

Extension of the Gyárfás-Sumner conjecture to signed graphs

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Abstract

The balanced chromatic number of a signed graph $\widehat{G}$ is the minimum number of balanced sets that cover all vertices of G . Studying structural conditions which implies bound on the balanced chromatic number of signed graphs then is among the most fundamental problems in graph theory. In this work, we initiate the study of coloring hereditary classes of signed graphs. More precisely we say that a set $F = \{\widehat{F}_1, \widehat{F}_2, \dots, \widehat{F}_l\}$ is a GS (for Gy  rf  s-Sumner) set if there exists a constant c such that signed graphs with no induced subgraph switching equivalent to a member of F admit a balanced c -coloring. The focus of this work then is to study GS sets of order 2. We show that if F is a GS set of order 2, then F_1 is either $(\widehat{K_3}, -)$ or $(\widehat{K_4}, -)$ and F_2 is a linear forest. In the case of $F_1 = (\widehat{K_3}, -)$ we show that any choice of a linear forest for F_2 works. In the case of $F_1 = (\widehat{K_4}, -)$ we show that if each connected component of F_2 is a path of length at most 4, then $\{F_1, F_2\}$ is a GS set.

1 Introduction

One of the key questions in the theory of proper coloring of graphs is: what structure conditions impose upper bounds on the chromatic number? After a series of constructions of triangle-free graphs of arbitrarily large chromatic number, P. Erd  s [4], in one of the earliest use of probabilistic methods, proved the following.

Theorem 1. *There exist graphs of arbitrarily large chromatic number and girth.*

This immediately implies that if given a finite set F of graphs, the class of graphs with no induced subgraph isomorphic to a graph in F , denoted $\text{Forb}_{\text{ind}}(F)$, has a bounded chromatic number, then F must contain at least one forest. Considering the class of complete graphs, any such set F must also contain a complete graph. Gy  rf  s and Sumner, independently, conjectured, in [7] and [14], that any such pair is enough for $\text{Forb}_{\text{ind}}(F)$ to have a bounded chromatic number:

Gy  rf  s-Sumner conjecture. *For any forest F and any complete graph K_t the class $\text{Forb}_{\text{ind}}\{F, K_t\}$ has a bounded chromatic number.*

The goal of this work is to consider potential extensions of this conjecture and related results to the class of signed graphs. To this end, we first introduce the notions and concepts.

1.1 Definitions and Notations

A *signed graph* is a pair (G, σ) where G is a graph and $\sigma : E(G) \rightarrow \{+, -\}$ is a mapping that assigns to each edge one of the two signs: positive or negative. If Σ is the set of negative edges, then (G, σ) can be equivalently presented as (G, Σ) . When σ is of little importance, we may write $\widehat{G}$ in the place of (G, σ) . The subgraph $(V(G), \Sigma)$ of $\widehat{G}$ is denoted by $\widehat{G}^-$.

The graph G is the *underlying graph* of $\widehat{G}$. A pair of parallel edges of different signs is called a *digon*. However, unless especially mentioned, in the rest of this work, we only consider signed simple graphs.

Switching a vertex v of (G, σ) is to multiply the sign of each edge incident to v by $-$. Observe that switching is an involution, and that the order doesn't matter when switching multiple vertices. Therefore, we may switch a set of vertices, meaning that we switch all of the vertices of the set in an arbitrary order. If (G, σ') is obtained from (G, σ) by switching some vertices then we say they are *switching equivalent* (see Figure 1).

The *sign of a structure* W in (G, σ) , denoted $\sigma(W)$, is the product of the signs of its edges, considering multiplicity. It is immediate that the sign of a closed walk, and in particular a cycle, will not change after switching.

A signed graph $\widehat{G}$ is said to be *balanced* if every cycle of it is positive. Hence, we set the following definition:

Definition 2. A *balanced set of a sign graph* $\widehat{G}$ is a subset of vertices $X \subseteq V(\widehat{G})$ such that every cycle in $\widehat{G}[X]$ is positive.

The signed graph on G where all edges are negative (respectively, positive) is denoted by $(G, -)$ (respectively, $(G, +)$). A signed graph switching equivalent to $(G, -)$ will be denoted by $\widehat{(G, -)}$.

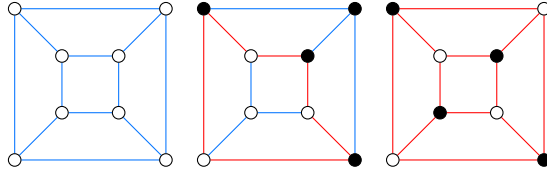


Figure 1: $(Q_3, +) \simeq (Q_3, \sigma) \simeq (Q_3, -)$
Switching black vertices in (Q_3, σ) , or in $(Q_3, -)$, results in $(Q_3, +)$.

The following lemma of Zaslavsky [15], extending a special case, first proven by Harary [9], characterizes switching equivalent classes of signed graphs.

Lemma 3. Two signed graphs (G, σ) and (G, σ') are switching equivalent ($\simeq$) if and only if $\sigma(C) = \sigma(C')$ for every cycle C of G .

Lemma 4. The following are equivalent:

- $\widehat{G}$ is balanced,
- $\widehat{G}$ is switching equivalent to $(G, +)$,
- the negative edges of $\widehat{G}$ form an edge-cut $(X, \overline{X})$ of G .

We then have a notion of balanced chromatic number for signed graphs.

Definition 5. The balanced chromatic number of a signed graph $\widehat{G}$, denoted $\chi_b(\widehat{G})$, is the minimum number of balanced sets needed to cover $V(\widehat{G})$.

This parameter was first introduced by Zaslavsky [16] under the name of *balanced partition number*. Zaslavsky has also introduced the notion of *0-free p -coloring of signed graphs*. That is, a coloring c of $V(G)$ with colors from the set $\{\pm 1, \pm 2, \dots, \pm p\}$ such that $c(x) \neq \sigma(xy)c(y)$ for each edge xy of (G, σ) . It can be easily verified that a balanced p -coloring of (G, σ) is equivalent to a 0-free p -coloring of $(G, -\sigma)$. This can also be viewed as a homomorphism to the signed graph on p vertices where there are both positive and negative edge between each pair of vertices and there is positive loop on each vertex. For more on homomorphisms of signed graphs we refer to [12].

The notion naturally extends to a family $\mathcal{G}$ of signed graphs by

$$\chi_b(\mathcal{G}) = \max_{\widehat{G} \in \mathcal{G}} \chi_b(\widehat{G}),$$

where $\chi_b(\mathcal{G}) = \infty$ if the maximum does not exit.

From here on, when we refer to a coloring of a graph, it will be a proper coloring. A coloring of a signed graph, on the other hand, will be a balanced coloring, which could be far from a proper coloring of the underlying graph. However, there is a tight relation between the balanced chromatic number of a signed graph and the chromatic number of its negative subgraph [16]:

Lemma 6. $\chi_b(\widehat{G}) \leq \chi(\widehat{G}^-) \leq 2\chi_b(\widehat{G})$

Proof. Let $c = \chi_b(\widehat{G})$ and consider a partition of $\widehat{G}$ into c balanced subgraphs: $(\widehat{G}_i)_{i \leq c}$. By Lemma 4, in each $\widehat{G}_i$, all negative edges form an edge cut. In other words, $\widehat{G}_i^-$ is 2-colorable. Thus, $\widehat{G}^-$ is $2c$ -colorable.

Conversely, any partition of G_σ^- into independent sets induces a partition of $\widehat{G}$ into balanced sets. Therefore, $\chi_b(\widehat{G}) \leq \chi(\widehat{G}^-)$. $\square$

This directly implies the following:

Corollary 7. A family $\mathcal{G}$ of signed graphs has bounded balanced chromatic number if and only if $\{\widehat{G}^- \mid \widehat{G} \in \mathcal{G}\}$ has bounded chromatic number.

