

The theory of planar ballistic SNS junctions

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Abstract

The paper presents the theory of planar ballistic SNS junctions with equal Fermi velocities and effective masses in all layers. The theory takes into account phase gradients in superconducting layers commonly ignored in the past. At $T = 0$ the current-phase relation was derived for any thickness L of the normal layer in the model of the steplike pairing potential model analytically. The obtained current-phase relation is essentially different from that in theory neglecting phase gradients, especially in the limit $L \rightarrow 0$ (short junction). The analysis resolves the problem with the charge conservation law in the steplike pairing potential model. The current-phase relation at temperatures exceeding the energy spacing between Andreev levels but less than the critical temperature was also calculated numerically. The current at these temperatures is temperature independent and decreases with growing L as $1/L^4$. The previous theory predicted the current exponentially decreasing with growing T and L . Possible implications of the analysis for planar junctions with non-equal Fermi velocities and for non-planar junctions (narrow normal bridge between two bulk superconductors) are also discussed,

Keywords: Andreev reflection, Andreev states, SNS junction, Current-phase relation of Josephson junction

1 Introduction

The ballistic SNS junction has already been investigated a half-century. The pioneer papers [1–3] and many subsequent ones used the self-consistent field method [4]. In this method an effective pairing potential is introduced, which

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transforms the second-quantization Hamiltonian with the effective interaction into an effective Hamiltonian quadratic in creation and annihilation electron operators. The effective Hamiltonian can be diagonalized by the Bogolyubov – Valatin transformation.

The effective Hamiltonian is not gauge invariant, and the theory using this Hamiltonian violates the charge conservation law. The charge conservation law is restored if one solves the Bogolyubov – de Gennes equations together with the self-consistency equation for the pairing potential. Starting from the original paper of Andreev [5], instead of solving the self-consistency equation, it was assumed that there is a gap Δ of constant modulus $\Delta_0 = |\Delta|$ in the superconducting layers and zero gap inside the normal layer. Thus, the proximity effect (penetration of the pairing potential into the normal metal) was ignored. We call it the steplike pairing potential model (or, shortly, the steplike model).

The steplike model was investigated for a special case, in which Fermi velocities and effective masses were assumed to be equal in the superconductors and in the normal metal. Under this assumption, there is no normal scattering, and Andreev scattering is the only scattering mechanism at SN interfaces.

Another assumption was that not only the absolute value Δ_0 but also phases were constant in superconducting leads (no phase gradients), although its values in two leads were different [1–3, 6]. The phase variation in space at this assumption is shown in Fig. 1(a). At this phase profile, the charge

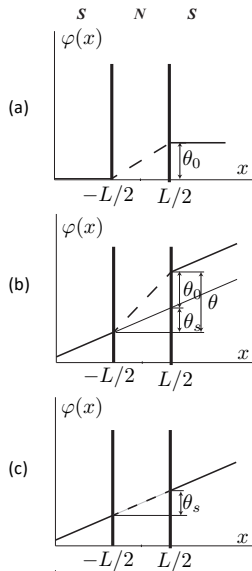


Fig. 1 The phase variation across the SNS junction. (a) The vacuum current produced by the vacuum phase θ_0 . The current is confined to the normal layer. (b) The superposition of the vacuum current and the condensate current determined by the superfluid phase $\theta_s = L\nabla\varphi$. The phase $\theta = \theta_0 + \theta_s$ is the Josephson phase. (c) The condensate current produced by the phase gradient $\nabla\varphi$ in the superconducting layers. In all layers the electric current is equal to env_s .

conservation law is violated since the current flows only inside the normal layer. But it was believed that the charge conservation law can be restored by so small phase gradients in leads that this cannot affect calculations ignoring the gradients. This suggestion is true for a weak link, inside which the phase varies much faster than in the leads.

Within the steplike model, the problem is reduced to quadratures. There is the *ab initio* exact expression for the current through the junction via the sum over all Andreev bound states and the integral over all continuum states. But these sum and integral are rather complicated for analytical and even numerical calculation because of oscillating integrand and necessity to calculate small difference of large terms. The rather sophisticated formalism of temperature Green function was used [1, 2, 6, 7].

Recently [8–10] it was demonstrated that if transverse cross-sections of all layers are the same, the SNS junction is not a weak link at zero temperature. Such junctions are called planar junctions. We shall use this name even for junctions in 1D wires when the cross-section plane becomes a point. In a planar junction the phase gradients in the leads do affect the current in the normal layer. Thus, one should determine currents in the normal and superconducting layers self-consistently.

In Refs. [8–10] and in the present paper the following gap profile with the constant gradient $\nabla\varphi$ in leads was considered [Fig. 1(b)]:

$$\Delta = \begin{cases} \Delta_0 e^{i\theta_0/2 + i\nabla\varphi x} & x > L/2 \\ 0 & -L/2 < x < L/2 \\ \Delta_0 e^{-i\theta_0/2 + i\nabla\varphi x} & x < -L/2 \end{cases} . \quad (1)$$

The phase profile is determined by two phases θ_0 and $\theta_s = L\nabla\varphi$. The total phase difference across the normal layer (Josephson phase) is a sum of two phases: $\theta = \theta_s + \theta_0$. There is a mathematically correct exact analytical solution of the Bogolyubov–de Gennes equations for any choice of θ_0 and θ_s . But we filter these solutions by the requirement of the charge conservation law. The strict conservation law was replaced by a softer condition that, at least, total currents deep in all layers are the same.

Introduction of a more complicated phase profile does not prevent the possibility to exactly solve the Bogolyubov–de Gennes equations for any θ_0 and $\nabla\varphi$ analytically in the steplike model (see Sec. 9.5 in Ref. [11]). In particular, there is the solution at $\theta_0 = 0$, which does not violate the charge conservation law. In this state the phase profile is similar to that in a uniform superconductor [Fig. 1(c)] and the current J in all layers is equal to the current flowing in a uniform superconductor:

$$J_s = env_s = J_0 \frac{\theta_s}{\pi}. \quad (2)$$

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Here n is the electron density, $v_s = \frac{\hbar}{2m} \nabla \varphi$ is the superfluid velocity,

$$J_0 = \frac{ev_f}{L} = \frac{\pi en \hbar}{2mL}, \quad (3)$$

and v_f is the Fermi velocity. The phase θ_s and the current J_s were called the superfluid phase and the condensate current respectively [8].

The state with the condensate current can be obtained by the Galilean transformation of the ground state without currents ($\theta_s = \theta_0 = 0$). For long ballistic SNS junctions Galilean invariance despite broken translational invariance has been already noticed by Bardeen and Johnson [3] and used in Ref. [8] in the steplike model. In Ref. [9] the Galilean invariance was demonstrated for an arbitrary gap profile under the condition that the ratio of the gap Δ_0 to the Fermi energy ε_f is small and the Andreev reflection is the predominant mechanism of scattering. This means that state with the condensate current exists beyond the steplike model for junctions with any L .

The uniform current state with constant phase gradients was confirmed by numerical calculations by Riedel *et al.* [12], They did not assume that the pairing potential is steplike, but calculated it solving the Bogolyubov–de Gennes equations together with the integral self-consistency equation for the gap. Riedel *et al.* obtained that although the pairing potential amplitude smoothly varied across the interface between the normal and superconducting layers, the phase gradient remained strictly constant along the whole junction as in Fig. 1(b). But nobody paid attention to the fact that this contradicts the existing theory. Recently new numerical calculations by Krekels *et al.* [13] taking into account the self-consistency equation also confirmed importance of phase gradients in leads.

The phase θ_0 and the current J_v produced by this phase were called vacuum phase and vacuum current respectively [8]. The current J_v flows only in the normal layer at all Andreev and continuum states being unoccupied. The charge conservation law is violated, and the current J_v must be compensated by the current produced by quasiparticles at Andreev levels, which also flows only in the normal layer. It was called excitation current J_q [8]. At zero temperature the excitation current appears at the critical current determined by the Landau criterion that the energy of the lowest Andreev level reaches 0 [10]. The current-phase relation (CPR) is derived from the condition that the total current flowing only in the normal layer vanishes: $J_v + J_q = 0$. Thus, our analysis demonstrated that taking into account phase gradients in the superconducting leads one can resolve the problem of the charge conservation law *within* the steplike model, contrary to what was believed before [6].

Despite an essential difference in the physical picture of the charge transport through the junction, for long junctions with $L \gg \zeta_0$ at zero temperature the both theories predicted the same saw-tooth CPR (Fig. 2), which at $-\pi < \theta < \pi$ is

$$J = J_0 \frac{\theta}{\pi}. \quad (4)$$

Here

$$\zeta_0 = \frac{\hbar v_f}{\Delta_0} \quad (5)$$

than the coherence length.

