REAL ZEROS OF $L'(s, \chi_d)$

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Abstract. Let ν be any positive function such that $\nu(x) \to \infty$ as $x \to \infty$. We prove that for almost all fundamental discriminants d, $L'(s, \chi_d)$ has at most $(\log\log|d|)(\log\log\log|d|)$ real zeros inside the interval $[1/2 + \nu(|d|)/\log|d|, 1]$. Combining this result with a recent work of Klurman, Lamzouri, and Munsch, shows that the number of these zeros equals $(\log\log|d|)(\log\log\log|d|)^{\theta}$ for almost all d, where $|\theta| \leqslant 1$. This comes close to proving a conjecture of Baker and Montgomery, which predicts $\asymp \log\log|d|$ real zeros of $L'(s, \chi_d)$ in the interval $[1/2, 1]$, for almost all d. Moreover, assuming a mild hypothesis on the low lying zeros of quadratic Dirichlet L-functions (which follows from GRH and the one level density conjecture of Katz and Sarnak), we fully resolve the Baker-Montgomery conjecture (up to the $\log\log\log|d|$ factor). We also show, under the same hypothesis, that for almost all d, 100% of the real zeros of $L'(s, \chi_d)$ on $[1/2, 1]$ lie to the right of $1/2 + \nu(|d|)/\log|d|$.

1. Introduction

Understanding the location and distribution of zeros of derivatives of L-functions has important and deep applications to the *horizontal* and *vertical* distributions of zeros of L-functions. One of the earliest and most striking links between the zeros of $\zeta'(s)$ (where $\zeta(s)$ is the Riemann zeta function) and the Riemann Hypothesis (RH) is Speiser's Theorem [20], which states that RH is equivalent to the assertion that $\zeta'(s)$ has no zeros to the left of the critical line. This was quantified by Levinson and Montgomery [12], and is the basis of Levinson's method which produces one third of the zeros of $\zeta(s)$ on the critical line. Furthermore, the works of Soundararajan [18], and Radziwiłł [15] show that the horizontal distribution of the zeros of $\zeta'(s)$ is also related to the vertical distribution of the zeros of $\zeta(s)$.

In [1], Baker and Montgomery studied the real zeros of $L'(s, \chi_d)$ on $[1/2, 1]$, where χ_d is the primitive quadratic character attached to the fundamental discriminant d, and $L(s, \chi_d)$ is the associated Dirichlet L-function. Baker and Montgomery's motivation was to study real zeros of Fekete polynomials, and sign changes of quadratic character sums. Let $F_d(z) := \sum_{n=1}^{|d|-1} \chi_d(n) z^n$ be the Fekete polynomial associated to d. Fekete observed that if F_d does not vanish on $(0, 1)$ then $L(s, \chi_d) > 0$ for all $s \in (0, 1)$, which in particular implies Chowla's conjecture that $L(1/2, \chi_d) \neq 0$, and refutes the existence of a possible Siegel zero. This follows from the following identity, obtained by a familiar inverse Mellin

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transform

$$L(s,\chi_d)\Gamma(s)=\int_0^1 \frac{(-\log u)^{s-1}}{u}\frac{F_d(u)}{1-u^d}du, \text{ for } \mathrm{Re}(s)>0. \tag{1.1}$$

Fekete conjectured that F_d does not vanish on $(0,1)$ if $|d|$ is large enough, but this was disproved shortly afterwards by Pólya [14], for a positive proportion of fundamental discriminants d. In [1], Baker and Montgomery proved that Fekete's hypothesis is false for 100% of fundamental discriminants. In fact, they proved the stronger result that for any fixed positive integer K, F_d has at least K zeros in $(0,1)$ for almost all fundamental discriminants d. Baker and Montgomery's approach consists in relating zeros of F_d on $(0,1)$ to sign changes of $\frac{L'}{L}(s,\chi_d)$ on $(1/2,1)$ via the following identity which is obtained from (1.1) by differentiating with respect to s:

$$L(s,\chi_d)\Gamma(s)\left(\frac{L'(s,\chi_d)}{L(s,\chi_d)}+\frac{\Gamma'(s)}{\Gamma(s)}\right)=\int_0^\infty F_d(e^{-t})(1-e^{-|d|t})^{-1}t^{s-1}(\log t)dt. \tag{1.2}$$

Indeed, if the left-hand side of (1.2) has K sign changes in $(1/2,1)$ (which implies in particular that $L'(s,\chi_d)$ has K zeros in this interval) then F_d has at least K zeros on $(0,1)$ by a lemma of a real analysis (see Lemma 4 of [1]).

Let $R_d(\sigma_1,\sigma_2)$ be the number of real zeros of $L'(s,\chi_d)$ on the interval $[\sigma_1,\sigma_2]$. Based on a heuristic argument inspired by their construction, Baker and Montgomery made the following conjecture.

Conjecture 1.1 ([1], Baker-Montgomery). *For almost all fundamental discriminants d, we have*

$$R_d\left(\frac{1}{2},1\right)\asymp \log\log|d|.$$

In [11], Klurman, Lamzouri, and Munsch proved that for almost all fundamental discriminants d we have

$$R_d\left(\frac{1}{2},1\right)\gg \frac{\log\log|d|}{\log_4|d|}, \tag{1.3}$$

where here and throughout $\log_k$ denotes the k-th iterate of the natural logarithm function. This comes close of establishing the lower bound in Conjecture 1.1.

Baker and Montgomery [1] (and later Conrey, Granville, Poonen, and Soundararajan [4]) made a similar conjecture about the number of real zeros of F_d on $(0,1)$, predicting that it should be $\asymp \log\log|d|$ for almost all d. Klurman, Lamzouri, and Munsch [11] established an analogous "localized" version of the lower bound (1.3) in this case, using appropriate variants of (1.3) concerning oscillations of $L'(s,\chi_d)$, coupled with a concentration result for the distribution of $L(s,\chi_d)$ in the vicinity of 1/2. However, the only partial result towards the conjectured upper bound for the number of real zeros of F_d was established in [11] and states that for at least $x^{1-\varepsilon}$ fundamental discriminants $|d|\leqslant x$, F_d has at most $O(x^{1/4+\varepsilon})$ zeros in $(0,1)$. This breaks the $O(\sqrt{x})$ bound which holds for all Littlewood polynomials by a result of Borwein, Erdélyi, and Kós [3], but is very far from the conjectured $\log\log x$ bound.

In this paper, we focus on the upper bound in Conjecture 1.1. For convenience, as in previous works on the moments and non-vanishing of $L(1/2, \chi_d)$, we restrict the modulus d to be of the form $8m$ where m is squarefree and odd. However, our methods would apply to fundamental discriminants in any fixed arithmetic progression. Here and throughout we define

$$\mathcal{D}(x) := \{d = 8m : m \text{ is squarefree and odd, and } x/2 \leqslant m \leqslant x\}.$$

Note that $|\mathcal{D}(x)| \asymp x$.

The real zeros of $L'(s, \chi_d)$ produced by the authors of [11] to prove (1.3) all lie in the interval $[1/2 + 1/(\log x)^{1/5}, 1]$. More precisely they showed that

$$R_d\left(\frac{1}{2} + \frac{1}{(\log x)^{1/5}}, 1\right) \gg \frac{\log\log x}{\log_4 x}, \tag{1.4}$$

for almost all[1] $d \in \mathcal{D}(x)$. The exponent of $\log x$ was not optimized in [11] (since this was not needed to establish (1.3)), but one can probably push their method to produce zeros in the interval $[1/2 + 1/(\log x)^{1/2}, 1/2 + 1/(\log x)^{1/5}]$ for almost all $d \in \mathcal{D}(x)$. Our main result shows that one can control the number of real zeros of $L'(s, \chi_d)$, almost getting the upper bound predicted by Conjecture 1.1, in a much larger interval, which we believe to be the limit of our method.

Theorem 1.2. *Let $\nu(x) \to \infty$ as $x \to \infty$. For almost all $d \in \mathcal{D}(x)$ we have*

$$R_d\left(\frac{1}{2} + \frac{\nu(x)}{\log x}, 1\right) \ll (\log\log x)(\log\log\log x).$$

Our approach relies on information about the distribution of values of the logarithmic derivative $-\frac{L'}{L}(s, \chi_d)$ at points s on the interval $[1/2 + \nu(x)/\log x, 1]$. This makes it unlikely to prove results to the left of $1/2 + c/\log x$ (where c is a positive constant) without some unproven hypothesis on the zeros of $L(s, \chi_d)$. Indeed, in the case of the Riemann zeta function, Goldston, Gonek, and Montgomery [7] showed (assuming the Riemann Hypothesis) that the second moment of $\frac{\zeta'}{\zeta}(\sigma + it)$ as t varies in $[T, 2T]$, and σ lies in the range $(\log T)^2/T < \sigma - 1/2 \ll 1/\log T$, is ultimately connected to correlations of the zeros of the Riemann zeta-function.

Assuming the following mild hypothesis on the low lying zeros of $L(s, \chi_d)$, for almost all d, we fully resolve Conjecture 1.1, up to the factor $\log\log\log |d|$. We also show that most of the real zeros of $L'(s, \chi_d)$ lie away from $1/2$, for almost all $d \in \mathcal{D}(x)$.

[1]Here and throughout, we say that almost all $d \in \mathcal{D}(x)$ have the property P if $|\{d \in \mathcal{D}(x) \colon d \text{ has property } P\}| \sim |\mathcal{D}(x)|$ as $x \to \infty$.

Hypothesis L_d**.** *Let $d \in \mathcal{D}(x)$. There exists a positive function ν such that $\nu(t) \to \infty$ and $\nu(t) \leqslant (\log\log t)^{1/5}$ as $t \to \infty$, for which $L(z, \chi_d)$ has no zeros inside the disc of center $1/2 + \nu(x)/\log x$ and radius $\nu(x)/\log x + 1/(\nu(x)^3 \log x)$.*

Theorem 1.3. *Let $\mathcal{D}_0(x)$ be the set of fundamental discriminants $d \in \mathcal{D}(x)$ for which Hypothesis L_d holds with function ν. Then there exists a subset $\mathcal{D}_1(x) \subset \mathcal{D}_0(x)$ such that $|\mathcal{D}_0(x) \setminus \mathcal{D}_1(x)| \ll x\sqrt{(\log \nu(x))/\nu(x)}$ and for all $d \in \mathcal{D}_1(x)$ we have*

$$R_d\left(\frac{1}{2}, 1\right) \ll (\log\log x)(\log\log\log x), \tag{1.5}$$

and

$$R_d\left(\frac{1}{2}, \frac{1}{2} + \frac{\nu(x)}{\log x}\right) = o\left(R_d\left(\frac{1}{2} + \frac{\nu(x)}{\log x}, 1\right)\right) \quad \text{as } x \to \infty. \tag{1.6}$$

An immediate consequence of this theorem is the following corollary.

