CHROMATIC NUMBERS OF RANK-TWO ABELIAN CAYLEY GRAPHS

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Abstract. A connected Cayley graph for an Abelian group generated by a finite symmetric subset S can be represented by an integer matrix, its Heuberger matrix. We call the number of columns of that matrix its *rank* and the number of rows its *dimension*. Several previous papers have dealt with the question of finding a formula for the chromatic number of an Abelian Cayley graph in terms of an associated Heuberger matrix. In this paper, we fully resolve this matter for all integer matrices of rank ≤ 2. Prior results provide such formulas when the rank is 1, as well as when the rank is 2 and the dimension is no more than 4. Here, we complete the picture for the rank-two case by showing that when the rank is 2 and the dimension is at least 5, then the chromatic number equals 3 unless the graph has loops (in which case it is uncolorable); the graph is bipartite (in which case the chromatic number is 2); or the matrix has a zero row (in which case, the chromatic number does not change when that row is deleted).

1. Introduction

1.1. **Overview.** We begin by recalling some basic definitions. We say a subset S of a group G is *symmetric* if $x^{-1} \in S$ whenever $x \in S$. The *Cayley graph* for a group G with a symmetric subset S, denoted $\mathrm{Cay}(G, S)$, is the graph whose vertices are the elements of G, where each vertex g is adjacent to every vertex of the form gs with $s \in S$. In this case we call the set S the *generating set*, and we refer to its elements as *generators*. When the group G is abelian, naturally enough we call the graph an *Abelian Cayley graph*. The *chromatic number* of a graph X, denoted $\chi(X)$, is the minimum number of colors needed to assign each vertex in the graph a color such that no two adjacent vertices have the same color.

The chromatic number of Cayley graphs is important in information theory (see, for example, [2]) as well as in the construction of expanders and concentrators (see, for example, [9]). There are also applications with random sets and graphs, as seen in [1] and [6].

This paper focuses on chromatic numbers of Abelian Cayley graphs. As discussed in [4], to every finite-degree, connected Abelian Cayley graph we may associate an integer matrix called a *Heuberger matrix*. Conversely, every integer matrix has an associated Abelian Cayley graph. In Section 1.2 we lay out the essentials of this correspondence.

We call the number of columns of the Heuberger matrix its *rank* and the the number of rows its *dimension*. The goal of this research program is to obtain the chromatic number of an Abelian Cayley graph from an associated Heuberger matrix. We may divvy into cases according to the rank and dimension.

In matrices where the rank is greater than the dimension, one of the columns will be in the $\mathbb{Z}$-span of the others, so it can be reduced to a square matrix that represents the same graph. (See Theorem 1.4 for more detail.) So, without loss of generality, we can assume that the number of rows is at least the number of columns.

1.1.1. *The past.* Previously, chromatic numbers of Abelian Cayley graphs have been found in cases where the rank and/or dimension is small. Specifically, for rank 1 the chromatic number is determined in [4]. (See also [3].) We state that result here as Theorem 1.9. For rank 2, dimensions 2 and 3, theorems completely determining the chromatic number appear in [5]. These are stated here as Theorem 2.4 and Theorem 2.7. For rank 2, dimension 4, this is accomplished in [7]. That theorem is stated here as Theorem 2.10.

1.1.2. *The present.* The main new result in this paper is for $m \times 2$ matrices where $m \geq 5$. In these cases, the chromatic number of the associated Cayley graph is 3, with a few standard exceptions. Recall that a *loop* in a graph is an edge from a vertex to itself. A graph with a loop cannot be properly colored. (We adopt the convention that the chromatic number of a graph with loops is ∞.) A graph is *bipartite* if its vertex set can be partitioned into two sets A and B such that every edge has one endpoint in A and one in B. A connected bipartite graph always has chromatic number is 2. We call a row of a matrix a *zero row* if all of its entries are zero.

Theorem 1.1. *Let $m \geq 5$ be an integer. Let M_X be an $m \times 2$ integer matrix and $X = M_X^{SACG}$ (the Abelian Cayley graph associated with M_X). Suppose M_X has no zero rows. If X is not bipartite and has no loops, then $\chi(X) = 3$.*

We remark that the case when M_X has a zero row is handled by Theorem 1.7, which ensures that we may delete all zero rows without changing the chromatic number, whereupon a theorem for a smaller number of rows (i.e., one of Theorems 1.1, 2.4, 2.7, or 2.10) can be applied.

1.1.3. *The future.* This paper settles our main question for matrices of rank 2, so one obvious direction for future work is in matrices with rank greater than 2. For rank 3, the square matrices include circulant graphs of degree six. While there are many partial results, a complete formula for the chromatic numbers of such circulant graphs is unknown.

