Proper kernels in microlocal sheaf theory

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Abstract

Let X and Y be real analytic manifolds and let $\Lambda \subseteq T^*X$ and $\Sigma \subseteq T^*Y$ be closed conic subanalytic singular isotropics. Given a sheaf $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ microsupported in $-\Lambda \times \Sigma$, consider the convolution functor $(-) * K \colon \mathrm{Sh}_\Lambda(X) \to \mathrm{Sh}_\Sigma(Y)$ from sheaves microsupported in Λ to sheaves microsupported in Σ. We show that the convolution functor $(-) * K$ preserves compact objects if and only if for each $x \in X$, the restriction $K|_{\{x\}\times Y} \in \mathrm{Sh}_\Sigma(Y)$ is a compact object. By a result of Kuo-Li [15], the functor sending a sheaf kernel K to the convlution functor $(-) * K$ is an equivalence between the category $\mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ of sheaves microsupported in $-\Lambda \times \Sigma$ and the category of cocontinuous functors from $\mathrm{Sh}_\Lambda(X)$ to $\mathrm{Sh}_\Sigma(Y)$. We therefore classify all cocontinuous functors that preserve compact objects between the two categories. Our approach is entirely categorical and requires minimal input from geometry: we introduce the notion of a proper object in a compactly generated stable ∞-category and study its properties under strongly continuous localizations to obtain the result. The main geometric input is the analysis of compact and proper objects of the category of P-constructible sheaves for a triangulation P of a manifold Z via the exit path category $\mathrm{Exit}(Z, P) \simeq P$. Along the way, we show that a sheaf $F \in \mathrm{Sh}_\Lambda(X)$ is proper if and only if it has perfect stalks, which is equivalent to a result of Nadler.

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1 Introduction

1.1 Motivation and background

Let X be a real analytic manifold and let $\Lambda \subseteq T^*X$ be a closed conic subanalytic singular isotropic. The work of Kuo-Li [15] shows that the category $\mathrm{Sh}_\Lambda(X)$ of sheaves microsupported in Λ is a *dualizable* stable ∞-category (cf. Definition 2.4). More precisely, let Y be a real analytic manifold and Σ be a closed conic subanalytic singular isotropic, it is shown there [15, Theorem 1.1, Theorem 1.2, Corollary 1.7] that the dual of $\mathrm{Sh}_\Lambda(X)$ is given by $\mathrm{Sh}_{-\Lambda}(X)$:

$$\mathrm{Sh}_\Lambda(X)^\vee \simeq \mathrm{Sh}_{-\Lambda}(X),$$

the Kunneth formula holds:

$$\mathrm{Sh}_\Lambda(X) \otimes \mathrm{Sh}_\Sigma(Y) \simeq \mathrm{Sh}_{\Lambda\times\Sigma}(X \times Y),$$

and that the equivalence

$$\mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y) \simeq \mathrm{Sh}_\Lambda(X)^\vee \otimes \mathrm{Sh}_\Sigma(Y) \simeq \mathrm{Fun}^L(\mathrm{Sh}_\Lambda(X), \mathrm{Sh}_\Sigma(Y))$$

is given by the assignment

$$K \mapsto (-) * K,$$

where

$$(-) * K := \pi_{2,!}(\pi_1^*(-) \otimes K)$$

is the convolution functor. Here $\mathrm{Fun}^L(-,-)$ denotes the category of cocontinuous functors, and $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are the projections.

In conclusion, cocontinuous functors between the sheaf categories $\mathrm{Sh}_\Lambda(X)$ and $\mathrm{Sh}_\Sigma(Y)$ are classified by sheaf kernels $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ on $X \times Y$ microsupported in $-\Lambda \times \Sigma$.

On the other hand, previous works [20, 18, 5] have drawn parallels between microlocal sheaf theory and the theory of Fukaya categories in various flavors. In particular, assuming $\Lambda \subseteq T^*X$ contains the zero section 0_X, it is shown in [5] that there is an equivalence between the category of sheaves of $\mathbb{Z}$-modules microsupported in Λ and the ind-completion of the partially wrapped Fukaya category of T^*X stopped at $-\Lambda_\infty$:

$$\mathrm{Sh}_\Lambda(X; \mathbb{Z}\text{-Mod}) \simeq \mathrm{Ind}\, \mathcal{W}(T^*X, -\Lambda_\infty).$$

Here Λ_∞ is the projection of $\Lambda - 0_X$ to the cosphere bundle S^*X. Taking compact objects on both sides, we have

$$\mathrm{Sh}_\Lambda(X; \mathbb{Z}\text{-Mod})^\omega \simeq \mathrm{Perf}\, \mathcal{W}(T^*X, -\Lambda_\infty),$$

where $\mathrm{Perf}\, \mathcal{W}(T^*X, -\Lambda)$ is the idempotent completion of $\mathcal{W}(T^*X, -\Lambda_\infty)$.[1] In [19], Nadler first introduced the notion of *wrapped sheaves*, which are by definition, compact objects in $\mathrm{Sh}_\Lambda(X)$. In

[1]Strictly speaking, it is only possible to compare the category on the left and that on the right *up to Morita equivalence*, since the identification of R-linear stable ∞-categories, dg-categories, and A_∞-categories requires the Morita model structure. In this sense, the equivalence only exists up to some replacement in the model category in the first place, and it is technically redundant to explictly mention idempotent completion.