Definition 8. A signed graph $\widehat{H}$ is said to be an induced subgraph of a signed graph $\widehat{G}$, denoted $\widehat{H} \subset_{ind} \widehat{G}$, if $\widehat{H}$ is isomorphic to a subgraph of $\widehat{G}$ obtained from $\widehat{G}$ by applying the following (commutative) operations:

- removal of a vertex (and all edges incident to it),
- switching at vertex.

For instance, in Figure 1 one can see that $(C_6, -) \subset_{ind} (Q_3, +)$.

A family $\mathcal{G}$ of signed graphs is said to be *hereditary* if it is closed under taking induced subgraphs. Given a family $\mathcal{F}$ of signed graphs, we denote by $\text{Forb}_{ind}(\mathcal{F})$ the class $\{\widehat{G} \mid \forall \widehat{H} \in \mathcal{F}, \widehat{H} \not\subset_{ind} \widehat{G}\}$. Note that $\text{Forb}_{ind}(\mathcal{F})$ is hereditary.

In this paper, we sometimes need to work with signed graphs where switching will not be considered. In such cases, considering negative edges as red and positive edges as blue, we rather refer to the signed graph in hand as a 2-edge-colored graph. We may also prefer to work with the underlying graph of a

signed graph in certain cases (where forbidding a subgraph is equivalent to forbidding all possible signatures). To capture the three notions together (graphs, 2-edge-colored graphs and signed graphs) we adopt the following notation. Given $\mathcal{F} = \{F_1, (F_2, \pi), \widehat{F}_3\}$, where F_1 is a graph, (F_2, π) a 2-edge-colored graph (with π the edge coloring function) and $\widehat{F}_3$ is a signed graph, the class $\text{Forb}_{\text{ind}}(\mathcal{F})$ is the class of signed graphs (G, σ) where no induced subgraph of G is isomorphic to F_1 , no induced subgraph of (G, σ) (as a 2-edge-colored graph) is isomorphic to (F_2, π) and no induced subgraph of (G, σ) is isomorphic to $\widehat{F}_3$ (where switching is permitted). In Figure 2 we have an example of (K_5, σ) which is in $\text{Forb}_{\text{ind}}\{P_3, (K_3, +), (\widehat{K_4}, -)\}$.

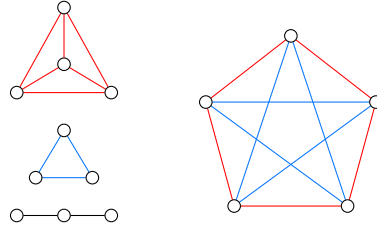


Figure 2: A graph in $\text{Forb}_{\text{ind}}\{P_3, (K_3, +), (\widehat{K_4}, -)\}$

Definition 9. A finite set $\mathcal{F}$ of signed graphs is a Gyárfás-Sumner set (or GS set) whenever $\chi_b(\text{Forb}_{\text{ind}}(\mathcal{F}))$ is bounded.

By Lemma 6, to determine if $\{\chi_b(\widehat{G}) \mid \widehat{G} \in \mathcal{G}\}$ is bounded it would be enough to consider the chromatic number of the family $\{\widehat{G}^- \mid \widehat{G} \in \mathcal{G}\}$ where $\widehat{G}^-$ is chosen for an arbitrary signature among all signatures equivalent to the signature in $\widehat{G}$.

The study can be compared to the notion of GS sets for dichromatic number of digraphs introduced in [2]. For more recent study on this subject, see [1] and references therein. For dichromatic number the vertices of a digraph are to be partitioned into sets none of which induces a directed cycle. While the two notions of dichromatic number and balanced chromatic number are quite similar, there are two essential differences between the two. The first is that being balanced is more restrictive for a color class in signed graphs than being acyclic is in digraphs. That is because among all orientations of a cycle only two are directed, while among all assignments of signs half are unbalanced. The second reason is that balanced coloring is arguably more suitable than even proper coloring (of graphs) to study the connection between minor theory and coloring. This is one of the main motivations for studying the balanced chromatic number. For further comments on this connection see [11].

2 Signed Gyárfás-Sumner sets

Given a graph G , the signed graph obtained from G by replacing each edge with a digon is denoted by $\widetilde{G}$. Independent sets in G correspond to balanced sets in $\widetilde{G}$. Hence, $\chi_b(\widetilde{G}) = \chi(G)$. Furthermore, observe that $\{\widetilde{G} \mid G \text{ is a graph}\} = \text{Forb}_{\text{ind}}\{(\widehat{K_2}, -)\}$. Thus, the Gyárfás-Sumner conjecture can be restated as follows.

Conjecture 10. For any forest F and any complete graph K_t the set $X = \{(\widehat{K_2}, -), \widetilde{K_t}, \widetilde{F}\}$ is a GS set.

In other words, the claim is that any minimal GS set containing $(\widehat{K_2}, -)$ is of order three except for the trivial cases: $\mathcal{F} = \{(\widehat{K_2}, -), \widetilde{K_2}\}$ and $\mathcal{F} = \{K_1\}$. Our question can be restated as finding minimal

GS sets including $\widetilde{K}_2$, but for simplicity we will not repeat this element, and only consider simple graphs. Toward characterizing such sets, the following is a key observation.

Observation 11. *If $\mathcal{G}$ is a hereditary class of signed graphs such that $\chi_b(\mathcal{G})$ is unbounded, then any GS set must contain a signed graph that is switching equivalent to a member of $\mathcal{G}$.*

Basic families of hereditary signed simple graphs with unbounded balanced chromatic number are the followings.

- $\mathcal{K} = \{(K_i, -) \mid i \geq 1\}$.
- $\mathcal{G}_k = \{(G, -) \mid G \text{ has girth at least } k\}$.
- $PC(\mathcal{G}_k) = \{(K_{|V(G)|}, E(G)) \mid G \text{ has girth at least } k\}$.

The first two classes are cliques and graphs of high girth. The third class is obtained from the second by replacing all non-edges with positive edges. We call this operation the *positive completion* (denoted $PC(\bullet)$). $PC(C_5)$ is drawn in Figure 2

Lemma 12. *The classes of signed graphs $\mathcal{K}$, $\mathcal{G}_k$ and $PC(\mathcal{G}_k)$ have unbounded balanced chromatic number.*

Proof. In $\mathcal{K}$, maximal balanced sets are of size 2, therefore $\chi_b(K_i, -) \geq \lceil \frac{i}{2} \rceil$ and the class has unbounded balanced chromatic number.

The class $\mathcal{G}_k$ and $PC(\mathcal{G}_k)$ have the same negative edges : graphs of girth at least k , which have unbounded chromatic number from Theorem 1. Hence, by Lemma 6, those two classes also have unbounded balanced chromatic number. $\square$

An immediate corollary is the following :

Corollary 13. *If $\mathcal{F}$ is a GS set, then there exists an integer n and two forests F_1, F_2 , such that*

$$\left\{ (\widehat{K_t}, -), PC(F_1), F_2 \right\} \subset \mathcal{F}$$

In the case that $t = 2$ or one of F_1 or F_2 has two vertices, the family $\mathcal{F}$ consists of empty graphs. Otherwise, if $t \in \{3, 4\}$, then $(\widehat{K_t}, -)$ and $PC(F_1)$ could be switch equivalent, in which case we can have a GS set of order 2. This work is then a first step toward characterizing GS sets of order 2.

Our contribution is depicted in Table 1, where a linear forest is a forest whose components are paths.

	$K_n = K_3$	$K_n = K_4$
F contains a vertex of degree 3	UNBOUNDED	
F is a linear forest and $P_5 \not\subseteq F$	BOUNDED	BOUNDED
F is a linear forest and $P_5 \subseteq F$		UNKNOWN

Table 1: Results on $\chi_b \left(\text{Forb}_{ind} \left\{ (\widehat{K_n}, -), F \right\} \right)$

3 Sequences of unbounded balanced chromatic number

In this section we present some sequences of graphs with unbounded balanced chromatic number excluding some particular red-blue induced subgraph.