Corollary 1.4. *Assume that Hypothesis L_d holds with function ν for almost all fundamental discriminants $d \in \mathcal{D}(x)$. Then for almost all $d \in \mathcal{D}(x)$ we have*

$$R_d\left(\frac{1}{2}, 1\right) \ll (\log\log x)(\log\log\log x),$$

and

$$R_d\left(\frac{1}{2}, \frac{1}{2} + \frac{\nu(x)}{\log x}\right) = o\left(R_d\left(\frac{1}{2} + \frac{\nu(x)}{\log x}, 1\right)\right) \quad \text{as } x \to \infty.$$

Note that Hypothesis L_d implies the non-vanishing of $L(1/2, \chi_d)$, which is not unconditionally known to hold for almost all d. The best result in this direction is due to Soundararajan [19] who showed that $L(1/2, \chi_d) \neq 0$ for at least $7/8$ of the fundamental discriminants $d \in \mathcal{D}(x)$. Assuming the Generalized Riemann Hypothesis (GRH) for $L(s, \chi_d)$, the Hypothesis L_d is equivalent to the non-vanishing of $L(s, \chi_d)$ on the vertical segment $\{1/2 + it, |t| \leqslant \eta\}$, where $\eta \asymp 1/(\nu(x) \log x)$. Since the conductor of our family is $\asymp x$, the average spacing of the zeros of $L(s, \chi_d)$ is $\asymp 1/\log x$, and hence we expect that Hypothesis L_d holds for almost all $d \in \mathcal{D}(x)$ since $\nu(x) \to \infty$. In fact, this follows from GRH together with the following assumption:

Low Lying Zeros Hypothesis (LLZ). *Let $\nu(x) \to \infty$ as $x \to \infty$. For a fundamental discriminant d, let $\gamma_{min}(d) = \min\{|\gamma| : L(\beta + i\gamma, \chi_d) = 0, \text{ and } 0 < \beta < 1\}$. Then we have*

$$\lim_{x\to\infty} \frac{1}{|\mathcal{D}(x)|} \# \left\{ d \in \mathcal{D}(x) : \gamma_{min}(d) \leqslant \frac{1}{\nu(x) \log x} \right\} = 0.$$

Hypothesis LLZ was used by Hough [8] to prove a conjecture of Keating and Snaith [10], which is an analogue of Selberg's central limit theorem for the distribution of $\log L(1/2, \chi_d)$ as d varies in $\mathcal{D}(x)$. A somewhat similar assumption (on the gaps between consecutive zeros) was used by Bombieri and Hejhal [2] (in addition to the GRH) to show that 100% of the zeros of a linear combination of primitive L-functions (satisfying certain natural conditions) lie on the critical line. We should also note that assuming GRH, Hypothesis LLZ follows from the one level density conjecture of Katz and Sarnak [9], which predicts that

$$\lim_{x\to\infty} \frac{1}{|\mathcal{D}(x)|} \sum_{d\in\mathcal{D}(x)} \sum_{\substack{\rho=1/2+i\gamma\\ L(\rho,\chi_d)=0}} \phi\left(\frac{\gamma \log x}{2\pi}\right) = \int_{-\infty}^{\infty} \phi(u)\left(1-\frac{\sin(2\pi u)}{2\pi u}\right) du, \tag{1.7}$$

for any real even Schwartz class test function, whose Fourier transform has compact support. This is known assuming GRH if $\widehat{\phi}$ has support in $(-2,2)$ by the work of Özlük and Snyder [13].

Thus, in summary one can replace the hypothesis in Corollary 1.4 by GRH and LLZ, or by GRH and the one level density conjecture (1.7). Finally, we should note that conditionally on GRH, it follows from the work of Özlük and Snyder (see the proof of [13, Corollary 3]) that the Hypothesis L_d holds for at least $7/8$ fundamental discriminants.

Notation. We will use standard notation in this paper. However, for the convenience of readers, we would like to highlight a few of them. Expressions of the form $f(x)=O(g(x))$, $f(x)\ll g(x)$, and $g(x)\gg f(x)$ signify that $|f(x)|\leqslant C|g(x)|$ for all sufficiently large x, where $C>0$ is an absolute constant. A subscript of the form $\ll_A$ means the implied constant may depend on the parameter A. The notation $f(x)\asymp g(x)$ indicates that $f(x)\ll g(x)\ll f(x)$. Next, we write $f(x)=o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x)=0$.

Organization of the paper. The paper is organized as follows. In Section 2 we prove several basic mean value estimates with quadratic characters. In Section 3 we use ideas of Selberg and zero density estimates to approximate $-\frac{L'}{L}(s,\chi_d)$ by short Dirichlet polynomials, for almost all $d\in\mathcal{D}(x)$, once $\mathrm{Re}(s)\geqslant 1/2+\nu(x)/\log x$. In Section 4 we establish a bound for the discrepancy between the distribution of $-\frac{L'}{L}(s,\chi_d)$ (normalized by $1/(s-1/2)$) and that of a corresponding random model, uniformly in the range $1/2+\nu(x)/\log x\leqslant s\leqslant 1$. In Section 5, we use Hypothesis L_d to bound the moments of $-\frac{L'}{L}(s,\chi_d)$ near the central point $1/2$. Finally, Section 6 is devoted to the proofs of Theorems 1.2 and 1.3.

2. Mean values of Dirichlet polynomials with quadratic characters

In this section we gather together several basic mean value estimates with quadratic characters. The first is an "orthogonality relation" for the family $\mathcal{D}(x)$.

Lemma 2.1. *For all $n\leqslant x$ we have*

$$\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)} \chi_d(n) = \begin{cases} \displaystyle\prod_{\substack{p\mid n\\ p>2}} \left(\frac{p}{p+1}\right) + O(x^{-1/5}) & \text{if } n \text{ is a square,}\\ O(x^{-1/5}) & \text{otherwise.}\end{cases} \tag{2.1}$$

Proof. This is a special case of Lemma 2.3 of [8], upon taking $\delta = 1$ and choosing $\gamma(\delta) = 1/5$ therein, which is admissible. $\square$

Let $\{\mathbb{X}(p)\}_{p \text{ prime}}$ be a sequence of independent random variables defined as: $\mathbb{X}(2) = 0$; and for $p > 2$, $\mathbb{X}(p)$ takes the values $\{-1, 0, 1\}$ with probabilities

$$\mathbb{P}\big(\mathbb{X}(p) = 1\big) = \mathbb{P}\big(\mathbb{X}(p) = -1\big) = \frac{p}{2(p+1)}, \quad \text{and} \quad \mathbb{P}\big(\mathbb{X}(p) = 0\big) = \frac{1}{p+1}.$$

We extend the $\mathbb{X}(p)$ multiplicatively by setting $\mathbb{X}(n) = \mathbb{X}(p_1)^{a_1} \cdots \mathbb{X}(p_k)^{a_k}$ if n has the prime factorization $n = p_1^{a_k} \cdots p_k^{a_k}$. Then one can write (2.1) as

$$\frac{1}{|\mathcal{D}(x)|} \sum_{d \in \mathcal{D}(x)} \chi_d(n) = \mathbb{E}(\mathbb{X}(n)) + O(x^{-1/5}), \tag{2.2}$$

for all $n \leqslant x$. As a consequence, we establish the following lemma.

Lemma 2.2. *Let $C > 0$ be a fixed constant. Let $b(n)$ be real numbers such that $|b(n)| \leqslant C$ for all $n \geqslant 1$. Then uniformly for $x \geqslant Y \geqslant 2$ and all positive integers $k \leqslant \log x / \log Y$ we have*

$$\frac{1}{|\mathcal{D}(x)|} \sum_{d \in \mathcal{D}(x)} \left(\sum_{n \leqslant Y} b(n) \chi_d(n) \right)^k = \mathbb{E}\left[\left(\sum_{n \leqslant Y} b(n) \mathbb{X}(n) \right)^k \right] + O\big(x^{-1/5} (CY)^k\big),$$

where the implicit constant in the error term is absolute.

Proof. We have

$$\begin{aligned} \frac{1}{|\mathcal{D}(x)|} \sum_{d \in \mathcal{D}(x)} \left(\sum_{n \leqslant Y} b(n) \chi_d(n) \right)^k &= \frac{1}{|\mathcal{D}(x)|} \sum_{d \in \mathcal{D}(x)} \Bigg(\sum_{n_1, n_2, \ldots, n_k \leqslant Y} \prod_{i=1}^k b(n_i) \chi_d(n_i) \Bigg) \\ &= \sum_{n_1, n_2, \ldots, n_k \leqslant Y} \prod_{i=1}^k b(n_i) \frac{1}{|\mathcal{D}(x)|} \sum_{d \in \mathcal{D}(x)} \chi_d\left(\prod_{i=1}^k n_i \right). \end{aligned}$$

By (2.2) and the fact that $|b(n)| \leqslant C$ for all $n \geqslant 1$, this sum equals

$$\begin{aligned} &\sum_{n_1, \ldots, n_k \leqslant Y} \prod_{i=1}^k b(n_i) \mathbb{E}\left(\prod_{i=1}^k \mathbb{X}(n_i) \right) + O\left(x^{-1/5} (CY)^k \right) \\ &= \mathbb{E}\left[\left(\sum_{n \leqslant Y} b(n) \mathbb{X}(n) \right)^k \right] + O\left(x^{-1/5} (CY)^k \right), \end{aligned}$$

as desired. $\square$

We end this section by proving upper bounds for the moments of certain quadratic character sums supported on prime powers.

Lemma 2.3. *Let $\{a(n)\}_{n\geqslant 1}$ be a sequence of complex numbers such that $|a(n)| \leqslant 1$ for all n. Let x be large and $10 \leqslant y \leqslant z$ be real numbers. Then for all positive integers k such that $k \leqslant \log x/(10 \log z)$ we have*

$$\frac{1}{|\mathcal{D}(x)|} \sum_{d\in\mathcal{D}(x)} \Big| \sum_{y\leqslant n\leqslant z} \frac{a(n)\Lambda(n)\chi_d(n)}{\sqrt{n}} \Big|^{2k}$$
$$\ll \left(20k \sum_{y\leqslant p\leqslant z} \frac{|a(p)|^2(\log p)^2}{p}\right)^k + \left(3 \sum_{\sqrt{y}\leqslant p\leqslant\sqrt{z}} \frac{|a(p^2)|\log p}{p}\right)^{2k} + \left(c_0 y^{-1/3}\right)^k,$$

for some positive constant c_0.