[14], Kuo showed that the category of wrapped sheaves can also be realized geometrically, without alluding to compactness in the ambient category.

Consequently, one can argue that studying compact objects in $\mathrm{Sh}_\Lambda(X)$ is not a purely academic pursuit, but is of intrinsic geometric interest from the viewpoint of Floer theory. Given Liouville manifolds M and N, and a Lagragian correspondence $\mathcal{L} \subseteq M^- \times N$, Gao [6] showed that, if $\mathcal{L} \to N$ is proper, under some genericity conditions, there is an induced A_∞-functor

$$\Theta_{\mathcal{L}} \colon \mathcal{W}(M) \to \mathcal{W}(N),$$

which on objects is given by geometric composition of Lagrangians:

$$M \supseteq L \mapsto L \circ \mathcal{L} \subseteq N.$$

A natural question to ask is what the sheaf-theoretic incarnation of the above functor is. Note that the category

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sh}_\Lambda(X)^\omega, \mathrm{Sh}_\Sigma(Y)^\omega)$$

of exact functors is equivalent to the subcategory of

$$\mathrm{Fun}^L(\mathrm{Sh}_\Lambda(X), \mathrm{Sh}_\Sigma(Y)) \simeq \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$$

spanned by functors that preserve compact objects. Therefore, an equivalent question to ask is:

Question 1.1. Under what conditions on the sheaf kernel $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$, does the convolution functor

$$(-) * K \colon \mathrm{Sh}_\Lambda(X) \to \mathrm{Sh}_\Sigma(Y)$$

preserve compact objects?

Our main result provides a complete answer to this question. As a consequence, we can verify the following special case, which was conjectured by Ganatra-Kuo-Li-Wu (see Remark 1.7).

Conjecture 1.2 (Ganatra-Kuo-Li-Wu)**.** Let $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ be a sheaf kernel. If $\mathrm{SS}(K) \to T^*Y$ is proper and K has perfect stalks, assuming Y is compact, then $(-) * K$ preserves compact objects.

Remark 1.3. Ganatra-Kuo-Li-Wu pursue a geometric approach to this question in [4], using techniques based on wrappings. Their work actually addresses the following more general conjecture.

Conjecture 1.4 (Ganatra-Kuo-Li-Wu)**.** Let $L \in \mathrm{Sh}(X \times Y)$ be a constructible sheaf with perfect stalks, not necessarily microsupported in $-\Lambda \times \Sigma$. Certain geometric constraints on $\mathrm{SS}(L)$ guarantee that $\mathfrak{M}^+_{-\Lambda\times\Sigma}(L) * (-)$ preserves compact objects. [2]

1.2 Main results and overview

In this section, fix real analytic manifolds X and Y and closed conic subanalytic singular isotropics $\Lambda \subseteq T^*X$ and $\Sigma \subseteq T^*Y$. Our main result is the following.

[2] Here $\mathfrak{M}^+_{-\Lambda\times\Sigma}$ is the positive wrapping functor introduced in [14], which is equivalent to the localization functor $\iota^*_{-\Lambda\times\Sigma} \colon \mathrm{Sh}(X) \to \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ in our notation.

Theorem 1.5 (Theorem 6.11)**.** Let $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ be a sheaf kernel. The convolution functor

$$(-) * K\colon \mathrm{Sh}_\Lambda(X) \to \mathrm{Sh}_\Sigma(Y)$$

preserves compact objects if and only if for every $x \in X$, the restriction $K|_{\{x\}\times Y} \in \mathrm{Sh}_\Sigma(Y)$ is a compact object. Consequently, there is an equivalence

$$\mathcal{P} \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sh}_\Lambda(X)^\omega, \mathrm{Sh}_\Sigma(Y)^\omega),$$

where $\mathcal{P} \subseteq \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ is the full subcategory spanned by such sheaf kernels.

From this, we can deduce a sufficient condition.

Corollary 1.6 (Corollary 6.12)**.** Let $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ be a sheaf kernel. If K has perfect stalks and $\mathrm{supp}(K|_{\{x\}\times Y})$ is compact for every $x \in X$, then convolution with K preserves compact objects.

Remark 1.7. If Y is compact, then $\mathrm{supp}(K|_{\{x\}\times Y})$ is always compact. In this case, if K has perfect stalks, then convolution with K preserves compact objects. In particular, Conjecture 1.2 is true.

