

Heisenberg's S-matrix program and Feynman's divergence problem

Lev Sakhnovich

*Retired from Courant Institute, NY,
99 Cove ave. Milford, CT 06461, USA,
E-mail: lsakhnovich@gmail.com*

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Abstract

In the present article, we assume that the first approximation of the scattering operator is given and that it has the logarithmic divergence. This first approximation allows us to construct the so called deviation factor. Using the deviation factor, we regularize all terms of the scattering operator's approximations. The infrared and ultraviolet cases as well as concrete examples are considered. Thus, for a wide range of cases, we provide a positive answer to the well-known problem of J. R. Oppenheimer regarding scattering operators in QED: "Can the procedure be freed of the expansion in ε and carried out rigorously?"

1 Introduction

In the present article, we assume that the first approximation of the scattering operator is given and it has the logarithmic divergence [1]. The so called deviation factor was constructed using this first approximation in our papers [24, 26] (see Appendix A for further details). Using the deviation factor, we regularize all the terms of the scattering operator's series representation.

Remark 1.1 *As was already stated by R. Feynman himself [9] and remained true afterwards “a strict basis for the rules of convergence is not known”. A closely related basic question was formulated by J.R. Oppenheimer in the following form [18]:*

*“Can the procedure be freed of the expansion in ε and carried out rigorously?”
In our article, we give a positive answer to this question.*

In particular, a mathematical justification of the Feynman’s theory [1, 7] is presented for the important case of the logarithmic divergence.

We start with the self-adjoint operators A, A_0, A_1 which are defined in Hilbert space H and are connected by the relation

$$A = A_0 + \varepsilon A_1, \quad (1.1)$$

Here, A is a perturbed operator, A_0 is an unperturbed operator, and A_1 is a perturbation operator. In section 2 on Coulomb type potentials, we consider concrete examples of A, A_0, A_1 , where deviation factor theory is applied to the radial Schrödinger equation and to the radial Dirac equation with Coulomb type potentials, as well as to the Friedrichs model with a discontinuous kernel. These examples provide heuristics for our further research. In the main scheme and the corresponding proofs, these examples are not used.

Section 3 is dedicated to the *secondary generalized scattering* and perturbation operators. (Note that *generalized scattering* operators used, for instance, in [22–26] are introduced in appendix A.) At the beginning of section 3, we consider the operator function

$$S(t, \tau, \varepsilon) = \exp(itA_0) \exp(-itA) \exp(i\tau A) \exp(-i\tau A_0), \quad (1.2)$$

which is closely related to the scattering operator $S(A, A_0)$ (see [21, 23]). The operators A and A_0 above satisfy (1.1). Hence, it follows from (1.2) that

$$\frac{\partial}{\partial t} S(t, \tau, \varepsilon) = -i\varepsilon V(t) S(t, \tau, \varepsilon), \quad \frac{\partial}{\partial \tau} S(t, \tau, \varepsilon) = i\varepsilon S(t, \tau, \varepsilon) V(\tau), \quad (1.3)$$

where

$$V(t) = \exp(itA_0) A_1 \exp(-itA_0), \quad S(t, t, \varepsilon) = I, \quad (1.4)$$

and I is the identity operator. According to (1.4), the self-adjoint operator function $V(t)$ may be considered (at each t) as a special representation of the perturbation operator A_1 .

Definition 1.2 *The self-adjoint operator function $V(t)$ is called the perturbation operator function.*

Further in section 3, we investigate the generalized systems (1.3), that is, we do not suppose that $V(t)$ has the form (1.4). We define the operator function $V(t)$ using the first approximation of the scattering operator (see (3.25)).

We do not use operators A , A_1 and A_0 . These operators are not given (and, may be, they do not exist at all). *Hence, the presented in this paper results follow the suggestions of Heisenberg's S-program [13].* In this way, our approach to quantum electrodynamics here is similar to the approach to classical mechanics from our paper [27].

In sections 4 (on ultraviolet divergence) and 5 (on concrete examples), we apply the results of section 3 to the well-known divergence problems in quantum electrodynamics.

The last section "Conclusion" is devoted to the interpretation of the results obtained in this article.

In appendix A we give a definition of the deviation factors when the operators A , A_0 , A_1 are known. Then, we generalize this notion for the case where the operators A , A_0 , A_1 are either unknown or do not exist.

In appendix B the concept and an important property of the multiplicative integrals are shortly introduced [5]. The case of the so called "time infinity" in our theory is considered in Appendix C.

Remark 1.3 *The well-known divergence problems in quantum electrodynamics are old, classical and very important. They have been studied by many outstanding physicists. These divergencies appear when the small parameter ε expansions are considered. We study the equation (1.3) without expanding its solution into a power series.*

2 Coulomb type potentials: classical cases

1. In this section, we study important cases of the explicit expressions for the deviation factors $W_0(t)$. The deviation factors as well as the generalized wave and scattering operators are introduced in appendix A (see Definitions A.1 and A.2).

Example 2.1 *Let us consider the radial Schrödinger operator*

$$Af = -\frac{d^2}{dr^2}f + \left(\frac{\ell(\ell+1)}{r^2} - \frac{2\varepsilon z}{r} + \varepsilon q(r) \right) f, \quad z = \bar{z} \quad (2.1)$$

with Coulomb type potentials $2\varepsilon z/r - \varepsilon q(r)$, where ε is a small parameter, $q(r)$ satisfies the condition $q(r) = \overline{q(r)}$ and

$$\int_0^1 |q^2| r^2 dr < \infty, \quad \int_1^\infty |q^n(r)| dr < \infty, \quad (n = 1, 2). \quad (2.2)$$

Here, A acts in $H = L^2(0, +\infty)$. The operators A_0 and A_1 in this example have the form:

$$A_0 f = -\frac{d^2}{dr^2} f + \frac{\ell(\ell+1)}{r^2} f, \quad A_1 f = \left(-\frac{2z}{r} + q(r) \right) f. \quad (2.3)$$

The following statement was proved in [21, section 6]:

Theorem 2.2 *Suppose that $q(r)$ satisfies (2.2). Then, the generalized wave operators $W_\pm(A, A_0)$ and the generalized scattering operator $S(A, A_0)$ exist. The corresponding deviation factor has the form*

$$W_0(t) = |t|^{i \operatorname{sgn}(t) \varepsilon z A_0^{-1/2}}. \quad (2.4)$$

A special case of Theorem 2.2, where $q(r) \equiv 0$, was proved by J.G. Dollard [4].