Moreover, the same bound holds for $\mathbb{E}\left(\Big| \sum_{y\leqslant n\leqslant z} \frac{a(n)\Lambda(n)\mathbb{X}(n)}{\sqrt{n}} \Big|^{2k}\right)$, for all integers $k \geqslant 1$.

Proof. We shall only prove the bound for the sum over d, since the proof of the corresponding bound for the random model is similar and simpler. First, we have

$$\sum_{y\leqslant n\leqslant z} \frac{a(n)\Lambda(n)\chi_d(n)}{\sqrt{n}} = \sum_{y\leqslant p\leqslant z} \frac{a(p)(\log p)\chi_d(p)}{\sqrt{p}} + \sum_{\substack{\sqrt{y}\leqslant p\leqslant\sqrt{z}\\ p\nmid d}} \frac{a(p^2)\log p}{p} + O\left(y^{-1/6}\right),$$

since the contribution of prime powers p^k with $k \geqslant 3$ is

$$\ll \sum_{k\geqslant 3} \sum_{p^k\geqslant y} \frac{\log p}{p^{k/2}} \ll y^{-1/6}.$$

Now, using the basic inequality $|a+b+c|^k \leqslant 3^k(|a|^k + |b|^k + |c|^k)$ (which is valid for all real numbers a, b, c and positive integers k), we obtain

$$\frac{1}{|\mathcal{D}(x)|} \sum_{d\in\mathcal{D}(x)} \Big| \sum_{y\leqslant n\leqslant z} \frac{a(n)\Lambda(n)\chi_d(n)}{\sqrt{n}} \Big|^{2k}$$
$$\ll \frac{9^k}{|\mathcal{D}(x)|} \sum_{d\in\mathcal{D}(x)} \Big| \sum_{y\leqslant p\leqslant z} \frac{a(p)(\log p)\chi_d(p)}{\sqrt{p}} \Big|^{2k} + \left(3 \sum_{\sqrt{y}\leqslant p\leqslant\sqrt{z}} \frac{|a(p^2)|\log p}{p}\right)^{2k} + \left(c_0 y^{-1/3}\right)^k,$$

for some positive constant c_0. Furthermore, we have

(2.3)
$$\sum_{d\in\mathcal{D}(x)} \Big| \sum_{y\leqslant p\leqslant z} \frac{a(p)(\log p)\chi_d(p)}{\sqrt{p}} \Big|^{2k}$$
$$= \sum_{d\in\mathcal{D}(x)} \sum_{y\leqslant p_1,\dots,p_{2k}\leqslant z} \frac{a(p_1)\cdots a(p_k)\overline{a(p_{k+1})}\cdots\overline{a(p_{2k})}(\log p_1)\cdots(\log p_{2k})\chi_d(p_1\cdots p_{2k})}{(p_1p_2\cdots p_{2k})^{1/2}}.$$

The diagonal terms $p_1 \cdots p_{2k} = \square$ contribute

$$\ll x\frac{(2k)!}{2^k k!}\left(\sum_{y\leqslant p\leqslant z} \frac{|a(p)|^2(\log p)^2}{p}\right)^k \leqslant x\left(2k \sum_{y\leqslant p\leqslant z} \frac{|a(p)|^2(\log p)^2}{p}\right)^k.$$

On the other hand, if $p_1p_2...p_{2k} \neq \square$ and $p_i \leqslant z$ then Lemma 2.1 gives

$$\sum_{d\in\mathcal{D}(x)} \chi_d(p_1p_2...p_{2k}) \ll x^{4/5},$$

since $p_1p_2\cdots p_{2k} \leqslant z^{2k} \leqslant x$. This implies that the contribution of these terms to (2.3) is

$$\ll x^{4/5}\left(\sum_{y\leqslant p\leqslant z}\frac{\log p}{\sqrt{p}}\right)^{2k} \ll x^{19/20},$$

by the prime number theorem, and using our assumption on z. Combining the above estimates completes the proof. $\square$

3. Approximating $-\frac{L'}{L}(s,\chi_d)$ by short Dirichlet polynomials

To shorten our notation, we define

$$\mathcal{L}_d(s) := -\frac{L'}{L}(s,\chi_d).$$

The goal of this section is to approximate $\mathcal{L}_d(s)$ by short Dirichlet polynomials. In order to do that, we will use ideas of Selberg from [16] and [17]. For $d\in\mathcal{D}(x)$ and $2\leqslant y\leqslant x$, we let

$$\sigma_{y,\,d} := \frac{1}{2} + 2\max_{\mathcal{G}_{y,d}}\left(\beta-\frac{1}{2}, \frac{2}{\log y}\right),$$

where

$$\mathcal{G}_{y,\,d} := \{\rho=\beta+i\gamma\colon L(\rho,\chi_d)=0, |\gamma-t|\leqslant y^{3(\beta-1/2)}/\log y\}.$$

Next, for $2\leqslant y\leqslant x$, we set

$$\Lambda_{y,\,d}(n) := \Lambda(n)\chi_d(n)w_y(n), \tag{3.1}$$

where

$$\omega_y(n) = \begin{cases} 1 & \text{if} \quad n\leqslant y,\\ \frac{\log^2(y^3/n)-2\log^2(y^2/n)}{2\log^2 y} & \text{if} \quad y\leqslant n\leqslant y^2,\\ \frac{\log^2(y^3/n)}{2\log^2 y} & \text{if} \quad y^2\leqslant n\leqslant y^3,\\ 0 & \text{if} \quad n>y^3.\end{cases}$$

Note that $0\leqslant w_y(n)\leqslant 1$ for all n. We shall use the following lemma due to Selberg [16].

Lemma 3.1. *Let $d\in\mathcal{D}(x)$ and $10\leqslant y\leqslant x$. We have*

$$\sum_{\rho}\frac{\sigma_{y,d}-\frac{1}{2}}{|\sigma_{y,d}-\rho|^2} \ll \log d + \left|\sum_{n\leqslant y^3}\frac{\Lambda_{y,d}(n)}{n^{\sigma_{y,d}}}\right|, \tag{3.2}$$

where the sum is over the non-trivial zeros of $L(s,\chi_d)$. Moreover, for $s=\sigma+it$ with $\sigma\geqslant\sigma_{y,\,d}$ and $|t|\leqslant 1$, we have

$$\mathcal{L}_d(s) = \sum_{n\leqslant y^3}\frac{\Lambda_{y,\,d}(n)}{n^s} + O\left(y^{(1/2-\sigma)/2}\left|\sum_{n\leqslant y^3}\frac{\Lambda_{y,\,d}(n)}{n^{\sigma_{y,d}+it}}\right| + y^{(1/2-\sigma)/2}\log d\right). \tag{3.3}$$

Proof. Selberg proved these estimates for the Riemann zeta function in pages 22-26 of [16]. The analogous estimates for Dirichlet L-functions hold mutatis mutandis (see Lemma 2.6 of [8]). $\square$

We now record the following zero density estimates for the family $\{L(s,\chi_d)\}_{d\in\mathcal{D}(x)}$ near the critical line, which follows from the work of Conrey and Soundararajan [5].

Lemma 3.2 (Theorem 2.7 of [8])**.** *Let x be large and $\delta>0$ be a small positive constant. There exists $\theta=\theta(\delta)>0$ such that uniformly in $1/2+4/\log x<\sigma<1$ and $10/\log x<T<x^\delta$ we have*

$$\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\#\Big\{\rho=\beta+i\gamma\colon L(\rho,\chi_d)=0,\beta>\sigma,|\gamma|\leqslant T\Big\}\ll x^{-\theta(\sigma-1/2)}T\log x.$$

Using this result we show that for almost all $d\in\mathcal{D}(x)$ we have $\sigma_{y,d}=1/2+4/\log y$ if $\log x/\log y\to\infty$. This will allow us to conclude that for complex numbers z in the range $1/2+4/\log y\leqslant \mathrm{Re}(z)\leqslant 1$ and $|\mathrm{Im}(z)|\leqslant 1$, the approximation (3.3) holds for almost all $d\in\mathcal{D}(x)$.

Lemma 3.3. *Let x be large and $10\leqslant y\leqslant x$ be such that $\log x/\log y\to\infty$ as $x\to\infty$. Define*

$$\mathcal{D}_y(x):=\{d\in\mathcal{D}(x)\colon \sigma_{y,d}=1/2+4/\log y\}.$$

Then, there exists a constant $C_0>0$ such that

$$\big|\mathcal{D}(x)\setminus\mathcal{D}_y(x)\big|\ll x\exp\left(-C_0\frac{\log x}{\log y}\right).$$

Proof. Let $\sigma=1/2+4/\log y$. By the definition of $\sigma_{y,\,d}$, if for $d\in\mathcal{D}(x)$ we have $\sigma_{y,d}>\sigma$, then there exists $\rho_0=\beta_0+i\gamma_0$ such that $L(\rho_0,\chi_d)=0$,

$$\beta_0>\frac{1}{2}+\frac{2}{\log y},\quad \text{and}\quad |\gamma_0|\leqslant\frac{y^{3(\beta_0-1/2)}}{\log y}.$$

Write $\sigma':=1/2+2/\log y$. Then, we have

$$\begin{aligned}&\frac{1}{|\mathcal{D}(x)|}\#\{d\in\mathcal{D}(x)\colon\sigma_{y,d}>\sigma\}\\ &\ll\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\#\{\exists\,\rho=\beta+i\gamma\colon L(\rho,\chi_d)=0,\ \beta>\sigma',\ |\gamma|\leqslant 2y^{3(\beta-1/2)}/\log y\}\\ &\ll\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\sum_{j=2}^{\log y}\#\{\exists\,\rho=\beta+i\gamma\colon L(\rho,\chi_d)=0,\ \beta-1/2>j/\log y,\ |\gamma|\leqslant 2e^{3(j+1)}/\log y\}.\end{aligned}$$

Applying Lemma 3.2, we see that the above quantity is

$$\ll\sum_{j=4}^{\log y}x^{-\theta j/\log y}e^{3(j+1)}\frac{\log x}{\log y}\ll\frac{\log x}{\log y}e^{-\theta\log x/\log y}\ll e^{-\frac{\theta}{2}\log x/\log y},$$

as desired. $\square$

For a complex number z with $\mathrm{Re}(z) > 1/2$, we define

$$V_z := \frac{1}{\mathrm{Re}(z) - 1/2}.$$

We also set

$$\mathcal{L}_{\mathrm{rand}}(z) := \sum_{n=1}^{\infty} \frac{\Lambda(n)\mathbb{X}(n)}{n^z}.$$

Note that this series converges almost surely in the half plane $\mathrm{Re}(z) > 1/2$ by Kolmogorov's three series theorem. We end this section by proving upper bounds for the moments of $\mathcal{L}_d(z)$ and $\mathcal{L}_{\mathrm{rand}}(z)$ when $(\mathrm{Re}(z) - 1/2)\log x \to \infty$ and $\mathrm{Im}(z)$ is bounded.