The core to our argument is the notion of a *proper* object in a compactly generated stable ∞-category.

Definition 1.8 (Definition 3.2)**.** Let $\mathcal{C}$ be a presentable stable ∞-category. We say $c \in \mathcal{C}$ is *proper* if the functor

$$\mathrm{map}_{\mathcal{C}}(-, c)\colon \mathcal{C}^{\omega,\mathrm{op}} \to \mathrm{Sp}$$

factors through $\mathrm{Sp}^\omega \subseteq \mathrm{Sp}$.

The following result on proper objects in $\mathrm{Sh}_\Lambda(X)$ can be also seen as a special case of the main theorem.

Theorem 1.9 (Theorem 6.7)**.** A sheaf $F \in \mathrm{Sh}_\Lambda(X)$ is proper if and only if F has perfect stalks.

The result above, albeit stated in a slightly different setting, was first proved as [19, Theorem 3.21] using arborealization, and later proved again as [5, Corollary 4.24] with a more direct argument. The main point of this paper is that, once we distill the essential ideas in its proof to categorical terms, the argument can be further simplified and adapted to a relative setting, allowing us to prove the main theorem of this paper.

1.3 Notations and conventions

For the sake of brevity and clarity, we will work exclusively with sheaves of spectra, unless otherwise specified.

Remark 1.10. All our arguments work mutatis mutandis if we replace the ∞-category Sp of spectra with any compactly generated rigid monoidal ∞-category $\mathcal{V}$, and argue in the context of $\mathcal{V}$-enriched categories instead. Much of the theory of $\mathcal{V}$-enriched category is developed in [7, 12, 11, 10, 2]. For a quick review on the theory of presentable and dualizable categories in the enriched setting directly applicable to this paper, see [23, §1].

Notation 1.11. Let $\mathcal{C}$ be a stable ∞-category. Throughout this paper, $\mathrm{map}_{\mathcal{C}}(-,-)$ denotes the mapping spectrum, while $\mathrm{Map}_{\mathcal{C}}(-,-)$ denotes the mapping space.

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2 Preliminaries

2.1 Exodromy

Consider a topological space X with a stratification P, or simply a *stratified space* (X, P).

Recollection 2.1. A *stratified space* (X, P) is a topological space X, together with a continuous map $X \to P$, where P is a poset equipped with the Alexandroff topology.

The *exodromy equivalence* (see [24, HA, 16, 21, 9]) states that under suitable assumptions on (X, P) and $\mathcal{V}$, the category

$$\mathrm{Cons}_P(X; \mathcal{V})$$

of P-constructible sheaves [3] on X valued in $\mathcal{V}$ is equivalent to the category

$$\mathrm{Fun}(\mathrm{Exit}(X, P), \mathcal{V})$$

of functors from the *exit-path* ∞*-category* $\mathrm{Exit}(X, P)$ to $\mathcal{V}$. While the construction of $\mathrm{Exit}(X, P)$ is quite involved in general, the only result we need in this paper is the following.

Proposition 2.2. Let P be a triangulation of a manifold X. Then $\mathrm{Exit}(X, P) \simeq P$.

Proof. This is a special case of [HA, Theorem A.6.10]. □

Corollary 2.3. Let P be a triangulation of a manifold X. Then $\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(P, \mathrm{Sp})$.

2.2 Dualizable stable ∞-categories

We give a quick recap of the theory of dualizable stable ∞-categories.

The ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of presentable stable ∞-categories and left adjoints admits a closed symmetric monoidal structure given by tensor product in Pr^L. The internal hom is given by $[\mathcal{C}, \mathcal{D}] = \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$.

[3] Technically, to state the most general result, one has to consider hyper-constructible hypersheaves. However, the distinction between hyper-constructible hypersheaves and constructible sheaves disappears when everything is hypercomplete, as is the case if X is a finite-dimensional manifold and all the strata are submanifolds.

The ∞-category $\mathrm{Cat}^{\mathrm{perf}}$ of small idempotent complete stable ∞-categories has a symmetric monoidal structure where the tensor product $\mathcal{C}_0 \otimes \mathcal{D}_0$ classifies bi-exact functors out of $\mathcal{C}_0 \times \mathcal{D}_0$. It is again a closed monoidal category with internal homs given by $[\mathcal{C}_0, \mathcal{D}_0] = \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_0, \mathcal{D}_0)$.

The ind-construction

$$\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

makes $\mathrm{Cat}^{\mathrm{perf}}$ a wide subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, with its essential image spanned by compactly generated stable ∞-categories and left adjoints that preserve compact objects.

Moreover, Ind is a symmetric monoidal fuctor. The ind construction and taking compact objects give inverse equivalences

$$\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{perf}} \leftrightarrows \mathrm{Pr}^{L,\omega}_{\mathrm{st}} \colon (-)^{\omega},$$

where $\mathrm{Pr}^{L,\omega}_{\mathrm{st}}$ is the ∞-category of compactly generated stable ∞-categories and left adjoints that preserve compact objects.