Coincidence of the CPRs in the two theories in the limit $L/\zeta_0 \rightarrow \infty$ led to the wrong conclusion that there is no difference between the condensate and the vacuum currents (see discussion in Refs. [14, 15]). There is a principal difference between two currents and two phases θ_s and θ_0 producing them. The condensate currents flows in all layers, while the vacuum current flows only in the normal layer violating the charge conservation law. At tuning the phase θ_0 Andreev levels move with respect to the gap sometimes entering into or exiting from the gap, i.e., either a new Andreev level appears, or an Andreev level existing before disappears. At tuning the phase θ_s Andreev levels move together with the gap and their respective positions do not vary.

The theory taking into account phase gradients in superconducting leads was extended on the whole diapason of L down to $L = 0$ [10] and is discussed in the present paper. Remarkably, at zero temperature it is possible to obtain a simple analytical expression for the CPR for any L . For small L (short junctions) the difference between theories taking or not taking into account phase gradients becomes very essential. In the limit $L \rightarrow 0$ the SNS junction becomes a uniform superconductor with a constant current produced by a constant phase gradient without phase jumps. The theory ignoring phase gradients fails to describe this natural behavior. Neglecting phase gradients, the maximum current through the junction (critical Josephson current) is always at the phase π , i.e., the CPR is forward-skewed (the current maximum is located at the phase exceeding the phase $\pi/2$ of the current maximum of the sinusoidal CPR). But taking into account phase gradients, in short junctions the CPR becomes backward-skewed (the current maximum is located at the phase less than $\pi/2$).

Krekels *et al.* [13] compared their numerical results with our analytical theory using the steplike model. The comparison supports the conclusion that the CPR becomes backward-skewed in short junction. This contradicts the previous theory predicting that the CPR is always forward-skewed. The backward-skewed CPR was observed in short InAs nanowire junctions by Span-ton *et al.* [16]. They also acknowledged disagreement with the existing theory,

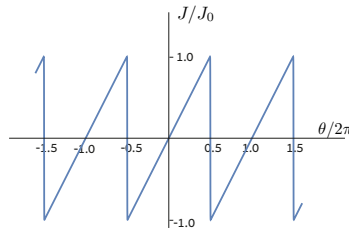


Fig. 2 The saw-tooth CPR at zero temperature. Here $J_0 = \frac{\pi \hbar}{2mL} en$ ($= \frac{ev_f}{L}$ in the 1D case).

but connected this with Coulomb interaction. Our analysis and numerical calculations [13] show that the backward-skewed CPR is possible also in the model without interaction.

The present paper considers also nonzero temperatures, when one cannot avoid complicated calculations of the sum and the integral determining the vacuum current. As in the past [3, 6], the $T \neq 0$ analysis focus on long junctions $L \gg \zeta_0$ when the temperature exceeds the Andreev level energy spacing, but still less than the gap Δ_0 . At short junctions $L \leq \zeta_0$ the temperature effect becomes important at temperatures close to critical. This requires an essential revision of existing theories, which is beyond the scope of the present paper.

At high temperatures the total current is very small, and the SNS junction becomes a weak link without essential effect of phase gradients in leads. So, the total current can be approximated by the sum of the vacuum and the excitation current as was done in the past. The main contribution $\propto 1/L$ to the excitation current is equal in amplitude but opposite in sign to the vacuum current in the limit $L \rightarrow \infty$. In the past the small current was determined by the contribution of Matsubara poles. This contribution takes into account a difference between the exact sum determining the excitation current in Andreev levels and the main term $\propto 1/L$ obtained by replacing the sum by an integral. The total current exponentially decreasing with T and L .

In Ref. [9] it was suggested that there are other possible corrections more important at high temperature than the Matsubara contribution. These are power law corrections $\propto 1/L^w$ to the vacuum current, which at $w > 1$ decrease with growing L faster than the main term $\propto 1/L$. However, corrections $\propto 1/L^{3/2}$ calculated in Ref. [9] and corrections $\propto 1/L^2$ calculated in the present work (App. A) vanish in the total current. Failing to find the power law correction analytically the present paper presents results of numerical calculation based on the exact *ab initio* expressions for currents. The calculation revealed a small power-law correction $\propto 1/L^4$, which is independent from temperature and therefore becomes more important than the exponentially small Matsubara term. As a result, the critical Josephson current dependence on T has a plateau at temperatures between the Andreev level energy spacing and critical one. At the plateau the current decreases with L following the power law $1/L^4$.

The analysis in the paper mostly addresses the 1D channel. For the extension of the calculation on multidimensional systems currents were integrated over transverse components of wave vectors.

2 Solution of the Bogolyubov – de Gennes equations for a moving condensate

The Bogolyubov – de Gennes equations for the Bogolyubov – de Gennes wave function

$$\psi(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}, \quad (6)$$

are

$$\begin{aligned}\varepsilon u &= -\frac{\hbar^2}{2m} (\nabla^2 + k_f^2) u + \Delta v, \\ \varepsilon v &= \frac{\hbar^2}{2m} (\nabla^2 + k_f^2) v + \Delta^* u.\end{aligned}\quad (7)$$

Here k_f is the Fermi wave number and $\Delta = \Delta_0(x)e^{i\varphi(x)}$ in general.

If the ratio Δ_0/ε_f is very small, among solutions of the Bogolyubov–de Gennes equations there are wave functions $\psi(x)$, which are superpositions of plane waves with wave numbers only close to either $+k_f$, or $-k_f$. Quasiparticles in these states will be called rightmovers (+) and leftmovers (−) respectively. After transformation of the wave function,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} e^{\pm i k_f x}, \quad (8)$$

the second order terms in gradients, $\nabla^2 \tilde{u}$ and $\nabla^2 \tilde{v}$, can be neglected, and the Bogolyubov–de Gennes equations are reduced to the equations of the first order in gradients [5]:

$$\begin{aligned}\varepsilon \tilde{u} &= \mp i \hbar v_f \nabla \tilde{u} + \Delta \tilde{v}, \\ \varepsilon \tilde{v} &= \pm i \hbar v_f \nabla \tilde{v} + \Delta^* \tilde{u}.\end{aligned}\quad (9)$$

The boundary conditions at interfaces between layers require the continuity of the wave function components, but not their gradients.

2.1 Delocalized continuum scattering states

For a rightmover quasiparticle and a leftmover quasihole incident from left and propagating from $x = -\infty$ to $x = \infty$ the wave function with the energy ε is

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u_0(\pm\xi)e^{-i\theta_0/4+i\nabla\varphi x/2} \\ v_0(\pm\xi)e^{i\theta_0/4-i\nabla\varphi x/2} \end{pmatrix} e^{i\left(\pm k_f + \frac{m\xi}{\hbar^2 k_f}\right)x} \\ &+ r(\pm\theta) \begin{pmatrix} u_0(\mp\xi)e^{-i\theta_0/4+i\nabla\varphi x/2} \\ v_0(\mp\xi)e^{i\theta_0/4-i\nabla\varphi x/2} \end{pmatrix} e^{i\left(\pm k_f - \frac{m\xi}{\hbar^2 k_f}\right)x}\end{aligned}\quad (10)$$

for $x < -L/2$,

$$\begin{pmatrix} u \\ v \end{pmatrix} = t(\pm\theta) \begin{pmatrix} u_0(\pm\xi)e^{i\theta_0/4+i\nabla\varphi x/2} \\ v_0(\pm\xi)e^{-i\theta_0/4-i\nabla\varphi x/2} \end{pmatrix} e^{i\left(\pm k_f + \frac{m\xi}{\hbar^2 k_f}\right)x}\quad (11)$$

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for $x > L/2$, and

$$\begin{pmatrix} u \\ v \end{pmatrix} = t(\pm\theta) \begin{pmatrix} u_0(\pm\xi)e^{i\theta/4 \pm \frac{im\varepsilon}{\hbar^2 k_f} x + \frac{im(\xi \mp \varepsilon)L}{2\hbar^2 k_f}} \\ v_0(\pm\xi)e^{-i\theta/4 \mp \frac{im\varepsilon}{\hbar^2 k_f} x + \frac{im(\xi \pm \varepsilon)L}{2\hbar^2 k_f}} \end{pmatrix} e^{\pm ik_f x} \quad (12)$$

inside the normal layer $-L/2 < x < L/2$. Here $\xi = \sqrt{\varepsilon_0^2 - \Delta_0^2}$ and

$$u_0(\xi) = \sqrt{\frac{\varepsilon_0 + \xi}{2\varepsilon_0}}, \quad v_0(\xi) = \sqrt{\frac{\varepsilon_0 - \xi}{2\varepsilon_0}} = u_0(-\xi). \quad (13)$$