2. Let us consider the radial Dirac equation.

Example 2.3 *The radial Dirac equation has the form:*

$$\left(\frac{d}{dr} + \frac{k}{r} \right) f(r) - \left(\lambda + m + \frac{\varepsilon z}{r} - \varepsilon q(r) \right) g(r) = 0, \quad (2.5)$$

$$\left(\frac{d}{dr} - \frac{k}{r} \right) g(r) + \left(\lambda - m + \frac{\varepsilon z}{r} - \varepsilon q(r) \right) f(r) = 0, \quad (2.6)$$

where $\lambda = \bar{\lambda}$, $m > 0$, $k = \bar{k}$, ε is a small parameter and

$$z = \bar{z} \neq 0, \quad |k| > \varepsilon |z|. \quad (2.7)$$

The corresponding Dirac operator A acts in the space $L_2^2(0, \infty)$ and has the form

$$AF = \left(J_1 \frac{d}{dr} + J_2 \frac{k}{r} + m J_3 + \varepsilon v(r) \right) F, \quad F(r) = \begin{bmatrix} f(r) \\ g(r) \end{bmatrix}, \quad (2.8)$$

where $v(r) = \frac{z}{r} - q(r)$, I_2 is the 2×2 identity matrix, and

$$J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2.9)$$

Here, the unperturbed operator A_0 and the perturbation operator A_1 are given by the formulas:

$$A_0 F = \left(J_1 \frac{d}{dr} + J_2 \frac{k}{r} + m J_3 \right) F, \quad A_1 F = v(r) F. \quad (2.10)$$

We assume that the following conditions are valid:

$$\int_0^a |rq(r)| dr + \int_a^\infty |q(r)| dr < \infty, \quad q(r) = \overline{q(r)}. \quad (2.11)$$

The following statement is proved in our paper [23]:

Theorem 2.4 *Suppose that $q(r)$ satisfies (2.2). Then, the generalized wave operators $W_\pm(A, A_0)$ and the generalized scattering operator $S(A, A_0)$ exist. The corresponding deviation factor has the form*

$$W_0(t) = |t|^{i \operatorname{sgn}(t) \varepsilon z \phi(A_0)}, \quad (2.12)$$

where

$$\phi(A_0) = A_0(A_0^2 - m^2 I)^{-1/2}. \quad (2.13)$$

3. Next, we consider Friedrichs model [10].

Example 2.5 *In this model, we have*

$$A f = x f(x) + \varepsilon \int_a^b f(s) K(x, s) ds, \quad A_0 f = x f(x), \quad (2.14)$$

where $f \in L_n^2[a, b]$ ($-\infty < a < b < +\infty$) and $K(x, s)$ is an $n \times n$ matrix valued function (matrix function) such that $K(x, s) = K^*(s, x)$.

We assume below that the kernel $K(x, s)$ is continuous at all points $x \neq s$ and has a discontinuity of the first kind for $x = s$. Since $K(x, s) = K^*(s, x)$, the matrix function

$$P(x) = i[K(x, x+0) - K(x, x-0)] \quad (2.15)$$

is self-adjoint. We suppose that

$$P(x) \geq 0. \quad (2.16)$$

Let us introduce the new integral kernels

$$K_0(x, s) = (i/2) \operatorname{sgn}(x - s) P^{1/2}(s) P^{1/2}(x) \quad (2.17)$$

and

$$K_1(x, s) = K(x, s) - K_0(x, s). \quad (2.18)$$

In [21, section 2], we proved the following statement:

Theorem 2.6 *Suppose P and K_1 satisfy Hölder conditions*

$$\|P(x_2) - P(x_1)\| \leq M|x_2 - x_1|^\alpha, \quad \|K_1(x, s_2) - K_1(x, s_1)\| \leq M|s_2 - s_1|^{\alpha_1} \quad (2.19)$$

for some $M > 0$, $\alpha > 0$ and $\alpha_1 > 1/2$. Then, the generalized wave operators $W_\pm(A, A_0)$ exist and the deviation factor has the form

$$W_0(t)f = |t|^{i\varepsilon P(x)} f \quad (2.20)$$

where $f \in L_n^2[a, b]$ ($-\infty < a < b < +\infty$).

4. Finally, consider the differential operator A below (acting in the space $H = L^2(-\infty, +\infty)$).

Example 2.7

$$Af = -\frac{d^2}{dx^2} f + \left(\frac{2\varepsilon z x}{x^2 + a^2} + \varepsilon q(x) \right) f, \quad (2.21)$$

where

$$q(x) = \overline{q(x)}, \quad q(x) \in L(-\infty, +\infty), \quad a > 0. \quad (2.22)$$

The corresponding operator A_0 has the form

$$A_0 f = -\frac{d^2}{dx^2} f. \quad (2.23)$$

The following statement is proved in [21, section 4].

Theorem 2.8 *Suppose that $q(x)$ satisfies (2.22). Then, the generalized wave operators $W_\pm(A, A_0)$ and the generalized scattering operator $S(A, A_0)$ exist. The corresponding deviation factor has the form*

$$W_0(t)f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikt} |t|^{z\varepsilon/k} g(k) dk, \quad (2.24)$$

where f is the Fourier transformation of g :

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikt} g(k) dk. \quad (2.25)$$

In the coordinate space, relation (2.24) may be rewritten in the form

$$W_0(t)f = |t|^{iz\varepsilon B_0^{-1}} f, \quad B_0 f = i \frac{d}{dx} f. \quad (2.26)$$

3 Secondary generalized scattering and perturbation operators, infrared divergence

1. It is well known that Coulomb type potentials generate infrared divergences (infrared catastrophe) . The problem has a long history starting in 1940 with the fundamental works of R. Feynman. Here, we construct a rigorous solution of the corresponding problem for a broad class of cases. Recall the operator function

$$S(t, \tau, \varepsilon) = \exp(itA_0) \exp(-itA) \exp(i\tau A) \exp(-i\tau A_0), \quad (3.1)$$

which was mentioned in Introduction. Recall also that the operators A , A_0 , \mathcal{A}_1 satisfy (1.1). It follows from (3.1) that

$$\frac{\partial}{\partial t} S(t, \tau, \varepsilon) = -i\varepsilon V(t) S(t, \tau, \varepsilon), \quad \frac{\partial}{\partial \tau} S(t, \tau, \varepsilon) = i\varepsilon S(t, \tau, \varepsilon) V(\tau), \quad (3.2)$$

where t and τ belong to the *real axis* \mathbb{R} and

$$V(t) = \exp(itA_0) A_1 \exp(-itA_0), \quad S(t, t, \varepsilon) = I. \quad (3.3)$$

2. Further in the sections, the operators A , A_1 , A_0 are unknown. Hence, we do not suppose that the operator $V(t)$ in (3.2) has the form (3.3). Therefore, in our further considerations, the operator $V(t)$ itself is called a perturbation operator. Using the method of successive approximation and relation (3.2), we obtain our next proposition.

Proposition 3.1 *Assume that $V(t)$ is a self-adjoint, continuous and bounded operator function in the domain $1 \leq |t| \leq T$. Then, there exists such $\varepsilon_T > 0$ that the series*

$$S(t, \tau, \varepsilon) = \sum_{p=0}^{\infty} S_p(t, \tau) \varepsilon^p \quad (S_0(t, \tau) \equiv I) \quad (3.4)$$

is convergent in the domains $\tau = 1$, $1 \leq t \leq T$ and $t = -1$, $-T \leq \tau \leq -1$ for $|\varepsilon| \leq \varepsilon_T$.

