor \alpha \rfloor} F_\alpha(t)$ can also be seen as a consequence of the result by Pollard [45] because the transformation $x = t^\alpha$ is a Bernstein function for $\alpha \in (0, 1)$.
Step 2. Moreover as H_α is continuous on $(0, +\infty)$, $H_\alpha(u) \overset{0}{\sim} u^{\alpha-1} \frac{\sin(\pi\alpha)}{\pi}$ and $H_\alpha(u) \overset{+\infty}{\sim} \frac{\sin(\pi\alpha)}{\pi u^{\alpha+1}}$.
It is clear that $H_\alpha \in \mathcal{L}^1_{\mathbb{R}_+}(\text{Leb}_1)$ and that both functions $u \mapsto uH_\alpha(u)$ and $u \mapsto u^{\alpha+1} H_\alpha(u)$ are bounded on $\mathbb{R}_+$. Thus, for every $t > 0$, $\int_0^{+\infty} e^{-tu} u H_\alpha(u) du < +\infty$ so that owing to a Lebesgue-type condition for differentiation under the integral sign, F_α is differentiable on $(0, +\infty)$ with

$$F'_\alpha(t) = -\int_0^{+\infty} e^{-tu} u H_\alpha(u) du, \quad t > 0. \tag{B.67}$$

The same rule applied k times shows that F_α is $\mathcal{C}^k$ for $k \in \mathbb{N}$, hence is infinitely differentiable and

$$F_\alpha^{(k)}(t) = \int_0^{+\infty} e^{-tu} H_\alpha^{(k)}(u)\, du \text{ with } H_\alpha^{(k)}(u) := (-1)^k u^k H_\alpha(u) = (-1)^k \frac{\sin(\alpha\pi)}{\pi} \frac{u^{\alpha-1+k}}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}. \tag{B.68}$$

$G_\alpha(t)$ is infinitely differentiable($\mathcal{C}^k$ for $k \in \mathbb{N}$) as product of such functions and by recurrence, we have:

$$\forall k \in \mathbb{N}, \quad G_\alpha^{(k)}(t) = \frac{2}{\alpha} \sum_{n=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \exp\left[t \cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right] \cos\left[t \sin\left(\frac{(2n+1)\pi}{\alpha}\right) - \frac{k(2n+1)\pi}{\alpha}\right]. \tag{B.69}$$

Claim (b) follows from the fact that $R_{\alpha,\lambda} = e_\alpha(\lambda^{1/\alpha}\cdot) = R_{\alpha,1}(\lambda^{1/\alpha}\cdot)$, hence infinitely differentiable on $(0,+\infty)$ from B.68 and B.69. The representation of $f_{\alpha,\lambda}$ follows from B.67 and B.69.

It follows from (5.47) and (B.66) that $H_\alpha(u) \leq \frac{u^{\alpha-1}\sin(\pi\alpha)}{\pi\sin^2(\pi\alpha)} = \frac{u^{\alpha-1}}{\pi\sin(\pi\alpha)}$. Hence, for every $t \geq 0$,

$$R_{\alpha,1}(t) = e_\alpha(t) = F_\alpha(t) + G_\alpha(t) < \frac{1}{\pi\sin(\pi\alpha)}\int_0^{+\infty} e^{-tu}u^{\alpha-1}du + \frac{2}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\exp\left[t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right]$$
$$\leq \frac{\Gamma(\alpha)}{\pi\sin(\pi\alpha)}t^{-\alpha} + \frac{2}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\exp\left[t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right] \leq \frac{\Gamma(\alpha)}{\pi\sin(\pi\alpha)}t^{-\alpha} + \frac{\lfloor\alpha+1\rfloor}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}$$

where the last inequality comes from the fact that $\cos(x)$ is non-increasing on $[0,\pi]$ so that $R_{\alpha,1} \in \mathcal{L}^\gamma(\mathrm{Leb}_1)$ for every $\gamma > \frac{1}{\alpha}$ where α is such that $\cos\left(\frac{\pi}{\alpha}\right) < 0$ i.e. $\alpha \in (0,2]$. This extends to $R_{\lambda,\alpha}$ by scaling. For the $L^{2\beta}$-integrability of $f_{\alpha,\lambda}$, once noted that $f_{\alpha,\lambda} = \lambda^{1/\alpha}f_{\alpha,1}(\lambda^{1/\alpha}\cdot)$ so that $\int_0^{+\infty} f^{2\beta}_{\alpha,\lambda}(t)dt = \lambda^{\frac{2\beta-1}{\alpha}}\int_0^{+\infty} f^{2\beta}_{\alpha,1}(t)dt$, it is clear that it is enough to prove that $f_{\alpha,1}$ is $\mathcal{L}^{2\beta}$-integrable.

By the same argument as above, it follows from (B.67) and (B.69) that for every $t > 0$

$$f_{\alpha,1}(t) < \frac{1}{\pi\sin(\pi\alpha)}\int_0^{+\infty} e^{-tu}u^{\alpha}du - \frac{2}{\alpha}\sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}\exp\left[t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)\right] \leq \frac{\Gamma(\alpha+1)}{t^{\alpha+1}\pi\sin(\pi\alpha)} - \frac{\lfloor\alpha+1\rfloor}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}.$$

Thus $f_{\alpha,1} \in \mathcal{L}^{2\beta}([1,+\infty),\mathrm{Leb}_1)\quad \forall\beta > 0$ provided that $\cos\left(\frac{\pi}{\alpha}\right) < 0$ i.e. $\alpha \in (0,2)$. On the other hand $f_{\alpha,\lambda}(t) = -R'_{\alpha,\lambda}(t) = \alpha\lambda t^{\alpha-1}E'_\alpha(-\lambda t^\alpha) = \lambda t^{\alpha-1}\sum_{k\geq 0}(-1)^k\lambda^k\frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}$ so that $f_{\alpha,1}(t) \overset{0}{\sim} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. As $t \mapsto \frac{1}{t^{1-\alpha}} \in \mathcal{L}^{2\beta}((0,1],\mathrm{Leb}_1)$ for any $\beta \in \left(\frac{1}{2(1-\alpha)},+\infty\right)$, we conclude that $f_{\alpha,1} \in \mathcal{L}^{2\beta}(\mathrm{Leb}_1)\quad\forall\beta > 0$ provided that $\cos\left(\frac{\pi}{\alpha}\right) < 0$ i.e. $\alpha \in (0,2)$.