Theorem 1.1 tells us that, for the rank-two case, the behavior of the chromatic number stabilizes starting at dimension 5. An inspection of the method of proof suggests that something similar may occur for higher-rank cases as well. The proofs of Theorems 2.7, 2.10 and 1.1 all stem from a single central idea. Namely, we obtain a graph homomorphism from a graph with an $m \times r$ Heuberger matrix to one with an $(m-1) \times r$ Heuberger matrix whenever we merge two rows by adding or subtracting them — see Theorem 1.6. In this way we obtain upper bounds on chromatic numbers of graphs with larger matrices from those with smaller matrices. There are $2\binom{m}{2}$ such homomorphisms. With each homomorphism comes the potential to establish an upper bound of 3 on the chromatic number of the original graph. Fixing r and letting m increase, we have more and more such potential. We therefore offer the following conjecture:

Conjecture 1.2. *Let r be a positive integer. Then for all sufficiently large m, if M_X is an $m \times r$ integer matrix and $X = M_X^{SACG}$ (the Abelian Cayley graph associated with M_X), and M_X has no zero rows and X is not bipartite and has no loops, then $\chi(X) = 3$.*

We remark that Theorem 1.2 holds for both $r = 1$ and $r = 2$, as shown in Theorem 1.9 and Theorem 1.1, respectively.

1.2. **Preliminaries.** In this section we quickly sketch the basic correspondence between Abelian Cayley graphs and integer matrices. For more details, see [4].

Let G be a finitely generated Abelian group, and let $S = \{\pm s_1, \pm s_2, \ldots, \pm s_m\}$ be a symmetric subset of G that generates G. (In this paper, for Abelian groups we always use additive notation.) Let e_k denote the kth standard basis vector in $\mathbb{Z}_m$, that is, the vector whose kth component is 1 and with every other component equal to 0. We define a homomorphism $\phi : \mathbb{Z}^m \to G$ where $e_k \mapsto s_k$. If $H = \ker \phi$, then ϕ induces a natural graph homomorphism from $X = Cay(\mathbb{Z}^m/H, \{H \pm e_1, \ldots, H \pm e_m\})$ to $Cay(G, S)$. Suppose that we have $y_1, \ldots, y_r \in \mathbb{Z}^m$ such that $H = \langle y_1, \ldots, y_r \rangle$. Then, to represent the graph X we can write the $m \times r$ matrix M_X where the jth column is y_j. We say that M_X is a (non-unique) *Heuberger matrix* of X. This construction can also go the other way: given any integer matrix M, there exists an Abelian Cayley graph X with M as a Heuberger matrix. In this case we write $X = M^{SACG}$, and we refer to X as the SACG associated to M. (SACG stands for "standardized Abelian Cayley graph.")

Lemma 1.3. *Let M be an integer matrix, and let $X = M^{SACG}$. Then X has loops if and only if for some i the standard basis vector e_i is in the $\mathbb{Z}$-span of the columns of M.*

Lemma 1.4. *Let X and X' be Abelian Cayley graphs with Heuberger matrices M_X and $M_{X'}$, respectively.*

(1) If $M_{X'}$ is obtained by permuting columns of M_X, then $X = X'$.

(2) If $M_{X'}$ is obtained by multiplying a column of M_X by -1, then $X = X'$.

(3) If $M_{X'}$ is obtained by replacing the jth column of M_X with $y_j + ay_i$ $(i \neq j)$ for some $a \in \mathbb{Z}$, then $X = X'$.

(4) If $M_{X'}$ is obtained by deleting a column of M_X that is in the $\mathbb{Z}$-span of the other columns, then $X = X'$.

(5) If $M_{X'}$ is obtained by permuting rows of M_X, then X is isomorphic to X'.

(6) If $M_{X'}$ is obtained by multiplying a row of M_X by -1, then X is isomorphic to X'.

For integer matrices M_X and $M_{X'}$, we write $M_X \cong M_{X'}$ to indicate that their associated SACGs are isomorphic as graphs.

An integer matrix is *unimodular* if it is square and has determinant ± 1. A *signed permutation matrix* is a square matrix such that for every row and for every column, there is exactly one nonzero entry, which is ± 1. Beginning with an integer matrix M_X and performing a finite sequence of actions as in (1), (2), and (3) in Theorem 1.4 is equivalent to multiplying M_X on the right by a unimodular matrix. Beginning with an integer matrix M_X and performing a finite sequence of actions as in (5) and (6) in Theorem 1.4 is equivalent to multiplying M_X on the left by a signed permutation matrix.

Recall that a *graph homomorphism* is a function f from the vertex set of a graph X to the vertex set of a graph Y such that if v and w are adjacent in X, then $f(v)$ and $f(w)$ are adjacent in Y. The following lemma is well-known.

Lemma 1.5. *If there is a graph homomorphism from X to Y, then $\chi(X) \leq \chi(Y)$.*

We remark that Theorem 1.5 applies even in this case where Y has loops; but then we get that $\chi(X) \leq \infty$, which furnishes no useful information.

We now provide some standard graph homomorphisms between Abelian Cayley graphs.