Here, for $p \geq 0$ we have

$$S_{p+1}(t, \tau) = -i \int_{\tau}^t V(t_1) S_p(t_1, \tau) dt_1 = -i \int_{\tau}^t S_p(t, \tau_1) V(\tau_1) d\tau_1. \quad (3.5)$$

In the present article, we consider the operator function V of the form

$$V(t) = \frac{1}{t} B_+ + u(t) \quad (t \geq 1), \quad V(t) = \frac{1}{t} B_- + u(t) \quad (t \leq -1), \quad (3.6)$$

where B_{\pm} are self-adjoint, bounded operators and $u(t)$ is a self-adjoint, continuous operator function such that

$$\|u(t)\| = O(|t|^{-\nu}) \quad (\nu > 1) \quad \text{for} \quad |t| \rightarrow \infty. \quad (3.7)$$

Relations (3.4)–(3.6) imply that

$$S_1(t, 1) = -i \left(B_+ \ln(t) + \int_1^t u(s) ds \right) \quad \text{for} \quad t \geq 1, \quad (3.8)$$

$$S_1(-1, \tau) = i \left(B_- \ln(|\tau|) + \int_{\tau}^{-1} u(s) ds \right) \quad \text{for} \quad \tau \leq -1. \quad (3.9)$$

The following deviation factors were constructed for the perturbation operators $V(t)$ of the form (3.6) in [26]:

$$W_0(t, \varepsilon) = t^{i\varepsilon B_+} \quad (t \geq 1), \quad W_0(t, \varepsilon) = (-t)^{i\varepsilon B_-} \quad (t \leq -1). \quad (3.10)$$

Remark 3.2 *Physically meaningful values of t and τ are $t \geq 0$ and $\tau \leq 0$. Therefore, we will need the values of $V(t)$ and $W_0(t)$ for $|t| < 1$ in further considerations. We set*

$$V(t) = u(t) \quad (|t| < 1), \quad W_0(t) = I \quad (|t| < 1). \quad (3.11)$$

Since we are interested in the asymptotics of $S(t, \tau, \varepsilon)$ at infinity, relations (3.11) are less important than (3.6) and (3.10).

Let us introduce the operator function S^R (regularized S):

$$S^R(t, \tau, \varepsilon) = W_0(t, \varepsilon) S(t, \tau, \varepsilon) W_0^{-1}(\tau, \varepsilon) \quad (3.12)$$

It follows from (3.2), (3.6) and (3.10)–(3.12) that

$$\frac{\partial}{\partial t} S^R(t, \tau, \varepsilon) = -i\varepsilon U(t, \varepsilon) S^R(t, \tau, \varepsilon), \quad S^R(t, t, \varepsilon) = I, \quad (3.13)$$

$$\frac{\partial}{\partial \tau} S^R(t, \tau, \varepsilon) = i\varepsilon S^R(t, \tau, \varepsilon) U(\tau, \varepsilon), \quad S^R(\tau, \tau, \varepsilon) = I, \quad (3.14)$$

where

$$U(t, \varepsilon) = W_0(t, \varepsilon) u(t) W_0^{-1}(t, \varepsilon) \quad (t \in \mathbb{R}). \quad (3.15)$$

Clearly, we have

$$S^R(0, 0, \varepsilon) = I \quad \text{and} \quad S^R(t, \tau, \varepsilon) = S(t, \tau, \varepsilon), \quad U(t, \varepsilon) = u(t) \quad (3.16)$$

for $t, \tau \in [-1, 1]$. The next theorem easily follows from the relations of this section.

Theorem 3.3 *Let the operator function $u(t)$ be bounded and continuous on \mathbb{R} . Then, $S^R(t, \tau, \varepsilon)$ given by (3.13)–(3.15) admits (for $t \geq 1$ and $\tau \leq -1$) the representation*

$$S^R(t, \tau, \varepsilon) = S^R(t, 1, \varepsilon) S^R(1, -1, \varepsilon) S^R(-1, \tau, \varepsilon). \quad (3.17)$$

Proof. Using multiplicative integrals (see appendix B) and (3.13) we obtain

$$S^R(t, \tau, \varepsilon) = \int_{\tau}^t \widehat{e^{-i\varepsilon U(s, \varepsilon) ds}} \quad (t \geq \tau). \quad (3.18)$$

Hence, for $t \geq 1$ and $\tau \leq -1$ we have

$$S^R(t, \tau, \varepsilon) = \int_1^t \widehat{e^{-i\varepsilon U(s, \varepsilon) ds}} \int_{-1}^1 \widehat{e^{-i\varepsilon U(s, \varepsilon) ds}} \int_{\tau}^{-1} \widehat{e^{-i\varepsilon U(s, \varepsilon) ds}}. \quad (3.19)$$

Formula (3.17) immediately follows from (3.18) and (3.19). \square

Formula (3.1) implies that the operators $S(t, \tau, \varepsilon)$ are unitary for the case, where the self-adjoint operators A_0 and A_1 exist. In our general case, it follows from (3.6) and (3.10) that $W_0(t)$ is unitary. Hence, (3.15) implies that $U(t, \varepsilon)$ is self-adjoint. Now, relation (3.18) yields the unitarity assertion:

Proposition 3.4 *The operators*

$$S^R(t, \tau, \varepsilon), S^R(t, 1, \varepsilon), S^R(1, -1, \varepsilon), S^R(-1, \tau, \varepsilon), \quad (3.20)$$

where $t \geq 1$ and $\tau \leq -1$, are unitary.

In fact, it is easy to see that all the operators $S^R(t, \tau, \varepsilon)$ are unitary (without the restriction $t \geq 1$ and $\tau \leq -1$). However, only the unitarity of the operators given in (3.20) is used further in the text.

Remark 3.5 *Clearly, formula (3.17) (under conditions of Theorem 3.3) may be rewritten as*

$$S^R(t, \tau, \varepsilon) = S^R(t, 1, \varepsilon)S^R(1, 0, \varepsilon)S^R(0, -1, \varepsilon)S^R(-1, \tau, \varepsilon), \quad (3.21)$$

which is sometimes more convenient in the case of a special behaviour at zero. We also have

$$S^R(t, 0, \varepsilon) = S^R(t, 1, \varepsilon)S^R(1, 0, \varepsilon). \quad (3.22)$$

All the factors in (3.21) and (3.22) are again unitary.

Next we formulate the main result of this section, which follows from Theorem 3.3, formula (3.16), Propositions 3.4 and B.1, and Corollary B.2.