STEP 3. As for the $\mathcal{L}^2(\mathbb{R}_+)$-$\vartheta$-Hölder continuity of $f_{\alpha,\lambda}$, one may again assume w.l.g. that $\lambda = 1$. Let $\delta > 0$. One has

$$f_{\alpha,1}(t+\delta) - f_{\alpha,1}(t) = (F'_\alpha(t) - F'_\alpha(t+\delta)) + (G'_\alpha(t) - G'_\alpha(t+\delta)) = (F'_\alpha(t) - F'_\alpha(t+\delta)) + \sum_{n=0}^{\lfloor\frac{\alpha-1}{2}\rfloor}(G'^n_\alpha(t) - G'^n_\alpha(t+\delta))$$

However, bearing in mind that $0 \leq \frac{\pi}{\alpha} \leq \frac{(2n+1)\pi}{\alpha} \leq \pi$ for $\alpha \in \mathbb{R}^+\setminus\mathbb{N}$ and $0 \leq n \leq \lfloor\frac{\alpha-1}{2}\rfloor$, we have:

$$\begin{aligned}
&G'^n_\alpha(t) - G'^n_\alpha(t+\delta) = \\
&\frac{2}{\alpha}e^{t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)}\left(\cos\left[t\sin\left(\frac{(2n+1)\pi}{\alpha}\right) - \frac{(2n+1)\pi}{\alpha}\right] - e^{\delta\cos\left(\frac{(2n+1)\pi}{\alpha}\right)}\cos\left[(t+\delta)\sin\left(\frac{(2n+1)\pi}{\alpha}\right) - \frac{(2n+1)\pi}{\alpha}\right]\right) \\
&= \frac{2}{\alpha}e^{t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)}\Bigg(\cos\left[t\sin\left(\frac{(2n+1)\pi}{\alpha}\right) - \frac{(2n+1)\pi}{\alpha}\right] - \cos\left[(t+\delta)\sin\left(\frac{(2n+1)\pi}{\alpha}\right) - \frac{(2n+1)\pi}{\alpha}\right] \\
&\qquad + \left(1 - e^{\delta\cos\left(\frac{(2n+1)\pi}{\alpha}\right)}\right)\cos\left[(t+\delta)\sin\left(\frac{(2n+1)\pi}{\alpha}\right) - \frac{(2n+1)\pi}{\alpha}\right]\Bigg) \\
&\leq \frac{2}{\alpha}e^{t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)}\left(2\sin\left[\frac{\delta}{2}\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right]\sin\left[-(t+\frac{\delta}{2})\sin\left(\frac{(2n+1)\pi}{\alpha}\right) + \frac{(2n+1)\pi}{\alpha}\right] + (1 - e^{\delta\cos\left(\frac{(2n+1)\pi}{\alpha}\right)})\right) \\
&\leq \frac{2}{\alpha}e^{t\cos\left(\frac{(2n+1)\pi}{\alpha}\right)}\left(2\sin\left[\frac{\delta}{2}\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right] + (1 - e^{\delta\cos\left(\frac{(2n+1)\pi}{\alpha}\right)})\right) \\
&\leq \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\left(2\left(\frac{\delta}{2}\pi\right)^\theta + (1 - e^{-\delta})\right) \leq \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\left(2^{1-\theta}\pi^\theta\delta^\theta + \delta^\theta\right) = \frac{2}{\alpha}e^{t\cos\left(\frac{\pi}{\alpha}\right)}\left(2^{1-\theta}\pi^\theta + 1\right)\delta^\theta.
\end{aligned}$$

The penultimate inequality follows from the fact that $0 \le \frac{\pi}{\alpha} \le \frac{(2n+1)\pi}{\alpha} \le \pi$, which leads to two key observations. On one hand, we have $1 - e^{\delta \cos\left(\frac{(2n+1)\pi}{\alpha}\right)} \le 1 - e^{-\delta}$, and on the other hand, by applying Lemma B.1 (2), we obtain the following inequality:

$$\sin\left[\frac{\delta}{2}\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right] \le \left(\frac{\delta}{2}\sin\left(\frac{(2n+1)\pi}{\alpha}\right)\right)^{\theta} \le \left(\frac{\delta}{2}\left(\frac{(2n+1)\pi}{\alpha}\right)\right)^{\theta} \le \left(\frac{\delta}{2}\pi\right)^{\theta}.$$

Where the final inequality follows from Lemma B.1 (1). Consequently, Hölder regularity with exponent ϑ for the function $f_{\alpha,\lambda}$ can be achieved provided that $\cos\left(\frac{\pi}{\alpha}\right) < 0$, i.e., $\alpha \in (0,2)$.

Now, about the α-fractional kernels with $1 < \alpha < 2$, it follows from Proposition5.1, (see also [23]) that:

$$e_\alpha(t) = F_\alpha(t) + G_\alpha(t) = \int_0^{+\infty} e^{-tu} H_\alpha(u)\, du + \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \cos\left(t \sin\left(\frac{\pi}{\alpha}\right)\right), \quad 1 < \alpha < 2, \quad t \ge 0.$$

Note that, in this case ($1 < \alpha < 2$), the function $H_\alpha(u)$ is negative for all u (and thus -F is completely monotone and hence infinitely differentiable on $\mathbb{R}_+^+$) since $1 < \alpha < 2$ implies $\sin(\alpha\pi) < 0$ and we have the following inequality: $u^{2\alpha} + 2u^\alpha \cos(\pi\alpha) + 1 \ge 1 - \cos^2(\alpha\pi) = \sin^2(\alpha\pi) > 0$ ($or\ \ge (u^\alpha - 1)^2 > 0$). $\square$

Proof of Proposition 5.2. (a) follows from the first claim of Proposition 5.1 since $R_{\alpha,\lambda} = e_\alpha(\lambda^{1/\alpha}\cdot) = R_{\alpha,1}(\lambda^{1/\alpha}\cdot)$, hence infinitely differentiable on $(0,+\infty)$ from B.68 and B.69. All will extend to $R_{\alpha,\lambda}$ by scaling. It follows from (5.47) and (B.66) that $H_\alpha(u) \le \frac{u^{\alpha-1}\sin(\pi\alpha)}{\pi \sin^2(\pi\alpha)} = \frac{u^{\alpha-1}}{\pi\sin(\pi\alpha)}$. Hence, for every $t \ge 0$,

$$R_{\alpha,1}(t) = e_\alpha(t) = F_\alpha(t) + G_\alpha(t) \le \frac{1}{\pi\sin(\pi\alpha)} \int_0^{+\infty} e^{-tu} u^{\alpha-1} du + \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} = \frac{\Gamma(\alpha)}{\pi\sin(\pi\alpha)} t^{-\alpha} + \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)}.$$

so that $R_{\alpha,1} \in \mathcal{L}^\gamma(\mathrm{Leb}_1)$ for every $\gamma > \frac{1}{\alpha}$ as $\cos\left(\frac{\pi}{\alpha}\right) < 0$, $\forall\, \alpha \in (1,2)$ and in particular $R_{\alpha,1}(t) \le 1 \quad \forall t \ge 0$ since $\sin(\pi\alpha) \le 0$. The representation of $f_{\alpha,\lambda}$ in (b) follows from (B.67) and (5.49).

