ADDITIVETORICVARIETIES: A MACAULAY2 PACKAGE FOR WORKING WITH ADDITIVE COMPLETE TORIC VARIETIES

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Abstract. We introduce the `AdditiveToricVarieties` package for *Macaulay2*, a software system for algebraic geometry and commutative algebra, with methods for working with additive group actions on complete toric varieties. More precisely, we implement algorithms, based on results by Arzhantsev, Dzhunusov and Romaskevich, to determine whether a complete toric variety admits an action of the commutative unipotent group and whether it is unique or not. We also observe that every smooth complete toric variety of Picard rank two is additive. We apply our methods to the class of smooth Fano toric varieties and notably determine all such varieties of dimension up to 6 admitting an additive action.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, let $\mathbb{G}_a = (\mathbb{K}, +)$ be its additive group, and let $\mathbb{G}_a^n = \mathbb{G}_a \times \cdots \times \mathbb{G}_a$ (n times) be the corresponding commutative unipotent group. For an irreducible algebraic variety X over $\mathbb{K}$ of dimension n, an *additive action* on X is a regular action $\mathbb{G}_a^n \times X \to X$ with an open orbit. If X is complete, this is equivalent to X being a completion of the affine space $\mathbb{A}^n$ that is equivariant with respect to the group of translations by elements of $\mathbb{A}^n$.

Additive actions naturally appear as a bridge between affine geometry and the geometry of projective varieties. They provide compactifications of the affine space that are equivariant with respect to translations, and hence are related to classical classification problems studied by Hirzebruch [Hir54]. Beyond their geometric interest, varieties with additive actions defined over number fields satisfy the Batyrev–Manin–Peyre principle, as shown by Chambert-Loir and Tschinkel [CLT02], which gives a precise description of the distribution of rational points of bounded height. From a differential-geometric perspective, they also relate to positivity questions: while the ampleness and nefness of the tangent bundle of smooth projective varieties are well understood through the results of Mori [Mor79] and the Campana–Peternell conjecture (see e.g. [MnOSC$^+$15]), Liu proved in [Liu23] that every variety with an additive action has a big tangent bundle. For further properties and techniques used in the study of additive actions, we kindly refer the reader to the survey by Arzhantsev and Zaitseva [AZ22].

An additive action on a toric variety X_Σ is *normalized* if it is normalized by the acting torus $\mathbb{T}$ (i.e., if $\mathbb{T}$ is a subgroup of the normalizer of $\mathbb{G}_a^n$ in the automorphism group of X_Σ). Arzhantsev and Romaskevich [AR17] prove that any two normalized additive actions on X_Σ are isomorphic, and that, if X_Σ is complete, then the existence of an additive action on X_Σ implies the existence of a normalized one.

We say X_Σ is *additive* if it admits an additive action and *uniquely additive* if any such action is isomorphic to the normalized one.

Macaulay2 [GSE93] is a software system that is widely used for research in algebraic geometry and commutative algebra. It contains a variety of existing methods which may be used alongside our package. In particular, it implements polyhedral fans and normal toric varieties.

Our paper is organized in the following way:

In Section 2 we give brief and minimal preliminaries, including previous work due to Arzhantsev, Romaskevich and Dzhunusov [AR17, Dzh21, Dzh22] about additive actions on complete and projective toric varieties, along with classification results for Gorenstein Fano and smooth Fano toric varieties primarily due to Batyrev, K. Watanabe, M. Watanabe, Sato and Øbro [Bat81, WW82, Bat99, Sat00, Øb07].

In Section 3 we prove that every smooth complete toric variety X of dimension $\dim(X) \geq 2$ and Picard number $\rho(X) = 2$ is additive, and also study certain projective bundles over Hirzebruch surfaces.

In Section 4 we describe the methods included in the *Macaulay2* [GSE93] and give examples of usage.

In Section 5 we apply our methods to classify smooth Fano toric varieties X of dimension $\dim(X) \leq 6$ as (uniquely) additive or not, and match these results with existing classifications for $\dim(X) \leq 4$. This is noteworthy, since existential results for additivity of smooth Fano toric varieties had previously only been obtained for the case in which $\dim(X) \leq 3$ by Huang and Montero [HM20], who used more geometric techniques.

Finally, in Section 6 we suggest some further directions and prove an enumeration result.

2. Preliminaries

2.1. **Complete toric varieties.**

We refer the reader to [CLS11] for details about the general theory of toric varieties.

Let $\mathbb{T} \cong (\mathbb{K}^\times)^n$ be a torus, let $M = \mathrm{Hom}(\mathbb{T}, \mathbb{K}^\times)$ be the lattice of characters of $\mathbb{T}$, and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice of M. Let $M_\mathbb{Q}$ and $N_\mathbb{Q}$ be the corresponding rational vector spaces, and let $\langle \cdot, \cdot \rangle : N_\mathbb{Q} \times M_\mathbb{Q} \to \mathbb{Q}$ be the corresponding non-degenerate duality pairing. Unless specified otherwise, $M \cong \mathbb{Z}^n$.

For a fan Σ in $N_\mathbb{Q}$, we denote by $\Sigma(1)$ the set of rays and by X_Σ the corresponding toric variety of dimension n. For each ray $\rho \in \Sigma(1)$, we denote by p_ρ the unique primitive lattice vector on ρ.

From now until the end of the paper, all varieties are assumed to be irreducible, and all fans Σ and toric varieties X_Σ are assumed to be *complete*.

Definition 2.1. For each ray $\rho \in \Sigma(1)$, define the set

$$\mathfrak{R}_\rho := \{m \in M : \quad \langle p_\rho, m\rangle = -1 \text{ and } \langle p_{\rho'}, m\rangle \geq 0 \text{ for all } \rho' \in \Sigma(1) \setminus \{\rho\}\}.$$

Define also $\mathfrak{R} := \bigsqcup_{\rho \in \Sigma(1)} \mathfrak{R}_\rho$. The elements of $\mathfrak{R}$ are called the **Demazure roots** of Σ.

Proposition 2.2. *Let $\rho \in \Sigma(1)$. Define the set*

$$\mathfrak{R}'_\rho := \{m \in M_\mathbb{Q} : \quad \langle p_\rho, m\rangle = -1 \textit{ and } \langle p_{\rho'}, m\rangle \geq 0 \textit{ for all } \rho' \in \Sigma(1) \setminus \{\rho\}\}.$$

Then, $\mathfrak{R}'_\rho$ is bounded. In particular, the set $\mathfrak{R}$ of Demazure roots of Σ is finite.

Proof. In [Dem70, §4, Lemme 7], Demazure proves that $\mathfrak{R}$ is finite under a related hypothesis, namely that the fan Σ is *semi-complete.* This is equivalent to Σ being *complete* for toric varieties defined over a field. In any case, we adapt his proof to show that $\mathfrak{R}'_\rho$ is bounded if Σ is complete.

Assume $\mathfrak{R}'_\rho$ is not bounded. Then, since it is a rational polyhedron, it contains an affine ray $x + \mathbb{Q}^{\geq 0}\, a$, where $x \in \mathfrak{R}'_\rho$ and $a \in N_{\mathbb{Q}}$. But $\langle p_\rho, x + \mathbb{Q}^{\geq 0}\, a\rangle = \langle p_\rho, x\rangle + \mathbb{Q}^{\geq 0}\langle p_\rho, a\rangle = -1$, so $\langle p_\rho, a\rangle = 0$ and $\langle p_{\rho'}, a\rangle \geq 0$ for all $\rho' \in \Sigma(1) \setminus \{\rho\}$. Then Σ is not complete. $\square$

Definition 2.3. Let $S = \{e_1, \dots, e_n\} \subseteq \mathfrak{R}$. If there exists a linear order $\rho_1, \dots, \rho_{\#\Sigma(1)}$ on $\Sigma(1)$ such that $\langle p_i, e_j\rangle = -\delta_{ij}$ for all $i, j = 1, \dots, n$, then S is called a **complete collection** of Demazure roots of Σ.