The energy $\varepsilon_0 > \Delta_0$ is the energy at resting condensate ($\nabla\varphi = 0$), which is connected with the energy ε at moving condensate ($\nabla\varphi \neq 0$) by the relation

$$\varepsilon = \varepsilon_0 \pm \frac{\hbar v_f}{2} \nabla\varphi = \varepsilon_0 \pm \frac{\hbar v_f}{2L} \theta_s = \varepsilon_0 \pm v_s \hbar k_f, \quad (14)$$

following from the Galilean invariance [9]. Amplitudes $t(\pm\theta)$ and $r(\pm\theta)$ of transmission and reflection are determined from the continuity of spinor components at $x = \pm L/2$:

$$\begin{aligned} t(\pm\theta) &= \frac{e^{-\frac{im\xi L}{\hbar^2 k_f}}}{\cos\left(\frac{\varepsilon mL}{\hbar^2 k_f} \mp \frac{\theta}{2}\right) - i\frac{\varepsilon_0}{\xi} \sin\left(\frac{\varepsilon mL}{\hbar^2 k_f} \mp \frac{\theta}{2}\right)} \\ &= \frac{e^{-\frac{im\xi L}{\hbar^2 k_f}}}{\cos\left(\frac{\varepsilon_0 mL}{\hbar^2 k_f} \mp \frac{\theta_0}{2}\right) - i\frac{\varepsilon_0}{\xi} \sin\left(\frac{\varepsilon_0 mL}{\hbar^2 k_f} \mp \frac{\theta_0}{2}\right)}, \end{aligned} \quad (15)$$

$$\begin{aligned} r(\pm\theta) &= \frac{\Delta_0}{\xi_0} \frac{e^{-\frac{im\xi L}{\hbar^2 k_f}} i \sin\left(\frac{\varepsilon mL}{\hbar^2 k_f} \mp \frac{\theta}{2}\right)}{\cos\left(\frac{\varepsilon mL}{\hbar^2 k_f} \mp \frac{\theta}{2}\right) - i\frac{\varepsilon_0}{\xi} \sin\left(\frac{\varepsilon mL}{\hbar^2 k_f} \mp \frac{\theta}{2}\right)} \\ &= \frac{\Delta_0}{\xi_0} \frac{e^{-\frac{im\xi L}{\hbar^2 k_f}} i \sin\left(\frac{\varepsilon_0 mL}{\hbar^2 k_f} \mp \frac{\theta_0}{2}\right)}{\cos\left(\frac{\varepsilon_0 mL}{\hbar^2 k_f} \mp \frac{\theta_0}{2}\right) - i\frac{\varepsilon_0}{\xi} \sin\left(\frac{\varepsilon_0 mL}{\hbar^2 k_f} \mp \frac{\theta_0}{2}\right)}. \end{aligned} \quad (16)$$

These expressions demonstrate that motion of the condensate has no effect on scattering parameters. The scattering parameters for continuum states were determined by Bardeen and Johnson [3] without phase gradients and in Refs. [8, 11, 17] with phase gradients.

The reflection and the transmission probabilities $R(\theta_0) = |t(\theta_0)|^2$ and $\mathcal{T}(\theta_0) = |t(\theta_0)|^2$ are

$$\begin{aligned} R(\theta_0) &= \frac{\Delta_0^2 \left[1 - \cos \left(\frac{2\varepsilon_0 mL}{\hbar^2 k_f} - \theta_0 \right) \right]}{2\varepsilon_0^2 - \Delta_0^2 - \Delta_0^2 \cos \left(\frac{2\varepsilon_0 mL}{\hbar^2 k_f} - \theta_0 \right)}, \\ \mathcal{T}(\theta_0) &= \frac{2(\varepsilon_0^2 - \Delta_0^2)}{2\varepsilon_0^2 - \Delta_0^2 - \Delta_0^2 \cos \left(\frac{2\varepsilon_0 mL}{\hbar^2 k_f} - \theta_0 \right)}. \end{aligned} \quad (17)$$

The expressions are valid also for a quasihole incident from right. For a quasi-particle incident from right and a hole incident from left the reflection and the transmission probabilities are $R(-\theta_0)$ and $\mathcal{T}(-\theta_0)$.

The scattering parameters satisfy the equality $R + \mathcal{T} = 1$ following from the conservation law for the number of quasiparticles, but not for the charge, which is not conserved in the steplike model [8].

2.2 Andreev bound states

The wave function for Andreev bound states at the energy $0 < \varepsilon_0 < \Delta_0$ is:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{\frac{N}{2}} \begin{pmatrix} e^{\pm \frac{i\eta}{2} + i\nabla\varphi x/2 + i\theta_0/4} \\ e^{\mp \frac{i\eta}{2} - i\nabla\varphi x/2 - i\theta_0/4} \end{pmatrix} e^{\pm ik_f x - (x-L/2)/\zeta} \quad (18)$$

inside the superconducting layer at $x > L/2$,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{\frac{N}{2}} \begin{pmatrix} e^{\pm \frac{i\eta}{2} + i\nabla\varphi x/2 - i\theta_0/4} \\ e^{\pm \frac{3i\eta}{2} - i\nabla\varphi x/2 + i\theta_0/4} \end{pmatrix} e^{\pm ik_f x + i\theta/2 \mp \frac{im\varepsilon L}{\hbar^2 k_f} + (x+L/2)/\zeta}, \quad (19)$$

inside the superconducting layer at $x < -L/2$, and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{\frac{N}{2}} \begin{pmatrix} e^{\pm \frac{i\eta}{2} + i\theta/4 \pm \frac{im\varepsilon}{\hbar^2 k_f} (x-L/2)} \\ e^{\mp \frac{i\eta}{2} - i\theta/4 \mp \frac{im\varepsilon}{\hbar^2 k_f} (x-L/2)} \end{pmatrix} e^{\pm ik_f x}. \quad (20)$$

inside the normal layer $-L/2 < x < L/2$. Here

$$\eta = \arccos \frac{\varepsilon_0}{\Delta_0}. \quad (21)$$

The normalization constant

$$N = \frac{1}{L + \zeta} \quad (22)$$

takes into account that the bound states penetrate into the superconducting layers on the penetration length

$$\zeta = \zeta_0 \frac{\Delta_0}{\sqrt{\Delta_0^2 - \varepsilon_0^2}}. \quad (23)$$

The length diverges when ε_0 approaches to the gap Δ_0 .

The wave function for Andreev states in Eqs. (18)–(20) satisfy the boundary conditions (continuity of wave function components) at $x = L/2$. The boundary conditions at $x = -L/2$ are satisfied at the Bohr–Sommerfeld condition, which determines the energy spectrum of the Andreev states:

$$\varepsilon_{\pm}(s) = \frac{\hbar v_f}{2L} \left[2\pi s + 2 \arccos \frac{\varepsilon_{0\pm}(s)}{\Delta_0}, \pm \theta \right]. \quad (24)$$

or

$$\varepsilon_{0\pm}(s) = \frac{\hbar v_f}{2L} \left[2\pi s + 2 \arccos \frac{\varepsilon_{0\pm}(s)}{\Delta_0} \pm \theta \right]. \quad (25)$$

Here s is an integer varying from zero to maximal value satisfying the condition that $\varepsilon_{0\pm} < \Delta_0$.