Lemma 3.4. *Let x be large and $\nu(x) \to \infty$ as $x \to \infty$. Let z be a complex number such that $1/2 + \nu(x)/\log x \leqslant \mathrm{Re}(z) \leqslant 1$ and $|\mathrm{Im}(z)| \leqslant 1$. Let $y = \exp\big(10V_z \log(\log x/V_z)\big)$, and $k \leqslant (\log x)/(30 \log y)$ be a positive integer. Define*

$$\mathcal{D}_z(x) := \{d \in \mathcal{D}(x) \colon \sigma_{y,d} = 1/2 + 4/\log y\}.$$

Then, there exist constants $C_1, C_2 > 0$ such that

$$\sum_{d \in \mathcal{D}_z(x)} |\mathcal{L}_d(z)|^{2k} \ll x(C_1 k V_z^2)^k \quad \text{and} \quad \mathbb{E}(|\mathcal{L}_{rand}(z)|^{2k}) \ll (C_2 k V_z^2)^k.$$

Proof. We will only establish the desired bound for the $2k$-th moment of $\mathcal{L}_d(z)$, since the corresponding bound for the random model follows along the same lines. If $d \in \mathcal{D}_z(x)$ and $\sigma_{y,d} \leqslant \mathrm{Re}(z) \leqslant 1$, then by Lemma 3.1 we have

$$\mathcal{L}_d(z) = \sum_{n \leqslant y^3} \frac{\Lambda_{y,d}(n)}{n^z} + O\left(y^{-1/(2V_z)} \left| \sum_{n \leqslant y^3} \frac{\Lambda_{y,d}(n)}{n^{\sigma_{y,d}+it}} \right| + y^{-1/(2V_z)} \log d\right),$$

where $t = \mathrm{Im}(z)$. Therefore, using the basic inequality $|a+b+c|^{2k} \leqslant 3^{2k}(|a|^{2k} + |b|^{2k} + |c|^{2k})$ we infer from Lemma 2.3 that

(3.4)
$$\begin{aligned}
\sum_{d \in \mathcal{D}_z(x)} |\mathcal{L}_d(z)|^{2k} &\ll 9^k \sum_{d \in \mathcal{D}_z(x)} \left| \sum_{n \leqslant y^3} \frac{\Lambda_{y,d}(n)}{n^z} \right|^{2k} + 9^k y^{-k/V_z} \sum_{d \in \mathcal{D}_z(x)} \left| \sum_{n \leqslant y^3} \frac{\Lambda_{y,d}(n)}{n^{\sigma_{y,d}+it}} \right|^{2k} \\
&\quad + 9^k x y^{-k/V_z} (\log x)^{2k} \\
&\ll x\left(200k \sum_{p \leqslant y^3} \frac{(\log p)^2}{p^{2\mathrm{Re}(z)}}\right)^k + x\left(30 \sum_{p \leqslant y^{3/2}} \frac{\log p}{p^{2\mathrm{Re}(z)}}\right)^{2k} + 9^k x y^{-k/V_z} (\log x)^{2k} \\
&\quad + x y^{-k/V_z} \left(200k \sum_{p \leqslant y^3} \frac{(\log p)^2}{p}\right)^k + x y^{-k/V_z} \left(30 \sum_{p \leqslant y^{3/2}} \frac{\log p}{p}\right)^{2k} \\
&\ll x(C_1 k V_z^2)^k,
\end{aligned}$$

for some positive constant C_1, by our assumptions on z and k, and since

$$\sum_p \frac{(\log p)^2}{p^{2\mathrm{Re}(z)}} \asymp V_z^2 \quad \text{and} \quad \sum_p \frac{\log p}{p^{2\mathrm{Re}(z)}} \asymp V_z, \tag{3.5}$$

by partial summation and the prime number theorem. □

4. A discrepancy bound for the distribution of $\mathcal{L}_d$

Throughout this section we let ν be a positive function such that $\nu(x) \to \infty$ as $x \to \infty$. Let x be large and z be a real number such that $1/2 + \nu(x)/\log x \leqslant z \leqslant 1$. Put $y = \exp\big(20V_z \log(\log x/V_z)\big)$, and define

$$\mathcal{D}_z(x) := \{d \in \mathcal{D}(x) \colon \sigma_{y,d} = 1/2 + 4/\log y\}.$$

Then $|\mathcal{D}_z(x)| \sim |\mathcal{D}(x)|$ by Lemma 3.3. Moreover, for any real number u, we define

$$\Phi_{x,z}(u) := \frac{1}{|\mathcal{D}_z(x)|} \sum_{d \in \mathcal{D}_z(x)} \exp\left(2\pi i u \frac{\mathcal{L}_d(z)}{V_z}\right),$$

and

$$\Phi_{\mathrm{rand},z}(u) = \mathbb{E}\left[\exp\left(2\pi i u \frac{\mathcal{L}_{\mathrm{rand}}(z)}{V_z}\right)\right].$$

Furthermore, we define the "discrepancy" between the distribution functions of $\mathcal{L}_d(z)/V_z$ and $\mathcal{L}_{\mathrm{rand}}(z)/V_z$ as

$$D(z) := \sup_{t \in \mathbb{R}} \left| \frac{1}{|\mathcal{D}_z(x)|} |\{d \in \mathcal{D}_z(x) \colon \mathcal{L}_d(z)/V_z \leqslant t\}| - \mathbb{P}\big(\mathcal{L}_{\mathrm{rand}}(z)/V_z \leqslant t\big) \right|.$$

The goal of this section is to prove the following theorem

Theorem 4.1. *Let $1/2 + \nu(x)/\log x \leqslant z \leqslant 1$ with $\nu(x) \to \infty$ as $x \to \infty$. Then, we have*

$$D(z) \ll \left(\frac{V_z \log(\log x/V_z)}{\log x}\right)^{1/2}.$$

We start by proving the following lemma.

Lemma 4.2. *Let x, ν, z and y be as above. Then, for all real numbers u such that $(V_z/\log x)^2 \leqslant |u| \leqslant (\log x/V_z)^5$, we have*

$$\Phi_{x,z}(u) = \frac{1}{|\mathcal{D}(x)|} \sum_{d \in \mathcal{D}(x)} \exp\left(2\pi i \frac{u}{V_z} \sum_{n \leqslant y} \frac{\Lambda(n)\chi_d(n)}{n^z}\right) + O\left(|u| \frac{V_z^5}{(\log x)^5}\right).$$

Proof. By Lemma 3.1, we have

$$\sum_{d \in \mathcal{D}_z(x)} \exp\left(2\pi i u \frac{\mathcal{L}_d(z)}{V_z}\right) = \sum_{d \in \mathcal{D}_z(x)} \exp\left(2\pi i \frac{u}{V_z} \sum_{n \leqslant y^3} \frac{\Lambda_{y,d}(n)}{n^z}\right) + E_1,$$

where

$$E_1 \ll \frac{|u|}{V_z} y^{-1/(2V_z)} \left(\sum_{d \in \mathcal{D}_z(x)} \left| \sum_{n \leqslant y^3} \frac{\Lambda_{y,\,d}(n)}{n^{\sigma_{y,d}}} \right| + x \log x \right).$$

By the Cauchy-Schwarz inequality and Lemma 2.3, we have

$$\sum_{d\in\mathcal{D}_z(x)}\left|\sum_{n\leqslant y^3}\frac{\Lambda_{y,\,d}(n)}{n^{\sigma_{y,d}}}\right| \leqslant x^{1/2}\Bigg(\sum_{d\in\mathcal{D}(x)}\left|\sum_{n\leqslant y^3}\frac{\Lambda_{y,\,d}(n)}{n^{\sigma_{y,d}}}\right|^2\Bigg)^{1/2}$$
$$\ll x\Bigg(\sum_{p\leqslant y^3}\frac{(\log p)^2}{p^{2\sigma_{y,d}}}\Bigg)^{1/2}+x\Bigg(\sum_{p\leqslant y^{3/2}}\frac{\log p}{p^{2\sigma_{y,d}}}\Bigg)\ll x\log x,$$

since $2\sigma_{y,d}>1$. Hence, we get

$$\sum_{d\in\mathcal{D}_z(x)}\exp\left(2\pi i u\frac{\mathcal{L}_d(z)}{V_z}\right)=\sum_{d\in\mathcal{D}_z(x)}\exp\left(2\pi i\frac{u}{V_z}\sum_{n\leqslant y^3}\frac{\Lambda_{y,d}(n)}{n^z}\right)+O\left(x|u|\Big(\frac{V_z}{\log x}\Big)^9\right),$$

since $y^{-1/V_z}=(V_z/\log x)^{20}$. Next, we write

$$\sum_{d\in\mathcal{D}_z(x)}\exp\left(2\pi i\frac{u}{V_z}\sum_{n\leqslant y^3}\frac{\Lambda_{y,d}(n)}{n^z}\right)=\sum_{d\in\mathcal{D}_z(x)}\exp\left(2\pi i\frac{u}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\right)+E_2,$$

where

$$E_2\ll\frac{|u|}{V_z}\sum_{d\in\mathcal{D}_z(x)}\left|\sum_{n\leqslant y^3}\frac{\Lambda_{y,d}(n)}{n^z}-\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\right|$$
$$\ll\frac{|u|}{V_z}\sum_{d\in\mathcal{D}(x)}\left|\sum_{y<n\leqslant y^3}\frac{\Lambda_{y,d}(n)}{n^z}\right|,$$

using the definition of $\Lambda_{y,\,d}$. Applying the Cauchy-Schwarz inequality and Lemma 2.3 we obtain

$$\sum_{d\in\mathcal{D}(x)}\left|\sum_{y<n\leqslant y^3}\frac{\Lambda_{y,d}(n)}{n^z}\right|\leqslant x^{1/2}\Bigg(\sum_{d\in\mathcal{D}(x)}\left|\sum_{y<n\leqslant y^3}\frac{\Lambda_{y,\,d}(n)}{n^z}\right|^2\Bigg)^{1/2}$$
$$\ll x\Bigg(\sum_{y<p\leqslant y^3}\frac{(\log p)^2}{p^{2z}}\Bigg)^{1/2}+x\Bigg(\sum_{\sqrt{y}<p\leqslant y^{3/2}}\frac{\log p}{p^{2z}}\Bigg)+xy^{-1/6}$$
$$\ll y^{-(z-1/2)/3}\log x\ll\frac{V_z^6}{(\log x)^5},$$

since

$$\sum_{p>y}\frac{(\log p)^2}{p^{2z}}\ll\frac{\log y}{y^{2z-1}(z-1/2)}+\frac{1}{(z-1/2)^2y^{2z-1}},\tag{4.1}$$

and

$$\sum_{p>\sqrt{y}}\frac{\log p}{p^{2z}}\ll\frac{1}{y^{z-1/2}(z-1/2)},\tag{4.2}$$

by partial summation and the prime number theorem. Finally, we note that

$$\frac{1}{|\mathcal{D}_z(x)|}\sum_{d\in\mathcal{D}_z(x)}\exp\left(2\pi i\frac{u}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\right)=\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\exp\left(2\pi i\frac{u}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\right)+E_3,$$

where

$$E_3 \ll \frac{|\mathcal{D}(x) \setminus \mathcal{D}_z(x)|}{|\mathcal{D}(x)|} \ll \exp\Big(-\theta \frac{\log x}{\log y}\Big),$$

by Lemma 3.3. Collecting the above estimates completes the proof. □

Proposition 4.3. *Let x, ν, z and y be as above. There exists a constant $c_1 > 0$ such that for all real numbers u with $(V_z/\log x)^2 \leqslant |u| \leqslant c_1\sqrt{\log x/(V_z \log(\log x/V_z))}$, we have*

$$\Phi_{x,z}(u) = \Phi_{\mathrm{rand},z}(u) + O\bigg(|u|\frac{V_z^4}{(\log x)^4}\bigg).$$

