

Cycle lengths in graphs of given minimum degree

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Abstract

In a graph, k cycles are *admissible* if their lengths form an arithmetic progression with common difference one or two. Let G be a 2-connected graph with minimum degree at least $k \geq 4$. We prove that

- (1) G contains k admissible cycles, unless $G \cong K_{k+1}$ or $K_{k,n-k}$;
- (2) G contains cycles of lengths ℓ modulo k for all even ℓ , unless $G \cong K_{k+1}$ or $K_{k,n-k}$;
- (3) G contains cycles of lengths ℓ modulo k for all ℓ , unless $G \cong K_{k+1}$ or G is bipartite.

In addition, we show that if k is even and G is 2-connected with minimum degree at least $k - 1$ and order at least $k + 2$, then G contains cycles of lengths ℓ modulo k for all even ℓ . These findings provide a stability analysis of the main results on cycle lengths in graphs of given minimum degree in [J. Gao, Q. Huo, C. Liu, J. Ma, A unified proof of conjectures on cycle lengths in graphs, International Mathematics Research Notices 2022 (10) (2022) 7615–7653]. As a corollary, we determine the maximum number of edges in a graph that does not contain a cycle of length 0 modulo k for all odd k .

1 Introduction

The distribution of cycle lengths in graphs of given basic parameters, such as minimum degree, edge number, etc, has been extensively studied in the literature and remains one of the most active and fundamental research areas in graph theory.

For a cycle (or a path) C , its *length*, denoted by $|C|$, is the number of edges in C . Let $k \geq 1$ and $\ell \geq 0$ be two integers. An $(\ell \bmod k)$ -*cycle* refers to a cycle whose length is congruent to ℓ modulo k . In a graph, k cycles $C_1, \dots, C_k$ are *admissible* if $|C_1|, \dots, |C_k|$ form an arithmetic progression with common difference one or two; and k paths $P_1, \dots, P_k$ are *admissible* if $|P_1| \geq 2$ and $|P_1|, \dots, |P_k|$ form an arithmetic progression with common difference one or two. In 2022, Gao, Huo, Liu and Ma [13] established the following remarkable results.

Theorem 1.1 (Gao, Huo, Liu and Ma [13]). *Let G be a graph with minimum degree at least $k + 1$ and $k \geq 3$. The following statements hold:*

- (1) G contains k admissible cycles.
- (2) G contains cycles of lengths ℓ modulo k for all even ℓ .
- (3) G contains cycles of lengths ℓ modulo k for all ℓ if G is non-bipartite and 2-connected.

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Theorem 1.2 (Gao, Huo, Liu and Ma [13]). *Let G be a k -connected graph. The following statements hold:*

- (1) *If $k \geq 3$, then G contains a cycle of length 0 modulo k .*
- (2) *If $k \geq 6$, then G contains cycles of lengths ℓ modulo k for all even $\ell \not\equiv 2$ modulo k .*

Theorem 1.1 (1) confirms a conjecture proposed by Liu and Ma [17] in 2018. Theorem 1.1 (2) and (3) confirm two conjectures proposed by Thomassen [20] in 1983. Theorem 1.2 (1) confirms a conjecture proposed by Dean [7] in 1988. Note that the complete graph K_{k+1} and the complete bipartite graph $K_{k,n-k}$ have minimum degree k but do not contain k admissible cycles. Note also that, for each integer k , the graph K_{k+1} does not contain a $(2 \bmod k)$ -cycle; and for each odd k , the graph $K_{k,n-k}$ does not contain a $(2 \bmod k)$ -cycle. Thus, the minimum degree conditions in the above results are tight in general.

In this paper, we investigate the stability of Theorems 1.1 and 1.2. Regarding Theorem 1.1 (1) and (2), we prove the following two results.

Theorem 1.3. *Let G be a 2-connected graph with minimum degree at least $k \geq 4$. Then G contains k admissible cycles, unless $G \cong K_{k+1}$ or $K_{k,n-k}$.*

Theorem 1.4. *Let G be a 2-connected graph with minimum degree at least $k \geq 4$. Then G contains cycles of lengths ℓ modulo k for all even ℓ , unless $G \cong K_{k+1}$ or $K_{k,n-k}$.*

Theorems 1.3 and 1.4 do not hold for $k = 3$, as the Petersen graph is a counterexample. By results of Chen and Saito [5], and Dean et al. [8], the Petersen graph is the unique counterexample for $k = 3$ in the context of Theorem 1.4 (see Theorem 4.6 in Section 4). We do not know whether there are additional counterexamples other than the Petersen graph for $k = 3$ when considering Theorem 1.3.

By Theorem 1.4, we can get that the statements in Theorem 1.2 still hold for 2-connected graphs with minimum degree at least k for every $k \geq 4$.

Corollary 1.5. *Let G be a 2-connected graph with minimum degree at least $k \geq 4$. Then G contains cycles of lengths ℓ modulo k for all even $\ell \not\equiv 2$ modulo k .*

Note that $K_{k-1,n-k+1}$ does not contain a cycle of length 0 modulo k if k is odd. Thus, the minimum degree in Corollary 1.5 cannot be reduced for odd k . But for even k , we can improve the result as follows.

Theorem 1.6. *Let G be a 2-connected graph with minimum degree at least $k - 1$ and order at least $k + 2$, where $k \geq 4$ is even. Then G contains cycles of lengths ℓ modulo k for all even ℓ .*

Regarding Theorem 1.1 (3), we show the following result.

Theorem 1.7. *Let G be a 2-connected non-bipartite graph with minimum degree at least $k \geq 3$. Then G contains cycles of lengths ℓ modulo k for all ℓ , unless $G \cong K_{k+1}$, or $k = 3$ and G is a Petersen graph.*

In view of graphs containing a cut-vertex such that each block is isomorphic to the complete graph K_{k+1} (or K_k when consider Theorem 1.6), one can see that 2-connectivity cannot be dropped in the above theorems.

By Theorem 1.4, we can also determine the maximum number of edges in an n -graph that does not contain a cycle of length 0 modulo k for odd k .

Let $\mathcal{F}$ be a set of graphs. A graph is $\mathcal{F}$ -free if it contains no subgraph isomorphic to any member of $\mathcal{F}$. The classic Turán problem in graphs seeks to determine the maximum number of edges in $\mathcal{F}$ -free graphs on n vertices. This maximum number is referred to as the *Turán number* for $\mathcal{F}$ and is commonly denoted by $ex(n, \mathcal{F})$, i.e.,

$$ex(n, \mathcal{F}) = \max\{|E(G)| : |V(G)| = n, G \text{ is } \mathcal{F}\text{-free}\}.$$

For two integers $k > \ell \geq 0$, denote the set of all $(\ell \bmod k)$ -cycles by $\mathcal{C}_{\ell \bmod k}$. The problem of determining the Turán number for $(\ell \bmod k)$ -cycles has a research history spanning nearly half a century. In 1976, Burr and Erdős [10] proposed this problem and conjectured that $ex(n, \mathcal{C}_{\ell \bmod k}) = \Theta(n)$ for $k > \ell \geq 0$ when $k\mathbb{Z} + \ell$ contains an even integer. In view of the complete balanced bipartite graphs, one can see that if $k\mathbb{Z} + \ell$ contains no even integers, then $ex(n, \mathcal{C}_{\ell \bmod k}) \geq \lfloor n^2/4 \rfloor$. Thus, the condition that $k\mathbb{Z} + \ell$ contains an even integer cannot be dropped. Bollobás [2] confirmed the above conjecture and demonstrated that $ex(n, \mathcal{C}_{\ell \bmod k}) \leq ((k+1)^k - 1)n/4k$. Subsequently, Erdős proposed the problem of determining the smallest constant $c_{\ell,k}$ such that every n -vertex graph with at least $c_{\ell,k} \cdot n$ edges contains an $(\ell \bmod k)$ -cycle. For $k > \ell \geq 3$, Sudakov and Verstraëte [19] showed that

$$\frac{ex(k, \mathcal{C}_{\ell})}{k} \leq c_{\ell,k} \leq 96 \cdot \frac{ex(k, \mathcal{C}_{\ell})}{k}.$$

It follows that for even $\ell \geq 4$, determining $c_{\ell,k}$ is at least as hard as the famous extremal problem of determining the Turán number for the even cycle C_{ℓ} .

To the best of our knowledge, precise values of $c_{\ell,k}$ are known for very few k, ℓ . If G contains no cycles, or alternatively, no $(0 \bmod 1)$ -cycles, then one can see that $e(G) \leq n - 1$ and the equality holds if and only if G is a tree. It is not difficult to see that if G contains no $(0 \bmod 2)$ -cycles then $e(G) \leq 3(n - 1)/2$ and, for $2|(n - 1)$, the equality holds if and only if each block of G is a triangle. The above two results imply that $c_{0,1} = 1$ and $c_{0,2} = 3/2$. The first non-trivial result on $c_{\ell,k}$ was obtained by Chen and Saito [5], who showed that $c_{0,3} = 2$. It has been shown in [5] that if G contains no $(0 \bmod 3)$ -cycles then $e(G) \leq 2(n - 2)$ for $n \geq 3$ and the equality holds if and only if G is isomorphic to $K_{2,n-2}$.

Theorem 1.8 (Chen and Saito [5]). *If G is an n -vertex graph such that every vertex (with at most one exception) has degree at least 3, then G contains a $(0 \bmod 3)$ -cycle. As a corollary, for $n \geq 3$,*

$$ex(n, \mathcal{C}_{0 \bmod 3}) = 2(n - 2).$$

Dean, Kaneko, Ota and Toft [8] and, independently, Saito [18] showed that if G contains no $(2 \bmod 3)$ -cycles then $e(G) \leq 3(n - 3)$ for $n \geq 5$, and the equality holds if and only if G is isomorphic to $K_{3,n-3}$. Gao, Li, Ma and Xie [14] showed that if G contains no $(2 \bmod 4)$ -cycles then $e(G) \leq 5(n - 1)/2$ and, for $4|(n - 1)$, the equality holds if and only if each block of G is isomorphic to K_5 . Bai, Li, Pan and Zhang [1] showed that if G contains no $(1 \bmod 3)$ -cycles then $e(G) \leq 5(n - 1)/3$ and, for $9|(n - 1)$, the equality holds if and only if each block of G is isomorphic to the Petersen graph.

For the case of $(0 \bmod 4)$ -cycles, Győri et al. [15] obtained the following result.

Theorem 1.9 (Győri et al. [15]). *If G is an n -vertex graph without $(0 \bmod 4)$ -cycles, then $e(G) \leq \lfloor 19(n - 1)/12 \rfloor$ and the bound is tight for all $n \geq 1$, i.e.,*

$$ex(n, \mathcal{C}_{0 \bmod 4}) = \left\lfloor \frac{19}{12}(n - 1) \right\rfloor.$$

It follows that all precise values of $c_{\ell,k}$ for $k \leq 4$ are fixed, as also shown in Table 1. However, prior to this work, not too much was known for the value of $c_{\ell,k}$ for large k . In this paper, we bound the maximum number of edges in a graph that does not contain a cycle of length 0 modulo k for all $k \geq 3$. In view of the complete bipartite graph $K_{k-1,n-k+1}$, one can see that the result is tight for every odd k .

Theorem 1.10. *Let $k \geq 3$ be an integer and G a graph on $n \geq 2k - 3$ vertices. If $e(G) > (k - 1)(n - k + 1)$, then G contains a $(0 \bmod k)$ -cycle.*

$c_{0,2} = 3/2$ (trivial)	—	—
$c_{0,3} = 2$ ([5])	$c_{1,3} = 5/3$ ([1])	$c_{2,3} = 3$ ([8, 18])
$c_{0,4} = 19/12$ ([15])	—	$c_{2,4} = 5/2$ ([14])

Table 1: Precise values of $c_{\ell,k}$ for $k \leq 4$.

As a corollary, $c_{0,k} \leq k - 1$ for all $k \geq 3$ and, together with Theorem 1.8, we have $c_{0,k} = k - 1$ for all odd k . Note that the existence of a $(0 \bmod k)$ -cycle is equivalent to the existence of a k -cycle when $n < 2k$. Therefore, Theorem 1.10 implies the following result. We remark that Füredi and Gunderson [11] determined $ex(n, C_k)$ for odd k .

Corollary 1.11. *Let $k \geq 3$ be an odd integer. Then*

$$ex(n, \mathcal{C}_{0 \bmod k}) = \begin{cases} ex(n, C_k), & n < 2k - 3; \\ (k - 1)(n - k + 1), & n \geq 2k - 3. \end{cases}$$

The same arguments as in the proof of Theorem 1.10 together with Theorem 1.1 (2), show that for an odd integer $k \geq 3$,

$$ex(n, \mathcal{C}_{2 \bmod k}) = \begin{cases} ex(n, C_{k+2}), & n < 2k; \\ k(n - k), & n \geq 2k. \end{cases}$$

This can also be obtained from the results of Gao, Huo, Liu and Ma in [13].

The rest of this paper is organized as follows. In Section 2, we give some necessary notations and definitions as well as some useful results needed in our proofs. In Section 3 we prove Theorem 1.3, in Section 4 Theorems 1.4 and 1.6, in Section 5 Theorem 1.7, while Section 6 is devoted to the proof of Theorem 1.10.

2 Notations and preliminaries

All graphs considered in this paper are finite, undirected and simple. Let G be a graph and $v \in V(G)$. The *degree* of v , denoted by $d(v)$, is the number of its neighbors in G , whose set is denoted by $N(v)$. For a vertex set $X \subseteq V(G)$, we denote $N_X(v) = N(v) \cap X$ and $d_X(v) = |N_X(v)|$. Additionally, for a set $U \subseteq V(G)$, by $N_X(U)$ we denote the union of $N_X(v)$ over all $v \in U$. The minimum degree of a graph G , denoted by $\delta(G)$, is the minimum number among all the degrees of the vertices in G .

A *bipartite* graph is a graph whose vertex set can be partitioned into two parts such that the two end-vertices of each edge of the graph are in different parts. For a vertex set $X \subseteq V(G)$, denote by $D[X]$ the subgraph of G induced by X . For two disjoint vertex sets $X, Y \subseteq V(G)$, denote by $G(X, Y)$ the subgraph induced by all the edges between X and Y in G . A Θ -*graph* is graph consisting of 3 paths having common end-vertices and disjoint internal vertices.

For a cycle C in a graph G and two vertices $x, y \in V(C)$, by $d_C(x, y)$ we denote the distance between x and y in C . If we fix an orientation of C , then by $\vec{C}[x, y]$ we denote the path from x to y along the orientation of C , and by $\overleftarrow{C}[x, y]$ the path from y to x along the orientation of C . For a path P in G and two vertices $x, y \in V(P)$, by $P[x, y]$ we denote the path from x to y contained in P .

A graph is *connected* if there exists a path between any two vertices in the graph. For a positive integer k , call a graph *k -connected* if it has order at least $k + 1$ and the deletion of any set of at most $k - 1$ vertices results in a

connected subgraph. The *connectivity* of a graph G is the maximum k such that G is k -connected. A *component* of a graph G is a maximal connected subgraph of G . A subset S of $V(G)$ is a *vertex cut* of G if $G - S$ has more components than G . If a vertex cut consists of exactly one vertex x , then x is a *cut-vertex* of G . A *block* of G is a maximum connected subgraph of G with no cut-vertex. In other words, a block is an isolated vertex, a cut-edge or a maximal 2-connected subgraph. An *end block* of G is a block of G containing exactly one cut-vertex of G . An *inner-vertex* in an end block is a vertex distinct with the cut-vertex contained in the end block. A graph is a *block-chain* if it has exactly two end blocks. We view a 2-connected graph as its unique end block. In particular, every vertex in a 2-connected graph is an inner-vertex.

For two vertices $x, y \in V(G)$, an (x, y) -*path* is a path with two end-vertices x, y . For two subset $X, Y \subset V(G)$, an (X, Y) -*path* is an (x, y) -path with $x \in X, y \in Y$, and all internal vertices in $V(G) \setminus (X \cup Y)$. An X -*path* is a path with two end-vertices in X and all internal vertices in $V(G) \setminus X$. For an integer ℓ , we use ℓ -*path* (cycle) to denote a path (cycle) with length ℓ . The following results on admissible paths will be useful in our proofs.

Theorem 2.1 (Gao, Huo, Liu and Ma [13]). *Let $k \geq 1$ be an integer and G a graph with two distinct vertices x, y such that $G + xy$ is 2-connected. If $d(v) \geq k + 1$ for each $v \in V(G) \setminus \{x, y\}$, then G contains k admissible (x, y) -paths.*

The following result shows that we can have one more vertex with degree less than $k + 1$ for the existence of k admissible paths in a 2-connected graph, which relaxes the degree condition in Theorem 2.1.

Theorem 2.2 (Chiba, Ota and Yamashita [6]). *Let $k \geq 1$ be an integer and G a graph on at least four vertices with $x, y, z \in V(G)$ such that $x \neq y$ and $G + xy$ is 2-connected. If $d(v) \geq k + 1$ for each $v \in V(G) \setminus \{x, y, z\}$, then G contains k admissible (x, y) -paths.*

We first show the following two useful lemmas, which roughly state that, if a 2-connected graph G with bounded minimum degree does not contain k admissible cycles or a cycle of length ℓ modulo k for some even ℓ , then the connectivity of G increases with k .