Lemma 1.6. *The following operations on integer matrices induce graph homomorphisms on the corresponding Abelian Cayley graphs:*

(1) *Any of the transformations from Theorem 1.4*

(2) *Reducing a column by a common integer factor*

(3) *Collapsing the top two rows by adding them*

(4) *Appending an arbitrary column*

(5) *Appending a zero row*

(6) *Any composition of the above*

Lemma 1.7. *Let M_X and $M_{X'}$ be integer matrices with associated SACGs X and X', respectively. Suppose that $M_{X'}$ is obtained from M_X by deleting a zero row. Then $\chi(M_X) = \chi(M_Y)$.*

In many cases, we indicate the existence of a homomorphism from X to Y as in Lemma 1.6 without explicitly writing it out. In such cases we write $M_X \xrightarrow{\circledcirc} M_Y$, where M_X and M_Y are Heuberger matrices of X and Y, respectively. One common situation in which we do so occurs when we either add or subtract two rows. In this case, we will use ε_i (for some index i) to represent either 1 or -1. For example,

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \xrightarrow{\circledcirc} \begin{pmatrix} x_{11} + \varepsilon_1 x_{21} & x_{12} + \varepsilon_1 x_{22} \end{pmatrix}$$

describes a homomorphism where we either add the first and second row or multiply the second row by -1 and then add the rows.

The next few results describe the chromatic number for very simple Abelian Cayley graphs in terms of Heuberger matrices.

Lemma 1.8. *Let M be an integer matrix, and let $X = M^{SACG}$. Then X is bipartite if and only if all column sums of M are even.*

Theorem 1.9 (Tomato Cage Theorem)**.** *Let $y_1 = (y_{11}, \dots, y_{m1})^T \in \mathbb{Z}^m$ and let X be the SACG defined by $(y_1)_X^{SACG}$. If $y_1 = \pm e_i$ for some i, then X has loops and cannot be properly colored. If that's not the case, then*

$$\chi(X) = \begin{cases} 2 & \text{if } y_1 \text{ has an even number of odd entries} \\ 3 & \text{if } y_1 \text{ has an odd number of odd entries} \end{cases}$$

2. Previous Results

2.1. 1×2 **Heuberger matrices.** The result for a 1×2 matrix follows from a stronger result from [5].

Lemma 2.1. *Let $y_1,\dots,y_r$ be integers, not all 0. Suppose X is a SACG defined by $(y_1,\dots,y_r)_X^{SACG}$. Let $e = \gcd(y_1,\dots,y_r)$ Then X has loops if and only if $e = 1$. If there are no loops, then $\chi(X) = 2$ if e is even and $\chi(X) = 3$ if e is odd.*

2.2. 2×2 **Heuberger matrices.**

Definition 2.2. Let a, b, n be relatively prime integers where $n \neq 0$, $n \nmid a$, and $n \nmid b$. then, the graph $\mathrm{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$ is a *Heuberger circulant*, written as $C_n(a,b)$.

The relative primeness is so the graph is connected and the condition where $n \nmid a, b$ is so there are no loops.

The chromatic numbers for Heuberger circulants was determined by Heuberger:

Theorem 2.3 ([8, Theorem 3])**.** *Let $C_n(a,b)$ be a Heuberger circulant.*

$$\chi(C_n(a,b)) = \begin{cases} 2 \text{ if } a,b \text{ are both odd, but } n \text{ is even} \\ 5 \text{ if } n = \pm 5 \text{ and } a \equiv \pm 2b \mod 5 \\ 4 \text{ if } n = \pm 13 \text{ and } a \equiv \pm 5b \mod 13 \\ 4 \text{ if } (i)\ n \neq \pm 5 \text{ and } (ii)\ 3 \nmid n \text{ and } (iii)\ a \equiv \pm 2b \mod n \text{ or } b \equiv \pm 2a \mod n \\ 3 \text{ otherwise} \end{cases}$$

The following comes from [5]: Suppose M_X is a 2×2 integer matrix, and that $X = M_X^{SACG}$. We can always perform row and column operations as in Theorem 1.4 to create a lower-triangular 2×2 matrix $M_{X'}$ with non-negative diagonal entries such that $X' = M_{X'}^{SACG}$ is isomorphic to X.

With this, the chromatic number for any 2×2 matrix M_X can be computed.

Theorem 2.4 ([5])**.** *Let X be an SACG defined by*

$$\begin{pmatrix} y_{11} & 0 \\ y_{21} & y_{22} \end{pmatrix}_X^{SACG}$$

Also, suppose that $y_{11} \geq 0$ and $y_{22} \geq 0$. Let $d = \gcd(y_{11}, y_{21})$ and $e = \gcd(y_{11}, y_{21}, y_{22})$.

(1) If either (i) $y_{22} = 1$ or (ii) $y_{11} = 1$ and $y_{22} | y_{21}$ or (iii) $y_{11} = 0$ and $\gcd(y_{21}, y_{22}) = 1$, then X has loops and is not colorable.