(c) Let us prove the $L^{2\beta}$-integrability of $f_{\alpha,\lambda}$. Once noted that $f_{\alpha,\lambda} = \lambda^{1/\alpha} f_{\alpha,1}(\lambda^{1/\alpha}\cdot)$ so that $\int_0^{+\infty} f^{2\beta}_{\alpha,\lambda}(t)dt = \lambda^{\frac{2\beta-1}{\alpha}} \int_0^{+\infty} f^{2\beta}_{\alpha,1}(t)dt$, it is clear that it is enough to prove that $f_{\alpha,1}$ is $\mathcal{L}^{2\beta}$-integrable.

It follows from (B.67) and (5.49) that for every $t > 0$,

$$f_{\alpha,1}(t) = -e'_\alpha(t) = -F'_\alpha(t) - G'_\alpha(t) \le \frac{1}{\pi\sin(\pi\alpha)} \int_0^{+\infty} e^{-tu} u^\alpha du + \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} = \frac{\Gamma(\alpha+1)}{t^{\alpha+1}\pi\sin(\pi\alpha)} + \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)}.$$

Thus $f_{\alpha,1} \in \mathcal{L}^{2\beta}([1,+\infty), \mathrm{Leb}_1) \quad \forall \beta > 0$. On the other hand $f_{\alpha,\lambda}(t) = \lambda t^{\alpha-1} \sum_{k\ge0} (-1)^k \lambda^k \frac{t^{\alpha k}}{\Gamma(\alpha(k+1))}$ so that $f_{\alpha,1}(t) \overset{0}{\sim} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. As $t \mapsto \frac{1}{t^{1-\alpha}} \in \mathcal{L}^{2\beta}((0,1], \mathrm{Leb}_1)$ for any $\beta \in \left(\frac{1}{2(1-\alpha)}, +\infty\right) \cap \mathbb{R}_+^* = \mathbb{R}_+^*$, we conclude that $f_{\alpha,1} \in \mathcal{L}^{2\beta}(\mathrm{Leb}_1) \quad \forall \beta > 0$ and in particular $\forall \beta > 1$. Another consequence is that, for every $t \ge 1$, $R_{\alpha,1}(t) = e_\alpha(t) = \int_t^{+\infty} f_{\alpha,1}(s)\, ds \le C'_\alpha\, t^{-\alpha} + C''_\alpha\, e^{t\cos\left(\frac{\pi}{\alpha}\right)}$, so that $R_{\alpha,1} \in L^2(\mathrm{Leb}_1)$.

As for the $\mathcal{L}^2(\mathbb{R}_+)$-$\vartheta$-Hölder continuity of $f_{\alpha,\lambda}$, one may again assume w.l.g. that $\lambda = 1$. Let $\delta > 0$. One has $f_{\alpha,1}(t+\delta) - f_{\alpha,1}(t) = (F'_\alpha(t) - F'_\alpha(t+\delta)) + (G'_\alpha(t) - G'_\alpha(t+\delta))$ and following the same reasoning as above while bearing in mind that $\cos\left(\frac{\pi}{\alpha}\right) \le 0$, $\sin\left(\frac{\pi}{\alpha}\right) \ge 0$ for $\alpha \in (1,2)$, we have:

$$\begin{aligned}
G'_\alpha(t) - G'_\alpha(t+\delta) &= \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left(\cos\left[t\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right] - e^{\delta\cos\left(\frac{\pi}{\alpha}\right)} \cos\left[(t+\delta)\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right]\right) \\
&= \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left((\cos\left[t\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right] - \cos\left[(t+\delta)\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right]) + (1 - e^{\delta\cos\left(\frac{\pi}{\alpha}\right)}) \cos\left[(t+\delta)\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right]\right) \\
&\le \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left(2\sin\left[\frac{\delta}{2}\sin\left(\frac{\pi}{\alpha}\right)\right] \sin\left[-(t+\frac{\delta}{2})\sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right] + (1 - e^{\delta\cos\left(\frac{\pi}{\alpha}\right)})\right) \\
&\le \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left(2\sin\left[\frac{\delta}{2}\sin\left(\frac{\pi}{\alpha}\right)\right] + (1 - e^{\delta\cos\left(\frac{\pi}{\alpha}\right)})\right) \le \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left(2\left(\frac{\delta}{2}\frac{\pi}{\alpha}\right)^{\theta} + (1 - e^{-\delta})\right) \\
&\le \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left(2^{1-\theta}(\frac{\pi}{\alpha})^\theta \delta^\theta + \delta^\theta\right) = \frac{2}{\alpha} e^{t\cos\left(\frac{\pi}{\alpha}\right)} \left(2^{1-\theta}(\frac{\pi}{\alpha})^\theta + 1\right) \delta^\theta.
\end{aligned}$$

Where the penultimate inequality follows from the fact that $\frac{\pi}{2} \leq \frac{\pi}{\alpha} \leq \pi$, so that $1 - e^{\delta \cos\left(\frac{\pi}{\alpha}\right)} \leq 1 - e^{-\delta}$, and $\sin\left[\frac{\delta}{2}\sin\left(\frac{\pi}{\alpha}\right)\right] \leq \left(\frac{\delta}{2}\sin\left(\frac{\pi}{\alpha}\right)\right)^{\theta} \leq \left(\frac{\delta}{2}\left(\frac{\pi}{\alpha}\right)\right)^{\theta}$ owing to Lemma B.1 (2). The final inequality follows from Lemma B.1 (1). Moreover, for the term $F'_\alpha(t) - F'_\alpha(t+\delta) := \int_0^{+\infty} e^{-tu}(1-e^{-\delta u})uH_\alpha(u)du$, we may write

$$F'_\alpha(t) - F'_\alpha(t+\delta) \leq \int_0^{+\infty} e^{-tu}(1-e^{-\delta u})^{\vartheta} uH_\alpha(u)du \leq \int_0^{+\infty} e^{-tu}\delta^{\vartheta}u^{1+\vartheta}H_\alpha(u)du.$$

