NIVEN NUMBERS ARE AN ASYMPTOTIC BASIS OF ORDER 3

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Abstract. A base-g Niven number is a natural number divisible by the sum of its base-g digits. We show that, for any $g \geqslant 3$, all sufficiently large natural numbers can be written as the sum of three base-g Niven numbers. We also give an asymptotic formula for the number of representations of a sufficiently large integer as the sum of three integers with fixed, close to average, digit sums.

Contents

1. Introduction

A central question in additive number theory is to establish whether a given set of integers $\mathcal{S}$ is an *asymptotic basis* for the integers, that is, to determine whether there exists a natural number k such that any sufficiently large integer can be written as the sum of k elements of $\mathcal{S}$. Here, k denotes the *order* of the basis.

Famously, Lagrange's theorem gives that the squares are a basis of order 4, and Waring's problem, solved by Hilbert, shows that k^{th} powers are also an additive basis. Some interesting variants of Waring's problem consider k^{th} powers of integers which have restrictions on their digits in some base. For example, Pfeiffer and Thuswaldner [9] show that the k^{th} powers of integers with certain congruence conditions on their sums of digits in different bases is an asymptotic basis. More recently, Green [4] established that, given any two digits which are coprime, the integers whose base-g expansions consists of only these digits satisfy Waring's problem. Further references for additive bases coming from sets of integers with digit restrictions are given in the introduction of [10].

A *base-g Niven number* is a natural number that is divisible by its base-g sum of digits. Such integers are also referred to as *Harshad numbers*. It is shown in [2], and independently in [5], that the number of base-g Niven numbers less than x is asymptotically $\eta_g x/\log x$ for some constant $\eta_g > 0$.

It is conjectured that the set of base-g Niven numbers is an asymptotic basis of order 2. In [10] Sanna established, conditionally upon a certain generalisation of the Riemann Hypothesis, that the set of base-g Niven numbers is an asymptotic basis with order growing linearly in g. Our result is the following unconditional statement.

Theorem 1.1. *For any $g \geqslant 3$, the set of base-g Niven numbers is an asymptotic basis of order 3.*

We first count, via the circle method, the number of representations of a sufficiently large integer M as the sum of three integers with a near-average digit sum. By showing that a proportion of such representations have each summand being a Niven number, we provide a lower bound for the number of representations of M as the sum of three Niven numbers. To state these results more precisely, we need the following notation.

Let $s_g(n)$ denote the base-g digit sum of n. Let $K \geqslant 1$ be a sufficiently large integer, and let $M \in (g^{K-1}, g^K]$. For $k \in \mathbf{N}$, let

$$S_g(k) := \{n < g^K : s_g(n) = k\} \text{ and } \mathcal{N}_g(k) := \{n \in S(k) : k \mid n\}.$$

Thus $\mathcal{N}_g(k)$ is the set of Niven numbers in $S_g(k)$, that is, Niven numbers of a certain size with fixed digit sum. Let

$$\mu_K := \frac{(g-1)K}{2},$$

which is the average digit sum of $n < g^K$. For a fixed choice of $k_1, k_2, k_3 \in \mathbf{N}$, let $S_i := S_g(k_i)$ for $i = 1, 2, 3$, and let $\mathcal{N}_i := \mathcal{N}_g(k_i)$ for $i = 1, 2, 3$. Suppose that $k_1, k_2, k_3 \in \mathbf{N}$ are such that

$$|k_i - \mu_K| \leqslant C_g \text{ and } k_1 + k_2 + k_3 \equiv M (\bmod\ g-1), \tag{1}$$

where $C_g := g(g-1)\prod_{p \leqslant 10g^2} p$, and the product is over primes. The specific choice of constant here is unimportant, and any sufficiently large value would do, but this value is large enough to ensure that k_i with further desired properties exist. Let $r_{S_1+S_2+S_3}(M)$ be the number of representations of $M = s_1 + s_2 + s_3$, for $s_i \in S_i$. Our main result is the following theorem, from which we can later deduce the corresponding result for $\mathcal{N}_i \subset S_i$.

Theorem 1.2. *Let g, K and M be integers such that $g \geqslant 3$, K is sufficiently large in terms of g, and $M \in (g^{K-1}, g^K]$. Suppose that k_1, k_2, k_3 satisfy* (1). *Then*

$$r_{S_1+S_2+S_3}(M) = \frac{(g-1)M^2}{2(2\pi\sigma^2 K)^{3/2}}(1 + O_g((\log K)^4 K^{-1/4})),$$

where $\sigma^2 = (g^2-1)/12$.

Let $r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M)$ be the number of representations of $M = n_1 + n_2 + n_3$, where $n_i \in \mathcal{N}_i$ for $i = 1, 2, 3$. In order to relate the quantity $r_{S_1+S_2+S_3}(M)$ to $r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M)$, we require some further conditions on the choice of k_1, k_2, k_3. Throughout, (a, b) denotes the greatest common divisor of a and b. Suppose that $k_1, k_2, k_3 \in \mathbf{N}$ also satisfy, in addition to (1),

$$(k_i, k_j) = 1 \text{ and } (k_i, g) = 1 \text{ for } i, j = 1, 2, 3 \text{ and } i \neq j. \tag{2}$$

Let

$$c_g(k_1, k_2, k_3) := \frac{4}{(g-1)^2} \prod_{i=1}^{3} (g-1, k_i).$$

Then we have the following theorem, from which Proposition 1.1 is a corollary after showing that such a choice of k_1, k_2 and k_3 exist.

Theorem 1.3. *Let g, K and M be integers such that $g \geqslant 3$, K is sufficiently large in terms of g, and $M \in (g^{K-1}, g^K]$. Suppose that k_1, k_2, k_3 fulfil* (1) *and* (2)*, then*

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{(g-1)^2 c_g(k_1, k_2, k_3)}{4k_1k_2k_3} r_{S_1+S_2+S_3}(M) + O_g(M^2 K^{-29/6}).$$

In particular,

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = c_g(k_1, k_2, k_3) \frac{M^2}{(2\pi\sigma^2)^{3/2} K^{9/2}} + O_g(M^2 (\log K)^4 K^{-19/4}),$$

where $\sigma^2 = (g^2 - 1)/12$.

Our methods can be adapted to show the analogous versions of Proposition 1.1, Proposition 1.2 and Proposition 1.3 for base 2, however there is one main technical difference. For readability, we do not give the details here, but comment on the necessary changes in Section 5.

1.1. **Notation.** Throughout, we consider g to be fixed. Let $\|x\|_{\mathbf{R}/\mathbf{Z}}$ denote the distance of a real number x to the nearest integer. We write $e(x)$ for $\exp(2\pi i x)$, and $e_{g-1}(x)$ for $e(x/(g-1))$. We require the quantity ℓ, defined to be

$$\ell := \lceil 384 g^3 \log K \rceil. \tag{3}$$

Throughout, $\sigma = \sqrt{(g^2-1)/12}$.

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2. An overview of the proof

This section outlines the key steps to showing Proposition 1.1. The main work is to establish Proposition 1.2, and then to show that a sufficient number of these representations are of integers which are actually Niven numbers.

The primary tool for showing Proposition 1.2 is the circle method. Notably in our application there are few major arcs; to show the base-g result, we take $g-1$ major arcs, these are short intervals around the rationals $j/(g-1)$ for $j = 0, \dots, g-2$. The need to consider rationals of this form stems from the relation $s_g(n) \equiv n (\mathrm{mod}\ g-1)$, which is the only congruence obstruction to Proposition 1.2. The contribution from these major arcs is handled in Section 4, giving the main term in Proposition 1.2. We establish a uniform pointwise bound for the minor arcs; existing results cover a subset of the minor arcs, but are not sufficiently strong at certain minor arc points to give Proposition 1.2. This bound is proved in Section 5, and relies on a local limit theorem, the necessary consequences of which are given in Section 3.

In Section 6, we show Proposition 1.3 by linking the number of representations of an integer as the sum of three integers with near-average digit sum, to representations where each summand is also a Niven number.

Finally a choice of k_1, k_2 and k_3 which will allow us to deduce Proposition 1.1 from Proposition 1.3 is given in Section 6.2.

2.1. **Counting the number of representations of an integer in $S_1 + S_2 + S_3$.** The main congruence obstruction to finding solutions to $M = s_1 + s_2 + s_3$ comes from the following fact. For all $n \in \mathbf{Z}$ and any base g, $g \geqslant 2$,

$$s_g(n) \equiv n (\text{mod } g-1) \tag{4}$$

Thus in order for $M = s_1 + s_2 + s_3$ to have solutions with $s_i \in S_i$, we must have

$$k_1 + k_2 + k_3 \equiv M (\text{mod } g-1). \tag{5}$$

Let $\mu_K := (g-1)K/2$, then we also require that

$$|k_i - \mu_K| \leqslant C_g. \tag{6}$$

Note that μ_K is the average base-g digit sum for $n < g^K$. By restricting to target digit sums k_i that are close to the average value, we ensure that the sets $|S_i|$ are large. It is shown in [5, Lemma 3] that $|S_g(k)|$ is unimodal in k, with maximum size when $k = \lfloor \mu_K \rfloor$. We require an asymptotic bound for the sizes of the sets S_i. The results of Mauduit and Sárközy [7, Theorem 1], and Fouvry and Mauduit [3, Theorem 1.1] bound the size of $|S_g(k)|$, including for more general ranges of k relative to μ_K, however the error terms stated for these results are too large for our purposes; we need something that is $o_g(g^K K^{-1})$. As we only require a bound for $|S_i|$ when k_i satisfies (6), we are able to use the local limit theorem stated in Section 3 to get the following bound in this range.

Corollary 2.1. *For k_i satisfying* (6),

$$|S_i| = \frac{g^K}{\sqrt{2\pi\sigma^2 K}} + O_g(g^K K^{-3/2}),$$

where $\sigma^2 = (g^2 - 1)/12$.

Now we outline the proof of Proposition 1.2. Let $f_i(\theta)$ denote the Fourier transform $\widehat{\mathbf{1}_{S_i}}(\theta)$, for $i = 1, 2, 3$, so

$$f_i(\theta) := \sum_{n \in S_i} e(n\theta).$$

Then by orthogonality,

$$r_{S_1+S_2+S_3}(M) = \int_{\mathbf{R}/\mathbf{Z}} f_1(\theta) f_2(\theta) f_3(\theta) e(-M\theta)\, \mathrm{d}\theta. \tag{7}$$

We take the following simple major arcs:

$$\mathfrak{M} := \bigcup_{j=0}^{g-2} \Big[\frac{j}{g-1} - \varepsilon, \frac{j}{g-1} + \varepsilon \Big],$$

where $\varepsilon := K^{3/4}g^{-K}/(g-1)$ throughout. Let the minor arcs be the remaining points, $\mathfrak{m} := (\mathbf{R}/\mathbf{Z}) \setminus \mathfrak{M}$. Intuition for these major arcs can be provided by the results of [2, 5, 7]. These works show that the sets S_i are well-distributed across the possible residue classes for a given modulus m, with the quality of relevant error terms depending on the size of m, and whether it is coprime to g and $g-1$. The restriction to certain residue classes comes from (4), but this is the only congruence restriction. As such, we expect cancellation in $\sum_{n\in S_i} e(n\theta)$ unless θ is very close to a multiple of $1/(g-1)$. At these points, there cannot be cancellation due to the following relationship, which holds for any $\theta \in \mathbf{R}$ and $x \in \mathbf{Z}$,

$$f_i(\theta + x/(g-1)) = e_{g-1}(k_i x) f_i(\theta). \tag{8}$$

As such, on intervals around multiples of $1/(g-1)$, $|f_i(\theta)|$ behaves identically to around $\theta = 0$, hence these intervals are included in our major arcs. The relation (8) follows immediately from (4), as

$$f_i\Big(\theta + \frac{x}{g-1}\Big) = \sum_{n\in S_i} e\Big(n\Big(\theta + \frac{x}{g-1}\Big)\Big) = \sum_{n\in S_i} e(n\theta)e_{g-1}(s_g(n)x) = e_{g-1}(xk_i)f_i(\theta).$$

In Section 4 we evaluate the contribution to (7) from the major arcs, as stated in the following proposition.

Proposition 2.2. *Let $K, M \geqslant 1$ be integers such that K is sufficiently large and $M \in (g^{K-1}, g^K]$. Then*

$$\int_{\mathfrak{M}} \prod_{i=1}^{3} f_i(\theta)e(-M\theta) = \frac{(g-1)M^2}{2(2\pi\sigma^2 K)^{3/2}} + O_g(M^2(\log K)^4 K^{-7/4}),$$

where $\sigma^2 = (g^2-1)/12$.

We show in Section 5 that $f_i(\theta)$ is sufficiently small at $\theta \in \mathfrak{m}$ so that the contribution to (7) from the minor arcs is subsumed into the error term of Proposition 1.2. Here is a precise statement.

Proposition 2.3. *For all $\theta \in \mathfrak{m}$,*

$$f_i(\theta) \ll_g g^K K^{-5/4}.$$

Via Parseval's identity and the bound for $|S_i|$ given by Proposition 2.1, this is sufficient to prove that the minor arcs contribute only to the error term in Proposition 1.2. In this work, we use a bound on $|f_i(\theta)|$ due to Fouvry and Mauduit, [3], however we remark that one can find other bounds on exponential sums over sets of integers with fixed digit sums in [2, 6, 11].

We apply the aforementioned result of Fouvry and Mauduit [3] to bound $f_i(\theta)$ whenever θ is such that $(g-1)\theta$ has many non-zero digits in its *centred* base-g expansion. We define the notion of the centred base-g expansion in Section 2.4; this expansion shifts the range of digits to be centred around zero. This leaves the task of bounding $f_i(\theta)$ for θ bounded away from translates of $1/(g-1)$, and with the specific form

$$\theta = \frac{1}{g-1}\Big(\frac{\varepsilon_{m_1}}{g^{m_1}} + \ldots + \frac{\varepsilon_{m_t}}{g^{m_t}} + \eta\Big),$$

where $m_1 < \ldots < m_t \leqslant K$, $t \leqslant \ell$, $\varepsilon_{m_i} \in (-\frac{g}{2}, \frac{g}{2}] \cap \mathbf{Z}$ and $|\eta| < g^K$. We use the fact that $t \leqslant \ell$ to approximate the value of $e(n\theta)$ for $n \in S_i$. It turns out that very little information about n is actually needed for this task. Obviously the digits of n must sum to k_i, as $n \in S_i$. We show that besides this, we also need to know the value of a very small number of digits of n, and the residue class modulo $g-1$ of the sum of a fixed subset of digits.

By modelling the digits of n as independent copies of a uniform random variable, we replace the condition that $n \in S_i$ by the probability that the random variables modelling the digits sum to k_i. We can estimate this term using the local limit theorem. This application of the local limit theorem uses that the target digit sums, k_i, are within a constant of the actual average digit sum $\mu_K = (g-1)K/2$, so that the bounds produced for $f_1(\theta), f_2(\theta)$ and $f_3(\theta)$ are identical.

Given Proposition 2.2 and Proposition 2.3, we can deduce Proposition 1.2.

Proof of Proposition 1.2. From (7),

$$r_{S_1+S_2+S_3}(M) = \int_{\mathbf{R}/\mathbf{Z}} \prod_{i=1}^{3} f_i(\theta) e(-M\theta)\, \mathrm{d}\theta. \tag{9}$$

Proposition 2.2 gives that the contribution to the above integral from the major arcs provides the main term in Proposition 1.2. We show that the contribution from the minor arcs $\mathfrak{m}$ is $O_g(g^{2K}K^{-7/4})$, and thus these points only contribute to the error term in Proposition 1.2. We have

$$\Big| \int_{\mathfrak{m}} \prod_{i=1}^{3} f_i(\theta) e(-M\theta) \Big| \leqslant \sup_{\theta \in \mathfrak{m}} |f_1(\theta)| \int_{\mathbf{R}/\mathbf{Z}} |f_2(\theta)||f_3(\theta)|\, \mathrm{d}\theta.$$

By Cauchy-Schwarz and Parseval, this is

$$\leqslant |S_2|^{1/2} |S_3|^{1/2} \sup_{\theta \in \mathfrak{m}} |f_1(\theta)|.$$

Proposition 2.1 gives $|S_i| \ll_g g^K K^{-1/2}$ for $i = 1, 2$, and combining this with Proposition 2.3 gives

$$\Big| \int_{\mathfrak{m}} \prod_{i=1}^{3} f_i(\theta) e(-M\theta) \Big| \ll_g g^{2K} K^{-7/4}. \qquad \square$$

2.2. **Restricting to Niven numbers with fixed digit sums.** To deduce Proposition 1.3 from Proposition 1.2, we show that for "good" choices of k_1, k_2, k_3, the number of representations of M as $s_1 + s_2 + s_3$ with $k_i \mid s_i$ for each i is roughly a $(k_1k_2k_3)^{-1}$ proportion of the total number of representations. Note that the conditions for the k_i stated in the introduction give such a choice.

It is shown by De Koninck, Doyon and Kátai in [2] and independently, Mauduit, Pomerance and Sárközy in [5, Theorem C], that

$$\mathcal{N}_i \sim \frac{(k_i, g-1)}{k_i} S_i. \tag{10}$$

The $(k_i, g-1)$ term arises from the congruence relation between n and $s_g(n)$ given in (4). We show that indeed the expected proportion of representations are of sums of three Niven numbers, by showing that for appropriate k_1, k_2 and k_3,

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = r_{S_1+S_2+S_3}(M) \prod_{i=1}^{3} \frac{(k_i, g-1)}{k_i}(1 + o_g(1)).$$

To show this, we exploit the fact that Proposition 1.2 counts the number of ways to write M as the sum of three integers with *fixed* near-average digit sums. Let $g_i(\theta)$ be the Fourier transform of $\mathcal{N}_i$,

$$g_i(\theta) \coloneqq \sum_{n\in\mathcal{N}_i} e(n\theta).$$

By considering fixed target digit sums k_i, we gain in that we can relate $g_i(\theta)$ to $f_i(\theta)$, detecting the condition that $k_i \mid n$ by orthogonality. This reduces the problem to one of understanding the Fourier transform $f_i(\theta)$ along translates of frequencies by multiples of $1/k_i$.

We show in Section 5 that a strong bound for $f_i(\theta)$ is available whenever $(g-1)\theta$ has many digits in its centred base-g expansion. As such, a "good" choice of k_1, k_2, k_3 requires that their reciprocals, and certain multiples thereof, have many non-zero digits in base g. This ensures that $g_1(\theta)$, $g_2(\theta)$ and $g_3(\theta)$ are only simultaneously large for $\theta \in \mathfrak{M}$, at which point we can use the results of Section 5 to conclude the proof of Proposition 1.3.