Theorem 2.4. [AR17, Theorem 1] *Complete collections of Demazure roots of Σ are in one-to-one correspondence with normalized additive actions on X_Σ.*

Theorem 2.5. [AR17, Theorem 2] *Any two normalized actions on X_Σ are isomorphic.*

Theorem 2.6. [AR17, Theorem 3] *The following are equivalent:*

1. *There exists an additive action on X_Σ.*
2. *There exists a normalized additive action on X_Σ.*

The following proposition will reduce the search space of the problem of finding complete collections of Demazure roots, since it involves a notion defined in terms of subsets of $\Sigma(1)$ instead of linear orders.

Definition 2.7. Let $S = \{s_1, \dots, s_{\#S}\} \subset N_{\mathbb{Q}}$ be a finite subset. The **negative octant** of S is the cone of non-positive linear combinations of elements of S

$$\left\{ \sum_{i=1}^{\#S} a_i s_i : \quad a_i \in \mathbb{Q} \text{ and } a_i \leq 0 \right\} \subseteq N_{\mathbb{Q}}.$$

Proposition 2.8. [Dzh21, Proposition 1, Lemma 2] *The following are equivalent:*

1. *X_Σ is additive.*
2. *There exist rays $\rho_1, \dots, \rho_n \in \Sigma(1)$ such that $B = \{p_1, \dots, p_n\}$ is a basis of the lattice N and the remaining rays in $\Sigma(1)$ are contained in the negative octant of B. In this case, $-p_i^\vee \in \mathfrak{R}_i$ for all $i = 1, \dots, n$ and $-B^\vee = \{-p_1^\vee, \dots, -p_n^\vee\} \subset \mathfrak{R}$ is a complete collection of Demazure roots of Σ. Here, $B^\vee$ is the* linear algebraic *dual basis of B.*

Theorem 2.9. [Dzh22] *The following are equivalent:*

1. *X_Σ is uniquely additive.*

2. *There exist rays $\rho_1, \dots, \rho_n \in \Sigma(1)$ such that $B = \{p_1, \dots, p_n\}$ is a basis of the lattice N, the remaining rays in $\Sigma(1)$ are contained in the negative octant of B, and $\mathfrak{R}_i = \{-p_i^\vee\}$ for all $i = 1, \dots, n$.*

The following proposition will further reduce the aforementioned search space. It is inspired by [AR17, Theorem 4] (here, Theorem 2.15).

Proposition 2.10. *Assume there exist rays $\rho_1, \dots, \rho_n \in \Sigma(1)$ such that $B = \{p_1, \dots, p_n\}$ is a basis of the lattice N and the remaining rays in $\Sigma(1)$ are contained in the negative octant of B. Then, $\rho_1, \dots, \rho_n$ are the extremal rays of a maximal cone $\sigma \in \Sigma$.*

Proof. Let $u \in \mathrm{RelInt}(\mathrm{Cone}(p_1, \dots, p_n)) \cap N$ such that u is in the relative interior of some maximal cone $\sigma = \mathrm{Cone}(p_1, \dots, p_m, p'_1, \dots, p'_M) \in \Sigma$ with $m \leq n$ maximal with this property, which is possible since Σ is complete. We may write $u = \sum_{i=1}^n \lambda_i p_i$ and $u = \sum_{i=1}^m \mu_i p_i + \sum_{j=1}^M \nu_j p'_j$ with $\lambda_i, \mu_i, \nu_i > 0$. Note that necessarily we have that $m = n$ because the rightmost sum contributes non-positively to the coefficients of the p_i, since each p'_j is in the negative octant of B. If $\sigma \neq \mathrm{Cone}(p_1, \dots, p_n)$, then σ contains both p'_1 and $-p'_1$, which contradicts its strict convexity. The result follows. □

We may now state the two characterisations that we will implement in the core method of our *Macaulay2* package (see Subsection 4.2).

Corollary 2.11. *The following are equivalent:*

1. *X_Σ is additive.*
2. *There exists a maximal cone $\sigma \in \Sigma$ such that the primitive vectors on the extremal rays of σ_P form a basis $B = \{p_1, \dots, p_n\}$ of the lattice N and the remaining rays in $\Sigma_P(1)$ are contained in the negative octant of B.*

Corollary 2.12. *The following are equivalent:*

1. *X_Σ is uniquely additive.*
2. *There exists a maximal cone $\sigma \in \Sigma$ such that the primitive vectors on the extremal rays of σ_P form a basis $B = \{p_1, \dots, p_n\}$ of the lattice N, the remaining rays in $\Sigma_P(1)$ are contained in the negative octant of B, and $\mathfrak{R}_i = \{-p_i^\vee\}$ for all $i = 1, \dots, n$.*

2.2. **Projective toric varieties.**

We refer the reader to [CLS11] for details about the interplay between polytopes and projective toric varieties.

From now until the end of the paper, a polytope $P \subset M_\mathbb{Q}$ is a convex lattice polyhedron that is bounded, n-dimensional and very ample. We denote by Σ_P its (complete) normal fan in $N_\mathbb{Q}$ and by X_P its corresponding projective toric variety. We also denote by $\mathcal{V}(P)$, $\mathcal{E}(P)$ and $\mathcal{F}(P)$ its set of vertices, edges and facets, respectively.

Definition 2.13. A polytope $P \subset M_{\mathbb{Q}}$ is **inscribed in a rectangle** if there exists a vertex $v_0 \in \mathcal{V}(P)$ such that:

1. The primitive lattice vectors on the edges $E_i \in \mathcal{E}(P)$ of P starting at v_0 form a basis $e_1, \ldots, e_n$ of M.
2. $\langle p_F, e_i \rangle \leq 0$ for all $i = 1, \ldots, n$ and $\rho_F \in \Sigma(1)$ corresponding to a facet $F \in \mathcal{F}(P)$ such that $v_0 \notin F$.

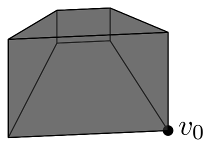

FIGURE 1. The polytope corresponding to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$.

Proposition 2.14. *Let $P \subset M_{\mathbb{Q}}$ be a polytope. For a vertex $v \in \mathcal{V}(P)$, let C_v be the cone generated by the subset $(P \cap M) - v \subset M_{\mathbb{Q}}$ and let $\sigma_v = (C_v)^{\vee}$. Then:*

1. *For a vertex $v \in \mathcal{V}(P)$, the set of rays parallel to the edges in $\mathcal{E}(P)$ starting at v is equal to the set of extremal rays of C_v.*
2. *There is a one-to-one correspondence between vertices in $\mathcal{V}(P)$ and maximal cones in Σ mapping $v \in \mathcal{V}(P)$ to $\sigma_v \in \Sigma_P$.*

Proof. 1. This follows from the definition of C_v.

2. This follows from [CLS11, Proposition 2.3.7, Proposition 2.3.8].

□

Theorem 2.15. [AR17, Theorem 4] *Let $P \subset M_{\mathbb{Q}}$ be a polytope. The following are equivalent:*

1. *X_P is additive.*
2. *P is inscribed in a rectangle.*

2.3. **Smooth Fano toric varieties.**

Definition 2.16. Let X be an algebraic variety over $\mathbb{K}$.

1. Assume X is normal and complete. X is **Gorenstein Fano** if the anticanonical divisor $-K_X$ is Cartier and ample (in particular, X is projective).
2. Assume X is normal and complete. X is **smooth Fano** if it is Gorenstein Fano and smooth.