1. Owing to Fubini-Tonelli's theorem in the first line to interwind the order of integration, we have: $\int_0^{+\infty}\left(F'_\alpha(t+\delta) - F'_\alpha(t)\right)dt \leq \int_{(0,+\infty)} u^{1+\vartheta}H_\alpha(u)\int_0^{+\infty} e^{-tu}dt\,du\,\delta^{\vartheta} = \left[\int_{(0,+\infty)} u^{\vartheta}H_\alpha(u)\,du\right]\delta^{\vartheta}$ and

$$\int_0^{+\infty}\left(G'_\alpha(t) - G'_\alpha(t+\delta)\right)dt \leq \frac{2}{\alpha}\left(2^{1-\theta}(\frac{\pi}{\alpha})^{\theta}+1\right)\delta^{\theta}\int_0^{+\infty} e^{t\cos\left(\frac{\pi}{\alpha}\right)}dt = \left[\frac{-2}{\alpha\cos\left(\frac{\pi}{\alpha}\right)}\left(2^{1-\theta}(\frac{\pi}{\alpha})^{\theta}+1\right)\right]\delta^{\vartheta}.$$

It follows that, $\int_0^{+\infty}\left(f_{\alpha,1}(t+\delta) - f_{\alpha,1}(t)\right)dt \leq \left[\int_{\mathbb{R}_+} u^{\vartheta}H_\alpha(u)\,du + \frac{2}{\alpha}\left(2^{1-\theta}(\frac{\pi}{\alpha})^{\theta}+1\right)\right]\delta^{\vartheta}$.

Now, we derive form (5.47) that: $H_\alpha(u) \overset{0}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{\alpha-1}$ and $H_\alpha(u) \overset{+\infty}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{-(\alpha+1)}$. Consequently

$$u^{\vartheta}H_\alpha(u) \overset{0}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{\alpha-1+\vartheta} \quad \text{and} \quad u^{\vartheta}H_\alpha(u) \overset{+\infty}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{-(1+\alpha-\vartheta)},$$

which implies that $\int_{(0,+\infty)} u^{\vartheta}H_\alpha(u)\,du < +\infty$ if and only if $2-\alpha < \vartheta < \alpha$.

2. Secondly, as: $\left(f_{\alpha,1}(t+\delta) - f_{\alpha,1}(t)\right)^2 \leq 2\left((F'_\alpha(t) - F'_\alpha(t+\delta))\right)^2 + 2\left((G'_\alpha(t) - G'_\alpha(t+\delta))\right)^2$ with:

$$\begin{aligned}\int_0^{+\infty}\left(F'_\alpha(t+\delta) - F'_\alpha(t)\right)^2dt &\leq \int_0^{+\infty}\int_0^{+\infty} e^{-tu}\delta^{\vartheta}u^{1+\vartheta}H_\alpha(u)du\int_0^{+\infty} e^{-tv}\delta^{\vartheta}v^{1+\vartheta}H_\alpha(v)dv \\ &\leq \int_{(0,+\infty)^2}(uv)^{1+\vartheta}H_\alpha(u)H_\alpha(v)\int_0^{+\infty} e^{-t(u+v)}dt\,du\,dv\,\delta^{2\vartheta} = \int_{(0,+\infty)^2}\frac{(uv)^{1+\vartheta}}{u+v}H_\alpha(u)H_\alpha(v)\,du\,dv\,\delta^{2\vartheta} \\ &\leq \tfrac{1}{2}\int_{(0,+\infty)^2}(uv)^{\frac{1}{2}+\vartheta}H_\alpha(u)H_\alpha(v)\,du\,dv\,\delta^{2\vartheta} = \tfrac{1}{2}\left[\int_{(0,+\infty)} u^{\frac{1}{2}+\vartheta}H_\alpha(u)\,du\right]\delta^{2\vartheta}.\end{aligned}$$

where we used Fubini-Tonelli's theorem in the first line to interwind the order of integration and the elementary inequality $\sqrt{uv} \leq \frac{1}{2}(u+v)$ when $u,\, v \geq 0$ in the penultimate line. Furthermore,

$$\int_0^{+\infty}\left(G'_\alpha(t+\delta) - G'_\alpha(t)\right)^2dt \leq \frac{4}{\alpha^2}\left(\left(2^{1-\theta}(\frac{\pi}{\alpha})^{\theta}+1\right)\delta^{\theta}\right)^2\int_0^{+\infty} e^{2t\cos\left(\frac{\pi}{\alpha}\right)}dt = \left[\frac{-2}{\alpha^2\cos\left(\frac{\pi}{\alpha}\right)}\left(2^{1-\theta}(\frac{\pi}{\alpha})^{\theta}+1\right)^2\right]\delta^{2\vartheta}.$$

It follows that , $\int_0^{+\infty}\left(f_{\alpha,1}(t+\delta) - f_{\alpha,1}(t)\right)^2dt \leq \left[\int_{\mathbb{R}_+} u^{\frac{1}{2}+\vartheta}H_\alpha(u)\,du + \frac{4}{\alpha^2}\left(2^{1-\theta}(\frac{\pi}{\alpha})^{\theta}+1\right)^2\right]\delta^{2\vartheta}$.

Now, we derive form (5.47) that: $H_\alpha(u) \overset{0}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{\alpha-1}$ and $H_\alpha(u) \overset{+\infty}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{-(\alpha+1)}$, Consequently

$$u^{\frac{1}{2}+\vartheta}H_\alpha(u) \overset{0}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{\alpha-\frac{1}{2}+\vartheta} \quad \text{and} \quad u^{\frac{1}{2}+\vartheta}H_\alpha(u) \overset{+\infty}{\sim} \frac{\sin(\pi\alpha)}{\pi}u^{-(-\frac{1}{2}+\alpha-\vartheta)},$$

which implies that $\int_{\mathbb{R}_+} u^{\frac{1}{2}+\vartheta}H_\alpha(u)\,du < +\infty$ iff $\vartheta < \alpha - \frac{1}{2}$. One concludes when $\lambda > 0$ by scaling. □

Lemma B.2. *Let $\alpha \in (1, \frac{3}{2})$. For every $k \geq 1$,*

1. $\forall l \geq 1 \quad \forall a \geq 1, \quad B(\alpha\ell, \alpha(k-\ell+a)) \geq \frac{1}{(\alpha(k+a)-1)2^{\alpha k+2(a-1)}} \geq \frac{1}{\alpha(k+a)2^{\alpha k+2(a-1)}}$.
2. $(a*b)_k \leq \frac{2^{\alpha k}}{\Gamma(\alpha(k+1))}\left(1+(k+1)(1+\log k)\right)$.
3. $(b^{*2})_k \leq \frac{(\alpha(k+2)-1)(k+1)2^{\alpha k+2}}{\Gamma(\alpha(k+2))}$.