Our first construction is based on the notion of shift graphs studied in [5]. The second construction is based on the notions of line graphs and arc graphs. Ultimately, the line graph construction will be stronger than the shift graph one, but since the shift graph construction is self contained and more explicit, we believe that both of them might be interesting to the reader.

3.1 Signed shift graphs

Given positive integers k and n , $k \leq n$, the shift graph $S_{k,n}$ has as its vertices all increasing sequences $(s_1, s_2, \dots, s_k)$, $1 \leq s_1 \leq s_2 \leq \dots \leq s_k \leq n$, where two sequences are adjacent if they are of the form $(s_1, s_2, \dots, s_k)$ and $(s_2, s_3, \dots, s_{k+1})$.

Given an integer k , the family of shift graphs $\{S_{k,n} \mid n \in \mathbb{N}\}$, is an example of a family of triangle-free graphs of unbounded chromatic number. For the sake of completeness, we present a proof of this fact first proved in [6].

Lemma 14. *Given any positive integers k and c , there exists an integer n such that $\chi(S_{k,n}) > c$.*

Proof. We prove this by induction on k . For $k = 1$, $S_{1,n} = K_n$ and the claim is immediate. The claim follows if we show that $\chi(S_{k,n}) \leq 2^{\chi(S_{k+1,n+1})}$.

Let ϕ be a c -coloring of $S_{k+1,n+1}$. For each vertex $s = (s_1, s_2, \dots, s_k)$ of $S_{k,n}$ let $\psi(s)$ be the set of all colors assigned to the vertices $(s_1, s_2, \dots, s_k, t)$ of $S_{k+1,n+1}$ such that $s_k < t \leq n+1$. We claim that ψ is a (proper) coloring of $S_{k,n}$. That is because given a pair $u' = (u_1, u_2, \dots, u_k)$ and $u'' = (u_2, u_3, \dots, u_k, u_{k+1})$ of adjacent vertices in $S_{k,n}$, we consider the vertex $u = (u_1, u_2, \dots, u_k, u_{k+1})$ of $S_{k+1,n+1}$ and observe that $\phi(u) \in \psi(u')$, but $\phi(u) \notin \psi(u'')$ because every extension of u'' is adjacent to u . Thus $\psi(u') \neq \psi(u'')$. $\square$

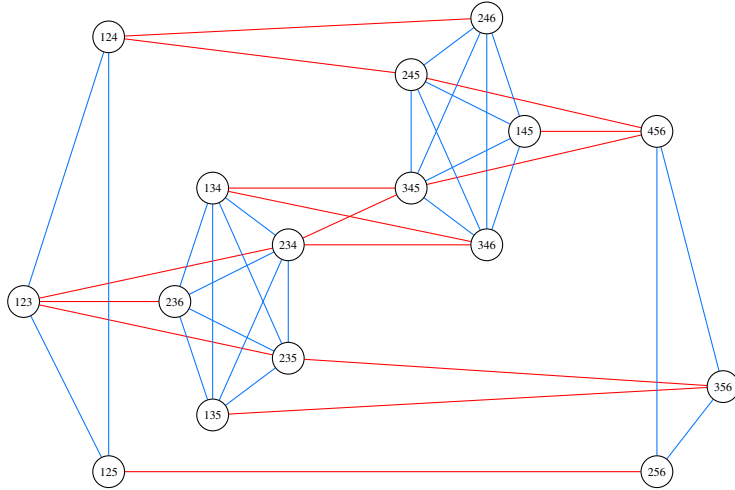


Figure 3: The biggest connected component of the signed shift graph $\widehat{S}_{3,6}$

We now consider the signed graph $\widehat{S}_{3,n}$ built on $S_{3,n}$ as follows. The negative edges of $\widehat{S}_{3,n}$ are the edges of $S_{3,n}$. Each pair of vertices, (a, b, c) and (a', b, c') , is connected by a positive edge. The main

connected component of $\widehat{S}_{3,6}$ is depicted in Figure 3. Using the signed graph $\widehat{S}_{3,n}$ we can prove the following:

Theorem 15. *The set $\{(\widehat{K_3}, -), K_{1,4}\}$ is not a GS set.*

Proof. By Corollary 7 and Lemma 14, the class of signed graphs $\widehat{S}_{3,n}$ has unbounded balanced chromatic number. It remains to prove that each signed graph $\widehat{S}_{3,n}$ is in $\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -), K_{1,4}\}$. Observe that the set of positive edges induces a disjoint union of cliques (each corresponding to a middle value of the triplets). Furthermore, the negative neighbors of a vertex (a, b, c) are partitioned into two types: $(\bullet, a, b)$ and $(b, c, \bullet)$, each of which induces a clique where all the edges are positive. Thus, there can only be a maximum of three independent neighbors of a vertex (a, b, c) : a positive neighbor $(\bullet, b, \bullet)$, two negative neighbors $(\bullet, a, b)$ and $(b, c, \bullet)$. Furthermore, since $S_{3,n}$ is triangle-free and since positive edges induce unions of cliques, $\widehat{S}_{3,n}$ has no negative triangle. $\square$

3.2 Signed line graphs

The line graph of a graph G , denoted $L(G)$, has as vertices the edges of G , where edges with common end point in G are adjacent in $L(G)$. Various characterizations of line graphs are given in the literature, among which is Beineke's characterization [3]: a graph H is isomorphic to the line graph of a simple graph G if and only if it has no induced subgraph isomorphic to one of the nine graphs known as Beineke's. The first of those is the claw $(K_{1,3})$, which means, in particular, that line graphs are claw free.

Given an orientation D of G , the arc graph of D , denoted $A(D)$, is the graph whose vertices are arcs of D and where the arcs uv and vw are adjacent. A relation between the chromatic number of $A(D)$ and the chromatic number of the graph G was given in [10, 13].

Theorem 16. *Given a graph G and an orientation D of it, we have*

$$\min \{k \mid \chi(G) \leq 2^k\} \leq \chi(A(D)) \leq \min \left\{ k \mid \chi(G) \leq \binom{k}{\lfloor \frac{k}{2} \rfloor} \right\}.$$

Definition 17. *Given a graph G and an orientation D of it, we define the signed line graph of D , denoted $(L(G), \sigma_D)$ to be a signed graph on $L(G)$ whose negative edges are the edges of $A(D)$.*

Observe that a triangle in $L(G)$ corresponds to one of two possibilities: either a K_3 in G , or a $K_{1,3}$ in G . Any triangle in $L(G)$ which corresponds to a $K_{1,3}$ in G is of positive sign in $(L(G), \sigma_D)$. That is because either the three edges of $K_{1,3}$ are all oriented the same way, in which case they induce three positive edges in $(L(G), \sigma_D)$, or exactly two of them are in the same direction, in which case we have a triangle with exactly two negative edges in $(L(G), \sigma_D)$.

If the graph G is selected to be triangle-free, then all of the triangles of $(L(G), \sigma_D)$ are of the second type and. Hence, every triangle is positive.

By taking G to be a triangle-free graph of arbitrarily large chromatic number (for example using Theorem 1), then applying Theorem 16, we conclude that the negative edges of $(L(G), \sigma_D)$ induce a graph of high chromatic number. Hence, from Lemma 6, we conclude that the balanced chromatic number of $(L(G), \sigma_D)$ can be arbitrarily large.

Noting that line graphs are in particular claw free, we conclude that:

Theorem 18. *The set $\{(\widehat{K_3}, -), K_{1,3}\}$ is not a GS set.*

4 Negative triangle and linear forest

So far we have observed that, for a set $\mathcal{F} = \{\widehat{F_1}, \widehat{F_2}\}$ to be a GS set, one of $\widehat{F_1}, \widehat{F_2}$, say $\widehat{F_1}$, must be switching equivalent to either $(K_3, -)$ or $(K_4, -)$ and the other must be a forest. If the forest F_2 has a vertex of degree 3 or more, then $\text{Forb}_{\text{ind}}(\mathcal{F})$ contains $\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -), K_{1,3}\}$ and, therefore, its balanced chromatic number is not bounded. Thus, for $\mathcal{F}$ to be a GS set of order two, besides the fact that the underlying graph of F_1 must be K_3 or K_4 , the second, F_2 , must be a linear forest: that is a forest where each component is a path. In this section, then, we show that for each linear forest F , $\mathcal{F} = \{(\widehat{K_3}, -), F\}$ is a GS set. To that end, we first show that it is enough to only consider the cases when F is a path, i.e., a connected linear forest. The disjoint union of two graphs G and H is denoted $G + H$.