Definition 2.17. Let P be a polytope in $M_{\mathbb{Q}}$ or $N_{\mathbb{Q}}$.

1. P is **smooth** if, for every vertex $v \in \mathcal{V}(P)$, the primitive lattice vectors on the edges $e \in \mathcal{E}(P)$ starting at v form a basis of the lattice.

2. Assume $0 \in \mathrm{RelInt}(P)$. P is **reflexive** if its dual $P^\vee$ is also a lattice polytope.
3. Assume $0 \in \mathrm{RelInt}(P)$. P is **smooth Fano** if the vertices of every facet form a basis of the lattice.

Proposition 2.18. [CLS11, Exercise 8.3.6 (a), (b)] *Let $P \subset N_{\mathbb{Q}}$ be a polytope such that $0_{N_{\mathbb{Q}}} \in \mathrm{RelInt}(P)$. If P is smooth Fano, then P is reflexive and $P^\vee$ is smooth and reflexive.*

Theorem 2.19. *Up to isomorphism on both sides:*

1. *There is a one-to-one correspondence between pairs (X_P, D_P), where X_P is a smooth projective toric variety and D_P is a torus-invariant Cartier divisor on X_P, and smooth polytopes $P \subset M_{\mathbb{Q}}$.*
2. *There is a one-to-one correspondence between Gorenstein Fano toric varieties X_P and reflexive polytopes $P \subset M_{\mathbb{Q}}$. Here, the torus-invariant Cartier divisor on X_P is $-K_{X_P}$.*
3. *There is a one-to-one correspondence between smooth Fano toric varieties $X_{P^\vee}$ and smooth Fano polytopes $P \subset N_{\mathbb{Q}}$. Here, $P^\vee \subset M_{\mathbb{Q}}$ is the dual polytope of P and the torus-invariant Cartier divisor on $X_{P^\vee}$ is $-K_{X_{P^\vee}}$.*

Proof. 1. This is [CLS11, Theorem 2.4.3]
2. This is [CLS11, Theorem 8.3.4]
3. This follows from [CLS11, Exercise 8.3.6 (b), (c)].

□

Remark 2.20. A potential source of confusion is that, in [CLS11], "smooth Fano" polytopes and varieties are simply called "Fano". We follow the more common convention by calling them "smooth Fano".

Note that smooth Fano polytopes are not necessarily smooth, but their duals are.

Note also that smoothness implies condition 1. in Definition 2.13 at all vertices $v_0 \in \mathcal{V}(P)$. In particular, for each vertex $v_0 \in \mathcal{V}(P)$, the primitive vectors on the extremal rays of the corresponding maximal cone $\sigma \in \Sigma$ form a basis of the lattice N.

Finally, note that Demazure roots of normal fans Σ_P of reflexive polytopes $P \subset M_{\mathbb{Q}}$ correspond to points in the relative interior of the facets in $\mathcal{F}(P)$.

It is well-known that there are finitely many isomorphism classes of reflexive polytopes, then also of Gorenstein Fano toric varieties. They were classified algorithmically by Kreuzer and Skarke [KS04].

Smooth Fano toric surfaces are easily classified by inspecting the corresponding reflexive polygons.

Smooth Fano threefolds were classified by Mori and Mukai [MM03]. Smooth Fano toric threefolds were classified by K. Watanabe and M. Watanabe [WW82], and independently by Batyrev [Bat81], who also classified smooth Fano toric fourfolds [Bat99] along with Sato [Sat00].

Higher-dimensional smooth Fano toric varieties were all classified algorithmically by Øbro [Øb07]. The data up to dimension 6 is available on the GRDB [BK09].

Remark 2.21. The sequence $(a_n)_{n\geq 1}$, where a_n is the number of isomorphism classes of reflexive polytopes, starts with $1, 16, 4319, 473800776, \ldots$

On the other hand, the sequence $(b_n)_{n\geq 1}$, where b_n is the number of isomorphism classes of smooth Fano polytopes, starts with $1, 5, 18, 124, 866, 7622, \ldots$

Since a_n grows much faster than b_n, it is much more feasible to determine existence and uniqueness of additive actions on higher-dimensional smooth Fano toric varieties than it is on Gorenstein Fano toric varieties.

2.4. **Betti numbers.**

We now recall and give identities involving invariants of complete toric varieties with *simplicial* fans. Further specializations will allow us to match our results on smooth Fano toric varieties with existing classifications.

In this subsection we assume that $P \subset N_{\mathbb{Q}}$ is simplicial and that $0_{N_{\mathbb{Q}}} \in \mathrm{RelInt}(P)$.

Definition 2.22. For $i = 0, \ldots, 2n$, let $H^i(X_{P^\vee}, \mathbb{Q})$ be the i-th rational cohomology group of $X_{P^\vee}$. The i-th **Betti number** of $X_{P^\vee}$ is $b_i := \dim(H^i(X_{P^\vee}, \mathbb{Q}))$ (in particular, b_2 is the rank of the Picard group when $X_{P^\vee}$ is smooth). For $p = 0, \ldots, n$, define $h_p := b_{2p}$.

The i-th **face number** f_i of P is its number of faces of dimension i (in particular, $f_0 = \#\mathcal{V}(P)$ and $f_{n-1} = \#\mathcal{F}(P)$). We also set $f_{-1} = 1$.

Finally, we say a Betti number of $X_{P^\vee}$ is **constant** if it is equal to an expression which does not depend on any face number.

Proposition 2.23. *1. For each $p = 0, \ldots, n$, $h_p = h_{n-p}$.*
2. If $i = 0, \ldots, 2n$ is odd, then $b_i = 0$.

Proof. 1. This follows from Poincaré duality [Ful93, Section 5].
2. This is the fact that $H^i(X_{P^\vee}, \mathbb{Q})$ is trivial if i is odd [Ful93, Section 5].

□

Proposition 2.24. *For each $p = 0, \ldots, n$,*

$$h_p = b_{2p} = \sum_{i=p}^{d} (-1)^{i-p} \binom{i}{p} f_{d-i-1}.$$

In particular:

1. The constant even Betti numbers of $X_{P^\vee}$ are $h_0 = b_0 = \sum_{i=0}^{n} (-1)^i f_{n-i-1} = h_n = b_{2n} = 1$.

2. $h_1 = b_2 = \sum_{i=1}^{n}(-1)^{i-1} i f_{n-i-1} = h_{n-1} = b_{2n-2} = f_0 - n = \#\mathcal{V}(P) - n.$

Proof. This follows from [Ful93, Section 5]. □

3. Toric varieties with Picard number equal to two

Let us recall the classification of smooth complete toric varieties X with Picard rank $\rho(X) = 2$ due to Kleinschmidt (compare with [CLS11, Theorem 7.3.7]).

Theorem 3.1. [Kle88, Theorem 1] *Let X be a smooth complete toric variety of dimension $n \geq 2$ with $\rho(X) = 2$. Then there are positive integers $r, s \geq 1$ such that $r + s = n$ and that*

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r)),$$

where $0 \leq a_1 \leq \cdots \leq a_r$.

The main result of this section is the following:

Proposition 3.2. *Let X be a smooth complete toric variety of dimension $n \geq 2$ with $\rho(X) = 2$. Then, X is additive. Furthermore, X is uniquely additive if and only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let $s, r \geq 1$ and $0 \leq a_1 \leq \cdots \leq a_r$ be integers such that

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r)),$$

where $n = r + s$. It follows from [Kle88, §2] (compare with [CLS11, Example 7.3.5]) that X is the toric variety corresponding to a fan $\Sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ whose rays $\rho_1, \ldots, \rho_{n+2}$ are generated by the following $n + 2$ primitive lattice vectors:

- $u_i := e_i$ for $i = 1, \ldots, r$,
- $u_{r+1} := -\sum_{i=1}^{r} e_i$,
- $u_{r+j+1} := e_{r+j}$ for $j = 1, \ldots, s$,
- $u_{n+2} := \sum_{i=1}^{r} a_i e_i - \sum_{j=1}^{s} e_{r+j}$,

where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$. Let $e_1^\vee, \ldots, e_n^\vee \in M$ be the basis of M that is dual to $e_1, \ldots, e_n \in N$, and define

- $m_i := e_r^\vee - e_i^\vee$ for $i = 1, \ldots, r-1$,
- $m_{r+1} := e_r^\vee$,
- $m_{r+1+j} := -e_{r+j}^\vee$ for $j = 1, \ldots, s$.