Definition 2.4. A presentable stable ∞-category $\mathcal{C}$ is *dualizable* if there exists $\mathcal{C}^\vee$ so that the functor

$$- \otimes \mathcal{C}\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}.$$

is left adjoint to

$$- \otimes \mathcal{C}^\vee\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}.$$

Example 2.5 ([SAG, Proposition D.7.2.3])**.** If $\mathcal{C}$ is a compactly generated stable ∞-category, then $\mathcal{C}$ is dualizable. The dual is given by $\mathcal{C}^\vee \simeq \mathrm{Ind}(\mathcal{C}^{\omega,\mathrm{op}})$, and the evaluation map $\mathrm{ev} : \mathcal{C}^\vee \otimes \mathcal{C} \to \mathrm{Sp}$ is given by the left Kan extension of

$$\mathrm{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\omega,\mathrm{op}} \otimes \mathcal{C}^{\omega} \to \mathrm{Sp}.$$

along $\mathcal{C}^{\omega,\mathrm{op}} \otimes \mathcal{C}^{\omega} \to \mathcal{C}^\vee \otimes \mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\omega,\mathrm{op}} \otimes \mathcal{C}^{\omega})$.

Recollection 2.6. A left adjoint functor between presentable ∞-categories is called an *internal left adjoint* in Pr^L, if it is the left adjoint of an adjunction in the $(\infty,2)$-category $\mathbf{Pr}^L$ of presentable ∞-categories and left adjoints. Being an internal left adjoint is equivalent to being a *strongly cocontinuous functor*: a functor whose right adjoint admits a further right adjoint.

Remark 2.7. A left adjoint functor between compactly generated presentable stable ∞-categories is an internal left adjoint if and only if it preserves compact objects. Therefore, $\mathrm{Cat}^{\mathrm{perf}} \simeq \mathrm{Pr}^{L,\omega}_{\mathrm{st}}$ is equivalent to the ∞-category of compactly generated stable ∞-categories and internal left adjoints.

Definition 2.8. The ∞-category $\mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$ of dualizable stable ∞-categories is the wide subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ spanned by dualizable objects and internal left adjoints in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Remark 2.9. By Remark 2.7, $\mathrm{Cat}^{\mathrm{perf}} \simeq \mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$ is the full subcategory of $\mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$ spanned by compactly generated stable ∞-categories.

Theorem 2.10 ([SAG, Proposition D.7.3.1])**.** A presentable stable ∞-category is dualizable if and only if it is the retract of a compactly generated one in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

3 Compactness and properness

In this section, we discuss *compact* and *proper* objects in presentable stable ∞-categories. We will focus on how these objects behave under the inclusion of reflective and co-reflective subcategories.

Recollection 3.1. Let $\mathcal{C}$ be a presentable ∞-category. An object $c \in \mathcal{C}$ is *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathrm{Spc}$ commutes with filtered colimits. If $\mathcal{C}$ is stable, this is equivalent to that $\mathrm{map}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathrm{Sp}$ commutes with filtered colimits.

Definition 3.2. Let $\mathcal{C}$ be a presentable stable ∞-category. We say $c \in \mathcal{C}$ is *proper* if the functor

$$\mathrm{map}_{\mathcal{C}}(-, c)\colon \mathcal{C}^{\omega,\mathrm{op}} \to \mathrm{Sp}$$

factors through $\mathrm{Sp}^{\omega} \subseteq \mathrm{Sp}$. Denote by $\mathcal{C}^{\mathrm{pr}} \subseteq \mathcal{C}$ the full subcategory of proper objects.

Remark 3.3. If $\mathcal{C}$ is compactly generated and every compact object in $\mathcal{C}$ is proper, then $\mathcal{C}$ is a proper stable ∞-category in the sense of [SAG, Definition 11.1.0.1].

Proposition 3.4. Let $\mathcal{C}$ be a presentable stable ∞-category. Then $\mathcal{C}^{\mathrm{pr}}$ is an idempotent complete stable subcategory of $\mathcal{C}$.

Proof. Note that $\mathrm{Sp}^{\omega} \subseteq \mathrm{Sp}$ is closed under finite (co)limits and retracts, and for any $x \in \mathcal{C}^{\omega}$, the functor $\mathrm{map}_{\mathcal{C}}(x, -)\colon \mathcal{C} \to \mathrm{Sp}$ preserves finite (co)limits and retracts. It follows that $\mathcal{C}^{\mathrm{pr}}$ is closed under finite (co)limits and retracts. $\square$

Proposition 3.5. Let $\mathcal{C}$ be a compactly generated stable ∞-category. The spectral Yoneda embedding

$$c \mapsto \mathrm{map}_{\mathcal{C}}(-, c)$$

restricts to an equivalence

$$\mathcal{C}^{\mathrm{pr}} \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\omega,\mathrm{op}}, \mathrm{Sp}^{\omega}).$$

Here $\mathrm{Fun}^{\mathrm{ex}}(-, -)$ denotes the category of exact functors between two stable ∞-categories.