Lemma 2.3. *Let G be a 2-connected graph with minimum degree at least $k - r$, where $k \geq 2$ and $r \geq 0$ are integers. If G does not contain k admissible cycles, then the connectivity of G is at least $(k + 2)/2 - r$.*

Proof. Let V_0 be a minimum vertex cut of G with $n_0 = |V_0| \geq 2$. Assume the opposite that $n_0 \leq (k + 1)/2 - r$. Let Q_1, Q_2 be two components of $G - V_0$ and x, y two vertices in V_0 . We claim that $G[V(Q_1) \cup \{x, y\}] + xy$ is 2-connected. If not, say w is a cut-vertex in it, then $(V_0 \setminus \{x, y\}) \cup \{w\}$ is a smaller vertex cut of G , a contradiction. Note that every vertex other than x, y in $G[V(Q_1) \cup \{x, y\}]$ has degree at least $k - r - n_0 + 2$. Note also that $k - r - n_0 + 2 \geq 2$. By Theorem 2.1, there exist $k - r - n_0 + 1$ admissible (x, y) -paths in $G[V(Q_1) \cup \{x, y\}]$. Similarly, we can get $k - r - n_0 + 1$ admissible (x, y) -paths in $G[V(Q_2) \cup \{x, y\}]$. Concatenating these admissible paths, we get $2(k - r - n_0 + 1) - 1 \geq k$ admissible cycles in G , a contradiction. $\square$

Lemma 2.4. *Let G be a 2-connected graph with minimum degree at least $k - r$, where $k \geq 2$ and $r \geq 0$ are integers. If G does not contain cycles of length ℓ modulo k for some even ℓ , then the connectivity of G is at least $(k + 2)/2 - r$.*

Proof. Let V_0 be a minimum vertex cut of G with $n_0 = |V_0| \geq 2$. Assume the opposite that $n_0 \leq (k + 1)/2 - r$. Since G does not contain an $(\ell \bmod k)$ -cycle for some even ℓ , we get that G does not contain k cycles of consecutive lengths. By Lemma 2.3, G contains k admissible cycles of common difference two. If G is bipartite, then we are done as now G contains k cycles of consecutive even lengths. So G is non-bipartite. If k is odd, then G contains an $(\ell \bmod k)$ -cycle for any even ℓ , a contradiction. So k is even and $n_0 \leq k/2 - r$. If $r \geq (k - 2)/2$, then $(k + 2)/2 - r \leq 2$ and the assertion clearly holds. So assume that $r \leq (k - 4)/2$.

Let Q_1, Q_2 be two components of $G - V_0$. Similar to the analysis in the proof of Lemma 2.3, we have that for any two distinct vertices $x, y \in V_0$, there are $k - r - n_0 + 1$ admissible (x, y) -paths in $G[V(Q_i) \cup \{x, y\}]$, $i \in \{1, 2\}$. If the $k - r - n_0 + 1$ admissible (x, y) -paths in $G[V(Q_i) \cup \{x, y\}]$ are of consecutive lengths, then G have k cycles of consecutive lengths, a contradiction. So we conclude that for each two vertices $x, y \in V_0$ and $i \in \{1, 2\}$, $G[V(Q_i) \cup \{x, y\}]$ has $k - r - n_0 + 1 \geq (k + 2)/2$ admissible (x, y) -paths with common difference two.

If $G[V(Q_1) \cup V_0]$ contains an odd cycle C , then by G being 2-connected there exist two disjoint paths from C to $\{x, y\} \subseteq V_0$, implying that there are two (x, y) -paths in $G[V(Q_1) \cup V_0]$ with different parities. Note that $G[V(Q_2) \cup \{x, y\}]$ contains $(k + 2)/2$ admissible (x, y) -paths of lengths with common difference two. It follows that G contains $(k + 2)/2$ cycles of consecutive even lengths, a contradiction. So we conclude that $G[V(Q_1) \cup V_0]$ is bipartite, and similarly, $G[V(Q_2) \cup V_0]$ is bipartite.

If $n_0 \geq 3$, then there exist $x, y \in V_0$ such that x, y are in the same part of $G[V(Q_1) \cup V_0]$ if and only if x, y are in the same part of $G[V(Q_2) \cup V_0]$. This implies the existence of k admissible even cycles, a contradiction. So we conclude that $n_0 = 2$.

Let $V_0 = \{x, y\}$. Since $\delta(G) \geq k - r \geq (k + 4)/2 \geq 3$, we can assume w.l.o.g. that x has at least two neighbors in $G[V(Q_1) \cup \{x, y\}]$. Note that $G[V(Q_1) \cup \{x, y\}] + xy$ is 2-connected. If $G[V(Q_1) \cup \{x, y\}]$ is not 2-connected, then it is a block-chain such that x and y are inner-vertices of two distinct end blocks. In this case we let G_1 be the end block of $G[V(Q_1) \cup \{x, y\}]$ containing x (so $|V(G_1)| \geq 4$), y' be the cut-vertex of $G[V(Q_1) \cup \{x, y\}]$ contained in G_1 , and z be a neighbor of x in G_1 . If $G[V(Q_1) \cup \{x, y\}]$ is 2-connected, then let $G_1 = G[V(Q_1) \cup \{x, y\}]$, $y' = y$ and z be a neighbor of x in G_1 . Now G_1 is bipartite and every vertex in $V(G_1) \setminus \{x, y', z\}$ has degree at least $k - r$ in G_1 . By Theorem 2.2, G_1 has $k - r - 1$ admissible (x, z) -paths. Together with the edge xz , we can get $k - r - 1 \geq k/2$ admissible cycles, which are consecutive even cycles, a contradiction. $\square$

The following simple lemma will be used many times in our proof.

Lemma 2.5. *Let G be a graph, H a subgraph of G and X, Y two disjoint subsets of $V(H)$. Assume that $A = \{a_1, \dots, a_s\}$, $s \geq 1$, is a set of positive integers forming an arithmetic progression, such that for every $x \in X$, $y \in Y$, $i \in [1, s]$, H contains an (x, y) -path of length a_i . Assume also that $B = \{b_1, \dots, b_t\}$, $t \geq 1$, is a set of positive integers forming an arithmetic progression with $b_1 \geq 2$, such that for every $j \in [1, t]$, G contains an (X, Y) -path of length b_j with all internal vertices in $V(G) \setminus V(H)$.*

- (1) *If each of A and B has common difference 1 or 2, then G has $s + t - 1$ admissible cycles.*
- (2) *If the common differences of A and B are 2 and 1, respectively, then G has $2s + t - 2$ admissible cycles.*
- (3) *If the common differences of A and B are 1 and 2, respectively, then G has $s + 2t - 2$ admissible cycles.*

Proof. Consider an arbitrary (X, Y) -path P with all internal vertices in $V(G) \setminus V(H)$, say from $x \in X$ to $y \in Y$ and denote by $p = |P| \geq 2$. For any $i \in [1, s]$, since H contains an (x, y) -path of length a_i , by concatenating such a path with P , we obtain a cycle of length $p + a_i$, implying s cycles of lengths $p + a_1, p + a_2, \dots, p + a_s$. Recall that, for any $j \in [1, t]$, G contains an (X, Y) -path of length b_j with all internal vertices in $V(G) \setminus V(H)$. So we can obtain cycles of all lengths in $\{a_i + b_j : i \in [1, s], j \in [1, t]\} := L$. One can see that L is a an arithmetic progression with common difference 1 or 2, and

$$|L| = \begin{cases} s + t - 1, & \text{if the common differences of } A, B \text{ are both 1 or both 2,} \\ 2s + t - 2, & \text{if the common differences of } A, B \text{ are 2 and 1, respectively,} \\ s + 2t - 2, & \text{if the common differences of } A, B \text{ are 1 and 2, respectively.} \end{cases}$$

The statements then follow from the above observation and the fact that $2s + t - 2 \geq s + t - 1$ for $s \geq 1$ and $s + 2t - 2 \geq s + t - 1$ for $t \geq 1$. $\square$

Lemma 2.6. *Let G be a graph, H a subgraph of G , and $X \subseteq V(H)$. Assume that $A = \{a_1, \dots, a_s\}$, $s \geq 1$, is a set of positive integers forming an arithmetic progression, such that for every two distinct vertices $x, y \in X$ and $i \in [1, s]$, H contains an (x, y) -path of length a_i . Assume also that $B = \{b_1, \dots, b_t\}$, $t \geq 1$, is a set of positive integers forming an arithmetic progression with $b_1 \geq 2$, such that for every $j \in [1, t]$, G contains an X -path of length b_j with all internal vertices in $V(G) \setminus V(H)$.*

- (1) *If each of A, B has common difference 1 or 2, then G has $s + t - 1$ admissible cycles.*
- (2) *If the common differences of A, B are 2 and 1, respectively, then G has $2s + t - 2$ admissible cycles.*
- (3) *If the common differences of A, B are 1 and 2, respectively, then G has $s + 2t - 2$ admissible cycles.*

Proof. The proof is similar to that of Lemma 2.5, and we omit the details. $\square$

3 Admissible cycles

In this section, we set

$$\mathcal{F} := \{K_{s,t} : \min\{s, t\} \geq 2\} \cup \{K_{s,t}^- : \min\{s, t\} \geq 2 \text{ and } \max\{s, t\} \geq 3\},$$

where $K_{s,t}^-$ is obtained from $K_{s,t}$ by removing an arbitrary edge.

Lemma 3.1. *Let F be a graph in $\mathcal{F}$ with bipartition (S, T) . Set $s = |S|$, $t = |T|$ and $r = \min\{s, t\}$. For any two distinct vertices $u, v \in V(F)$, the following statements hold:*

- (1) *If $u, v \in S$ (or $u, v \in T$), then F has (u, v) -paths of consecutive even lengths from 2 to $2r - 2$. Moreover, if $s > r$ (or $t > r$), then F has a (u, v) -path of length $2r$ as well.*
- (2) *If $u \in S$ and $v \in T$, then F has (u, v) -paths of consecutive odd lengths from 1 to $2r - 1$, except that $uv \notin E(G)$, in which case F has (u, v) -paths of consecutive odd lengths from 3 to $2r - 1$.*

Proof. It is not difficult to check the assertion for $F \cong K_{s,t}$. So assume that $F \cong K_{s,t}^-$, say $xy \notin E(F)$ with $x \in S$ and $y \in T$. One can check that the assertion holds for $r \leq 3$. So assume that $r \geq 4$. Let $F' = F + xy$ and $P = w_0 w_1 \dots w_p$ an arbitrary (u, v) -path in F' , where $u = w_0$, $v = w_p$ and $p \geq 2$. It suffices to show the existence of a (u, v) -path of length p in F . If P does not pass through xy , then P itself is a required (u, v) -path. So assume that P passes through xy . Since $p \geq 2$, we can assume w.l.o.g. that $v \notin \{x, y\}$ and that y is the successor of x in P . Let x' be the successor of y . If $S \setminus \{x, x', u, v\} \neq \emptyset$, then let $x^* \in S \setminus \{x, x', u, v\}$. Replacing x with x^* (if $x^* \notin V(P)$) or exchanging x and x^* (if $x^* \in V(P)$) yields a required (u, v) -path. So assume that $S \setminus \{x, x', u, v\} = \emptyset$. It follows that $s = r = 4$ and $u, v \in S$. Thus, $T \setminus V(P) \neq \emptyset$. Let $y^* \in T \setminus V(P)$. Then replacing y with y^* in P yields a required (u, v) -path. $\square$

We recall the following result from [12] and prove a useful lemma.

Theorem 3.2 (Gao, Huo and Ma [12]). *Let G be a 2-connected graph with minimum degree at least $k \geq 2$. If G contains a triangle, then G contains k cycles of consecutive lengths, unless $G \cong K_{k+1}$.*

Lemma 3.3. *Let G be a graph with $x, y \in V(G)$ such that $G - \{x, y\}$ is connected and every end block of $G - \{x, y\}$ has an inner-vertex adjacent to either x or y . If there exist two disjoint edges from $\{x, y\}$ to $V(G) \setminus \{x, y\}$, then $G + xy$ is 2-connected.*

Proof. Clearly, $G + xy$ is connected. Assume that $G + xy$ is not 2-connected. Then $G + xy$ has a cut-vertex z in $\{x, y\}$ or in $V(G) \setminus \{x, y\}$. Since $G - \{x, y\}$ is connected and there exist two disjoint edges from $\{x, y\}$ to

$V(G) \setminus \{x, y\}$, we get that z is in $V(G) \setminus \{x, y\}$ and it is also a cut-vertex of $G - \{x, y\}$, a contradiction to the fact that any end block of $G - \{x, y\}$ has an inner-vertex adjacent to x or y . $\square$

The following lemma states that containing a subgraph $F \in \mathcal{F}$, for example a 4-cycle, is beneficial for finding admissible cycles.

Lemma 3.4. *Let G be a 2-connected graph with minimum degree at least $k \geq 3$ that contains F which is isomorphic to a graph in $\mathcal{F}$. Let (S, T) be the bipartition of $V(F)$ and set $s = |S|$, $t = |T|$, $r = \min\{s, t\}$. If each vertex in $V(G) \setminus V(F)$ has at most $r - 1$ neighbors in F , then G contains k admissible cycles, unless $G \cong K_{k+1}$ or $K_{k, n-k}$.*

Proof. We prove by contradiction. By Lemma 2.3 and $k \geq 3$, we get that G is 3-connected. If G contains a triangle, then by Theorem 3.2, G contains k cycles of consecutive lengths or $G \cong K_{k+1}$, a contradiction. So G contains no triangle. Let $F \in \mathcal{F}$ be a subgraph of G with minimum order satisfying the assumptions of the lemma. Recall that $F \cong K_{s,t}$ or $K_{s,t}^-$, implying that F is an induced subgraph of G .

If $V(G) = V(F)$, then by the minimum degree condition, we have that $r \geq k$ and $G = F$. If $r \geq k + 1$, then any edge $xy \in E(F)$ together with (x, y) -paths of consecutive odd lengths from 3 to $2r - 1$ (by Lemma 3.1), form $r - 1 \geq k$ admissible cycles in G , a contradiction. So $r = k$ and $G \cong K_{k, n-k}$ (by the minimum degree condition), a contradiction. So we assume that $V(G) \setminus V(F) \neq \emptyset$. Let Q be a component of $G - V(F)$. By the 2-connectivity of G , there are two disjoint edges from Q to F .

Claim 1. $r \geq 3$.

Proof. Assume the opposite, say, $s = 2$. Then every vertex in $V(G) \setminus V(F)$ has at most one neighbor in F . If $F \cong K_{2,t}^-$ for some $t \geq 3$, say $\{x_1, x_2\} = S$, $y_1 \in T$ and $x_1y_1 \notin E(F)$, then $x_2y_1 \in E(F)$. It follows that y_1 has exactly one neighbor in F and we can replace F with $F - y_1$ (which is isomorphic to $K_{2,t-1}$), a contradiction to the minimality of $V(F)$. So $F \cong K_{2,t}$ for some $t \geq 2$. Recall that G is 3-connected, implying that $N_T(Q) \neq \emptyset$.

First assume that $N_S(Q) \neq \emptyset$. Let G_1 be the graph obtained from Q by adding two vertices x^* and y^* , and for each vertex $v \in V(Q)$,

- (1) adding an edge vx^* if v has a neighbor in S , and
- (2) adding an edge vy^* if v has a neighbor in T .

Since G is triangle-free and 2-connected, there are two disjoint edges from Q to S and T , respectively, and every end block of Q has an inner-vertex adjacent to S or T . It follows that G_1 has two disjoint edges from Q to $\{x^*, y^*\}$ and every end block of Q has an inner-vertex adjacent to x^* or y^* . By Lemma 3.3, $G_1 + x^*y^*$ is 2-connected. Note that every vertex in G_1 other than x^*, y^* has degree at least k . By Theorem 2.1, G_1 has $k - 1$ admissible (x^*, y^*) -paths. It follows that G has $k - 1$ admissible (S, T) -paths with all internal vertices in Q . By Lemma 3.1, F has two (x, y) -paths of lengths 1 and 3 for any $x \in S$, $y \in T$. By Lemma 2.5, G has k admissible cycles, a contradiction.

Now assume that $N_S(Q) = \emptyset$, i.e., $N_F(Q) \subseteq T$. It follows that $t \geq 3$ since G is 3-connected. Thus, G has two disjoint edges from Q to T , say y_1, y_2 are two end-vertices in T of such two edges. Let (T_1, T_2) be a bipartition of T with $y_1 \in T_1$, $y_2 \in T_2$. Let G_2 be the graph obtained from Q by adding two vertices y_1^* and y_2^* , and for each vertex $v \in V(Q)$,

- (1) adding an edge vy_1^* if v has a neighbor in T_1 , and
- (2) adding an edge vy_2^* if v has a neighbor in T_2 .