(2) If both $y_{11} + y_{21}$ and y_{22} are even, then $\chi(X) = 2$

(3) If (i) *neither of the above holds and* (ii) $y_{11} = 0$ *or* $y_{22} = 0$ *or* $e > 1$ *or* $y_{22} | y_{21}$*, then* $\chi(X) = 3$*.*

(4) If none of the above hold, take $q \in \mathbb{Z}$ *such that* $\gcd(y_{11}, y_{21} + qy_{22}) = 1$*. Then,* $\chi(X) = \chi(C_n(a,b))$*, where* $a = -y_{21} - qy_{22}$*,* $b = y_{11}$*, and* $n = y_{11}y_{22}$*.*

2.3. 3×2 **Heuberger matrices.** We begin with a technical lemma about 3×2 matrices that we'll use later in the proof of the main theorem.

Lemma 2.5. *Suppose we have* $X = M_X^{SACG}$*, where*

$$M_X = \begin{pmatrix} y_{11} & 0 \\ y_{21} & 0 \\ y_{31} & y_{32} \end{pmatrix},$$

and that $y_{11}, y_{21}, y_{32} > 0$ *and* $-\frac{y_{32}}{2} \leq y_{31} \leq 0$*.*

Then:

(1) We have that X *has loops if and only if* $y_{32} = 1$*.*

(2) We have that $\chi(X) = 2$ *if and only if* $y_{11} + y_{21} + y_{31}$ *and* y_{32} *are both even.*

(3) We have that $\chi(X) = 4$ *if and only if* $y_{11} = y_{21} = -y_{31} = 1$ *and* $3 \nmid y_{32}$ *and* $y_{32} > 1$*.*

(4) Otherwise, $\chi(X) = 3$*.*

We now turn our attention to the statement of the theorem which gives the chromatic number for graphs associated with arbitrary 3×2 integer matrices.

Definition 2.6. Let

$$M = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}$$

be an integer matrix with no zero rows. Then, M is in *modified Hermite Normal Form* (MHNF) if all the following hold:

(1) $y_{11} > 0$

(2) $y_{12} = 0$

(3) $y_{11}y_{22} \equiv y_{11}y_{32} \mod 3$ (This is equivalent to $y_{11} \equiv 0$ or $y_{22} \equiv y_{32} \mod 3$)

(4) $y_{22} \leq y_{32}$

(5) $|y_{22}| \leq |y_{32}|$

(6) Either:

  (a) $y_{22} = 0$ and $-\frac{1}{2}|y_{32}| \leq y_{31} \leq 0$ or

  (b) $-\frac{1}{2}|y_{22}| \leq y_{21} \leq 0$

Lemma 2.12 in [5] gives a step-by-step method to turn any 3×2 integer matrix M_X without zero rows into a matrix $M_{X'}$ in MHNF in such a way that M_X^{SACG} is isomorphic to $M_{X'}^{SACG}$. We have a formula for the chromatic number of a 3×2 matrix in MHNF, namely:

Theorem 2.7 ([5])**.** *Let X be a standardized Abelian Cayley graph with Heuberger matrix*

$$M_X = \begin{pmatrix} y_{11} & 0 \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}$$

in MHNF.

(1) If the first column is e_1 or the second column is e_3, then X has loops and cannot be colored.

(2) If $y_{11} + y_{21} + y_{31}$ and $y_{22} + y_{32}$ are both even, then $\chi(X) = 2$.

(3) If

$$M_X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \pm 3k & 1+3k \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ \pm 3k & -1+3k \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ -1-3k & 2+3k \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -1 & -2 \\ -1+3k & -2+3k \end{pmatrix}$$

for some $k \in \mathbb{Z}^+$ or $M_X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3b & 2 \end{pmatrix}$ for some $b \in \mathbb{Z}$, or $M_X = \begin{pmatrix} 1 & 0 \\ -1 & a \\ -1 & a+3(k-1) \end{pmatrix}$ for $a \in \mathbb{Z}$ where $3 \nmid a$ and some $k \in \mathbb{Z}^+$, then $\chi(X) = 4$.

(4) If none of the above hold, then $\chi(X) = 3$.

We refer to the various possibilities in Theorem 2.7(3) as the *six exceptional cases.*

We say that a row of an integer matrix is *divisible by n* if all elements of the row are divisible by n. If a matrix has a row divisible by 3, then that property is preserved by the various row and column operations in Theorem 1.4. None of the six exceptional cases in Theorem 2.7 have a row divisible by 3. This yields the following corollary, which we will make use of in Section 3.

Corollary 2.8. *Let M_X be a 3×2 integer matrix, and let $X = M_X^{SACG}$. Suppose that M_X has no zero rows and that X does not have loops. If some row of M_X is divisible by 3, then $\chi(X) \le 3$.*

2.4. 4×2 **Heuberger matrices.** Before stating the main theorem for chromatic numbers of Abelian Cayley graphs with 4×2 Heuberger matrices, we first state a special case, which will be useful in the proof of this paper's main theorem in Section 3.