A direct computation shows that the set $\{m_1, \ldots, m_{r-1}, m_{r+1}, m_{r+2}, \ldots, m_{n+1}\}$ is a complete collection of Demazure roots for the fan Σ of X. More precisely, we have that $m_i \in \mathfrak{R}_{\rho_i}$ for every $i = 1, \ldots, r-1, r+1, \ldots, n+1$ and that [1]

$$\langle u_i, m_j \rangle = -\delta_{ij} \text{ for all } i, j = 1, \ldots, r-1, r+1, \ldots, n+1.$$

It follows from Theorem 2.4 that X is additive.

Furthermore, if there exists $a_i \neq 0$, then $\{e_r^\vee, e_r^\vee + e_{r+1}^\vee\} \subseteq \mathfrak{R}_{\rho_{r+1}}$ so, by Theorem 2.9, X is not uniquely additive. If all $a_i = 0$, then $X \cong \mathbb{P}^r \times \mathbb{P}^s$, which is uniquely additive if $r = s = 1$ and admits at least two non-isomorphic additive actions for all other values of r, s by [HT99]. □

Remark 3.3. In dimension 3 or higher, the assumption $\rho(X) = 2$ in Proposition 3.2 cannot be removed. Indeed, the smooth Fano toric threefold $\mathrm{III}_{25} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1))$ is not additive, since III_{25} is also obtained as the blow-up of $\mathbb{P}^3$ along two disjoint lines and it follows from [HT99, Proposition 3.3] that the invariant subvarieties for all the four additive actions on $\mathbb{P}^3$ are contained in the boundary projective plane.

The following well-known fact will be useful for computations. It follows in greater generality from Proposition 2.24 2.

Proposition 3.4. *Let X_Σ be a smooth complete toric variety. Then, $\rho(X) = \#\Sigma(1) - n$.*

3.1. **Projective bundles over Hirzebruch surfaces.**

Let $d \geq 0$ be a non-negative integer. Let us recall that the fan of the Hirzebruch surface $\Sigma_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ is given by

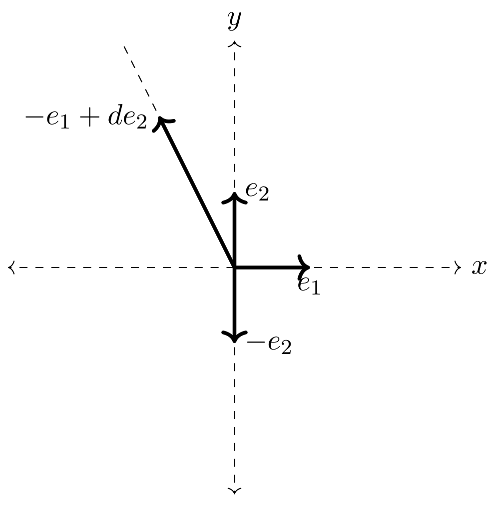

FIGURE 2. The fan of the Hirzebruch surface Σ_d.

It follows from [CLS11, Proposition 7.3.3], that if we consider $D \sim aF + b\xi$ with $a, b \in \mathbb{Z}$, then

$$\Sigma(d; a, b) := \mathbb{P}(\mathcal{O}_{\Sigma_d} \oplus \mathcal{O}_{\Sigma_d}(aF + b\xi))$$

[1]Note that we only require $\langle u_i, m_j \rangle \geq 0$ for $i = r$ and $i = n+2$.

is a smooth toric threefold, whose fan is given by

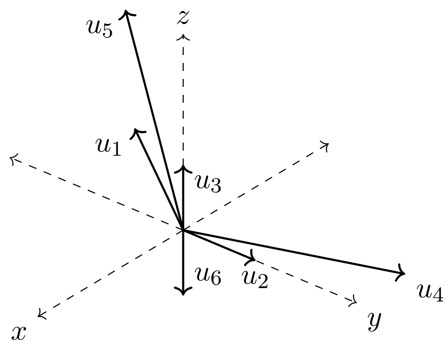

FIGURE 3. The fan of $\Sigma(d; a, b)$.

Here, the primitive generators of the rays of the fan of $\Sigma(d; a, b)$ are given by

$$u_1 = (1, 0, a), \ \ u_2 = (0, 1, 0), \ \ u_3 = (0, 0, 1), \ \ u_4 = (-1, d, 0), \ \ u_5 = (0, -1, b), \ \ u_6 = (0, 0, -1).$$

Moreover, by twisting by $\mathcal{O}_{\Sigma_d}(-D)$ if necessary, we may assume that $a \leq b$.

In order to apply the results in [AR17] to determine whether $\Sigma(d; a, b)$ is additive or not, we will first compute the set of Demazure roots associated to each ray of the fan. More precisely, for $i = 1, \ldots, 6$, consider

$$\mathfrak{R}_i := \{m = (p, q, r) \in \mathbb{Z}^3 \mid \langle u_i, m\rangle = -1 \text{ and } \langle u_j, m\rangle \geq 0 \text{ for all } j \neq i\}.$$

Then:

1. $\mathfrak{R}_1 = \{(-1, 0, 0)\}$.
2. $\mathfrak{R}_2 = \{(0, -1, 0)\}$.
3. $\mathfrak{R}_3 = \{m = (p, q, -1) \in \mathbb{Z}^3 \mid p \geq a, \ q \geq 0, \ qd \geq p, \ -b \geq q\}$. In particular, $\mathfrak{R}_3 \neq \varnothing$ if and only if $b \leq 0$.
4. $\mathfrak{R}_4 = \{(1, 0, 0)\}$.
5. $\mathfrak{R}_5 = \{(0, 1, 0)\}$.
6. $\mathfrak{R}_6 = \{m = (p, q, 1) \in \mathbb{Z}^3 \mid p \geq -a, \ q \geq 0, \ qd \geq p, \ b \geq q\}$. In particular, $\mathfrak{R}_6 \neq \varnothing$ if and only if $a + bd \geq 0$.

Proposition 3.5. *Let $d \geq 0$ and $a \leq b$ be integers. Then, the toric threefold*

$$\Sigma(d; a, b) := \mathbb{P}(\mathcal{O}_{\Sigma_d} \oplus \mathcal{O}_{\Sigma_d}(aF + b\xi))$$

is additive if and only if $b \leq 0$ or $a + bd \geq 0$.

Proof. We first remark that $\mathfrak{R}_3 \neq \varnothing$ and $\mathfrak{R}_6 \neq \varnothing$ if and only if $a \leq b \leq 0$ and $0 \leq (-b)d \leq a$, which implies that $a = b = 0$. On the other hand, $\Sigma(d; 0, 0) \cong \Sigma_d \times \mathbb{P}^1$ is additive.