2.3. **Probabilistic model for digits.** Throughout, we switch to a probabilistic model, viewing the digits of n as random variables to model the condition that $s_g(n) = k_i$ by a local limit theorem. To be precise, let us state some notation. Let Y be a random variable uniformly taking values in $\{0, \ldots, g-1\} - (g-1)/2$. Throughout, σ^2 will denote the variance of Y, $\sigma^2 = (g^2-1)/12$. The translation by $-(g-1)/2$ is to ensure that Y is mean-zero; we will account for this shift where appropriate, and thus note that when g is even, Y is not integer valued. Let $X_0, \ldots, X_{K-1}$ be independent and identically distributed copies of Y, and let

$$X \coloneqq \sum_{j=0}^{K-1} X_j g^j.$$

Then $X + (g^K - 1)/2$ is a uniform random integer supported on $\{0, \ldots, g^K - 1\}$, with j^{th} digit $X_j + (g-1)/2$. This follows by the uniqueness of base-g expansions, as

$$X + \frac{g^K-1}{2} = \sum_{j=0}^{K-1} \Big(X_j + \frac{g-1}{2}\Big) g^j \text{ and } X_j + \frac{g-1}{2} \in \{0, \ldots, \frac{g-1}{2}\}.$$

Moreover, this means that $s_g(X + (g^K-1)/2) = \sum_{j=0}^{K-1} X_j + \mu_K$. For convenience, we define the following mean-zero digit sum function for X. For $X = \sum_{j=0}^{K-1} X_j g^j$, let $s(X) \coloneqq \sum_{j=0}^{K-1} X_j$. Equivalently, $s(X) = s_g(X + (g^K-1)/2) - \mu_K$. Let

$$\xi_i \coloneqq k_i - \mu_K. \tag{11}$$

Now we may replace the sum over $n \in S_i$ by an average over X,

$$f_i(\theta) = g^K e\Big(\frac{g^K-1}{2}\theta\Big) \mathbb{E}_X e(X\theta) \mathbf{1}_{s(X)=\xi_i}. \tag{12}$$

Recasting $f_i(\theta)$ as an expectation is not formally needed, but this interpretation as an average is convenient for subsequent sections.

2.4. **Centred base-g expansion.** We also require the notion of a *centred* base-g expansion of a real number, as used by Green in [4], where more detail regarding such expansions can be found. The centred base-g expansion is closely linked to the regular base-g expansion, but shifts the range of permissible digits. Let

$$\mathcal{R}_g \coloneqq \{-\tfrac{g-1}{2},\dots,\tfrac{g-1}{2}\} \text{ for odd } g \text{ and } \mathcal{R}_g \coloneqq \{-\tfrac{g-2}{2},\dots,\tfrac{g}{2}\} \text{ for even } g,$$

and let

$$I_g \coloneqq (-1/2,1/2] \text{ for } g \text{ odd, and } I_g \coloneqq \Big(-\frac{g-2}{2(g-1)},\frac{g}{2(g-1)}\Big] \text{ for } g \text{ even.} \tag{13}$$

Then for $\alpha \in I_g$, if

$$\alpha = \sum_{i\geqslant 1}\alpha_i g^{-i}, \text{ with } \alpha_i \in \mathcal{R}_g \text{ for all } i, \tag{14}$$

we call this the centred base-g expansion of α. Note that I_g is the interval for which the centred base-g expansion of any element has no integer part, as opposed to $[0,1)$ for the regular base-g expansion. Let $\mathcal{R}_g^+ = \max\mathcal{R}_g$ and $\mathcal{R}_g^- = \min\mathcal{R}_g$. As in the regular expansion, the centred expansion of a real number is unique, except when it ends in an infinite sequence of digits all equal to $\mathcal{R}^+$, or all $\mathcal{R}^-$. In this case, we would choose the latter representation. The reason for using this alternate notion of expansion is to use the following function, as defined in [4].

Definition 2.4. *Let $w_K(\alpha)$ be the function counting the number of non-zero digits within the first K digits of the centred base-g expansion of α, after the radix point. For α with expansion given in (14),*

$$w_K(\alpha) = \sum_{i=1}^{K}\mathbf{1}_{\alpha_i\neq 0}.$$

3. Consequences of the local limit theorem

In this section, we state a required local limit theorem and use this along with the probabilistic digit model outlined in Section 2.3 to prove results required for Section 4 and Section 5.

The local limit theorem we use is a special case of a more general local limit theorem, such as Theorem 13 of [8, Ch. VII]. Let Y be a random variable uniformly taking values in $\{0,\dots,g-1\}-(g-1)/2$, and let $T \geqslant 0$ be an integer. For $\nu \in \{0,\dots,T(g-1)\}-\mu_T$, let $P(T,\nu)$ denote the probability that T i.i.d. copies of Y sum to ν.

Corollary 3.1. *For T,ν and $P(T,\nu)$ as defined above,*

$$P(T,\nu) = \frac{e^{-x^2/2}}{\sqrt{2\pi\sigma^2 T}} + O_g(T^{-3/2}),$$

where $x = \nu/\sqrt{\sigma^2 T}$, and $\sigma^2 = (g^2-1)/12$ is the variance of X. In particular, if $|x| < 1/2$,

$$P(T,\nu) = \frac{1}{\sqrt{2\pi\sigma^2 T}} + O_g(\max(x^2T^{-1/2},T^{-3/2})).$$

Proof. The first statement is a corollary of Theorem 13 of [8, Ch. VII], and we give a self-contained proof of this in Section B. The second statement follows immediately from expanding the exponential term. ∎

The second part of Proposition 3.1 gives the bound on the sets $|S_i|$ claimed in Proposition 2.1. Recall that $\mu_K = (g-1)K/2$ is the average digit sum for an integer $n \in [0, g^K)$.

Proof of Proposition 2.1. Note that $|S_i| = g^K P(K, k_i - \mu_K)$, therefore this is an immediate consequence of the second part of Proposition 3.1, using (6) to show $x := (k_i - \mu_K)/\sqrt{\sigma^2 K}$ satisfies $|x| < 1/2$ for sufficiently large K. $\square$

We now state some technical lemmas required for Section 5. These concern a function which we define below, which generalises the probability $P(K,\nu) = \mathbb{P}(X_0 + \ldots + X_{K-1} = \nu)$ to include powers of the $(g-1)^{\text{th}}$ roots of unity weighted by subsets of the random variables X_i. Here, the random variables X_i are i.i.d. copies of the random variable Y, as defined in Section 2.3.. For the rest of this section we assume that $g \geqslant 3$.

Let $\mathbf{a} = (a_0, \ldots, a_{g-2})$ be a $(g-1)$-tuple of non-negative integers. For any $\nu \in \{0, \ldots, (g-1)K\} - \mu_K$, the function $\Psi(\mathbf{a}; \nu)$ is defined to be:

$$\Psi(\mathbf{a};\nu) := \sum_{\substack{j_1,\ldots,j_{g-2} \\ \sum_{i=0}^{g-2} j_i = \nu}} \prod_{s=0}^{g-2} e_{g-1}(sj_s) P(a_s, j_s). \tag{15}$$

where the sum ranges over all tuples $(j_1, \ldots, j_{g-2})$ such that $P(a_0, \nu - \sum_{r=1}^{g-2} j_r) \prod_{s=1}^{g-2} P(a_s, j_s) > 0$. Note that $P(a_s, j_s) > 0$ for $j_s \in \{0, \ldots, a_s(g-1)\} - a_s(g-1)/2$, and $P(a_s, j_s) = 0$ otherwise, so certainly we have the bound

$$|j_s| \leqslant a_s(g-1)/2 \text{ for } s = 1, \ldots, g-2. \tag{16}$$

Recall from Section 2.3 that $X_0, \ldots, X_{K-1}$ are i.i.d. copies of Y, and that $X := \sum_{i=0}^{K-1} X_i g^i$. Then we have the relation

$$e_{g-1}(s(X)) = e_{g-1}(X) \tag{17}$$

as $X = \sum_{j=0}^{K-1} X_j g^j \equiv \sum_{j=0}^{K-1} X_j (\text{mod } g-1)$. We use this fact to state the following equivalent expression for $\Psi(\mathbf{a}; \nu)$. This is the form in which the function actually arises in calculations in Section 5, however the form stated in (15) is more convenient for the results in this section.

Lemma 3.2. *Let $\mathbf{a} = (a_0, \ldots, a_{g-2})$ for integers $a_i \geqslant 0$. For $j \in \{0, \ldots, g-2\}$ and $0 \leqslant i \leqslant a_j - 1$, let $Y_j := \sum_{i=0}^{a_j-1} Y_{j,i} g^i$, where the $Y_{j,i}$ are i.i.d. copies of Y. Then*

$$\Psi(\mathbf{a};\nu) = \mathbb{E}_{Y_0,\ldots,Y_{g-2}} e_{g-1}(Y_1 + \ldots + (g-2)Y_{g-2}) \mathbf{1}_{\sum_{j=0}^{g-2} s(Y_j) = \nu}.$$

Proof. From (17), we have

$$\begin{aligned} &\mathbb{E}_{Y_0,\ldots,Y_{g-2}} e_{g-1}(Y_1 + \ldots + (g-2)Y_{g-2}) \mathbf{1}_{\sum_{j=0}^{g-2} s(Y_j) = \nu} \\ &\qquad = \mathbb{E}_{Y_0,\ldots,Y_{g-2}} e_{g-1}(s(Y_1) + \ldots + (g-2)s(Y_{g-2})) \mathbf{1}_{\sum_{j=0}^{g-2} s(Y_j) = \nu}. \end{aligned} \tag{18}$$

Let t_j denote the possible values of $s(Y_j)$ for $j=1,\dots,g-2$. Then (18) equals

$$\mathbb{E}_{Y_0}\sum_{t_1,\dots,t_{g-2}}\prod_{j=1}^{g-2}e_{g-1}(jt_j)\mathbb{P}(s(Y_j)=t_j)\mathbf{1}_{s(Y_0)=\nu-\sum_{j=1}^{g-2}s(Y_j)}$$

$$=\sum_{t_1,\dots,t_{g-2}}P(a_0,\nu-\sum_{j=1}^{g-2}t_j)\prod_{j=1}^{g-2}e_{g-1}(jt_j)P(a_j,t_j)=\Psi(\mathbf{a};\nu). \qquad \square$$

Note that if $\mathbf{a}=(t,0,\dots,0)$, then $\Psi(\mathbf{a};\nu)=P(t,\nu)$ which can be estimated by Proposition 3.1. The following lemma generalises the second part of Proposition 3.1: it says that $\Psi(\mathbf{a};\nu)$ is approximately constant as ν varies, provided that ν is sufficiently small.

Lemma 3.3. *Suppose that $a_0,\dots,a_{g-2}\geqslant 0$ be integers with $\mathbf{a}=(a_0,\dots,a_{g-2})$ and let $\nu\in\mathbf{Z}$. Suppose further that $|\nu|\leqslant Ca_0^{1/4}$ for some $C>0$ and $a_0\geqslant a_s$ for $s=1,\dots,g-2$. Then*

$$\Psi(\mathbf{a};\nu)=\Psi(\mathbf{a};0)+O_{C,g}(\nu^2a_0^{-3/2}).$$

Proof. From the definition (15), we see that

$$\Psi(\mathbf{a};\nu)=\sum_{j_1,\dots,j_{g-2}}P(a_0,\nu-\sum_{r=1}^{g-1}j_r)\prod_{s=1}^{g-2}e_{g-1}(sj_s)P(a_s,j_s). \tag{19}$$

First, we use the local limit theorem to estimate the term $P(a_0,\nu-\sum_{r=1}^{g-2}j_r)$. Let $J:=\sum_{r=1}^{g-2}j_r$. Note that $|J|<g^2\max a_i$ from (16). From Proposition 3.1, we have

$$P(a_0,\nu-J)=(2\pi\sigma^2a_0)^{-1/2}e^{-(\nu-J)^2/2\sigma^2a_0}+O_g(a_0^{-3/2}). \tag{20}$$

Substituting this into (19) gives

$$\Psi(\mathbf{a};\nu)=(\sigma\sqrt{2\pi a_0})^{-1}e^{-\nu^2/2\sigma^2a_0}\sum_{j_1,\dots,j_{g-2}}e^{(2\nu J-J^2)/2\sigma^2a_0}\prod_{s=1}^{g-2}e_{g-1}(sj_s)P(a_s,j_s)+O_g(a_0^{-3/2}), \tag{21}$$

where the error term in (21) comes from that of (20), and the fact that

$$\sum_{j_1,\dots,j_{g-2}}\prod_{s=1}^{g-2}P(a_s,j_s)=1. \tag{22}$$

By expanding the term $e^{-\nu^2/2\sigma^2a_0}=1+O_{C,g}(\nu^2a_0^{-1})$ in (21), we will show that:

Claim 3.4.

$$\Psi(\mathbf{a};\nu)=(\sigma\sqrt{2\pi a_0})^{-1}\sum_{j_1,\dots,j_{g-2}}e^{(2\nu J-J^2)/2\sigma^2a_0}\prod_{s=1}^{g-2}e_{g-1}(sj_s)P(a_s,j_s)+O_{C,g}(\nu^2a_0^{-3/2}). \tag{23}$$

To establish the claim, we first show that $e^{(2\nu J-J^2)/2\sigma^2a_0}\ll_{C,g}1$ by considering the ranges $|\nu|\leqslant |J|/2$ and $|\nu|>|J|/2$ separately. In the former case, when $|\nu|\leqslant|J|/2$,

$$e^{(2\nu J-J^2)/2\sigma^2a_0}\leqslant e^{(2|\nu J|-J^2)/2\sigma^2a_0}\leqslant 1.$$

In the latter case, when $|\nu| > |J|/2$, we use the assumption that $|\nu| \leqslant Ca_0^{1/4}$ to obtain

$$e^{(2\nu J-J^2)/2\sigma^2 a_0} \leqslant e^{2\nu^2/\sigma^2 a_0} \leqslant e^{2C^2/\sigma^2 a_0^{1/2}} \ll_{C,g} 1.$$

Thus using that $e^{(2\nu J-J^2)/2\sigma^2 a_0} \ll_{C,g} 1$ and (22) gives

$$\sum_{j_1,\ldots,j_{g-2}} e^{(2\nu J-J^2)/2\sigma^2 a_0} \prod_{s=1}^{g-2} P(a_s, j_s) \ll_{C,g} 1.$$

This concludes the proof of Proposition 3.4.

Now we discard the terms with $|J| \geqslant \sigma^2 a_0/2|\nu|$ from (23) (if there are any), showing that the contribution from these terms is negligible. Indeed, as $\sigma^2 a_0/2|\nu| \leqslant |J| \leqslant g^2 \max a_s$, and $\max a_s \leqslant a_0$,

$$e^{(2\nu J-J^2)/2\sigma^2 a_0} \leqslant e^{g^2|\nu| \max a_s/\sigma^2 a_0 - \sigma^2 a_0/8\nu^2} \leqslant e^{Cg^2 a_0^{1/4}/\sigma^2 - \sigma^2 a_0^{1/2}/8C^2} \ll_{C,g} a_0^{-10}. \tag{24}$$

This additionally uses the assumption that $|\nu| \leqslant Ca_0^{1/4}$ for some $C > 0$. Bounding the term $|e_{g-1}(sj_s)P(a_s, j_s)| \leqslant 1$ for all s, j_s and using (24), we have

$$\Bigg| \sum_{\substack{j_1,\ldots,j_{g-2} \\ \sigma^2 a_0/2|\nu| \leqslant |J| \leqslant g^2 a_0}} e^{(2\nu J-J^2)/2\sigma^2 a_0} \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) \Bigg| \ll_{C,g} a_0^{-9},$$

hence the contribution to (23) from $(j_1, \ldots, j_{g-2})$ such that $|J|$ is large can be absorbed into the overall error term of $O_{C,g}(\nu^2 a_0^{-3/2})$. It remains to estimate the contribution to (23) from $(j_1, \ldots, j_{g-2})$ with $|J| \leqslant \sigma^2 a_0/2|\nu|$. To this end, we expand the term $e^{\nu J/\sigma^2 a_0}$ in (23), giving

$$\Psi(\mathbf{a};\nu) = \frac{1}{\sigma\sqrt{2\pi a_0}} \sum_{\substack{j_1,\ldots,j_{g-2} \\ |J| \leqslant \sigma^2 a_0/2|\nu|}} e^{-J^2/2\sigma^2 a_0} \Big(1 + \frac{\nu J}{\sigma^2 a_0} + O_g\Big(\frac{\nu^2 J^2}{a_0^2}\Big)\Big) \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) + O_{C,g}(\nu^2 a_0^{-3/2}). \tag{25}$$

First, we show that

$$\frac{1}{\sigma\sqrt{2\pi a_0}} \sum_{\substack{j_1,\ldots,j_{g-2} \\ |J| \leqslant \sigma^2 a_0/2|\nu|}} e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) = \Psi(\mathbf{a};0) + O_{C,g}(a_0^{-3/2}). \tag{26}$$

To do so, note we can undo the truncation on the range of summation in (26). If the contribution from the range $\sigma^2 a_0/2|\nu| \leqslant |J| \leqslant g^2 a_0$ is non-zero, then in this range:

$$e^{-J^2/2\sigma^2 a_0} \leqslant e^{-\sigma^2 a_0/8\nu^2} \leqslant e^{-\sigma^2 a_0^{1/2}/8C^2} \ll_{C,g} a_0^{-10}.$$

Using this and (22) we have

$$\sum_{\substack{j_1,\ldots,j_{g-2} \\ |J| > \sigma^2 a_0/2|\nu|}} e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) \ll_{C,g} a_0^{-10}. \tag{27}$$

We now work in the full range of summation for $(j_1, \dots, j_{g-2})$. We apply Proposition 3.1 to note that $e^{-J^2/2\sigma^2 a_0}/\sigma\sqrt{2\pi a_0} = P(a_0, -J) + O_g(a_0^{-3/2})$, hence

$$\begin{aligned}\frac{1}{\sigma\sqrt{2\pi a_0}} &\sum_{j_1,\dots,j_{g-2}} e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) \\ &= \sum_{j_1,\dots,j_{g-2}} P(a_0, -J) \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) + O_{C,g}(a_0^{-3/2}) \\ &= \Psi(\mathbf{a}; 0) + O_{C,g}(a_0^{-3/2}),\end{aligned}$$

using the definition in (15) to obtain the second equality. This establishes (26), giving the main term in Proposition 3.3.