Lemma 2.9 ([7])**.** *Let M_Y be a 4×2 integer matrix, and let $Y = M_Y^{SACG}$. Suppose that M_Y has no zero rows and that Y does not have loops. If some row of M_Y is divisible by 3, then $\chi(Y) \le 3$.*

Observe the similarity between Theorem 2.8 and Theorem 2.9. We refer to them as the 3×2 (respectively, 4×2) three-divisible row lemmas.

Theorem 2.10 ([7])**.** *Let M be a $4{\times}2$ integer matrix, and let $X = M^{SACG}$. Furthermore suppose that X is not bipartite; that X has no loops; and that M has no zero rows. Then $\chi(X) = 4$ if and only if there exists a signed permutation matrix P and unimodular matrix U such that*

$$PMU = \begin{pmatrix} 1 & a \\ 1 & b \\ 1 & c \\ 0 & 1 \end{pmatrix}$$

for $a, b, c \in \mathbb{Z}$ with $3 \mid a+b+c$. Otherwise, $\chi(X) = 3$.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We begin with the case where $m = 5$. This will then serve as the base of an induction that proves the theorem for all $m \ge 5$.

3.1. 5×2 **Heuberger matrices.** The main idea of the proof for $m = 5$ is to use Theorem 1.6(3) to obtain a graph homomorphism from our graph with a given 5×2 matrix to a graph with a matrix with fewer rows. Then we can take advantage of the main theorems for $m\times 2$ matrices with $m < 5$. In particular, Theorem 2.9 will set things in motion. To make sure the conditions of that result are met, we first require a lemma. Recall our convention that every variable ϵ_i equals ± 1.

Lemma 3.1. *In an arbitrary 5×2 integer matrix M_Z, then either some row of M_Z is divisible by 3, or there exist two distinct rows r_1, r_2 of M_Z such that $r_1 + \epsilon_1 r_2$ is divisible by 3.*

Proof. Suppose that no row of M_Z is divisible by 3. Let r_1, r_2, r_3, r_4, r_5 be the rows of M_Z. By a harmless (we hope) abuse of notation, also let r_i denote the reduction mod 3 of the ith row of M_Z. So r_1, r_2, r_3, r_4, r_5 are elements of $V = \mathbb{Z}_3^2 \smallsetminus \{(0,0)\}$, which has eight elements. Partition V into four sets, where each set contains a vector and its negative. That is, the four sets are

$$\{(1,0),(2,0)\},\{(0,1),(0,2)\},\{(1,1),(2,2)\},\{(1,2),(2,1)\}.$$

By Pigeonhole Principle, two of the five elements r_1, r_2, r_3, r_4, r_5 must belong to the same set in this partition. Therefore two rows are either equal mod 3 (in which case, let $\epsilon_1 = -1$) or else opposite mod 3 (in which case, let $\epsilon_1 = 1$). $\square$

Theorem 3.2. *Let M_X be a 5×2 integer matrix and $X = M_X^{SACG}$. Suppose that X is not bipartite; that X has no loops; and that M_X has no zero rows. Then $\chi(X) = 3$.*

Proof. Let

$$M_X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \end{pmatrix}$$

and assume that $\chi(X) \geq 4$. We claim that two rows of M_X are linearly dependent over $\mathbb{Q}$.

Without loss of generality, we can use Theorem 3.1 to collapse the first two rows to get $M_X \overset{\circledcirc}{\to} M_Y$ where some row of M_Y is divisible by 3. (That is: If some row of M_X is divisible by 3, then permute rows to move it to the third row, and then collapse the top two rows by adding them. Otherwise, take two rows that are either equal or opposite mod 3, permute rows so as to move them to the top, and then collapse them by either adding or subtracting to produce a row divisible by 3.) Let $Y = M_Y^{SACG}$. By Theorem 2.9 and Theorem 1.5, either the graph Y has loops or M_Y has a zero row. In the latter case, it's immediate that two rows of M_Y are linearly dependent over $\mathbb{Q}$.

So assume Y has loops. Let y_1 be the first column of M_Y, and let y_2 be the second column. By Theorem 1.3, some standard basis vector e_i must lie in the $\mathbb{Z}$-span of y_1 and y_2. That is, $\alpha y_1 + \beta y_2 = e_i$ for some integers α, β. Note that α, β are not both zero. But at least two of the bottom three entries of $\alpha y_1 + \beta y_2$ are zero. Further, these bottom three rows of M_Y are also rows of M_X. So those two rows give us a 2×2 minor of M_X with determinant 0.

We have now shown that two rows of M_Y are linearly dependent over $\mathbb{Q}$. Assume without loss of generality that these are the bottom two rows.