Assume first that $\mathfrak{R}_3 \neq \varnothing$ and $\mathfrak{R}_6 = \varnothing$. In this case, we obtain a complete collection of Demazure roots for the fan of $\Sigma(d; a, b)$ by considering

$$m_1 = (-1, 0, 0) \in \mathfrak{R}_1, \; m_2 = (0, 1, 0) \in \mathfrak{R}_5, \; m_3 = (a, -b, -1) \in \mathfrak{R}_3.$$

Secondly, we assume that $\mathfrak{R}_6 \neq \varnothing$ and $\mathfrak{R}_3 = \varnothing$. In this case, we obtain a complete collection of Demazure roots for the fan of $\Sigma(d; a, b)$ by considering

$$m_1 = (-1, 0, 0) \in \mathfrak{R}_1, \; m_2 = (0, 1, 0) \in \mathfrak{R}_5, \; m_3 = (-a, b, 1) \in \mathfrak{R}_6.$$

Therefore, in both cases we conclude from Theorem 2.4 that $\Sigma(d; a, b)$ is additive.

Finally, we note that if $\mathfrak{R}_3$ and $\mathfrak{R}_6$ are empty, then we have at most two linearly independent Demazure roots, and hence in that case $\Sigma(d; a, b)$ is not additive. □

4. Package description

We introduce `AdditiveToricVarieties`, a new *Macaulay2* [GSE93] package for working with additive actions on complete toric varieties, with methods based on the results we have discussed so far. We have followed package writing guidelines and standards described in the *Macaulay2* official website. In particular, we have included documentation and examples for each method, which are automatically generated as HTML files upon installation. We have also implemented automated tests which may be run at any time.

The package also includes a pre-processed database which classifies smooth Fano toric varieties X of dimension at most 6 as additive (or not) and uniquely additive (or not), as per the database included in the `NormalToricVarieties` package.

The package dependencies are packages `NormalToricVarieties` and `Polyhedra`. The exported methods are `demazureRoots`, `isAdditive`, `listAdditiveSmoothFanoToricVarieties` and `listUniquelyAdditiveSmoothFanoToricVarieties`.

The package and installation instructions may be found in the first author's *GitHub* repository: `https://github.com/iZafiro/AdditiveToricVarieties`.

4.1. **demazureRoots.**

This method takes as input a complete fan Σ (an object of type `Fan`) and optionally a primitive lattice vector p_i on a ray $\rho_i \in \Sigma(1)$ (an object of type `Matrix`). If only a fan Σ is given, it returns a hash table with keys the lattice vectors on all rays in $\Sigma(1)$ (objects of type `List`) and values the corresponding sets $\mathfrak{R}_i$ (objects of type `Set`). If additionally a vector p_i is given, it only returns the corresponding set $\mathfrak{R}_i$ (an object of type `List`).

Example 4.1. The projective plane has six Demazure roots:

```
i1 : F = fan toricProjectiveSpace 2;
i2 : demazureRootsF = demazureRoots F;
```

```
i3 : flatten values demazureRootsF

o3 = {{0, -1}, {1, -1}, {-1, 1}, {-1, 0}, {0, 1}, {1, 0}}

o3 : List
```

There are two roots corresponding to `e1`:

```
i4 : e1 = {1, 0};

i5 : demazureRootsF#e1

o5 = {{-1, 1}, {-1, 0}}

o5 : List
```

Alternatively, this can also be computed directly:

```
i6 : demazureRoots(F, e1)

o6 = {{-1, 1}, {-1, 0}}

o6 : List
```

4.2. additiveActions.

This method takes as input either a complete fan Σ (an object of type `Fan`), a complete toric variety X (an object of type `NormalToricVariety`), a polytope $P \subset M_{\mathbb{Q}}$ (an object of type `Polyhedron`) or a pair (n, i) with $n = 1, \ldots, 6$ and i an index in the database of smooth Fano toric varieties included in the `NormalToricVarieties` package.

The method may also take as optional inputs `getCompleteCollection` (a `Boolean`, whether the output must include a complete collection of Demazure roots), `checkIfUniquelyAdditive` (a `Boolean`, whether the output must include if the variety is uniquely additive), `fanIsComplete` (a `Boolean`, whether the fan is known to be complete) and `fanIsSmooth` (a `Boolean`, whether the fan is known to be smooth).

The output is a hash table with keys `"isAdditive"` (a `Boolean`, whether the variety is additive), `"completeCollection"` (an object of type `List`, a complete collection of Demazure roots, if the variety is additive and `getCompleteCollection` is true) and `"isUniquelyAdditive"` (a `Boolean`, whether the variety is uniquely additive, if `checkIfUniquelyAdditive` is true).

This method is very fast if the input is a pair (n, i), since then the data is fetched from the pre-processed database included in our package. It is also very fast if `getCompleteCollection` is false, the variety is smooth and the Picard number is equal to two, due to Proposition 3.2 and Proposition 3.4. Furthermore, setting `getCompleteCollection` or `checkIfUniquelyAdditive` to false, or `fanIsComplete` or `fanIsSmooth` to true or false (the former value more than the latter) may dramatically increase performance, especially if the variety is high-dimensional.

Example 4.2. The projective plane is additive but not uniquely additive:

```
i1 : PP2 = toricProjectiveSpace 2;

i2 : additiveActions PP2
```

```
o2 = HashTable{completeCollection => {{-1, 1}, {0, 1}}}
               isAdditive => true
               isUniquelyAdditive => false

o2 : HashTable
```

The projective variety corresponding to the del Pezzo polygon (see Definition 5.7) does not admit an additive action:

```
i3 : V = transpose matrix {{1, 0}, {0, 1}, {-1, 0}, {0, -1}, {1, 1}, {-1, -1}};

              2        6
o3 : Matrix ZZ  <--- ZZ

i4 : P = convexHull V;

i5 : additiveActions P

o5 = HashTable{completeCollection => {}   }
               isAdditive => false
               isUniquelyAdditive => false

o5 : HashTable
```

4.3. listAdditiveSmoothFanoToricVarieties.

This method takes as input an integer $n = 1, \ldots, 6$. It returns a list containing hash tables with keys `"databaseIndex"` (the index in the database of smooth Fano toric varieties included in the `NormalToricVarieties` package), `"isUniquelyAdditive"` (a `Boolean`, whether the variety is uniquely additive) and `"completeCollection"` (an object of type `List`, a complete collection of Demazure roots). This method is very fast, as the data is fetched from the pre-processed database included in our package.

Example 4.3. The following is the two-dimensional case:

```
i1 : listAdditiveSmoothFanoToricVarieties 2

o1 = {HashTable{completeCollection => {{-1, 1}, {0, 1}}},
                databaseIndex => 0
                isUniquelyAdditive => false
     ------------------------------------------------------------------------
     HashTable{completeCollection => {{-1, 0}, {0, -1}}},
               databaseIndex => 1
               isUniquelyAdditive => true
     ------------------------------------------------------------------------
     HashTable{completeCollection => {{-1, 0}, {0, 1}}},
               databaseIndex => 2
               isUniquelyAdditive => false
```

```
     ------------------------------------------------------------------------
     HashTable{completeCollection => {{-1, 0}, {0, 1}}}}
               databaseIndex => 3
               isUniquelyAdditive => true
o1 : List
```

The smooth Fano toric surface with database index 4 is not additive:

```
i2 : additiveActions(2, 4)
o2 = HashTable{completeCollection => {}   }
               isAdditive => false
               isUniquelyAdditive => false
o2 : HashTable
```

There are 470 additive smooth Fano toric fivefolds:

```
i3 : #(listAdditiveSmoothFanoToricVarieties 5)
o3 = 470
```

4.4. **listUniquelyAdditiveSmoothFanoToricVarieties.**

This method takes as input an integer $n = 1, \ldots, 6$. It returns a list containing hash tables with keys `"databaseIndex"` (the index in the database of smooth Fano toric varieties included in the `NormalToricVarieties` package) and `"completeCollection"` (an object of type `List`, a complete collection of Demazure roots). This method is very fast, as the data is fetched from the pre-processed database included in our package.