Theorem 3.6 *Let the conditions of Theorem 3.3 and relation (3.7) hold. Then, $S^R(t, \tau, \varepsilon)$ converges at each ε by norm (to the unitary operator $S^R(+\infty, -\infty, \varepsilon)$) when t tends to infinity and τ tends to minus infinity. Moreover, we have*

$$S^R(+\infty, -\infty, \varepsilon) = S^R(+\infty, 1, \varepsilon)S^R(1, -1, \varepsilon)S^R(-1, -\infty, \varepsilon), \quad (3.23)$$

where

$$S^R(+\infty, 1, \varepsilon) = \int_1^{+\infty} e^{-i\varepsilon U(s, \varepsilon)} ds, \quad S^R(-1, -\infty, \varepsilon) = \int_{-\infty}^{-1} e^{-i\varepsilon U(s, \varepsilon)} ds. \quad (3.24)$$

Remark 3.7 *The notion of the generalized scattering operator was introduced in [21] (see also appendix A) and fruitfully used in [22–26]. In this section, we introduce a more general notion of the secondary generalized scattering operator. Namely, we do not require the existence of the operators A, A_0, A_1 . Under conditions of Theorem 3.6, the operator $S^R(+\infty, -\infty, \varepsilon)$ is the secondary generalized scattering operator.*

Remark 3.8 *Theorem 3.6 gives a positive answer to Oppenheimer's question, which is cited in Remark 1.1.*

Our next proposition is important for the theory but easily follows from Proposition 3.1 and formulas (3.5) (for $p = 0$), (3.8) and (3.9).

Proposition 3.9 *Under conditions of Proposition 3.1 we have*

$$V(t) = i \frac{d}{dt} S_1(t, 1) \quad (t \geq 1), \quad V(\tau) = -i \frac{d}{d\tau} S_1(-1, \tau) \quad (\tau \leq -1). \quad (3.25)$$

Moreover, the perturbation operator $V(t)$ has the form (3.6) if and only if the first approximation S_1 has the form (3.7).

Remark 3.10 *Recall that in the classical quantum theory the perturbation operator $V(t)$ is defined by (3.3). In our theory, the operators A, A_1, A_0 are unknown and may not exist. In our case, the operator $V(t)$ is recovered from the first approximation $S_1(t, \tau)$ using (3.25).*

3. In a number of theoretical and applied problems, the values of S_p ($p > 1$) are required in addition to the value S_1 of the first approximation (see, e.g., [1, f-la (47.52)]). Finding S_p ($p > 1$) in a standard way is a complicated task. Below, we provide a simple solution for a wide range of problems of finding such S_p . Moreover, the expressions for such approximations are written down in an explicit form.

First, we set

$$C_{1,0} = \int_1^{+\infty} u(s) ds, \quad Q_1(t) = - \int_t^{\infty} u(s) ds \quad (t \geq 1). \quad (3.26)$$

The following theorem is valid.

Theorem 3.11 *Let the relations (3.6), (3.7) and (3.26) be fulfilled. Then,*

$$S_{p+1}(t, 1) = (-i)^{p+1} \left(\frac{B_+^{p+1}}{(p+1)!} \ln^{p+1}(t) + \sum_{k=0}^p C_{p+1,k} \ln^k(t) + Q_{p+1}(t) \right), \quad (3.27)$$

where $p \geq 1$, $t \geq 1$ and

$$C_{p+1,k} = B_+ C_{p,k-1}/k \quad (1 \leq k \leq p), \quad (3.28)$$

$$C_{p+1,0} = \int_1^{+\infty} \left([B_+/s + u(s)] Q_p(s) + u(s) \sum_{k=0}^{p-1} C_{p,k} \ln^k(s) \right) ds, \quad (3.29)$$

$$Q_{p+1}(t) = - \int_t^{+\infty} \left([B_+/s + u(s)] Q_p(s) + u(s) \sum_{k=0}^{p-1} C_{p,k} \ln^k(s) \right) ds, \quad (3.30)$$

$$\|Q_p(t)\| = O(t^{-\delta_p}) \quad (\delta_p > 0). \quad (3.31)$$

It follows from (3.8) and (3.26) that (3.27) holds for $p = 0$. Next, the proof of Theorem 3.11 using (3.5) and complete mathematical induction is straightforward. According to the formulas (3.7) and (3.31), the integrals in Theorem 3.11 are absolutely converging.

Now, we set

$$C_{1,0} = \int_{-\infty}^{-1} u(s) ds, \quad Q_1(\tau) = - \int_{-\infty}^{\tau} u(s) ds \quad (\tau \leq -1). \quad (3.32)$$

Theorem 3.12 *Let the conditions (3.6), (3.7) and (3.32) be fulfilled. Then,*

$$S_{p+1}(-1, \tau) = (i)^{p+1} \left(\frac{B_-^{p+1}}{(p+1)!} \ln^{p+1}(|\tau|) + \sum_{k=0}^p C_{p+1,k} \ln^k(|\tau|) + Q_{p+1}(\tau) \right), \quad (3.33)$$

where $p \geq 1$, $\tau \leq -1$ and

$$C_{p+1,k} = [C_{p,k-1}/k] B_- \quad (1 \leq k \leq p), \quad (3.34)$$

$$C_{p+1,0} = \int_{-\infty}^{-1} \left(Q_p(s) [B_-/s + u(s)] + \left[\sum_{k=0}^{p-1} C_{p,k} \ln^k(s) \right] u(s) \right) ds, \quad (3.35)$$

$$Q_{p+1}(\tau) = - \int_{-\infty}^{\tau} \left(Q_p(s) [B_-/s + u(s)] + \left[\sum_{k=0}^{p-1} C_{p,k} \ln^k(s) \right] u(s) \right) ds, \quad (3.36)$$

$$\|Q_p(\tau)\| = O(|\tau|^{-\delta_p}) \quad (\delta_p > 0). \quad (3.37)$$

It follows from (3.9) and (3.32) that (3.33) holds for $p = 0$. Next, the proof of Theorem 3.12 using (3.5) and complete mathematical induction is

straightforward. According to the formulas (3.7) and (3.37), the integrals in Theorem 3.12 are absolutely converging.

It is interesting that Heisenberg understood divergency problems encountered in the theory of elementary particles and apparently assumed that these problems could be overcome using his S-program [13].

4 Ultraviolet divergence

1. In physical studies of the ultraviolet case (see [26] and references therein), a matrix function $d(L, q, \varepsilon)$ ($L > 0$, $q \in \mathbb{R}^4$) is considered such that

$$d(L, q, \varepsilon) = 1 + \varepsilon(a_1(L, q)) + o(\varepsilon). \quad (4.1)$$

Here, $a_1(L, q)$ may be written down as an integral over the four dimensional sphere with radius L (in spherical coordinates):

$$a_1(L, q) = -i \int_0^L \int_0^{2\pi} \int_0^\pi \int_0^\pi F(p, q) r^3 (\sin^2 \phi_1) \sin \phi_2 d\phi_1 d\phi_2 d\phi_3 dr, \quad (4.2)$$

where $F(p, q)$ is a rational matrix function, $p = [p_1, p_2, p_3, p_4]$, $q = [q_1, q_2, q_3, q_4]$

($p, q \in \mathbb{R}^4$) and

$$p_1 = r \cos \phi_1, \quad p_2 = r \sin \phi_1 \cos \phi_2, \quad (4.3)$$

$$p_3 = r \sin \phi_1 \sin \phi_2 \cos \phi_3, \quad p_4 = r \sin \phi_1 \sin \phi_2 \sin \phi_3. \quad (4.4)$$

In the classical case, it is assumed that the limit of $d(L, q, \varepsilon)$ ($L \rightarrow \infty$) exists: $d(q, \varepsilon) = \lim_{L \rightarrow \infty} d(L, q, \varepsilon)$. We will study the case, where the limit of $a_1(L, q)$ (for $L \rightarrow \infty$) does not exist. In that case we have ultraviolet divergence. In particular, formula (4.2) is actively used in section 5.