Proof. By Lemma 4.2, we have

$$\Phi_{x,z}(u) = \frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)} \exp\bigg(2\pi i \frac{u}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\bigg) + O\bigg(|u|\frac{V_z^5}{(\log x)^5}\bigg).$$

Next, we deal with the main term in the above expression. Let $N = \lfloor(\log x)/(50\log y)\rfloor$. By applying the Taylor expansion of $e^{2\pi i t}$ for real t, we see that

$$\begin{aligned}&\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)} \exp\bigg(2\pi i \frac{u}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\bigg)\\ &\qquad= \sum_{k=0}^{2N-1}\frac{(2\pi i u)^k}{V_z^k k!}\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\bigg(\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\bigg)^k + E_4,\end{aligned}$$

where

$$\begin{aligned}E_4 &\ll \frac{(2\pi u)^{2N}}{V_z^{2N}(2N)!}\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\bigg(\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\bigg)^{2N}\\ &\ll \frac{(2\pi u)^{2N}}{V_z^{2N}(2N)!}\cdot (c_2 N V_z^2)^N \ll (c_3 u^2/N)^N \ll e^{-N},\end{aligned}$$

for some positive constants c_2, c_3, where the second inequality follows by the same calculations leading to (3.4), and the third from Stirling's formula. Therefore,

$$\Phi_{x,z}(u) = \sum_{k=0}^{2N-1}\frac{(2\pi i u)^k}{V_z^k k!}\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\bigg(\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\bigg)^k + O\bigg(|u|\frac{V_z^5}{(\log x)^5}\bigg). \tag{4.3}$$

On the other hand, by Lemma 2.2, we have

$$\begin{aligned}&\bigg|\sum_{k=0}^{2N-1}\frac{(2\pi i u)^k}{V_z^k k!}\bigg(\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}\bigg(\sum_{n\leqslant y}\frac{\Lambda(n)\chi_d(n)}{n^z}\bigg)^k - \mathbb{E}\bigg(\sum_{n\leqslant y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}\bigg)^k\bigg)\bigg|\\ &\qquad\ll x^{-1/5}\sum_{k=0}^{2N-1}\bigg(\frac{c_4 u y}{V_z k}\bigg)^k \ll x^{-1/5} N y^{2N} \ll x^{-1/10},\end{aligned} \tag{4.4}$$

which is negligible. Here we have used our assumptions on u and N to bound the sum over k.

We now handle the characteristic function of the random model. Let $\mathcal{A}$ denote the event

$$\Bigg|\sum_{n>y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}\Bigg| \leqslant B := \frac{V_z^6}{(\log x)^5}.$$

Let κ be a positive integer to be chosen. Then, by Markov's inequality and Lemma 2.3 (letting $z\to\infty$ therein) we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{A}^c) &\leqslant \frac{1}{B^{2\kappa}}\mathbb{E}\Bigg|\sum_{n>y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}\Bigg|^{2\kappa}\\ &\ll \left(c_5\frac{\kappa}{B^2}\sum_{p>y}\frac{(\log p)^2}{p^{2z}}\right)^{\kappa}+\left(c_6\sum_{p>\sqrt{y}}\frac{\log p}{p^{2z}}\right)^{2\kappa} \ll \left(c_7\frac{\kappa V_z\log y}{B^2y^{2z-1}}\right)^{k},\end{aligned}$$

for some positive constants c_5, c_6 and c_7, where the last bound follows from (4.1) and (4.2). Choosing $\kappa = \lfloor B^2y^{2z-1}/(ec_7V_z\log y)\rfloor$ and using that $y^{2z-1} = (\log x)^{40}/V_z^{40}$ we deduce that

$$\mathbb{P}(\mathcal{A}^c) \ll e^{-\kappa} \ll \exp\left(-\frac{\log x}{V_z}\right).$$

Letting $\mathbf{1}_{\mathcal{A}}$ denote the indicator function of the event $\mathcal{A}$, we therefore get

$$\begin{aligned}\Phi_{\text{rand},z}(u) &= \mathbb{E}\left[\mathbf{1}_{\mathcal{A}}\cdot\exp\left(2\pi iu\frac{\mathcal{L}_{\text{rand}}(z)}{V_z}\right)\right]+O\left(\exp\left(-\frac{\log x}{V_z}\right)\right)\\ &= \mathbb{E}\left[\mathbf{1}_{\mathcal{A}}\cdot\exp\left(\frac{2\pi iu}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}+O\left(\frac{|u|V_z^5}{(\log x)^5}\right)\right)\right]+O\left(\exp\left(-\frac{\log x}{V_z}\right)\right)\\ &= \mathbb{E}\left[\exp\left(\frac{2\pi iu}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}\right)\right]+O\left(\frac{|u|V_z^5}{(\log x)^5}\right).\end{aligned}\tag{4.5}$$

Next, by the same argument leading to (4.3) together Lemma 2.3, we obtain

$$\mathbb{E}\left[\exp\left(\frac{2\pi iu}{V_z}\sum_{n\leqslant y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}\right)\right] = \sum_{k=0}^{2N-1}\frac{(2\pi iu)^k}{V_z^kk!}\mathbb{E}\left(\sum_{n\leqslant y}\frac{\Lambda(n)\mathbb{X}(n)}{n^z}\right)^k+O(e^{-N}).\tag{4.6}$$

Combining (4.3), (4.4), (4.5) and (4.6) completes the proof. □

Next, we show that the characteristic function of $\mathcal{L}_{\text{rand}}(z)/V_z$ decays exponentially on $\mathbb{R}$, uniformly in $1/2 < z \leqslant 1$.

Lemma 4.4. *Let $1/2 < z \leqslant 1$. Then, there exists an absolute constant $C_0 > 0$ such that for all $u \in \mathbb{R}$ we have*

$$\Phi_{\text{rand},z}(u) \ll \exp\left(-C_0\frac{|u|^{1/z}}{\log(|u|+1)^{2-1/z}}\right).$$

Proof. Let A be a suitably large constant. Since $|\Phi_{\text{rand},z}(u)| \leqslant 1$ for all real numbers u, we may assume that $|u| > A$. First, note that

$$\mathcal{L}_{\text{rand}}(z) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\mathbb{X}(n)}{n^z} = \sum_p \log p \sum_{k=1}^{\infty} \frac{\mathbb{X}(p)^k}{p^{kz}} = \sum_p \frac{\mathbb{X}(p)\log p}{p^z - \mathbb{X}(p)},$$

and hence

$$\Phi_{\text{rand},z}(u) = \prod_{p>2} \mathbb{E}\Bigg[\exp\left(2\pi i u \frac{\mathbb{X}(p)\log p}{V_z(p^z - \mathbb{X}(p))}\right)\Bigg],$$

since $\{\mathbb{X}(p)\}_{p \text{ prime}}$ are independent and $\mathbb{X}(2) = 0$. Now for any odd prime p, by Taylor's expansion, we have

$$\begin{aligned}
&\exp\left(2\pi i u \frac{\mathbb{X}(p)\log p}{V_z(p^z - \mathbb{X}(p))}\right) \\
&= 1 + 2\pi i u \frac{\mathbb{X}(p)\log p}{V_z(p^z - \mathbb{X}(p))} - 2\pi^2 u^2 \frac{\mathbb{X}(p)^2(\log p)^2}{V_z^2(p^z - \mathbb{X}(p))^2} + O\left(|u|^3 \frac{(\log p)^3}{V_z^3 p^{3z}}\right) \\
&= 1 + 2\pi i u \frac{\mathbb{X}(p)\log p}{V_z p^z} + 2\pi i u \frac{\mathbb{X}(p)^2\log p}{V_z p^{2z}} - 2\pi^2 u^2 \frac{\mathbb{X}(p)^2(\log p)^2}{V_z^2 p^{2z}} \\
&\quad + O\left(|u| \frac{\log p}{V_z p^{3z}} + |u|^3 \frac{(\log p)^3}{V_z^3 p^{3z}}\right).
\end{aligned}$$

Since $\mathbb{E}(\mathbb{X}(p)) = 0$ and $\mathbb{E}(\mathbb{X}(p)^2) = 1 - 1/(p+1)$ we get

$$\mathbb{E}\left[\exp\left(2\pi i u \frac{\mathbb{X}(p)\log p}{V_z(p^z - \mathbb{X}(p))}\right)\right] = 1 - 2\pi^2 u^2 \frac{(\log p)^2}{V_z^2 p^{2z}} + O\left(|u| \frac{\log p}{V_z p^{2z}} + |u|^3 \frac{(\log p)^3}{V_z^3 p^{3z}}\right).$$

Let

$$U = \max\left(e^{AV_z}, (A|u|\log|u|)^{1/z}\right).$$

Then we have

(4.7)

$$\begin{aligned}
|\Phi_{\text{rand},z}(u)| &\leqslant \prod_{p \geqslant U} \left|\mathbb{E}\Bigg[\exp\left(2\pi i u \frac{\mathbb{X}(p)\log p}{V_z(p^z - \mathbb{X}(p))}\right)\Bigg]\right| \\
&\leqslant \exp\left(-2\pi^2 \frac{u^2}{V_z^2} \sum_{p>U} \frac{(\log p)^2}{p^{2z}} + O\left(\frac{|u|}{V_z} \sum_{p>U} \frac{\log p}{p^{2z}} + \frac{|u|^3}{V_z^3} \sum_{p>U} \frac{(\log p)^3}{p^{3z}}\right)\right).
\end{aligned}$$