Proposition 19. *If $\{(\widehat{K_3}, -), F_1\}$ and $\{(\widehat{K_3}, -), F_2\}$ are GS sets, then $\{(\widehat{K_3}, -), F_1 + F_2\}$ is also a GS set.*

Proof. Suppose $\chi_b(\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -), F_1\}) \leq s$ and $\chi_b(\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -), F_2\}) \leq t$. We now consider a signed graph $\widehat{G} \in \text{Forb}_{\text{ind}}\{(\widehat{K_3}, -), F_1 + F_2\}$ and claim that $\chi_b(\widehat{G}) \leq \max\{s, |F_1| + t\}$. Assume to the contrary, that $\chi_b(\widehat{G}) \geq \max\{s + 1, |F_1| + t + 1\}$.

Observe that being a $(\widehat{K_3}, -)$ -free signed graph is the same as the (closed) neighborhood of each vertex inducing a balanced signed graph. That is because given a vertex v , after switching at some neighbors if needed, we may have only positive edges incident to v . Then there is no negative triangle incident with v if and only if there is no negative edge induced by its neighbors.

As $\chi_b(\widehat{G}) > s$, and since there is no $(\widehat{K_3}, -)$, there must be an induced copy F' of F_1 . Consider the subgraph $\widehat{G}_1$ induced by F' and all its neighbors. We claim that $\chi_b(\widehat{G}_1) \leq |F_1|$. Indeed, $V(\widehat{G}_1) = \bigcup_{u \in F'} N[u]$, and each of these $|F_1|$ sets are balanced.

As $\chi_b(\widehat{G}) \geq |F_1| + t + 1$, we have $\chi_b(\widehat{G} - \widehat{G}_1) \geq t + 1$. As $\widehat{G} - \widehat{G}_1$ still has no $(\widehat{K_3}, -)$, it must have an induced subgraph F'' isomorphic to F_2 . See Figure 4 for an illustration. Then, F' and F'' together induce an isomorphic copy of $F_1 + F_2$, contradicting the choice of $\widehat{G}$. $\square$

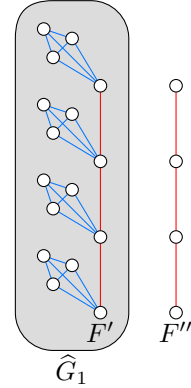


Figure 4: Illustration of the proof, with $F_1 = F_2 = P_4$

Theorem 20. *For every positive integer k , the set $\{(\widehat{K_3}, -), P_k\}$ is a GS set.*

The following proof is inspired from the classic argument from Gyárfás [8].

Proof. We prove, by induction on k , that $\chi_b(\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -), P_{k+2}\}) < 2^k$. In fact, we prove a stronger claim: given a connected signed graph $\widehat{G}$ in $\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -)\}$ whose balanced chromatic number is at least 2^k , for each vertex u of G there is an induced path of length $k + 1$ (i.e. P_{k+2}) starting at u .

For $k = 1$, if there is no path of length 2 starting at the vertex u , noting that G is connected, we conclude that $\widehat{G}$ is a switching of $(K_i, +)$ for some i . Thus, $\chi_b(\widehat{G}) = 1$.

Suppose that the claim holds for every value up to $k - 1$ and consider the statement for k . Let $\widehat{G}$ be a connected signed graph in $\text{Forb}_{\text{ind}}\{(\widehat{K_3}, -)\}$ which has balanced chromatic number at least 2^k and let

u be a vertex of $\widehat{G}$. Let $\widehat{G}_d$ be the subgraph of $\widehat{G}$ induced by the vertices at distance d from u . Assume, toward a contradiction, that there is no induced path of length $k + 1$ starting from u . That is to say: $\{u\}, \widehat{G}_1, \widehat{G}_2, \dots, \widehat{G}_k$ cover all of the vertices of $\widehat{G}$.

We claim that, for some i , we have $\chi_b(\widehat{G}_i) \geq 2^{k-i} + 1$. Observe that u and $\widehat{G}_1$ together induce a balanced set. If each $\widehat{G}_i$ is 2^{k-1} -colorable for $i = 2, \dots, k$, then $\chi_b(\widehat{G}) \leq 1 + 2^{k-1} + 2^{k-2} \dots + 2 = 2^k - 1$, contradicting the assumption that $\chi_b(\widehat{G}) \geq 2^k$. Thus, we assume that $\chi_b(\widehat{G}_i) \geq 2^{k-i} + 1$ for a fixed i , with $2 \leq i \leq k$.

In $\widehat{G}_i$, one of the connected components, say $\widehat{G}_i^1$ has balanced chromatic number at least $2^{k-i} + 1$. Let u_{i-1} be a vertex in $\widehat{G}_{i-1}$ that connects $\widehat{G}_i^1$ to u . Since the neighborhood of each vertex has balanced chromatic number 1. The signed graph $\widehat{G}_i^1 - N(u_{i-1})$ has balanced chromatic number at least 2^{k-i} . Thus, one of its connected components, say $\widehat{C}$, has balanced chromatic number at least 2^{k-i} . Let u_i be a vertex in $N(u_{i-1})$ which has a neighbor in $\widehat{C}$. The subgraph $\widehat{C}'$ induced by $\widehat{C}$ and u_i is connected, belongs to $\text{Forb}_{ind} \{(\widehat{K_3}, -)\}$, and has balanced chromatic number at least 2^{k-i} . Thus, by the inductive assumption, we have an induced path P of length $k - i + 1$ in $\widehat{C}'$ starting at u_i . By the choice of $\widehat{C}'$, u_i is the only neighbor of u_{i-1} in P . Hence, extending P to u_{i-1} and then, through a shortest path, to u , we have an induced path of length $k + 1$ with u as its starting point. $\square$

Corollary 21. *If F is linear forest consisting of l paths each of length at most k , then*

$$\chi_b \left(\text{Forb}_{ind} \left\{ (\widehat{K_3}, -), F \right\} \right) < 2^k + (l - 1)k.$$

5 $(\widehat{K_4}, -)$ and linear forest

We have already seen that, for $\{(\widehat{K_4}, -), F\}$ to be a GS set, F must be a linear forest. We conjecture that this necessary condition is also sufficient.

Conjecture 22. *For any linear forest F , the set $\{(\widehat{K_4}, -), F\}$ is a GS set.*

As an approach to this conjecture, we first show that it would be enough to prove the conjecture for when F is just a path. Then we verify it for $F = P_4$, concluding the conjecture for when each component of F is a path of length at most 3. The first step is an extension of Theorem 20 for signed graphs with no induced $(\widehat{K_4}, -)$. Observe that a key element in the proof of Theorem 20 is the fact that the (closed) neighborhood of each vertex is balanced, that is a consequence of the assumption that there is no induced $(\widehat{K_3}, -)$. When $(\widehat{K_4}, -)$ is forbidden instead, we do not know if the neighborhood of a vertex has bounded balanced chromatic number (see Conjecture 24). Thus, in order to extend Theorem 20, we add this as an assumption.

Theorem 23. *Given two positive integers b and k , with $k \geq 3$, let $\widehat{G} \in \text{Forb}_{ind} \{(\widehat{K_4}, -), P_k\}$ be a signed graph with the property that the closed neighborhood of each vertex admits a balanced b -coloring. Then $\chi_b(\widehat{G}) \leq b2^{k-3}$.*

Proof. We prove the claim by induction on k for the family of all such graphs. More precisely we claim the following. Let $\widehat{G}$ be a connected signed graph with no induced $(\widehat{K_4}, -)$ whose balanced chromatic number is larger than $b2^{k-3}$, but with the property that the closed neighborhood of each vertex admits a balanced b -coloring. Then for each vertex u of $\widehat{G}$ there is an induced path of length at least $k - 1$ starting at u .