Proof. 1.$\forall l \geq 1 \quad \forall a \geq 1$, we have:

$$B(\alpha\ell, \alpha(k-\ell+a)) = \int_0^1 u^{\alpha\ell-1}\,(1-u)^{\alpha(k-\ell+a)-1}\; du \geq \int_0^{\frac{1}{2}} u^{\alpha(k+a)-2}\,du + \int_{\frac{1}{2}}^1 (1-u)^{\alpha(k+a)-2}\;du$$
$$= 2\int_0^{\frac{1}{2}} u^{\alpha(k+a)-2}\,du \geq \frac{1}{(\alpha(k+a)-1)2^{\alpha(k+a)-2}} \geq \frac{1}{(\alpha(k+a)-1)2^{\alpha k+2(a-1)}}.$$

Where the last inequality comes from the fact that $\alpha < 2$.

2. Using the identity : $\forall a, b > 0 \quad \Gamma(a+1) = a\Gamma(a), \quad B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, we have for every $k \geq 1$

$$(a*b)_k = \sum_{\ell=0}^{k} \frac{1}{\Gamma(\alpha\ell+1)\Gamma(\alpha(k-\ell+1))} = \frac{1}{\Gamma(1)\Gamma(\alpha(k+1))} + \sum_{\ell=1}^{k}\frac{1}{\alpha\ell\Gamma(\alpha\ell)\Gamma(\alpha(k-\ell+1))}$$
$$= \frac{1}{\Gamma(\alpha(k+1))}\left[1+\frac{1}{\alpha}\sum_{\ell=1}^{k}\frac{1}{\ell}\frac{1}{B(\alpha\ell,\alpha(k-\ell+1))}\right] \leq \frac{2^{\alpha k}}{\Gamma(\alpha(k+1))}\,(1+(k+1)(1+\log k)).$$

where the last inequality comes from Lemma B.2(1) for $a = 1$ and the fact that $\frac{1}{2^\alpha} \leq 1$. 3. Likewise,

$$(b^{*2})_k = \sum_{\ell=0}^{k}\frac{1}{\Gamma(\alpha(\ell+1))\Gamma(\alpha(k-\ell+1))} = \frac{1}{\Gamma(\alpha(k+2))}\sum_{\ell=0}^{k}\frac{1}{B(\alpha(\ell+1),\alpha(k-\ell+1))}$$
$$\leq \frac{(\alpha(k+2)-1)(k+1)}{\Gamma(\alpha(k+2))}2^{\alpha k+2}.$$

Still owing to Lemma B.2 (1), now for $a = 2$. □

Proof of Proposition 5.2. Step 1. (1) comes from equation (5.51) and Lemma 3.9 (4).

Step 2. To establish statement (2), following the approach in [36], it is useful (though not strictly necessary) to transition to Laplace transforms. For simplicity, and as indicated in remark (5.53), we assume $c = \lambda = 1$ and proceed by rewriting the series expansions in (5.50). We define $R_\alpha := R_{\alpha,1}$ and $f_\alpha := f_{\alpha,1}$, as follows:

$$R_\alpha(t) = \sum_{k\geq 0}(-1)^k a_k t^{\alpha k},\; f_\alpha(t) = t^{\alpha-1}\sum_{k\geq 0}(-1)^k b_k t^{\alpha k} \quad \text{with} \quad a_k = \frac{1}{\Gamma(\alpha k+1)},\; b_k = \frac{1}{\Gamma(\alpha(k+1))},\; k\geq 0.$$

Now, using the Cauchy product of two series [15] and the fact that $L_{u^\gamma}(t) = t^{-(\gamma+1)}\Gamma(\gamma+1)$, we obtain the following Laplace transforms: $L_{R_\alpha f_\alpha}(t) = t^{-\alpha}\sum_{k\geq 0}(-1)^k(a*b)_k t^{-\alpha k}\Gamma(\alpha(k+1))$ and $L_{f_\alpha^2}(t) = t^{-2\alpha+1}\sum_{k\geq 0}(-1)^k(b^{*2})_k t^{-\alpha k}\Gamma(\alpha(k+2)-1)$, where for two sequences of real numbers $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$, the Cauchy product is defined as $(u*v)_k = \sum_{\ell=0}^{k} u_\ell v_{k-\ell}$. We define the sequences

$$\widetilde{b}_k = (b^{*2})_k\Gamma(\alpha(k+2)-1) \quad \text{and} \quad \widetilde{c}_k = c_k\Gamma(\alpha(k-1)+2),\; k\geq 0.$$

Assuming that $\varsigma_\alpha^2(t)$ (for $c = \lambda = 1$) takes the expected form (5.53), we have:

$$L_{\varsigma_\alpha^2}(t) = 2\sum_{k\geq 0}(-1)^k c_k t^{-(\alpha(k-1)+2)}\Gamma(\alpha(k-1)+2) = 2t^{\alpha-2}\sum_{k\geq 0}(-1)^k\widetilde{c}_k t^{-\alpha k}.$$

15. The Cauchy product of two series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ is given by the series $C(x) = A(x)\cdot B(x) = \sum_{n=0}^{\infty} c_n x^n$, where the coefficients c_n are defined by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Thus, by equating the coefficients from both sides of equation (3.26), we obtain the condition: $\forall k \geq 0, \quad (\widetilde{b} * \widetilde{c})_k = (a * b)_k \Gamma(\alpha(k+1))$.

Simple computations yield $c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)\Gamma(2-\alpha)}$, and for every $k \geq 1$,

$$c_k = \frac{\Gamma(\alpha)^2}{\Gamma(\alpha(k-1)+2)\Gamma(2\alpha-1)} \left[\Gamma(\alpha(k+1))(a*b)_k - \sum_{\ell=1}^{k} \Gamma(\alpha(\ell+2)-1)\Gamma(\alpha(k-\ell-1)+2)(b^{*2})_\ell c_{k-\ell} \right]. \tag{B.70}$$

Using standard identities such as $\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$ for $a, b > 0$, where $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du$, and $\Gamma(a+1) = a\Gamma(a)$, we arrive at the formulation of the c_k's provided in the proposition, which is more suitable for numerical computations.