We now show that the remaining terms in (25) contribute only to the error term in Proposition 3.3. The $O_g(\nu^2 J^2 a_0^{-2})$ term within the summation over $(j_1, \dots, j_{g-2})$ in (25) can be absorbed into the error term $O_{C,g}(\nu^2 a_0^{-3/2})$. Indeed, as $e^{-J^2/2\sigma^2 a_0} J^2 \leqslant 2\sigma^2 a_0/e$, the contribution from this term is bounded as follows:

$$\frac{\nu^2}{a_0^{5/2}} \sum_{\substack{j_1,\dots,j_{g-2} \\ |J| \leqslant \sigma^2 a_0/2|\nu|}} \prod_{s=1}^{g-2} P(a_s, j_s) e^{-J^2/2\sigma^2 a_0} J^2 \ll_g \frac{\nu^2}{a_0^{3/2}} \sum_{\substack{j_1,\dots,j_{g-2} \\ |J| \leqslant \sigma^2 a_0/2|\nu|}} \prod_{s=1}^{g-2} P(a_s, j_s) \ll_g \frac{\nu^2}{a_0^{3/2}},$$

additionally using (22) in the final inequality.

We have shown that

$$\Psi(\mathbf{a}; \nu) = \Psi(\mathbf{a}; 0) + \frac{\nu}{\sigma^3 a_0^{3/2}\sqrt{2\pi}} \sum_{j_1,\dots,j_{g-2}} e^{-J^2/2\sigma^2 a_0} J \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) + O_{C,g}(\nu^2 a_0^{-3/2}). \quad (28)$$

Finally we show that (28) implies the statement of the lemma. It suffices to prove

$$\sum_{j_1,\dots,j_{g-2}} e^{-J^2/2\sigma^2 a_0} J \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) \ll_{C,g} 1. \quad (29)$$

Before proving (29), we remark that the proof is essentially trivial in base 3. In this case, the equation on the left hand side of (29) equals

$$\sum_j (-1)^j j e^{-j^2/2\sigma^2 a_0} P(a_1, j) = \sum_j g(j) = 0,$$

as $g(j) := (-1)^j j e^{-j^2/2\sigma^2 a_0} P(a_1, j)$ is an odd function. For $g \geqslant 4$, the proof of (29) is rather more involved, and we make use of the cancellation coming from the $e_{g-1}(sj_s)$ terms instead of the sign of J. Let

$$g(j_1, \dots, j_{g-2}) := J e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} P(a_s, j_s). \quad (30)$$

We first work with the tuples $(j_1,\dots,j_{g-2})$ for which the following condition holds, in addition to the assumption throughout that $|j_s| \leqslant a_s(g-1)/2$. Suppose that $J = j_1+\dots+j_{g-2}$ is such that

$$\Big|\frac{2Jr+r^2}{2\sigma^2 a_0}\Big| \leqslant 1/2 \text{ for all } r\in\{0,\dots,g-2\}. \tag{31}$$

For such $(j_1,\dots,j_{g-2})$, the function $g(j_1,\dots,j_{g-2})$ doesn't vary too much when incrementing j_1 by a small amount, as shown in the following claim.

Claim 3.5. *For $(j_1,\dots,j_{g-2})$ such that* (31) *holds, and for all $r=0,\dots,g-2$,*

$$|g(j_1,\dots,j_{g-2})-g(j_1+r,j_2,\dots,j_{g-2})| \ll_g e^{-J^2/2\sigma^2 a_0}\Big(\frac{J^2}{a_0}(P(a_1,j_1)+a_1^{-3/2})+P(a_1,j_1)\Big)\prod_{s=2}^{g-2}P(a_s,j_s).$$

Proof of Proposition 3.5. First we note that the claim is trivial when $J=0$. In this case, $(j_1,\dots,j_{g-2}) = (0,\dots,0)$ and $g(0,\dots,0)=0$, and it follows from (30) that $g(r,0,\dots,0) \ll_g \prod_{s=1}^{g-1}P(a_s,j_s)$. Thus from now on, we assume that $|J|\geqslant 1/2$. From Proposition 3.1,

$$|P(a_1,j_1)-P(a_1,j_1+r)| \ll_g a_1^{-1/2}|e^{-a_1^2/2\sigma^2 a_1} - e^{-(a_1+r)^2/2\sigma^2 a_1}| + O_g(a_1^{-3/2}) \ll_g a_1^{-3/2}. \tag{32}$$

Here, we also use that the function e^{-x^2} is Lipschitz to obtain the final inequality. We also have, under the assumptions of (34),

$$\exp\Big(-\frac{2Jr+r^2}{2\sigma^2 a_0}\Big) = 1+O_g\Big(\frac{J}{a_0}\Big). \tag{33}$$

From the definition in (30),

$$\begin{aligned}&|g(j_1,\dots,j_{g-2})-g(j_1+r,j_2,\dots,j_{g-2})|\\ &= e^{-J^2/2\sigma^2 a_0}\prod_{s=2}^{g-2}P(a_s,j_s)|JP(a_1,j_1)-(J+r)e^{-(2Jr+r^2)/2\sigma^2 a_0}P(a_1,j_1+r)|.\end{aligned}$$

Expanding the term $e^{-(2Jr+r^2)/2\sigma^2 a_0}$ using (33), and using (32) to estimate $P(a_1+r,j_1)$, we obtain,

$$\begin{aligned}&|JP(a_1,j_1)-(J+r)e^{-(2Jr+r^2)/2\sigma^2 a_0}P(a_1,j_1+r)|\\ &\qquad = |JP(a_1,j_1)-(J+r)(1+O_g(J/a_0))(P(a_1,j_1)+O_g(a_1^{-3/2})))|\\ &\qquad \ll_g \frac{J^2}{a_0}(P(a_1,j_1)+a_1^{-3/2})+P(a_1,j_1).\end{aligned}$$

To simplify the final expression, we have used that $r\ll_g 1$, and that for any tuple $(j_1,\dots,j_{g-2})$, $J^2\geqslant |J|/2$. □

We use Proposition 3.5 to prove the following:

$$\sum_{j_1,\dots,j_{g-2}} e^{-J^2/2\sigma^2 a_0} J\prod_{s=1}^{g-2}e_{g-1}(sj_s)P(a_s,j_s) \ll_g 1. \tag{34}$$

We have

$$\sum_{j_1,\dots,j_{g-2}} e^{-J^2/2\sigma^2 a_0} J \prod_{s=1}^{g-2} e_{g-1}(sj_s) P(a_s, j_s) = \sum_{j_1,\dots,j_{g-2}} g(j_1,\dots,j_{g-2}) \prod_{s=1}^{g-2} e_{g-1}(sj_s)$$

To get the cancellation required, we use Proposition 3.5 to assert that $g(j_1+r,\dots,j_{g-2})$ is essentially constant as r varies in $\{0,\dots,g-2\}$, which allows us to get cancellation from the $\prod_{s=1}^{g-2} e_{g-1}(sj_s)$ term. Our aim is to split the range of summation into sets where j_1 has a fixed congruence modulo $g-1$. Note that if g is even and a_1 odd, the range of j_1 is not contained in the integers, as $j_1 \in \{0,\dots,a_1(g-1)\} - a_1(g-1)/2$. In this case, $j_1 + 1/2 \subset \mathbf{Z}$, so we can run the following argument by multiplying through by a factor of $e_{g-1}(1/2)$. If $a_1(g-1)/2 \in \mathbf{Z}$, let $x := -a_1(g-1)/2$, otherwise let $x := -a_1(g-1)/2 - 1/2$. Note that from (16), the range of j_1 is a multiple of $g-1$. Thus splitting the range of j_1 (compensating by a factor of $e_{g-1}(1/2)$ if necessary),

$$\begin{aligned}\sum_{j_1,\dots,j_{g-2}} g(j_1,\dots,j_{g-2}) \prod_{s=1}^{g-2} e_{g-1}(sj_s) &= \sum_{\substack{j_1,\dots,j_{g-2}\\ j_1\equiv x (\mathrm{mod}\ g-1)}} \prod_{s=1}^{g-2} e_{g-1}(sj_s) \sum_{r=0}^{g-2} e_{g-1}(r) g(j_1+r, j_2,\dots,j_{g-2}) \\ &= \sum_{\substack{j_1,\dots,j_{g-2}\\ j_1\equiv x (\mathrm{mod}\ g-1)}} \prod_{s=1}^{g-2} e_{g-1}(sj_s) \sum_{r=0}^{g-2} e_{g-1}(r) \Big(g(j_1, j_2,\dots,j_{g-2}) + O_g(E(J,a_0,a_1)) \Big)\end{aligned} \tag{35}$$

from (34), where

$$E(J,a_0,a_1) = e^{-J^2/2\sigma^2 a_0} \Big(\frac{J^2}{a_0} (P(a_1,j_1) + a_1^{-3/2}) + P(a_1,j_1) \Big) \prod_{s=2}^{g-2} P(a_s,j_s).$$

The first term in the sum over r is zero:

$$\begin{aligned}&\sum_{\substack{j_1,\dots,j_{g-2}\\ j_1\equiv x (\mathrm{mod}\ g-1)}} \prod_{s=1}^{g-2} e_{g-1}(sj_s) \sum_{r=0}^{g-2} e_{g-1}(r) g(j_1, j_2,\dots,j_{g-2}) \\ &= \sum_{\substack{j_1,\dots,j_{g-2}\\ j_1\equiv x (\mathrm{mod}\ g-1)}} \prod_{s=1}^{g-2} e_{g-1}(sj_s) g(j_1, j_2,\dots,j_{g-2}) \sum_{r=0}^{g-2} e_{g-1}(r) = 0.\end{aligned}$$

Finally, we show that the error term in (35) is $\ll_g 1$, that is, we show

$$\sum_{j_1,\dots,j_{g-2}} E(J,a_0,a_1) \ll_g 1. \tag{36}$$

First note that from (22),

$$\sum_{j_1,\dots,j_{g-2}} e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} P(a_s,j_s) \ll_g 1. \tag{37}$$

and further using that $\sup J^2 e^{-J^2/2\sigma^2 a_0} \ll_g a_0$ gives

$$\frac{1}{a_0} \sum_{j_1,\ldots,j_{g-2}} J^2 e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} P(a_s, j_s) \ll_g 1. \tag{38}$$

Moreover, a similar statement holds when replacing $P(a_1, j_1)$ by $a_1^{-3/2}$:

$$\begin{aligned} \frac{1}{a_0 a_1^{3/2}} \sum_{j_1,\ldots,j_{g-2}} J^2 e^{-J^2/2\sigma^2 a_0} \prod_{s=2}^{g-2} P(a_s, j_s) &\ll_g \sum_{j_1} \frac{1}{a_1^{3/2}} \sum_{j_2,\ldots,g-2} \prod_{s=2}^{g-2} P(a_s, j_s) \\ &= \sum_{j_1} \frac{1}{a_1^{3/2}} \ll_g \frac{1}{a_1^{1/2}}, \end{aligned} \tag{39}$$

where the last line uses that j_1 ranges over $|j_1| \leqslant a_1(g-1)/2$. Combining (37), (38) and (39) gives (36).

Assuming (34), it suffices to show that the contribution from $(j_1, \ldots, j_{g-2})$ such that (31) doesn't hold is bounded. Suppose for $(j_1, \ldots, j_{g-2})$ and $J = j_1 + \ldots + j_{g-2}$,

$$\left|\frac{2Jr + r^2}{2\sigma^2 a_0}\right| > \frac{1}{2}.$$

In particular, for such J, $|J| > \sigma^2 a_0/2 - (g-1)/2$, therefore

$$J^2 e^{-J^2/2\sigma^2 a_0} \ll_g a_0^2 e^{-\sigma^2 a_0/8} \ll_g a_0^{-10}.$$

Thus the contribution from $j_1, \ldots, j_{g-2}$ where $|J|$ is large is

$$\Big| \sum_{\substack{j_1,\ldots,j_{g-2} \\ |J| \geqslant \sigma^2 a_0/4}} J^2 e^{-J^2/2\sigma^2 a_0} \prod_{s=1}^{g-2} P(a_s, j_s) \Big| \ll_g a_0^{-10}. \qquad \square$$

Proposition 3.3 can be used to bound $\Psi(\mathbf{a}; 0)$ for certain $\mathbf{a}$.

Corollary 3.6. *Suppose $\mathbf{a} = (a_0, \ldots, a_{g-2})$ is such that $a_i \geqslant 0$ and $a_0 \geqslant a_i$ for $0 \leqslant i \leqslant g-2$. Furthermore, let $0 \leqslant m \leqslant C a_0^{1/4}$ be an integer, for some $C > 0$. Let $1 \leqslant t \leqslant g-2$. For $\mathbf{a}' = (a_0, \ldots, a_t + m, \ldots, a_{g-2})$, we have*

$$\Psi(\mathbf{a}'; 0) \leqslant \frac{1}{g^m} + O_{C,g}(m^2 a_0^{-3/2}).$$

Proof. Let $W_{j,i}$, for $0 \leqslant j \leqslant g-2, 0 \leqslant i \leqslant a_j - 1$, and Z_j, for $0 \leqslant j \leqslant m-1$, be i.i.d. copies of Y, and let $W_j \coloneqq \sum_{i=0}^{a_j-1} W_{j,i} g^i$ and $Z \coloneqq \sum_{i=0}^{m-1} Z_i g^i$. With the alternate definition of $\Psi(\mathbf{a}'; 0)$ as stated in Proposition 3.2, we have

$$\Psi(\mathbf{a}'; 0) = \mathbb{E}_{W_0,\ldots,W_{g-2},Z}\, e_{g-1}\Big(\sum_{r=1}^{g-2} r W_r + tZ\Big) \mathbf{1}_{\sum_{j=0}^{g-2} s(W_j) = -s(Z)}.$$

Separating out the contribution from Z and using the definition of $\Psi(\mathbf{a}; -s(Z))$ from Proposition 3.2,

$$\begin{aligned}\Psi(\mathbf{a}'; 0) &= \mathbb{E}_Z e_{g-1}(tZ)\mathbb{E}_{W_0,\dots,W_{g-2}} e_{g-1}\Big(\sum_{r=1}^{g-2} rW_r\Big)\mathbf{1}_{\sum_{j=0}^{g-2} s(W_j)=-s(Z)} \\ &= \mathbb{E}_Z e_{g-1}(tZ)\Psi(\mathbf{a}; -s(Z)).\end{aligned}$$

As $|s(Z)| \ll_g Ca_0^{1/4}$, Proposition 3.3 applies to give

$$\Psi(\mathbf{a}; -s(Z)) = \Psi(\mathbf{a}; 0) + O_{C,g}(m^2 a_0^{-3/2}).$$

Hence

$$\Psi(\mathbf{a}'; 0) = \mathbb{E}_Z e_{g-1}(tZ)\Psi(\mathbf{a}; 0) + O_{C,g}(m^2 a_0^{-3/2}).$$

From (17), that is, using that $s(Z) \equiv Z (\mathrm{mod}\ g-1)$, and the fact that the Z_j are i.i.d. copies of Y, we have

$$\mathbb{E}_Z e_{g-1}(tZ) = \prod_{j=0}^{m-1} \mathbb{E}_{Z_j} e_{g-1}(tZ_j) = \Big(\mathbb{E}_Y e_{g-1}(tY)\Big)^m \ll \frac{1}{g^m}.$$

Here we also use that $t \neq 0$ and that $Y + (g-1)/2$ uniformly takes values in $\{0, \dots, g-1\}$. Finally, using the fact that $|\Psi(\mathbf{a}; 0)| \leqslant 1$ for any tuple $\mathbf{a}$ gives the result. □

4. Major arcs contribution

In this section, we establish Proposition 2.2: that the contribution to (7) from the major arcs $\mathfrak{M}$ gives the main term in Proposition 1.2. Recall that we have the following major arcs $\mathfrak{M}$,

$$\mathfrak{M} := \bigcup_{j=0}^{g-2} [\frac{j}{g-1} - \varepsilon, \frac{j}{g-1} + \varepsilon]$$

for $\varepsilon := K^{3/4} g^{-K}/(g-1)$. We also have that $f_i(\theta + j/(g-1)) = e_{g-1}(jk_i)f_i(\theta)$ for $j \in \mathbf{Z}$ from (4). Therefore in order to evaluate the contribution from the major arcs to (7), it suffices to consider the contribution from $\mathfrak{M}$ around 0, as

$$\begin{aligned}\int_{\mathfrak{M}} \prod_{i=1}^{3} f_i(\theta)e(-M\theta)\,\mathrm{d}\theta &= \sum_{j=0}^{g-2} \int_{|\theta|\leqslant\varepsilon} \prod_{i=1}^{3} f_i(\theta + \tfrac{j}{g-1})e(-M(\theta + \tfrac{j}{g-1}))\,\mathrm{d}\theta \\ &= \Big(\sum_{j=0}^{g-2} e_{g-1}(j(k_1+k_2+k_3-M))\Big) \int_{|\theta|\leqslant\varepsilon} \prod_{i=1}^{3} f_i(\theta)e(-M\theta)\,\mathrm{d}\theta \\ &= (g-1)\int_{|\theta|\leqslant\varepsilon} \prod_{i=1}^{3} f_i(\theta)e(-M\theta)\,\mathrm{d}\theta \qquad \text{by (5).} \qquad (40)\end{aligned}$$

One can view the factor of $(g-1)$ in (40) as a very simple singular series, with the integral term being our singular integral. To evaluate this integral, we require the following lemma. This gives an asymptotic for $f_i(\theta)$ on a range around 0 which includes $[-\varepsilon, \varepsilon]$ as well as on some minor arc points; the asymptotic will be used to bound these points later. By combining Proposition 4.1 with (8), we can get an asymptotic for $f_i(\theta)$ for $\theta \in \mathfrak{M}$ more generally.

Lemma 4.1. *Let $\ell := \lfloor C_0 \log K \rfloor$ be as defined in* (3)*. For all θ such that $\|\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}/(g-1)$,*

$$f_i(\theta) = \frac{g^K}{\sqrt{2\pi\sigma^2 K}} \int_0^1 e(g^K \theta x)\,\mathrm{d}x + O_g\Big(\frac{g^K \ell^4}{K^{3/2}}\Big).$$

To prove Proposition 4.1, we switch to the probabilistic model for $n < g^K$ outlined in Section 2.3. Recall that X_j, $j = 0, \dots, K-1$, are i.i.d. copies of the uniform random variable taking values in $\{0, \dots, g-1\} - (g-1)/2$, with $X = \sum_{j=0}^{K_1} X_j g^j$. For an indexing set $\mathcal{S} \subseteq \{0, \dots, K-1\}$, let $X_{\mathcal{S}}$ denote the random variable

$$X_{\mathcal{S}} := \sum_{j \in \mathcal{S}} X_j g^j. \tag{41}$$

Proof of Proposition 4.1. We can rewrite $f_i(\theta)$ as the following expectation from (12),

$$f_i(\theta) = g^K e\Big(\frac{g^K - 1}{2}\theta\Big)\mathbb{E}_X e(X\theta) 1_{s(X)=\xi_i} \tag{42}$$

where $\xi_i := k_i - \mu_K$ is the distance of the target digit sum, k_i, from the average value. Note that the condition given in (6) implies that $|\xi_i| \ll_g 1$.