2. The results of section 3 may be applied in section 4, if we replace the variable t by L . Thus, the expression $a_1(L, q)$ is a version of the expression $S_1(t, \tau)$. The equality

$$V(L, q) = i \frac{d}{dL} a_1(L, q). \quad (4.5)$$

is a version of relation (3.25). It follows from (4.2) and (4.5) that

$$V(L, q) = \int_0^{2\pi} \int_0^\pi \int_0^\pi F(P, q) L^3 (\sin^2 \phi_1) \sin \phi_2 d\phi_1 d\phi_2 d\phi_3, \quad (4.6)$$

where $P = [P_1, P_2, P_3, P_4]$ and

$$\begin{aligned} P_1 &= L \cos \phi_1, & P_2 &= L \sin \phi_1 \cos \phi_2, \\ P_3 &= L \sin \phi_1 \sin \phi_2 \cos \phi_3, & P_4 &= L \sin \phi_1 \sin \phi_2 \sin \phi_3. \end{aligned}$$

It is known that the ultraviolet divergences may be removed using mass and charge renormalizations [1]. F.J. Dyson [7] stressed that it is important “to prove the convergence in the frame of the theory”. For a broad class of examples, it was done in our paper [26] using deviation factors. The results of [26] are valid only for the first approximation $a_1(q)$. In the present paper, we show that deviation factors and results of section 3 allow us to solve the divergence problems (logarithmic divergence case) for all approximations, in other words for the scattering operator.

Interesting results on the asymptotic behaviour of the Feynman integrals (4.2) are derived in [6] (see also some other papers and references in [2]).

Example 4.1 *Let us consider the case where $a_1(L, q)$ is a scalar function and*

$$a_1(L, q) = -i \left(\phi(q) \ln L + \int_1^L u(s, q) ds \right). \quad (4.7)$$

Here, $\phi(q)$ and $u(L, q)$ are bounded, self-adjoint operators (of multiplication by the functions of q) and

$$\|u(L, q)\| = O(L^{-\nu}) \quad (\nu > 1, \quad L \geq 1). \quad (4.8)$$

In the spirit of R. Feynman’s “space-time approach” [9] and according to (4.5) and (4.7), we have

$$V(L, q) = \frac{\phi(q)}{L} + u(L, q), \quad (B_+ f)(q) = \phi(q)f(q). \quad (4.9)$$

Using $V(L, q)$, we introduce the differential equation for the ultraviolet case in the form

$$\frac{d}{dL} S(L, q, \varepsilon) = -i\varepsilon V(L, q) S(L, q, \varepsilon) \quad (S(1, q, \varepsilon) = 1, \quad L \geq 1), \quad (4.10)$$

where $V(L, q)$ is given by (4.9). The solution of (4.10) may be presented as a series:

$$S(L, q, \varepsilon) = 1 - i\varepsilon \int_1^L V(s, q) ds + \dots$$

The corresponding deviation factor $W_0(L, q, \varepsilon)$ has the form

$$W_0(L, q, \varepsilon) = L^{i\varepsilon\phi(q)}. \quad (4.11)$$

A regularized operator function S^R is introduced by the formula

$$S^R(L, q, \varepsilon) = W_0(L, q, \varepsilon)S(L, q, \varepsilon) \quad (4.12)$$

It follows from (4.10)–(4.12) that

$$\frac{d}{dL}S^R(L, q, \varepsilon) = -i\varepsilon U(L, q, \varepsilon)S^R(L, q, \varepsilon), \quad S^R(1, q, \varepsilon) = 1, \quad (4.13)$$

where

$$U(L, q, \varepsilon) = W_0(L, q, \varepsilon)u(L, q)W_0^{-1}(L, q, \varepsilon), \quad L \geq 1. \quad (4.14)$$

Using multiplicative integrals (see Appendix B) and relation (4.13) we have

$$S^R(L, q, \varepsilon) = \int_1^L e^{-i\varepsilon U(s, q, \varepsilon)ds}, \quad L \geq 1. \quad (4.15)$$

Next, we formulate the main result in this section.

Theorem 4.2 *Let conditions (4.7) and (4.8) be fulfilled. Then,*

$$S^R(L, q, \varepsilon) \implies S^R(+\infty, q, \varepsilon) \quad \text{for } L \rightarrow +\infty, \quad (4.16)$$

where the symbol \implies denotes convergence by norm and

$$S^R(+\infty, q, \varepsilon) = \int_1^{+\infty} e^{-i\varepsilon U(s, q, \varepsilon)ds}. \quad (4.17)$$

Here, we call $S^R(+\infty, q, \varepsilon)$ the secondary generalized scattering operator.

Remark 4.3 *In a similar way, Theorem 4.2 may be formulated and proved in the case of the matrix function a_1 .*

5 The simplest concrete examples, Feynman's integrals

Integrals of the form (4.2) are studied and their physical interpretation is given in the famous paper [9] by R.P. Feynman. In the present section, we will illustrate our approach to the divergence problems (see sections 3 and 4) by concrete physical examples of the form (4.2) from [1]. Let us introduce the Pauli matrices σ_k and α_k :

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad (5.1)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.2)$$

The matrices β , γ_μ and \hat{p} are defined by the relations

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_j = \beta \alpha_j, \quad (j = 1, 2, 3), \quad \gamma_4 = \beta, \quad \hat{p} = \sum_{\mu=1}^4 p_\mu \gamma_\mu. \quad (5.3)$$

where I_k is the $k \times k$ identity matrix. Similar to \hat{p} , we have $\hat{q} = \sum_{\mu=1}^4 q_\mu \gamma_\mu$.

Example 5.1 *Let the first approximation $a_1(L, q, \mu)$ have the form (4.2), where $F(p, q, \mu)$ is a 4×4 matrix function:*

$$F(p, q, \mu) = \frac{1}{(2\pi)^4 p^2} \gamma_\mu \frac{i(\hat{q} - \hat{p}) - mI_4}{(q - p)^2 + m^2} \gamma_\mu, \quad (5.4)$$

$p = [p_1, p_2, p_3, p_4]$, $p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2$, and m is a constant (usually it is a mass). Here, $J_\mu(L, q) := a_1(L, q, \mu)$ are the so called Feynman's integrals.

