A COMMON GENERALIZATION TO STRENGTHENINGS OF DRISKO'S THEOREM FOR INTERSECTIONS OF TWO MATROIDS

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ABSTRACT. Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same ground set V. Let $A_1, \dots, A_{2n-1}$ be sets which are independent in both $\mathcal{M}$ and $\mathcal{N}$, satisfying $|A_i| \geq \min(i, n)$ for all i. We show that there exists a partial rainbow set of size n, which is independent in both $\mathcal{M}$ and $\mathcal{N}$. This is a common generalization of rainbow matching results for bipartite graphs by Aharoni, Berger, Kotlar, and Ziv, and for the intersection of two matroid by Kotlar and Ziv.

1. INTRODUCTION

Let $(M_1, \dots, M_n)$ be a collection (possibly with repetition) of matchings (vertex disjoint edges) in a bipartite graph G. A *partial rainbow matching* of size m is a set of pairwise vertex disjoint edges $e_1, \dots, e_m$ such that $e_1 \in M_{i_1}, \dots, e_m \in M_{i_m}$ where $1 \leq i_1 < \cdots < i_m \leq n$. The following conjecture of Aharoni and Berger from [2] on rainbow matchings is of significant interest (see also Conjecture 1.6 in [4]).

Conjecture 1.1 (Berger-Aharoni [2]). *A collection of n matchings of size n in a bipartite graph has a partial rainbow matching of size $n-1$.*

The fact that the number $n-1$ in Conjecture 1.1 cannot be replaced by n follows from the following example. Let n be even and let G be the complete bipartite graph with both partite sets equal to $[n] = \{1, 2, 3, \dots, n\}$. Note that we think of the edges in such a graph as ordered pairs, where (i, j) means we take the vertex i from the first partite set and the vertex j from the second partite set. The collection of n matchings M_i for $1 \leq i \leq n$, where M_i is given by the set of edges $(j, j+i)$ (taking addition modulo n), does not have a rainbow matching of size n.

Conjecture 1.1 is closely related to well-known open problems concerning transversals of latin squares by Ryser, Brualdi, and Stein [6, 12, 13]. See for instance [2] for the statements of these conjectures and how they are related to Conjecture 1.1.

One may naturally ask how many matchings of size n of a bipartite graph G are needed to guarantee a partial rainbow matching of size n. Interestingly, almost twice as many matchings are required.

Theorem 1.2 (Aharoni-Berger [2]). *A collection of $2n-1$ matchings of size n in a bipartite graph has a partial rainbow matching of size n.*

The following example shows that the number $2n-1$ in Theorem 1.2 cannot be replaced by a smaller number. Let G be the cycle on $2n$ vertices and take $n-1$ copies of one maximum matching and $n-1$ copies of the other maximum matching. There is no rainbow partial matching of size n for this collection of matchings.

Theorem 1.2 is a generalization of the following well-known theorem of Drisko. For the purpose of stating this result, we say that a *diagonal* of a matrix is a set of entries with at most one entry in each column and at most one entry in each row.

Theorem 1.3 (Drisko [8]). *Let X be an $n \times (2n-1)$ matrix such that each column contains each of the integers $1, \dots, n$. Then there is a diagonal of X that contains each of the integers $1, \dots, n$.*

To see that Theorem 1.2 implies Theorem 1.3, note that each column of the matrix X corresponds to a matching of the complete bipartite graph G with both partite sets equal to $[n]$, where (i, j) is an edge of the matching if j is the entry in the i'th row of the column. A rainbow matching of size n corresponds to a diagonal as in the conclusion of Theorem 1.3.

A matroid-theoretic generalization of Theorem 1.3 was proven soon after by Chappel (see Section 2 for background on matroids).

Theorem 1.4 (Chappel [7]). *Let X be an $n \times (2n-1)$ matrix whose entries are elements of the ground set V of a matroid $\mathcal{M}$. Then there is a diagonal of X of size n that is independent in $\mathcal{M}$*

Theorem 1.4 is easily seen to be a generalization of Theorem 1.3 by taking $\mathcal{M}$ to be the matroid with ground set $[n]$ where each subset is independent.

Kotlar and Ziv [9] took this one step further by proving the following theorem. Here, if $S_1, \dots, S_n$ are sets, a *partial rainbow set* is a selection of elements $x_1 \in S_{i_1}, \dots, x_m \in S_{i_m}$ where $1 \leq i_1 < \cdots < i_m \leq n$.