As $|\theta| \leqslant g^{-K+2\ell^2}/(g-1)$, the value of $e(X\theta)$ is determined mainly by the value of the random variables $X_{K-2\ell^2-\ell}, \dots, X_{K-1}$, up to a small error. Let

$$\mathcal{D} := \{K - 2\ell^2 - \ell, \dots, K-1\} \text{ and } \mathcal{E} = \{0, \dots, K-1\} \setminus \mathcal{D}.$$

Then as $|\sum_{j \notin \mathcal{D}} X_j g^j \theta| \leqslant g^{-\ell}$, we have $e(X\theta) = e(X_{\mathcal{D}}\theta) + O(g^{-\ell})$. Here, $X_{\mathcal{D}}$ is defined as in (41). Let $L := |\mathcal{D}| = 2\ell^2 + \ell$. Thus,

$$\begin{aligned}\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi_i} &= \mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta)\mathbb{E}_{X_{\mathcal{E}}}\mathbf{1}_{s(X)=\xi_i} + O(g^{-\ell}) \\ &= \mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta) P(K-L, \xi_i - s(X_{\mathcal{D}})) + O(g^{-\ell}),\end{aligned}$$

where the notation $P(T,t)$ is used to denote $\mathbb{P}(X_1 + \ldots + X_T = t)$. By Proposition 3.1,

$$P(K-L, \xi_i - s(X_{\mathcal{D}})) = \frac{e^{-x^2/2}}{\sqrt{2\pi\sigma^2(K-L)}} + O_g((K-L)^{-3/2}), \tag{43}$$

where $x = (\xi_i - s(X_{\mathcal{D}}))/\sqrt{\sigma^2(K-L)}$. Note that x has size approximately $L/\sqrt{K}$, coming from the fact that $|\xi_i - s(X_{\mathcal{D}})| \ll_g |\mathcal{D}|$ and $|\mathcal{D}| = L \ll l^2$. From expanding the exponential term in (43) and using that $(K-L)^{-1/2} - K^{-1/2} = O(LK^{-3/2})$, we can remove the dependence of the values of ξ_i and $s(X_{\mathcal{D}})$ from $P(K-L, \xi_i - s(X_{\mathcal{D}}))$, giving

$$P(K-L, \xi_i - s(X_{\mathcal{D}})) = \frac{1}{\sqrt{2\pi\sigma^2 K}} + O_g(\ell^4 K^{-3/2}).$$

Hence the expectation over all digits in X may be replaced by an average over the digits indexed by $\mathcal{D}$ only,

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi_i} = \frac{1}{\sqrt{2\pi\sigma^2 K}}\mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta) + O_g\big(\ell^4 K^{-3/2}\big). \tag{44}$$

Note that $X_{\mathcal{D}}g^{-K+L}+(g^L-1)/2$ takes values in $\{0,\dots,g^L-1\}$ uniformly at random, so we can write the expectation over $X_{\mathcal{D}}$ explicitly as a normalised sum,

$$\mathbb{E}_{X_{\mathcal{D}}}e(X_{\mathcal{D}}\theta)=\frac{1}{g^L}e(-\theta(g^K-1)/2)\sum_{j=0}^{g^L-1}e(g^{K-L}j\theta).$$

Evaluating this series and comparing it to the corresponding integral gives:

$$\begin{aligned}\frac{1}{g^L}e(-\theta(g^K-g^{K-L})/2)\sum_{j=0}^{g^L-1}e(g^{K-L}j\theta)&=\frac{1}{g^L}e(-\theta(g^K-g^{K-L})/2)\int_0^{g^L}e(g^{K-L}\theta x)\,\mathrm{d}x\\&=e(-\theta(g^K-g^{K-L})/2)\int_0^1 e(g^K\theta x)\,\mathrm{d}x.\end{aligned}$$

Combining this with (44) and multiplying through by $g^Ke(\theta(g^K-1)/2)$ gives

$$f_i(\theta)=e\Big(\frac{g^{K-L}-1}{2}\theta\Big)\frac{g^K}{\sqrt{2\pi\sigma^2K}}\int_0^1 e(g^K\theta x)\,\mathrm{d}x+O_g\Big(\frac{g^K\ell^4}{K^{3/2}}\Big).$$

To obtain the expression for $f_i(\theta)$ given in the statement of the lemma, note that,

$$e\Big(\frac{g^{K-L}-1}{2}\theta\Big)=1+O_g(g^{-\ell}).$$

This follows from the fact that $|\theta|\leqslant g^{-K+L-\ell}/(g-1)$. From the choice of ℓ given in (3), we have that $g^{K-\ell}=O_g(g^KK^{-3/2})$. □

We now prove Proposition 2.2. As Proposition 4.1 holds for $|\theta|\leqslant g^{-K+2\ell^2}/2$, this includes $|\theta|\leqslant\varepsilon$ as $\varepsilon=K^{3/4}g^{-K}/(g-1)$ and $\ell\geqslant\log K$ from (3). Hence we can apply Proposition 4.1 to estimate the contribution from the $f_i(\theta)$ for $|\theta|\leqslant\varepsilon$,

$$\prod_{i=1}^3 f_i(\theta)=g^{3K}(2\pi\sigma^2K)^{-3/2}\Big(\int_0^1 e(g^K\theta x)\,\mathrm{d}x\Big)^3+O_g(g^{3K}\ell^4K^{-5/2}).$$

Substituting into (40) and using the change of variable $\eta=g^K\theta$,

$$\begin{aligned}(g-1)\int_{\mathfrak{M}}\prod_{i=1}^3 f_i(\theta)e(-M\theta)\,\mathrm{d}\theta&=\frac{(g-1)g^{3K}}{(2\pi\sigma^2K)^{3/2}}\int_{|\theta|\leqslant\varepsilon}\Big(\int_0^1 e(g^K\theta x)\,\mathrm{d}x\Big)^3e(-M\theta)\,\mathrm{d}\theta+O_g\Big(\frac{g^{2K}\ell^4}{K^{7/4}}\Big)\\&=\frac{(g-1)g^{2K}}{(2\pi\sigma^2K)^{3/2}}\int_{|\eta|\leqslant g^K\varepsilon}\Big(\int_0^1 e(\eta x)\,\mathrm{d}x\Big)^3e(-Mg^{-K}\eta)\,\mathrm{d}\eta+O_g\Big(\frac{g^{2K}\ell^4}{K^{7/4}}\Big).\end{aligned}\tag{45}$$

To evaluate the integral, we follow the treatment of the singular integral in [1, Ch. 4], though it is considerably simpler than the case arising in Waring's problem. Firstly, we extend the range of integration of η to $(-\infty,\infty)$. This accrues error

$$\frac{g^{2K}}{K^{3/2}}\int_{|\eta|\geqslant g^K\varepsilon}\Big(\int_0^1 e(\eta x)\,\mathrm{d}x\Big)^3e(-\eta Mg^{-K})\,\mathrm{d}\eta\ll\frac{g^{2K}}{K^{3/2}}\int_{|\eta|\geqslant g^K\varepsilon}\Big(\frac{|e(\eta)-1|}{|\eta|}\Big)^3\,\mathrm{d}\eta\ll\frac{1}{K^{3/2}\varepsilon^2}.\tag{46}$$

As $K^{3/2}\varepsilon^2 = K^3 g^{-2K}/(g-1)^2$, extending the range of integration contributes error $O_g(g^{2K}K^{-3})$. To evaluate the extended integral, note that

$$\int_{-\infty}^{\infty}\Big(\int_0^1 e(\eta x)\,\mathrm{d}x\Big)^3 e(-\eta M g^{-K})\,\mathrm{d}\eta = \int_{-\infty}^{\infty}\widehat{h}(\eta)^3 e(-\eta M g^{-K})\,\mathrm{d}\eta = h*h*h(Mg^{-K}) \tag{47}$$

where $h = \mathbf{1}_{[0,1]}$. Using that $Mg^{-K}\in(1/g,1]$, we can explicitly calculate $h*h*h(Mg^{-K})$,

$$\begin{aligned} h*h*h(Mg^{-K}) &= \int_{-\infty}^{\infty}\int_0^1 \mathbf{1}_{y\in[z-1,z]}\mathbf{1}_{z\in[Mg^{-K}-1,Mg^{-K}]}\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{Mg^{-K}-1}^{Mg^{-K}} z\mathbf{1}_{z\in[0,1)} + (2-z)\mathbf{1}_{z\in[1,2)}\,\mathrm{d}z = M^2 g^{-2K}/2. \end{aligned} \tag{48}$$

This concludes the proof of Proposition 2.2; extending the integral in (45) and substituting (47) and (48) into the extended integral gives the stated contribution from the major arcs.

5. Minor arcs contribution

In this section, we prove Proposition 2.3, that is, showing that $f_i(\theta)$ is uniformly bounded by $\ll g^K K^{-5/4}$ on the minor arcs $\mathfrak{m}$. For a subset of the minor arcs, existing results give a stronger bound, which we demonstrate shortly. The remaining minor arc points have a specific structure, which we exploit to show the required bound on $f_i(\theta)$.

We have the following bound on $|f_i(\theta)|$ due to Fouvry and Mauduit [3].

Theorem 5.1 ([3]). *For $\theta\in\mathbf{R}/\mathbf{Z}$,*

$$|f_i(\theta)| \leqslant g^K \exp\Big(-\frac{1}{2g}\sum_{i=0}^{K-1}\|g^i(g-1)\theta\|^2_{\mathbf{R}/\mathbf{Z}}\Big).$$

This theorem as written above is not stated explicitly in [3], rather it follows immediately from the proof of [3, Theorem 1.2]. We sketch this in Section A.

Proposition 5.1 gives a strong saving over the bound required for Proposition 2.3 whenever θ is such that

$$\sum_{i=1}^{K}\|(g-1)g^i\theta\|^2_{\mathbf{R}/\mathbf{Z}} \geqslant 2gC\log K, \tag{49}$$

for large enough C. Thus it remains to prove Proposition 2.3 for θ such that (49) does not hold for sufficiently large C. In order to do so, we need to understand the structure of such θ, which we achieve by using the centred base-g expansion of $(g-1)\theta$. Recall from Section 2.4, that for real $\alpha\in I_g$, the centred base-g expansion of α is

$$\alpha = \sum_{i\geqslant 1}\varepsilon_i g^{-i},\ \text{with } \varepsilon_i \in \Big(-\frac{g}{2},\frac{g}{2}\Big]\cap\mathbf{Z} \text{ for all } i.$$

Given the centred expansion of α above, we define

$$w_K(\alpha) := \sum_{i=1}^{K}\mathbf{1}_{\varepsilon_i\neq 0},$$

which counts the number of non-zero digits within the first K digits of the centred expansion of α. The following lemma due to Green allows us to replace the sum over fractional parts in (49) with the function w_K.

Lemma 5.2. *[4, Lemma 7.2] For $g \geqslant 3$ and $\alpha \in \mathbf{R}$,*

$$\frac{w_K(\alpha)}{16g^2} \leqslant \sum_{i=0}^{K-1} \|g^i\alpha\|_{\mathbf{R}/\mathbf{Z}}^2 \leqslant w_K(\alpha).$$

As we define $\ell := \lceil 384g^3 \log K \rceil$ in (3), we have the following corollary to Proposition 5.1.

Corollary 5.3. *For ℓ as defined in* (3) *and θ such that $w_K((g-1)\theta) > \ell$,*

$$f_i(\theta) \ll_g g^K K^{-12}, \text{ for } i = 1, 2, 3.$$

Proof. Recall that $\ell = \lceil 384g^3 \log K \rceil$. From the lower bound in Proposition 5.2,

$$\exp\Big(-\frac{1}{2g}\sum_{i=1}^{K} \|(g-1)g^i\theta\|_{\mathbf{R}/\mathbf{Z}}^2\Big) \leqslant \exp(-w_K((g-1)\theta)/32g^3) \leqslant \exp(-\ell/32g^3) \leqslant K^{-12}.$$

Inserting this upper bound into the bound given by Proposition 5.1 gives the required bound for $f_i(\theta)$. $\square$

Remark. It is at this point that we have to restrict our results to base g, for $g \geqslant 3$. This is because Proposition 5.2 is not valid for base 2; discussion regarding why this is not the case is given in [4].

As a consequence, we cannot use the base-2 version of $w_K(\alpha)$ to model the function $\sum_{i=1}^{K} \|2^i\alpha\|_{\mathbf{R}/\mathbf{Z}}^2$. Instead of counting the number of non-zero digits within the first K digits of the expansion of α, we count the number of times consecutive digits alternate value. More precisely, if α has base-2 expansion $\alpha = \sum_{i\geqslant 1} \varepsilon_i 2^{-i}$, let $d_K(\alpha)$ denote

$$d_K(\alpha) := |\{(\varepsilon_i, \varepsilon_{i+1}) : \varepsilon_i \neq \varepsilon_{i+1}, i \in \{1, \ldots, K\}\}|.$$

One can show that $d_K(\alpha) \asymp \sum_{i=1}^{K} \|2^i\alpha\|_{\mathbf{R}/\mathbf{Z}}^2$, and use this to establish the base-2 versions of our results by replacing instances of w_K by d_K, and making some small technical adjustments.

We will show that Proposition 4.1 directly gives Proposition 2.3 for $\theta \in \mathfrak{m}$ with $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}$. Recall that $\mathfrak{m} := \{\theta \in \mathbf{R}/\mathbf{Z} : \|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} > K^{3/4}g^{-K}\}$.

Corollary 5.4. *Suppose that $K^{3/4}g^{-K} \leqslant \|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}$. Then*

$$f_i(\theta) \ll_g g^K K^{-5/4}.$$

Proof. Suppose first that $K^{3/4}g^{-K} \leqslant \|\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}/(g-1)$. Then as $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}$, Proposition 4.1 applies to give that

$$\begin{aligned} f_i(\theta) &= \frac{g^K}{\sqrt{2\pi\sigma^2 K}}\int_0^1 e(g^K\theta x)\,\mathrm{d}x + O_g\Big(\frac{g^K\ell^4}{K^{3/2}}\Big) \\ &= \frac{1}{(2\pi)^{3/2}\sigma K^{1/2} i\theta}(e(g^K\theta)-1) + O_g\Big(\frac{g^K\ell^4}{K^{3/2}}\Big) \ll_g \frac{g^K}{K^{5/4}}. \end{aligned}$$

Now suppose that $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}$, but that $\|\theta\|_{\mathbf{R}/\mathbf{Z}} > g^{-K+2\ell^2}/(g-1)$. In this case, there exists $j \in \{1,\dots,g-2\}$ such that $K^{3/4}g^{-K} \leqslant \|\theta - j/(g-1)\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-K+2\ell^2}/(g-1)$. Thus Proposition 4.1 gives, following the above argument,

$$f_i\Big(\theta - \frac{j}{g-1}\Big) \ll_g \frac{g^K}{K^{5/4}},$$

and from (8), $f_i(\theta) \ll_g g^K K^{-5/4}$ as well. □

It remains then to prove Proposition 2.3 for θ such that $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2\ell^2}$ and $w_K((g-1)\theta) \leqslant \ell$. We first recall the probabilistic model for digits set up in Section 2.3. Let $X_i : i = 0,\dots,K-1$ be i.i.d. copies of the random variable uniformly taking values in $\{0,\dots,g-1\} - (g-1)/2$. Let

$$X \coloneqq \sum_{i=0}^{K-1} X_i g^i,$$

and for $\mathcal{S} \subset \mathbf{R}$, let

$$X_{\mathcal{S}} \coloneqq \sum_{i\in\mathcal{S}\cap\{0,\dots,K-1\}} X_i g^i.$$

Recall from (12) that

$$f_i(\theta) = g^K e(\theta(g^K-1)//2)\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi_i}.$$

Here $\xi_i = \mu_K - k_i$, where $\mu_K \coloneqq (g-1)K/2$, and $s(X) = \sum_{j=0}^{K-1} X_j$. Our aim in this section is to show the following proposition.

Proposition 5.5. *Let $\xi \in \mathrm{Supp}(X)$ and let L, R be positive integers such that $\frac{3}{2}\log_g K \leqslant R \leqslant L$. Let $C > 0$ be such that $\max(|\xi|, LR) \leqslant CK^{1/4}$. Suppose that $\theta \in \mathbf{R}$ is such that $w_K((g-1)\theta) \leqslant L$ and $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2LR}$. Then*

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} \ll_{C,g} (|\xi| + LR)^2 K^{-3/2}.$$

This proposition allows us to now prove the minor arc bound, Proposition 2.3, for all $\theta \in \mathfrak{m}$.

Proof of Proposition 2.3. Proposition 5.3 and Proposition 5.4 give Proposition 2.3 for all $\theta \in \mathfrak{m}$ except those with $w_K((g-1)\theta) \leqslant \ell$ and $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2\ell^2}$. In this remaining case, Proposition 5.5 gives Proposition 2.3 upon taking $\xi = \xi_i$ and $R = L = \ell$; the choice of ℓ from (3) ensures that the assumptions in the statement of the proposition hold. The constant C in the statement of Proposition 5.5 can be taken to be some constant depending only on g coming from (3) and (6). □

In our application of Proposition 5.5, the parameters L and R are both taken to be $\ell \asymp_g \log K$. Despite this, it is convenient to separate the roles of L and R in the proof of Proposition 5.5. The quantity L controls the number of non-zero digits in the centred expansion of $(g-1)\theta$, and the quality of this approximation is controlled by R. By assuming that $LR \leqslant CK^{1/4}$, we are able to approximate $e(X\theta)$ by a small number ($\ll LR$ many) of the random variables X_i.