Proof. Since $\mathcal{C}$ is compactly generated, there are equivalences

$$\mathcal{C} \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\omega,\mathrm{op}}, \mathrm{Spc}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\omega,\mathrm{op}}, \mathrm{Sp}).$$

Here $\mathrm{Fun}^{\mathrm{lex}}(-, -)$ denotes the category of left exact[4] functors. By definition, the composite functor is the spectral Yoneda embedding. Restricting to proper objects gives the desired equivalence. $\square$

Corollary 3.6. If $\mathcal{C}$ is a compactly generated stable ∞-category, then $\mathcal{C}^{\mathrm{pr}}$ is a small ∞-category. $\square$

Compact and proper objects behave in a very controllable way, under *strongly cococontinuous localizations*, which we now introduce.

Recollection 3.7. A functor between presentable ∞-categories is called *strongly cocontinuous*, if its right adjoint admits a further right adjoint. Alternatively, it is an internal left adjoint in Pr^{L}.

[4]Recall that a functor is called *left exact* if it preserves finite limits.

Observation 3.8. Let $\mathcal{C}$ be a presentable ∞-category, and $\iota_*\colon \mathcal{D} \hookrightarrow \mathcal{C}$ be a full subcategory. Suppose that $\mathcal{D}$ is closed under small limits and colimits. By the ∞-categorical reflection theorem [22], the inclusion ι_* participates in a triple adjunction:

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook, "\iota_*"] & \mathcal{C}. \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^\flat"] \end{tikzcd}$$

In particular, the localization functor ι^* is strongly cocontinuous.

Notation 3.9. In the above situation, we say $\iota^*\colon \mathcal{C} \to \mathcal{D}$ is a *strongly cocontinuous localization*, and $\iota_*\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is a *bi-reflective subcategory*. In this case, we always use $\iota^* \dashv \iota_* \dashv \iota^\flat$ to refer to the adjoint triple.

Proposition 3.10. Let $\mathcal{C}$ be a compactly generated presentable ∞-category. Suppose $\iota_*\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is a bi-reflective subcategory. Then:

(1) The inclusion ι_* detects compact objects: if $\iota_* d$ is compact in $\mathcal{C}$, then d is compact in $\mathcal{D}$.

(2) The ∞-category $\mathcal{D}$ is compactly generated by $\iota^*\mathcal{C}^\omega$.

(3) The $\mathcal{D}^\omega$ is the smallest replete subcategory of $\mathcal{D}$ containing $\iota^*\mathcal{C}^\omega$ and closed under finite colimits and retracts.

Proof. To prove point (1), note that there is a natural equivalence

$$\mathrm{Map}_{\mathcal{D}}(d, -) \simeq \mathrm{Map}_{\mathcal{C}}(\iota_* d, \iota_* -)$$

and that ι_* preserves all colimits, in particular filtered colimits. Therefore if $\iota_* d$ is compact in $\mathcal{C}$, then $\mathrm{Map}_{\mathcal{D}}(d, -)$ commutes with filtered colimits, and hence d is compact in $\mathcal{D}$.

To prove points (2) and (3), first note that ι^* preserves compact objects: indeed, its right adjoint ι_* preserves all colimits, in particular filtered colimits [HTT, Proposition 5.5.7.2].

Now let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ generated under colimits by $\iota^*\mathcal{C}^\omega$. Consider the full subcategory

$$(\iota^*)^{-1}\mathcal{D}' := \{c \in \mathcal{C} \mid \iota^* c \in \mathcal{D}'\} \subseteq \mathcal{C}.$$

Since ι^* preserves colimits, it is immediate that $(\iota^*)^{-1}(\mathcal{D}')$ is closed under colimits. However, by definition of $\mathcal{D}'$ we have

$$\mathcal{C}^\omega \subseteq (\iota^*)^{-1}\iota^*\mathcal{C}^\omega \subseteq (\iota^*)^{-1}\mathcal{D}'.$$

As $\mathcal{C}^\omega$ generates $\mathcal{C}$ colimits, we must have $(\iota^*)^{-1}\mathcal{D}' = \mathcal{C}$. Therefore, we obtain $\mathcal{D}' = \mathcal{D}$: for any $d \in \mathcal{D}$, we have $d \simeq \iota^*\iota_* d \in \iota^*(\mathcal{C}) = \iota^*(\iota^*)^{-1}\mathcal{D}' \subseteq \mathcal{D}'$. Consequently, $\iota^*\mathcal{C}^\omega$ generates $\mathcal{D}$ under colimits. $\square$

Proposition 3.11. Let $\mathcal{C}$ be a compactly generated stable ∞-category. Let $\iota_*\colon \mathcal{D} \hookrightarrow \mathcal{C}$ be a bi-reflective subcategory. Then $d \in \mathcal{D}$ is proper in $\mathcal{D}$ if and only if $\iota_* d \in \mathcal{C}$ is proper in $\mathcal{C}$.