Theorem 1.5 (Kotlar-Ziv [9]). *Let $\mathcal{M}$ and $\mathcal{N}$ be matroids on the same ground set V. For any $2n-1$ subsets of V of size n that are independent in both $\mathcal{M}$ and $\mathcal{N}$, there is a partial rainbow set of size n that is independent in both $\mathcal{M}$ and $\mathcal{N}$.*

Theorem 1.4 follows from Theorem 1.5 by the following argument. Let V be the ground set of a matroid $\mathcal{M}$. Let $\mathcal{M}'$ be the matroid on the ground set $V \times [n]$ whose independent sets are those sets with elements that have distinct values in their first coordinate and whose image under projection onto the first coordinate is independent in $\mathcal{M}$. Let $\mathcal{N}$ be the matroid on $V \times [n]$ whose independent sets are those sets with elements that have distinct values in their second coordinate. Given a matrix X as in Theorem 1.4, define the set A_i for $1 \leq i \leq 2n-1$ to be the set of elements (v, j) where $v = X_{ji}$. Then A_i is independent in $\mathcal{M}'$ and $\mathcal{N}$ for all i and thus has a partial rainbow set of size n that is independent in both $\mathcal{M}'$ and $\mathcal{N}$. This corresponds to the diagonal as in the conclusion of Theorem 1.4.

Theorem 1.2 also follows from Theorem 1.5 by the following observation. Let G be a bipartite graph with partite sets A and B. Let V be the edge set of G and let $\mathcal{M}$ be the matroid with ground set V whose independent sets are the sets of edges whose endpoints in A are all distinct. Define the matroid $\mathcal{N}$ similarly where instead the endpoints in B are considered. Then a matching of G is precisely a set of edges that is independent in both $\mathcal{M}$ and $\mathcal{N}$.

Theorem 1.2 was further generalized by Aharoni, Berger, Kotlar, and Ziv in [4] by showing that the size of some the matchings can be reduced while the same conclusion still holds.

Theorem 1.6 (Aharoni et. al. [4]). *Let $(M_1, \dots, M_{2n-1})$ be matchings in a bipartite graph G satisfying $|M_i| \geq \min(i, n)$ for all i. Then there is a partial rainbow matching of size n.*

Additionally, it was shown in [4] that Theorem 1.6 is tight in the following sense. If the matchings $M_1, \dots, M_{2n-1}$ are ordered in nondecreasing order in terms of their size, then the conclusion of Theorem 1.6 does not hold in general if $|M_i| < \min(i, n)$ for some i.

The main theorem of this paper is a common generalization of Theorems 1.5 and 1.6.

Theorem 1.7 (Main theorem). *Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same ground set V. Let $A_1, \dots, A_{2n-1}$ be sets which are independent in both $\mathcal{M}$ and $\mathcal{N}$, satisfying $|A_i| \geq \min(i, n)$ for all i. Then there exists a partial rainbow set of size n, which is independent in both $\mathcal{M}$ and $\mathcal{N}$.*

2. Preliminaries

In this section, we introduce the necessary background and supporting results needed to prove Theorem 1.7. A hypergraph $H = (V, E)$ consists of a *ground set* V (for our purposes V will be finite) and an *edge set* E consisting of subsets of V. A *matroid* is a hypergraph whose edge set E satisfies the following properties:

(1) $\emptyset \in E$,
(2) If $T \in E$ and $S \subseteq T$, then $S \in E$,
(3) If $S, T \in E$ and $|S| < |T|$, then there exists $x \in T \setminus S$ such that $S \cup \{x\} \in E$.

The *rank* in $\mathcal{M}$ of a set $S \subseteq V$ is the size of the largest edge of $\mathcal{M}$ contained in S, and we denote it as $\rho_{\mathcal{M}}(S)$. The rank of $\mathcal{M}$, denoted $\rho(\mathcal{M})$, is defined to be $\rho_{\mathcal{M}}(V)$. A *basis* of $\mathcal{M}$ is an edge e of $\mathcal{M}$ such that $|e|$ is equal to the rank of $\mathcal{M}$. A *flat* F of $\mathcal{M}$ is a subset of V satisfying $\rho_{\mathcal{M}}(F \cup \{x\}) > \rho_{\mathcal{M}}(F)$ for all $x \in V \setminus F$. A *loop* is an element $x \in V$ such that $\{x\} \notin E$. The set of *circuits* of $\mathcal{M}$, which we denote $\mathrm{circ}(\mathcal{M})$, are the minimal sets (with respect to inclusion) that are not an edge of $\mathcal{M}$. A *coloop* is an element that does not belong to any circuit.