At small energy $\varepsilon_{0\pm}(s) \ll \Delta_0$ (small s)

$$\varepsilon_{0\pm}(s) = \frac{\hbar v_f}{2L + \zeta_0} \left[2\pi \left(s + \frac{1}{2} \right) \pm \theta_0 \right], \quad (26)$$

or

$$\varepsilon_{\pm}(s) = \frac{\hbar v_f}{2L + \zeta_0} \left[2\pi \left(s + \frac{1}{2} \right) \pm \theta \right]. \quad (27)$$

For Andreev levels close to the gap one can expand the arcsin function in Eq. (25) in $\Delta_0 - \varepsilon_{0\pm}$ transforming it to

$$\Delta_0 - \varepsilon_{0\pm} - \frac{\zeta_0}{L} \sqrt{2\Delta_0(\Delta_0 - \varepsilon_{0\pm})} = \pi(t + \alpha) \frac{\zeta_0}{L} \Delta_0. \quad (28)$$

Here α ($0 < \alpha < 1$) is the parameter of incommensurability, which is the fractional part of the ratio of the gap Δ_0 to the Andreev level energy spacing,

$$\frac{\Delta_0 L}{\pi \hbar v_f} = \frac{L}{\pi \zeta_0} = s_m + \alpha, \quad (29)$$

s_m is the maximal integer s less than the ratio, and $t = s_m - s$. Solution of Eq. (28) quadratic with respect to $\sqrt{\Delta_0 - \varepsilon_{0\pm}}$ yields

$$\varepsilon_{0\pm} = \Delta_0 - \frac{\pi \zeta_0}{L} \Delta_0 \left[\sqrt{t + \alpha \mp \frac{\theta_0}{2\pi} + \frac{\zeta_0}{2\pi L}} - \sqrt{\frac{\zeta_0}{2\pi L}} \right]^2,$$

$$\sqrt{\Delta_0^2 - \varepsilon_{0\pm}^2} = \Delta_0 \sqrt{\frac{2\pi\zeta_0}{L}} \left[\sqrt{t + \alpha \mp \frac{\theta_0}{2\pi} + \frac{\zeta_0}{2\pi L}} - \sqrt{\frac{\zeta_0}{2\pi L}} \right]. \quad (30)$$

In the limit $L = 0$ the SNS junction becomes a uniform superconductor without normal layer. Still there is the only bound state of leftmovers with the energy determined from Eq. (26) at $s = 0$:

$$\varepsilon_{ps} = \varepsilon_{0-}(0) = \Delta_0 \cos \frac{\theta_0}{2}. \quad (31)$$

This “Andreev” bound state describes a phase slip center in a uniform superconductor with a phase jump at $x = 0$. Later on it will be called the phase slip state. In this state there are two evanescent plane waves at $x > 0$ and $x < 0$ [Eqs. (18) and (19) at $L = 0$]. Its energy ε_{0-} in Eq. (31) directly follows from the continuity of the wave function at $x = 0$ ($\theta_s \propto L \rightarrow 0$, $\theta = \theta_0$):

$$\begin{pmatrix} e^{\pm \frac{i\eta}{2} + i\theta_0/4} \\ e^{\mp \frac{i\eta}{2} - i\theta_0/4} \end{pmatrix} = \begin{pmatrix} e^{\mp \frac{i\eta}{2} - i\theta_0/4} \\ e^{\pm \frac{i\eta}{2} + i\theta_0/4} \end{pmatrix} e^{\pm i\eta + \theta_0/2}. \quad (32)$$

3 *Ab initio* expressions for currents

In this section we summarize *ab initio* expressions for the vacuum and the excitation currents in the normal layer. The expression Eq. (2) for the condensate current directly follows from Galilean invariance and does not need further discussion.

3.1 Vacuum current in continuum states

One can transform the expression for $\mathcal{T}$ in Eq. (17) revealing its dependence on the incommensurability parameter α introduced in Eq. (29):

$$\mathcal{T}(\theta_0) = \frac{2(\varepsilon_0^2 - \Delta_0^2)}{2\varepsilon_0^2 - \Delta_0^2 - \Delta_0^2 \cos \left[\frac{2(\varepsilon_0 - \Delta_0)mL}{\hbar^2 k_f} + 2\pi\alpha - \theta_0 \right]}. \quad (33)$$

The transmission probability rapidly oscillates as a functions of energy.

Collecting together all contributions from rightmovers and leftmovers, quasiparticles and quasiholes, the *ab initio* expression for the continuum vacuum current is [8]

$$J_{vC} = \frac{e}{\pi\hbar} \int_{\Delta_0}^{\infty} [\mathcal{T}(-\theta_0) - \mathcal{T}(\theta_0)] d\xi. \quad (34)$$

At $L \rightarrow 0$ the current J_{vC} vanishes since $\mathcal{T}(-\theta_0) = \mathcal{T}(\theta_0)$ in this limit.

3.2 Vacuum current in bound Andreev states

The current in the Andreev state is determined by the canonical relation connecting it with the derivative of the energy with respect to the phase:

$$j_{\pm}(s) = \frac{2e}{\hbar} \frac{\partial \varepsilon_{0\pm}(s)}{\partial \theta_0} = \pm \frac{e}{\pi \hbar} \frac{\partial \varepsilon_{0\pm}(s)}{\partial s} = \pm \frac{ev_f}{L + \zeta} = \pm \frac{ev_f}{L} \frac{\sqrt{1 - \frac{\varepsilon_{0\pm}(s)^2}{\Delta_0^2}}}{\sqrt{1 - \frac{\varepsilon_{0\pm}(s)^2}{\Delta_0^2}} + \frac{\zeta_0}{L}}. \quad (35)$$

The factor 2 takes into account that θ_0 is the phase of a Cooper pair but not of a single electron.

The current $j_{\pm}(s)$ is a current produced by a quasiparticle created at the s th state. If the state is not occupied, the vacuum current is two times less than $j_{\pm}(s)$, and has an opposite sign. Taking this into account together with two spin states, the *ab initio* expression for the vacuum current in bound states is

$$\begin{aligned} J_{vA} &= -\frac{e}{\pi \hbar} \sum_s \left\{ \frac{\partial \varepsilon_{0+}(s)}{\partial s} \text{H}[\varepsilon_{0+}(s)] \text{H}[\Delta_0 - \varepsilon_{0+}(s)] \right. \\ &\quad \left. - \frac{\partial \varepsilon_{0-}(s)}{\partial s} \text{H}[\varepsilon_{0-}(s)] \text{H}[\Delta_0 - \varepsilon_{0-}(s)] \right\} \\ &= -\frac{ev_f}{L} \sum_s \left\{ \frac{\sqrt{1 - \frac{\varepsilon_{0+}(s)^2}{\Delta_0^2}}}{\sqrt{1 - \frac{\varepsilon_{0+}(s)^2}{\Delta_0^2}} + \frac{\zeta_0}{L}} \text{H}[\varepsilon_{0+}(s)] \text{H}[\Delta_0 - \varepsilon_{0+}(s)] \right. \\ &\quad \left. - \frac{\sqrt{1 - \frac{\varepsilon_{0-}(s)^2}{\Delta_0^2}}}{\sqrt{1 - \frac{\varepsilon_{0-}(s)^2}{\Delta_0^2}} + \frac{\zeta_0}{L}} \text{H}[\varepsilon_{0-}(s)] \text{H}[\Delta_0 - \varepsilon_{0-}(s)] \right\}. \quad (36) \end{aligned}$$

Here $\text{H}(q)$ are Heaviside step functions, which ensure that summation over s extends only on states with energies $0 < \varepsilon_{0\pm} < \Delta_0$ inside the gap.

In the limit $L \rightarrow 0$ (uniform superconductor without normal layer) the only contribution to the vacuum current is the current in the phase slip state with the energy ε_{ps} given by Eq. (31):

$$J_v = J_{vA} = -\frac{2e}{\hbar} \frac{\partial \varepsilon_{ps}(s)}{\partial \theta_0} = \frac{e\Delta_0}{\hbar} \sin \frac{\theta_0}{2}. \quad (37)$$

3.3 Excitation current

The general expression for the excitation current in bound states is

$$\begin{aligned} J_{vA} &= \sum_s \{ 2j_+(s)f_+(s)\text{H}[\varepsilon_{0+}(s)]\text{H}[\Delta_0 - \varepsilon_{0+}(s)] \\ &\quad + 2j_-(s)f_-(s)\text{H}[\varepsilon_{0-}(s)]\text{H}[\Delta_0 - \varepsilon_{0-}(s)] \}, \quad (38) \end{aligned}$$

where $f_{\pm}(s)$ are occupation numbers of Andreev levels for rightmovers (+) and leftmovers (-). At nonzero temperature the occupation numbers are determined by the Fermi distribution

$$f_{\pm}(s) = \frac{1}{e^{\varepsilon_{\pm}(s)/T} + 1}. \quad (39)$$

At $T = 0$ at the level with zero energy the occupation number $f_{\pm}(s)$ is a free parameter in the interval from 0 to 1.

As well as in previous literature, at high temperatures (much higher than the Andreev level energy spacing $\pi\hbar v_f/L$, but still much lower than critical), the present analysis focuses on the case of the long junction. Then one can ignore excitations in continuum states and replace sums for a large but finite number of Andreev states by infinite sums.

$$J_q = \sum_{s=0}^{\infty} \frac{2j_{+}(s)}{e^{\varepsilon_{+}(s)/T} + 1} + \sum_{s=0}^{\infty} \frac{2j_{-}(s)}{e^{\varepsilon_{-}(s)/T} + 1}. \quad (40)$$

This expression is for a moving condensate, but one use $j_{\pm}(s)$ derived for the condensate at rest [Eq. (35)]. This is because a quasiparticle created in an Andreev state does not change the electron density [9].