Example 4.4. The following is the two-dimensional case:

```
i1 : listUniquelyAdditiveSmoothFanoToricVarieties 2
o1 = {HashTable{completeCollection => {{-1, 0}, {0, -1}}},
                databaseIndex => 1
     ------------------------------------------------------------------------
     HashTable{completeCollection => {{-1, 0}, {0, 1}}}}
               databaseIndex => 3
o1 : List
```

There are 4 uniquely additive smooth Fano toric fivefolds:

```
i2 : #(listUniquelyAdditiveSmoothFanoToricVarieties 5)
o2 = 4
```

5. Applications to smooth Fano toric varieties

In this section we apply our methods to determine existence and uniqueness of additive actions on smooth Fano toric varieties X of dimension $\dim(X) \leq 6$. Most data was obtained from the GRDB [BK09], who in turn obtained it from Øbro [Øb07]. The case in which $\dim(X) = 2$ can be easily done by inspecting the corresponding polygons, but we include it for the sake of completeness. Existence in the $\dim(X) = 3$ case was previously done in [HM20], who used more geometric techniques. Uniqueness in the $\dim(X) = 3$ case, along with existence and uniqueness in all higher-dimensional cases, are new results.

5.1. Surfaces.

Proposition 5.1. *Let $P \subset N_{\mathbb{Q}}$ be a smooth Fano polygon. Then:*

1. The only non-constant Betti number of $X_{P^\vee}$ is

$$h_1 = b_2 = \rho(X_{P^\vee}) = \#\mathcal{V}(P) - 2.$$

2. $\#\mathcal{V}(P) = \#\mathcal{F}(P)$.

Proof. 1. This follows from Proposition 2.23 and from expanding the identity in Proposition 2.24 for $p = 1$.

2. This follows from expanding the identity in Proposition 2.24 for $p = 0, 2$.

□

There are 5 isomorphism classes of smooth Fano toric surfaces, of which 1 is not additive, 2 are additive but not uniquely additive, and 2 are uniquely additive.

Table 1 contains all (uniquely) additive smooth Fano toric surfaces. These are matched to the data in the GRDB [BK09]. Non-additive surfaces are omitted. The column labels indicate, in order:

1. **ID**: The ID in the GRDB.
2. **Degree**: The degree $(-K_X)^2$.
3. $\boldsymbol{b_2}$: The second Betti number, calculated using Proposition 5.1.
4. $\boldsymbol{\#\mathcal{V}(P), \#\mathcal{F}(P)}$: The number of vertices and facets of the smooth Fano polygon $P \subset N_{\mathbb{Q}}$, respectively.
5. **Variety**: The corresponding smooth Fano toric surface $X_{P^\vee}$.
6. **Uniquely additive?**: *Yes* if the variety is uniquely additive, *no* if it is not.

ID	**Degree**	$\boldsymbol{b_2}$	$\boldsymbol{\#\mathcal{V}(P)}$	$\boldsymbol{\#\mathcal{F}(P)}$	**Variety**	**Uniquely additive?**
1	7	3	5	5	$\mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$	Yes
3	8	2	4	4	$\mathrm{Bl}_p(\mathbb{P}^2)$	No
4	8	2	4	4	$\mathbb{P}^1 \times \mathbb{P}^1$	Yes
5	9	1	3	3	$\mathbb{P}^2$	No

Table 1: Additive smooth Fano toric surfaces.

5.2. Threefolds.

Proposition 5.2. *Let $P \subset N_{\mathbb{Q}}$ be a smooth Fano polytope of dimension 3. Then, the non-constant Betti numbers of $X_{P^\vee}$ are*

$$h_1 = b_2 = h_2 = b_4 = \rho(X_{P^\vee}) = \#\mathcal{V}(P) - 3.$$

Proof. This follows from Propositions 2.23 and 2.24. □

There are 18 isomorphism classes of smooth Fano toric threefolds, of which 4 are not additive, 12 are additive but not uniquely additive, and 2 are uniquely additive.

Tables 2 and 3 contain all additive smooth Fano toric threefolds. These are matched to the data in the GRDB [BK09] and to the classifications in [MM03] and [HM20]. Non-additive threefolds are omitted. The column labels indicate, in order:

1. **Nº**: The nº in [MM03].
2. **ID**: The ID in the GRDB.
3. **Degree**: The degree $(-K_X)^3$, obtained from [MM03] and the GRDB.
4. $\boldsymbol{b_2}$: The second Betti number, obtained from [MM03] and calculated using Proposition 5.2.
5. $\#\boldsymbol{\mathcal{V}(P)}, \#\boldsymbol{\mathcal{F}(P)}$: The number of vertices and facets of the smooth Fano polytope $P \subset N_{\mathbb{Q}}$, obtained from the GRDB.
6. **Notation**: The notation in [HM20].
7. **Uniquely additive?**: *Yes* if the variety is uniquely additive, *no* if it is not.
8. **Variety** (Table 3): The corresponding smooth Fano toric threefold $X_{P^\vee}$, obtained from [MM03].

Nº	**ID**	**Degree**	$\boldsymbol{b_2}$	$\#\boldsymbol{\mathcal{V}(P)}$	$\#\boldsymbol{\mathcal{F}(P)}$	**Notation**	**Uniquely additive?**
-	23	64	1	4	4	$\mathbb{P}^3$	No
33	22	54	2	5	6	II_{33}	No
34	19	54	2	5	6	II_{34}	No
35	20	56	2	5	6	II_{35}	No
36	7	62	2	5	6	II_{36}	No
26	16	46	3	6	8	III_{26}	No
27	21	48	3	6	8	III_{27}	Yes
28	17	48	3	6	8	III_{28}	No

29	6	50	3	6	8	III_{29}	No
30	12	50	3	6	8	III_{30}	No
31	11	52	3	6	8	III_{31}	No
10	14	42	4	7	10	IV_{10}	Yes
11	10	44	4	7	10	IV_{11}	No
12	8	46	4	7	10	IV_{12}	No

Table 2: Additive smooth Fano toric threefolds.

Nº	**ID**	**Notation**	**Variety**
27	21	III_{27}	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
10	14	IV_{10}	$\mathbb{P}^1 \times \mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$

TABLE 3. Uniquely additive smooth Fano toric threefolds.

5.3. **Fourfolds.**

Proposition 5.3. *Let $P \subset N_{\mathbb{Q}}$ be a smooth Fano polytope of dimension 4. Then, the non-constant Betti numbers of $X_{P^\vee}$ are*

$$h_1 = b_2 = h_3 = b_6 = \rho(X_{P^\vee}) = \#\mathcal{V}(P) - 4$$

and

$$h_2 = b_4 = -2\#\mathcal{V}(P) + \#\mathcal{F}(P) + 6.$$

Proof. Proposition 2.24 2. implies that $h_1 = b_2 = f_2 - 2f_1 + 3\#\mathcal{V}(P) - 4 = h_3 = b_6 = \#\mathcal{V}(P) - 4$. Then,

$$f_2 - 2f_1 + 2\#\mathcal{V}(P) = 0. \tag{1}$$

On the other hand, Proposition 2.24 1. implies that $h_0 = b_0 = \#\mathcal{F}(P) - f_2 + f_1 - \#\mathcal{V}(P) + 1 = h_4 = b_8 = 1$. Then,

$$\#\mathcal{F}(P) - f_2 + f_1 - \#\mathcal{V}(P) = 0. \tag{2}$$

Adding (1) and (2) gives $f_1 = \#\mathcal{V}(P) + \#\mathcal{F}(P)$, thus the identity in Proposition 2.24 implies that $h_2 = b_4 = f_1 - 3\#\mathcal{V}(P) + 6 = -2\#\mathcal{V}(P) + \#\mathcal{F}(P) + 6$. □

There are 124 isomorphism classes of smooth Fano toric fourfolds, of which 45 are not additive, 75 are additive but not uniquely additive, and 4 are uniquely additive.