Note that we may also view $\mathrm{circ}(\mathcal{M})$ as a hypergraph with ground set V. We may abuse notations by writing $\mathrm{circ}(\mathcal{M})$ instead of $(V, \mathrm{circ}(\mathcal{M}))$. A well-known property of matroids is that they satisfy the *circuit elimination axiom*.

- (circuit elimination axiom) For any two circuits C_1 and C_2 with $x \in C_1 \cap C_2$ and $y \in C_1 \setminus C_2$, there exists a circuit C_3 such that $C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$ and $y \in C_3$.

See [11] for a more detailed account of matroids.

For a general hypergraph $H = (V, E)$, we call a set $S \subseteq V$ independent if there is no edge $e \in E$ such that $e \subseteq S$. We denote by $\mathcal{I}(H)$ the hypergraph on the same ground set whose edges are the independent sets of H. Note that for a matroid $\mathcal{M}$ we have $\mathcal{M} = \mathcal{I}(\mathrm{circ}(\mathcal{M}))$. For an edge $e \in E$, we define

$$H - e = (V, E \setminus \{e\}).$$

For a set $S \subseteq V$, we define

$$H/S = (V \setminus S, \{e \setminus S \mid e \in E,\, e \not\subseteq S\})$$

For two hypergraphs $H_1 = (V, E_1), H_2 = (V, E_2)$ on the same ground set V, we denote $H_1 \cap H_2$ as the hypergraph $(V, E_1 \cap E_2)$. For a matroid $\mathcal{M}$, we define $\mathcal{M}/S = \mathcal{I}(\mathrm{circ}(\mathcal{M})/S)$. Note that $\mathcal{M}/S$ coincides with the usual notion of contraction in matroids, and hence is itself a matroid.

We will also require some topological preliminaries. Homology groups will be taken with $\mathbb{Q}$ coefficients. A *simplicial complex* is a collection of finite sets that is closed under taking subsets, and a set of the simplicial complex may be called a simplex. For a simplicial complex $\mathcal{C}$, $\eta(\mathcal{C})$ is the smallest integer k such that the reduced homology $\tilde{H}_{k-1}(\mathcal{C})$ is not zero. If $\mathcal{C}$ is empty we take $\eta(\mathcal{C}) = 0$ and if all homology groups of $\mathcal{C}$ vanish, then we take $\eta(\mathcal{C}) = \infty$.

The following result will allow us to obtain lower bounds on η and is key to the proof of the main theorem.

Theorem 2.1. *Let H be a hypergraph, and let e be an edge that does not contain another edge. Then*

$$\eta(\mathcal{I}(H)) \geq \min\Big(\eta\big(\mathcal{I}(H - e)\big), \eta\big(I(H/e)\big) + |e| - 1\Big).$$

Theorem 1.3 was proven by Meshulam in [10] for graphs, but the proof holds for general hypergraphs as well. It was used implicitly in [3] and explicitly in [5].

Another result that is necessary for our proof is the following theorem of Aharoni and Berger that provides a necessary condition for the existence of a set that is both a basis of a matroid and an element of a simplicial complex both with the same ground set V. For a matroid $\mathcal{M}$ and a set $S \subseteq V$, we write $\mathcal{M}.S$ for the matroid consisting of the sets $e \subseteq S$ such that $e \cup f$ is an edge of $\mathcal{M}$ for all edges f with $f \cap S = \emptyset$. For a simplicial complex $\mathcal{C}$ and a set S contained in its ground set, we write $\mathcal{C}|S$ for the simplicial complex given by the collection of sets of $\mathcal{C}$ contained in S.