Since $\varepsilon_{0\pm}(s) \ll \Delta_0$ one may use Eq. (27), and the excitation current is

$$J_q = \frac{2ev_f}{L + \zeta_0} \sum_0^{\infty} \left[\frac{1}{e^{\beta(s + \frac{\pi + \theta}{2\pi})} + 1} - \frac{1}{e^{\beta(s + \frac{\pi - \theta}{2\pi})} + 1} \right], \quad (41)$$

where

$$\beta = \frac{\pi\hbar v_f}{(L + \zeta_0)T}. \quad (42)$$

While the vacuum current depends on the vacuum phase θ_0 , the excitation current is determined by the total Josephson phase $\theta = \theta_0 + \theta_s$.

4 CPR at zero temperature

At zero temperature the CPR can be derived analytically without complicated calculations of sums and integrals for the vacuum currents [10].

At the Josephson phase θ less than critical one (see below) the total current reduces to the same condensate current in all layers as in a uniform superconductor. This conclusion is valid beyond the steplike model for any thickness L of the normal layer.

The condensate current is the only current flowing through the junction until the Landau criterion is satisfied. The Landau criterion is violated when the energy of the lowest Andreev level $s = 0$ for leftmovers (states with the excitation current flowing in the direction opposite to the total current direction) vanishes. According to Eqs. (14) and (24), this happens when the phase

$\theta = \theta_s$ reaches the critical value θ_{cr} determined by the equation

$$\theta_{cr} = \frac{2L\varepsilon_{0-}(0,0)}{\hbar v_f} = \frac{2mL\Delta_0}{\hbar^2 k_f} \cos \frac{\theta_{cr}}{2}. \quad (43)$$

At $\theta > \theta_{cr}$ partial occupation of the lowest Andreev level starts and nonzero phase θ_0 appears.

The CPR $J(\theta)$ at $\theta > \theta_{cr}$ is derived from Eq. (14) at $\varepsilon_- = 0$, Eq. (25) at $s = 0$, and the relation Eq. (2) connecting the phase θ_s with the total current J :

$$J = J_{cr} \cos \frac{\theta}{2}, \quad (44)$$

where

$$J_{cr} = \frac{2e\Delta_0}{\pi\hbar} \quad (45)$$

is the critical current in the superconducting leads determined from the Landau criterion (depairing current).

Thus, there are two branches of the CPR. At $\theta < \theta_{cr}$ the condensate current is the only current through the junction, and the phase distribution is the same as in a uniform superconductor. We call it the condensate current branch. Along this branch $\theta_0 = 0$ and $\theta = \theta_s$. At $\theta > \theta_{cr}$ the vacuum current and the excitation currents appear, but their sum vanishes, as required by the charge conservation law. At this branch $\theta_0 \neq 0$ and $\theta_s \neq 0$. Along the branch the phase slip occurs when the phase difference across the junction loses 2π . So, the branch can be called phase slip branch.

The CPR at $L = 0$, when the SNS becomes a uniform superconductor, is shown in Fig. 3(a) by a solid line. Along the condensate current branch

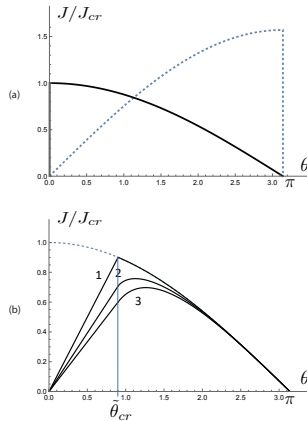


Fig. 3 CPRs at $T = 0$. (a) $L = 0$. The solid line shows the CPR valid for any dimensionality of the junction. The current phase relation in the theory neglecting phase gradients in leads is shown by the dashed line. (b) $L = \tilde{\zeta}/2$. The curves 1, 2, and 3 are the current phase relations for 1D, 2D, and 3D junctions respectively. In the 1D case the length $\tilde{\zeta}$ coincides with ζ_0 .

(vertical segment of the curve) the phase θ is equal to zero because there is no phase jump in a uniform superconductor in the subcritical regime.

The CPR for a 1D junction with $L/\zeta_0 = 1/2$ is shown in Fig. 3(b) (curve 1). When the normal layer thickness L grows the slope of the condensate current branch decreases and the branch approaches to the horizontal axis. The critical phase θ_{cr} approaches to π , while the phase slip branch becomes vertical. Eventually, the CPR is described by the saw-tooth current-phase curve (Fig. 2) for a very long junction. At growing L the Josephson critical current (the maximal current across the junction) decreases from the bulk critical current J_{cr} down to the very small current ev_f/L .

As was already mentioned, the derivation of the CPR does not require calculation of the vacuum current. This calculation is necessary only if one wants to know the occupation number at the phase slip state. We check occupation numbers required for the condition $J_v + J_q = 0$ for the cases when there are simple expressions for the vacuum current J_v . In the limit $L = 0$ the whole vacuum current is equal to the current in the phase slip state given by Eq. (37). The current $J_v + J_q$ vanishes if the occupation number is $1/2$ along the whole phase slip branch. In the opposite limit $L \rightarrow \infty$ the total vacuum current is

$$J_v = \frac{ev_f}{L} \frac{\theta_0}{\pi}. \quad (46)$$

On the other hand, the excitation current in the lowest Andreev level for leftmovers is (see Sec. 3.3)

$$J_q = 2j_-(0)f = -\frac{2ev_f}{L}f. \quad (47)$$

Thus, $J_v + J_q = 0$ if $f = \theta_0/2\pi$. Since along the phase slip branch θ_0 varies from 0 at $\theta = \theta_{cr}$ to π at $\theta = \pi$, the occupation number varies from 0 to $1/2$. At θ crossing π the current changes its direction. Correspondingly, in this point the lowest Andreev level for leftmovers changes from half-occupied to empty, while the lowest Andreev level for rightmovers changes from empty to half-occupied.

For multidimensional systems currents calculated for a single 1D channel must be integrated over the space of wave vectors $\mathbf{k}_\perp$ transverse to the current direction keeping in mind that $k_f = \sqrt{k_F^2 - k_\perp^2}$. Here k_F is the radius of the Fermi sphere in a multidimensional system. The integration operation is $\int_{-k_F}^{k_F} \frac{dk_\perp}{2\pi} \dots$ in the 2D case and $\int_0^{k_F} \frac{k_\perp dk_\perp}{2\pi} \dots$ in the 3D case. After integration currents become current densities.

At the condensate current branch the linear CPR remains valid after integration, but one should replace the 1D density n by the 2D or 3D density. The condensate current branch extends up the phase $\tilde{\theta}_{cr}$ determined by the equation

$$\tilde{\theta}_{cr} - \frac{2L}{\tilde{\zeta}} \cos \frac{\tilde{\theta}_{cr}}{2} = 0, \quad (48)$$

which is similar to Eq. (43) for 1D junctions with k_f replaced by k_F and the 1D coherence length ζ_0 [Eq. (5)] replaced by the coherence length

$$\tilde{\zeta} = \frac{\hbar^2 k_F}{m \Delta_0} \quad (49)$$

for multidimensional junctions. At $\theta = \tilde{\theta}_{cr}$ the transition to the phase slip branch occurs in the channel with the maximal $k_f = k_F$. In all other channels with $k_f < k_F$, the condensate current branch extends up to phases larger than $\tilde{\theta}_{cr}$. Thus, at $\theta > \tilde{\theta}_{cr}$ we have a mixture of channels with the condensate current branch at $k_f < k_c$ and with the phase slip branch at $k_f > k_c$. Here

$$k_c = \frac{2L \cos \frac{\theta}{2}}{\tilde{\zeta} \theta} k_F. \quad (50)$$

Finally, integration over all channels yields

$$\begin{aligned} J &= \frac{2e\Delta_0}{\pi^2 \hbar} \cos \frac{\theta}{2} \int_0^{\sqrt{k_F^2 - k_c^2}} dk + \frac{e\hbar}{\pi^2 m L} \theta \int_{\sqrt{k_F^2 - k_c^2}}^{k_F} \sqrt{k_F^2 - k^2} dk \\ &= J_{cr} \theta \left(\frac{\cos \frac{\theta}{2}}{2\theta} \sqrt{1 - \frac{4L^2 \cos^2 \frac{\theta}{2}}{\tilde{\zeta}^2 \theta^2}} + \frac{\tilde{\zeta}}{4L} \arctan \frac{\frac{2L \cos \frac{\theta}{2}}{\tilde{\zeta} \theta}}{\sqrt{1 - \frac{4L^2 \cos^2 \frac{\theta}{2}}{\tilde{\zeta}^2 \theta^2}}} \right) \end{aligned} \quad (51)$$