Clearly, G_2 has two disjoint edges from Q to $\{y_1^*, y_2^*\}$, and each end block of Q has an inner-vertex adjacent to y_1^* or y_2^* . By Lemma 3.3, $G_2 + y_1^*y_2^*$ is 2-connected. Note that every vertex in G_2 other than y_1^*, y_2^* has degree at least

k . By Theorem 2.1, G_2 has $k - 1$ admissible (y_1^*, y_2^*) -paths, implying that G has $k - 1$ admissible (T_1, T_2) -paths with all internal vertices in Q . Recall that $t \geq 3$, F has two (y_1, y_2) -paths of lengths 2 and 4 for any $y_1 \in T_1$ and $y_2 \in T_2$. By Lemma 2.5, G has k admissible cycles, a contradiction. $\square$

Recall that Q is adjacent to S or T . Now we show that Q is adjacent to only one of S, T .

Claim 2. *Either $N_S(Q) = \emptyset$ or $N_T(Q) = \emptyset$.*

Proof. Assume the opposite that Q is adjacent to both S and T (see Figure 1 (i)). If $F \cong K_{s,t}$, then we define G_1 as in the proof of Claim 1. Similarly to the analysis in the proof of Claim 1, we have that $G_1 + x^*y^*$ is 2-connected. Now every vertex of G_1 other than x^*, y^* has degree at least $k - r + 2$; implying that G_1 has $k - r + 1$ admissible (x^*, y^*) -paths. It follows that G has $k - s + 1$ admissible (S, T) -paths with all internal vertices in Q . By Lemma 3.1, F has (x, y) -paths of consecutive odd lengths from 1 to $2s - 1$ for any $x \in S$, $y \in T$. By Lemma 2.5, G has k admissible cycles, a contradiction.

So assume that $F \cong K_{s,t}^-$, say $x_1y_1 \notin E(F)$, where $x_1 \in S$ and $y_1 \in T$. Recall that G is 3-connected, implying that $N_F(Q) \setminus \{x_1, y_1\} \neq \emptyset$. We can assume w.l.o.g. that $N_T(Q) \setminus \{y_1\} \neq \emptyset$. Let G_3 be the graph obtained from Q by adding two vertices x^* and y^* , and for each vertex $v \in V(Q)$,

- (1) adding an edge vx^* if v has a neighbor in S , and
- (2) adding an edge vy^* if v has a neighbor in $T \setminus \{y_1\}$.

Since $N_S(Q) \neq \emptyset$, $N_T(Q) \setminus \{y_1\} \neq \emptyset$ and G is triangle-free, we see that Q has two disjoint edges from Q to S and $T \setminus \{y_1\}$, respectively. By G being 3-connected, every end block of Q has an inner-vertex adjacent to $V(F) \setminus \{y_1\}$. By Lemma 3.3, $G_3 + x^*y^*$ is 2-connected. Note that every vertex in G_3 other than x^*, y^* has degree at least $\min\{k - 1, k - r + 2\} = k - r + 2$ (since $r \geq 3$). By Theorem 2.1, G_3 has $k - r + 1$ admissible (x^*, y^*) -paths, implying that G has $k - r + 1$ admissible $(S, T \setminus \{y_1\})$ -paths with all internal vertices in Q . By Lemma 3.1, F has (x, y) -paths of consecutive odd lengths from 1 to $2s - 1$ for any $x \in S$, $y \in T \setminus \{y_1\}$. By Lemma 2.5, G has k admissible cycles, a contradiction. $\square$

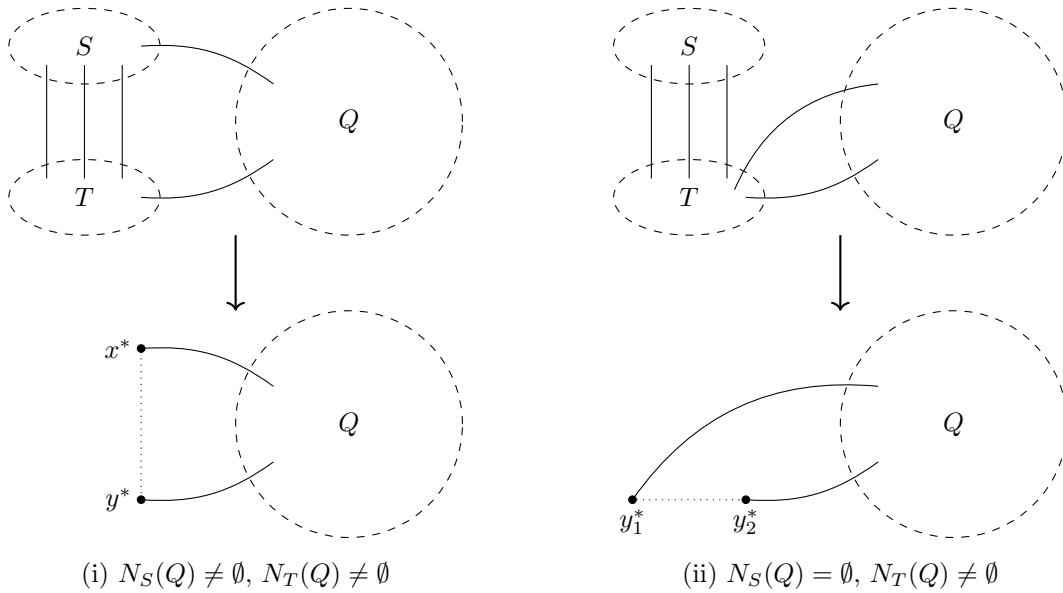


Figure 1: Constructions of S, T, Q and G_i for $i \in \{1, 2, 3, 4\}$ in Lemma 3.4.

By Claim 2, we can assume w.l.o.g. that $N_S(Q) = \emptyset$ (see Figure 1 (ii)). By G being 2-connected, G has two disjoint edges from Q to T , say y_1, y_2 are the end-vertices of such two edges in T . Let (T_1, T_2) be a bipartition of T with $y_1 \in T_1, y_2 \in T_2$.

Let G_4 be the graph obtained from Q by adding two vertices y_1^* and y_2^* , and for each vertex $v \in V(Q)$,

- (1) adding an edge vy_1^* if $d_{T_1}(v) \geq 1$ or $d_T(v) \geq 2$, and
- (2) adding an edge vy_2^* if $d_{T_2}(v) \geq 1$ or $d_T(v) \geq 2$.

We remark that if $v \in V(Q)$ has at least two neighbors in T , then both $vy_1^*, vy_2^* \in E(G_4)$. Recall that G has two disjoint edges from Q to T_1, T_2 , respectively, implying that G_4 has two disjoint edges from Q to $\{y_1^*, y_2^*\}$. Also note that every end block of Q has an inner-vertex adjacent to T in G , implying that every end block of Q has an inner-vertex adjacent to y_1^* or y_2^* in G_4 . By Lemma 3.3, $G_4 + y_1^*y_2^*$ is 2-connected. Note that every vertex in G_4 other than y_1^*, y_2^* has degree at least $\min\{k, k - (r - 1) + 2\} = k - r + 3$. By Theorem 2.1, G_4 has $k - s + 2$ admissible (y_1^*, y_2^*) -paths, implying that G has $k - s + 2$ admissible T -paths with all internal vertices in Q . By Lemma 3.1, F has (y'_1, y'_2) -paths of consecutive even lengths from 2 to $2r - 2$ for each two distinct vertices $y'_1, y'_2 \in T$. By Lemma 2.6, G has k admissible cycles, a contradiction. $\square$

Lemma 3.5. *Let G be a 2-connected graph with minimum degree at least $k \geq 3$. If G contains a 4-cycle, then G contains k admissible cycles, except that $G \cong K_{k+1}$ or $K_{k,n-k}$.*

Proof. If G contains a triangle, then, by Theorem 3.2, G contains k admissible cycles or $G \cong K_{k+1}$, as desired. So assume that G does not contain a triangle. Assume also that G does not contain k admissible cycles. Let s be the maximum integer such that G contains a $K_{s,s}$. Then $s \geq 2$ and G does not contain a $K_{s+1,s+1}$.

Suppose first that G contains a subgraph $F \cong K_{s+1,s+1}^-$ with bipartition (S, T) . By the absence of $K_{s+1,s+1}$, every vertex in $V(G) \setminus V(F)$ has at most s neighbors in S and at most s neighbors in T . If a vertex in $V(G) \setminus V(F)$ has neighbors in both S and T , then it has exactly $2 \leq s$ neighbors in $V(F)$ since G is triangle-free. This implies that every vertex in $V(G) \setminus V(F)$ has at most s neighbors in $V(F)$. By Lemma 3.4, G contains k admissible cycles or G is isomorphic to K_{k+1} or $K_{k,n-k}$, a contradiction. So G does not contain a $K_{s+1,s+1}^-$.

Suppose now that G contains a $K_{s,s+1}$. Let $F \cong K_{s,t}$ be a subgraph of G with bipartition (S, T) having maximum $t = |T|$, where $t \geq s + 1$. By G being triangle-free, every vertex in $V(G) \setminus V(F)$ cannot have neighbors in both S and T . By the maximality of t , every vertex in $V(G) \setminus V(F)$ has at most $s - 1$ neighbors in S . If a vertex in $V(G) \setminus V(F)$ has s neighbors in T , then G contains a $K_{s+1,s+1}^-$, a contradiction. So every vertex in $V(G) \setminus V(F)$ has at most $s - 1$ neighbors in T . It follows that every vertex in $V(G) \setminus V(F)$ has at most $s - 1$ neighbors in $V(F)$. By Lemma 3.4, G contains k admissible cycles or G is isomorphic to K_{k+1} or $K_{k,n-k}$, a contradiction. So G does not contain a $K_{s,s+1}$.

Finally, let $F \cong K_{s,s}$ be a subgraph of G with bipartition (S, T) . By G being triangle-free, every vertex in $V(G) \setminus V(F)$ cannot have neighbors in both S and T . If a vertex in $V(G) \setminus V(F)$ has s neighbors in S (or in T), then G contains a $K_{s,s+1}$, a contradiction. This implies that every vertex in $V(G) \setminus V(F)$ has at most $s - 1$ neighbors in $V(F)$. By Lemma 3.4, G contains k admissible cycles or G is isomorphic to K_{k+1} or $K_{k,n-k}$, a contradiction. $\square$

By Theorem 3.2 and Lemma 3.4 in order to prove Theorem 1.3, we may assume that G contains neither a triangle nor a 4-cycle. Moreover, by Lemma 2.3 the connectivity of G is at least $(k + 2)/2 \geq 3$. Therefore, the following results imply that for $k \geq 6$ Theorem 1.3 holds.

Theorem 3.6 (Gao, Huo, Liu and Ma [13]). *Let $k \geq 5$ be an integer and G a 3-connected bipartite graph of minimum degree at least k . If G contains no 4-cycles, then G contains k admissible cycles.*

Theorem 3.7 (Gao, Huo, Liu and Ma [13]). *Let G be a 3-connected non-bipartite graph with minimum degree at least $k \geq 6$. If G does not contain a triangle, then G contains k admissible cycles. Furthermore, if $k \geq 8$, then G contains k cycles of consecutive lengths; and if $k \in \{6, 7\}$, then G contains cycles of all lengths modulo k .*

To prove Theorem 1.3 in the remaining case $k \in \{4, 5\}$, we use the following results.

Lemma 3.8. *Let G be a 3-connected graph, C a cycle of G , and Q a component of $G - V(C)$. If G' is the graph obtained from G by contracting all edges in Q , then G' is 3-connected.*

Proof. Denote by w the vertex of G' obtained by contracting Q . Suppose that $\{x, y\}$ is a 2-cut of G' . If $w \notin \{x, y\}$, then $\{x, y\}$ is also a 2-cut of G , a contradiction. So we have that $w \in \{x, y\}$. It follows that $G - Q = G' - w$ has a cut-vertex, a contradiction. $\square$

Theorem 3.9 (Gao, Li, Ma and Xie [14]). *Let G be a graph with $x, y \in V(G)$ such that $G + xy$ is 2-connected. If every vertex of G other than x and y has degree at least k , and any edge $uv \in E(G)$ with $\{u, v\} \cap \{x, y\} = \emptyset$ has degree sum $d(u) + d(v) \geq 2k + 1$, then G has k admissible (x, y) -paths of consecutive lengths or $k - 1$ admissible (x, y) -paths whose lengths form an arithmetic progression with common difference two.*

Lemma 3.10. *Let $k \in \{4, 5\}$. Let G be a graph, H a subgraph of G and X, Y two disjoint subsets of $V(H)$. Assume that a_1, a_2 are two integers with $a_2 - a_1 = 1$ or 4 , such that for every $x \in X, y \in Y, i \in \{1, 2\}$, H has an (x, y) -path of length a_i . Assume also that G has $k - 1$ admissible (X, Y) -paths of consecutive lengths or $k - 2$ admissible (X, Y) -paths whose lengths form an arithmetic progression with common difference two, with all internal vertices in $V(G) \setminus V(H)$. Then G has k admissible cycles.*

Proof. If $a_2 - a_1 = 1$, then the assertion is deduced by Lemma 2.5. Now assume that $a_2 - a_1 = 4$. Consider an arbitrary (X, Y) -path P with all internal vertices in $V(G) \setminus V(H)$, say of length p and with end-vertices $x \in X, y \in Y$. Together with two (x, y) -paths in H of lengths a_1 and $a_2 = a_1 + 4$, respectively, we can get two cycles of G of lengths $p + a_1$ and $p + a_1 + 4$.

Suppose first that G has $k - 1$ admissible (X, Y) -paths of consecutive lengths, say of lengths $p + i$ for $i = 0, \dots, k - 2$. If $k = 4$, then G has cycles of lengths $p + a_1, p + a_1 + 2, p + a_1 + 4, p + a_1 + 6$. If $k = 5$, then G has cycles of lengths $p + a_1, p + a_1 + 1, p + a_1 + 2, p + a_1 + 3, p + a_1 + 4$. Suppose now that G has $k - 2$ admissible (X, Y) -paths whose lengths form an arithmetic progression with common difference two, say of lengths $p + 2i$, $i = 0, \dots, k - 3$. If $k = 4$, then G has cycles of lengths $p + a_1, p + a_1 + 2, p + a_1 + 4, p + a_1 + 6$. If $k = 5$, then G has cycles of lengths $p + a_1, p + a_1 + 2, p + a_1 + 4, p + a_1 + 6, p + a_1 + 8$. In each case, G has k admissible cycles, as desired. $\square$

Let C be a cycle. If C is even, then we say two vertices $x, y \in V(C)$ are *diagonal* if $d_C(x, y) = |C|/2$; *quasi-diagonal* if $d_C(x, y) = |C|/2 - 1$; and *sub-quasi-diagonal* if $d_C(x, y) = |C|/2 - 2$. While if C is odd, then we say $x, y \in V(C)$ are *quasi-diagonal* if $d_C(x, y) = (|C| - 1)/2$. Note that for any vertex $x \in V(C)$, $|C| \geq 5$, x has exactly two quasi-diagonal vertices; and if $|C|$ is even, then x has exactly two sub-quasi-diagonal vertices.

Lemma 3.11. *Let G be a 2-connected graph with minimum degree at least k , where $k \in \{4, 5\}$. Then G contains k admissible cycles, unless $G \cong K_{k+1}$ or $K_{k,n-k}$.*

Proof. We prove by contradiction. We first deal with the case that G has a 2-cut (a vertex cut of two vertices).

Claim 1. *G is 3-connected.*

Proof. Let $\{x, y\}$ be a 2-cut of G , and let Q_1, Q_2 be two components of $G - \{x, y\}$. Set $G_i = G[V(Q_i) \cup \{x, y\}]$, $i = 1, 2$. Since G is 2-connected, we see that $G_i + xy$ is 2-connected. Note that every vertex of G_i other than

x, y has degree at least k . By Theorem 2.1, G_i contains $k - 1$ admissible (x, y) -paths. By Lemma 2.5, G contains $2(k - 1) - 1 \geq k$ admissible cycles, a contradiction. $\square$

By Theorem 3.2 and Lemma 3.5, we can assume that G does not contain a triangle or a 4-cycle. We now show that G does not contain a 5-cycle.

Claim 2. G contains no 5-cycle.

Proof. Assume the opposite that G contains a 5-cycle. Define F_r to be the graph obtained by joining two vertices x_1, x_2 with r paths of length 3 and one path of length 2 such that all the paths are pairwise internally disjoint, see Figure 2 for an illustration. Note that F_1 is a 5-cycle.

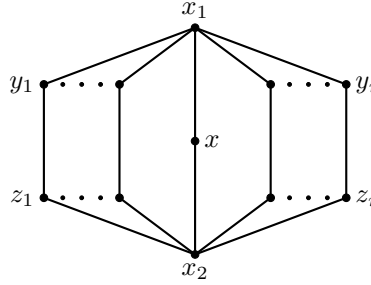


Figure 2: Graph F_r .

Claim 2.1. If C is a 5-cycle of G , then

- (1) C is an induced cycle of G ,
- (2) every vertex outside C has at most one neighbor in C , and
- (3) if P is a $V(C)$ -path in G of length 3, then its two end-vertices have distance exactly 2 in C (that is, $G[V(C) \cup V(P)] \cong F_2$).