Theorem 2.2 (Aharoni-Berger [1]). *Let $\mathcal{M}$ be a matroid and $\mathcal{C}$ be a simplicial complex with the same ground set V. If $\eta(\mathcal{C}|S) \geq \rho(\mathcal{M}.S)$ for all $S \subseteq V$, then there is a basis of $\mathcal{M}$ in the edge set of $\mathcal{M} \cap \mathcal{C}$. In fact, it suffices to assume the condition holds for sets S that are complements of a flat of $\mathcal{M}$.*

3. Proof of Theorem 1.7

We are now ready to prove the main theorem. We will need Lemma 3.1 below to provide lower bounds on η, which will allow us to later apply Theorem 2.2. Given sets $X_1 \dot\cup \cdots \dot\cup X_m$ ($\dot\cup$ denotes disjoint union), the *partition matroid* P is the matroid whose edges are all partial rainbow sets of the X_i.

Lemma 3.1. *Let $\ell \geq 1$ and $m \geq 2\ell - 1$ be integers. Let P be the partition matroid on $X_1 \dot\cup \cdots \dot\cup X_m$. Let $\mathcal{N}$ be a matroid on $V \supseteq X_1 \cup \cdots \cup X_m$, and assume that there are indices $i_1, \ldots, i_{2\ell-1}$ such that $\rho_{\mathcal{N}}(X_{i_j}) \geq \min(j, \ell)$. Then $\eta(P \cap \mathcal{N}) \geq \ell$.*

Proof. We proceed by induction on ℓ; for the base case $\ell = 1$ we only need to show that there exists some non-empty set in $P \cap \mathcal{N}$, and this follows from the condition $\rho_{\mathcal{N}}(X_{i_1}) \geq 1$.

We now need to show the induction step. For notational purposes, we will assume without loss of generality that $i_j = j$. Furthermore, by replacing $\mathcal{N}$ with $\mathcal{N}|(X_1 \cup \cdots \cup X_m)$ if necessary, we can assume that $X_1 \cup \cdots \cup X_m = V$.

Let H be the hypergraph on the ground set V whose edges are the circuits of $\mathcal{N}$ and the 2-element subsets $\{x, y\}$ where $x, y \in X_i$ for some i. Note that $P \cap \mathcal{N} = \mathcal{I}(H)$.

Let $x \in X_1$ be a non-loop. The existence of such a non-loop follows from the assumption $\rho_{\mathcal{N}}(X_1) \geq 1$. Let $C_1, \dots, C_r$ be the circuits of $\mathcal{N}$ that contain x in any order. We apply Theorem 2.1 iteratively to H with the edges $C_1, \dots, C_r$ one at a time. Let $H_0 = H$, and for $i \geq 1$, let $H_i = H_{i-1} - C_i$ or $H_i = H_{i-1}/C_i$ depending on which one provides a lower bound for $\eta(\mathcal{I}(H_{i-1}))$ as in Theorem 2.1. If $H_i = H_{i-1}/C_i$ for some i then we stop the process. Assume first that there exists some i such that $H_i = H_{i-1}/C_i$. We claim that $\mathcal{I}(H_i) = \mathcal{I}(H/C_i)$. It is clear that $\mathcal{I}(H_i) \supseteq \mathcal{I}(H/C_i)$ since the edge set of H_i is contained in the edge set of H/C_i. If there exists an edge $e \in \mathcal{I}(H_i) \setminus \mathcal{I}(H/C_i)$, then e contains a subset of the form $C_j \setminus C_i$ for some $j < i$. However, by the circuit elimination property, there exists a circuit C' such that $x \notin C'$, $C' \subseteq C_j \cup C_i$, and $C' \nsubseteq C_i$. Hence, $C' \setminus C_i \subseteq C_j \setminus C_i$, contradicting the fact that $e \in \mathcal{I}(H_i)$. Therefore, $\eta(\mathcal{I}(H)) \geq \eta(\mathcal{I}(H_i)) + |C_i| - 1 = \eta(\mathcal{I}(H/C_i)) + |C_i| - 1$. Let $k = \rho_{\mathcal{N}}(C_i)$. If $k \geq \ell$, then $|C_i| \geq \ell + 1$ and we are done. Otherwise, for the indices j for which $X_j \cap C_i = \emptyset$ we have that $\rho_{\mathcal{N}/C_i}(X_j) \geq j - k$ for all j. In particular, there are $2(\ell - k) - 1$ indices $i_1, \dots, i_{2(\ell-k)-1}$ such that $\rho_{\mathcal{N}/C_i}(X_{i_j}) \geq j - k$ for all j. Thus, by induction applied to $\mathcal{N}/C_i$ and the X_j with $X_j \cap C_i = \emptyset$, we have $\eta(\mathcal{I}(H/C_i)) \geq \ell - k$, which implies that $\eta(\mathcal{I}(H)) \geq \ell$.