for the 2D junction and

$$\begin{aligned} J &= \frac{e\Delta_0}{\pi^2 \hbar} \cos \frac{\theta}{2} \int_0^{\sqrt{k_F^2 - k_c^2}} k dk + \frac{e\hbar}{2\pi^2 m L} \theta \int_{\sqrt{k_F^2 - k_c^2}}^{k_F} \sqrt{k_F^2 - k^2} 2\pi k dk \\ &= J_{cr} \cos \frac{\theta}{2} \left(1 - \frac{4L^2 \cos^2 \frac{\theta}{2}}{3\tilde{\zeta}^2 \theta^2} \right) \end{aligned} \quad (52)$$

for the 3D junction. In the limit $L = 0$ the expression for the ratio J/J_{cr} in multidimensional junctions does not differ from that in 1D junctions, and the plot J/J_{cr} vs. θ [solid line in 3(a)] describes the CPR for junctions of any dimensionality. But the critical current given by Eq. (45) for 1D junctions must be replaced by the critical current densities for multidimensional junctions:

$$J_{cr} = \begin{cases} \frac{2e\Delta_0 k_F}{\pi^2 \hbar} & 2D \text{ case} \\ \frac{e\Delta_0 k_F^2}{2\pi^2 \hbar} & 3D \text{ case} \end{cases}. \quad (53)$$

The CPRs for 2D and 3D junctions at $L/\tilde{\zeta} = 1/2$ are shown in Fig. 3(b) (curves 2 and 3) together with the CPR for a 1D junction (curve 1). There is

a cusp in the 1D CPR in the critical phase $\theta = \tilde{\theta}_{cr}$, which is smeared out in the 2D and 3D cases. In 2D and 3D junctions the first derivative (slope) of the current-phase curve is continuous at $\theta = \tilde{\theta}_{cr}$, but the critical point is still non-analytic with jumps in a higher derivative.

The CPR taking into account phase gradients in leads is essentially different from that obtained ignoring these gradients [6, 7]. The latter is shown by a dashed line in Fig. 3(a) at $L = 0$. The theory without gradients fails to reproduce the evident behavior of a uniform superconductor without normal layer. This theory rules out the backward-skewed CPR revealed in numerical calculations [13] and experimentally [16].

5 CPR of long SNS junction at high temperatures

We consider high temperatures exceeding the Andreev level energy spacing $\pi\zeta_0/L$. Though the temperatures are called high, they are still much lower than critical ($T \ll \Delta_0$). This temperature diapason is possible only in long junctions when $\pi\zeta_0/L \ll \Delta_0$.

As discussed in Introduction, possible temperature independent corrections to the current decreasing faster than $1/L$ with growing L are more important [9] than the exponentially decreasing with T and L current predicted by the previous theory. Corrections $\propto 1/L^{3/2}$ were revealed in Ref. [9] for the vacuum currents in the bound and the continuum states separately, but in their sum they vanished compensating one another. Next corrections proportional to $1/L^2$ were calculated in the vacuum current and the excitation current in bound states in the present work (see App. A). Again, they were essential for both currents separately, but vanished in their sum. An analytical calculation of correction of higher order in $1/L$ is cumbersome, and it was decided to use numerical methods. They revealed that the corrections to main terms $\propto 1/L$ decrease very fast, as $1/L^4$.

5.1 Calculations of the vacuum current in bound Andreev states

At numerical calculation of the sum in Eq. (36) for vacuum current in bound Andreev states, energies of the states were determined by numerical solutions of Eq. (25). Numerical calculations of the sum with Mathematica encountered with problems at large numbers of Andreev levels (large s_m and L), and some tricks were used. For example, the sum was divided on two or three sums calculated separately. The vacuum current in Andreev states $J_{vA}(\theta_0)$ is shown in Fig. 4(a) for $s_m = 30$ for two values of α . The cusp at $\theta_0 = 2\pi\alpha$ is connected with the entrance or the exit of an Andreev level to or from the gap, which changes the number of Andreev levels from even to odd, or *vice versa* (parity effect [9]). The cusp becomes a sharp current jump in the limit $L \rightarrow \infty$.

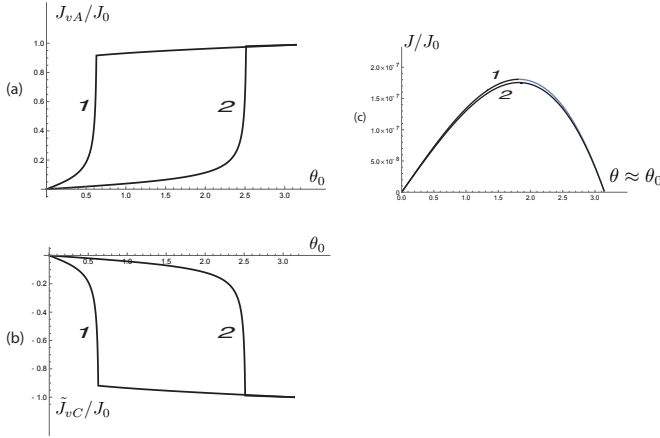


Fig. 4 The CPRs for 31 Andreev levels at $\alpha = 0.1$ (1) and $\alpha = 0.4$ (2). (a) The vacuum current J_{vA} in Andreev bound states. (b) The reduced vacuum current $\tilde{J}_{vA}$ in continuum states [Eq. (62)]. (c) The total current J at high temperatures.

5.2 Calculation of vacuum current in continuum states

For calculation of the continuum vacuum current we used the method described in Appendix B of Ref. [9]. In Ref. [9] calculation was analytical taking into account only the main terms $\propto 1/L$ and $\propto 1/L^{3/2}$. Now calculation is numerical using the exact *ab initio* expression [Eqs. (33) and (34)].

The continuum vacuum current is a difference of contributions from right- and leftmovers:

$$J_{vC} = J_+ - J_-, \quad J_{\pm} = -\frac{e}{\pi\hbar} \int_{\Delta_0}^{\infty} \mathcal{T}(\pm\theta_0) d\xi, \quad (54)$$

where $\mathcal{T}(\pm\theta_0)$ is given by Eq. (33).

Calculating $J_{\pm}$ we introduce a new variable $z = (\varepsilon_0 - \Delta_0)/\Delta_0$:

$$J_{\pm} = -\frac{e\Delta_0}{\pi\hbar} \int_0^{x_m} \frac{2\sqrt{2z+z^2}(1+z) dz}{4z+2z^2+1-\cos(2Lz/\zeta_0+\gamma_{\pm})}, \quad (55)$$

where

$$\gamma_{\pm} = 2\pi\alpha \mp \theta_0. \quad (56)$$

The integrals for $J_{\pm}$ diverge, but their difference does not. So, we introduced a large x_m as an upper cutoff assuming that in the end $x_m \rightarrow \infty$. Integrands of these integrals are rapidly oscillating functions. This complicates their analytical and numerical calculation. at large L .

We divide the whole interval of integration in Eq. (55) on intervals of the length equal to the oscillation period $\pi\zeta_0/L$ of the integrand:

$$J_{\pm} = -\frac{e\Delta_0}{\pi\hbar} \left[I_0(\gamma_{\pm}) + \sum_{s=1}^{s_{xm}} I(s, \gamma_{\pm}) + I_m(\gamma_{\pm}) \right], \quad (57)$$

where

$$I(s, \gamma_{\pm}) = \int_{z_{s\pm} - \frac{\pi\zeta_0}{2L}}^{z_{s\pm} + \frac{\pi\zeta_0}{2L}} \frac{2\sqrt{2z + z^2}(1+z) dz}{4z + 2z^2 + 1 - \cos(2Lz/\zeta_0 + \gamma_{\pm})}. \quad (58)$$

Here

$$z_{s\pm} = \frac{\zeta_0}{2L}(2\pi s - \gamma_{\pm}) \quad (59)$$

is the coordinate of the period center, where the cosine argument is $2\pi s$, and s_{xm} is the maximal integer s satisfying the condition that the both upper limits $z_{s\pm} + \frac{\pi\zeta_0}{2L}$ are smaller than x_m . The integrand in Eq. (58) is equal to 1 at very large z . At large L and small z the integrand becomes a sharp peak at $z = z_{s\pm}$. These peaks are peaks of the transmission probability (transmission resonances) [3, 18, 19]. The energy spacing between transmission resonances is the same as that for bound Andreev levels.