Proof. Fix d so that $\iota_* d$ is proper in $\mathcal{C}$. We need to show that $\mathrm{map}_{\mathcal{D}}(-,d)$ sends $\mathcal{D}^\omega$ to Sp^ω. Consider the full subcategory

$$\mathcal{D}' := \{x \in \mathcal{D} \mid \mathrm{map}_{\mathcal{D}}(x,d) \in \mathrm{Sp}^\omega\} \subseteq \mathcal{D}.$$

Clearly, $\mathcal{D}'$ is closed under finite (co)limits and retracts. For any $y \in \mathcal{C}^\omega$, we have

$$\mathrm{map}_{\mathcal{D}}(\iota^* y, d) \simeq \mathrm{map}_{\mathcal{C}}(y, \iota_* d) \in \mathrm{Sp}^\omega.$$

Therefore, $\iota^*\mathcal{C}^\omega \subseteq \mathcal{D}'$. By Proposition 3.10, $\mathcal{D}^\omega$ is generated by $\iota^*\mathcal{C}^\omega$, so $\mathcal{D}^\omega \subseteq \mathcal{D}'$ as $\mathcal{D}'$ is closed under retracts and finite colimits. Thus by definition of $\mathcal{D}'$, d is proper in $\mathcal{D}$.

Conversely, assume d is proper in $\mathcal{C}$. Then for any $y \in \mathcal{C}^\omega$, we have

$$\mathrm{map}_{\mathcal{D}}(\iota^* y, d) \simeq \mathrm{map}_{\mathcal{C}}(y, \iota_* d) \in \mathrm{Sp}^\omega$$

and thus $\iota_* d$ is proper in $\mathcal{C}$ □

4 $\mathcal{V}$-properness

Throughout this section, let $\mathcal{V}$ be a compactly generated stable ∞-category.

Notation 4.1. Let $\mathcal{C}$ be a dualizable stable ∞-category. Denote by

$$e\colon \mathcal{C} \otimes \mathcal{V} \xrightarrow{\simeq} \mathrm{Fun}^L(\mathcal{C}^\vee, \mathcal{V})$$

the evaluation functor.

Definition 4.2. Let $\mathcal{C}$ be a compactly generated stable ∞-category. An object $F \in \mathcal{C} \otimes \mathcal{V}$ is *$\mathcal{V}$-proper*, if the functor

$$e(F)\colon \mathcal{C}^\vee \to \mathcal{V}$$

sends $(\mathcal{C}^\vee)^\omega \simeq \mathcal{C}^{\omega,\mathrm{op}}$ to $\mathcal{V}^\omega$.

Remark 4.3. An object $x \in \mathcal{C}$ is proper in the sense of Definition 3.2, precisely if $x \in \mathcal{C} \simeq \mathcal{C} \otimes \mathrm{Sp}$ is Sp-proper.

Example 4.4. Let $x \in \mathcal{C}$ be a proper object and $v \in \mathcal{V}$ a compact object. Then $x \boxtimes v \in \mathcal{C} \otimes \mathcal{V}$ is $\mathcal{V}$-proper. Here $- \boxtimes -\colon \mathcal{C} \times \mathcal{V} \to \mathcal{C} \otimes \mathcal{V}$ is the universal bi-cocontinuous functor.

Observation 4.5. Let $\mathcal{C}$ be a compactly generated stable ∞-category. Suppose $\iota_*\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is a bi-reflective subcategory (cf. Notation 3.9):

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook, "\iota_*" description] & \mathcal{C}. \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^\flat"] \end{tikzcd}$$

Since $\iota^* \dashv \iota_*$ is an adjunction internal to Pr^L, tensoring with $\mathcal{V}$ gives rise to a bi-reflective subcategory

$$\begin{tikzcd} \mathcal{D} \otimes \mathcal{V} \arrow[r, hook, "\iota_*^{\mathcal{V}}" description] & \mathcal{C} \otimes \mathcal{V}. \arrow[l, bend right, "\iota_{\mathcal{V}}^*"'] \arrow[l, bend left, "\iota_{\mathcal{V}}^\flat"] \end{tikzcd}$$

Here $\iota_{\mathcal{V}}^* = \iota^* \otimes \mathcal{V}$, $\iota_*^{\mathcal{V}} = \iota_* \otimes \mathcal{V}$, and $\iota_{\mathcal{V}}^\flat$ is the right adjoint to $\iota_* \otimes \mathcal{V}$.