Tables 4 and 5 contain all additive smooth Fano toric fourfolds. These are matched to the data in the GRDB [BK09] and to the classification in [Bat99] (see also [Sat00]). Non-additive fourfolds are omitted. The column labels indicate, in order:

1. **Nº**: The nº in [Bat99].
2. **ID**: The ID in the GRDB.
3. **Degree**: The degree $(-K_X)^4$, obtained from [Bat99] and the GRDB.
4. $\boldsymbol{b_2, b_4}$: The second and fourth Betti numbers, obtained from [Bat99] and calculated using Proposition 5.3.
5. $\boldsymbol{\#\mathcal{F}(P), \#\mathcal{V}(P)}$: The number of facets and vertices of the smooth Fano polytope $P \subset N_{\mathbb{Q}}$, obtained from the GRDB.
6. **Notation**: The notation in [Bat99].
7. **Uniquely additive?**: *Yes* if the polytope is uniquely additive, *no* if it is not.
8. **Variety** (Table 5): The corresponding smooth Fano toric fourfold $X_{P^\vee}$, obtained from [Bat99].

Nº	**ID**	**Degree**	$\boldsymbol{b_2}$	$\boldsymbol{b_4}$	$\boldsymbol{\#\mathcal{F}(P)}$	$\boldsymbol{\#\mathcal{V}(P)}$	**Notation**	**Uniquely additive?**
1	147	625	1	1	5	5	$\mathbb{P}^4$	No
2	25	800	2	2	8	6	B_1	No
3	139	640	2	2	8	6	B_2	No
4	144	544	2	2	8	6	B_3	No
5	145	512	2	2	8	6	B_4	No
6	138	512	2	2	8	6	B_5	No
7	44	594	2	3	9	6	C_1	No
8	141	513	2	3	9	6	C_2	No
9	70	513	2	3	9	6	C_3	No
10	146	486	2	3	9	6	C_4	No
11	24	605	3	3	11	7	E_1	No
12	128	489	3	3	11	7	E_2	No
13	127	431	3	3	11	7	E_3	No
14	30	592	3	4	12	7	D_1	No
15	31	576	3	4	12	7	D_2	No
16	49	560	3	4	12	7	D_3	No
17	35	560	3	4	12	7	D_4	No
18	42	496	3	4	12	7	D_5	No
19	129	496	3	4	12	7	D_6	No

20	97	486	3	4	12	7	D_7	No
21	134	480	3	4	12	7	D_8	No
22	66	464	3	4	12	7	D_9	No
23	132	464	3	4	12	7	D_{10}	No
24	117	459	3	4	12	7	D_{11}	No
25	140	448	3	4	12	7	D_{12}	No
26	143	432	3	4	12	7	D_{13}	No
27	133	432	3	4	12	7	D_{14}	No
28	135	432	3	4	12	7	D_{15}	No
29	68	432	3	4	12	7	D_{16}	No
33	41	529	3	5	13	7	G_1	No
34	40	450	3	5	13	7	G_2	No
35	64	433	3	5	13	7	G_3	No
36	60	417	3	5	13	7	G_4	No
37	69	406	3	5	13	7	G_5	No
38	137	401	3	5	13	7	G_6	No
39	26	558	4	5	15	8	H_1	No
40	45	505	4	5	15	8	H_2	No
41	28	478	4	5	15	8	H_3	No
42	118	447	4	5	15	8	H_4	No
43	123	415	4	5	15	8	H_5	No
46	124	378	4	5	15	8	H_8	No
49	74	480	4	6	16	8	L_1	No
50	75	464	4	6	16	8	L_2	No
51	83	448	4	6	16	8	L_3	No
52	105	432	4	6	16	8	L_4	No
53	95	416	4	6	16	8	L_5	No
54	112	400	4	6	16	8	L_6	No
55	106	384	4	6	16	8	L_7	No
56	142	384	4	6	16	8	L_8	Yes
57	130	384	4	6	16	8	L_9	No
62	33	496	4	6	16	8	I_1	No
63	29	463	4	6	16	8	I_2	No
64	47	442	4	6	16	8	I_3	No

65	38	433	4	6	16	8	I_4	No
67	93	411	4	6	16	8	I_6	No
68	37	400	4	6	16	8	I_7	No
69	115	384	4	6	16	8	I_8	No
70	94	390	4	6	16	8	I_9	No
71	111	389	4	6	16	8	I_{10}	No
72	59	384	4	6	16	8	I_{11}	No
74	126	368	4	6	16	8	I_{13}	No
77	61	385	4	7	17	8	M_1	No
78	50	417	4	7	17	8	M_2	No
79	58	369	4	7	17	8	M_3	No
80	57	369	4	7	17	8	M_4	No
81	110	364	4	7	17	8	M_5	No
84	71	442	5	8	20	9	Q_1	No
85	79	405	5	8	20	9	Q_2	No
86	73	394	5	8	20	9	Q_3	No
87	77	405	5	8	20	9	Q_4	No
88	81	373	5	8	20	9	Q_5	No
89	84	368	5	8	20	9	Q_6	No
90	91	363	5	8	20	9	Q_7	No
91	90	352	5	8	20	9	Q_8	No
93	102	336	5	8	20	9	Q_{10}	No
94	120	336	5	8	20	9	Q_{11}	Yes
105	89	332	5	9	21	9	R_1	No
117	62	307	5	11	23	9	See Remark 5.4	Yes
119	98	294	6	11	25	10	$\mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right) \times \mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$	Yes

Table 4: Additive smooth Fano toric fourfolds.

Remark 5.4. There is a small misprint in Batyrev's classification. Namely, the types of polytopes Nº 117 and 118 are exchanged. The correct statement is that polytope Nº 117 is

Nº	ID	Notation	Variety
56	142	L_8	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
94	120	Q_{11}	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$
117	62	See Remark 5.4	See Remark 5.4
119	98	$\mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right) \times \mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$	$\mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right) \times \mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$

TABLE 5. Uniquely additive smooth Fano toric fourfolds.

isomorphic to the convex hull of the subset

$$\left\{ \pm e_1, \ldots, \pm e_n, -\sum_{i=1}^{n} e_i \right\} \subset N_{\mathbb{Q}}.$$

This can be seen, for example, by using Proposition 5.3 to calculate the corresponding invariants. The corresponding fourfold is isomorphic to $\mathrm{Bl}_p\left(\left(\mathbb{P}^1\right)^4\right)$, the blow-up of $\left(\mathbb{P}^1\right)^4$ at one torus-invariant point.

5.4. **Fivefolds and sixfolds.**

Data for fivefolds and sixfolds can be accessed through the `listAdditiveSmoothFanoToricVarieties` and `listUniquelyAdditiveSmoothFanoToricVarieties` methods of the *Macaulay2* package, see Subsections 4.3 and 4.4.

There are 866 isomorphism classes of smooth Fano toric fivefolds, of which 396 are not additive, 466 are additive but not uniquely additive, and 4 are uniquely additive.

There are 7622 isomorphism classes of smooth Fano toric sixfolds, of which 4194 are not additive, 3420 are additive but not uniquely additive, and 8 are uniquely additive.

5.5. **Uniquely additive varieties.**

Definition 5.5. Let $P, Q \subset N_{\mathbb{Q}}$ be two polytopes. The **free sum** $P \oplus Q \subset N_{\mathbb{Q}} \times N_{\mathbb{Q}}$ is the polytope defined as the convex hull of the subset

$$(P \times \{0_{N_{\mathbb{Q}}}\}) \cup (\{0_{N_{\mathbb{Q}}}\} \times Q) \subset N_{\mathbb{Q}} \times N_{\mathbb{Q}}.$$

Proposition 5.6. [CLS11] *Let $P, Q \subset N_{\mathbb{Q}}$ be two polytopes such that $0_{N_{\mathbb{Q}}} \in \mathrm{RelInt}(P), \mathrm{RelInt}(Q)$. Then:*

1. $(P \oplus Q)^{\vee} = P^{\vee} \times Q^{\vee}$.