The term $I_m(\gamma_{\pm})$ takes into account the integration interval between the upper limit x_m in the integral Eq. (55) and the upper border of the last $s = s_{xm}$ period. Since the integrand at large z goes to 1, the term can be calculated exactly:

$$I_m(\gamma_{\pm}) = x_m - \frac{(2\pi s_{xm} + \pi - \gamma_{\pm})\zeta_0}{2L}. \quad (60)$$

The term $I_0(\gamma_{\pm})$ is the contribution of the integration interval between $z = 0$ and the lower border of the $s = 1$ interval. The value of $I_0(\gamma_{\pm})$ is determined by the integral in Eq. (58) at $s = 0$ with the lower limit replaced by 0.

Finally, the continuum vacuum current is

$$J_{vC} = \tilde{J}_{vC} - \frac{e\Delta_0}{\pi\hbar} [I_m(\gamma_+) - I_m(\gamma_-)] = \tilde{J}_{vC} + \frac{ev_f \theta_0}{L \pi}, \quad (61)$$

where only the reduced continuum vacuum current

$$\tilde{J}_{vC} = -\frac{e\Delta_0}{\pi\hbar} \left\{ I_0(\gamma_+) - I_0(\gamma_-) + \sum_{s=1}^{s_{xm}} [I(s, \gamma_+) - I(s, \gamma_-)] \right\} \quad (62)$$

must be calculated numerically. Below we shall see that the largest term $\propto 1/L$ in Eq. (61) is compensated in the total current by the similar term in the excitation current with the opposite sign.

The reduced vacuum current in continuum states $\tilde{J}_{vC}(\theta_0)$ is shown in Fig. 4(b) for $s_m = 30$ for two values of α . The cusp at $\theta_0 = 2\pi\alpha$ has an opposite sign to the cusp in bound states, and there is no cusp in the total current (see Sec. 5.4).

5.3 Calculation of excitation current

For temperatures much larger than the energy spacing between Andreev levels but much smaller than the gap, the sum in the expression Eq. (41) for the excitation current can be calculated analytically [3, 9]:

$$J_q = -\frac{ev_f}{L + \zeta_0} \frac{\theta}{\pi} + \frac{8eT}{\hbar} e^{-2\pi TL/\hbar v_f} \sin \theta. \quad (63)$$

The first term is the expression Eq. (41) with the sum replaced by integration over s . The second exponential term is the contribution of Matsubara poles emerging at the exact calculation of the sum.

Since we subtracted the term $\propto 1/L$ from the vacuum current, we should also subtract the same term with opposite sign in the excitation current. Thus, we introduce the reduced excitation current

$$\tilde{J}_q = J_q + \frac{ev_f}{L} \frac{\theta}{\pi} = \frac{ev_f}{L} \frac{\zeta_0}{L + \zeta_0} \frac{\theta}{\pi}. \quad (64)$$

The exponentially small Matsubara term in Eq. (63) was neglected.

Since the final value $\propto 1/L^4$ of the current is very small, subtraction of larger terms $\propto 1/L$ from sums makes the problem of small difference between large terms easier, but did not eliminate it completely. In calculated sums terms much larger than the total current still remain..

5.4 CPR

At high temperatures the SNS junction is a weak link, and the superfluid phase θ_s is very small ($\theta \approx \theta_0$). Then phase gradients in leads can be ignored, as was done in the past. The total current is determined by currents flowing in the normal layer:

$$J = J_{vA} + \tilde{J}_{vC} + \tilde{J}_q. \quad (65)$$

The numerically calculated current J is shown in Fig. 4(c) for $s_m = 30$. The total current J is by the factor about $\sim 10^{-7}$ smaller than the currents in the sum determining J . This illustrates the problem of small difference between large terms mentioned above.

According to Fig. 4(c), the dependence on α is rather weak. It is difficult to decide whether it really exists but numerically small, or is connected with numerical inaccuracy. Anyway, this dependence can be ignored. The CPR is close (but not identical) to sinusoidal CPR

$$J = J_J \sin \theta. \quad (66)$$

The Josephson critical current J_J vs. the number s_m is shown in Fig. 5. The plot is close to $J_J = 0.005 J_0 / s_m^3$. According to Eqs. (3) and (29) for J_0 and

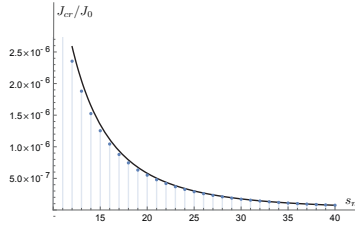


Fig. 5 Josephson critical current vs. number of Andreev levels s_m for $\alpha = 0.4$ (discrete plot). The continuous solid line shows the power law $0.005/s_m^3$.

s_m , the Josephson critical current is

$$J_J = 0.15 \frac{ev_f}{L} \frac{\zeta_0^3}{L^3} = 0.15 \frac{e\hbar^3}{\Delta_0^3} \frac{v_f^4}{L^4}. \quad (67)$$

At large L and high temperatures the temperature independent power law current exceeds the exponentially decreasing Matsubara term in Eq. (63). Thus, the prediction of exponentially small critical current at high temperature in numerous publications starting from Bardeen and Johnson [3] must be revised for planar junctions. The exponential law is still valid for not very high temperatures less than T^* [9]. Comparing the power law current in Eq. (66) with the exponentially small term in Eq. (63) the temperature T^* is

$$T^* = \frac{3\hbar v_f}{2\pi L} \ln \frac{L\Delta_0}{\hbar v_f}. \quad (68)$$

In this expression we ignored numerical factors in the logarithm argument as not important in the limit $L \rightarrow \infty$.

6 Beyond the steplike model for planar junctions with equal Fermi velocities and effective masses

As already discussed above, the condensate current branch is obtained from Galilean invariance and is not conditioned by assumptions of the steplike model. Using a more realistic model would affect only the phase slip branch of the current-phase curve.

If Fermi velocities or effective masses in superconductors and a normal metal are not equal, the normal scattering cannot be ignored in general. But in the limit $L \rightarrow 0$ when the normal layer disappears, the CPR cannot depend on the Fermi velocity and the effective mass of the absent normal metal. This means that the CPR shown in Fig. 3(a) by solid line is valid in this case. The case of non-equal Fermi velocities for planar junctions was calculated by Kupriyanov [20] neglecting phase gradients. As expected, at $L \rightarrow 0$ he obtained

the CPR different from that in a uniform superconductors. This means that phase gradients should not be ignored also for non-equal Fermi velocities.

Let us discuss possible implications of the present analysis for a non-planar junction: a narrow normal bridge between massive superconducting leads considered by Kulik and Omel'yanchouk [21] using the Green function formalism. In contrast to a planar junction, in the limit $L \rightarrow 0$ the bridge junction becomes not a uniform superconductor, but a superconductor divided by a thin impermeable partition wall with a small orifice. It is possible to explain within our approach, why, despite this difference, Eq. (37) is valid for both cases.

In the limit $L \rightarrow 0$ the vacuum current is the current in the phase slip bound state, which is described by two evanescent waves in superconductors on both sides from the point with the phase jump (small orifice in the case of Kulik and Omel'yanchouk). In a planar junction evanescent waves are plane waves, but in the orifice case the evanescent waves must be cylindrical waves in the 2D case or spherical waves in the 3D case. Kulik and Omel'yanchouk considered orifices of diameter much large than the interparticle distance $\sim 1/k_f$ (but less than the coherence length). In this case cylindrical and spherical waves are described by asymptotic expressions for zero orbital moment (no angular dependence): $e^{(ik_f - 1/\zeta)r}/\sqrt{r}$ in the 2D case and $e^{(ik_f - 1/\zeta)r}/r$ in the 3D case. Here r is the distance from the small orifice. Divergence at $r \rightarrow 0$ can be cut off by the size of the orifice. Replacing plane waves by cylindrical or spherical waves does not affect the condition of the continuity of two evanescent waves [Eq. (32)]. So, our simple approach presents a clear physical picture of the vacuum current through a short non-planar junction, which is not evident in the sophisticated Green function analysis [7, 21].

Kulik and Omel'yanchouk [21] ignored phase gradients in leads as was common at that time. Far from the orifice phase gradients are very small indeed. But approaching the orifice they grow as $1/r$ in the 2D case and as $1/r^2$ in the 3D case. Close to the orifice the current in the bridge requires the same gradients as in a planar junction. The analysis of Kulik and Omel'yanchouk [21] widely accepted up to now (see, e.g., Eq. (7) in Ref. [6], Eq. (177) in Ref. [7], or Eq. (3.3) in Ref. [22]) rules out the existence of the condensate current branch with zero phase jump at the orifice.