We note that the integrals $J_\mu(L, q)$ play an important role in electron collision problems [1]. In [26], we derived the equalities

$$J_\mu(L, q) = \frac{1}{(2\pi)^4} \left(im\pi^2(2 \ln L - 1)I_4 - \frac{\pi^2}{2} \gamma_\mu \hat{q} \gamma_\mu - im\pi^2(\ln(B(q))I_4) \right) + R_\mu(L, q) \quad \text{for } \mu = 1, 2, 3; \quad (5.5)$$

$$J_4(L, q) = -\frac{1}{(2\pi)^4} \left(im\pi^2(2 \ln L - 1)I_4 + \frac{\pi^2}{2} \gamma_4 \hat{q} \gamma_4 - im\pi^2(\ln(B(q))I_4) \right) + R_4(L, q), \quad (5.6)$$

where B and R_μ satisfy relations

$$\ln(B(q)) = \int_0^1 \ln(\ell(p, v) - q^2 v^2) dv, \quad \ell(p, v) = (p^2 + m^2)v; \quad (5.7)$$

$$\lim_{L \rightarrow +\infty} R_\mu(L, q) = 0 \quad (\mu = 1, 2, 3, 4). \quad (5.8)$$

It follows from (3.25) that

$$V(L, q, \mu) = i \frac{d}{dL} a_1(L, q, \mu). \quad (5.9)$$

Hence, taking into account (5.5)–(5.7) we obtain:

$$V(L, q, \mu) = \frac{B_+}{L} + u(L, q, \mu), \quad (5.10)$$

where

$$B_+ = \frac{m}{2(2\pi)^2}, \quad u(L, q, \mu) = \frac{d}{dL} R_\mu(L, q). \quad (5.11)$$

In the case of Example 5.1, $\frac{d}{dL} R_\mu(L, q)$ is rational with respect to L and admits Laurent series representation at $L = \infty$. Hence, taking into account (5.8) one can see that $R_\mu(L, q)$ admits Laurent series representation $\sum_{k \geq 1} C_k(q)/L^k$. In this way, one obtains

$$\frac{d}{dL} R_\mu(L, q) = \sum_{k \geq 1} (-k C_k(q))/L^{k+1}.$$

Now, in view of (5.11), it is easy to show that condition (4.8) is fulfilled with $\nu = 2$.

Finally, let us rewrite formula (4.2) in the form

$$\begin{aligned} a_1(L, q, \mu) = & -i \int_1^L \int_0^{2\pi} \int_0^\pi \int_0^\pi F(p, q, \mu) r^3 (\sin^2 \phi_1) \sin \phi_2 d\phi_1 d\phi_2 d\phi_3 dr \\ & - i\psi_\mu(q), \end{aligned}$$

where

$$\psi_\mu(q) = \int_0^1 \int_0^{2\pi} \int_0^\pi \int_0^\pi F(p, q, \mu) r^3 (\sin^2 \phi_1) \sin \phi_2 d\phi_1 d\phi_2 d\phi_3 dr.$$

The deviation factor $W_0(L, q, \varepsilon, \mu)$ is defined by the formula

$$W_0(L, q, \varepsilon, \mu) = L^{i\varepsilon\phi} e^{i\varepsilon\psi_\mu(q)}, \quad \phi = \frac{m}{8\pi^2}. \quad (5.12)$$

Using formulas (4.12)–(4.15), we can show that Theorem 4.2 and Remark 4.3 are valid in the case of Example 5.1.

Example 5.2 *Let the first approximation $a_1(L, q)$ have the form (4.2) with*

$$F(p, q) = \frac{p_\sigma p_\tau}{(p^2 - 2pq + \ell(q))^3} \quad (\ell(q) > q^2), \quad (5.13)$$

where σ and τ are fixed positive integers ($1 \leq \sigma, \tau \leq 4$) and $pq = \sum_{k=1}^4 p_k q_k$.

The following expression for $a_1(L, q)$ was obtained in [1, (47.10)]:

$$a_1(L, q) = \frac{i\pi^2}{4} \delta_{\sigma\tau} \left(\ln \frac{L^2}{\ell(q) - q^2} - \frac{3}{2} \right) + \frac{i\pi^2}{2} \frac{q_\sigma q_\tau}{\ell(q) - q^2} + iR(L, q), \quad (5.14)$$

where $\delta_{\sigma\tau}$ is the Kronecker-delta and

$$\lim_{L \rightarrow +\infty} R(L, q) = 0 \quad (R(L, q) \in \mathbb{R}). \quad (5.15)$$

Using (5.9) and (5.15) we get

$$V(L, q) = \frac{B_+}{L} + u(L, q), \quad (5.16)$$

where

$$B_+ = \phi, \quad u(L, q) = \frac{d}{dL} R(L, q), \quad \phi = \frac{\pi^2}{2}. \quad (5.17)$$

Here, in a way similar to Example 5.1 one may show that condition (4.8) is fulfilled for $\nu = 2$. Similar to Example 5.1, we rewrite (4.2) in the form

$$a_1(L, q) = -i \int_1^L \int_0^{2\pi} \int_0^\pi \int_0^\pi F(p, q) r^3 (\sin^2 \phi_1) \sin \phi_2 d\phi_1 d\phi_2 d\phi_3 dr - i\psi(q), \quad (5.18)$$

where

$$\psi(q) = \int_0^1 \int_0^{2\pi} \int_0^\pi \int_0^\pi F(p, q) r^3 (\sin^2 \phi_1) \sin \phi_2 d\phi_1 d\phi_2 d\phi_3 dr. \quad (5.19)$$

In view of (4.1), (5.17) and (5.18), the deviation factor $W_0(L, q, \varepsilon)$ is defined by the formula

$$W_0(L, q, \varepsilon) = L^{i\varepsilon\phi} e^{i\varepsilon\psi(q)}, \quad (5.20)$$

where ψ and ϕ are given by (5.13), (5.19) and (5.17), respectively. Using again formulas (4.12)–(4.15), we can show that Theorem 4.2 is valid in the case of Example 5.2.

Example 5.3 *Let the first approximation $a_1(L, q)$ have the form (4.2) where*

$$F(p, q) = \frac{p_\sigma}{(p^2 - 2pq + \ell(q))^2} \quad (\ell(q) > q^2). \quad (5.21)$$

The following expression for $a_1(L, q)$ was obtained in [1, (47.12)]:

$$a_1(L, q) = i\pi^2 q_\sigma \left(\ln \frac{L^2}{\ell(q) - q^2} - \frac{3}{2} \right) + iR(L, q). \quad (5.22)$$

where

$$\lim_{L \rightarrow +\infty} R(L, q) = 0 \quad (R(L, q) \in \mathbb{R}). \quad (5.23)$$

Hence, we obtain

$$V(L, q) = \frac{B_+}{L} + u(L, q), \quad (5.24)$$

where

$$B_+ = 2\pi^2 q_\sigma = \phi. \quad (5.25)$$

In view of (4.1), (5.18) and (5.25), the deviation factor $W_0(L, q, \varepsilon)$ is defined by the formula

$$W_0(L, q, \varepsilon) = L^{i\varepsilon\phi} e^{i\varepsilon\psi(q)}, \quad (5.26)$$

where ψ and ϕ are given by (5.19), (5.21) and (5.25), respectively.