Proposition 4.6 (cf. Proposition 3.11)**.** In above situation, an object $F \in \mathcal{D} \otimes \mathcal{V}$ is $\mathcal{V}$-proper if and only if $\iota_*^{\mathcal{V}}(F)$ is $\mathcal{V}$-proper.

Proof. By [HA, Proposition 4.8.1.17], there is a commutative diagram

$$\begin{tikzcd} \mathrm{Pr}^R \arrow[r, "-\otimes\mathcal{V}"] \arrow[d, "\simeq"'] & \mathrm{Pr}^R \arrow[d, equal] \\ \mathrm{Pr}^{L,\mathrm{op}} \arrow[r, "\mathrm{Fun}^R(-^{\mathrm{op}},\mathcal{V})"] & \mathrm{Pr}^R \end{tikzcd}.$$

Restricting the domain of the functors to compactly generated stable ∞-categories and compact preserving left adjoints (resp. their right adjoints), we have

$$-\otimes\mathcal{V} \simeq \mathrm{Fun}^R(-^{\mathrm{op}}, \mathcal{V}) \simeq \mathrm{Fun}^{\mathrm{ex}}((-)^{\omega,\mathrm{op}}, \mathcal{V}).$$

Under this equivalence, the right adjoint $\iota_*^{\mathcal{V}}$ corresponds to

$$-\circ\iota^*\colon \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{\omega,\mathrm{op}}, \mathcal{V}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\omega,\mathrm{op}}, \mathcal{V}).$$

Suppose $F\colon \mathcal{D}^{\omega,\mathrm{op}} \to \mathcal{V}$ factors through $\mathcal{V}^\omega$. It follows that $\iota_*^{\mathcal{V}}(F) = F\circ\iota^*$ also factors through $\mathcal{V}^\omega$. On the other hand, suppose $\iota_*^{\mathcal{V}}(F)$ factors through $\mathcal{V}^\omega$, and let $\mathcal{K} \subseteq \mathcal{D}^\omega$ be the full subcategory spanned by objects $d \in \mathcal{D}^\omega$ such that $F(d) \in \mathcal{V}^\omega$. By assumptions on $\iota_*^{\mathcal{V}}(F)$, we have $F(\iota^*c) = \iota_*^{\mathcal{V}}(F)(c) \in \mathcal{V}^\omega$, and therefore $\iota^*\mathcal{C}^\omega \subseteq \mathcal{K}$. Because $\mathcal{K}$ is closed under finite colimits and retracts, Proposition 3.10 shows that $\mathcal{K} = \mathcal{D}^\omega$. $\square$

5 Proper objects in functor categories

Let P be a triangulation of a manifold X. By the exodromy equivalence (cf. Corollary 2.3), we have

$$\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(P, \mathrm{Sp}).$$

In this section, we study proper objects in $\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(P, \mathrm{Sp})$. First we note that, if P is a triangulation of X, then P as a poset is *locally finite*.

Definition 5.1. A poset P is *locally finite* if for every $p \in P$, the poset $P_{p/}$ is finite.

Proposition 5.2 ([1, Lemma 4.4.10])**.** Let P be a locally finite poset. Then F is compact in $\mathrm{Fun}(P, \mathrm{Sp})$ if and only if F is finitely supported and $F(p)$ is a finite spectrum for every $p \in P$.

Theorem 5.3. Let P be a locally finite poset. Then $F \in \mathrm{Fun}(P, \mathrm{Sp}) \otimes \mathcal{V} \simeq \mathrm{Fun}(P, \mathcal{V})$ is $\mathcal{V}$-proper if and only if $F(p) \in \mathcal{V}^\omega$ for every $p \in P$.

Proof. Write

$$\begin{aligned} よ\colon P^{\mathrm{op}} &\to \mathrm{Fun}(P, \mathrm{Spc}) \to \mathrm{Fun}(P, \mathrm{Sp}) \\ p &\mapsto \mathrm{Map}_P(p, -) \mapsto \Sigma_+^\infty \mathrm{Map}_P(p, -) \end{aligned}$$

for the stable (co)Yoneda embedding.

By [17, Proposition 2.2.3], $\mathrm{Fun}(P, \mathrm{Sp})$ is compactly generated by the collection $\{よ(p)\}_{p\in P}$. Recall that the equivalence

$$\mathrm{Fun}(P, \mathrm{Sp}) \otimes \mathcal{V} \simeq \mathrm{Fun}(P, \mathcal{V})$$

can be obtained as the composite

$$\mathrm{Fun}(P, \mathrm{Sp}) \otimes \mathcal{V} \simeq \mathrm{Fun}^R(\mathrm{Fun}(P, \mathrm{Sp})^{\mathrm{op}}, \mathcal{V}) \simeq \mathrm{Fun}^R(\mathrm{Fun}(P, \mathrm{Spc})^{\mathrm{op}}, \mathcal{V}) \simeq \mathrm{Fun}(P, \mathcal{V}).$$

It follows that under the above equivalence, the continuos functor

$$\tilde{e}(F)\colon\ \mathrm{Fun}(P, \mathrm{Sp})^{\mathrm{op}} \to \mathcal{V}$$

classified by an object $F \in \mathrm{Fun}(P, \mathcal{V})$ sends $よ(p)$ to $F(p) \in \mathcal{V}$.