2. The product $P^{\vee} \times Q^{\vee}$ corresponds to the toric variety $X_{P^{\vee}} \times X_{Q^{\vee}}$ with the Segre embedding.

Definition 5.7. Let $n \in \mathbb{Z}^+$ be even.

1. The **del Pezzo polytope** is the polytope defined as the convex hull of the subset

$$\left\{ \pm e_1, \ldots, \pm e_n, \pm\sum_{i=1}^{n} e_i \right\} \subset N_{\mathbb{Q}}.$$

2. The **pseudo-del Pezzo polytope** is the polytope defined as the convex hull of the subset
$$\left\{\pm e_1, \ldots, \pm e_n, -\sum_{i=1}^{n} e_i\right\} \subset N_{\mathbb{Q}}.$$

Remark 5.8. The toric variety corresponding to the del Pezzo polygon (see Definition 5.7) is isomorphic to both $\mathrm{Bl}_p\left(\left(\mathbb{P}^1\right)^2\right)$ and $\mathrm{Bl}_{p,q}\left(\mathbb{P}^2\right)$.

Proposition 5.9. [Cas03] *Let $P \subset N_{\mathbb{Q}}$ be a smooth Fano polytope with n distinct unordered pairs (v_1, v_2) of vertices such that $v_1 = -v_2$. Then, P is isomorphic to a free sum of intervals $[-1, 1]$, del Pezzo polytopes and pseudo-del Pezzo polytopes.*

Proposition 5.10. *Let $P \subset N_{\mathbb{Q}}$ be a polytope.*

1. *If P is isomorphic to a del Pezzo polytope, then the corresponding toric variety $X_{P^\vee}$ is not additive.*
2. *If P is isomorphic to a pseudo-del Pezzo polytope, then the corresponding toric variety $X_{P^\vee}$ is uniquely additive.*

Proof. 1. The dual polytope is
$$P^\vee = \{(x_1, \ldots, x_n) \in M_{\mathbb{Q}} : \quad |x_i| \leq 1 \text{ for all } i = 1, \ldots, n \text{ and } |x_1 + \cdots + x_n| \leq 1\}.$$
It is easy to see that all facets of $P^\vee$ have empty relative interior, so the set of Demazure roots of $P^\vee$ is empty (see Remark 2.20).

2. The dual polytope is
$$P^\vee = \{(x_1, \ldots, x_n) \in M_{\mathbb{Q}} : \quad |x_i| \leq 1 \text{ for all } i = 1, \ldots, n \text{ and } -x_1 - \cdots - x_n \leq 1\}.$$
There exist rays $\rho_1, \ldots, \rho_n \in \Sigma_{P^\vee}(1)$ such that $B = \{p_1, \ldots, p_n\} = \{e_1, \ldots, e_n\}$ is a basis of the lattice N (here, $\{e_1, \ldots, e_n\}$ is the canonical basis of N), the remaining rays in $\Sigma(1)$ are contained in the negative octant of B, and $\mathfrak{R}_i = \{-p_i^\vee\} = \{-e_i^\vee\}$ for all $i = 1, \ldots, n$. The result then follows by Theorem 2.9.

□

The following theorem is a consequence of applying our methods to the data in the GRDB [BK09].

Theorem 5.11. *Let $X = X_{P^\vee}$ be a smooth Fano toric variety such that* $\dim(X) \leq 6$. *The following are equivalent:*

1. *X is uniquely additive.*
2. *X is isomorphic to the product of projective lines $\mathbb{P}^1$, blow-ups of $\left(\mathbb{P}^1\right)^n$ with n even at one torus-invariant point $\mathrm{Bl}_p\left(\left(\mathbb{P}^1\right)^n\right)$ and, if* $\dim(X_P) = 6$, *$X_{\mathfrak{P}^\vee}$, where $\mathfrak{P} \subset N_{\mathbb{Q}}$ is the polytope with* ID 5816 *in the* GRDB.

3. $P \subset N_{\mathbb{Q}}$ is isomorphic to a free sum of intervals $[-1,1]$, pseudo-del Pezzo polytopes and, if $\dim(P) = 6$, the polytope $\mathfrak{P}$ with ID 5816 on the GRDB.

In particular, the sequence $(c_n)_{n\geq 1}$, where c_n is the number of isomorphism classes of uniquely additive smooth Fano toric varieties of dimension n, starts with $1, 2, 2, 4, 4, 8, \ldots$

6. Further directions

We welcome any contributions and feature requests for the *Macaulay2* package.

Since the enumeration of classes uniquely additive smooth Fano toric varieties is very simple up to $n = 6$ (see Theorem 5.11), it would be interesting to characterise and enumerate these classes for higher or arbitrary values of n.

More generally, it would also be interesting to enumerate classes of additive complete toric varieties. In order to get finite values, one may consider fans whose primitive ray generators have bounded coordinates. To this effect, Proposition 2.8 may be a good starting point. For example:

Proposition 6.1. *Let $(d_n)_{n\geq 1}$ be the sequence counting the number of isomorphism classes of complete toric varieties X of dimension n such that:*

1. There exists a fan Σ in $N_{\mathbb{Q}}$ with $X \cong X_{\Sigma}$.

2. All primitive lattice vectors $p_1, \ldots, p_{\#\Sigma(1)}$ on the rays $\rho_1, \ldots, \rho_{\#\Sigma(1)} \in \Sigma(1)$ are contained in $[-1,1]^n$.

3. For all $i = 1, \ldots, n$, $p_i = e_i$.

4. For all $i = n+1, \ldots, \#\Sigma(1)$, p_i is in the negative octant of the standard basis of $N_{\mathbb{Q}}$.

Then, all such varieties X are additive, and, for all $n \in \mathbb{Z}_+$,

$$1 = d_1 < \cdots < d_n \leq \mathrm{A055621}(n),$$

where A055621 *is the* OEIS [OEI25, A055621] *sequence counting the number of hypergraphs on n unlabelled vertices with no isolated vertices.*

Proof. The first statement is a consequence of Proposition 2.8. We now prove the second statement.

For all $i = n+1, \ldots, \#\Sigma(1)$, we map p_i to the set given by its support. For example, $(0, -1, 0, -1)$ is mapped to $\{2, 4\}$. The image of $\{p_i\}_{i=n+1,\ldots,\#\Sigma(1)}$ is a hypergraph $G = (V, E)$ on the vertices $V = \{1, \ldots, n\}$ with no isolated vertices (since Σ is complete). The symmetric group $\mathfrak{S}_n$ acts on fans Σ in $N_{\mathbb{Q}}$ by permuting coordinates and on hypergraphs $G = (\{1, \ldots, n\}, E)$ by permuting vertices. Since permutation matrices have determinant ± 1, all fans Σ in $N_{\mathbb{Q}}$ in a given orbit belong to the same isomorphism class. On the other hand, the orbit of a hypergraph $G = (\{1, \ldots, n\}, E)$ is the underlying hypergraph G' on n unlabelled vertices. □

Proposition 6.1 is inspired by the following conjecture of Ewald, although it is *not* true that all additive smooth Fano toric varieties satisfy hypotheses 1. to 4. – counterexamples may be found computationally already for $n \leq 4$.

Conjecture 6.2 (Ewald)**.** *Let $P \subset N_{\mathbb{Q}}$ be a smooth Fano polytope of dimension n. Then, there exists $P' \subset [-1,1]^n \subset N_{\mathbb{Q}}$ such that $P \cong P'$.*

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