Since $U \geqslant e^{AV_z}$ (and A is suitably large) it follows by partial summation and the prime number theorem that

$$\sum_{p>U} \frac{(\log p)^2}{p^{2z}} \asymp \frac{V_z \log U}{U^{2z-1}}, \quad \sum_{p>U} \frac{\log p}{p^{2z}} \asymp \frac{V_z}{U^{2z-1}}, \quad \text{and} \quad \sum_{p>U} \frac{(\log p)^3}{p^{3z}} \asymp \frac{(\log U)^2}{U^{3z-1}}.$$

Inserting these estimates in (4.7) implies that

$$\begin{aligned}
|\Phi_{\text{rand},z}(u)| &\ll \exp\left(-C_1 \frac{u^2 \log U}{V_z U^{2z-1}} \left(1 + O\left(\frac{V_z}{|u|\log U} + \frac{|u|\log U}{V_z^2 U^z}\right)\right)\right) \\
&\ll \exp\left(-\frac{C_1}{2} \frac{u^2 \log U}{V_z U^{2z-1}}\right) \ll \exp\left(-\frac{C_1}{2} \frac{u^2}{U^{2z-1}}\right),
\end{aligned}$$

for some positive constant C_1, by our choice of U. The result follows upon noting that $U^{2z-1} \asymp_A 1$ if $U = e^{AV_z}$, and $U^{2z-1} \asymp_A (|u|\log|u|)^{2-1/z}$ otherwise. $\square$

It follows from Lemma 4.4 that uniformly in $1/2 < z \leqslant 1$ we have

$$\Phi_{\mathrm{rand},z}(u) \ll \exp\left(-C_0\frac{|u|}{\log|u|}\right)$$

for all $u \in \mathbb{R}$. Thus, by Fourier inversion, the random variable $\mathcal{L}_{\mathrm{rand}}(z)/V_z$ is absolutely continuous, and has a uniformly bounded density function. In particular, for any $\varepsilon > 0$ we have

$$\mathbb{P}\left(\mathcal{L}_{\mathrm{rand}}(z)/V_z \in [-\varepsilon, \varepsilon]\right) \ll \varepsilon, \tag{4.8}$$

where the implied constant is absolute. We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let

$$T(z) := c_1\sqrt{\frac{\log x}{V_z\log(\log x/V_z)}},$$

where c_1 in the constant in the statement of Proposition 4.3. Since $\mathcal{L}_{\mathrm{rand}}(z)/V_z$ has a uniformly bounded density function, it follows from the Berry-Esseen Theorem (see Theorem 7.16 of [21]) that

$$D(z) \ll \frac{1}{T(z)} + \int_{-T(z)}^{T(z)} \frac{|\Phi_{x,z}(u) - \Phi_{\mathrm{rand},z}(u)|}{u}\,du.$$

Note that if $|u| \leqslant 1/T(z)$, then by Taylor's expansion and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Phi_{x,z}(u) - \Phi_{\mathrm{rand},z}(u) &= \frac{1}{|\mathcal{D}_z(x)|}\sum_{d\in\mathcal{D}_z(x)} \exp\left(2\pi i u\frac{\mathcal{L}_d(z)}{V_z}\right) - \mathbb{E}\Big[\exp\left(2\pi i u\frac{\mathcal{L}_{\mathrm{rand}}(z)}{V_z}\right)\Big]\\
&\ll |u|\left(\frac{1}{|\mathcal{D}_z(x)|}\sum_{d\in\mathcal{D}_z(x)}\frac{|\mathcal{L}_d(z)|}{V_z} + \mathbb{E}\left(\frac{|\mathcal{L}_{\mathrm{rand}}(z)|}{V_z}\right)\right)\\
&\leqslant |u|\left(\left(\frac{1}{|\mathcal{D}_z(x)|}\sum_{d\in\mathcal{D}_z(x)}\frac{|\mathcal{L}_d(z)|^2}{V_z^2}\right)^{1/2} + |u|\mathbb{E}\left(\frac{|\mathcal{L}_{\mathrm{rand}}(z)|^2}{V_z^2}\right)^{1/2}\right)\\
&\ll |u|
\end{aligned}$$

by Lemma 3.4. Therefore, we obtain

$$D(z) \ll \frac{1}{T(z)} + \int_{1/T(z)\leqslant|u|\leqslant T(z)} \frac{|\Phi_{x,z}(u) - \Phi_{\mathrm{rand},z}(u)|}{u}\,du.$$

By invoking Proposition 4.3, we infer that

$$\int_{1/T(z)\leqslant|u|\leqslant T(z)} \frac{|\Phi_{x,z}(u) - \Phi_{\mathrm{rand},z}(u)|}{u}\,du \ll T(z)\frac{V_z^4}{(\log x)^4} \ll \frac{1}{T(z)},$$

which completes the proof. $\square$

5. Moments of $\mathcal{L}_d(s)$ near the central point

Let $d \in \mathcal{D}(x)$. The completed L-function associated to $L(s,\chi_d)$ is

$$\Lambda(s,\chi_d) = \left(\frac{d}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi_d),$$

since $\chi_d(-1) = 1$. The completed L-function satisfies the self-dual functional equation

$$\Lambda(s,\chi_d) = \Lambda(1-s,\chi_d),$$

and its zeros are precisely the non-trivial zeros of $L(s,\chi_d)$. We start by recording the following standard lemma.

Lemma 5.1. *Let $s \in \mathbb{C}$ be such that $1/4 < \mathrm{Re}(s) \leqslant 5/4$, and s does not coincide with a non-trivial zero of $L(s,\chi_d)$. Then we have*

$$\mathcal{L}_d(s) = \frac{1}{2}\log\left(\frac{d}{\pi}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - \sum_{\rho} \frac{1}{s-\rho}, \tag{5.1}$$

where the sum is over all non-trivial zeros of $L(s,\chi_d)$. We also have

$$(\mathcal{L}_d)'(s) = \sum_{\rho} \frac{1}{(s-\rho)^2} + O(1). \tag{5.2}$$

Proof. The identity (5.1) follows from the Hadamard product formula for $\Lambda(s,\chi_d)$ (see for example Eq. (17) and (18) of [6, Chapter 12]). While the second estimate follows by taking the derivative of (5.1) with respect to s. $\square$

Throughout this section we let ν be a positive function such that $\nu(x) \to \infty$ as $x \to \infty$. Let $y = x^{4/\nu(x)}$ and $\mathcal{D}_y(x)$ be the set in the statement of Lemma 3.3, namely

$$\mathcal{D}_y(x) := \{d \in \mathcal{D}(x) \colon \sigma_{y,d} = 1/2 + 4/\log y\}.$$

Then it follows from Lemma 3.3 that $|\mathcal{D}(x) \backslash \mathcal{D}_y(x)| \ll x e^{-C_0 \nu(x)}$, for some positive constant C_0.

Proposition 5.2. *Let $\widetilde{\mathcal{D}}_y(x)$ be the set of fundamental discriminants $d \in \mathcal{D}_y(x)$ such that Hypothesis L_d holds (with function ν). Let $s_0 = 1/2 + \nu(x)/\log x$ and $\mathcal{C}_0$ be the circle of center s_0 and radius $r_0 := s_0 - 1/2 + 1/(2\nu(x)^3 \log x)$. Uniformly for all s with $|s - s_0| \leqslant r_0$ and all positive integers $k \leqslant \nu(x)/20$ we have*

$$\frac{1}{|\mathcal{D}(x)|} \sum_{d \in \widetilde{\mathcal{D}}_y(x)} |\mathcal{L}_d(s)|^{2k} \ll \nu^{4k}(x) \left(c_1 k (\log x)^2\right)^k.$$

Proof. Let $d \in \widetilde{\mathcal{D}}_y(x)$. By (3.3), we have

$$\mathcal{L}_d(\sigma_{y,d}) \ll \log d + |A_d(y)|, \tag{5.3}$$

where

$$A_d(y) := \sum_{n \leqslant y^3} \frac{\Lambda_{y,d}(n)}{n^{\sigma_{y,d}}},$$

and $\Lambda_{y,d}$ is as defined in (3.1). Let s be a complex number such that $|s-s_0|\leqslant r_0$. Since Hypothesis L_d holds we have

$$\min_{\rho}|s-\rho|\gg\frac{1}{\nu(x)^3\log x},\tag{5.4}$$

where the minimum runs over the non-trivial zeros of $L(s,\chi_d)$. Furthermore, using the identity

$$-\frac{1}{s-\rho}=-\frac{1}{\sigma_{y,d}-\rho}+\frac{s-\sigma_{y,d}}{(\sigma_{y,d}-\rho)^2}+\frac{(s-\sigma_{y,d})^2}{(\sigma_{y,d}-\rho)^2(s-\rho)}.$$

together with (5.1) and (5.2) we obtain (a similar estimate was derived by Selberg for the Riemann zeta function, see Eq. (12) of [17])

$$\mathcal{L}_d(s)=\mathcal{L}_d(\sigma_{y,d})+(s-\sigma_{y,d})(\mathcal{L}_d)'(\sigma_{y,d})+\sum_{\rho}\frac{(s-\sigma_{y,d})^2}{(\sigma_{y,d}-\rho)^2(s-\rho)}+O(1),$$

where ρ runs over the non-trivial zeros of $L(s,\chi_d)$. Therefore, combining (3.2), (5.2), (5.3) and (5.4) we get

$$|\mathcal{L}_d(s)|\ll(\log d+|A_d(s)|)\left(1+\frac{|s-\sigma_{y,d}|}{\sigma_{y,d}-1/2}+\frac{|s-\sigma_{y,d}|^2\nu(x)^3\log x}{\sigma_{y,d}-1/2}\right).$$