Proof. All three statements can be deduced from the fact that G has neither a triangle nor a 4-cycle. $\square$

Claim 2.2. G contains no F_3 .

Proof. Suppose that $F \cong F_r$ is a subgraph of G with r maximum and $r \geq 3$. We label the vertices of F as in Figure 2. By Claim 2.1, F is an induced subgraph of G and any vertex outside F can have at most one neighbor in any 5-cycle in F . We further claim that any vertex outside F can have at most one neighbor in F . If $v \in V(G) \setminus V(F)$ is adjacent to both y_1 and y_2 , then a 4-cycle appears; if v is adjacent to both y_1 and z_2 , then we can get a contradiction by finding five cycles of lengths from 5 to 9, namely,

$$x_1 y_1 z_1 x_2 x x_1, x_1 y_1 v z_2 x_2 x x_1, x_1 y_1 v z_2 x_2 z_3 y_3 x_1, x_1 y_2 z_2 v y_1 z_1 x_2 x x_1, x_1 y_2 z_2 v y_1 z_1 x_2 z_3 y_3 x_1,$$

a contradiction. The other cases can be proved similarly. So we conclude that every vertex outside F has at most one neighbor in F .

Set $T = \{y_i, z_i : 1 \leq i \leq r\}$. Recall that G is 3-connected. There is a path from x to T avoiding x_1, x_2 . It follows that $G - V(F)$ has a component Q such that $x \in N(Q)$ and $N_T(Q) \neq \emptyset$.

We claim that every end block of Q has an inner-vertex adjacent to $\{x\} \cup T$. Suppose otherwise that B is an end block of Q , c is the cut-vertex of Q contained in B , and $B - c$ is not adjacent to $\{x\} \cup T$. By Claim 1, $N_F(B - c) = \{x_1, x_2\}$. Recall that every vertex outside F has at most one neighbor in F . There are two

disjoint edges from $B - c$ to $\{x_1, x_2\}$. Let $G_0 = G[V(B) \cup \{x_1, x_2\}]$. It follows that $G_0 + x_1x_2$ is 2-connected and every vertex in $B - c$ has degree at least k in G_0 . By Theorem 2.2, G_0 has $k - 1$ admissible (x_1, x_2) -paths. Note that F has two (x_1, x_2) -paths of lengths 2 and 3, respectively. By Lemma 2.5, G has k admissible cycles, a contradiction. Now we conclude that every end block of Q has an inner-vertex adjacent to $\{x\} \cup T$.

Let G_1 be the graph obtained from $G[V(Q) \cup \{x\}]$ by adding a vertex x^* and all edges in $\{vx^* : v \in N_Q(T)\}$. By Lemma 3.3, $G_1 + xx^*$ is 2-connected. Recall that every vertex in Q has at most one neighbor in F . If there is an edge v_1v_2 such that $v_1x_1, v_2x_2 \in E(G)$, then $F \cup x_1v_1v_2x_2$ is a F_{r+1} , contradicting the choice of F . It follows that every vertex in Q has degree at least $k - 1$ in G_1 , and every pair of adjacent vertices in Q has degree sum at least $2k - 1$ in G_1 .

By Theorem 3.9, G_1 has $k - 1$ admissible (x, x^*) -paths of consecutive lengths or $k - 2$ admissible (x, x^*) -paths whose lengths form an arithmetic progression with common difference 2. This implies that G has $k - 1$ admissible (x, T) -paths of consecutive lengths or $k - 2$ admissible (x, T) -paths whose lengths form an arithmetic progression with common difference 2, with all internal vertices in Q . Note that F contains two (x, x') -paths of lengths 2 and 3 for each $x' \in T$. By Lemma 3.10, G contains k admissible cycles, a contradiction. $\square$

Claim 2.3. *G contains no Petersen graph.*

Proof. Let F be a subgraph of G isomorphic to the Petersen graph and let Q be a component of $G - V(F)$. By Claim 1, $N_F(Q)$ has at least three vertices, and two of them are nonadjacent. Thus, there exists $x \in N_F(Q)$ such that for $T := V(F) \setminus (\{x\} \cup N_F(x))$ it holds $N_T(Q) \neq \emptyset$.

Note that each two vertices of F are contained in a common 5-cycle. By Claim 2.1, every vertex outside F has at most one neighbor in F . Now we claim that for each two adjacent vertices outside F , at least one of them has no neighbor in F . Suppose that $v_1v_2 \in E(Q)$ such that $v_1x_1, v_2x_2 \in E(G)$ with $x_1, x_2 \in V(F)$. Since G has neither a triangle nor a 4-cycle, x_1, x_2 are at distance 2 in F . Then $F \cup x_1v_1v_2x_2$ contains a F_3 , contradicting Claim 2.2. So we conclude that for each two adjacent vertices outside F , at least one of them has no neighbor in F .

Now we claim that every end block of Q has an inner-vertex adjacent to $\{x\} \cup T$. Suppose that B is an end block of Q , c is the cut-vertex of Q contained in B , such that $B - c$ is not adjacent to $\{x\} \cup T$. Then $N_F(B - c) \subseteq N_F(x)$. Set $N_F(x) = \{y_1, y_2, y_3\}$ such that $y_1 \in N_F(B - c)$. Let G_0 the graph obtained from $G[V(B) \cup \{y_1\}]$ by adding a vertex y^* and all edges in $\{vy^* : v \in N_B(y_2) \cup N_B(y_3)\}$. It follows that $G_0 + y_1y^*$ is 2-connected and every vertex in $B - c$ has degree at least k in G_0 . By Theorem 2.2, G_0 has $k - 1$ admissible (y_1, y^*) -paths, implying that G has $k - 1$ admissible $(y_1, \{y_2, y_3\})$ -paths with all internal vertices in B . Note that F contains two (y_1, y_i) -paths of lengths 2 and 3, respectively, for each $i = 2, 3$. By Lemma 2.5, G has k admissible cycles, a contradiction. Now we conclude that every end block of Q has an inner-vertex adjacent to $\{x\} \cup T$.

Let G_1 be the graph obtained from $G[V(Q) \cup \{x\}]$ by adding a vertex x^* and all edges in $\{vx^* : v \in N_Q(T)\}$. By Lemma 3.3, $G_1 + xx^*$ is 2-connected. Recall that every vertex in Q has at most one neighbor in F , and that for every two adjacent vertices in Q , at most one of them has a neighbor in F . It follows that every vertex in Q has degree at least $k - 1$ in G_1 , and every pair of adjacent vertices in Q has degree sum at least $2k - 1$ in G_1 . By Theorem 3.9, G_1 has $k - 1$ admissible (x, x^*) -paths of consecutive lengths or $k - 2$ admissible (x, x^*) -paths whose lengths form an arithmetic progression with common difference 2. It follows that G has $k - 1$ admissible (x, T) -paths of consecutive lengths or $k - 2$ admissible (x, T) -paths whose lengths form an arithmetic progression with common difference 2, with all internal vertices in Q . Note that F contains two (x, x') -paths of lengths 2 and 3 for each $x' \in T$. By Lemma 3.10, G contains k admissible cycles, a contradiction. $\square$

Now we let C be a 5-cycle of G , say $C = x_1x_2x_3x_4x_5x_1$.

Claim 2.4. *There are two vertices $y_5, z_2 \in V(G) \setminus V(C)$ such that $x_5y_5, y_5z_2, z_2x_2 \in E(G)$.*

Proof. Suppose otherwise. Since G is 3-connected, there is a component Q of $G - V(C)$ such that $x_1 \in N_C(Q)$ and $\{x_3, x_4\} \cap N_C(Q) \neq \emptyset$. Let G_1 be the graph obtained from $G[V(Q) \cup \{x_1\}]$ by adding a vertex x^* and all edges in $\{vx^* : v \in N_Q(\{x_3, x_4\})\}$. Similarly to the proof in Claim 2.2, we see that $G_1 + x_1x^*$ is 2-connected. Recall that every vertex in Q has at most one neighbor in C , and at least one of each pair of adjacent vertices in Q is not adjacent to $\{x_2, x_5\}$. It follows that every vertex in Q has degree at least $k - 1$ in G_1 , and every pair of adjacent vertices in Q has degree sum at least $2k - 1$ in G_1 . By Theorem 3.9, G_1 has $k - 1$ admissible (x_1, x^*) -paths of consecutive lengths or $k - 2$ admissible (x_1, x^*) -paths whose lengths form an arithmetic progression with common difference 2. It follows that G has $k - 1$ admissible $(x_1, \{x_3, x_4\})$ -paths of consecutive lengths or $k - 2$ admissible $(x_1, \{x_3, x_4\})$ -paths whose lengths form an arithmetic progression with common difference two, with all internal vertices in Q . Note that C has two (x_1, x') -paths of lengths 2 and 3 for each $x' \in \{x_3, x_4\}$. By Lemma 3.10, G contains k admissible cycles, a contradiction. $\square$

By symmetry and Claim 2.4, we obtain that for each $i = 1, \dots, 5$, there are two vertices $y_i, z_{i+2} \in V(G) \setminus V(C)$ such that $x_iy_i, y_iz_{i+2}, z_{i+2}x_{i+2} \in E(G)$, where the indices are taken modulo 5. We remark that possibly $y_i = z_i$ (and for distinct i, j , $\{y_i, z_i\} \cap \{y_j, z_j\} = \emptyset$).

We define the hypo-Petersen graphs as graphs depicted in Figure 3. We say a hypo-Petersen graph is a proper one if it is not a Petersen graph. We remark that every proper hypo-Petersen graph contains cycles of consecutive lengths from 5 to 9, see Figure 4.

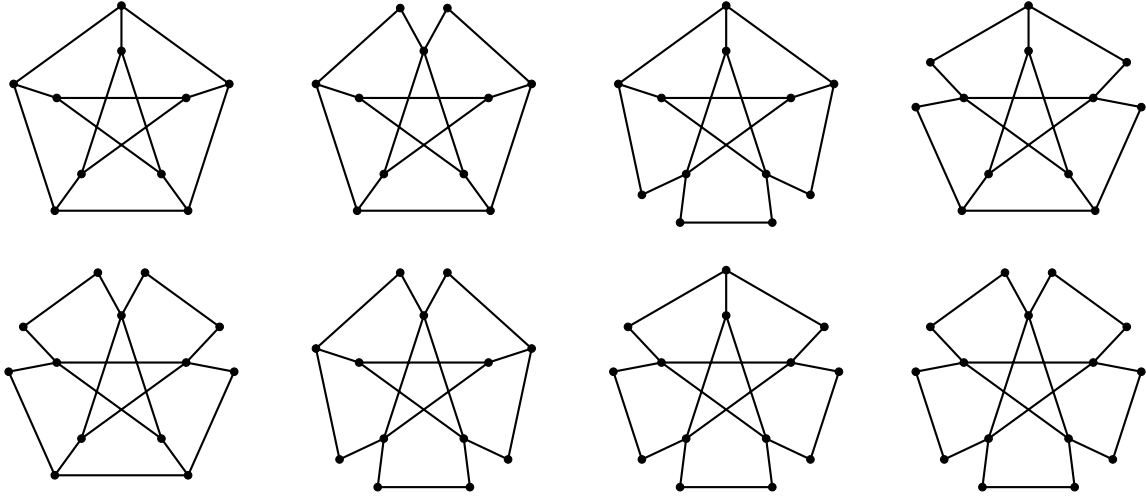


Figure 3: Hypo-Petersen graphs.

Now by Claims 2.4 and 2.3, $H = G[\{x_i, y_i, z_i : i = 1, \dots, 5\}]$ is a proper hypo-Petersen graph. It follows that H , and then, G , has cycles of lengths 5, 6, 7, 8, 9, a contradiction. $\square$

Let C be a shortest cycle of G . By Claim 2, $|C| \geq 6$. Note that any vertex outside C has at most one neighbor in C as otherwise it creates a cycle of length at most $|C|/2 + 2 < |C|$, contradicting the minimality of C . Since G is 3-connected, every end block of a component of $G - V(C)$ has at least two neighbors in C .

Claim 3. *Let Q be a component of $G - V(C)$ and $x_1, x_2 \in N_C(Q)$, say $x_1y_1, x_2y_2 \in E(G)$ with $y_1, y_2 \in V(Q)$.*

(1) *If Q is 2-connected, then G has $k - 2$ admissible (x_1, x_2) -paths with all internal vertices in Q .*

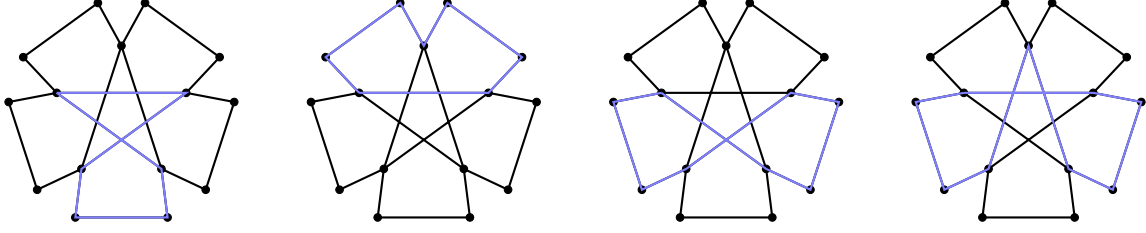


Figure 4: The 6-, 7-, 8-, 9-cycles in a hypo-Petersen graph.

- (2) If Q is not 2-connected, B is an end block of Q and $y_1, y_2 \in V(B)$, then G has $k-2$ admissible (x_1, x_2) -paths with all internal vertices in B .
- (3) If y_1, y_2 are inner-vertices of two distinct end blocks of Q , then G has $2k-5$ admissible (x_1, x_2) -paths with all internal vertices in Q .

Proof. Note that every vertex in Q has at most one neighbor in C . It follows that $y_1 \neq y_2$ and every vertex in Q has degree at least $k-1$ in Q .

(1) Set $G_1 = G[V(Q) \cup \{x_1, x_2\}]$. Then $G_1 + x_1x_2$ is 2-connected, and every vertex in Q has degree at least $k-1$ in G_1 . By Lemma 2.1, G_1 contains $k-2$ admissible (x_1, x_2) -paths, as desired.

(2) Let c be the cut-vertex of Q contained in B . Set $G_1 = G[V(B) \cup \{x_1, x_2\}]$. Then $G_1 + x_1x_2$ is 2-connected, and every vertex in $B-c$ has degree at least $k-1$ in G_1 . By Lemma 2.2, G_1 contains $k-2$ admissible (x_1, x_2) -paths, as desired.

(3) Let B_i be the end block of Q containing y_i , let c_i be the cut-vertex of Q contained in B_i , and let $G_i = G[V(B_i) \cup \{x_i\}]$, $i = 1, 2$. So $G_i + x_i c_i$ is 2-connected, and every vertex in $B_i - c_i$ has degree at least $k-1$ in G_i . By Lemma 2.1, G_i contains $k-2$ admissible (x_i, c_i) -paths, $i = 1, 2$. Together with a (c_1, c_2) -path in Q , we can get $2k-5$ admissible (x_1, x_2) -paths of G with all internal vertices in Q , as desired. $\square$

Claim 4. *Each component of $G - V(C)$ is 2-connected.*

Proof. Let Q be a component of $G - V(C)$ that is not 2-connected. If $k = 5$, then by Claim 3 (3), G has $2k-5 \geq 5$ admissible (x_1, x_2) -paths with all internal vertices in Q , where x_1, x_2 are two vertices in C as in Claim 3 (3). Together with an (x_1, x_2) -path in C , we get 5 admissible cycles in G , a contradiction. So we assume that $k = 4$.

Let B_1, B_2 be two end blocks of Q , and let c_1, c_2 be the cut-vertices of Q contained in B_1, B_2 , respectively. We choose B_1, B_2 such that the distance of c_1, c_2 in Q is as small as possible.

Claim 4.1. *$G - V(C)$ has only one component Q , and if Q has a third end block other than B_1, B_2 , then Q has only one cut-vertex $c_1 (= c_2)$.*

Proof. If $G - V(C)$ has a second component Q' , then let $B' = Q'$ if Q' is 2-connected, and let B' be an end block of Q' if Q' is not 2-connected. If Q has a third end block other than B_1, B_2 and c_1 is not the only cut-vertex of Q , then we can choose an end block B' of Q that is disjoint with B_1, B_2 . Let P be a shortest path of Q from c_1 to c_2 . By the choice of B_1, B_2 , we see that P and B' are disjoint.

By the 2-connectivity of G , let $x_1, x_2 \in V(C)$ that has neighbors in $B_1 - c_1$ and $B_2 - c_2$, respectively, $x_1 \neq x_2$; and let x'_1, x'_2 be two neighbors of B' in C . By Claim 3, there are $2k-5 = 3$ admissible (x_1, x_2) -paths with all internal vertices in $B_1 \cup B_2 \cup P$, and there are $k-2 = 2$ admissible (x'_1, x'_2) -paths with all internal vertices in B' . Together with two disjoint paths in C from $\{x_1, x_2\}$ to $\{x'_1, x'_2\}$, we can get four admissible cycles in G , a contradiction. $\square$

Claim 4.2. Q has exactly two end blocks B_1, B_2 .