Therefore, there exist $\alpha, \beta \in \mathbb{Z}$ (not both 0) such that

$$\alpha \begin{pmatrix} x_{41} \\ x_{51} \end{pmatrix} + \beta \begin{pmatrix} x_{42} \\ x_{52} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reducing by a common factor if need be, we can assume that α and β are relatively prime, so there exist integers k, ℓ such that $\alpha k + \beta \ell = 1$.

Multiply M_X on the right by the unimodular matrix

$$\begin{pmatrix} \alpha & -\ell \\ \beta & k \end{pmatrix}$$

to get

$$M_X \cong \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \\ 0 & w_{42} \\ 0 & w_{52} \end{pmatrix} \cong \begin{pmatrix} w_{42} & 0 \\ w_{52} & 0 \\ w_{12} & w_{11} \\ w_{22} & w_{21} \\ w_{32} & w_{31} \end{pmatrix}$$

for some integers w_{ij}.

Claim: $|w_{42}| = |w_{52}| = 1$.

Proof of claim: Consider

$$\begin{pmatrix} w_{42} & 0 \\ w_{52} & 0 \\ w_{12} & w_{11} \\ w_{22} & w_{21} \\ w_{32} & w_{31} \end{pmatrix} \xrightarrow{\circledcirc} \begin{pmatrix} w_{42} & 0 \\ w_{52} & 0 \\ w_{12} + \epsilon_5 w_{22} + \epsilon_6 w_{32} & w_{11} + \epsilon_5 w_{21} + \epsilon_6 w_{31} \end{pmatrix} = M_{Q_{\epsilon_5,\epsilon_6}},$$

where $Q_{\epsilon_5,\epsilon_6} = (M_{Q_{\epsilon_5,\epsilon_6}})^{SACG}$. Moreover, by Lemma 1.4 we have that

$$M_{Q_{\epsilon_5,\epsilon_6}} \cong \begin{pmatrix} |w_{42}| & 0 \\ |w_{52}| & 0 \\ w & |w_{11} + \epsilon_5 w_{21} + \epsilon_6 w_{31}| \end{pmatrix}$$

for some integer w such that the conditions of Theorem 2.5 are satisfied. Using Theorem 2.5, one can then show that if $Q_{\epsilon_5,\epsilon_6}$ has loops, then $w_{11} + \epsilon_5 w_{21} + \epsilon_6 w_{31} = \pm 1$. Solving the resulting system of equations, we then find that $Q_{\epsilon_5,\epsilon_6}$ has loops for all choices of ϵ_5 and ϵ_6 if and only if $(w_{11}, w_{21}, w_{31})^t = e_i$ for some $i = 1, 2, 3$. But then X has loops, contrary to our starting assumptions. Hence there is a choice of ϵ_5 and ϵ_6 such that $Q_{\epsilon_5,\epsilon_6}$ does not have loops. By Theorem 1.5, we have that $\chi(Q_{\epsilon_5,\epsilon_6}) \geq 4$. One more use of Theorem 2.5 then proves the claim.

Multiplying one or both of the top two rows by -1 as needed, we may assume without loss of generality that $w_{42} = w_{52} = 1$. That is:

$$M_X \cong \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ w_{12} & w_{11} \\ w_{22} & w_{21} \\ w_{32} & w_{31} \end{pmatrix}.$$

From here, we add the top two rows to get:

$$M_X \xrightarrow{\circledcirc} \begin{pmatrix} 2 & 0 \\ w_{12} & w_{11} \\ w_{22} & w_{21} \\ w_{32} & w_{31} \end{pmatrix} = M_W.$$

Let q be the number of elements of the set $\{w_{11}, w_{21}, w_{31}\}$ that are congruent to 0 mod 3. We now condition on the value of q.

Case $q = 3$**:** In this case we have that $w_{11} = 3t_1, w_{21} = 3t_2, w_{31} = 3t_3$ for some integers t_1, t_2, t_3. We can collapse the middle two rows to get

$$\begin{pmatrix} 2 & 0 \\ w_{12} & 3t_1 \\ w_{22} & 3t_2 \\ w_{32} & 3t_3 \end{pmatrix} \xrightarrow{\circledcirc} \begin{pmatrix} 2 & 0 \\ w_{12} + w_{22} & 3(t_1 + t_2) \\ w_{32} & 3t_3 \end{pmatrix} = M_Z.$$

Then M_Z satisfies conditions (1)-(3) of MNHF. We can perform row and column operations on M_Z to obtain a new matrix in MHNF, with the same top row as M_Z, whose associated graph is isomorphic to $Z = M_Z^{SACG}$. Because the top left entry is 2, this is not one of the six exceptional cases in Theorem 2.7. So, the only way it can fail to have chromatic number 3 is for Z to have loops. This occurs precisely when e_i is in the $\mathbb{Z}$-span of the columns of M_Z for some $i = 1, 2, 3$. However, it is straightforward to show that that is not possible.

Case $q = 2$**:** Assume without loss of generality that w_{11} and w_{21} are divisible by 3.