Since $|s-\sigma_{y,d}|\ll\nu(x)/\log x\asymp(\sigma_{y,d}-1/2)$ we deduce that

$$|\mathcal{L}_d(s)|\ll\nu(x)^4(\log x+|A_d(s)|).$$

Finally, by the same calculation leading to (3.4) we infer from Lemma 2.3 that for all positive integers $k\leqslant\nu(x)/20$ we have

$$\begin{aligned}\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\widetilde{\mathcal{D}}_y(x)}|\mathcal{L}_d(s)|^{2k}&\ll\nu^{8k}(x)(\log x)^{2k}+\nu^{8k}(x)\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\mathcal{D}(x)}|A_d(s)|^{2k}\\&\ll\nu^{8k}(x)\left(c_8k(\log x)^2\right)^k,\end{aligned}$$

for some positive constant c_8. This completes the proof. □

6. Real zeros of $L'(s,\chi_d)$: Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We may suppose that $\nu(x)\leqslant\log\log x$, otherwise we replace $\nu(x)$ by $\nu_1(x)=\min(\nu(x),\log\log x)$ throughout the proof. For $1\leqslant j\leqslant J:=\lfloor\frac{1}{\log 3}(\log\log x-\log\nu(x))\rfloor$, we define

$$z_j:=\frac{1}{2}+\frac{1}{3^j},\quad r_j:=\frac{1}{2\cdot3^j},\quad\text{and } R_j:=\frac{5}{4}r_j.$$

We also let $\mathcal{C}_j$ and $\widetilde{\mathcal{C}}_j$ be the concentric circles of center z_j and radii r_j and R_j, respectively (see Figure 1). One can observe that

$$\mathcal{I}:=\left[\frac{1}{2}+\frac{\nu(x)}{\log x},1\right]\subset\bigcup_{j=1}^{J}\{z\in\mathbb{C}:|z-z_j|\leqslant r_j\}.$$

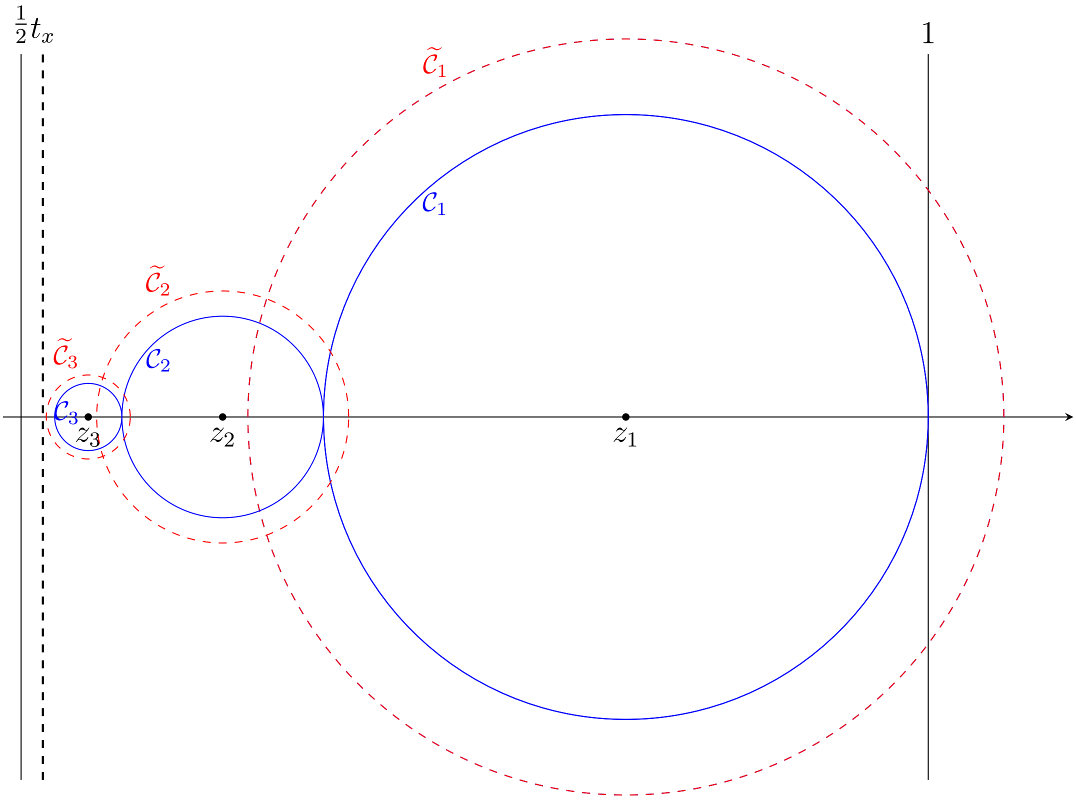

FIGURE 1. Circles covering $[t_x, 1]$, where $t_x = 1/2 + \nu(x)/\log x$.

Let $\widetilde{\mathcal{D}}(x) \subset \mathcal{D}(x)$ be the set of fundamental discriminants such that $L(s, \chi_d)$ has no zeros in the discs $|z - z_j| \leqslant \frac{7}{4} r_j$ for all $j \leqslant J$. Since for each $j \leqslant J$, such a disc is contained in the square $\{z : z_j - 7r_j/4 \leqslant \mathrm{Re}(z) \leqslant z_j + 7r_j/4 \text{ and } |\mathrm{Im}(z)| \leqslant 7r_j/4\}$, it follows from Lemma 3.2 that for some absolute positive constant c_9 we have

$$|\mathcal{D}(x) \setminus \widetilde{\mathcal{D}}(x)| \ll x \log x \sum_{j=1}^{J} \frac{x^{-c_9/3^j}}{3^j} \ll x \sum_{j=1}^{J} x^{-c_9/(2\cdot 3^j)} \ll x \exp(-c_{10}\nu(x)), \tag{6.1}$$

for some positive constant c_{10} since $(\log x)3^{-j} \ll \exp(\frac{c_9}{2}(\log x)3^{-j})$, for all $j \leqslant J$.

Let $d \in \widetilde{\mathcal{D}}(x)$. Then $\mathcal{L}_d$ is analytic on the open disc $|z - z_j| < 7r_j/4$ for all $j \leqslant J$, and moreover the number of zeros of $\mathcal{L}_d(s)$ in $\mathcal{I}$ is bounded by

$$\sum_{j=1}^{J} N_j(\mathcal{L}_d), \tag{6.2}$$

where $N_j(\mathcal{L}_d)$ is the number of zeros of $\mathcal{L}_d(s)$ inside the circle $\mathcal{C}_j$. Since $\mathcal{L}_d$ is analytic inside $\widetilde{\mathcal{C}}_j$, it follows from Jensen's formula that

$$N_j(\mathcal{L}_d) \leqslant \frac{\log\big(M_{j,d}/\mathcal{L}_d(z_j)\big)}{\log(R_j/r_j)} = \frac{1}{\log(5/4)}\Big(\log\big(M_{j,d}/V_j\big) - \log\big(\mathcal{L}_d(z_j)/V_j\big)\Big), \tag{6.3}$$

where

$$M_{j,d} := \max_{s\in\widetilde{\mathcal{C}}_j} |\mathcal{L}_d(s)|, \text{ and } V_j := \frac{1}{z_j - 1/2} = 3^j.$$

Note that we normalized both $M_{j,d}$ and $\mathcal{L}_d(z_j)$ by "the standard deviation" V_j. Therefore, in order to bound the sum on (6.2) we would like to show that for almost all fundamental discriminants $d \in \widetilde{\mathcal{D}}(x)$ we have

1. $\max_{j\leqslant J} M_{j,d}/V_j$ is not too large (namely $\ll (\log\log x)^2$ say).
2. $\min_{j\leqslant J} |\mathcal{L}_d(z_j)|/V_j$ is not too small (namely $\gg (\log\log x)^{-2}$ say).

We start by handling the first condition. Let $1 \leqslant j \leqslant J$. Since $\mathcal{L}_d(s)$ is analytic on the open disc of center z_j and radius $\frac{7}{5}R_j$ for all $d \in \widetilde{\mathcal{D}}(x)$, it follows from Cauchy's formula that

$$\mathcal{L}_d(s)^2 = \frac{1}{2\pi i}\int_{|z-z_j|=\frac{7}{6}R_j} \frac{\mathcal{L}_d(z)^2}{z-s}dz,$$

for all $s \in \widetilde{\mathcal{C}}_j$. This implies

$$M_{j,d}^2 = \max_{s\in\widetilde{\mathcal{C}}_j} |\mathcal{L}_d(s)|^2 \ll V_j \int_{|z-z_j|=\frac{7}{6}R_j} |\mathcal{L}_d(z)|^2|dz|, \tag{6.4}$$

since $|z-s| \geqslant |z-z_j| - |s-z_j| = R_j/6 \asymp 1/V_j$. Let L be a positive parameter to be chosen, and define $\mathcal{E}_1(x)$ to be the set of fundamental discriminants $d \in \widetilde{\mathcal{D}}(x)$ such that $\max_{j\leqslant J} M_{j,d}/V_j \geqslant L$. The proportion of $d \in \mathcal{E}_1(x)$ is

$$\begin{aligned}
&\leqslant \sum_{j=1}^{J} \frac{1}{(LV_j)^2}\frac{1}{|\widetilde{\mathcal{D}}(x)|}\sum_{d\in\widetilde{\mathcal{D}}(x)} M_{j,d}^2 \ll \sum_{j=1}^{J}\frac{1}{L^2V_j}\int_{|z-z_j|=\frac{7}{6}R_j}\left(\frac{1}{|\widetilde{\mathcal{D}}(x)|}\sum_{d\in\widetilde{\mathcal{D}}(x)}|\mathcal{L}_d(z)|^2\right)|dz|,\\
&\ll \sum_{j=1}^{J}\frac{1}{L^2V_j}\int_{|z-z_j|=\frac{7}{6}R_j} V_z^2|dz|
\end{aligned} \tag{6.5}$$

by (6.4), Lemma 3.4 and the fact that $|\widetilde{\mathcal{D}}(x)| \asymp x$. Furthermore, since $\int_{|z-z_j|=\frac{7}{6}R_j}|dz| \asymp 1/V_j$ and $V_z \leqslant 4V_j$ for all complex numbers z with $|z-z_j| = \frac{7}{6}R_j$ (since $\mathrm{Re}(z) \geqslant z_j - \frac{7}{6}R_j \geqslant \frac{1}{2} + \frac{1}{4V_j}$), we deduce that the right hand side of (6.5) is $\ll J/L^2$. We now choose $L = (\log\log x)^2$. This implies that the proportion of fundamental discriminants $d \in \mathcal{E}_1(x)$ is

$$\ll J(\log\log x)^{-4} \ll (\log\log x)^{-3}. \tag{6.6}$$

We now handle the second condition. Let $\varepsilon = 1/(\log\log x)^2$ and $\mathcal{E}_2(x)$ be the set of fundamental discriminants $d \in \widetilde{\mathcal{D}}(x)$ such that $\min_{j\leqslant J}|\mathcal{L}_d(z_j)/V_j| \leqslant \varepsilon$. Then by Theorem