In the case of Example 5.3 condition (4.7) is fulfilled and Theorem 4.2 is again valid. Some other concrete examples are contained in the article [26].

6 Conclusion

Feynman's theory assumes that some information about the scattering operator (namely, its first approximation) is given, and does not assume the existence of the perturbed operator A , unperturbed operator A_0 and perturbation operator A_1 . According to [8], "in a scattering problem this over-all

view of the complete scattering process is similar to the S-matrix view-point of Heisenberg”.

In the present article, we use the well-known equation (1.3). In the classical case (see, e.g., [1]), the operator function $V(t)$ in (1.3) is defined via (1.4) using the operators A , A_0 and A_1 . Here, we define the operator function $V(t)$ in (3.25) using the first approximation of the scattering operator. Similar to the Feynman’s theory, we do not use the operators A and A_1 and A_0 . Those operators are not given, and, may be, do not exist. Let us discuss some of the useful developments in this paper.

1. The first approximation of the scattering operator may tend to infinity when one of the parameters tends to infinity, and Feynman’s theory gives a numerically precise method for discarding terms tending to infinity. A rigorous justification of this method has long been an open problem.

In the present article, we give a rigorous approach to the divergence problems in QED for logarithmic divergences.

2. In our paper [26], a rigorous procedure for regularizing the first approximation of the scattering operator is given for various concrete examples and so called deviation factors are used for this purpose (see Appendix A). In the present article, the deviation factors are used for the general case of logarithmic divergences. Moreover, not only the first but all approximations are derived in the explicit form (see(3.27)).

Thus, the presented article proposes a new and fruitful approach to the realization of the Heisenberg’s S-program. In the framework of this approach, we solve the problem of the rigorous treatment of the divergencies. Moreover, the divergences generate the physically meaningful deviation factors, and so the divergencies are physically meaningful as well. In addition to the examples considered here, the self-energy examples with logarithmic divergence from the seminal paper [9] (see also [3]) will be considered in our next paper. We also assume that the ideas, which are presented here, are suitable for other types of divergences (e.g., for the cases of divergencies appearing in Example 4.9 and section 5 of our paper [26]).

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A Generalized wave operators and deviation factors

Wave operators play an essential role in many problems of mathematical physics. However, the wave operators do not exist, when the initial and/or final states of the system may not be regarded as free. In these cases, one has to consider the *generalized wave operators* (see, e.g., [4, 21, 22, 24–26]).

Let A and A_0 be linear self-adjoint (not necessary bounded) operators acting in some Hilbert space H . Denote the absolutely continuous subspace of the operator A (i.e., the subspace corresponding to the absolutely continuous spectrum of A) by G and the absolutely continuous subspace of A_0 by G_0 . Below, the notions of the generalized wave operators $W_{\pm}(A, A_0)$ mapping G_0 into G and of the unitary deviation factor (operator function) $W_0(t)$ acting in G_0 are introduced [21, 22, 26].

Definition A.1 *Operator functions $W_{\pm}(A, A_0)$ and $W_0(t)$ are called the generalized wave operators and a deviation factor, respectively, if the following conditions are fulfilled:*

1. *The limits*

$$W_{\pm}(A, A_0) = s - \lim_{t \rightarrow \pm\infty} [e^{iAt} e^{-iA_0 t} W_0^{-1}(t)] \quad (\text{A.1})$$

exist in the sense of strong convergence.

2. *The operators $W_0(t)$ and $W_0^{-1}(t)$ acting in G_0 are unitary for all t and*

$$\lim_{t \rightarrow \pm\infty} W_0(t + \tau) W_0^{-1}(t) = I_{G_0} \quad \text{for all } \tau = \bar{\tau}, \quad (\text{A.2})$$

where I_{G_0} is the identity operator in G_0 .

3. *The following commutation relations hold for the arbitrary values of t and τ :*

$$W_0(t) A_0 = A_0 W_0(t), \quad W_0(t) W_0(t + \tau) = W_0(t + \tau) W_0(t). \quad (\text{A.3})$$

If $W_0(t) \equiv I_{G_0}$, then the operators $W_{\pm}(A, A_0)$ are usual wave operators.

Definition A.2 *The generalized scattering operator $S(A, A_0)$ is defined by the formula:*

$$S(A, A_0) = W_+^*(A, A_0) W_-(A, A_0), \quad (\text{A.4})$$

where $W_{\pm}(A, A_0)$ are generalized wave operators.

Here, the operator W_+^* is adjoint to W_+ . The operator $S(A, A_0)$ is a unitary mapping of G_0 onto itself and

$$A_0 S(A, A_0) = S(A, A_0) A_0. \quad (\text{A.5})$$

In our theory, the operators A , A_1 and A_0 do not exist. Hence, the generalised wave operators do not exist too. In this case, we define the deviation factor with the help of the first approximation of S (see section 3).

Remark A.3 *If we generalize $W_0(t)$ for the case where A , A_1 and A_0 do not exist, the following properties of the deviation factor from Definition A.1 seem the most important:*

1. *The operators $W_0(t)$ and $W_0^{-1}(t)$ are unitary for all t and the limits*

$$s - \lim_{t \rightarrow \pm\infty} W_0(t + \tau) W_0^{-1}(t) = I \quad (\tau = \bar{\tau}). \quad (\text{A.6})$$

exist in the sense of strong convergence.

2. *The commutation relation*

$$W_0(t) W_0(t + \tau) = W_0(t + \tau) W_0(t) \quad (\text{A.7})$$

holds for arbitrary values of t and τ .

In particular, the condition 1. is fulfilled for the generalized deviation factor in the case of the logarithmic divergence (see the present work and our paper [26]).

Generalized wave operators were also used in [15]. Interesting papers on the infrared case [12, 14] are closely related to the Faddeev-Kulish construction [15].

B Multiplicative integrals

In our article, multiplicative integrals (see, e.g., [5, 11, 19]) are actively used. In this appendix, we present the corresponding basic definitions and some assertions, which we need in the main text. The right and left multiplicative

integrals $\overset{b}{\int_a} e^{F(t)dt}$ and $\overset{b}{\int_a} e^{F(t)dt}$ are defined, respectively, by the formulas

$$\overset{b}{\int_a} e^{F(t)dt} = \lim_{\max \Delta_j \rightarrow 0} [e^{F(t_1)\Delta_1} e^{F(t_2)\Delta_2} \dots e^{F(t_n)\Delta_n}], \quad (\text{B.1})$$

$$\overset{b}{\int_a} e^{F(t)dt} = \lim_{\max \Delta_j \rightarrow 0} [e^{F(t_n)\Delta_n} e^{F(t_{n-1})\Delta_{n-1}} \dots e^{F(t_1)\Delta_1}]. \quad (\text{B.2})$$

Here $-\infty < a = t_0 < \dots < t_n = b < \infty$, $\Delta_j = t_j - t_{j-1}$ and $F(t)$ is a continuous operator function. When $a \rightarrow -\infty$ or $b \rightarrow +\infty$ and the corresponding limits exist we obtain improper multiplicative integrals. For the right multiplicative integrals we have

$$\overset{b}{\int_a} e^{F(t)dt} \implies \overset{b}{\int_{-\infty}} e^{F(t)dt} \quad (a \rightarrow -\infty), \quad \overset{b}{\int_a} e^{F(t)dt} \implies \overset{\infty}{\int_a} e^{F(t)dt} \quad (b \rightarrow \infty). \quad (\text{B.3})$$

where the symbol \implies denotes convergence by norm. Similar notations are used for the left improper multiplicative integrals.