If a row of M_W is divisible by 3, we apply Theorem 2.9 to conclude that M_W has loops. The only way for this to have loops is if $w_{11} = w_{21} = 0$ and $w_{31} = \pm 1$, in which case X has loops, contrary to assumption.

Now suppose that no row of M_W is divisible by 3. Then w_{12} and w_{22} are ± 1 mod 3. Hence we have

$$M_W \xrightarrow{\circledcirc} \begin{pmatrix} 2 & 0 \\ w_{12} + \epsilon_2 w_{22} & w_{11} + \epsilon_2 w_{21} \\ w_{32} & w_{31} \end{pmatrix},$$

where ϵ_2 is chosen so that $w_{12} + \epsilon_2 w_{22} \equiv 0 \mod 3$. By Theorem 2.8, the only way the graph associated to the latter matrix can fail to have chromatic number 3 is for the second column to be e_3, so $w_{31} = \pm 1$ and $w_{11} = \epsilon_3 w_{21}$, where $\epsilon_3 = -\epsilon_2$. Thus we have that

$$M_X \cong \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ w_{12} & \epsilon_3 w_{21} \\ w_{22} & w_{21} \\ w_{32} & \epsilon_4 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ w_{12} - \epsilon_3\epsilon_4 w_{21} w_{32} & \epsilon_4\epsilon_3 w_{21} \\ w_{22} - \epsilon_4 w_{21} w_{32} & \epsilon_4 w_{21} \\ 0 & 1 \end{pmatrix}.$$

Then, the first and third row can be added or subtracted to get

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ w_{12} - \epsilon_3\epsilon_4 w_{21} w_{32} & \epsilon_4\epsilon_3 w_{21} \\ w_{22} - \epsilon_4 w_{21} w_{32} & \epsilon_4 w_{21} \\ 0 & 1 \end{pmatrix} \xrightarrow{\circledcirc} \begin{pmatrix} \epsilon_5 + w_{12} - \epsilon_3\epsilon_4 w_{21} w_{32} & \epsilon_4\epsilon_3 w_{21} \\ 1 & 0 \\ w_{22} - \epsilon_4 w_{21} w_{32} & \epsilon_4 w_{21} \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Because $w_{21} \equiv 0 \mod 3$, our ϵ_5 can be chosen such that $\epsilon_5 + w_{12} \equiv 0 \mod 3$, so the top row is divisible by 3. By Theorem 2.9, the graph associated to the latter matrix must have loops. Thus e_i is in the $\mathbb{Z}$-span of its columns for some $i = 1, 2, 3, 4$. From this it follows that either the first column is e_2 or the second column is e_4. The first column cannot be e_2 because w_{22} is not $0 \mod 3$. So the second column is e_4, which implies that $w_{21} = 0$. But then from (1) we see that X has loops, a contradiction.

Case $q = 1$: Without loss of generality, suppose that 3 divides w_{31}, and that $w_{11} \equiv \epsilon_1 \mod 3$ and $w_{21} \equiv \epsilon_2 \mod 3$. Writing the reduction of our matrices mod 3 rather than the matrices themselves, we have

$$\begin{pmatrix} 2 & 0 \\ w_{12} & \epsilon_1 \\ w_{22} & \epsilon_2 \\ w_{32} & 0 \end{pmatrix} \xrightarrow{\circledcirc} \begin{pmatrix} 2 & 0 \\ \epsilon_1 w_{12} - \epsilon_2 w_{22} & \epsilon_1^2 - \epsilon_2^2 = 0 \\ w_{32} & 0 \end{pmatrix}.$$

The contradiction here is now the same as in the $q = 3$ case above.

Case $q = 0$**:** Choose $\epsilon_1, \epsilon_2, \epsilon_3$ to be congruent mod 3 to w_{11}, w_{21}, and w_{31}, respectively. We then have

$$\begin{pmatrix} 2 & 0 \\ w_{12} & \epsilon_1 \\ w_{22} & \epsilon_2 \\ w_{32} & \epsilon_3 \end{pmatrix} \xrightarrow{\circledcirc} \begin{pmatrix} 2 & 0 \\ -(\epsilon_1 w_{12} + \epsilon_2 w_{22}) & -(\epsilon_1^2 + \epsilon_2^2) \\ \epsilon_3 w_{32} & 1 \end{pmatrix} \equiv \begin{pmatrix} 2 & 0 \\ -(\epsilon_1 w_{12} + \epsilon_2 w_{22}) & 1 \\ \epsilon_3 w_{32} & 1 \end{pmatrix}.$$

The argument going forward is similar to that in the $q = 3$ and $q = 1$ cases. The matrix on the right satisfies conditions (1)–(3) of MHNF. By the same logic as in the $q = 3$ case, the graph associated to that matrix must have loops. So e_i is in the $\mathbb{Z}$-span of its columns for some $i = 1, 2, 3$. We cannot have $i = 1$, as the top row is divisible by 2. So the second column is e_2 or e_3. If it were e_2, there is a contradiction because $1 \not\equiv 0 \mod 3$ (from the bottom right entry). If it were e_3, there is a contradiction from the middle right entry.