More precisely, we will partition $\{0,\dots,K-1\}$ into sets $\mathcal{D}$ and $\mathcal{E}$ which depend on the location of the non-zero digits in the centred base-g expansion of $(g-1)\theta$. The digits indexed by $\mathcal{D}$ are those which have indices close to those of the non-zero digits of $(g-1)\theta$; these determine $e(X\theta)$ up to an error determined by R. The remaining digits, indexed by $\mathcal{E}$, vary randomly according to the constraint that $s(X) = s(X_{\mathcal{D}}) + s(X_{\mathcal{E}}) = \xi$. Crucially, the set $\mathcal{D}$ only indexes a small number of the digits of X, as $|\mathcal{D}| \ll LR$, so the assumption that $|\xi| + LR \ll_{C,g} K^{1/4}$ ensures that we can replace the condition $s(X_{\mathcal{E}}) = \xi - s(X_{\mathcal{D}})$ by the simpler condition $s(X_{\mathcal{E}}) = 0$. This allows us to decouple the averages over $X_{\mathcal{D}}$ and $X_{\mathcal{E}}$, roughly giving the following:

$$\begin{aligned}\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} &= \mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta)\mathbb{E}_{X_{\mathcal{E}}} e(X_{\mathcal{E}}\theta)\mathbf{1}_{s(X_{\mathcal{D}})+s(X_{\mathcal{E}})=\xi} \\ &\approx \big(\mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta)\big)\big(\mathbb{E}_{X_{\mathcal{E}}} e(X_{\mathcal{E}}\theta)\mathbf{1}_{s(X_{\mathcal{E}})=0}\big). \end{aligned} \tag{50}$$

This requires the local limit theorem, specifically the application in Proposition 3.3. We then show that either the average $\mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta)$ has sufficient cancellation to give Proposition 5.5, or that $(g-1)\theta$ has an even more specific structure. In the latter case, we find that the average over $X_{\mathcal{E}}$ exhibits lots of cancellation, which requires the assumption that $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2LR}$.

In order to have more control over the centred base-g expansion of θ, we prove Proposition 5.5 for $\theta \in \frac{1}{g-1} I_g$. From the definition of the interval I_g, given in (13), this ensures that the centred base-g expansion of $(g-1)\theta$ has no integer part. Note this proves Proposition 5.5 in full generality: we can shift any $\theta \in I_g$ by an integer multiple of $1/(g-1)$ so that the translate lies in $\frac{1}{g-1} I_g$, and from (8), shifting by a multiple of $1/(g-1)$ doesn't affect the absolute value of $|\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)}|$. Let the centred base-$g$ expansion of $(g-1)\theta$ be the following,

$$(g-1)\theta = \sum_{j=1}^{\infty} \varepsilon_j g^{-j} = \sum_{1\leqslant j\leqslant K} \varepsilon_j g^{-j} + \eta, \tag{51}$$

where $\eta := \sum_{j>K} \varepsilon_j g^{-j}$, giving $|\eta| < g^{-K}$. Let $\{n_j : 1 \leqslant j \leqslant w_K((g-1)\theta)+1\}$ index the first $w_K((g-1)\theta)+1$ non-zero digits of $(g-1)\theta$ after the radix point, so that

$$1 \leqslant n_1 < n_2 \dots < n_{w_K((g-1)\theta)} \leqslant K < n_{w_K((g-1)\theta)+1}, \tag{52}$$

with $\varepsilon_{n_j} \neq 0$ for $1 \leqslant j \leqslant w_K((g-1)\theta)+1$ and $\varepsilon_j = 0$ for all other j, $1 \leqslant j \leqslant n_{w_K((g-1)\theta)+1} - 1$. Define $\mathcal{D}$ to be the set of indices which are within R of the n_j, more precisely,

$$\mathcal{D} = \bigcup_{j=1}^{w_K((g-1)\theta)+1} [n_j - R, n_j - 1] \cap \{0,\dots,K-1\}. \tag{53}$$

These are the indices of X_i which, up to a sufficiently small error, actually determine the value of $e(X\theta)$. From the assumption that $w_K((g-1)\theta) \leqslant L$, we have $|\mathcal{D}| \ll LR \leqslant CK^{1/4}$. For $0 \leqslant r \leqslant g-2$, let

$$\mathcal{E}_r = \{i \notin \mathcal{D} : \sum_{j\leqslant i} \varepsilon_j \equiv r (\mathrm{mod}\ g-1)\}. \tag{54}$$

The following lemma allows us to rigorously carry out the "decoupling" step sketched in (50).

Lemma 5.6. *Let $\xi \in \mathrm{Supp}(X)$, and let L, R be positive integers such that $\frac{3}{2}\log_g K \leqslant R \leqslant L$. Let $C > 0$ be such that $\max(|\xi|, LR) \leqslant CK^{1/4}$. Suppose that $\theta \in I_g$ is such that $w_K((g-1)\theta) \leqslant L$ and suppose that $(g-1)\theta$ has centred base-g expansion given by (51). Let the sets $\mathcal{D}$ and $\mathcal{E}_r$ be as defined by (53) and (54) respectively for $0 \leqslant r \leqslant g-2$. Let $x \in \{0, \dots, g-2\}$ be such that $\mathcal{E}_x$ is maximal among the sets $\mathcal{E}_r$, and define the following $(g-1)$-tuple of integers,*

$$\mathbf{a} := (|\mathcal{E}_x|, |\mathcal{E}_{x+1}|, \dots, |\mathcal{E}_{g-2}|, |\mathcal{E}_0|, |\mathcal{E}_1|, \dots, |\mathcal{E}_{x-1}|). \tag{55}$$

Then

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} = e_{g-1}(x\xi)\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big)\Psi(\mathbf{a};0) + O_{C,g}((|\xi| + LR)^2 K^{-3/2}).$$

The definition of $\Psi(\mathbf{a};0)$ is given in (15), and this term accounts for the average over $X_{\mathcal{E}}$ presented in the sketch (50). Recall that $\Psi(\mathbf{a};0)$ is a generalisation of the probability $P(T,0)$ to include certain $(g-1)^{\text{th}}$ roots of unity, where $P(T,0)$ is the probability that T i.i.d. copies of the uniform random variable taking values in $\{0, \dots, g-1\} - (g-1)/2$ sum to zero. In this instance, the value of T is taken to be $K - |\mathcal{D}|$.

We prove Proposition 5.6 in Section 5.1. The next lemma will be used to show that there is cancellation in the average over $X_{\mathcal{D}}$ if there is a non-zero digit in $(g-1)\theta$ followed by a string of zeros, and preceded by digits which have a sum congruent to x modulo $g-1$, for the value x defined by Proposition 5.6.

Lemma 5.7. *For $\theta \in \frac{1}{g-1}I_g$ suppose that $(g-1)\theta$ has centred base-g expansion $\sum_{i=1}^{\infty} \varepsilon_i g^{-i}$. Suppose further that there is an index m, $1 \leqslant m \leqslant K$, such that $\varepsilon_m \neq 0$, and $\varepsilon_{m+1} = \ldots = \varepsilon_{m+T} = 0$, for some integer $T \geqslant 0$. Let $r \in \{0, \dots, g-2\}$ be such that $\sum_{1 \leqslant j \leqslant m} \varepsilon_j \equiv r (\mathrm{mod}\ g-1)$. Then*

$$\mathbb{E}_Y e\Big(g^{m-1} Y\Big(\theta - \frac{r}{g-1}\Big)\Big) \ll g^{-T},$$

where Y is a random variable uniformly taking values in $\{0, \dots, g-1\} - (g-1)/2$.

This will be proved in Section 5.2. The next lemma will be used to show there is cancellation in the average over $X_{\mathcal{D}}$ when a rather different structure is present in the expansion of $(g-1)\theta$. In this case, we look for a non-zero digit preceded by a string of zeros, such that the digits preceding this have a sum that is not congruent to x modulo $g-1$.

Lemma 5.8. *For $\theta \in \frac{1}{g-1}I_g$ suppose that $(g-1)\theta$ has centred base-g expansion $\sum_{i=1}^{\infty} \varepsilon_i g^{-i}$. Let T be an integer such that $0 \leqslant T \leqslant K$, and suppose that $\varepsilon_{m-T} = \ldots = \varepsilon_{m-1} = 0$ for some index m, $T \leqslant m \leqslant K$. Let $r \in \{0, \dots, g-2\}$ be such that $\sum_{1 \leqslant j \leqslant m} \varepsilon_j \not\equiv r (\mathrm{mod}\ g-1)$. Then*

$$\mathbb{E}_{Y_{m-T}, \dots, Y_{m-1}} e\Big(\sum_{j=m-T}^{m-1} Y_j g^j \Big(\theta - \frac{r}{g-1}\Big)\Big) \ll_g g^{-T},$$

where $Y_{m-T}, \dots, Y_{m-1}$ are i.i.d. uniform random variables taking values in $\{0, \dots, g-1\} - (g-1)/2$.

This lemma will also be proved in Section 5.2. Assuming these lemmas, we may now prove Proposition 5.5.

Proof of Proposition 5.5. As noted in the preceding sketch, we may assume that $\theta \in \frac{1}{g-1} I_g$. Therefore from Proposition 5.6, we have

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} = e_{g-1}(x\xi)\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big)\Psi(\mathbf{a};0) + O_{C,g}((|\xi| + LR)^2 K^{-3/2}). \tag{56}$$

for some $x \in \{0, \ldots, g-2\}$ and $\mathbf{a}$ as defined by (55). To prove the proposition, it remains to show that the purported main term on the right hand side of (56) is also bounded by $O_{C,g}((|\xi| + LR)^2 K^{-3/2})$. We will show that at least one of the following two inequalities always holds, so that either

$$\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big) \ll_g g^{-R} \tag{57}$$

or

$$\Psi(\mathbf{a};0) \ll_{C,g} R^2 K^{-3/2}. \tag{58}$$

Note that as both $\Psi(\mathbf{a};0)$ and the average over $X_{\mathcal{D}}$ are trivially bounded by 1, either bound is sufficient to give Proposition 5.5, using that $R \geqslant \frac{3}{2}\log_g K$ to bound g^{-R}. Let us first consider when (57) holds. As the X_i are independent,

$$\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big) = \prod_{j\in\mathcal{D}} \mathbb{E}_{X_j} e\Big(g^j X_j\Big(\theta - \frac{x}{g-1}\Big)\Big). \tag{59}$$

If the centred expansion of $(g-1)\theta$ is such that the assumptions of Proposition 5.7 are fulfilled for any integers m, T with $1 \leqslant m \leqslant K$ and $T \geqslant R$, and with the value of r in the statement of Proposition 5.7 equal to x, then

$$\mathbb{E}_{X_{m-1}} e\Big(X_{m-1} g^{m-1}\Big(\theta - \frac{x}{g-1}\Big)\Big) \ll g^{-R}.$$

Note that $X_{m-1} \in \mathcal{D}$ in this case, as $\varepsilon_m \neq 0$, so this average over X_{m-1} appears in (59). Therefore under these circumstances, we can use Proposition 5.7 to show (57).

Similarly, we can use Proposition 5.8 to show (57). Suppose that the centred base-g expansion of $(g-1)\theta$ is such that there exists an integer m with $R \leqslant m \leqslant K$ for which the assumptions in the statement of Proposition 5.8 are fulfilled with $T = R$ and $r = x$. In this case, Proposition 5.8 applies with $Y_i = X_i$ for $i = m-R, \ldots, m-1$ to give

$$\mathbb{E}_{X_{[m-R,m-1]}} e\Big(X_{[m-R,m-1]}\Big(\theta - \frac{x}{g-1}\Big)\Big) \ll_g g^{-R}.$$

In order to use this to bound (59), we require that the random variables $X_{m-R}, \ldots, X_{m-1}$ are all contained in $\mathcal{D}$. Thus we also require that $\varepsilon_m \neq 0$ here.

The next claim shows that if (57) doesn't hold, then the expansion of $(g-1)\theta$ has a very specific structure. Recall that the indexes labelled n_j below are those defined by (52), which index the location of the non-zero digits within the first K digits in the centred expansion of $(g-1)\theta$.

Claim 5.9. *If*

$$\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big) \gg_g g^{-R} \tag{60}$$

then $n_{j+1} - n_j \leqslant R - 1$ *for* $j = 1, \ldots, w_K((g-1)\theta) - 1$. *Moreover,* $\sum_{j=1}^K \varepsilon_j \not\equiv x (\mathrm{mod}\ g-1)$.

Proof of Proposition 5.9. We can assume there is at least one non-zero digit within the first K digits of $(g-1)\theta$, as otherwise $\theta \in \mathfrak{M}$. Suppose that there are consecutive non-zero digits within the first K digits of the centred expansion of $(g-1)\theta$ indexed by v, w such that $w - v \geqslant R$. By assumption we have that $\varepsilon_{v+1} = \ldots = \varepsilon_{w-1} = 0$. Let $a \coloneqq \sum_{r \leqslant v} \varepsilon_r$. If $a \equiv x (\text{mod } g-1)$, then we can apply Proposition 5.7 with $m = v$, $T = w - v \geqslant R$ and $r = x$ to obtain $\mathbb{E}_{X_{v-1}} e(X_{v-1} g^{v-1}(\theta - x/(g-1))) \ll g^{-R}$. As $\varepsilon_v \neq 0$, we have $v - 1 \in \mathcal{D}$, so this gives sufficient cancellation in the average over $X_{\mathcal{D}}$ to contradict (60). Therefore, $a \not\equiv x (\text{mod } g-1)$.

However, if $a \not\equiv x (\text{mod } g-1)$ for a as defined above, the assumptions of Proposition 5.8 are now fulfilled with $m = w$, $T = R$ and $r = x$. This gives that $\mathbb{E}_{X_{[w-R,w-1]}} e(X_{[w-R,w-1]}(\theta - x/(g-1))) \ll_g g^{-R}$. Note that the interval $\{w-R, \ldots, w-1\}$ is contained in $\mathcal{D}$ by definition (53), as $\varepsilon_w \neq 0$, so this again provides enough cancellation in the average over $X_{\mathcal{D}}$ to contradict (60).

Therefore we must have that each non-zero digit within the first K digits of the centred expansion of $(g-1)\theta$ occurs within $R-1$ digits of another non-zero digit. Let $t \coloneqq n_{w_K((g-1)\theta)}$ be the largest index of a non-zero digit occurring among the first K digits in the centred expansion of $(g-1)\theta$.

To establish the second part of the claim, suppose that $\sum_{j=1}^{t} \varepsilon_j \equiv x (\text{mod } g-1)$. If $K - t \geqslant R$, then we can apply Proposition 5.7 with $m = t$ and $T = R$, contradicting (60). On the other hand, if $K - t < R$, then $n_1 > K - L(R-1)$ from the bound

$$t - n_1 = \sum_{j=1}^{w_K((g-1)\theta)-1} n_{j+1} - n_j \leqslant (L-1)(R-1). \tag{61}$$

As $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} < g^{-n_1+1}$, this contradicts the assumption that $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2LR}$, so we must have $\sum_{j=1}^{t} \varepsilon_j \not\equiv x (\text{mod } g-1)$. $\square$

Proposition 5.9 means that if there is insufficient cancellation in the average over $X_{\mathcal{D}}$ to show (57), then all the non-zero digits within the first K digits of $(g-1)\theta$ occur very close together. In this case, we use this structure to show (58) holds, which proves Proposition 5.5. To simplify notation, let

$$s \coloneqq n_1, \qquad t \coloneqq n_{w_K((g-1)\theta)} \qquad \text{and} \qquad u \coloneqq n_{w_K((g-1)\theta)+1}.$$

Suppose from now on that (57) doesn't hold, and recall that x is such that $|\mathcal{E}_x|$ is maximal among the $|\mathcal{E}_r|$. Let $a \coloneqq \sum_{j=s}^{t} \varepsilon_j$; from Proposition 5.9, $a \not\equiv x (\text{mod } g-1)$. Let $y \in \{0, \ldots, g-2\}$ be such that $a \equiv y (\text{mod } g-1)$. Then the set $\mathcal{E}$ is partitioned into two intervals, $\mathcal{E}_0 = \{0, \ldots, s-R\}$ and $\mathcal{E}_y = \{t, \ldots, \min(K-1, u-R-1)\}$, with $\mathcal{E}_r = \emptyset$ for $1 \leqslant r \leqslant g-2$, $r \neq y$. Thus x must either be 0 or y.

First note that $y \neq 0$. If not, we would have $\mathcal{E}_0 = \{0, \ldots, K-1\} \setminus \mathcal{D}$, and so $x = 0$, as $|\mathcal{E}_0| \geqslant K - LR$ and $|\mathcal{E}_r| = 0$ for $r \neq 0$. However this gives $a \equiv x (\text{mod } g-1)$, contradicting the second part of Proposition 5.9.

Additionally, if $x = y$ then this contradicts the fact that $a \not\equiv x (\text{mod } g-1)$. Therefore we must have $x = 0$, whence $|\mathcal{E}_0| \geqslant |\mathcal{E}_y|$ by definition of x.

We use the assumption that $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2LR}$ to show that $|\mathcal{E}_y| \geqslant R$. As $|\mathcal{E}_y| = |\{t, \ldots, \min(K-1, u-R-1)\}|$, if $|\mathcal{E}_y| < R$ then $t > K - 2R$. Recall from (61) and the definition of s

and t that $t-s \leqslant (L-1)(R-1)$. Combining these bounds, we see that $s > K-2R-(L-1)(R-1)$. As $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \leqslant g^{-s+1}$, this contradicts the assumption that $\|(g-1)\theta\|_{\mathbf{R}/\mathbf{Z}} \geqslant g^{-K+2LR}$.

Therefore $|\mathcal{E}_0| \geqslant |\mathcal{E}_y| \geqslant R$. Recall the definition of the tuple $\mathbf{a}$ from (55); as $x = 0$ we have that

$$\mathbf{a} = (|\mathcal{E}_0|, 0, \dots, 0, |\mathcal{E}_y|, 0, \dots, 0).$$

Our aim is to bound $\Psi(\mathbf{a}; 0)$ using Proposition 3.6 with $m = R$ and $t = y$. To satisfy the assumptions of Proposition 3.6 we need $R \ll_C |\mathcal{E}_0|^{1/4}$, which follows from the fact that $|\mathcal{E}_0| \geqslant (K - |\mathcal{D}|)/2 \geqslant (K - LR)/2$. As $LR \leqslant CK^{1/4}$ by assumption, we have that $|\mathcal{E}_0| \gg_C K$, and certainly $R \leqslant CK^{1/4}$, giving $R \ll_C |\mathcal{E}_0|^{1/4}$. Applying Proposition 3.6 gives the bound

$$\Psi(\mathbf{a}; 0) \ll_{C,g} g^{-R} + R^2 K^{-3/2}.$$

This gives (58), using that $R \geqslant \frac{3}{2}\log_g K$. □

5.1. **Decoupling the averages over $X_{\mathcal{D}}$ and $X_{\mathcal{E}}$.** In this section we prove the decoupling result, Proposition 5.6. This lemma allows us to replace the condition on the digits of X, $s(X_{\mathcal{D}}) + s(X_{\mathcal{E}}) = \xi$, with the condition $s(X_{\mathcal{E}}) = 0$, even as the digits in $X_{\mathcal{D}}$ vary.