For $k = 3$, if there is no P_3 starting at u , then $N[u]$ spans the whole graph and, therefore, by assumption, $\chi_b(\widehat{G}) \leq b$. Given $k \geq 4$, assume that the claim holds for every value up to $k - 1$ and consider a signed graph $\widehat{G}$ satisfying the condition for k . Suppose that there is a vertex u of $\widehat{G}$ not satisfying the conclusion. That is to say: $\{u\}, \widehat{G}_1, \widehat{G}_2, \dots, \widehat{G}_{k-3}$ cover all of the vertices of $\widehat{G}$, where $\widehat{G}_i$ is the subgraph of $\widehat{G}$ induced by the vertices at distance i from u . Since $\chi_b(\widehat{G}) > b2^{k-3}$, one of these subgraphs, say $\widehat{G}_i$, has balanced chromatic number larger than $b2^{k-i-3} + b$. Then, as in the proof of Theorem 20, by taking a suitable component of $\widehat{G}_i$, selecting a vertex u_i with a neighbor u_{i-1} in $\widehat{G}_{i-1}$, deleting all but one of the neighbors of u_{i-1} in $\widehat{G}_i$, then applying induction, we get an induced path of length $k - i - 1$ in $\widehat{G}_i$ with only one connection to u_{i-1} . This connection then can be extended with a shortest path connecting u_{i-1} to u to produce an induced path of length $k - 1$. $\square$

In light of this development, the following relaxation of Conjecture 22 seems to be essential for resolving the conjecture.

Conjecture 24. *For any positive integer k , there exists a positive integer b_k such that for any signed graph $\widehat{G} \in \text{Forb}_{\text{ind}} \{(\widehat{K_4}, -), P_k\}$ and any vertex u of it, the closed neighborhood of u has balanced chromatic number at most b_k .*

In fact the two conjectures are equivalent: If Conjecture 22 holds, then we can take the bound on the balanced chromatic number of $\text{Forb}_{\text{ind}} \{(\widehat{K_4}, -), P_k\}$ to be b_k . Conversely, if Conjecture 24 holds, then we apply Theorem 23 to get a bound on the balanced chromatic number of $\text{Forb}_{\text{ind}} \{(\widehat{K_4}, -), P_k\}$.

Therefore, we aim at understanding better the conditions on the neighborhood. To that end, considering a vertex u of $\widehat{G} \in \text{Forb}_{\text{ind}} \{(\widehat{K_4}, -), P_k\}$, we first switch at each neighbor of u which is adjacent to it by a negative edge. That renders us with a vertex u all whose neighbors are connected to it by positive edges. The fact that $\widehat{G} \in \text{Forb}_{\text{ind}} \{(\widehat{K_4}, -), P_k\}$ has two implications. The first is that the open neighborhood $\widehat{G}_u$ of the vertex u is P_k -free, that is because it is an induced subgraph of $\widehat{G}$. The second is that $\widehat{G}_u$ has neither $(K_3, -)$ nor (K_4, M) as induced subgraph. Here (K_4, M) is the signed graph on K_4 with a matching of size two being the negative edges (see Figure 5). We note that these (induced) subgraphs are forbidden only with the given signature. The first is forbidden, as otherwise, together with u , it would form a subgraph switching equivalent to $(\widehat{K_4}, -)$. The second is itself switching equivalent to $(\widehat{K_4}, -)$. Moreover, this is the only switching equivalent copy of $(K_4, -)$ which has no $(K_3, -)$ subgraph. Overall Conjecture 24 can be reformulated as follows.

Conjecture 25. *For any positive integer k , there exists a positive integer b_k such that any signed graph in $\text{Forb}_{\text{ind}} \{(K_3, -), (K_4, M), P_k\}$ has balanced chromatic number at most b_k .*

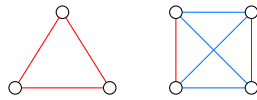


Figure 5: $(K_3, -)$ and (K_4, M)

Proposition 26. *For any given integer k , Conjecture 24 and Conjecture 25 are equivalent.*

Proof. Given a signed graph (G, σ) , let $(G, \sigma)^*$ be the signed graph obtained from (G, σ) by adding a universal vertex which is adjacent to all of the vertices with positive edges. The claim of the proposition follows from the fact that $(\widehat{G, \sigma})^* \in \text{Forb}_{\text{ind}} \{(\widehat{K_4}, -), P_k\}$ if and only if $(G, \sigma) \in \text{Forb}_{\text{ind}} \{(K_3, -), (K_4, M), P_k\}$

and noting that the values of b_k in Conjecture 24 will be at most one more than its value in Conjecture 25. $\square$

Given vertex-disjoint graphs G_1 and G_2 , their *full join*, denoted $G_1 \bowtie G_2$, is a graph on $V(G_1) \cup V(G_2)$ where each of the two sets induces the corresponding graph and each pair of vertices $x \in V(G_1)$ and $y \in V(G_2)$ is adjacent. The notion of full join is crucial in the study of P_4 -free graphs.

Lemma 27. *Assume that $(G_1 \bowtie G_2, \sigma) \in \text{Forb}_{\text{ind}} \{(K_3, -), (K_4, M), P_k\}$. If one of $\widehat{G}_1$ or $\widehat{G}_2$ contains a negative edge, then the subgraph of the other induced by its negative edges has chromatic number at most 3.*

Proof. Assume that uv is a negative edge in $\widehat{G}_1$. Consider the partition of $V(\widehat{G}_2)$ into:

$$N_{\widehat{G}_2}^-(u), N_{\widehat{G}_2}^-(v), N_{\widehat{G}_2}^+(u) \cap N_{\widehat{G}_2}^+(v).$$

Since $(G_1 \bowtie G_2, \sigma)$ has no induced $(K_3, -)$, neither of $N_{\widehat{G}_2}^-(u)$ and $N_{\widehat{G}_2}^-(v)$ induces a negative edge. As $(G_1 \bowtie G_2, \sigma)$ does not induce any (K_4, M) , there is no negative edge in $N_{\widehat{G}_2}^+(u) \cap N_{\widehat{G}_2}^+(v)$. Thus, we have a 3-coloring of $\widehat{G}_2^-$ as claimed. See Figure 6 for an illustration. $\square$

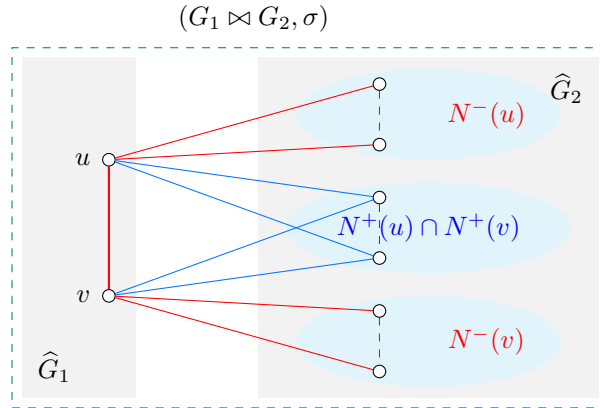


Figure 6: Illustration of the proof of Lemma 27

Corollary 28. *If $(G_1 \bowtie G_2, \sigma) \in \text{Forb}_{\text{ind}} \{(K_3, -), (K_4, M), P_k\}$ and each of $\widehat{G}_1$ and $\widehat{G}_2$ contains a negative edge, then the negative edges induce a 6-colorable graph, and hence $\chi_b(G_1 \bowtie G_2, \sigma) \leq 6$.*

Proof. We have that $\chi((G_1 \bowtie G_2, \sigma)^-) \leq \chi(\widehat{G}_1^-) + \chi(\widehat{G}_2^-)$. By lemma 27, the graph induced by the negative edges on each side has chromatic number at most 3. $\square$

The following lemma generalises the result of Proposition 19 in the perspective of using it with $(K_4, -)$. To better understand its statement, one can think of $\mathcal{G} = \text{Forb}_{\text{ind}} \{(K_3, -), (K_4, M)\}$.