Proof. Suppose otherwise. Let $B_1, B_2, \dots, B_s$, $s \geq 3$, be the end blocks of Q . By Claim 4.1, $c_1 = c_2 := c$ is the only cut-vertex of G .

Let $x_1, x_2 \in V(C)$ be two quasi-diagonal vertices. We claim that $N_Q(x_1) \cup N_Q(x_2) \subseteq V(B_i)$ for some $i \in [1, s]$. Since $\delta(G) \geq 4$ and Q is the only component of $G - V(C)$, every vertex in C has at least two neighbors in Q , and at least one neighbor in $Q - c$. Suppose $x_1y_1, x_2y_2 \in E(G)$ with $y_1 \in V(B_i) \setminus \{c\}$ and $y_2 \in V(B_j) \setminus \{c\}$, $i \neq j$. By Claim 3 (3), G has $2k - 5 = 3$ admissible (x_1, x_2) -paths with all internal vertices in Q . Note that C has two admissible (x_1, x_2) -paths. By Lemma 2.5, G has four admissible cycles, a contradiction. Thus, as we claimed, for each two quasi-diagonal vertices in C , their neighbors in Q are contained in one end block of Q .

Define $Qdi(C)$ the graph on $V(C)$ such that two vertices are adjacent in $Qdi(C)$ if and only if they are quasi-diagonal in C . If C is odd or $|C| \equiv 0 \pmod{4}$, then $Qdi(C)$ is a cycle; and if $|C| \equiv 2 \pmod{4}$, then $Qdi(C)$ is the union of two disjoint cycles of length $|C|/2$. By the analysis above, we see that for each component D of $Qdi(C)$, $N_Q(V(D)) \subseteq V(B_i)$ for some end block B_i of Q . It follows that $|C| \equiv 2 \pmod{4}$ and Q has exactly two end blocks, a contradiction. $\square$

Claim 4.3. Either $c_1 = c_2$ or $c_1c_2 \in E(Q)$ is a cut-edge of Q .

Proof. G is 2-connected, so it contains two disjoint edges $x_1y_1, x_2y_2 \in E(G)$ with $x_1, x_2 \in V(C)$, $y_1 \in V(B_1) \setminus \{c_1\}$ and $y_2 \in V(B_2) \setminus \{c_2\}$. Suppose that $c_1 \neq c_2$ and c_1c_2 is not a cut-edge of Q . Then $V(Q) \setminus (V(B_1) \cup V(B_2)) \neq \emptyset$.

Let $G_1 = G[V(B_1) \cup \{x_1\}]$, $G_2 = G[V(B_2) \cup \{x_2\}]$, and $G_3 = G[(V(Q) \setminus (V(B_1) \cup V(B_2))) \cup \{c_1, c_2\}]$. Note that $G_1 + x_1c_1$ and $G_2 + x_2c_2$ are 2-connected. By Claim 4.2, Q is a block-chain, implying that $G_3 + c_1c_2$ is 2-connected. Since every vertex in Q has degree at least 3, G_1 has two admissible (x_1, c_1) -paths, G_3 has two admissible (c_1, c_2) -paths, and G_2 has two admissible (c_2, x_2) -paths. It follows that G has four admissible (x_1, x_2) -paths with all internal vertices in Q . Together with an (x_1, x_2) -path in C , we get four admissible cycles of G , a contradiction. $\square$

We set $U_i = V(B_i) \cup \{c_1, c_2\}$, $i = 1, 2$. We claim that for each two quasi-diagonal vertices $x_1, x_2 \in V(C)$, $N_Q(x_1) \cup N_Q(x_2) \subseteq U_i$ for some $i = 1, 2$. Recall that every vertex in C has at least two neighbors in Q and G does not contain a triangle. It follows that every vertex in C has at least one neighbor in $V(Q) \setminus \{c_1, c_2\}$. Suppose $x_1y_1, x_2y_2 \in E(G)$ with $y_i \in V(B_i) \setminus \{c_i\}$, $i = 1, 2$. By Claim 3 (3), G has three admissible (x_1, x_2) -paths with all internal vertices in Q . Since C has two admissible (x_1, x_2) -paths, by Lemma 2.5, G has four admissible cycles, a contradiction. Thus, as we claimed, $N_Q(x_1) \cup N_Q(x_2) \subseteq U_i$. It follows, as in the proof of Claim 4.2, that for each component D of $Qdi(C)$, $N_Q(V(D)) \subseteq U_i$ for some $i = 1, 2$. This implies that $Qdi(Q)$ has two components D_1, D_2 (and $|C| \equiv 2 \pmod{4}$), such that $N_Q(V(D_1)) \subseteq U_1$ and $N_Q(V(D_2)) \subseteq U_2$.

Let x_1, x_2 be two sub-quasi-diagonal vertices in C , say $x_1 \in V(D_1)$, $x_2 \in V(D_2)$. It follows that x_1 has a neighbor $y_1 \in V(B_1) \setminus \{c_1\}$ and x_2 has a neighbor $y_2 \in V(B_2) \setminus \{c_2\}$. By Claim 3 (3), G has three admissible (x_1, x_2) -paths with all internal vertices in Q . Note that C has two (x_1, x_2) -paths whose lengths differ by four. By Lemma 3.10, G has four admissible cycles, a contradiction. $\square$

Claim 5. $|C| \geq 7$.

Proof. Suppose that $|C| = 6$. Let Q be a component of $G - V(C)$. By Claim 4, Q is 2-connected. First suppose that $N_C(Q)$ contains two adjacent vertices (that is, two sub-quasi-diagonal vertices). Let $x \in N_C(Q)$, $T = N_C(x)$ such that $N_T(Q) \neq \emptyset$. Let G_1 be the graph obtained from $G[V(Q) \cup \{x\}]$ by adding one vertex x^* and all edges in $\{vx^* : v \in V(Q), N_T(v) \neq \emptyset\}$. Clearly $G_1 + xx^*$ is 2-connected, and every vertex in Q has degree at least $k - 1$ in G_1 . Moreover, for every edge $v_1v_2 \in E(Q)$, either v_1 or v_2 has no neighbors in $V(C) \setminus (\{x\} \cup T)$; otherwise a

cycle of length at most 5 appears. This implies that v_1, v_2 have degree sum at least $2k - 1$ in G_1 . By Theorem 3.9, G_1 has $k - 1$ admissible (x, x^*) -paths of consecutive lengths or $k - 2$ admissible (x, x^*) -paths whose lengths form an arithmetic progression with common difference two. It follows that G has $k - 1$ admissible (x, T) -paths of consecutive lengths or $k - 2$ admissible (x, T) -paths whose lengths form an arithmetic progression with common difference two, with all internal vertices in Q . Notice that C has two (x, x') -paths of lengths 1 and 5, respectively, for each $x' \in T$. By Lemma 3.10, G has k admissible cycles, a contradiction.

Now suppose that $N_C(Q)$ contains no two adjacent vertices. Since G is 3-connected, $|N_C(Q)| \geq 3$. It follows that $N_C(Q)$ consists of three vertices in C each two of which have distance 2 (and thus quasi-diagonal in C). Set $N_C(Q) = \{x_1, x_2, x_3\}$. Let G_2 be the graph obtained from $G[V(Q) \cup \{x_1\}]$ by adding one vertex x^* and all edges in $\{vx^* : v \in N_Q(x_2) \cup N_Q(x_3)\}$. It follows that $G_2 + xx^*$ is 2-connected, and every vertex in Q has degree at least k in G_2 . Thus, G_2 has $k - 1$ admissible (x, x^*) -paths, implying that G has $k - 1$ admissible $(x, \{x_2, x_3\})$ -paths. Note that for each $i \in \{2, 3\}$, C has two (x_1, x_i) -paths of lengths 2 and 4 respectively. By Lemma 2.5, G has k admissible cycles, a contradiction. $\square$

Claim 6. *Let Q be a component of $G - V(C)$ and $x, y \in N_C(Q)$. Then $G[V(Q) \cup \{x, y\}]$ has either $k - 1$ admissible (x, y) -path of consecutive lengths, or $k - 2$ admissible (x, y) -paths whose lengths form an arithmetic progression with common difference two.*

Proof. By Claim 5, $|C| \geq 7$. Note that every vertex in Q has degree at least $k - 1$ in Q . For each edge $v_1v_2 \in E(Q)$, either $N_C(v_1) = \emptyset$ or $N_C(v_2) = \emptyset$; otherwise we find a cycle shorter than C . It follows that every two adjacent vertices in Q has degree sum at least $2k - 1$ in Q . Now the assertion can be deduced by Theorem 3.9. $\square$

Claim 7. *If C is odd, then let $x_1, x_2 \in V(C)$ be two quasi-diagonal vertices, while if C is even, then let $x_1, x_2 \in V(C)$ be two sub-quasi-diagonal vertices. For a component Q of $G - V(C)$, either $N_Q(x_1) = \emptyset$ or $N_Q(x_2) = \emptyset$.*

Proof. Suppose that $N_Q(x_1) \neq \emptyset$ and $N_Q(x_2) \neq \emptyset$. Let $G_1 = G[V(Q) \cup \{x_1, x_2\}]$. By Claim 6, G_1 has either $k - 1$ admissible (x_1, x_2) -path of consecutive lengths, or $k - 2$ admissible (x_1, x_2) -path whose lengths form an arithmetic progression with common difference two. Note that C contains two (x_1, x_2) -paths whose lengths differ by 1 in the odd case and 4 in the even case. By Lemma 3.10, G has k admissible cycles, a contradiction. $\square$

Claim 8. *$G - V(C)$ has exactly two components.*

Proof. If $G - V(C)$ has only one component, then every vertex in C has a neighbor this component, contradicting Claim 7.

Now suppose that $G - V(C)$ has at least three components, say Q_1, Q_2, Q_3 . Let G_0 be the graph obtained from G by contracting all edges in Q_1, Q_2 and Q_3 . By Lemma 3.8, G_0 is 3-connected. Let w_i be the vertex in G_0 obtained by contracting Q_i , $i = 1, 2, 3$. It follows that G_0 has a cycle C_0 containing w_1, w_2, w_3 . Let x_i and y_i be the predecessor and successor of w_i in C_0 , $i = 1, 2, 3$. So $x_i, y_i \in N_C(Q_i)$. Let $G_i = G[V(Q_i) \cup \{x_i, y_i\}]$, $i = 1, 2, 3$. By Claim 3, G_i has $k - 2$ admissible (x_i, y_i) -paths. Together with the three paths in $C_0 - \{w_1, w_2, w_3\}$, we can get $3k - 8 \geq k$ admissible cycles of G , a contradiction. $\square$

By Claim 8, let Q_1, Q_2 be the two components of $G - V(C)$. We fix an orientation of C .

Claim 9. *C is even and there exist $y_1, y_2, z_1, z_2 \in V(C)$ in this order along C such that $y_1, z_1 \in N_C(Q_1)$, $y_2, z_2 \in N_C(Q_2)$ and $|\vec{C}[y_1, y_2]| + |\vec{C}[z_1, z_2]| = |C|/2 - 1$.*

Proof. By Claim 7, every vertex in C has neighbors in exactly one of Q_1, Q_2 . Set $C = x_1x_2 \dots x_{|C|}x_1$. For convenience, we color all the vertices of $N_C(Q_1)$ red and all the vertices in $N_C(Q_2)$ blue.

If C is odd, then by Claim 7, any two quasi-diagonal vertices have distinct colors. Since for any i , vertices x_i and x_{i+1} (the indices are taken modulo $|C|$) have a common quasi-diagonal vertex, they have the same color. Thus, all vertices have the same color, contradicting Claim 7.

Similarly, if C is even, then by Claim 7, for any i , vertices x_i and x_{i+4} are of the same color, because they have a common sub-quasi-diagonal vertex.

If $|C| \equiv 2 \pmod{4}$, then all vertices x_i with i odd have the same color, say red, and all other vertices in C have the same color blue. We can choose $y_1 = x_1, y_2 = x_2, z_1 = x_{|C|/2}, z_2 = x_{|C|-2}$. One can compute that y_1, y_2, z_1, z_2 meet the conclusion of the assertion. So we assume that $|C| \equiv 0 \pmod{4}$.

If $|C| \equiv 4 \pmod{8}$, then y_1 and $y_{|C|/2+3}$ have the same color. However, y_1 and $y_{|C|/2+3}$ are sub-quasi-diagonal in C , a contradiction.

Now assume that $|C| \equiv 0 \pmod{8}$. Set $X_i = \{x_j : j \equiv i \pmod{4}\}$, $i = 1, 2, 3, 4$. So all vertices in X_i have the same color. Since $x_1 \in X_1$ and $x_{|C|/2+3} \in X_3$ are sub-quasi-diagonal, the vertices of X_1 and of X_3 have distinct colors, and similarly, the vertices of X_2 and of X_4 have distinct colors. We can assume w.l.o.g. that the vertices of $X_1 \cup X_2$ are red and of $X_3 \cup X_4$ are blue. Let $y_1 = x_2, y_2 = x_3, z_1 = x_{|C|/2+2}, z_2 = x_{|C|}$. One can compute that y_1, y_2, z_1, z_2 meet the conclusion of the assertion. $\square$

Let $y_1, y_2, z_1, z_2 \in V(C)$ be vertices as in Claim 9. Set $G_1 = G[V(Q_1) \cup \{y_1, z_1\}]$ and $G_2 = G[V(Q_2) \cup \{y_2, z_2\}]$. Then G_i contains $k - 2$ admissible (y_i, z_i) -paths for $i = 1, 2$. For a (y_1, z_1) -path P_1 in G_1 and a (y_2, z_2) -path P_2 in G_2 , we can get two cycles $P_1 \cup P_2 \cup \vec{C}[y_1, y_2] \cup \vec{C}[z_1, z_2]$ and $P_1 \cup P_2 \cup \vec{C}[y_2, z_1] \cup \vec{C}[z_2, y_1]$, which are of lengths $|P_1| + |P_2| + |C|/2 - 1$ and $|P_1| + |P_2| + |C|/2 + 1$, respectively. It follows that we can get k admissible cycles of G , a contradiction.

This finishes the proof of Lemma 3.11. $\square$

Proof of Theorem 1.3. By Lemma 2.3, we can assume that the connectivity of G is at least $(k + 2)/2 \geq 3$. If G has a triangle or a 4-cycle, then we are done by Theorem 3.2 or Lemma 3.5. If $k \in \{4, 5\}$, then we are done by Lemma 3.11; and if $k \geq 6$, then we are done by Theorems 3.6 and 3.7. $\square$

4 Cycles of lengths ℓ modulo k for all even ℓ

In this section, we prove Theorems 1.4 and 1.6. Corollary 1.5 can be deduced from Theorem 1.4. The following results will be used in our proofs.

Theorem 4.1 (Bondy and Vince [4]). *With the exception of K_1 and K_2 , every connected graph having at most two vertices of degree less than 3 contains two admissible cycles.*

Theorem 4.2 (Gao, Li, Ma and Xie [14]). *Every 3-connected graph with order at least 6 contains two cycles of consecutive even lengths.*

Theorem 4.3 (Dean, Lesniak and Saito [9]). *Every graph with minimum degree at least 2 and with at most two vertices of degree 2 contains a cycle of length 0 modulo 4.*

Lemma 4.4. *Let G be a graph and x, y two vertices of G . If $G + xy$ is 2-connected and every vertex of G other than x, y has degree at least 3, then at least one of the following holds:*

- (1) G contains a cycle of length 2 modulo 4;
- (2) G contains two (x, y) -paths whose lengths differ by 2 modulo 4; or

(3) G has the construction $L_i(x, y)$ for some $i = 1, \dots, 4$ (see Figure 5).

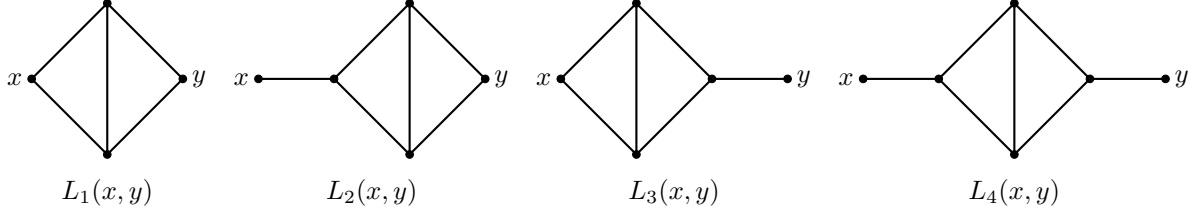


Figure 5: Constructions of $L_i(x, y)$ ($i = 1, \dots, 4$) in Lemma 4.4.

Proof. Suppose that G is a counterexample with the smallest number of edges. Since $G + xy$ is 2-connected, we have that G is 2-connected or a block-chain.

Claim 1. $xy \notin E(G)$.