Proof of Proposition 5.6. Let $(g-1)\theta$ have centred base-g expansion given by (51). Dividing through by $(g-1)$ in this expansion gives

$$\theta = \frac{1}{g-1}\sum_{j=1}^{w} \varepsilon_{n_j} g^{-n_j} + \frac{\eta}{g-1},$$

where $|\eta| < g^{-K}$ and $w := w_K((g-1)\theta)$. Thus we can separate the contribution to $e(X\theta)$ from each non-zero digit as follows,

$$e(X\theta) = e\Big(X\frac{\eta}{g-1}\Big)\prod_{j=1}^{w} e\Big(X\frac{\varepsilon_{n_j}}{(g-1)g^{n_j}}\Big). \tag{62}$$

We have that

$$\Big|X_{[0,n_j-R-1]}\frac{\varepsilon_{n_j}}{(g-1)g^{n_j}}\Big| \leqslant g^{-R} \text{ and } X_{[n_j,K-1]}\frac{\varepsilon_{n_j}}{g^{n_j}} \equiv X_{[n_j,K-1]}\varepsilon_{n_j} (\mathrm{mod}\ g-1),$$

where the latter statement uses (8). Thus the contribution to $e(X\theta)$ from the ${n_j}^{\text{th}}$ digit of $(g-1)\theta$ is

$$e\Big(X\frac{\varepsilon_j}{(g-1)g^{n_j}}\Big) = e\Big(X_{[n_j-R,n_j-1]}\frac{\varepsilon_{n_j}}{(g-1)g^{n_j}}\Big)e_{g-1}(\varepsilon_{n_j}X_{[n_j,K-1]}) + O(g^{-R}). \tag{63}$$

The η term gives a similar contribution: let $u := n_{w_K((g-1)\theta)+1} - R$, then

$$e(X\eta/(g-1)) = e(X_{[u,K-1]}\eta/(g-1)) + O(g^{-R}). \tag{64}$$

If $u \geqslant K$, then the interval $[u, K-1] \cap \{0, \ldots, K-1\}$ is empty and $e(X\eta/(g-1)) = 1 + O(g^{-R})$. From (62), (63) and (64), we have

$$\begin{aligned}
\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} &= \mathbb{E}_X e\Big(X\frac{\eta}{g-1}\Big)\prod_{i=1}^{w} e\Big(X\frac{\varepsilon_{n_i}}{(g-1)g^{n_i}}\Big)\mathbf{1}_{s(X)=\xi} \\
&= \mathbb{E}_X e\Big(X_{[u,K-1]}\frac{\eta}{g-1}\Big)\prod_{j=1}^{w} e\Big(X_{[n_j-R,n_j-1]}\frac{\varepsilon_{n_j}}{(g-1)g^{n_j}}\Big) \\
&\qquad \times e_{g-1}(X_{[n_j,K-1]}\varepsilon_{n_j})\mathbf{1}_{s(X)=\xi} + O_{C,g}(Lg^{-R}). \qquad (65)
\end{aligned}$$

Here, we have used that $L \leqslant CK^{1/4}$ and $R \geqslant \frac{3}{2}\log_g K$ to obtain this error term. Let $\mathcal{E} := \{0, \ldots, K-1\} \setminus \mathcal{D}$, and note that $\mathcal{E}$ is partitioned into $\mathcal{E} = \mathcal{E}_0 \cup \ldots \cup \mathcal{E}_{g-2}$ for $\mathcal{E}_r$ defined by (54). With this notation, we can rewrite (65) as

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} = \mathbb{E}_{X_\mathcal{D}} e(X_\mathcal{D}\theta)\mathbb{E}_{X_\mathcal{E}}\prod_{j=1}^{w} e_{g-1}(X_{[n_j,K-1]\cap\mathcal{E}}\varepsilon_j)\mathbf{1}_{s(X)=\xi} + O_{C,g}(Lg^{-R}). \qquad (66)$$

As the terms n_j index precisely the non-zero digits of $(g-1)\theta$, each random variable X_u for $u \in \mathcal{E}$ occurs in the above product once for each non-zero digit with index $n_j \leqslant u$, weighted by the value of that non-zero digit, ε_{n_j}. Rearranging the above product gives

$$\prod_{j=1}^{w} e_{g-1}(X_{[n_j,K-1]\cap\mathcal{E}}\varepsilon_{n_j}) = \prod_{u\in\mathcal{E}} e_{g-1}(X_u g^u \sum_{r\leqslant u}\varepsilon_r) = \prod_{v=1}^{g-2} e_{g-1}(vX_{\mathcal{E}_v}),$$

where the final equality follows from the definition of the sets $\mathcal{E}_v$ given by (54). Hence from (66) and the above equation, $\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi}$ equals

$$\mathbb{E}_{X_\mathcal{D}} e(X_\mathcal{D}\theta)\mathbb{E}_{X_\mathcal{E}} e_{g-1}(X_{\mathcal{E}_1} + 2X_{\mathcal{E}_2} + \ldots + (g-2)X_{\mathcal{E}_{g-2}})\mathbf{1}_{s(X_\mathcal{E})=\xi-s(X_\mathcal{D})} + O_{C,g}(Lg^{-R}). \qquad (67)$$

Let $\mathbf{a}_0 := (|\mathcal{E}_0|, \ldots, |\mathcal{E}_{g-2}|)$. From Proposition 3.2 the average over $X_\mathcal{E}$ equals

$$\mathbb{E}_{X_\mathcal{E}} e_{g-1}(X_{\mathcal{E}_1} + 2X_{\mathcal{E}_2} + \ldots + (g-2)X_{\mathcal{E}_{g-2}})\mathbf{1}_{s(X_\mathcal{E})=\xi-s(X_\mathcal{D})} = \Psi(\mathbf{a}_0; \xi - s(X_\mathcal{D})),$$

and thus (67) equals

$$\mathbb{E}_{X_\mathcal{D}} e(X_\mathcal{D}\theta)\Psi(\mathbf{a}_0; \xi - s(X_\mathcal{D})) + O_{C,g}(Lg^{-R}).$$

Our aim is to remove the dependence on $X_\mathcal{D}$ from the term $\Psi(\mathbf{a}_0; \xi - s(X_\mathcal{D}))$ by applying Proposition 3.3. However, in order to apply Proposition 3.3 with the tuple $\mathbf{a}_0$, we require that the first coordinate of $\mathbf{a}_0$, $|\mathcal{E}_0|$, satisfies $|\mathcal{E}_0| \geqslant |\mathcal{E}_j|$ for all $1 \leqslant j \leqslant g-2$, which may not be the case. To circumvent this issue, we exploit the fact that multiplying (67) through by factors of $e_{g-1}(X)$ increases the coefficients of each $X_{\mathcal{E}_r}$, essentially allowing us to cycle the coordinates of $\mathbf{a}_0$. To do this, we use the following relation. As $X \equiv s(X) (\mathrm{mod}\ g-1)$, we have $\mathbb{E}_X e_{g-1}(X)\mathbf{1}_{s(X)=\xi} = e_{g-1}(\xi)$, and thus for any integer m,

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)} = e_{g-1}(m\xi)\mathbb{E}_X e(X\theta)e_{g-1}(-mX)\mathbf{1}_{s(X)=\xi}. \qquad (68)$$

Let x be defined as in the statement of Proposition 5.6, that is, $x \in \{0, \ldots, g-2\}$ is such that $\mathcal{E}_x$ is maximal among the $|\mathcal{E}_r|$, and let $\mathbf{a}$ be as defined by (55),

$$\mathbf{a} := (|\mathcal{E}_x|, |\mathcal{E}_{x+1}|, \ldots, |\mathcal{E}_{g-2}|, |\mathcal{E}_0|, |\mathcal{E}_1|, \ldots, |\mathcal{E}_{x-1}|).$$

From (68), multiplying (67) through by $e_{g-1}(-xX_{\mathcal{E}})$ gives

$$\begin{aligned}\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} &= e_{g-1}(x\xi)\mathbb{E}_X e(X\theta)e_{g-1}(-xX)\mathbf{1}_{s(X)=\xi} \\ &= e_{g-1}(x\xi)\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big)\mathbb{E}_{X_{\mathcal{E}}} \prod_{j=0}^{g-2} e_{g-1}((j-x)X_{\mathcal{E}_j})\mathbf{1}_{s(X_{\mathcal{E}})=\xi - s(X_{\mathcal{D}})} \\ &= e_{g-1}(x\xi)\mathbb{E}_{X_{\mathcal{D}}} e\Big(X_{\mathcal{D}}\Big(\theta - \frac{x}{g-1}\Big)\Big)\Psi(\mathbf{a}; \xi - s(X_{\mathcal{D}})), \end{aligned} \tag{69}$$

using Proposition 3.2 and the definition of $\mathbf{a}$ for the final equality. As $|\mathcal{E}_x|$ is maximal among the sets $\mathcal{E}_r$, the tuple $\mathbf{a}$ fulfils the requirement that its first coordinate is the largest.

It remains to check that the other assumption for Proposition 3.3 holds for the tuple $\mathbf{a}$ and $\nu := \xi - s(X_{\mathcal{D}})$, namely that $|\nu| \leqslant C'|\mathcal{E}_x|^{1/4}$ for some $C' > 0$. We have that $|\nu| \leqslant |\xi| + g|\mathcal{D}|/2 \ll_{C,g} CK^{1/4}$, using the bound $|\mathcal{D}| \ll LR$ and the assumption $\max(|\xi|, LR) \leqslant CK^{1/4}$. Moreover, $|\mathcal{E}_x| \geqslant (K - |\mathcal{D}|)/(g-1)$ by definition of x, and thus $|\mathcal{E}_x| \gg_{C,g} K$. Therefore $|\nu| \ll_{C,g} |\mathcal{E}_x|^{1/4}$, and Proposition 3.3 applies to give the bound

$$\Psi(\mathbf{a}; \xi - s(X_{\mathcal{D}})) = \Psi(\mathbf{a}; 0) + O_{C,g}((|\xi| + LR)^2 K^{-3/2}). \tag{70}$$

Combining (68), (69) and (70) gives

$$\mathbb{E}_X e(X\theta)\mathbf{1}_{s(X)=\xi} = e_{g-1}(x\xi)\mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}\theta)e_{g-1}(-X_{\mathcal{D}}x)\Psi(\mathbf{a}; 0) + O_{C,g}(Lg^{-R} + (|\xi| + LR)^2 K^{-3/2}).$$

As $R \geqslant \frac{3}{2}\log_g K$, the Lg^{-R} can be absorbed into the $(|\xi| + LR)^2 K^{-3/2}$ term in the error term, giving the bound in the statement of the lemma. $\square$

5.2. **Showing cancellation in the average over $X_{\mathcal{D}}$.** In this section, we prove Proposition 5.7 and Proposition 5.8, which are used in the proof of Proposition 5.5 to give criteria on the structure of the centred base-g expansion of $(g-1)\theta$ to get cancellation in $\mathbb{E}_{X_{\mathcal{D}}} e(X_{\mathcal{D}}(\theta - x/(g-1)))$. Let us restate Proposition 5.7.

Proposition 5.7. *For $\theta \in \frac{1}{g-1}I_g$ suppose that $(g-1)\theta$ has centred base-g expansion $\sum_{i=1}^{\infty} \varepsilon_i g^{-i}$. Suppose further that there is an index m, $1 \leqslant m \leqslant K$, such that $\varepsilon_m \neq 0$, and $\varepsilon_{m+1} = \ldots = \varepsilon_{m+T} = 0$, for some integer $T \geqslant 0$. Let $r \in \{0, \ldots, g-2\}$ be such that $\sum_{1 \leqslant j \leqslant m} \varepsilon_j \equiv r (\mathrm{mod}\ g-1)$. Then*

$$\mathbb{E}_Y e\Big(g^{m-1}Y\Big(\theta - \frac{r}{g-1}\Big)\Big) \ll g^{-T},$$

where Y is a random variable uniformly taking values in $\{0, \ldots, g-1\} - (g-1)/2$.

Proof. As $\varepsilon_{m+1} = \ldots = \varepsilon_{m+T} = 0$ we have

$$\theta = \frac{a}{(g-1)g^m} + t, \tag{71}$$

where $a := \sum_{1 \leqslant j \leqslant m} \varepsilon_j g^{m-j}$ and $t := \sum_{j \geqslant m+T+1} \varepsilon_j g^{-j}$. Note that $a \in \mathbf{Z}$ and $a \equiv r (\mathrm{mod}\ g-1)$. We also have $|t| < g^{-m-T}$. Thus

$$\theta - \frac{r}{g-1} = \frac{a - g^m x}{(g-1)g^m} + t = \frac{a'}{g^m} + t,$$

where $a' \in \mathbf{Z}$. Crucially, $a' \not\equiv 0 (\text{mod } g)$ To see this, note that $a' \equiv a \equiv \varepsilon_m (\text{mod } g)$, and by assumption, $\varepsilon_m \neq 0$. Using this expression for θ, we have

$$e\Big(g^{m-1}Y\Big(\theta - \frac{r}{g-1}\Big)\Big) = e\Big(Y\frac{a'}{g}\Big)e(Ytg^{m-1}) = e\Big(Y\frac{a'}{g}\Big) + O(g^{-T}). \tag{72}$$

As Y uniformly takes values in $\{0, \ldots, g-1\}$ and $a' \not\equiv 0 (\text{mod } g)$, averaging (72) over Y gives the bound stated in the lemma, as $\mathbb{E}_Y e(Ya'/g) = 0$. □

We now prove Proposition 5.8, which we also restate for convenience.

Proposition 5.8. *For $\theta \in \frac{1}{g-1}I_g$ suppose that $(g-1)\theta$ has centred base-g expansion $\sum_{i=1}^{\infty} \varepsilon_i g^{-i}$. Let T be an integer such that $0 \leqslant T \leqslant K$, and suppose that $\varepsilon_{m-T} = \ldots = \varepsilon_{m-1} = 0$ for some index m, $T \leqslant m \leqslant K$. Let $r \in \{0, \ldots, g-2\}$ be such that $\sum_{1 \leqslant j \leqslant m} \varepsilon_j \not\equiv r (\text{mod } g-1)$. Then*

$$\mathbb{E}_{Y_{m-T}, \ldots, Y_{m-1}} e\Big(\sum_{j=m-T}^{m-1} Y_j g^j \Big(\theta - \frac{r}{g-1}\Big)\Big) \ll_g g^{-T},$$

where $Y_{m-T}, \ldots, Y_{m-1}$ are i.i.d. uniform random variables taking values in $\{0, \ldots, g-1\} - (g-1)/2$.

Proof. The assumptions on the digits of $(g-1)\theta$ ensure that θ has the following form:

$$\theta = \frac{a}{(g-1)g^{m-T-1}} + t,$$

where $a \not\equiv r (\text{mod } g-1)$, and $|t| < g^{-m}$. To obtain this, take $a = \sum_{1 \leqslant j \leqslant m-T-1} \varepsilon_j g^j$, and $t = \sum_{j \geqslant m} \varepsilon_j g^{-j}$. Then we have

$$\theta - \frac{r}{g-1} = \frac{a'}{(g-1)g^{m-T-1}} + t, \tag{73}$$

where $a' = a - g^{m-T-1}r$. All we will require about a' is that $a' \not\equiv 0 (\text{mod } g-1)$, which follows from the fact that $a' \equiv a - r (\text{mod } g-1)$. Let $Y := \sum_{j=m-T}^{m-1} Y_j g^j$; this uniformly takes values in $g^{m-T}\{0, \ldots, g^T - 1\} - g^{m-T}(g^T-1)/2$. Let

$$\gamma := e\Big(-g^{m-T}\frac{g^T-1}{2}\Big(\theta - \frac{r}{g-1}\Big)\Big).$$

As Y is uniformly distributed, we have

$$\begin{aligned}
\mathbb{E}_Y e\Big(Y\Big(\theta - \frac{r}{g-1}\Big)\Big) &= \frac{\gamma}{g^T}\sum_{b=0}^{g^T-1} e\Big(g^{m-T}b\Big(\theta - \frac{r}{g-1}\Big)\Big) \\
&= \frac{\gamma}{g^T}\sum_{b=0}^{g^T-1} e\Big(g^{m-T}b\Big(\frac{a'}{(g-1)g^{m-T-1}} + t\Big)\Big) \\
&= \frac{\gamma}{g^T}\sum_{b=0}^{g^T-1} e\Big(\frac{ga'b}{g-1} + tg^{m-T}b\Big) \\
&= \frac{\gamma}{g^T}\frac{e\big(\frac{g^{T+1}a'}{g-1} + g^m t\big) - 1}{e\big(\frac{ga'}{g-1} + g^{m-T}t\big) - 1}.
\end{aligned} \tag{74}$$

As $ga' \not\equiv 0 (\text{mod } g-1)$, we have that $|e(ga'/(g-1)) - 1| \geqslant |1 - e(1/(g-1))| =: \delta_g$. Therefore, for K sufficiently large,

$$\Big|e\Big(\frac{ga'}{g-1} + g^{m-T}t\Big) - 1\Big| = \Big|e\Big(\frac{ga'}{g-1}\Big)(1 + O(g^{-T})) - 1\Big| \geqslant \delta_g/2.$$

Thus from (74),

$$\Big|\mathbb{E}_Y e\Big(Y\Big(\theta - \frac{r}{g-1}\Big)\Big)\Big| \leqslant \frac{2}{g^T|e\big(\frac{ga'}{g-1} + g^{m-T}t\big) - 1|} \leqslant \frac{4}{\delta_g g^T}.$$

□

6. Restricting to sums of Niven numbers

In this section we establish Proposition 1.3 as a consequence of Proposition 1.2, by showing that the expected proportion of representations of M as the sum of three integers with near-average digit sum are of sums of three Niven numbers. At the end of the section, we deduce Proposition 1.1 from Proposition 1.3 by giving an explicit choice of k_1, k_2 and k_3 which satisfy the assumptions of Proposition 1.3.