Lemma 29. *Let $\mathcal{G}$ be a hereditary class of signed graphs, F_1 and F_2 two forests and c a positive integer such that: Signed graphs in $\mathcal{G}$ with no induced F_1 (respectively F_2) have balanced chromatic number at most c . Furthermore, assume that for each signed graph $\widehat{G} \in \mathcal{G}$ and for each induced copy F_1' of F_1 in $\widehat{G}$, the common neighbors of F_1' (with arbitrary signature) induce a c -colorable signed graph. Then for some integer $\phi(c)$, any signed graph in $\mathcal{G}$ with no induced copy of $F_1 + F_2$ is $\phi(c)$ -colorable.*

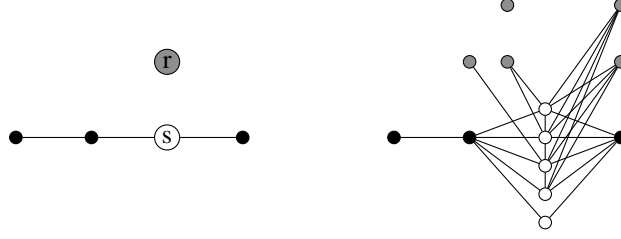


Figure 7: Illustration of the proof of Lemma 29 with $F_1 = F_2 = P_4$

On the left: F_1^* , On the right: $\widehat{G}$ with $Ext_{T \rightarrow S}(T')$ and $Ext_{T \rightarrow R}(T')$ highlighted.
 T (resp T') is in black. $S = T \cup \{s\}$, $R = T \cup \{r\}$

Proof. Throughout the proof $\widehat{G}$ is an element of $\mathcal{G} \cap \text{Forb}_{ind}\{F_1 + F_2\}$. Given a subset X of $V(G)$, the subgraph of $\widehat{G}$ induced by X is denoted by $\widehat{G}[X]$ (which must also be a member of $\mathcal{G} \cap \text{Forb}_{ind}\{F_1 + F_2\}$). Note that, in this proof, the graphs we consider are signed. However, since we mostly deal with forests, the signature will not play a big role. We introduce several notations before proceeding with the proof of the lemma.

- F_1^* is the forest $F_1 + K_1$ with the added isolated vertex named h_* .
- A T -piece in $\widehat{G}$ is a pair (T', f) where T' is an induced subgraph of $\widehat{G}$ isomorphic to T and f is an isomorphism mapping T to T' .
- For an induced subforest T of F_1 , a 1-extension of T (in F_1) is an induced subgraph S of F_1 , such that $V(S) = V(T) \cup \{s\}$ with $s \notin V(T)$.
- Given a T -piece (T', f) of $\widehat{G}$ and a 1-extension S of T in F_1 , with $V(S) = V(T) \cup \{s\}$, the set $Ext_{T \rightarrow S}(T', f)$, consists of the vertices s' of $\widehat{G}$ such that $(V(T') \cup \{s'\}, f')$ is an S -piece, where f' is the extension of f mapping s to s' . When it is obvious from the context, we write $Ext_{T \rightarrow S}(T')$. Formally,

$$Ext_{T \rightarrow S}(T', f) = \bigcap_{\substack{u \in T \\ u \sim s}} N(f(u)) \cap \bigcap_{\substack{v \in T \\ u \not\sim s}} \overline{N(f(v))}. \quad (*)$$

See Figure 7 for an illustration showcasing two different extenders.

We prove the following statement by induction:

Claim: For any T -piece (T', f) of F_1 , there exists a 1-extension S of T in F_1^* , such that $\chi_b(\widehat{G}[Ext_{T \rightarrow S}(T')]) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |T|)}$.

The base of induction will be $T = F_1$, then assuming it is true for T -pieces on i vertices, we prove that the statement is true for T -pieces on $i - 1$ vertices. At the end of the inductive process we conclude with T being the empty piece. In this case the set S has a single vertex and the set of S -extenders is all the vertices of $\widehat{G}$. This will give the desired result.

For the base of induction, assume $T = F_1$ and fix an F_1 -piece (T', f) . The only extension of F_1 inside F_1^* is F_1^* itself. The F_1^* -extenders of (T', f) is the set of vertices disjoint from F_1' . Since $\widehat{G}$ doesn't contain an induced copy of $F_1 + F_2$, this set of extenders doesn't contain an induced copy of F_2 and, by the assumption on F_2 , it has balanced chromatic number bounded by $c = c(|F_1|(|F_1| + 1))^{2(|F_1| - |F_1|)}$.

For the inductive part, assume the claim holds for every subforest of F_1 on at least i vertices. Consider an induced subforest T of F_1 on $i - 1$ vertices and let (T', f) be a T -piece in $\widehat{G}$. Consider a 1-extension

S of T inside F_1^* with $V(S) = V(T) \cup \{s\}$ and $s \neq h_*$, noting that we have a choice for s because T has less vertices than F_1 . If $\chi_b(\widehat{G}[Ext_{T \rightarrow S}(T')]) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |T|)}$, then we are done. Hence we assume that $\chi_b(\widehat{G}[Ext_{T \rightarrow S}(T')]) > c(|F_1|(|F_1| + 1))^{2(|F_1| - |T|)}$.

For any $s' \in Ext_{T \rightarrow S}(T')$, we have an S -piece $(S_{s'}, g)$ for which the claim applies (by the induction hypothesis). We get that there exists an $r_{s'}$ such that the extension $U = F_1^*[V(S) \cup \{r_{s'}\}] = F_1^*[V(T) \cup \{s, r_{s'}\}]$ verifies $\chi_b(\widehat{G}[Ext_{S \rightarrow U}(S')]) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)}$.

Note that there are less than $|F_1|$ 1-extensions of S in F_1^* . Hence, there is an r such that

$$\chi_b(\widehat{G}[a \in Ext_{T \rightarrow S}(T'), r_a = r]) > c(|F_1|(|F_1| + 1))^{2(|F_1| - |T|)} / |F_1|. \quad (**)$$

Call this subgraph $\widehat{A}$, fix such an r , and let $R = F_1^*[V(T) \cup \{r\}]$. Depending on the adjacency between s and r , we consider two cases.

- **Case $s \sim r$.** Since $\chi(\widehat{A}) > c$, we can find a copy F'_1 of F_1 inside $\widehat{A}$. Now consider the R -extenders of T in $\widehat{G}$. $Ext_{T \rightarrow R}(f)$ can be partitioned into:

- The vertices disjoint from F'_1 : This set has chromatic number bounded by c because it is F_2 -free (otherwise it would create a copy of $F_1 + F_2$ inside $\widehat{G}$).
- The vertices adjacent to a given vertex $a \in V(F'_1)$: This set correspond exactly to $Ext_{S \rightarrow U}(S', f')$.

Where $V(S') = V(T) \cup \{a\}$ and $f'(u) = \begin{cases} f(u) & \text{if } u \in V(T) \\ a & \text{if } u = s. \end{cases}$ Indeed, using formula (*),

we have $Ext_{S \rightarrow U}(S', f') = Ext_{T \rightarrow S}(T', f) \cap N(f'(s)) = Ext_{T \rightarrow S}(T', f) \cap N(a)$.

By definition of $r_a = r$ we get that $\chi(\widehat{G}[Ext_{S \rightarrow U}(S')]) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)}$.

We have successfully partitioned $\widehat{A}$ into $(|F_1| + 1)$ parts each with balanced chromatic number at most $c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)}$. Therefore, $\chi_b(\widehat{A}) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)} \times (|F_1| + 1) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |T|)} / |F_1|$, contradicting (**).