Proposition B.1 *If the operator function $F(t)$ is continuous and*

$$\int_a^\infty \|F(t)\| dt < \infty, \quad (\text{B.4})$$

then the corresponding left multiplicative integrals converge by norm for b tending to infinity

$$\overset{b}{\int_a} e^{F(t)dt} \implies \overset{\infty}{\int_a} e^{F(t)dt} \quad (b \rightarrow \infty). \quad (\text{B.5})$$

Relation (B.5) easily follows from the inequality

$$\left\| \overset{t}{\int_\tau} e^{F(t)dt} \right\| \leq \exp \left(\int_\tau^t \|F(t)\| dt \right), \quad \text{where } \tau < t. \quad (\text{B.6})$$

However, in spite of being easy to derive, Proposition B.1 proves quite useful in this work.

Corollary B.2 *If the operator function $F(t)$ is continuous and*

$$\int_{-\infty}^b \|F(t)\| dt < \infty,$$

then the corresponding left and right multiplicative integrals converge by norm for $a \rightarrow -\infty$:

$$\int_a^b e^{F(t)dt} \Longrightarrow \int_{-\infty}^b e^{F(t)dt}, \quad \int_a^b e^{F(t)dt} \Longrightarrow \int_{-\infty}^b e^{F(t)dt} \quad (a \rightarrow -\infty). \quad (\text{B.7})$$

C Time infinity

1. Recall that the generalized wave operators have been discussed in Appendix A. In the simplest classical case, we assume that the self-adjoint operators A and A_0 exist, $G = G_0 = H$, $W_0(t) = I_H$ and the wave operators $W_{\pm}(A, A_0)$ and $W_{\pm}(A_0, A)$ given by (A.1) (where $W_0(t) = I_H$) exist as well. It follows from (C.2) that

$$W_{\pm}(A_0, A) = W_{\pm}^{-1}(A, A_0) = W_{\pm}^*(A, A_0). \quad (\text{C.1})$$

For some initial vector $\psi_0 \in H$, we set

$$\psi_+ = W_+(A_0, A)\psi_0, \quad \psi_- = W_-(A_0, A)\psi_0. \quad (\text{C.2})$$

According to (C.1) and (C.2), we have

$$\psi_+ = W_+(A_0, A)W_-(A, A_0)\psi_- = W_+^*(A, A_0)W_-(A, A_0)\psi_- \quad (\text{C.3})$$

or

$$\psi_+ = S(A, A_0)\psi_-, \quad (\text{C.4})$$

where $S(A, A_0)$ is the scattering operator.

2. Next we assume that $G = G_0 = H$ and the generalized wave operators $W_{\pm}(A, A_0)$ and $W_{\pm}(A_0, A)$ given by (A.1) (where $W_0(t)$ is not necessarily

the identity operator) exist. Similar to [20, Ch. XI, Sect. 3] (and under the corresponding conditions), one may show that

$$W_{\pm}(A_0, A) = W_{\pm}^{-1}(A, A_0) = W_{\pm}^*(A, A_0). \quad (\text{C.5})$$

We set

$$\phi_+ = W_+(A_0, A)\psi_0, \quad \phi_- = W_-(A_0, A)\psi_0. \quad (\text{C.6})$$

Similar to (C.3), it follows from (C.5) and (C.6) that

$$\phi_+ = W_+(A_0, A)W_-(A, A_0)\phi_- = W_+^*(A, A_0)W_-(A, A_0)\phi_-. \quad (\text{C.7})$$

Thus, taking into account (A.4) we obtain

$$\phi_+ = S(A, A_0)\phi_-, \quad (\text{C.8})$$

where $S(A, A_0)$ is the generalized scattering operator. For $t \rightarrow +\infty$ and $\tau \rightarrow -\infty$ (according to (A.1) and (C.8)), we have

$$\phi_+ \sim (e^{iAt}e^{-iA_0t}W_0^{-1}(t))^* (e^{iA\tau}e^{-iA_0\tau}W_0^{-1}(\tau))\phi_-, \quad (\text{C.9})$$

where the relation $F(t, \tau) \sim G(t, \tau)$ means that

$$s - \lim (F(t, \tau) - G(t, \tau)) = 0 \quad (t \rightarrow +\infty, \tau \rightarrow -\infty). \quad (\text{C.10})$$

Setting

$$\psi_+(t) = W_0^{-1}(t)\phi_+, \quad \psi_-(\tau) = W_0^{-1}(\tau)\phi_-, \quad (\text{C.11})$$

and recalling that the operators $W_0(t)$ are unitary, we derive from (C.9) that

$$\psi_+(t) \sim (e^{iAt}e^{-iA_0t})^* (e^{iA\tau}e^{-iA_0\tau})\psi_-(\tau) \quad \text{for } t \rightarrow +\infty, \tau \rightarrow -\infty. \quad (\text{C.12})$$

Clearly, relations (C.11) and (C.12) are also valid in the classical case, where $W_0(t) = I_H$.

3. Finally, let us consider the secondary generalized case. For this purpose, we use the relations (3.4)–(3.12). For some initial vector $\psi_0 \in H$, we set

$$\phi_+ = s - \lim S^R(t, 0, \varepsilon)\psi_0, \quad \phi_- = s - \lim (S^R(0, \tau, \varepsilon))^*\psi_0 \quad (\text{C.13})$$

for $t \rightarrow +\infty$ and $\tau \rightarrow -\infty$. It follows from the considerations of section 3 that S^R is a unitary operator. Hence, using (C.13) we derive

$$\phi_+ = s - \lim (S^R(t, 0, \varepsilon)S^R(0, \tau, \varepsilon))\phi_- \quad (t \rightarrow +\infty, \tau \rightarrow -\infty). \quad (\text{C.14})$$

Taking into account (3.13)–(3.15), we obtain

$$(S^R(t, 0, \varepsilon))^* = S^R(0, t, \varepsilon), \quad S^R(t, 0, \varepsilon)S^R(0, \tau, \varepsilon) = S^R(t, \tau, \varepsilon). \quad (\text{C.15})$$

It follows from (C.14) and (C.15) that

$$\psi_+(t) \sim S(t, \tau)\psi_-(\tau) \quad \text{for } t \rightarrow +\infty, \tau \rightarrow -\infty, \quad (\text{C.16})$$

where

$$\psi_+(t) = W_0^{-1}(t)\phi_+, \quad \psi_-(\tau) = W_0^{-1}(\tau)\phi_-. \quad (\text{C.17})$$

Remark C.1 *It would be very interesting to check relations (C.16), (C.17) experimentally.*

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