Step 3. Using standard methods, as in [10] or Appendix A of [36] (in the case $\alpha \in (\frac{1}{2}, 1)$), we show that the radius of convergence ρ_α of the power series defined by the coefficients c_k is infinite. Firstly, let us prove by induction that there exists $A > 2^{\alpha+2}$ and $K > 1$ such that,

$$\forall k \geq 0 \quad |c_k| \leq \frac{KA^k}{\Gamma(\alpha(k-1)+2)}. \tag{B.71}$$

By the triangle inequality, we get the bound :

$$|c_k| \leq \frac{\Gamma(\alpha)^2\Gamma(\alpha(k+1))}{\Gamma(\alpha(k-1)+2)\Gamma(2\alpha-1)} \left[(a*b)_k + \alpha(k+1)\sum_{\ell=1}^{k} B\left(\alpha(\ell+2)-1, \alpha(k-\ell-1)+2\right)(b^{*2})_\ell |c_{k-\ell}| \right]. \tag{B.72}$$

Initialisation: For $k = 0$, $c_0 = \frac{\Gamma(\alpha)^2}{\Gamma(2-\alpha)\Gamma(2\alpha-1)} \leq \frac{K}{\Gamma(2-\alpha)}$ since $K > 1$ and by log-convexity $\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)} < 1$.
Heredity: Now let $k \geq 1$ and assume that c_ℓ satisfies the inequality (B.71) for every $\ell = 0, \ldots, k-1$. Then, for every $\ell = 1, \ldots, k$,

$$B\left(\alpha(\ell+2)-1, \alpha(k-\ell-1)+2\right)(b^{*2})_\ell |c_{k-\ell}| \leq \frac{\Gamma(\alpha(\ell+2)-1)\Gamma(\alpha(k-\ell-1)+2)}{\Gamma(\alpha(k+1)+1)\,\Gamma(\alpha(k-\ell-1)+2)} \times KA^{k-\ell}(b^{*2})_\ell$$
$$\leq \frac{KA^{k-\ell}\Gamma(\alpha(\ell+2)-1)}{\Gamma(\alpha(k+1)+1)} \frac{(\alpha(l+2)-1)(l+1)2^{\alpha l+2}}{\Gamma(\alpha(l+2))} \leq \frac{KA^{k-\ell}}{\Gamma(\alpha(k+1)+1)} \frac{(\alpha(l+2)-1)(l+1)2^{\alpha l+2}}{(\alpha(\ell+2)-1)}$$
$$\leq K\frac{(l+1)2^{\alpha l+2}A^{k-\ell}}{\alpha(k+1)\Gamma(\alpha(k+1))}.$$ Inserting this bound into the inequality (B.72) for c_k gives:

$$|c_k| \leq \frac{\Gamma(\alpha)^2}{\Gamma(\alpha(k-1)+2)\Gamma(2\alpha-1)} \left[\Gamma(\alpha(k+1))(a*b)_k + K\,A^k \frac{1}{\Gamma(\alpha(k+1))} \sum_{\ell=1}^{k} (\ell+1)\rho^\ell \right].$$

where we set $\rho = \rho(A) := \frac{2^{\alpha+2}}{A}$. Next, dividing the above inequality by KA^k and using the upper bound for $(a*b)_k$ from Lemma B.2(2):

$$\frac{|c_k|}{KA^k} \leq \frac{1}{\Gamma(\alpha(k-1)+2)} \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)} \left[\frac{\rho^k}{K}\left(1 + (k+1)(1+\log k)\right) + \frac{1}{(1-\rho)^2} \right].$$

Owing to the elementary inequality: $\forall\, \rho \in (0,1), \quad \sum_{l \geq 1} l\rho^{l-1} \leq \frac{1}{(1-\rho)^2}$. Let $\epsilon > 0$ and let $A = A_\epsilon$ be large enough so that $\sup_{k\geq 1}\left(\rho^k + \rho^k(k+1)(1+\log k)\right) < \epsilon$ and $\frac{1}{(1-\rho)^2} < 1 + \epsilon$. Due to the log-convexity of the Gamma function, $\log\Gamma(\alpha) \leq \frac{1}{2}\log\Gamma(2\alpha-1) + \log\Gamma(1) = \frac{1}{2}\log\Gamma(2\alpha-1)$, so that $\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)} < 1$. Thus, it is possible to choose ϵ small enough and K large enough such that:
$$\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)} \left[\frac{\rho^k}{K}\left(1 + (k+1)(1+\log k)\right) + \frac{1}{(1-\rho)^2} \right] \leq \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)}\left(\frac{\epsilon}{K} + 1 + \epsilon\right) < 1.$$

Consequently, $|c_k| \le \frac{KA^k}{\Gamma(\alpha(k-1)+2)}$. And thus the Cauchy-Hadamard's formula for the radius of convergence together with Stirling's formula give:

$$\limsup_{k\to\infty} |c_k|^{\frac{1}{k}} \le \limsup_{k\to\infty} \left(\frac{K\,A^k}{\Gamma(\alpha k+2-\alpha)}\right)^{1/k} \sim \lim_{k\to\infty} A\frac{K^{\frac{1}{k}}}{e^{-\alpha}(\alpha(k-1)+2)^{\alpha}} = 0.$$

Proof of Proposition 5.3. From equation (6.58), there exists an analytic function $\tilde{g}_\alpha : \mathbb{C} \to \mathbb{C}$ such that

$$\forall t \ge 0, \quad g_{\alpha,\rho,\lambda}(t) = e^{-2\rho t} t^{1-\alpha} \tilde{g}_\alpha(\lambda\, t^\alpha) \quad \textit{and} \quad \tilde{g}_\alpha(0) = 2\,c\,\lambda\,c_0 > 0. \tag{B.73}$$

Step 1. *Case $\alpha \le 1$:* The class of completely monotone (CM) functions is a convex cone, thus is stable under pointwise positive summation, product, and also convolution. Differentiating both sides of equation (6.59) and using the fact that $g_{\alpha,\rho,\lambda}(0) = 0$ yields

$$2c\lambda^2 f_{\alpha,\rho,\lambda}(t) R_{\alpha,\rho,\lambda}(t) = \int_0^t f^2_{\alpha,\rho,\lambda}(t-s)\, g'_{\alpha,\rho,\lambda}(s)\, ds, \quad \forall\, t \ge 0.$$

Here $f_{\alpha,\rho,\lambda}(t) := e^{-\rho t} f_{\alpha,\lambda}(t)$, which is CM as the product of two CM functions. Hence $R_{\alpha,\rho,\lambda}$ is CM, and consequently both $2c\lambda^2 f_{\alpha,\rho,\lambda} R_{\alpha,\rho,\lambda}$ and $f^2_{\alpha,\rho,\lambda}$ are CM functions. Since $g_{\alpha,\rho,\lambda}(0) = 0$, we deduce from [4, Theorem 5.5.5] that $g_{\alpha,\rho,\lambda}(t) = \int_0^t g'_{\alpha,\rho,\lambda}(s)\, ds \ge 0, \quad \forall\, t \ge 0$.