All possible cases lead to a contradiction, so $\chi(X) < 4$. By assumption, X is colorable and not bipartite, so $\chi(X) = 3$. $\square$

3.2. $m \times 2$ **Heuberger matrices with** $m > 5$**.** In this section, prove the main theorem of this paper.

Proof of Theorem 1.1. The proof is by induction, with Theorem 3.2 as a base case. Let $m \geq 6$. We assume that Theorem 1.1 has been proven for $(m-1) \times 2$ matrices, and we now prove it for $m \times 2$ matrices.

Let

$$M_X = \begin{pmatrix} x_{11} & x_{12} \\ \vdots & \vdots \\ x_{m1} & x_{m2} \end{pmatrix}.$$

Suppose $\chi(X) \geq 4$. Consider the graph homomorphisms

$$M_X \xrightarrow{\circledcirc} \begin{pmatrix} x_{11} & x_{12} \\ \vdots & \vdots \\ x_{m-2,1} & x_{m-2,2} \\ x_{m-1,1} + x_{m1} & x_{m-1,2} + x_{m2} \end{pmatrix} = M_{W_1} \text{ and}$$

$$M_X \xrightarrow{\circledcirc} \begin{pmatrix} x_{11} & x_{12} \\ \vdots & \vdots \\ x_{m-2,1} & x_{m-2,2} \\ x_{m-1,1} - x_{m1} & x_{m-1,2} - x_{m2} \end{pmatrix} = M_{W_2}$$

obtained by adding (respectively, subtracting) the bottom two rows of M_X.

By Theorem 1.5 and the inductive hypothesis, it follows that $W_1 = M_{W_1}^{SACG}$ and $W_2 = M_{W_2}^{SACG}$ both have loops. Thus we have that

$$\alpha_1 y_1 + \beta_1 y_2 = e_i \tag{2}$$

$$\alpha_2 z_1 + \beta_2 z_2 = e_j \tag{3}$$

for some $i, j = 1, \ldots, m-1$, where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are integers; y_1 and y_2 are the first and second columns of M_{W_1}, respectively; and z_1 and z_2 are the first and second columns of M_{W_2}, respectively.

Let $r_1, \ldots, r_{m-2}$ be the first $m-2$ rows of M_X. These are also the first $m-2$ rows of M_{W_1} as well as of M_{W_2}. Note that $m-2 \geq 4$. Hence there exist at least two distinct indices $a, b \notin \{i, j\}$. Without loss of generality, suppose that $a = 1$ and $b = 2$. By (2) and (3) we get that several inner products vanish, namely,

$$(\alpha_1, \beta_1) \cdot r_1 = (\alpha_2, \beta_2) \cdot r_1 = 0.$$

By assumption, r_1 is not a zero row. Moreover, the ith (respectively, jth) component in (2) (respectively, (3)) is 1, so neither (α_1, β_1) nor (α_2, β_2) is the zero vector and moreover, $\gcd(\alpha_1, \beta_1) = \gcd(\alpha_2, \beta_2) = 1$. It follows that $\alpha_2 = \epsilon_1 \alpha_1$ and $\beta_2 = \epsilon_1 \beta_1$ for some $\epsilon_1 = \pm 1$. From the $(m-1)$th components of (2) and (3), we then get that

$$\begin{aligned} \alpha_1(x_{m-1,1} + x_{m1}) + \beta_1(x_{m-1,2} + x_{m2}) &= \eta_1 \\ \epsilon_1 \alpha_1(x_{m-1,1} - x_{m1}) + \epsilon_1 \beta_1(x_{m-1,2} - x_{m2}) &= \eta_2, \end{aligned}$$

where $\eta_1 = 1$ if $i = m-1$ and $\eta_1 = 0$ if $i \neq m-1$, and $\eta_1 = 1$ if $i = m-1$ and $\eta_1 = 0$ if $i \neq m-1$. From this we find that

$$\alpha_1 x_{m-1,1} + \beta_1 x_{m-1,2} = (\eta_1 + \epsilon_1 \eta_2)/2 \tag{4}$$

$$\alpha_1 x_{m,1} + \beta_1 x_{m,2} = (\eta_1 - \epsilon_1 \eta_2)/2 \tag{5}$$

The right-hand sides of (4) and (5) are integers, so η_1 and η_2 are either both 0 or both 1. In the former case, we have that $i < m-1$. In the latter case, we have that $i = j = m-1$ and that the right-hand side of one of (4) or (5) is 0, while the other is 1. In either case, we get that $\alpha_1 x_1 + \beta_1 x_2 = e_k$ for some $k = 1, \ldots, m$, by considering (2) for the first $m-2$ rows, and (4) and (5) for the last two rows. But this implies that X has loops, contrary to assumption. $\square$

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