Recall that $S_i = \{n < g^K : s_g(n) = k_i\}$ for $i = 1, 2, 3$ and integers k_1, k_2 and k_3, where $s_g(n)$ denotes the base-g digit sum of n. Throughout this section, we assume that k_1, k_2 and k_3 satisfy the conditions given by (1) and (2), which we restate here:

$$k_1 + k_2 + k_3 \equiv M (\text{mod } g-1), \quad |k_i - \mu_K| \leqslant C_g,$$
$$(k_i, g) = 1 \text{ for } i = 1, 2, 3 \text{ and } (k_i, k_j) = 1 \text{ for } i, j \in \{1, 2, 3\}, i \neq j.$$

The set of Niven numbers less than g^K which have digit sum precisely k_i is denoted by $\mathcal{N}_i$; these are the element of S_i which are divisible by k_i. Let $g_i(\theta)$ be the Fourier transform $\widehat{\mathbf{1}_{\mathcal{N}_i}}(\theta)$, so that

$$g_i(\theta) = \sum_{n \in \mathcal{N}_i} e(n\theta).$$

The number of representations of M as $n_1 + n_2 + n_3$ is

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \int_0^1 g_1(\theta) g_2(\theta) g_3(\theta) e(-M\theta)\, \mathrm{d}\theta. \tag{75}$$

To obtain an expression for $r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M)$ in terms of $r_{S_1+S_2+S_3}(M)$, we use the following expression to link $g_i(\theta)$ and $f_i(\theta)$. By orthogonality, we can write $\mathbf{1}_{k|n}$ as

$$\mathbf{1}_{k|n} = \frac{1}{k} \sum_{j=0}^{k-1} e\big(\frac{jn}{k}\big).$$

From the definition of $\mathcal{N}_i$ and using the above equation we have

$$g_i(\theta) = \sum_{\substack{n \in S_i \\ k_i | n}} e(n\theta) = \frac{1}{k_i} \sum_{j=0}^{k_i-1} f_i\big(\theta + \frac{j}{k_i}\big). \tag{76}$$

Here, $f_i(\theta)$ is the Fourier transform $\widehat{\mathbf{1}_{S_i}}(\theta)$. From (75) and (76), we obtain

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{1}{k_1k_2k_3} \sum_{\substack{j_1,j_2,j_3\\ 0\leqslant j_i<k_i}} \int_0^1 \prod_{i=1}^{3} f_i\big(\theta + \frac{j_i}{k_i}\big) e(-M\theta)\,\mathrm{d}\theta. \tag{77}$$

Let $G(j_1, j_2, j_3)$ denote the following,

$$G(j_1, j_2, j_3) \coloneqq \int_0^1 \prod_{i=1}^{3} f_i\big(\theta + \frac{j_i}{k_i}\big) e(-M\theta)\,\mathrm{d}\theta,$$

and let

$$\mathcal{J} \coloneqq \{(j_1, j_2, j_3) \in \mathbf{Z}^3 : j_i \in \{0, \tfrac{k_i}{g-1}, \ldots, \tfrac{(g-2)k_i}{g-1}\} \text{ for } i = 1, 2, 3\}. \tag{78}$$

For all tuples $(j_1, j_2, j_3) \in \mathcal{J}$, we will show that

$$G(j_1, j_2, j_3) = r_{S_{k_1}+S_{k_2}+S_{k_3}}(M).$$

This is immediate for $G(0, 0, 0)$ from (7), and for other $(j_1, j_2, j_3) \in \mathcal{J}$ (if they exist) we use the relation (8). Suppose $j_i = a_ik_i/(g-1)$ for $i = 1, 2, 3$ and integers a_i. Then from (8),

$$f_i\Big(\theta + \frac{a_i}{g-1}\Big) = e_{g-1}(a_ik_i) f_i(\theta) = f_i(\theta),$$

as $(g-1) \mid a_ik_i$. Thus

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{|\mathcal{J}|}{k_1k_2k_3} r_{S_1+S_2+S_3}(M) + \frac{1}{k_1k_2k_3} \sum_{\substack{0\leqslant j_i<k_i\\ (j_1,j_2,j_3)\notin\mathcal{J}}} G(j_1, j_2, j_3). \tag{79}$$

The next lemma is used to show that the remaining tuples $(j_1, j_2, j_3) \notin \mathcal{J}$ contribute a negligible amount to (79).

Lemma 6.1. *Let k_1, k_2, k_3 be integers such that conditions* (1) *and* (2) *hold. Let j_1, j_2, j_3 be integers such that $(j_1, j_2, j_3) \notin \mathcal{J}$ and $0 \leqslant j_i \leqslant k_i - 1$ for $i = 1, 2, 3$. Then*

$$G(j_1, j_2, j_3) \ll_g g^{2K} K^{-29/6}.$$

To achieve this, we show that at any point $\theta \in [0, 1]$, there exists $i \in \{1, 2, 3\}$ such that $f_i(\theta + j_i/k_i)$ is very small, enough to kill off the contribution from the other terms. We are now ready to prove Proposition 1.3 using Proposition 1.2, and assuming Proposition 6.1.

Proof of Proposition 1.3. From (79),

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{|\mathcal{J}|}{k_1k_2k_3} r_{S_1+S_2+S_3}(M) + \frac{1}{k_1k_2k_3} \sum_{\substack{0\leqslant j_i<k_i\\ (j_1,j_2,j_3)\notin\mathcal{J}}} G(j_1, j_2, j_3).$$

Proposition 6.1 gives $G(j_1, j_2, j_3) \ll_g g^{2K}K^{-29/6}$ for $(j_1, j_2, j_3) \notin \mathcal{J}$. As there are at most $k_1k_2k_3$ such tuples, we have

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{|\mathcal{J}|}{k_1k_2k_3} r_{S_1+S_2+S_3}(M) + O_g(g^{2K}K^{-29/6}). \tag{80}$$

This proves the first part of Proposition 1.3, as

$$|\mathcal{J}| = \prod_{i=1}^{3} (g-1, k_i) \tag{81}$$

from the definition of $\mathcal{J}$ given in (78). To establish the second part of Proposition 1.3, we use Proposition 1.2. This gives that

$$r_{S_1+S_2+S_3}(M) = \frac{(g-1)M^2}{2(2\pi\sigma^2 K)^{3/2}} + O_g(g^{2K}(\log K)^4 K^{-7/4}),$$

and plugging this into (80) gives

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{(g-1)|\mathcal{J}|M^2}{2k_1k_2k_3(2\pi\sigma^2 K)^{3/2}} + O_g\Big(\frac{|\mathcal{J}|g^{2K}(\log K)^4}{k_1k_2k_3K^{7/4}} + \frac{g^{2K}}{K^{29/6}}\Big).$$

As $|k_i - \mu_K| \ll_g 1$ from (1), and $|\mathcal{J}| \leqslant (g-1)^3$, this error term is $O_g(g^{2K}(\log K)^4 K^{-19/4})$. Additionally using that $\mu_K = (g-1)K/2$, we have

$$\frac{1}{k_1k_2k_3} = \frac{8}{((g-1)K)^3} + O_g(K^{-4}).$$

Thus

$$r_{\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3}(M) = \frac{4|\mathcal{J}|M^2}{(g-1)^2(2\pi\sigma^2)^{3/2}K^{9/2}} + O_g\Big(\frac{g^{2K}(\log K)^4}{K^{19/4}}\Big).$$

Finally, using (81) we recover the main term stated in Proposition 1.3. □

6.1. **Translates of points with few non-zero digits.** In order to prove Proposition 6.1, we isolate the following subset of $\mathbf{R}/\mathbf{Z}$,

$$\mathcal{B} := \{\theta \in \mathbf{R}/\mathbf{Z} : w_K((g-1)\theta) \leqslant \ell\}, \tag{82}$$

which includes the majors arcs, as well as some of the minor arc points. Here, $\ell \asymp_g \log K$ is given by (3). Recall that for $\theta \notin \mathcal{B}$, Proposition 5.3 gives the bound

$$f_i(\theta) \ll_g g^K K^{-12}.$$

The next proposition states that translates of θ by certain multiples of k_i^{-1} cannot all simultaneously lie in $\mathcal{B}$.

Proposition 6.2. *Let k_1, k_2, k_3 be integers satisfying* (1) *and* (2)*, and let j_1, j_2, j_3 be integers such that $0 \leqslant j_i \leqslant k_i - 1$ and $(j_1, j_2, j_3) \notin \mathcal{J}$. Then for $\mathcal{B}$ as defined by* (82)*,*

$$\Big(\mathcal{B} - \frac{j_1}{k_1}\Big) \cap \Big(\mathcal{B} - \frac{j_2}{k_2}\Big) \cap \Big(\mathcal{B} - \frac{j_3}{k_3}\Big) = \emptyset.$$

Assuming Proposition 6.2, we can now prove Proposition 6.1.

Proof of Proposition 6.1. Assume that k_1, k_2 and k_3 fulfil (1) and (2). By two applications of Hölder's inequality,

$$|G(j_1, j_2, j_3)| \leqslant \sup_{\theta \in [0,1]} \Big(\prod_{i=1}^{3} \Big| f_i\Big(\theta + \frac{j_i}{k_i}\Big)\Big|\Big)^{1/3} \Big(\prod_{i=1}^{3} \int_0^1 |f_i(\theta)|^2\Big)^{1/3}.$$

By Parseval's identity, and using the size estimates for each set $|S_i|$ from Proposition 2.1, we have

$$G(j_1, j_2, j_3) \ll_g \frac{g^K}{\sqrt{K}} \sup_{\theta \in [0,1]} \Big(\prod_{i=1}^{3} \Big| f_i\Big(\theta + \frac{j_i}{k_i}\Big)\Big|\Big)^{1/3}.$$

Hence it suffices to show that for $(j_1, j_2, j_3) \notin \mathcal{J}$,

$$\sup_{\theta \in [0,1]} \prod_{i=1}^{3} \Big| f_i\Big(\theta + \frac{j_i}{k_i}\Big)\Big| = O_g(g^{3K} K^{-13}). \tag{83}$$

For any $\theta \in [0,1]$, Proposition 6.2 gives that at most two of translates $\theta + j_1/k_1, \theta + j_2/k_2, \theta + j_3/k_3$ can lie in $\mathcal{B}$. Suppose, without loss of generality, that these potential two bad translates are j_1/k_1 and j_2/k_2, so that $\theta + j_1/k_1, \theta + j_2/k_2$ can potentially lie in $\mathcal{B}$ and $\theta + j_3/k_3 \notin \mathcal{B}$.

We use the the trivial bound of $|S_i| \asymp_g g^K K^{-1/2}$ from Proposition 2.1 to bound $|f_i(\theta + j_i/k_i)|$ for $i = 1, 2$. As $\theta + j_3/k_3 \notin \mathcal{B}$, we use Proposition 5.3 to bound $|f_3(\theta + j_3/k_3)| \ll_g g^K K^{-12}$; combining these bounds gives (83). $\square$

Our strategy for proving Proposition 6.2 is as follows. We first show that certain rationals have many non-zero digits in their centred base-g expansions. Then we show that if α and β both have very few non-zero digits in their centred base-g expansions, so must $\alpha - \beta$. The conditions on the k_i given by (2) are required precisely to ensure that rationals of the form

$$\frac{(g-1)(j_s k_t - j_t k_s)}{k_s k_t}$$

have many non-zero digits in their centred expansions, provided that j_s and j_t are not both integer multiples of $k_s/(g-1)$ or $k_t/(g-1)$ respectively. This is driven by the following lemma.

Lemma 6.3. *Let $j, k \in \mathbf{N}$ be such that $(k, g) = 1$ and $j \not\equiv 0 (\mathrm{mod}\ k)$. Then*

$$w_K(j/k) > \frac{K}{\lceil \log_g k \rceil} - 1.$$

Proof. First, note the following fact. Suppose that a, m are integers with $m > 0$ and $a < g^m$. Then for j, k as in the statement of the lemma, we have

$$\Big\| \frac{j}{k} - \frac{a}{g^m} \Big\|_{\mathbf{R}/\mathbf{Z}} \geqslant \frac{1}{k g^m}. \tag{84}$$

From the assumption that $j \not\equiv 0 (\mathrm{mod}\ k)$, j/k must have at least one non-zero digit after the radix point in its centred expansion. Moreover, there are infinitely many non-zero digits in the expansion, otherwise j/k would be equal to a rational with a denominator that is a power of g, contradicting the fact $(k, g) = 1$. Let $n_0 = 0$, and let n_i denote the index of the i^{th} non-zero digit after the radix point in the centred expansion of j/k. The proof of the lemma will follow from the next claim.

Claim 6.4. *For all $i \geqslant 0$, $n_i \leqslant i \lceil \log_g k \rceil$.*

Assume that Proposition 6.4 holds, and let M be the number of non-zero digits before the $(K+1)^{\text{th}}$ digit, such that $n_M \leqslant K < n_{M+1}$. From Proposition 6.4, $K < n_{M+1} \leqslant (M+1)\lceil \log_g k \rceil$, giving $M > K/\lceil \log_g k \rceil - 1$.

To prove Proposition 6.4 we induct on i, with the $i=0$ case following immediately by definition of n_0. Suppose the claim holds for $i = 0, \dots, t-1$, but that $n_t > t\lceil \log_g k \rceil$. This gives that $n_t - n_{t-1} > \lceil \log_g K \rceil$. Let $a/g^{n_{t-1}}$ be the rational obtained by truncating the centred base-g expansion of j/k at the ${n_{t-1}}^{\text{th}}$ digit. By definition of the n_i, all the digits of j/k strictly between the ${n_{t-1}}^{\text{th}}$ and the ${n_t}^{\text{th}}$ are all zero, thus

$$\Big\| \frac{j}{k} - \frac{a}{g^m} \Big\|_{\mathbf{R}/\mathbf{Z}} \leqslant \frac{1}{g^{n_t}} < \frac{1}{kg^{n_{t-1}}},$$

where we have used that $n_t > n_{t-1} + \lceil \log_g K \rceil$. The above approximation for j/k contradicts (84), proving Proposition 6.4. $\square$

We now show that the function w_K is roughly additive.

Lemma 6.5. *For all $\alpha, \beta \in \mathbf{R}$,*

$$w_K(\alpha - \beta) \ll_g w_K(\alpha) + w_K(\beta).$$

Proof. From the upper bound on w_K given by Proposition 5.2,

$$w_K(\alpha - \beta) \asymp_g \sum_{i=1}^{K} \|g^i(\alpha-\beta)\|^2_{\mathbf{R}/\mathbf{Z}}.$$

Furthermore, we have that

$$\|\alpha - \beta\|^2_{\mathbf{R}/\mathbf{Z}} \leqslant (\|\alpha\|_{\mathbf{R}/\mathbf{Z}} + \|\beta\|_{\mathbf{R}/\mathbf{Z}})^2 \leqslant 3\|\alpha\|^2_{\mathbf{R}/\mathbf{Z}} + 3\|\beta\|^2_{\mathbf{R}/\mathbf{Z}}. \tag{85}$$

for all $\alpha, \beta \in \mathbf{R}$. Thus

$$\sum_{i=1}^{K} \|g^i(\alpha-\beta)\|^2_{\mathbf{R}/\mathbf{Z}} \leqslant 3\sum_{i=1}^{K}(\|g^i\alpha\|^2_{\mathbf{R}/\mathbf{Z}} + \|g^i\beta\|^2_{\mathbf{R}/\mathbf{Z}}) \ll_g w_K(\alpha) + w_K(\beta),$$

now using the lower bound on w_K from Proposition 5.2. Note that the implicit constant in the above inequality can be determined from Proposition 5.2, and only depends on g. $\square$

Recall from (78) that

$$\mathcal{J} := \{(j_1, j_2, j_3) \in \mathbf{Z}^3 : j_i \in \{0, \tfrac{k_i}{g-1}, \dots, \tfrac{(g-2)k_i}{g-1}\} \cap \mathbf{Z} \text{ for } i = 1,2,3\},$$

and from (82) that $\mathcal{B} := \{\theta \in \mathbf{R}/\mathbf{Z} : w_K((g-1)\theta) \leqslant \ell\}$, where $\ell \asymp_g \log K$. We may now prove Proposition 6.2.

Proof of Proposition 6.2. Assume that k_1, k_2 and k_3 fulfil conditions (1) and (2), but that there exist integers j_1, j_2, j_3 with $0 \leqslant j_i \leqslant k_i - 1$ for $i = 1,2,3$, and $(j_1, j_2, j_3) \notin \mathcal{J}$ with

$$\Big(\mathcal{B} - \frac{j_1}{k_1}\Big) \cap \Big(\mathcal{B} - \frac{j_2}{k_2}\Big) \cap \Big(\mathcal{B} - \frac{j_3}{k_3}\Big) \neq \emptyset. \tag{86}$$

As $(j_1, j_2, j_3) \notin \mathcal{J}$, there exists at least one index i, $i \in \{1,2,3\}$, such that

$$j_i \notin \Big\{0, \frac{k_i}{g-1}, \dots, \frac{(g-2)k_i}{g-1}\Big\} \cap \mathbf{Z}. \tag{87}$$

Without loss of generality, assume that this term is j_1. We will show, without any assumptions on j_2, that

$$\Big(\mathcal{B}-\frac{j_1}{k_1}\Big)\cap\Big(\mathcal{B}-\frac{j_2}{k_2}\Big)=\emptyset. \tag{88}$$

Suppose that (88) doesn't hold, so that there exists some θ such that $\theta\in(\mathcal{B}-j_1/k_1)\cap(\mathcal{B}-j_2/k_2)$. Then the following two translates of θ, $\theta+j_1/k_1, \theta+j_2/k_2$, both lie in $\mathcal{B}$ or in other words, both translates of θ simultaneously have few non-zero digits:

$$w_K\Big((g-1)\Big(\theta+\frac{j_1}{k_1}\Big)\Big)\leqslant \ell \text{ and } w_K\Big((g-1)\Big(\theta+\frac{j_2}{k_2}\Big)\Big)\leqslant \ell. \tag{89}$$

Let $\alpha_1 \coloneqq (g-1)(\theta+j_1/k_1)$ and $\alpha_2 \coloneqq (g-1)(\theta+j_2/k_2)$, and note that

$$\alpha_1=\alpha_2+(g-1)\frac{j_1k_2-j_2k_1}{k_1k_2}. \tag{90}$$

From Proposition 6.5, and (89), we have

$$w_K\Big((g-1)\frac{j_1k_2-j_2k_1}{k_1k_2}\Big)=w_K(\alpha_1-\alpha_2)\ll_g w_K(\alpha_1)+w_K(\alpha_2)\ll_g \log K. \tag{91}$$

Here we have used that $\ell \asymp_g \log K$ from (3). We will use this to derive a contradiction by showing that $(g-1)(j_1k_2-j_2k_1)/k_1k_2$ has many non-zero digits in its centred expansion from Proposition 6.3.