For simplification, we set $g_{\alpha,\rho,\lambda} \equiv g_\alpha$. One shows, as in [36], by contradiction that $g''_\alpha \le 0$ on $(0, +\infty)$, i.e. g_α is concave. Using the product and chain rules, we have that $g'_\alpha(t) = e^{-2\rho t}\left((-2\rho t^{1-\alpha} + (1-\alpha)t^{-\alpha})\tilde{g}_\alpha(\lambda t^\alpha) + \lambda\,\alpha\,\tilde{g}'_\alpha(\lambda t^\alpha)\right)$. Since $\alpha < 1$, $\lim_{t\to 0^+} \frac{t^{-\alpha}}{t^{1-\alpha}} = \lim_{t\to 0^+} \frac{1}{t}$ $= +\infty$, we have $g'_\alpha(t) \underset{t\to 0^+}{\sim} (1-\alpha)t^{-\alpha}\tilde{g}_\alpha(0) + \lambda\,\alpha \tilde{g}'_\alpha(0)$ so that $\lim_{t\to 0^+} g'_\alpha(t) = +\infty$. Moreover, by Tauberian Final Value Theorem if $\lim_{t\to+\infty} \tilde{g}'_\alpha(t)$ exists, then

$$\lim_{t\to+\infty} \tilde{g}'_\alpha(t) = \lim_{z\to 0} z L_{\tilde{g}'_\alpha}(z)(z) = \lim_{z\to 0} \left(z^2 L_{\tilde{g}_\alpha}(z) - z\tilde{g}_\alpha(0)\right) = \lim_{z\to 0} \left(z^2 L_{\tilde{g}_\alpha}(0) - z\tilde{g}_\alpha(0)\right) = 0$$

since $\tilde{g}_\alpha$ is integrable and thus have a finite Laplace transform. Consequently, $\lim_{t\to+\infty} g'_\alpha(t) = 0$. Finally, $\lim_{t\to 0^+} g'_\alpha(t) = +\infty$, $\lim_{t\to+\infty} g'_\alpha(t) = 0$ and g'_α is non-increasing on $(0, +\infty)$ ($g''_\alpha \le 0$), it follows that $g'_\alpha(t) \ge 0 \quad \forall t \in (0, +\infty)$. Hence g_α is concave, non-decreasing and non-negative on $(0, +\infty)$.

step 2. Case $\alpha > 1$: We have $\lim_{t\to 0^+} g_{\alpha,\rho,\lambda} = +\infty$ and $\lim_{t\to+\infty} g_{\alpha,\rho,\lambda} > 0$. Hence, there exists $t_0, t_1 > 0$ such that $g_{\alpha,\rho,\lambda} \ge 0$ at least on the small intervals $(0, t_0) \cup (t_1, +\infty)$ with $t_0 = \inf\{t : g_{\alpha,\rho,\lambda}(t) < 0\}$ and $t_1 = \sup\{t : g_{\alpha,\rho,\lambda}(t) < 0\}$. By continuity of $g_{\alpha,\rho,\lambda}$ it is clear that $g_{\alpha,\rho,\lambda}(t_0) = g_{\alpha,\rho,\lambda}(t_1) = 0$ and $g_{\alpha,\rho,\lambda} \ge 0$ on $[0, t_0] \cup [t_1, +\infty)$. While numerical computations suggest that $g_{\alpha,\rho,\lambda}$ is positive on $\mathbb{R}_+$ (i.e. $t_0 = t_1 = \infty$), establishing this positivity analytically turns out to be quite challenging. We shall, however, establish that if $T^{\alpha,\lambda,\rho}$ is the first zero of the resolvent $R_{\alpha,\rho,\lambda}$ (see [24, Proposition 3.13.] for all zeros of the functions E_α), then, since $R^2_{\alpha,\rho,\lambda}$ decreases strictly on $(0, T^{\alpha,\lambda,\rho})$, the function $g_{\alpha,\rho,\lambda}$ remains non-negative over that interval.

Let's assume that $t_0 \in (0, T^{\alpha,\lambda,\rho})$ and thus $g_{\alpha,\rho,\lambda} \le 0$ on a small interval $[t_0, t_0 + \eta] \subset (0, T^{\alpha,\lambda,\rho})$ for some $\eta > 0$. Then, for every $t \in (t_0, t_0 + \eta]$, there exists $\tau > 0$ such that $t = t_0 + \tau$. Let $\delta \in (0, \frac{\tau}{2})$, and set $c := -\max_{s\in[t_0+\delta, t_0+\tau]} g_{\alpha,\rho,\lambda}(s)$. By continuity $c > 0$ and $g_{\alpha,\rho,\lambda}(s) \le -c$ for all $s \in [t_0 + \delta, t_0 + \tau]$. For simplification, we set $f_{\alpha,\rho,\lambda} \equiv f_\alpha$ and $R_{\alpha,\rho,\lambda} \equiv R_\alpha$, $g_{\alpha,\rho,\lambda} \equiv g_\alpha$. Then, we have:

$$\begin{aligned}(f^2_\alpha * g_\alpha)(t_0+\tau) - (f^2_\alpha * g_\alpha)(t_0) &= \int_0^{t_0} \underbrace{\left(f^2_\alpha(t_0+\tau-s) - f^2_\alpha(t_0-s)\right)}_{\ge\approx 0} \underbrace{g_\alpha(s)}_{\ge 0}\, ds + \int_{t_0}^{t_0+\delta} f^2_\alpha(t_0+\tau-s)\, \underbrace{g_\alpha(s)}_{\le 0}\, ds \\ &+ \int_{t_0+\delta}^{t_0+\tau} f^2_\alpha(t_0+\tau-s)\, \underbrace{g_\alpha(s)}_{\le 0}\, ds \le I_1 - I_2 - c\left(\int_0^{\tau-\delta} f^2_\alpha(u)\, du\right).\end{aligned}$$

where $I_2 := -\int_{t_0}^{t_0+\delta} f_\alpha^2(t_0+\tau-s)\, g_\alpha(s)\, ds \geq 0$ and $I_1 := \int_0^{t_0}(f_\alpha^2(t_0+\tau-s) - f_\alpha^2(t_0-s))\, g_\alpha(s)\, ds \geq 0$

However, as I_1 is nonnegative and close to zero, for an adequate choice of $\delta \in (0, \frac{\tau}{2})$, the upper bound above is strictly negative. On the other hand, $(f_\alpha^2 * g_\alpha)(t_0+\tau) - (f_\alpha^2 * g_\alpha)(t_0) = c\lambda^2(R_\alpha^2(t_0) - R_\alpha^2(t_0+\tau)) > 0$, which yields a contradiction. Hence, for every large enough $n \geq 0$, there exists $t_n^+ \in (t_0, t_0 + \frac{1}{n}]$ such that $g_\alpha(t_n^+) > 0$. On the other hand, by the very definition of t_0, there exists a sequence $t_n^- > t_0$, $n \geq 1$, such that $g_\alpha(t_n^-) < 0$. One then builds by induction a sequence $(\tau_n)_{n\geq 1}$ such that $g_\alpha(\tau_{2n+1}) < 0$ and $g_\alpha(\tau_{2n}) > 0$, with $\tau_n \to t_0$ as $n \to +\infty$, $\tau_n > t_0$. In turn this implies, by the intermediate value theorem, the existence of a sequence $(\tilde{\tau}_n)_{n\geq 1}$ such that $\tilde{g}_\alpha(\lambda \tilde{\tau}_n^\alpha) = g_\alpha(\tilde{\tau}_n) = 0$, $\lambda \tilde{\tau}_n^\alpha > \lambda t_0^\alpha$ and $\lambda \tilde{\tau}_n^\alpha \to \lambda t_0^\alpha$ by the continuity of g_α. As $\tilde{g}_\alpha$ is analytic, it implies that $\tilde{g}_\alpha$ is everywhere zero. Hence a contradiction since $\tilde{g}_\alpha(0) > 0$.