4.1 we obtain

$$
\begin{aligned}
\frac{|\mathcal{E}_2(x)|}{|\widetilde{\mathcal{D}}(x)|} &= \frac{1}{|\widetilde{\mathcal{D}}(x)|}\left|\bigcup_{j=1}^{J}\left\{d \in \widetilde{\mathcal{D}}(x) : \mathcal{L}_d(z_j)/V_j \in [-\varepsilon,\varepsilon]\right\}\right| \\
&\leqslant \sum_{j=1}^{J}\frac{1}{|\widetilde{\mathcal{D}}(x)|}\left|\left\{d \in \widetilde{\mathcal{D}}(x) : \mathcal{L}_d(z_j)/V_j \in [-\varepsilon,\varepsilon]\right\}\right| \\
&\ll \sum_{j=1}^{J}\left(\mathbb{P}\big(\mathcal{L}_{\mathrm{rand}}(z_j)/V_j \in [-\varepsilon,\varepsilon]\big) + \frac{\sqrt{V_j \log\big(\log x/V_j\big)}}{\sqrt{\log x}}\right) \\
&\ll \frac{1}{\log\log x} + \sum_{j=1}^{J}\frac{\sqrt{3^j \log\big(\log x/3^j\big)}}{\sqrt{\log x}},
\end{aligned}
\tag{6.7}
$$

by (4.8). To bound the sum over j we split it in two parts $1 \leqslant j \leqslant J_0$ and $J_0 < j \leqslant J$, where $J_0 = \lfloor \frac{1}{\log 3}(\log\log x - 4\log\nu(x))\rfloor$. In the first part we use that $\log(\log x/3^j) \leqslant (\log x/3^j)^{1/2}$, while for the second we use that $\log(\log x/3^j) \ll \log\nu(x)$. This implies

$$
\sum_{j=1}^{J}\frac{\sqrt{3^j \log\big(\log x/3^j\big)}}{\sqrt{\log x}} \ll \sum_{1\leqslant j\leqslant J_0}\left(\frac{3^j}{\log x}\right)^{1/4} + \sqrt{\frac{\log\nu(x)}{\log x}}\sum_{J_0<j\leqslant J}3^{j/2} \ll \sqrt{\frac{\log\nu(x)}{\nu(x)}}.
$$

Inserting this bound in (6.7) shows that $|\mathcal{E}_2(x)| \ll x\sqrt{\log\nu(x)}/\sqrt{\nu(x)}$. To finish the proof, we let $\mathcal{D}_2(x) = \widetilde{\mathcal{D}}(x) \setminus (\mathcal{E}_1(x) \cup \mathcal{E}_2(x))$. Then combining our estimate on $\mathcal{E}_2(x)$ with (6.1) and (6.6) we deduce that

$$
|\mathcal{D}(x) \setminus \mathcal{D}_2(x)| \ll x\sqrt{\frac{\log\nu(x)}{\nu(x)}},
$$

and for all $d \in \mathcal{D}_2(x)$ we have $\max_{j\leqslant J} M_{j,d}/V_j \leqslant (\log\log x)^2$ and $\min_{j\leqslant J}|\mathcal{L}_d(z_j)|/V_j \geqslant (\log\log x)^{-2}$. Thus, if $d \in \mathcal{D}_2(x)$ then (6.3) implies that the number of real zeros of $\mathcal{L}_d$ on $\mathcal{I}$ is

$$
\ll J(\log\log\log x) \ll (\log\log x)(\log\log\log x),
$$

as desired. □

Proof of Theorem 1.3. By Theorem 1.2 it suffices to bound the number of real zeros of $\mathcal{L}_d$ in the interval $[1/2, 1/2 + \nu(x)/\log x]$. Let $s_0 = 1/2 + \nu(x)/\log x$ and consider the concentric circles $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ of center s_0 and radii r_0, r_1, r_2, and r_3 respectively, where $r_0 = s_0 - 1/2$, $r_1 = r_0 + 1/(4\nu^3(x)\log x)$, $r_2 = r_0 + 1/(2\nu^3(x)\log x)$, and $r_3 = r_0 + 3/(4\nu^3(x)\log x)$.

Recall that, by our hypothesis, $\mathcal{D}_0(x)$ is the set of fundamental discriminants $d \in \mathcal{D}(x)$ for which the Hypothesis L_d holds with function ν. Let $d \in \mathcal{D}_0(x)$. Then $\mathcal{L}_d$ is analytic inside the circle $\mathcal{C}_3$ and hence by Jensen's formula the number of real zeros of $\mathcal{L}_d$ in the

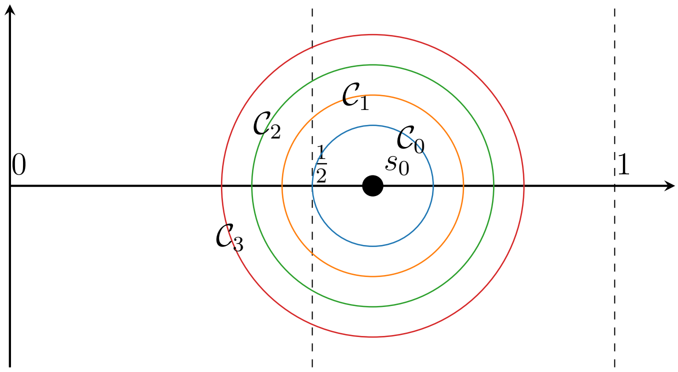

FIGURE 2. Four concentric circles $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.

interval $[1/2, 1/2 + \nu(x)/\log x]$ is bounded by

$$\begin{aligned}(6.8)\qquad &\frac{\log\left(\max_{s\in\mathcal{C}_1}|\mathcal{L}_d(s)|/|\mathcal{L}_d(s_0)|\right)}{\log(r_1/r_0)}\\ &=\frac{1}{\log(r_1/r_0)}\Big(\log\big(\max_{s\in\mathcal{C}_1}|\mathcal{L}_d(s)|/\log x\big)-\log\big(|\mathcal{L}_d(s_0)|/\log x\big)\Big)\end{aligned}$$

since $[1/2, 1/2+\nu(x)/\log x] \subset \{z \in \mathbb{C} : |z - s_0| \leqslant r_0\}$. Moreover, by Cauchy's formula, for all $s \in \mathcal{C}_1$, we have

$$\mathcal{L}_d(s)^2 = \frac{1}{2\pi i}\int_{z\in\mathcal{C}_2}\frac{\mathcal{L}_d(z)^2}{z-s}dz,$$

This implies

$$\max_{s\in\mathcal{C}_1}|\mathcal{L}_d(s)|^2 \ll \nu(x)^3\log x\int_{z\in\mathcal{C}_2}|\mathcal{L}_d(z)|^2|dz|, \tag{6.9}$$

since $|z-s| \geqslant r_2 - r_1 = 1/(4\nu^3(x)\log x)$ for all $z \in \mathcal{C}_2$ and $s \in \mathcal{C}_1$. By Lemma 3.3 and Proposition 5.2 there exists a subset $\widetilde{\mathcal{D}}_0(x) \subset \mathcal{D}_0(x)$ such that $|\mathcal{D}_0(x)\setminus\widetilde{\mathcal{D}}_0(x)| \ll xe^{-C_0\nu(x)}$ for some positive constant C_0 and

$$\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\widetilde{\mathcal{D}}_0(x)}|\mathcal{L}_d(z)|^2 \ll \nu^8(x)(\log x)^2, \tag{6.10}$$

uniformly for all $z \in \mathcal{C}_2$. Moreover, combining (6.9) and (6.10) we get

$$\begin{aligned}(6.11)\qquad \frac{1}{|\mathcal{D}(x)|}\sum_{d\in\widetilde{\mathcal{D}}_0(x)}\max_{s\in\mathcal{C}_1}|\mathcal{L}_d(s)|^2 &\ll \nu(x)^3\log x\int_{z\in\mathcal{C}_2}\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\widetilde{\mathcal{D}}_0(x)}|\mathcal{L}_d(z)|^2|dz|\\ &\ll \nu(x)^{12}(\log x)^2,\end{aligned}$$

since $\int_{z\in\mathcal{C}_2}|dz| \asymp \nu(x)/\log x$. We now define $\mathcal{E}_3(x)$ to be the set of fundamental discriminants $d \in \widetilde{\mathcal{D}}_0(x)$ such that $\max_{s\in\mathcal{C}_1}|\mathcal{L}_d(s)|/\log x \geqslant \nu(x)^{10}$. Then it follows from (6.11) that

$$\frac{|\mathcal{E}_3(x)|}{|\mathcal{D}(x)|} \leqslant \frac{1}{\nu(x)^{20}(\log x)^2}\frac{1}{|\mathcal{D}(x)|}\sum_{d\in\widetilde{\mathcal{D}}_0(x)}\max_{s\in\mathcal{C}_1}|\mathcal{L}_d(s)|^2 \ll \frac{1}{\nu(x)}. \tag{6.12}$$

Next, we let $\mathcal{E}_4(x)$ be the set of fundamental discriminants $d \in \widetilde{\mathcal{D}}_0(x)$ such $|\mathcal{L}_d(s_0)|/\log x \leqslant \varepsilon = 1/\nu(x)$. Then it follows from Theorem 4.1 together with (4.8) that

$$\frac{|\mathcal{E}_4(x)|}{|\mathcal{D}(x)|} = \frac{1}{|\mathcal{D}(x)|}\left|\left\{d \in \widetilde{\mathcal{D}}_0(x) : \mathcal{L}_d(s_0)/V_{s_0} \in [-\varepsilon, \varepsilon]\right\}\right|$$
$$\ll \Big(\mathbb{P}\big(\mathcal{L}_{\mathrm{rand}}(s_0)/V_{s_0} \in [-\varepsilon, \varepsilon]\big)\Big) + \sqrt{\frac{\log \nu(x)}{\nu(x)}} \ll \sqrt{\frac{\log \nu(x)}{\nu(x)}}.$$

Finally, we let $\mathcal{D}_1(x) = \widetilde{\mathcal{D}}_0(x)\setminus\big(\mathcal{E}_3(x)\cup\mathcal{E}_4(x)\big)$. Then $|\mathcal{D}_0(x)\setminus\mathcal{D}_1(x)| \ll x\sqrt{(\log \nu(x))/\nu(x)}$. Moreover, by (6.8), for all $d \in \mathcal{D}_1(x)$, the number of zeros of $\mathcal{L}_d$ on the interval $[1/2, 1/2 + \nu(x)/\log x]$ is

$$\ll \log(\nu(x))/\log(r_1/r_0) \ll \nu(x)^4 \log(\nu(x)).$$

Combining this estimate with Theorem 1.2 and (1.4) and using our assumption that $\nu(x) \leqslant (\log\log x)^{1/5}$ completes the proof. $\square$

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