Proof. Suppose that $xy \in E(G)$. Let $G_0 = G - xy$. Then G_0 satisfies the condition of the theorem. Note that $|E(G_0)| < |E(G)|$. It follows that G_0 satisfies (1), (2) or (3). If G_0 contains a cycle of length 2 modulo 4, or G_0 contains two (x, y) -paths whose lengths differ by 2 modulo 4, then so does G , a contradiction. Now assume that G_0 has the construction $L_i(x, y)$ for some $i = 1, \dots, 4$. If $i = 1, 2$ or 3 , then G_0 has an (x, y) -path of length 3, together with xy , we get two (x, y) -paths in G of lengths differing by 2 modulo 4, a contradiction. So $i = 4$. Then G_0 has an (x, y) -path of length 5, together with xy , we get a cycle of length 6 in G , a contradiction. $\square$

Claim 2. If C is an even cycle of G and P^x, P^y two disjoint paths from C to $\{x, y\}$, say with origins x', y' , respectively, then $d_C(x', y')$ is even.

Proof. Since G has no cycles of length 2 modulo 4, we have $|C| \equiv 0 \pmod{4}$. Fix an orientation of C . If $d_C(x', y')$ is odd, then both $\vec{C}[x', y']$ and $\overleftarrow{C}[x', y']$ are odd. We can assume w.l.o.g. that $|\vec{C}[x', y']| \equiv 1 \pmod{4}$ and $|\overleftarrow{C}[x', y']| \equiv 3 \pmod{4}$. Now $P^x x' \vec{C}[x', y'] y' P^y$ and $P^x x' \overleftarrow{C}[x', y'] y' P^y$ are two (x, y) -paths of lengths differing by 2 modulo 4, a contradiction. $\square$

We claim that G contains an even cycle. If not, then G is 2-connected or a block-chain such that each block is an odd cycle or an edge. This implies that G contains a vertex, different from x, y , with degree 2, a contradiction. Let G^* be the graph obtained from G by adding a new vertex w and two edges xw, yw . By $G + xy$ being 2-connected, G^* is 2-connected. Let C be an even cycle of G such that the component H of $G^* - V(C)$ containing w is as large as possible and subject to this, $|C|$ is as small as possible.

Now let P^x, P^y be two disjoint paths from C to $\{x, y\}$, say with origins x', y' , respectively. We remark that possibly $x = x'$ or $y = y'$ or both.

Claim 3. Every two vertices in $N_C(H)$ have an even distance in C .

Proof. By Claim 2, $d_C(x', y')$ is even. In the following, we will show that every vertex in $N_C(H)$ has an even distance to x' or to y' in C , following which the assertion can be deduced directly.

Let $z \in N_C(H)$. Then there is a path P from z to $P^x \cup P^y - \{x', y'\}$ with all internal vertices in H . Let z' be the end-vertex of P and assume w.l.o.g. that $z' \in V(P^x) \setminus \{x'\}$. Now $Pz'P^x[z', x]$ and P^y are two disjoint paths from C to $\{x, y\}$. By Claim 2, $d_C(z, y')$ is even. $\square$

Claim 4. C has at most one chord.

Proof. Suppose otherwise that C has at least two chords. If $|C| = 4$, then $G[V(C)] \cong K_4$, implying that there are two (x', y') -paths in $G[V(C)]$ of lengths 1 and 3, respectively. Together with P^x, P^y , we get two (x, y) -paths of lengths differing by 2 modulo 4, a contradiction. Now assume that $|C| \geq 8$. Then there is a Θ -graph T with $V(T) \subset V(C)$, which contains an even cycle C_1 . Hence $|C_1| < |C|$ and the component of $G^* - V(C_1)$ containing w will also contain H , contradicting the choice of C . $\square$

Claim 5. $G^* - V(C)$ has only one component.

Proof. Let Q be a component of $G^* - V(C)$ other than H .

Claim 5.1. $|N_C(Q)| \geq 3$.

Proof. Assuming otherwise, by the 2-connectivity of G , we have $|N_C(Q)| = 2$, say $N_C(Q) = \{x_1, y_1\}$. Let $G_1 = G[V(Q) \cup \{x_1, y_1\}]$. Then $G_1 + x_1y_1$ is 2-connected and every vertex of G_1 other than x_1, y_1 has the same degree as in G . It follows that G_1 satisfies the conclusion (1), (2) or (3). If G_1 contains a cycle of length 2 modulo 4, then so does G , a contradiction. If G_1 contains two (x_1, y_1) -paths of lengths differing by 2 modulo 4, then together with two disjoint paths from $\{x_1, y_1\}$ to $\{x, y\}$, we get two (x, y) -path in G of lengths differing by 2 modulo 4, a contradiction. Now we conclude that G_1 has construction $L_i(x_1, y_1)$ for some $i = 1, \dots, 4$.

Let C_1 be the 4-cycle of G_1 . If $\{x_1, y_1\} \neq \{x', y'\}$, or if G_1 has construction $L_i(x_1, y_1)$ with $i \in \{2, 3, 4\}$, then the component of $G^* - V(C_1)$ containing w will contain H and at least one vertex in $\{x', y'\}$, contradicting the choice of C . So we conclude that $\{x_1, y_1\} = \{x', y'\}$ and G_1 has construction $L_1(x_1, y_1)$.

Let $G_2 = G - Q$. If $x' \neq x$, then x' has two neighbors in C and one neighbor in P^x . If $y' \neq y$, then y' has two neighbors in C and one neighbor in P^y . In any case we see that every vertex of G_2 other than x, y has degree at least 3 in G_2 . It follows that G_2 satisfies the conclusion (1), (2) or (3). If G_2 contains a cycle of length 2 modulo 4, or G_2 contains two (x, y) -paths of lengths differing by 2 modulo 4, then so does G , a contradiction. Now we conclude that G_2 has construction $L_i(x, y)$ for some $i = 1, \dots, 4$.

Note that C is an even cycle in G_2 , so $|C| = 4$ and C has a chord zz' , where $\{z, z'\} = V(C) \setminus \{x', y'\}$. Also note that G_1 has construction $L_1(x', y')$, i.e., G_1 is a 4-cycle with a chord $z_1z'_1$, where $\{z_1, z'_1\} = V(G_1) \setminus \{x', y'\}$. Now $x'zz'y'z_1z'_1x'$ is a 6-cycle of G , a contradiction. $\square$

Claim 5.2. $|N_C(Q)| = 3$ and $N_C(H) = \{x', y'\} \subset N_C(Q)$.

Proof. If $|N_C(Q)| \geq 4$, or $N_C(H) \setminus N_C(Q) \neq \emptyset$, then there are three vertices $u_1, u_2, u_3 \in N_C(Q)$ and one vertex $z \in N_C(H) \setminus \{u_1, u_2, u_3\}$. Assume w.l.o.g. that u_1, u_2, u_3, z appear in this order along C . It follows that there are three internally-disjoint paths P_1, P_2, P_3 from some $u \in V(Q)$ to u_1, u_2, u_3 , respectively, with all internal vertices in Q . Now $P_1 \cup P_2 \cup P_3 \cup \vec{C}[u_1, u_3]$ is a Θ -graph, which contains an even cycle C_1 . The component of $G^* - C_1$ containing w will contain H and z , a contradiction. So we conclude that $|N_C(Q)| = 3$ and $N_C(H) \subseteq N_C(Q)$.

If $|N_C(H)| = 3$, say $N_C(H) = \{x', y', z\}$, then $N_C(Q) = \{x', y', z\}$ as well. We can assume w.l.o.g. that x', y', z appear in this order along C . By Claim 3, all three paths $\vec{C}[x', y']$, $\vec{C}[y', z]$, $\vec{C}[z, x']$ are even. Now let P_1, P_2, P_3 be three internally-disjoint paths from some $u \in V(Q)$ to x', y', z , respectively, with all internal vertices in Q . It follows that one of the three paths P_1uP_2 , P_2uP_3 , P_3uP_1 is even, implying that one of the three cycles $P_2uP_1x'\vec{C}[x', y']$, $P_3uP_2y'\vec{C}[y', z]$, $P_1uP_3z\vec{C}[z, x']$ is even. Let C_1 be such an even cycle. The component of $G^* - C_1$ containing w will contain H and one vertex in $\{x', y', z\}$, a contradiction. Hence we conclude that $|N_C(H)| = 2$, i.e., $N_C(H) = \{x', y'\}$. $\square$

By Claim 5.2, we can assume that $N_C(Q) = \{x', y', z\}$. We assume w.l.o.g. that $z \in V(\vec{C}[x', y'])$.

Claim 5.3. Q is the only component of $G^* - V(C)$ other than H .

Proof. Let Q' be a component of $G^* - V(C)$ other than H and Q . By the analysis in Claims 5.1 and 5.2, $N_C(Q') = \{x', y', z'\}$ for some $z' \in V(C) \setminus \{x', y'\}$. Let P be an (x', z) -path with all internal vertices in Q ; and let P' be an (x', z') -path with all internal vertices in Q' .

First we assume that $z' \in V(\vec{C}[x', y'])$. We can assume w.l.o.g. that x', z, z', y' appear in this order along C . Now $\vec{C}[x', z'] \cup P \cup P'$ is a Θ -graph, which contains an even cycle C_1 . The component of $G^* - C_1$ containing w will contain H and y' , a contradiction.

Now we assume that $z' \in V(\overleftarrow{C}[x', y'])$. If the cycle $\vec{C}[x', z]zPx'$ is of even length, then the component of $G^* - (V(\vec{C}[x', z]) \cup V(P))$ containing w will contain H and y' , contradicting the choice of C . So we conclude that $\vec{C}[x', z]zPx'$ is odd, implying that $Pz\vec{C}[z, y']$ is an odd (x', y') -path. By a similar analysis we see that $P'z'\overleftarrow{C}[z', y']$ is an odd (x', y') -path, contradicting Claim 2 for the cycle $x'Pz\vec{C}[z, y']\vec{C}[y', z']z'P'x'$. $\square$

Claim 5.4. $|C| = 4$ and C has exactly one chord zz' , where $\{z'\} = V(C) \setminus \{x', y', z\}$.

Proof. By Claim 4, C has at most one chord. By Claims 5.2 and 5.3, every vertex in $V(C) \setminus \{x', y', z\}$ has no neighbors outside C . If $|C| \geq 8$, then some vertices in $V(C) \setminus \{x', y', z\}$ are not incident to a chord of C and have degree 2, a contradiction. So we conclude that $|C| = 4$.

Set $C = x'zy'z'x'$. Notice that $z' \notin N_C(H) \cup N_C(Q)$. We have that z' is incident to the chord of C , i.e., $zz' \in E(G)$. $\square$

Let $G_3 = G - z'$. By Claim 5.4, z has two neighbors x', y' and one neighbor in Q . If $x' \neq x$, then x' has one neighbor z , one neighbor in P^x and one neighbor in Q ; if $y' \neq y$, then y' has one neighbor z , one neighbor in P^y and one neighbor in Q . In each case we have that every vertex of G_3 other than x and y has degree at least 3 in G_3 . Thus, G_3 satisfies the conclusion (1), (2) or (3). If G_3 contains a cycle of length 2 modulo 4, or G_3 contains two (x, y) -paths of lengths differing by 2 modulo 4, then so does G , a contradiction. Now we conclude that G_3 has construction $L_i(x, y)$ for some $i = 1, \dots, 4$. It follows that Q has exactly one vertex, say u , and $ux', uy', uz \in E(G)$. Now $P^x x' zy' P^y$ and $P^x x' z' zu y' P^y$ are two (x, y) -paths of lengths differing by 2 modulo 4, a contradiction. $\square$

Claim 6. $|C| = 4$ and C has exactly one chord zz' , where $\{z, z'\} = V(C) \setminus \{x', y'\}$.

Proof. By Claim 4, C has at most one chord. By Claims 3 and 5, there are at least $|C|/2$ vertices in C that have no neighbors outside C . If $|C| \geq 8$, then some vertices in $V(C) \setminus N_C(H)$ are not incident to a chord of C and have degree 2, a contradiction. So we conclude that $|C| = 4$.

Set $C = x'zy'z'x'$. Notice that $z, z' \notin N_C(H)$. We have that zz' is a chord of C . $\square$

Claim 7. $G - \{z, z'\}$ contains no path from P^x to P^y .

Proof. Suppose otherwise that $G - \{z, z'\}$ contains a path from P^x to P^y .

Claim 7.1. Every path of $G - \{z, z'\}$ from P^x to P^y is even.

Proof. Suppose that P is an odd path of $G - \{z, z'\}$ from P^x to P^y . Let x_1, y_1 be the end-vertices of P . Notice that either $P^x[x_1, x']x'zy'P^y[y', y_1]$ or $P^x[x_1, x']x'zz'y'P^y[y', y_1]$ is an odd (x_1, y_1) -path, together with P we get an even cycle, say C_1 . Now $P^x[x_1, x]$ and $P^y[y_1, y]$ are two disjoint paths from C_1 to $\{x, y\}$ such that $d_{C_1}(x_1, y_1)$ is odd, contradicting Claim 2. $\square$

We may assume that P^x and P^y are disjoint paths from x to x' and y to y' , respectively, such that the distance between P^x and P^y in $G - \{z, z'\}$ is the shortest. Let P be the shortest path of $G - \{z, z'\}$ from P^x to P^y , say from x_1 to y_1 . By Claim 7.1, P is even, implying that $V(P) \setminus \{x_1, y_1\} \neq \emptyset$. Let Q be the component of $G - (\{z, z'\} \cup V(P^x) \cup V(P^y))$ containing $P - \{x_1, y_1\}$.

Claim 7.2. $N_{P^x \cup P^y}(Q) = \{x_1, y_1\}$.

Proof. Suppose that $|N_{P^x \cup P^y}(Q)| \geq 3$, say $u \in N_{P^x \cup P^y}(Q) \setminus \{x_1, y_1\}$. We can assume w.l.o.g. that $u \in V(P^x)$. There is a path P' from a vertex $u' \in V(P) \setminus \{x_1, y_1\}$ to u with all internal vertices in Q . Now let P_1^x be the path obtained from P^x by using $P[x_1, u']u'P'$ instead of $P^x[x_1, u]$. Then P_1^x and P^y are two disjoint paths of G from C to $\{x, y\}$, and $P[u', y_1]$ is a path from P_1^x to P^y in $G - \{z, z'\}$. Clearly $|P[u', y_1]| < |P|$, contradicting the choice of P^x and P^y . $\square$

Let $G_4 = G[V(Q) \cup \{x_1, y_1\}]$. By Claim 7.2, $G_4 + x_1y_1$ is 2-connected and every vertex of G_4 other than x_1, y_1 has the same degree as that in G . It follows that G_4 satisfies the conclusion (1), (2) or (3). If G_4 contains a cycle of length 2 modulo 4, then so does G , a contradiction. If G_4 contains two (x_1, y_1) -paths of lengths differing by 2 modulo 4, then together with the two paths $P^x[x_1, x]$ and $P^y[y_1, y]$, we get two (x, y) -paths in G of lengths differing by 2 modulo 4, a contradiction. Now we conclude that G_4 has construction $L_i(x_1, y_1)$ for some $i = 1, \dots, 4$. It follows that G_4 contains an odd path from P^x to P^y , contradicting Claim 7.1. $\square$

Claim 8. *Either $x = x'$ or $N(x) = \{x'\}$. Similarly, either $y = y'$ or $N(y) = \{y'\}$.*

Proof. Suppose that $x \neq x'$. By Claim 7, x' is a cut-vertex of G separating x with $C - x'$. Let H' be the component of $G - x'$ containing x and let $G_5 = G[V(H') \cup \{x'\}]$. If $N(x) \neq \{x'\}$, then $|V(G_5)| \geq 3$, and every vertex of G_5 other than x, x' has degree at least 3 in G_5 . It follows that G_5 satisfies the conclusion (1), (2) or (3). If G_5 contains a cycle of length 2 modulo 4, then so does G , a contradiction. If G_5 contains two (x, x') -paths of lengths differing by 2 modulo 4, then together with $\vec{C}[x', y']y'P^yy$, we get two (x, y) -paths in G of lengths differing by 2 modulo 4, a contradiction. Now we conclude that G_5 has construction $L_i(x, x')$ for some $i = 1, \dots, 4$. It follows that G_5 contains two (x, x') -paths of lengths differing by 1. Notice that $G[V(C)]$ contains two (x', y') -paths of lengths differing by 1. It follows that G contains two (x, y) -paths of lengths differing by 2, a contradiction.

The second assertion can be proved similarly. $\square$

By Claims 5, 6 and 8, we see that G has construction $L_i(x, y)$ for some $i = 1, \dots, 4$, a contradiction. $\square$

Lemma 4.5. *Let G be a 2-connected graph on at least 6 vertices with minimum degree at least 3. Then G contains a cycle of length 2 modulo 4.*

Proof. Assume that G does not contain a $(2 \bmod 4)$ -cycle. By Theorem 4.2, the connectivity of G is 2, say $\{x, y\}$ is a vertex cut of G . Let Q_1, Q_2 be two of the components of $G - \{x, y\}$. We can choose the cut $\{x, y\}$ such that the component Q_1 is as small as possible. Denote by $G_i = G[V(Q_i) \cup \{x, y\}]$, where $i = 1, 2$. By G being 2-connected, $G_i + xy$ is 2-connected, and every vertex of G_i other than x, y has degree at least 3 in G_i , $i = 1, 2$. Notice that G_i contains no cycle of length 2 modulo 4.