Kulik and Omel'yanchouk assumed that the gap Δ_0 is constant everywhere ignoring its possible suppression near the orifice. At this assumption one may expect a current state of an ideal incompressible fluid flowing through the orifice, which is not accompanied by phase jump at the orifice. Thus, even in the orifice case one may expect in the limit $L \rightarrow 0$ the backward-skewed CPR [solid line in Fig. 3(a)] rather than the forward-skewed CPR obtained ignoring the phase gradients in leads [dashed line in Fig. 3(a)]. Observation of the backward-skewed CPR in short bridge non-planar SNS junctions [16] is evidence in favor of this scenario.

This discussion of possible implications of our analysis beyond planar junctions with equal Fermi velocities and effective masses was restricted only by the limit $L \rightarrow 0$. The quantitative analysis for any finite L requires taking

into account normal scattering and cannot rely upon the Galilean invariance broken in non-planar junctions. This is beyond the scope of the present work.

7 Conclusions

The paper analyzes the ballistic planar SNS junctions at $T = 0$ taking into account phase gradients in the superconducting leads commonly ignored in the past. Remarkably, adding phase gradients to the theoretical model has made the theory not more complicated but simpler. At $T = 0$ the CPR was derived analytically for any normal layer thickness L in the steplike pairing potential model. This derivation relied on the Galilean invariance in the SNS junction without normal scattering and the Landau criterion for bound Andreev states. The theory taking into account phase gradients in leads resolves the problem of charge conservation law in the steplike model refuting the opinion [6] that the charge conservation law cannot be restored without solving the self-consistency equation. The difference between the CPRs obtained taking or not taking phase gradients in leads is especially essential for short junctions. Taking into account phase gradients, the CPR for short junctions becomes backward-skewed, while according to the previous theory only forward-skewed CPR is possible. The backward-skewed CPR in short junctions was confirmed by numerical calculations [13] and experiment [16].

The paper revised also the existing theory for the long ballistic SNS junction for temperatures much higher than the Andreev level energy spacing (but still much lower than critical). Instead of exponential decrease of the maximal current through the SNS junction with growing L and T predicted by the previous theory, our analysis predicts the temperature independent power law $1/L^4$ decrease with growing L .

Although the quantitative analysis was done for planar SNS junctions with equal Fermi velocities and effective masses in all layers, it was argued that the analysis has implications also beyond this case: planar SNS junctions with non-equal Fermi velocities or even non-planar junctions (narrow normal bridges between bulk superconducting leads). The paper explains experimental observation of backward-skewed CPR in short InAs nanowire bridge junctions [16].

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Appendix A Corrections $\propto 1/L^2$ to the vacuum current in bound states

Calculating the vacuum current in bound Andreev states we expand the current in $1/\sqrt{L}$:

$$J_{vA} = J_1 + J_{3/2} + J_2, \quad (\text{A1})$$

where J_1 , $J_{3/2}$, and J_2 are terms $\propto 1/L$, $\propto 1/L^{3/2}$, and $\propto 1/L^2$ respectively.

In Ref. [8] only the terms J_1 and $J_{3/2}$ were calculated:

$$J_1 = \frac{ev_f}{L} H(-\gamma_+), \quad (\text{A2})$$

$$J_{3/2} = \frac{e\Delta_0\zeta_0^{3/2}}{\sqrt{2\pi\hbar}L^{3/2}} \left[\sum_{t=0}^{\infty} \left(\frac{1}{\sqrt{t + \frac{\gamma_+}{2\pi}}} - \frac{1}{\sqrt{t + \frac{\gamma_-}{2\pi}}} \right) - \sqrt{\frac{2\pi}{\gamma_+}} H(-\gamma_+) \right] \quad (\text{A3})$$

The both contributions to the current are from Andreev levels close to the gap, where one can use the expression Eq. (30) for $\varepsilon_{0\pm}(s)$. The convergence of the sum in Eq. (A3) is very fast, and an infinite upper limit of the sum is possible within the accuracy of the calculation in Ref. [8].

For calculation of corrections $J_2 \propto 1/L^2$ contributions from low Andreev levels are also important. We shall divide the sum determining the vacuum current in bound states on two parts:

$$J_{vA} = J_{<} + J_{>}, \quad J_{<} = -\frac{e}{\pi\hbar} \sum_{s=0}^{s_i} \left[\frac{\partial\varepsilon_{0+}(s)}{\partial s} - \frac{\partial\varepsilon_{0-}(s)}{\partial s} \right],$$

$$J_{>} = -\frac{e}{\pi\hbar} \left[\sum_{s=s_i+1}^{s_{m+}} \frac{\partial\varepsilon_{0+}(s)}{\partial s} - \sum_{s=s_i+1}^{s_m} \frac{\partial\varepsilon_{0-}(s)}{\partial s} \right], \quad (\text{A4})$$

where the upper limit $s_{m+} = s_m - H(\theta_0 - 2\pi\alpha)$ takes into account that the energy of the highest Andreev state crosses the gap at $\theta_0 = 2\pi\alpha$ and the bound state becomes a continuum state. The number s_i is chosen so that the energy $\varepsilon_{0\pm}(s_i)$ satisfies to two inequalities: $\Delta_0\zeta_0/L \ll \Delta_0 - \varepsilon_{0\pm} \ll \Delta_0$. At these conditions there is a large number of Andreev levels in the interval between s_i and s_m but this number, nevertheless, is much smaller than the total number s_m of Andreev levels.

Terms in the sum for the current $J_{<}$ slowly vary with s , and summation maybe replaced by integration:

$$J_{<} = -\frac{e}{\pi\hbar} \int_{s=0}^{s_i} \left[\frac{\partial\varepsilon_{0+}(s)}{\partial s} - \frac{\partial\varepsilon_{0-}(s)}{\partial s} \right] ds,$$

$$= -\frac{e}{\pi\hbar} [\varepsilon_{0+}(s_i) - \varepsilon_{0+}(0) - \varepsilon_{0-}(s_i) + \varepsilon_{0-}(0)]. \quad (\text{A5})$$

According to Eq. (26),

$$\varepsilon_{0+}(0) - \varepsilon_{0-}(0) = \frac{\hbar v_f}{L + \zeta} \theta_0 \approx \left(\frac{1}{L} - \frac{\zeta_0}{L^2} \right) \hbar v_f \theta_0 \quad (\text{A6})$$

According to Eq. (30) and taking into account that $t_i \gg 1$,

$$\varepsilon_{0+}(s_i) - \varepsilon_{0-}(s_i) = \left(\frac{1}{L} - \sqrt{\frac{\zeta_0}{2\pi t_i L^3}} \right) \hbar v_f \theta_0 \quad (\text{A7})$$

Finally,

$$J_{<} = -ev_f \left(\frac{\zeta_0}{L^2} - \sqrt{\frac{\zeta_0}{2\pi t_i L^3}} \right) \frac{\theta_0}{\pi}. \quad (\text{A8})$$

The correction $\propto 1/L^2$ to the current $J_{>}$ in high Andreev levels takes into account that the upper limit of the sum in Eq. (A3) must be $t_i = s_m - s_i$ but not ∞ . Terms $t > t_i$ must be subtracted from the sum. Then the correction to the current is

$$J_{>2} = -\frac{e\Delta_0\zeta_0^{3/2}}{\sqrt{2\pi}\hbar L^{3/2}} \sum_{t=t_i}^{\infty} \left(\frac{1}{\sqrt{t + \frac{\gamma_+}{2\pi}}} - \frac{1}{\sqrt{t + \frac{\gamma_-}{2\pi}}} \right). \quad (\text{A9})$$

Since t_i is a large integer one can expand the sum in $\gamma_{\pm}$ and replace the sum by an integral:

$$J_{>} = -ev_f \sqrt{\frac{\zeta_0}{2\pi t_i L^3}} \frac{\theta_0}{\pi}. \quad (\text{A10})$$

Summing the contributions $\propto 1/L^2$ from the two energy intervals one obtains the $\propto 1/L^2$ term in the bound states vacuum current:

$$J_2 = J_{<} + J_{>} = -\theta_0 \frac{e\Delta_0\zeta_0^2}{\pi\hbar L^2} = -ev_f \frac{\zeta_0}{L^2} \frac{\theta_0}{\pi}.. \quad (\text{A11})$$

This contribution to the vacuum current in bound states is compensated by the contribution $\propto 1/L^2$ to the excitation current [Eq. (63)] bearing in mind that we consider high temperatures when θ_s is small and $\theta \approx \theta_0$. Thus, there is no correction $\propto 1/L^2$ to the total current.

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