Let $j \coloneqq (g-1)(j_1k_2-j_2k_1)$, and $k \coloneqq k_1k_2$. To see that the assumptions of Proposition 6.3 hold for the rational j/k, first note that we have $(k,g)=1$ from (2). Furthermore, $j\not\equiv 0(\text{mod } k)$, as otherwise we would have $(g-1)j_1k_2\equiv 0(\text{mod } k_1)$. As $(k_1,k_2)=1$, this would imply that $(g-1)j_1\equiv 0(\text{mod } k_1)$, contradicting (87). Therefore from Proposition 6.5,

$$w_K\Big(\frac{(g-1)(j_1k_2-j_2k_1)}{k_1k_2}\Big)>\frac{K}{\lceil \log_g k_1k_2\rceil}-1\gg_g \frac{K}{\log K},$$

where we have also used that $k_1k_1\asymp_g K^2$ from (1). This contradicts (91) for sufficiently large K. □

6.2. **An explicit choice of k_1,k_2 and k_3.** We may now prove Proposition 1.1. This follows immediately from Proposition 1.3, provided that an appropriate choice of k_1,k_2 and k_3 exist for all sufficiently large K and M. Recall that k_1,k_2,k_3 are integers fulfilling the conditions given by (1) and (2). We restate these conditions here:

$$k_1+k_2+k_3\equiv M(\text{mod } g-1),\quad |k_i-\mu_K|\leqslant C_g,$$
$$(k_i,g)=1 \text{ for } i=1,2,3 \text{ and } (k_i,k_j)=1 \text{ for } i,j\in\{1,2,3\}, i\neq j,$$

where

$$C_g \coloneqq g(g-1)\prod_{p\leqslant 10g^2} p.$$

Note that the product in the above definition is over primes. We show that such a choice of k_1,k_2 and k_3 exists for all g,K and M. Let K_0 be such that

$$K_0\equiv 0(\text{mod } C_g) \text{ and } |K_0-\mu_K|\leqslant C_g/2. \tag{92}$$

We define $k_i \coloneqq r_i+K_0$ for $i=1,2,3$, with r_i to be determined by the following process. Let $r_1=1$, and let r_2 be the smallest prime greater than g. Clearly $|k_1-\mu_K|\leqslant C_g/2+1$, and we also have

$|k_2 - \mu_K| \leqslant C_g/2 + 2g + 2$. Furthermore, as $k_i \equiv r_i (\text{mod } g)$ by definition of C_g, both k_1 and k_2 are coprime to g for these choices of r_1 and r_2.

It remains to choose r_3. As $k_1+k_2+k_3 \equiv r_1+r_2+r_3 (\text{mod } g-1)$, we need $r_3 \equiv M-r_2-1 (\text{mod } g-1)$. Let $a \in \{0, \ldots, g-2\}$ be such that $a \equiv M - r_2 - 1 (\text{mod } g-1)$. We choose $\lambda \geqslant 0$ such that $r_3 \coloneqq a + \lambda(g-1)$ and so that the remaining conditions on k_3 are satisfied.

If $a \neq 1$, let $\lambda_0 \coloneqq a - 1$, otherwise let $\lambda_0 \coloneqq g$. In either case, let $\lambda_1 \coloneqq \lambda_0 + g$. We cannot have $r_2 \mid a + \lambda_1(g-1)$ and $r_2 \mid (a + \lambda_0(g-1))$, as this implies $r_2 \mid g(g-1)$, which contradicts the fact that $r_2 > g$ and r_2 is prime. Let λ be either λ_0 or λ_1 as appropriate to ensure that $r_2 \nmid r_3$.

By construction, $k_1+k_2+k_3 \equiv M (\text{mod } g-1)$. As $-1 \leqslant \lambda \leqslant 2g$, $|k_3 - \mu_K| \leqslant C_g/2 + a + |\lambda|(g-1) \leqslant C_g$. Moreover, $k_3 \equiv r_3 \equiv a - \lambda (\text{mod } g)$. In all cases, $\lambda \equiv a - 1 (\text{mod } g)$, and so $k_3 \equiv 1 (\text{mod } g)$, which ensures that $(k_3, g) = 1$.

It remains to show that k_i, k_j are coprime. If not, any prime q dividing both k_i and k_j must be bounded by $|k_i - k_j|$. However this is in turn bounded by $\max(|r_1 - r_2|, |r_2 - r_3|, |r_1 - r_3|) < 10g^2$, so we must have $q \mid C_g$. Hence $q \mid K_0$, and thus q divides r_i and r_j, contradicting the fact that r_1, r_2, r_3 are pairwise coprime by construction.

Appendix A. Proof of Proposition 5.1

The direct statement of the exponential sum bound, Proposition 5.1, does not appear as a named theorem in [3]. Instead, it appears in the proof of their result [3, Theorem 1.2]. For completeness, we sketch the proof of Proposition 5.1 here, following the necessary parts of the proof of [3, Theorem 1.2]. Let us restate the result.

Proposition 5.1. *For $\theta \in \mathbf{R}/\mathbf{Z}$,*

$$|f_i(\theta)| \leqslant g^K \exp\Big(- \frac{1}{2g} \sum_{i=0}^{K-1} \|g^i (g-1)\theta\|^2_{\mathbf{R}/\mathbf{Z}} \Big).$$

Proof. By using orthogonality to detect the condition that $s_g(n) = k_i$ for $n \in S_i$, we may rewrite $f_i(\theta)$ as follows

$$f_i(\theta) = \sum_{n \in S_i} e(n\theta) = \int_0^1 e(-xk_i) \sum_{n < g^K} e(n\theta) e(s(n)x) \, \mathrm{d}x.$$

To bound $|f_i(\theta)|$, it suffices to bound

$$|f_i(\theta)| \leqslant \Big| \int_0^1 \sum_{n < g^K} e(n\theta + s(n)x) \, \mathrm{d}x \Big|. \tag{93}$$

Note that from now on, the specific target digit sum does not appear in the proof; the only relevance of the exact value of k_i, in so far as this bound is concerned, is the strength of the bound relative to the trivial bound. To control the right hand side of (93), we use the recursive structure of the sum of digits function to isolate the contribution coming from each of the K digits of $n < g^K$. We have

$$\sum_{n < g^K} e(n\theta + s(n)x) = \prod_{\nu=0}^{K-1} \sum_{j=0}^{g-1} e(jg^\nu \theta + jx),$$

and thus

$$|f_i(\theta)| \leqslant \int_0^1 \prod_{\nu=0}^{K-1} \Big| \sum_{j=0}^{g-1} e(jg^\nu\theta + jx) \Big| \,\mathrm{d}x = \int_0^1 \prod_{\nu=0}^{K-1} |U(g^\nu\theta + x)| \,\mathrm{d}x \tag{94}$$

where, following the notation of [3], $U(\alpha) \coloneqq \sum_{j=0}^{g-1} e(j\alpha)$. The result [3, Lemma 3.3] gives that for any real t, t_0,

$$|U(t)U(t+t_0)| \leqslant g^2 \exp\big(- \frac{1}{g}\|t_0\|^2\big). \tag{95}$$

We will apply (95) with $t = g^\nu\theta + x$, and $t_0 = (g-1)g^\nu\theta$ for $\nu = 0, \dots, K-2$, so that $t + t_0 = g^{\nu+1}\theta + x$. Crucially, applying (95) removes the dependency on the variable x. We have that

$$\begin{aligned}\prod_{\nu=0}^{K-1} |U(g^\nu\theta + x)|^2 &= |U(\theta + x)U(g^{K-1}\theta + x)| \prod_{\nu=0}^{K-2} |U(g^\nu\theta + x)U(g^{\nu+1}\theta + x)| \\ &\leqslant g^{2(K-2)} |U(\theta + x)U(g^{K-1}\theta + x)| \exp\Big(- \frac{1}{g} \sum_{\nu=0}^{K-2} \|(g-1)g^\nu\theta\|^2_{\mathbf{R}/\mathbf{Z}} \Big). \end{aligned} \tag{96}$$

We obtain

$$\begin{aligned}\prod_{\nu=0}^{K-1} |U(g^\nu\theta + x)| &\leqslant g^{K-2} |U(\theta + x)U(g^{K-1}\theta + x)|^{1/2} \exp\Big(- \frac{1}{2g} \sum_{\nu=0}^{K-2} \|(g-1)g^\nu\theta\|^2_{\mathbf{R}/\mathbf{Z}} \Big) \\ &\leqslant g^K \exp\Big(- \frac{1}{2g} \sum_{\nu=0}^{K-1} \|(g-1)g^\nu\theta\|^2_{\mathbf{R}/\mathbf{Z}} \Big). \end{aligned} \tag{97}$$

The first inequality follows from taking the square root of the expression obtained in (96). The final equality comes from the fact that $|U(\alpha)| \leqslant g$ for any α, and that $g \exp(-\|g^{K-1}(g-1)\theta\|^2_{\mathbf{R}/\mathbf{Z}}/2g) \geqslant ge^{-1/8g} \geqslant 1$ for $g \geqslant 2$. As this expression no longer depends on x, combining (94) and (97) gives the bound stated in the theorem. □

Appendix B. Local limit theorem

In Section 3 and Section 5 we use the following local limit theorem statement, Proposition 3.1. Recall that $X_0, \dots, X_{K-1}$ are i.i.d. copies of Y, which uniformly takes values in $\{0, \dots, g-1\} - (g-1)/2$.

Proposition 3.1 (First part)**.** *Let ν be an integer. Then*

$$\mathbb{P}(X_0 + \ldots + X_K = \nu) = \frac{e^{-x^2/2}}{\sqrt{2\pi\sigma^2 K}} + O_g(K^{-3/2}),$$

where $x = \nu/\sqrt{\sigma^2 K}$, and $\sigma^2 = (g^2-1)/12$ is the variance of Y.

This can be deduced immediately from far more general local limit theorems, such as Theorem 13 of [8, Ch. VII]. The theorems in [8, Ch. VII] allow one to obtain explicit expressions for arbitrarily many lower order terms and allow for much more general i.i.d. random variables. In order to show Proposition 3.1 from one such more general theorem, it is necessary to calculate the cumulants of the random variable Y to obtain the error term we require. As we are only concerned with a very specific family of uniform distributions, and don't require lower order terms, we can prove

Proposition 3.1 directly in this special case. We do so here, following the proofs of the more general theorems given in [8].

We start by analysing the characteristic function of Y. Let $\varphi_Y(z)$ denote the characteristic function of Y,

$$\begin{aligned}\varphi_Y(z) &= \mathbb{E}_Y e^{izY} \\ &= 1 + \frac{2}{g}\sum_{j=1}^{(g-1)/2} \cos(jz) \text{ for odd } g, \text{ and } \frac{2}{g}\sum_{j=0}^{(g-2)/2} \cos\Big(\frac{2j+1}{2}z\Big) \text{ for even } g. \qquad (98)\end{aligned}$$

Around $z = 0$, $\varphi_Y(z)$ has the following expansion.

Lemma B.1. *There exists $\varepsilon_g > 0$ such that for $|z| \leqslant \varepsilon_g$, we have $\varphi_Y(z) \geqslant 1/2$ and*

$$\varphi_Y(z) = 1 - \frac{\sigma^2 z^2}{2} + O_g(z^4).$$

Proof. Let $\varepsilon_g > 0$ be sufficiently small such that

$$\cos(jz) \geqslant 1/2 \text{ for } j \in [1, g/2].$$

For odd g, using (98) we obtain:

$$\varphi_Y(Y) = \frac{1}{g}\Big(1 + 2\sum_{j=1} \cos(jz)\Big) \geqslant \frac{1}{g}\Big(1 + \frac{g-1}{2}\Big) \geqslant \frac{1}{2},$$

and for even g,

$$\varphi_Y(z) = \frac{2}{g}\sum_{j=0}^{(g-2)/2} \cos\Big(\frac{2j+1}{2}z\Big) \geqslant \frac{1}{2},$$

thus proving the first part of the claim. Moreover, ε_g only depends on g, and so uniformly in z, depending only on g, we can expand $\cos(jz)$ in this range

$$\cos(jz) = 1 - \frac{j^2 z^2}{2} + O_g(z^4), \text{ for } j \in [1, g/2].$$

Plugging this expansion into (98) and using that $\sigma^2 = (g^2-1)/12$, we see that for odd g,

$$\varphi_Y(z) = \frac{1}{g} + \frac{2}{g}\sum_{j=1}^{(g-1)/2} \cos(jz) = 1 - \frac{z^2}{g}\sum_{j=1}^{(g-1)/2} j^2 + O_g(z^4) = 1 - \frac{\sigma^2 z^2}{2} + O_g(z^4),$$

and for even g,

$$\begin{aligned}\varphi_Y(z) = \frac{2}{g}\sum_{j=0}^{(g-2)/2} \cos\Big(\frac{2j+1}{2}z\Big) &= 1 - \frac{z^2}{g}\sum_{j=0}^{(g-2)/2} \Big(\frac{2j+1}{2}\Big)^2 + O_g(z^4) \\ &= 1 - \frac{\sigma^2 z^2}{2} + O_g(z^4). \qquad \square\end{aligned}$$

We may now prove Proposition 3.1.

Proof of Proposition 3.1. By orthogonality,

$$\mathbb{P}(X_1 + \ldots + X_K = \nu) = \mathbb{E}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it\nu} e^{it(X_1+\ldots+X_K)}\,\mathrm{d}t\Big) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it\nu}\varphi_Y(t)^K\,\mathrm{d}t$$
$$= \frac{1}{2\pi\sigma K^{1/2}}\int_{-\pi\sigma K^{1/2}}^{\pi\sigma K^{1/2}} e^{-itx}\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K\,\mathrm{d}t, \tag{99}$$

where $x = \nu/\sigma K^{1/2}$. To establish Proposition 3.1 then, it suffices to show that

$$\int_{-\pi\sigma K^{1/2}}^{\pi\sigma K^{1/2}} e^{-itx}\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K\,\mathrm{d}t = \sqrt{2\pi}e^{-x^2/2} + O_g(K^{-1}). \tag{100}$$

We approximate the characteristic function $\varphi_Y(t/\sigma K^{1/2})$ close to 0 to obtain the main term in Proposition 3.1, and bound the remaining contribution by $K^{-3/2}$. Let ε_g be as in Proposition B.1. We show that for $|t| < K^{1/4}\varepsilon_g$,

$$\Big|\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K - e^{-t^2/2}\Big| \ll_g e^{-t^2/2}t^4K^{-1}. \tag{101}$$

To show (101), we expand $\log\varphi_Y(z)$ in terms of z. For $|z| \leqslant \varepsilon_g$, we have $\varphi_Y(z) \geqslant 1/2$ and in particular, $\varphi_Y(z)$ is positive in this range so $\log\varphi_Y(z)$ is defined. For $|z| \leqslant \varepsilon_g$, let $w = 1 - \varphi_Y(z)$. We have that $|w| \leqslant 1/2$ and thus

$$\log\varphi_Y(z) = \log(1-w) = -w + O_g(w^2).$$

From Proposition B.1 and the definition of w, we have

$$\log\varphi_Y(z) = -\frac{\sigma^2 z^2}{2} + O_g(z^4) \tag{102}$$

for $|z| \leqslant \varepsilon_g$. We apply (102) with $z = t/\sigma K^{1/2}$ in the range $|t| \leqslant K^{1/4}\varepsilon_g$. Note that $\sigma^{-2}K^{-1/4} \leqslant 1$ for all $g \geqslant 2$ and $K \geqslant 1$, so in this range, $|t/\sigma K^{1/2}| \leqslant \varepsilon_g$.

Hence we have

$$\log\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K = -\frac{t^2}{2} + O_g(t^4K^{-1})$$

which gives

$$\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K = e^{-t^2/2}\exp\big(O_g(t^{-4}K^{-1})\big).$$

Using that $e(O_g(t^{-4}K^{-1})) = 1 + O_g(t^{-4}K^{-1})$, as $|t| \leqslant K^{1/4}\varepsilon_g$, gives the bound stated in (101). We can now estimate (99) by evaluating the following integrals:

$$I_1 := \int_{-\infty}^{\infty} e^{itx}e^{-t^2/2}\,\mathrm{d}t, \;\; I_2 := \int_{|t|>K^{1/4}\varepsilon_g} e^{itx}e^{-t^2/2}\,\mathrm{d}t$$
$$I_3 := \int_{|t|<K^{1/4}\varepsilon_g}\Big|\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K - e^{-t^2/2}\Big|\,\mathrm{d}t, \;\; I_4 = \int_{K^{1/4}\varepsilon_g\leqslant|t|\leqslant\sigma K^{1/2}}\Big|\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)\Big|^K\,\mathrm{d}t.$$

Thus

$$\int_{-\pi\sigma K^{1/2}}^{\pi\sigma K^{1/2}} e^{-itx}\varphi_Y\Big(\frac{t}{\sigma K^{1/2}}\Big)^K\,\mathrm{d}t = I_1 + O_g(I_2 + I_3 + I_4).$$

The integral I_1 is

$$\int_{-\infty}^{\infty} e^{itx}e^{-t^2/2}\,\mathrm{d}t = \sqrt{2\pi}e^{-x^2/2}.$$

Thus in order to show (100), it remains to bound the other integral terms. We have

$$I_2 \leqslant \int_{|t|>K^{1/4}\varepsilon_g} e^{-t^2/2}\,\mathrm{d}t \ll_g K^{-10} \text{ and } I_3 \ll \frac{1}{K}\int_{|t|<K^{1/4}\varepsilon_g} e^{-t^2/2}t^4\,\mathrm{d}t \ll_g \frac{1}{K},$$

using (101) for I_3. Finally we turn to I_4. For $t \in [K^{1/4}\varepsilon_g, \sigma K^{1/2}]$, we have that $|\varphi_Y(t/\sigma K^{1/2})| \leqslant \varphi_Y(\varepsilon_g \sigma K^{1/4})$. To see that $\varphi_Y(\varepsilon_g\sigma K^{1/4}) > 0$, we use Proposition B.1 to expand the characteristic function

$$\varphi_Y(\varepsilon_g\sigma K^{1/4}) = 1 - \frac{\varepsilon_g^2}{2K^{1/2}} + O_g(K^{-1}) \leqslant 1 - \frac{\varepsilon_g^2}{4K^{1/2}}$$

for K sufficiently large in terms of g. Thus

$$I_4 \leqslant \int_{|t|\in[K^{1/4}\varepsilon_g, \sigma K^{1/2}]} \Big(1 - \frac{\varepsilon_g}{4\sigma^2 K^{1/2}}\Big)^K \leqslant \sigma K^{1/2}\Big(1 - \frac{\varepsilon_g}{4\sigma^2 K^{1/2}}\Big)^K \ll_g K^{-10}. \qquad \square$$

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