We suppose first that G_1 contains two (x, y) -paths of lengths differing by 2 modulo 4, say P_1, P_2 . If G_2 contains an odd cycle C , then G_2 contains two (x, y) -paths of different parities. Let P be an (x, y) -path of G_2 such that P, P_1 have lengths of the same parity. Now $P \cup P_1$ and $P \cup P_2$ are two even cycles of lengths differing by 2 modulo 4, one of which is of length 2 modulo 4, a contradiction. Now we assume that G_2 contains no odd cycles, implying that G_2 is bipartite. By Theorem 4.1, G_2 contains two admissible cycles, which have consecutive even lengths, and one of which is a $(2 \bmod 4)$ -cycle, a contradiction. So we conclude that G_1 contains no two (x, y) -paths of lengths differing by 2 modulo 4.

By Lemma 4.4, G_1 has construction $L_i(x, y)$ for some $i = 1, \dots, 4$. By a similar analysis we can prove that G_2 has construction $L_j(x, y)$ for some $j = 1, \dots, 4$. Recall that we choose $\{x, y\}$ such that Q_1 is as small as possible.

So G_1 has construction $L_1(x, y)$ and it contains two (x, y) -paths of lengths 2 and 3, respectively. Also note that G_2 has an (x, y) -path of length either 3 or 4. It follows that G contains a cycle of length 6, a contradiction. $\square$

The following result will be used in our proof.

Theorem 4.6. *Let G be a 2-connected graph with minimum degree at least 3. Then*

- (1) (Chen and Saito [5]) G has a cycle of length 0 modulo 3;
- (2) (Dean et al. [8]) G has a cycle of length 1 modulo 3 unless G is a Petersen graph;
- (3) (Dean et al. [8]) G has a cycle of length 2 modulo 3 unless G is isomorphic to K_4 or $K_{3,n-3}$.

Now we are ready to prove Theorems 1.4 and 1.6.

Proof of Theorem 1.4. If k is odd, then the existence of k admissible cycles implies the existence of $(\ell \bmod k)$ -cycles for all even ℓ . The assertion follows by Theorem 1.3. So assume that k is even. By Theorem 4.3 and Lemma 4.5, the assertion holds for $k = 4$. So assume that $k \geq 6$.

Assume that G does not contain an $(\ell \bmod k)$ -cycle for some even ℓ and G is isomorphic to neither K_{k+1} nor $K_{k,n-k}$. By Lemma 2.4, the connectivity of G is at least $(k+2)/2 \geq 4$. By Theorem 1.3, G contains k admissible cycles whose lengths form an arithmetic progression of common difference two. If G is non-bipartite, then a contradiction is caused by Theorems 3.2 and 3.7. So G is bipartite. Recall that k and ℓ are even. It clearly holds that any k admissible cycles in a bipartite graph contain cycles of all even lengths modulo k , a contradiction. $\square$

Proof of Theorem 1.6. Let G be a counterexample to the theorem. By Theorem 4.3 and Lemma 4.5, $k \geq 6$. By Lemma 2.4, the connectivity of G is at least $k/2 \geq 3$. Let H be a bipartite subgraph of G such that the number of edges of H is maximal. It is not difficult to see that H is connected and has minimum degree at least $\lceil (k-1)/2 \rceil = k/2$.

Claim 1. H is not 2-connected.

Proof. Assume the opposite that H is 2-connected. If $k = 6$, then by Theorem 4.6, H has cycles of lengths 0, 1, 2 modulo 3 (which are of lengths 0, 4, 2 modulo 6 since H is bipartite), or $H \cong K_{3,n-3}$. If $k \geq 8$, then by Theorem 1.3, H contains $k/2$ admissible cycles (which are of lengths ℓ modulo k for all even ℓ) or $H \cong K_{k/2,n-k/2}$.

Suppose now $H \cong K_{k/2,n-k/2}$, say with bipartition (X, Y) such that $|X| = k/2$ and $|Y| = n - k/2$. This implies the existence of $(\ell \bmod k)$ -cycles for any even $\ell \not\equiv 2 \pmod k$. By $n \geq k + 2$, we have $|Y| \geq k/2 + 2$. Since $\delta(G) \geq k - 1$, there exists $y \in Y$ that has $k/2 - 1 \geq 2$ neighbors in $G[Y]$ (recall that $k \geq 6$). Let y_1 and y_2 be two neighbors of y in Y . By $H \cong K_{k/2,n-k/2}$ and $n \geq k + 2$, there exists a (y_1, y_2) -path of length k in $H - y$. Together with the edges yy_1 and yy_2 , we get a cycle of length 2 modulo k in G , a contradiction. $\square$

Claim 2. Let B be an end block of H and c the cut-vertex of H contained in B . Then $B - c$ is 2-connected.

Proof. Suppose that $B - c$ has a cut-vertex z . Let Q_1, Q_2 be two components of $B - \{c, z\}$, and for $i = 1, 2$, let $H_i = H[V(Q_i) \cup \{c, z\}]$. Note that $H_i + cz$ is 2-connected and every vertex in Q_i has degree at least $k/2$ in H_i . By Theorem 2.1, H_i contains $k/2 - 1$ admissible (c, z) -paths, $i = 1, 2$. It follows that B contains $k - 3 \geq k/2$ admissible cycles, which are even cycles since H is bipartite. Thus, G contains a cycle of length ℓ modulo k , a contradiction. $\square$

Claim 3. H has exactly one cut-vertex.

Proof. If H has at least two cut vertices, then H contains two disjoint end blocks. Let B_1, B_2 be two disjoint end blocks of H , and let c_1, c_2 two cut vertices of H contained in B_1, B_2 , respectively.

Claim 3.1. G contains two disjoint paths P^1, P^2 from B_1 to B_2 such that $B_1 \cup B_2 \cup P^1 \cup P^2$ is bipartite.

Proof. Recall that G is 3-connected. Let P_0^1, P_0^2, P_0^3 be three disjoint paths from B_1 to B_2 . Let u_i^j be the end-vertex of P_0^j in B_i , $j = 1, 2, 3$, $i = 1, 2$. Then B_i contains three internally disjoint paths from $u_i \in V(B_i)$ to u_i^1, u_i^2, u_i^3 , respectively, say P_i^j , $j = 1, 2, 3$, $i = 1, 2$. Now $\bigcup\{P_i^j : i \in \{0, 1, 2\}, j \in \{1, 2, 3\}\}$ is a Θ -graph, which contains an even cycle. We assume w.l.o.g. that $u_1 P_1^1 u_1^1 P_0^1 u_2^1 P_2^1 u_2 P_2^2 u_2^2 P_0^2 u_1^2 P_1^2 u_1$ is an even cycle. Since both B_1 and B_2 are bipartite, we see that $B_1 \cup B_2 \cup P_0^1 \cup P_0^2$ is bipartite, as desired. $\square$

Let P^1, P^2 be two disjoint paths of G from B_1 to B_2 as in Claim 3.1, and let u_i^j be the end-vertex of P^j in B_i , $j = 1, 2$, $i = 1, 2$. Set $G_1 = B_1 \cup B_2 \cup P^1 \cup P^2$. Note that every vertex of B_i other than c_i has degree at least $k/2$. By Theorem 2.2, B_i contains $k/2 - 1$ admissible (u_i^1, u_i^2) -paths. It follows that G_1 contains $k - 3 \geq k/2$ admissible cycles, which are $(\ell \bmod k)$ -cycles for all even ℓ since G_1 is bipartite, a contradiction. $\square$

By Claims 1 and 3, we can assume that c is the unique cut-vertex of H . So c is contained in every block of H .

Claim 4. H has exactly two blocks.

Proof. Suppose that H has at least three blocks. Since G is 3-connected, there exist blocks B_0, B_1, B_2 such that $E(G) \setminus E(H)$ contains two nonadjacent edges from $B_0 - c$ to $B_1 - c$ and $B_2 - c$, respectively. Let $u_1 v_1, u_2 v_2$ be such two edges, and let P be a path of $B_0 - c$ from u_1 to u_2 . Notice that $G_2 := B_1 \cup B_2 \cup P \cup u_1 v_1 \cup u_2 v_2$ is a bipartite graph.

By Theorem 2.1, B_i contains $k/2 - 1$ admissible (v_i, c) -paths, for $i = 1, 2$. It follows that G_2 contains $k - 3 \geq k/2$ admissible cycles, which are $(\ell \bmod k)$ -cycles for all even ℓ , a contradiction. $\square$

By Claim 4, we can assume that B_1, B_2 are the two blocks of H . Recall that G is 3-connected, there are two non-adjacent edges in $E(G) \setminus E(H)$ from $B_1 - c$ to $B_2 - c$, say $u_1 u_2, v_1 v_2$ with $u_1, v_1 \in V(B_1) \setminus \{c\}$, $u_2, v_2 \in V(B_2) \setminus \{c\}$. Let $H' = H \cup \{u_1 u_2, v_1 v_2\} - \{cz : z \in N_{B_1}(c)\}$. Clearly H' is bipartite. By Claim 2, $B_1 - c$ is 2-connected. It follows that H' is 2-connected.

Suppose first that $k \geq 8$. Note that every vertex in $B_1 - c$ has degree at least $k/2 - 1$. By Theorem 2.1, $B_1 - c$ contains $k/2 - 2$ admissible (u_1, v_1) -paths. Note that every vertex of B_2 other than c has degree at least $k/2$ in B_2 . By Theorem 2.2, B_2 contains $k/2 - 1$ admissible (u_1, v_1) -paths. It follows that H' contains $(k/2 - 2) + (k/2 - 1) - 1 = k - 4 \geq k/2$ admissible cycles, which are of length ℓ modulo k for all even ℓ , a contradiction. Thus, we conclude that $k = 6$.

If $d_{B_1}(c) = 2$, then $e(H') = e(H)$. Recall that H' is 2-connected. By the analysis as in Claim 1, G contains cycles of lengths ℓ modulo k , a contradiction. So we conclude that $d_{B_1}(c) \geq 3$ and analogously $d_{B_2}(c) \geq 3$. That is, $\delta(B_1) \geq 3$ and $\delta(B_2) \geq 3$.

By Theorem 4.6, B_i ($i = 1, 2$) contains cycles of lengths 0, 1, 2 modulo 3 (which are of lengths 0, 4, 2 modulo 6), unless $B_i \cong K_{3, n_i - 3}$, in which case B_i contains no $(2 \bmod 6)$ -cycle, where $n_i = |V(B_i)| \geq 6$. Let B'_1 be a 4-cycle in $B_1 - c$ containing u_1, v_1 , and B'_2 be a $K_{3,3}$ in B_2 containing u_2, v_2 . So $H'' = B'_1 \cup B'_2 \cup \{u_1 u_2, v_1 v_2\}$ is a subgraph of H' . One can check that H'' contains a 8-cycle, a contradiction. $\square$

5 Cycles of lengths ℓ modulo k for all ℓ

In this section, we prove Theorem 1.7. The following lemma is needed.

Lemma 5.1 (Gao, Li, Ma and Xie [14]). *Let G be a 3-connected graph and D a connected subgraph of G such that $G - V(D)$ contains an even cycle. If C is an even cycle of $G - V(D)$ such that the component of $G - V(C)$*

containing D is maximal, then $G - V(C)$ is connected. Moreover, such a C has at most one chord or $G[V(C)]$ is isomorphic to K_4 , and if u_0v_0 is a chord of C , then $d_C(u_0, v_0)$ is even.

Let C be an even cycle. Recall that we say two vertices $u, v \in V(C)$ are diagonal if $d_C(u, v) = |C|/2$; and u, v are quasi-diagonal if $d_C(u, v) = |C|/2 - 1$. Note that if u, v are quasi-diagonal on C , then the two paths $\vec{C}[u, v]$ and $\overleftarrow{C}[u, v]$ have lengths differing by 2.

Lemma 5.2. *Let G be a graph, C_1 an even cycle, and C_2 an odd cycle of G . If one of the following statements holds, then G contains two cycles of consecutive odd lengths.*

- (1) C_1 and C_2 are disjoint and there exist two disjoint paths P^1, P^2 from C_1 to C_2 such that their origins in C_1 are quasi-diagonal.
- (2) C_1 and C_2 have exactly one common vertex u and there exist a path P , from $C_1 - u$ to $C_2 - u$ avoiding u such that its origin in C_1 is quasi-diagonal to u .

Proof. For (1), let u_i^j be the end-vertex of P^j on C_i , where $j = 1, 2$ and $i = 1, 2$. Since $|C_2|$ is odd, the lengths of the two paths $R_1 = P^1 \cup P^2 \cup \vec{C}_2[u_2^1, u_2^2]$ and $R_2 = P^1 \cup P^2 \cup \overleftarrow{C}_2[u_2^1, u_2^2]$ differ in parity. Then one of these paths, say R_1 , satisfies $|R_1| \equiv |C_1|/2 \pmod{2}$. Since u_1^1, u_1^2 are quasi-diagonal in C_1 , the two cycles $R_1 \cup \vec{C}_1[u_1^1, u_1^2]$ and $R_1 \cup \overleftarrow{C}_1[u_1^1, u_1^2]$ are of consecutive odd lengths. The proof for (2) is identical to (1). $\square$

Lemma 5.3. *Let G be a 3-connected graph and C_1, C_2 two cycles of G . If one of the following statements holds, then G contains two cycles of lengths 1 and 3 modulo 4, respectively.*

- (1) C_1 is even, C_2 is odd, and C_1, C_2 are disjoint.
- (2) C_1 is even, C_2 is odd, and C_1, C_2 intersect at exactly one vertex.
- (3) C_1 and C_2 are disjoint and both odd, and G is not a Petersen graph.

Proof. (1) By Lemma 5.1, we can suppose that $G - V(C_1)$ is connected, C_1 has at most one chord, except that $G[V(C_1)] \cong K_4$, and if u_0v_0 is a chord of C_1 , then $d_{C_1}(u_0, v_0)$ is even.

If $|C_1| = 4$, then there are three disjoint paths from C_1 to C_2 , two of which have their origins quasi-diagonal in C_1 . By Lemma 5.2, there are two consecutive odd cycles, which are of lengths 1 and 3 modulo 4, as desired. So we conclude that $|C_1| \geq 6$, which follows that C_1 has at most one chord.

Set $H = G - V(C_1)$, and let B be the block of H containing C_2 (if H is 2-connected, then $B = H$). Note that for each two vertices $v_1, v_2 \in V(B)$, there are two disjoint paths from $\{v_1, v_2\}$ to C_2 in B . For every vertex $v \in V(B)$, we let H_v be the component of $H - E(B)$ containing v . Possibly $V(H_v) = \{v\}$ (in the case v is an inner-vertex of B). We call the graphs $H_v, v \in V(B)$, the B -branches.

We claim that for each two vertices $u_1, u_2 \in V(H)$, there are two disjoint paths from $\{u_1, u_2\}$ to C_2 in H if and only if u_1, u_2 are contained in different B -branches. Firstly, if u_1, u_2 are contained in a common B -branch H_v , then every path from $\{u_1, u_2\}$ to C_2 passes through v . Secondly, suppose that u_1, u_2 are contained in B -branches H_{v_1}, H_{v_2} , respectively, where $v_1 \neq v_2$. Let P_i be a (u_i, v_i) -path in H_{v_i} , $i = 1, 2$, and let P'_1, P'_2 be two disjoint paths from $\{v_1, v_2\}$ to C_2 in B . Now $P_1v_1P'_1$ and $P_2v_2P'_2$ are two disjoint paths from $\{u_1, u_2\}$ to C_2 in H .

For two quasi-diagonal vertices x_1, x_2 in C_1 , if x_1 has a neighbor u_1 in H , x_2 has a neighbor u_2 in H such that u_1, u_2 are in different B -branches, then there are two disjoint paths from C_1 to C_2 with origins x_1, x_2 . By Lemma 5.2, there are two cycles of consecutive odd lengths. So we can assume that if both x_1, x_2 have neighbors in H , then their neighbors are contained in a common B -branch.

As in the proof of Lemma 3.11, we define the graph $Qdi(C_1)$ on $V(C_1)$ such that two vertices are adjacent in $Qdi(C_1)$ if and only if they are quasi-diagonal in C_1 . Note that if $|C_1| \equiv 0 \pmod{4}$, then $Qdi(C_1)$ is a cycle, and if $|C_1| \equiv 2 \pmod{4}$, then $Qdi(C_1)$ is the union of two odd cycles. We consider two cases as follows.

First we assume that $|C_1| \equiv 2 \pmod{4}$. If C_1 has a chord u_1u_2 , then $d_{C_1}(u_1, u_2)$ is even. It follows that the two cycles $C'_1 = \overrightarrow{C_1}[u_1, u_2]u_2u_1$ and $C'_2 = \overleftarrow{C_1}[u_1, u_2]u_2u_1$ are odd. Since $|D_1| + |D_2| = |C_1| + 2 \equiv 0 \pmod{4}$, the two cycles C'_1, C'_2 have lengths 1 and 3 modulo 4, as desired. So we conclude that C_1 has no chord. By G being 3-connected, every vertex in C_1 has a neighbor in H . Let D_1, D_2 be the two components of $Qdi(C_1)$. It follows that all vertices in $N_H(V(D_1))$ are in a common B -branch H_{v_1} , and all vertices in $N_H(V(D_2))$ are in a common B -branch H_{v_2} . Now $\{v_1, v_2\}$ is a vertex cut of G , a contradiction.