Now, assume instead that $H_i = H_{i-1} - C_i$ for all i, and let $H' = H_r$. Let $e_1, \dots, e_s$ be the edges of the form $\{x, y\}$ where $y \neq x$ and $y \in X_1$ in any order. Define H'_i for $0 \leq i \leq s$ in a similar way as before. If $H'_i = H'_{i-1} - e_i$ for all i, or if no such edge exists, then x is not contained in any edge of H'_s. This implies that $\mathcal{I}(H'_s)$ is a cone and is contractable as a simplicial complex, which means $\eta(\mathcal{I}(H)) \geq \eta(\mathcal{I}(H'_s)) = \infty$. Thus, we may assume that $H'_i = H'_{i-1}/e_i$ for some i, and let $e_i = \{x, y\}$. Similarly to before, we have $\mathcal{I}(H'_i) = \mathcal{I}(H'/e_i)$. Let $\mathcal{N}'$ be the matroid obtained by removing all circuits of $\mathcal{N}$ containing x. Since x is a coloop in $\mathcal{N}'$, we have $\rho_{\mathcal{N}'/\{x,y\}}(X_j) \geq j - 1$ for all j. Thus, by induction applied to $\mathcal{N}'/\{x, y\}$ and $X_2, \dots, X_m$, $\eta(\mathcal{I}(H'/e_i)) \geq \ell - 1$. Therefore, $\eta(\mathcal{I}(H)) \geq \eta(\mathcal{I}(H')) \geq \eta(\mathcal{I}(H'/e_i)) + 1 \geq \ell$.

□

Proof of Theorem 1.7. We will assume that $\mathcal{M}$ is a matroid of rank n, which we can do without loss of generality since we can always replace $\mathcal{M}$ with the matroid whose independent sets consist of the independent sets of $\mathcal{M}$ with size at most n.

Let $A_1, \dots, A_{2n-1}$ be independent sets of $\mathcal{M}$ and $\mathcal{N}$ as in the statement of the theorem. Let $\mathcal{C}$ be the simplicial complex with vertex set V' given by the set of pairs $\{(x, i) \mid x \in A_i\}$ and with simplices of the form $\{(x_1, i_1), \dots, (x_m, i_m)\}$ where the i_j's, as well as the x_j's, are pairwise distinct, and $\{x_1, \dots, x_m\}$ is independent in $\mathcal{N}$. Let $\mathcal{M}'$ be the matroid on the same ground set V' whose independent sets are sets of the form $\{(x_1, i_1), \dots, (x_m, i_m)\}$ where the x_j's are pairwise distinct and $\{x_1, \dots, x_m\}$ is independent in $\mathcal{M}$ (the i_j's do not have to be distinct here). Define $\mathcal{N}'$ similarly.

Notice that the existence of a set that is simultaneously a basis of $\mathcal{M}'$ and a simplex of $\mathcal{C}$ gives us a partial rainbow set as in the conclusion of Theorem 1.7 and hence, would complete the proof. We will apply Theorem 2.2 to prove the existence of such a set.

Let F' be a flat of $\mathcal{M}'$ of rank k and let S' be its complement. Notice that there is a corresponding flat F of $\mathcal{M}$ such that $F' = \{(x, i) \mid x \in F, 1 \leq i \leq 2n - 1\}$. It is clear from the definition of $\mathcal{M}'.S'$ that $\rho(\mathcal{M}'.S') \leq n - k$, so it suffices to prove that $\eta(\mathcal{C}|S') \geq n - k$. Let S be the complement of F. We have $\rho_{\mathcal{M}}(A_i \cap S) \geq \rho_{\mathcal{M}}(A_i) - k$, which implies that $|A_i \cap S| \geq |A_i| - k$. Therefore, the lower bound on $\eta(\mathcal{C}|S)$ follows from applying Lemma 3.1 to the matroid $\mathcal{N}'|S'$ and the sets $X_i = \{(x, i) \mid x \in A_i \cap S\}$ for $1 \leq i \leq 2n - 1$. This completes the proof.

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