From the above steps, we have $\forall t \geq 0 \quad g_{\alpha,\rho,\lambda}(t) \geq 0$ on an interval $I \subseteq (0, +\infty)$ so that the function $\sqrt{g_{\alpha,\rho,\lambda}}$ is well-defined on I. □

Proof of Proposition 6.1. (a) We consider the function $R_{\alpha,\rho,\lambda}(t) = 1 + \sum_{k\geq 1}(-1)^k \frac{\lambda^k}{\Gamma(k\alpha)} I_k(t)$ where $I_k(t) = \int_0^t e^{-\rho s} s^{k\alpha-1}\, ds$. Given that for all $k \geq 1$, the function $s \mapsto e^{-\rho s}s^{k\alpha-1}$ is measurable and locally integrable on $(0,t)$, the map $t \mapsto \int_0^t e^{-\rho s}s^{k\alpha-1}\, \mathrm{d}s$ is differentiable. Moreover, the series of derivatives $\sum_{k\geq 1}(-1)^k \frac{\lambda^k}{\Gamma(k\alpha)} e^{-\rho t} t^{k\alpha-1}$ converges absolutely locally uniformly in $t > 0$. Hence, by the dominated convergence theorem (or Lebesgue's theorem on differentiation under the integral sign), term-by-term differentiation is justified, and $R_{\alpha,\rho,\lambda}(t)$ is differentiable for $t > 0$, with its derivative given by: $R'_{\alpha,\rho,\lambda}(t) = \sum_{k\geq 1}(-1)^k \frac{\lambda^k}{\Gamma(k\alpha)} e^{-\rho t} t^{k\alpha-1} =: f_{\alpha,\rho,\lambda}(t)$, One could argue similarly to show that $R_{\alpha,\rho,\lambda}$ is infinitely differentiable, i.e., $\mathcal{C}^\infty$ on $(0,+\infty)$. Alternatively, observe that for all $t > 0$, we have $f_{\alpha,\rho,\lambda}(t) = e^{-\rho t} f_{\alpha,\lambda}(t)$, which is $\mathcal{C}^\infty$ as the product of such functions, by virtue of the first claim in Proposition 5.2.

(b) The representation of $f_{\alpha,\rho,\lambda}$ follows by definition and from the claim (b) of Proposition 6.1.

(c) Let us prove the $L^{2\beta}$-integrability of $f_{\alpha,\rho,\lambda}$. Once noted that $f_{\alpha,\rho,\lambda} = e^{-\rho t} f_{\alpha,\lambda}$ so that

$$\int_0^{+\infty} f_{\alpha,\rho,\lambda}^{2\beta}(t)dt = \int_0^{+\infty} e^{-2\beta\rho t} f_{\alpha,\lambda}^{2\beta}(t)dt \leq \int_0^{+\infty} f_{\alpha,\lambda}^{2\beta}(t)dt,$$

it is clear that it is enough to have that $f_{\alpha,\lambda}$ is $\mathcal{L}^{2\beta}$-integrable.

It follows from [36, Proposition 5.1] and Proposition 5.2 that $f_{\alpha,\rho,\lambda}$ is $\mathcal{L}^{2\beta}$-integrable $\forall \beta \in \left(0, \frac{1}{2(1-\alpha)}\right)$ if $\alpha < 1$ and $\forall \beta \in \mathbb{R}_+^*$ if $\alpha > 1$. As for the $\mathcal{L}^2(\mathbb{R}_+)$-$\vartheta$-Hölder continuity of $f_{\alpha,\rho,\lambda}$, let $\delta > 0$. One has

$$f_{\alpha,\rho,\lambda}(t+\delta) - f_{\alpha,\rho,\lambda}(t) = e^{-\rho(t+\delta)}\left(f_{\alpha,\lambda}(t+\delta) - f_{\alpha,\lambda}(t)\right) + f_{\alpha,\lambda}(t)\left(e^{-\rho(t+\delta)} - e^{-\rho t}\right).$$

Then, for $i \in \{1,2\}$, we write:

$$|f_{\alpha,\rho,\lambda}(t+\delta) - f_{\alpha,\rho,\lambda}(t)|^i \leq 2^{i-1}\left(e^{-i\rho(t+\delta)}\left|f_{\alpha,\lambda}(t+\delta) - f_{\alpha,\lambda}(t)\right|^i + e^{-i\rho t}\left|f_{\alpha,\lambda}(t)\right|^i (1 - e^{-i\rho\delta})^i\right).$$

Integrating both side and using again Lemma B.1, one may deduce

$$\int_0^\infty |f_{\alpha,\rho,\lambda}(t+\delta) - f_{\alpha,\rho,\lambda}(t)|^i\, dt \leq 2^{i-1}\left(e^{-i\rho\delta}\int_0^\infty |f_{\alpha,\lambda}(t+\delta) - f_{\alpha,\lambda}(t)|^i\, dt + (\rho\delta)^{i\vartheta}\int_0^\infty |f_{\alpha,\lambda}(t)|^i\, dt\right).$$

Consequently, since $f_{\alpha,\lambda} \in L^2(\mathrm{Leb}_1)$

$$\begin{aligned}\left(\int_0^\infty |f_{\alpha,\rho,\lambda}(t+\delta) - f_{\alpha,\rho,\lambda}(t)|^i\, dt\right)^{1/i} &\leq e^{-\rho\delta}\left(\int_0^\infty |f_{\alpha,\lambda}(t+\delta) - f_{\alpha,\lambda}(t)|^i\, dt\right)^{1/i} + (\rho\delta)^\vartheta\left(\int_0^\infty |f_{\alpha,\lambda}(t)|^i\, dt\right)^{\frac{1}{i}}\\ &\leq e^{-\rho\delta} C_{\vartheta,\lambda}\delta^\vartheta + C_{f_\lambda}\delta^\vartheta := C_{\vartheta,\rho,\lambda}\delta^\vartheta.\end{aligned}$$

where the last inequality is a direct application of Proposition 5.2 and we are done. □