Second we assume that $|C_1| \equiv 0 \pmod{4}$. Recall that $Qdi(C_1)$ is a cycle. If C_1 has no chord, then all vertices in $N_H(V(C_1))$ are in a common B -branch H_v , and v is a cut-vertex of G , a contradiction. So we conclude that C_1 has a chord u_1u_2 . Since $d_{C_1}(u_1, u_2)$ is even, we have $u_1u_2 \notin E(Qdi(C_1))$. If $N_H(u_1) \cup N_H(u_2) \neq \emptyset$, then all the neighbors in H of vertices in C_1 are in a common B -branch, implying that G has a cut-vertex, a contradiction. So assume that u_1 and u_2 have no neighbor in H . Let D_1, D_2 be the two components of $Qdi(C_1) - \{u_1, u_2\}$. Then D_1, D_2 are two paths. If both of the paths are nontrivial, then $N_H(V(D_1)) \subseteq V(H_{v_1})$, and $N_H(V(D_2)) \subseteq V(H_{v_2})$ for some B -branches H_{v_1}, H_{v_2} . Now $\{v_1, v_2\}$ is a vertex cut of G , a contradiction. Now assume w.l.o.g. that D_2 is trivial, say $V(D_2) = \{u_0\}$. It follows that $N_H(V(D_1)) \subseteq V(H_{v_1})$ for a B -branch H_{v_1} . Recall that u_1, u_2 have no neighbor in H . We have that $\{v_1, u_0\}$ is a vertex cut of G , a contradiction.

(2) Let u be the common vertex of C_1 and C_2 . We choose C_1 such that the component of $G - V(C_1)$ containing $C_2 - u$ is maximal. By Lemma 5.1, $G - V(C_1) =: H$ is connected.

Let u_1, u_2 be the two vertices that are quasi-diagonal with u in C_1 , and let u_0 be the vertex that is diagonal with u in C_1 . If $N_H(u_1) \cup N_H(u_2) \neq \emptyset$, then there is a path from $C_1 - u$ to $C_2 - u$ avoiding u with origin in $\{u_1, u_2\}$. By Lemma 5.2, G has two consecutive odd cycles, as desired. So we conclude that $N_H(u_1) \cup N_H(u_2) = \emptyset$. If $|C_1| = 4$, then $\{u, u_0\}$ is a vertex cut of G , contradicting the fact that G is 3-connected. So we assume that $|C_1| \geq 6$.

By G being 3-connected, we have that u_1u_2 is the unique chord of C_1 . Let u_3 be a neighbor of u on C_1 . Then $N_H(u_3) \neq \emptyset$, and there is a path P from $C_1 - u$ to $C_2 - u$ avoiding u with origin u_3 . Thus, $C_2 \cup P \cup \{uu_3\}$ is a Θ -graph, which contains an even cycle disjoint with the odd cycle $u_0u_1u_2u_0$. Now we are done by (1).

(3) We choose the two odd cycles C_1, C_2 such that the sum of its lengths is as small as possible. So neither C_1 nor C_2 has a chord.

If there is a component H of $G - V(C_1) \cup V(C_2)$, then $|N_{C_1 \cup C_2}(H)| \geq 3$ since G is 3-connected. It follows that H has at least two neighbors in C_i for some $i = 1, 2$. Thus, there is a path P with two end-vertices in C_i and all internal vertices in H . We see that $C_i \cup P$ is a Θ -graph, which contains an even cycle disjoint with C_{3-i} , and we are done by (1). Now we assume that $G - V(C_1) \cup V(C_2)$ has no component, which is, $V(G) = V(C_1) \cup V(C_2)$.

Let u be an arbitrary vertex in C_1 . If u has two neighbors in C_2 , then $G[V(C_2) \cup \{u\}]$ contains a Θ -graph, which contains an even cycle intersecting with C_1 at u . Then we are done by (2). So we conclude that every vertex in C_1 has exactly one neighbor in C_2 , and similarly, every vertex in C_2 has exactly one neighbor in C_1 . It follows that $|C_1| = |C_2|$, and the edges between C_1 and C_2 form a perfect matching of G .

Let u_1, u_2 be two adjacent vertices in C_1 , and u'_1, u'_2 their neighbors in C_2 . If $u'_1u'_2 \in E(C_2)$, then C_1 and $(C_1 - \{u_1u_2\}) \cup u_1u'_1u'_2u_2$ are two consecutive odd cycles, as desired. So we assume that $d_{C_2}(u'_1, u'_2) \geq 2$.

If $|C_1| = |C_2| = 3$, then every two vertices in C_2 have distance 1, a contradiction. If $|C_1| = |C_2| = 5$, then for each two adjacent vertices in C_1 , their neighbors in C_2 are of distance exactly 2, implying that G is a Petersen graph, a contradiction. Therefore, $|C_1| = |C_2| \geq 7$.

We choose two adjacent vertices $u_1, u_2 \in V(C_1)$ such that $d_{C_2}(u'_1, u'_2)$ is as small as possible. We assume w.l.o.g. that $\overrightarrow{C_2}[u'_1, u'_2]$ is even and $\overleftarrow{C_2}[u'_1, u'_2]$ is odd. Now $C'_1 = u_1u'_1\overrightarrow{C_2}[u'_1, u'_2]u'_2u_2u_1$ is an odd cycle. Since $\overleftarrow{C_2}[u'_1, u'_2]$ is odd and u'_1 and u'_2 are not adjacent, there are two adjacent vertices v'_1, v'_2 in $\overleftarrow{C_2}[u'_1, u'_2]$ different than u'_1 and u'_2 . Let v_1, v_2 be the neighbors of v'_1, v'_2 in C_1 . Clearly $\{v_1, v_2\} \cap \{u_1, u_2\} = \emptyset$. Let P be the (v_1, v_2) -path in C_1 avoiding u_1, u_2 . Then $C'_2 = v'_1v_1Pv_2v'_2v'_1$ is a cycle disjoint with C'_1 .

If C'_2 is even, then we are done by (1). So we assume that C'_2 is odd. By the choice of C_1, C_2 , we see that $V(G) = V(C'_1) \cup V(C'_2)$. It follows that $d_{C_2}(u'_1, u'_2) = 3$ (i.e., $\overleftarrow{C}_2[u'_1, u'_2] = u'_1 v'_1 v'_2 u'_2$), and v_1, v_2 are the two vertices adjacent to u_1, u_2 in C_1 . Now $u_1 v_1 \in E(C_1)$ or $u_1 v_2 \in E(C_1)$ and their neighbors in C_2 have distance less than 3, contradicting the choice of u_1, u_2 . $\square$

Lemma 5.4. *Let G be a 3-connected non-bipartite graph. Then G contains two cycles of lengths 1 and 3 modulo 4, respectively, unless G is a K_4 or a Petersen graph.*

Proof. By G being non-bipartite, let C be the shortest odd cycle of G . Then C has no chord. If $G - V(C)$ has a cycle, then we are done by Lemma 5.3. So we assume that $G - V(C)$ is a forest. Let T be a component of $G - V(C)$ with order as large as possible. Then T is a tree.

Suppose first that $|C| = 3$. If T is trivial, say $V(T) = \{u\}$, then $N(u) = V(C)$. By the choice of T , every component of $G - V(C)$ is trivial. If $G - V(C)$ has only one component T , then $G \cong K_4$, as desired. So let u' be the vertex of a component of $G - V(C)$ other than T . We have $N(u') = V(C)$ as well. Clearly $G[V(C) \cup \{u, u'\}]$ contains a 3-cycle and a 5-cycle, as desired.

Now assume that T is nontrivial. Let u_1, u_2 be two leaves of T and let P be the (u_1, u_2) -path of T . Since G is 3-connected, u_i has two neighbors in $V(C)$, for $i = 1, 2$. It follows that u_1, u_2 have a common neighbor in C , say v_1 . If P is even, then $v_1 u_1 P u_2 v_1$ is an even cycle that intersects with C at exactly one vertex, and we are done by Lemma 5.3. So assume that P is odd. Let $V(C) = \{v_1, v_2, v_3\}$ and assume v_2 is a second neighbor of u_2 in C . Then $C_1 = v_1 u_1 P u_2 v_1$ and $C_2 = v_1 u_1 P u_2 v_2 v_3 v_1$ are two consecutive odd cycles, as desired.

Suppose now that $|C| \geq 5$. We claim that if a vertex in T has two neighbors in C , then the two neighbors have distance exactly 2 in C . To show this, we assume that $u \in V(T)$ has two neighbors $v_1, v_2 \in V(C)$. If $v_1 v_2 \in E(C)$, then $u v_1 v_2 u$ is a triangle, contradicting the choice of C . Now assume $d_C(v_1, v_2) \geq 3$. Note that either $\overrightarrow{C}[v_1, v_2]$ or $\overleftarrow{C}[v_1, v_2]$ is odd, say $\overrightarrow{C}[v_1, v_2]$ is odd. Since $|\overrightarrow{C}[v_1, v_2]| \geq 3$, $u v_1 \overrightarrow{C}[v_1, v_2] v_2 u$ is an odd cycle shorter than C , a contradiction. Thus, as we claimed, for every vertex in T , its two neighbors in C have distance exactly 2 in C . It follows that every vertex in T has at most two neighbors in C . If T is trivial, then the vertex in T has three neighbors in C , a contradiction. So we assume that T is nontrivial.

Let u_1 be a leaf of T . Since G is 3-connected, u_1 has two neighbors in $V(C)$. Let v_1, v_2 be the two neighbors of u_1 in C , and let v_0 be the common neighbor of v_1, v_2 in C (recall that $d_C(v_1, v_2) = 2$). Suppose that there is a vertex $v_3 \in V(C) \setminus \{v_0, v_1, v_2\}$ that has a neighbor u_3 in T . We can assume w.l.o.g. that v_1, v_2, v_3 appear in this order along C . Let P be the (u_1, u_3) -path in T . Note that either $C_1 = \overrightarrow{C}[v_1, v_3] v_3 u_3 P u_1 v_1$ or $C_2 = \overleftarrow{C}[v_1, v_3] v_3 u_3 P u_1 v_1$ is odd. If C_1 is odd, then C_1 and $C'_1 = \overrightarrow{C}[v_2, v_3] v_3 u_3 P u_1 v_2$ are two consecutive odd cycles, as desired. While, if C_2 is odd, then C_2 and $C'_2 = \overleftarrow{C}[v_2, v_3] v_3 u_3 P u_1 v_2$ are two consecutive odd cycles, as desired. So we conclude that there is no vertex in $V(C) \setminus \{v_0, v_1, v_2\}$ that has a neighbor in T . It follows that $N_C(T) = \{v_0, v_1, v_2\}$.

Let u_2 be a leaf of T other than u_1 and let P be the (u_1, u_2) -path in T . Recall that u_2 has two neighbors in C which have distance exactly 2, and that $N_C(T) = \{v_0, v_1, v_2\}$. This implies that $N_C(u_2) = \{v_1, v_2\}$. If P is even, then $v_1 u_1 P u_2 v_1$ is an even cycle that intersects with C at exactly one vertex, and we are done by Lemma 5.3. So assume that P is odd. Then $C_1 = v_1 u_1 P u_2 v_1$ and $C_2 = v_1 u_1 P u_2 v_2 v_0 v_1$ are two consecutive odd cycles, as desired. $\square$

We would like to remark that while writing this paper, we noticed that Lin, Wang and Zhou [16] have showed that every 2-connected non-bipartite graph with minimum degree at least $k \geq 2$ contains $\lceil (k-1)/2 \rceil$ cycles of consecutive odd lengths. Although Lemma 5.4 can be deduced from this result, our proof is different. For the sake of completeness, we kept the proof of Lemma 5.4 in this paper.

Proof of Theorem 1.7. Note that when k is odd, the existence of cycles of all lengths modulo k is equivalent to the existence of cycles of all even lengths modulo k . If $k = 3$, then the assertion holds by Theorem 4.6. So assume that $k \geq 4$. By Theorem 1.4, it suffices to consider the case that k is even and ℓ is odd.

If G is 3-connected, then for $k = 4$ the assertion holds by Lemma 5.4. For $k \geq 6$, if G contains a triangle, then we are done by Theorem 3.2, while if G does not contain a triangle, then we are done by Theorem 3.7. Therefore, we may assume that G has a vertex cut with two vertices, say $\{x, y\}$.

Denote by Q_1, Q_2 two components of $G - \{x, y\}$. Let $G_i = G[V(Q_i) \cup \{x, y\}]$, $i = 1, 2$. By Theorem 2.1, there exist $k - 1$ admissible (x, y) -paths in G_i for each $i \in \{1, 2\}$. If G_i contains $k - 1$ admissible (x, y) -paths of consecutive lengths for some $i \in \{1, 2\}$, then we can get $2(k - 1) - 1 \geq k$ cycles of consecutive lengths, which contain $k/2$ cycles of consecutive odd lengths in G , as desired. So G_i has $k - 1$ admissible (x, y) -paths of lengths forming an arithmetic progression with common difference two for any $i \in \{1, 2\}$.

If G_i contains an odd cycle for some $i \in \{1, 2\}$, then there exist two (x, y) -paths with different parities in G_i . By concatenating the $k - 1$ admissible (x, y) -paths with common difference two in G_{3-i} , we get $k - 1$ cycles of consecutive odd lengths, as desired. So G_i is bipartite for each $i \in \{1, 2\}$. Since G is non-bipartite, by symmetry, we can assume that x, y are in different parts in G_1 and in the same part in G_2 . It follows that G_1 has $k - 1$ admissible (x, y) -paths of consecutive odd lengths and G_2 has $k - 1$ admissible (x, y) -paths of consecutive even lengths, implying the existence of $2k - 3 \geq k$ cycles of consecutive odd lengths, as desired. $\square$

6 Maximum number of edges

The following two theorems on cycles in graphs will be used in the proof of Theorem 1.10.

Theorem 6.1 (Woodall [21]). *Every n -vertex graph with more than $\lfloor n^2/4 \rfloor$ edges contains cycles of lengths from 3 to $(n + 3)/2$.*

Theorem 6.2 (Bondy [3]). *Every n -vertex graph with minimum degree more than $n/2$ is pancyclic, i.e., contains cycles of lengths from 3 to n .*

Proof of Theorem 1.10. We present the proof by induction on $n = |V(G)|$. If $n = 2k - 3$, then n is odd and

$$e(G) > (k - 1)(n - k + 1) = \frac{n + 1}{2} \cdot \frac{n - 1}{2} = \left\lceil \frac{n}{2} \right\rceil \cdot \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

By Theorem 6.1, we get that G contains cycles of all lengths in $[3, k]$, which includes a $(0 \bmod k)$ -cycle, so the base case of our statement is proved. Assume that $n > 2k - 3$ and the statement holds for any graph with order in $[2k - 3, n - 1]$. Assume the opposite that G contains no $(0 \bmod k)$ -cycles.

Claim 1. $\delta(G) \geq k$.

Proof. Assume the opposite that $\delta(G) \leq k - 1$. Let v be a vertex with minimum degree in G . Then

$$e(G - v) = e(G) - d(v) > (k - 1)(n - k + 1) - (k - 1) = (k - 1)[(n - 1) - k + 1].$$

Note that $|V(G - v)| \geq 2k - 3$. By induction hypothesis, $G - v$ contains a $(0 \bmod k)$ -cycle, so does G , a contradiction. $\square$

Claim 2. G is 2-connected.

Proof. If G is disconnected, then we can consider the connected graph obtained from G by adding some cut edges. Notice that the added edges are not contained in any cycles. So we only deal with the case that G has connectivity 1. Let c be a cut vertex of G . Then G is the union of two subgraphs G_1, G_2 with $V(G_1) \cap V(G_2) = \{c\}$. Set $n_i = |V(G_i)|$, where $n + 1 = n_1 + n_2$.

By Claim 1, $\delta(G) \geq k$, implying that $n_i \geq k + 1$. If $n_i < 2k - 1$, then $\delta(G_i - c) \geq k - 1 > (n_i - 1)/2$. By Theorem 6.2, $G_i - c$ contains a k -cycle, and we are done. So we assume that $n_i \geq 2k - 1$. By induction hypotheses, $e(G_i) \leq (k - 1)(n_i - k + 1)$. It follows that

$$e(G) = e(G_1) + e(G_2) \leq (k - 1)(n_1 - k + 1) + (k - 1)(n_2 - k + 1) = (k - 1)(n - 2k + 3),$$

a contradiction to the assumption that $e(G) > (k - 1)(n - k + 1)$. $\square$

By Claims 1 and 2, G is 2-connected and has minimum degree at least k . By Theorem 1.4, G contains a $(0 \bmod k)$ -cycle, a contradiction. $\square$

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