- **Case $s \not\sim r$.** The reasoning is very similar, swapping adjacencies for non-adjacencies.

Since $\chi_b(\widehat{A}) > c$, we can find a copy F'_1 of F_1 inside $\widehat{A}$. Now consider the R -extenders of T in $\widehat{G}$. $Ext_{T \rightarrow R}(T')$ can be partitioned into:

- The vertices joined to all of F'_1 : This set has chromatic number bounded by c by the statement of the lemma.
- The vertices non-adjacent to a given vertex $a \in V(F'_1)$: This set corresponds exactly to $Ext_{S \rightarrow U}(S', f')$. Where $V(S') = V(T) \cup \{a\}$ and $V(S') = V(T') \cup \{a\}$,

and $f'(u) = \begin{cases} f(u) & \text{if } u \in V(T) \\ a & \text{if } u = s. \end{cases}$ By definition of $r_a = r$, we get that this set has balanced chromatic number at most $c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)}$.

We have successfully partitioned $\widehat{A}$ into $(|F_1| + 1)$ parts each with balanced chromatic number at most $c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)}$. Therefore, $\chi_b(\widehat{A}) \leq c(|F_1|(|F_1| + 1))^{2(|F_1| - |S|)} \times (|F_1| + 1) = c(|F_1|(|F_1| + 1))^{2(|F_1| - |T|)} / |F_1|$ which contradicts (**).

See Figure 7 for an illustration of the proof. □

Theorem 30. *The class $\text{Forb}_{ind} \{(K_3, -), (K_4, M), P_4\}$ has balanced chromatic number at most 6.*

Proof. In fact, we will prove a stronger statement that, for each element $\widehat{G}$ of $\text{Forb}_{ind} \{(K_3, -), (K_4, M), P_4\}$, the negative edges of $\widehat{G}$ induce a 6-colorable graph.

Consider a signed graph $\widehat{G}$ in $\text{Forb}_{ind} \{(K_3, -), (K_4, M), P_4\}$. As all signatures on P_4 are switching equivalent, the underlying graph G of $\widehat{G}$ has no induced P_4 , in other words G is a cograph. A classic characterization of P_4 -free graphs is that any P_4 -free graph G , except K_1 , has nontrivial disjoint subgraphs G_1 and G_2 such that either $G = G_1 \bowtie G_2$ or $G = G_1 + G_2$.

We may assume that G is connected, as otherwise we may work with each component separately. Hence, there is partition of $V(G) = A \cup B$ such that $G = G[A] \bowtie G[B]$. If each of $\widehat{G}[A]_-$ and $\widehat{G}[B]_-$ is a 3-colorable subgraph of $\widehat{G}_-$, then $\widehat{G}_-$ is 6-colorable and we are done. Thus, we assume one of them, say $\widehat{G}[B]_-$, is not 3-colorable. Then, by Lemma 27, $\widehat{G}[A]$ has no negative edge.

Let A' be a maximal set of vertices containing A satisfying the following conditions:

1. It induces no negative edge,
2. Each connected component of $G \setminus A'$ is a module, that is to say: each vertex outside a component X is either adjacent to all of the vertices of X or to none of them.

Let $\widehat{H}_1, \widehat{H}_2, \dots, \widehat{H}_\ell$ be the connected components of $\widehat{G} \setminus A'$. Our goal is to present a 6-coloring of $\widehat{G}_-$ where all of the vertices in A' are colored with the same color. To achieve this goal, an $\widehat{H}_i$ satisfying $\chi(\widehat{H}_i) \leq 5$ poses no problem. Thus we assume $\chi(\widehat{H}_i) \geq 6$ for each i . Since each H_i is a connected P_4 -free graph, it is a full join of two subgraphs, say H'_i and H''_i . We first claim that each of $\widehat{H}'_{i-}$ and $\widehat{H}''_{i-}$ is 3-chromatic.

To that end, it is enough to show that each of them is 3-colorable. By symmetry, suppose $\widehat{H}'_i$ is not 3-colorable. Let A'_i be the subset of vertices in A' which are adjacent to every vertex in H_i . Applying Lemma 27 to the subgraph induced by H_i and A' , we conclude that the set $A'_i \cup V(H''_i)$ induces no negative edge. We now consider $A'' = A' \cup V(H''_i)$. As we observed, A'' induces no negative edge. The components of $G \setminus A''$ are either $\widehat{H}_j$, $j \neq i$, or the components of H'_i . Thus each of them is a module in G . But this contradicts the maximality of A' , which proves our claim.

Next, we claim that there is no negative edge connecting A'_i to an $\widehat{H}_i$. Consider an arbitrary vertex x of A' . Since G induces no $(K_3, -)$, the negative neighborhood of x in $\widehat{H}_i$, denoted $N_i^-(x)$, is an independent set of $\widehat{H}_{i-}$. Thus $\widehat{H}_{i-} \setminus N_i^-(x)$ is of chromatic number 5 or 6. In either case at least one of the two subgraphs, $\widehat{H}'_{i-} \setminus N_i^-(x)$ or $\widehat{H}''_{i-} \setminus N_i^-(x)$, is of chromatic number 3, that is to say it induces an odd cycle. Suppose $\widehat{H}'_{i-} \setminus N_i^-(x)$ induces an odd cycle C and let v be a vertex in $V(\widehat{H}''_{i-}) \cap N_i^-(x)$ (assuming it exists). Since there is no $(K_3, -)$, $N_i^-(v) \cap C$ is an independent set and since C is odd, there is an edge uv of C which has no negative connection to v . Then those edges must be positive because v is fully joined to C . But then the vertices x, v, u , and w induce a (K_4, M) , we have a contradiction, which is shown in Figure 8. This implies that x has no negative edge connecting it to $\widehat{H}''_{i-}$, but then noting that $\chi(\widehat{H}''_{i-}) = 3$ and taking C to be an odd cycle in $\widehat{H}''_{i-}$, we can repeat the same argument to conclude that x has no negative connection to $\widehat{H}'_{i-}$ either.

In conclusion, the negative edges of each component of $\widehat{G} \setminus A'$ either induces a 5-colorable graph in which case we use colors 1, 2, 3, 4, 5 to color it, or it induces a 6-chromatic graph with no negative edge connecting it to A' . In this case we use colors 1, 2, 3, 4, 5, 6 to color it. Finally we can use color 6 on the vertices of A' , concluding with a 6-coloring of $\widehat{G}_-$. $\square$

Applying Proposition 26 we have the following.

Corollary 31. *The class $\text{Forb}_{ind} \{(\widehat{K_4}, -), P_4\}$ is a GS set.*

Combining this corollary with Lemma 29, we have the following conclusion.

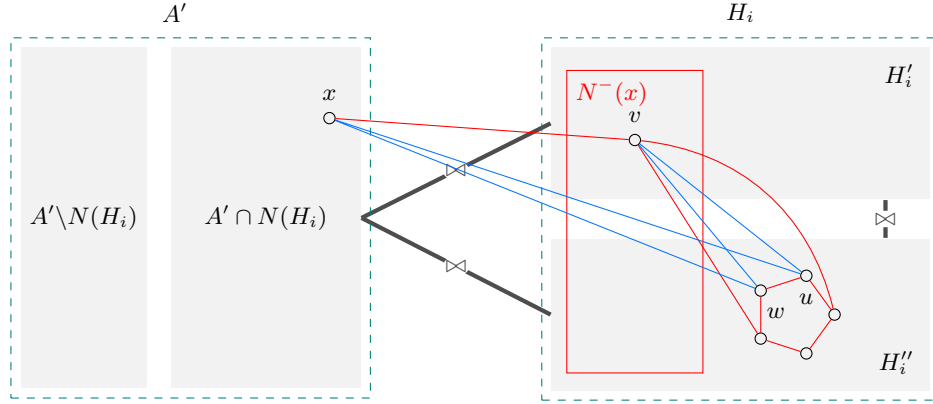


Figure 8: Illustration of the proof of Theorem 30

Theorem 32. *If each of the connected components of a linear forest F has at most 4 vertices, then $\text{Forb}_{\text{ind}}\{(\widehat{K_4}, -), F\}$ is a GS set.*

For each of the three results Theorem 30, Corollary 31, Theorem 32, we do not know what is the best possible upper bound. In Theorem 30 we provided an upper bound of 6, but in fact this was the upper bound on the chromatic number of the subgraph induced by the negative edges. While we do not know if 6 is the best bound for the balanced chromatic number of the family, in the following we provide an example for which the negative edges induces a 6-chromatic graph. Applying this upper bound of 6, for Corollary 31 we get an upper bound 2^7 . For Theorem 32, the upper bound is a function of the number of components of F .

Proposition 33. *There is a signed graph in $\text{Forb}_{\text{ind}}((K_3, -), (K_4, M), P_4)$ whose negative edges induce a graph of chromatic number 6.*

Proof. The basic element of the construction is the signed graph of Figure 9 called *envelope* and denoted $\widehat{EN}$.

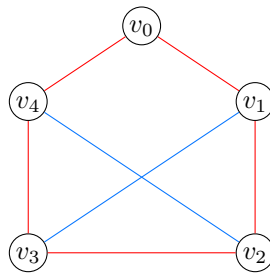


Figure 9: The envelope

Claim 1. *Given an integer $c \geq 3$, if $k \geq 2c - 3$, for any c -coloring ϕ of the disjoint union of k 5-cycles, there exist three disjoint independent sets X, Y and Z such that*

$$|\phi(X) \cap \phi(Y) \cap \phi(Z)| \geq 3.$$

Proof of the claim. We first claim that there are three colors, say a, b, c , such that each color appears in at least three of the C_5 's. If not, after removing the two colors that appear in the most C_5 's, each C_5 , being 3-chromatic, has a color that is shared with at most one other C_5 . Thus the number of colors, c , is at least $\lceil \frac{k}{2} \rceil + 2$, contradicting the assumption that $k \geq 2c - 3$.

Let A be a set of three vertices from three distinct C_5 all colored a . Similarly we choose B and C each consisting of three vertices, colored b and c respectively. We now consider the incidence graph between $\{A, B, C\}$ and the C_5 's. Observe that this is a bipartite graph of maximum degree 3. Thus it admits a 3-edge-coloring. Let X, Y, Z be the three color classes in one such coloring. One can view each of X, Y , and Z as a 3-subset of the vertices of C_5 's, spanning the three colors a, b, c and coming from different C_5 's. Thus X, Y, Z are disjoint independent sets and $|\phi(X) \cap \phi(Y) \cap \phi(Z)| \geq 3$. $\diamond$

We want to construct a signed graph $\widehat{G}$ in $\mathcal{G} = \text{Forb}_{\text{ind}}((K_3, -), (K_4, M), P_4)$ such that the subgraph $\widehat{G}_-$ has chromatic number 6. To that end we work with red-independent sets that are independent sets of $\widehat{G}_-$. First observe that the envelope is in $\mathcal{G}$. Start with the signed graph $\widehat{R}$ which is the union of 7 disjoint envelopes. Construct the signed graph $\widehat{L}$ as follows. For every possible 5-coloring ϕ of $\widehat{R}_-$, add a disjoint envelope $\widehat{EN}_\phi$ in $\widehat{L}$ fully joined with $\widehat{R}$. Applying Claim 1 to $\widehat{R}_-$, we find disjoint independent sets X, Y, Z (of $\widehat{R}_-$). As $\widehat{R}_-$ is a union of disjoint C_5 's, hence 2-regular, each vertex not in $X \cup Y \cup Z$ can be assigned to one of the sets while they remain independent sets. Thus we may assume the $X \cup Y \cup Z = V(\widehat{R}_-)$, and note that we still have the property $|\phi(X) \cap \phi(Y) \cap \phi(Z)| \geq 3$.

The signs of the edges in the full join of $\widehat{EN}_\phi$ to $\widehat{R}$ are chosen as follows. The copy of v_0 in $\widehat{EN}_\phi$ is adjacent to every vertex in X by a negative edge, copies of v_1, v_3 in $\widehat{EN}_\phi$ are both adjacent to every vertex in Y by negative edges, and copies of v_2, v_4 in $\widehat{EN}_\phi$ are both adjacent to all vertices in Z by negative edges. All other edges in the full join are positive. The resulting signed graph is called $\widehat{LR}$.

Claim 2. $\widehat{LR} \in \mathcal{G}$.

The underlying graph is obtain from a full join of disjoint unions of envelopes which are cographs. Therefore, it is itself a cograph, i.e. P_4 -free. For every vertex u in $V(\widehat{L})$, $N_{\widehat{R}}^-(u)$ does not induce a negative edge because X (resp Y, Z) is an independent set in $\widehat{R}_-$. For every pair u, v of vertices in $\widehat{L}$ connected by a negative edge, we have 1. $N_{\widehat{R}}^-(u) \cap N_{\widehat{R}}^-(v) = \emptyset$ 2. $N^+(u) \cap N^+(v)$ contains no negative edge. The former is simply because the three sets X, Y , and Z are disjoint. The latter is because $N^+(u) \cap N^+(v) = R \setminus (N^-(u) \cup N^-(v))$ which is one of the X, Y, Z and hence is an independent set in $\widehat{R}_-$.

Those conditions, together with the fact that there is no triangle with two positive edge in an envelope, we are ensured that there is neither $(K_3, -)$ nor (K_4, M) in $\widehat{LR}$.

Claim 3. The graph $\widehat{LR}_-$ has chromatic number 6.

Suppose ϕ is a 5-coloring of $\widehat{LR}_-$. Let ϕ_R be the coloring induced on $\widehat{R}_-$ and consider $\widehat{EN}_{\phi_R} \in \widehat{L}$ and the partition X_ϕ, Y_ϕ, Z_ϕ of $V(\widehat{R})$ associated to ϕ . Since $|\phi(X) \cap \phi(Y) \cap \phi(Z)| \geq 3$, there are at least 3 colors not used on $\phi(\widehat{EN}_{\phi_R})$. But $\widehat{EN}_{\phi_R}$ (which is a C_5) needs at least three more colors, which is a contradiction. $\square$

6 Conclusion

In this work, we initiated the study of balanced chromatic number on the hereditary classes of signed graphs.

The notion of balanced chromatic number generalizes the classic chromatic number of graphs. Given a graph G , the balanced chromatic number of the signed graph $\tilde{G}$ is the same as the chromatic number of G . The class of all signed graphs of the form $\tilde{G}$ is the hereditary class $\text{Forb}_{\text{ind}} \{(\widehat{K_2}, -)\}$. That is the class of signed graphs where no pair of vertices induces a simple edge. Thus the Gyárfás-Sumner conjecture can be restated in the language of signed graphs as follows.

Gyárfás-Sumner conjecture (Restated). *Any minimal (finite) GS set containing $(\widehat{K_2}, -)$ has three elements.*

In this work, then, we studied minimal (finite) GS sets that do not contain $\tilde{K}_2$. It is observed that any such set must be of order at least 3 (including $\tilde{K}_2$). Our focus has been to classify those sets of order 3. We showed that any such set must contain either $(\widehat{K_3}, -)$ or $(\widehat{K_4}, -)$ and a linear forest. For $(\widehat{K_3}, -)$ we showed that all such sets are GS sets. In the case of $(\widehat{K_4}, -)$ we showed that, as long as each component of the linear forest is of order at most 4, we have a GS set. We conjectured that for every linear forest F the set $\{\tilde{K}_2, (\widehat{K_4}, -), F\}$ is a GS set.

Thus, unlike the case where $(\widehat{K_2}, -)$ is forbidden, in the case where $\tilde{K}_2$ is forbidden, we already have minimal GS sets of order at least 4, and perhaps we can build minimal GS sets of larger size. In a follow-up work we study minimal GS sets of order 4.

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