On the other hand, unwinding definitions (cf. the proof of [SAG, Proposition D.7.2.3]), the evaluation

$$e(F)\colon\ \mathrm{Fun}(P, \mathrm{Sp})^{\vee} \to \mathcal{V}$$

is the ind extension of

$$\tilde{e}(F)|_{\mathrm{Fun}(P,\mathrm{Sp})^{\omega,\mathrm{op}}}\colon\ \mathrm{Fun}(P, \mathrm{Sp})^{\omega,\mathrm{op}} \to \mathcal{V}.$$

Since $よ(p)$ is compact in $\mathrm{Fun}(P, \mathrm{Sp})$, it follows that

$$e(F)(よ(p)) \simeq \tilde{e}(F)(よ(p)) \simeq F(p).$$

As $\{よ(p)\}_{p\in P}$ generate $\mathrm{Fun}(P, \mathrm{Sp})^{\omega}$ under small colimits and retracts, the result follows. □

Corollary 5.4. If P is a locally finite poset, then $\mathrm{Fun}(P, \mathrm{Sp})$ is a proper stable ∞-category: every compact object is proper. The compact objects are precisely those functors that factor through Sp^{ω} and are supported on a finite subset of P. □

6 Proper objects in sheaf categories

In this section, we prove the main theorem. We do this by reducing to the category of P-constructible sheaves for a triangulation P, studied in the previous section.

To this end, we first recall some facts on the geometry of stratifications and microlocal sheaf theory.

Notation 6.1. Let P be a C^1-stratification of a C^1-manifold X. We write

$$N^*P := \cup_{p\in P} N^*X_p \subseteq T^*X$$

for the union of the conormals of the strata in X.

Recollection 6.2 ([13, Corollary 8.3.22], [3])**.** Let X be a real analytic manifold and $\Lambda \subseteq T^*X$ be a closed conic subanalytic singular isotropic. There is a C^{∞} Whitney stratification S of X so that $\Lambda \subseteq N^*S$. Moreover, S can be refined to a C^p Whi:etney triangulation P for any $p \geq 1$.

Recollection 6.3 ([13, Proposition 8.4.1])**.** Let P be a C^1 Whitney stratification of X. Then

$$\mathrm{Cons}_P(X) \simeq \mathrm{Sh}_{N^*P}(X).$$

Recollection 6.4 ([8, Proposition 3.4])**.** Let $\Lambda \subseteq T^*X$ be a closed conic isotropic. The ∞-category $\mathrm{Sh}_\Lambda(X)$ is closed under limits and colimits in $\mathrm{Sh}(X)$.

The results mentioned so far can be combined and summarized as follows.

Observation 6.5. Let X be a real analytic manifold and $\Lambda \subseteq T^*X$ a closed conic subanalytic isotropic. Then there exists a C^1 Whitney triangulation P of X, so that $\Lambda \subseteq N^*P$. And the inclusion

$$\iota_{\Lambda,P,*} \colon \mathrm{Sh}_\Lambda(X) \hookrightarrow \mathrm{Sh}_{N^*P}(X) \simeq \mathrm{Cons}_P(X) \simeq \mathrm{Fun}(P, \mathrm{Sp})$$

is closed under both limits and colimits, *i.e.*, a bi-reflective subcategory (cf. Notation 3.9). In particular, it participates in an adjoint triple $\iota^*_{\Lambda,P} \dashv \iota_{\Lambda,P,*} \dashv \iota^\flat_{\Lambda,P}$.

For the rest of this section, fix X, Λ, and P as above.

Corollary 6.6. If $F \in \mathrm{Sh}_\Lambda(X)$ is compactly supported and has perfect stalks, then F is compact.

Proof. By Proposition 5.2, F is compact in $\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(P, \mathrm{Sp})$ and thus also compact in $\mathrm{Sh}_\Lambda(X)$ by Proposition 3.10. □

Theorem 6.7. A sheaf $F \in \mathrm{Sh}_\Lambda(X)$ is proper if and only if F has perfect stalks.

To prove this, we need a lemma about calculating stalks in $\mathrm{Cons}_P(X)$.

Lemma 6.8. Let (X, P) be an exodromic stratified space, and $\mathcal{V}$ a dualizable stable ∞-category[5] Then the stalk functor at x

$$(-)_x \colon \mathrm{Cons}_P(X; \mathcal{V}) \to \mathcal{V}$$

is canonically equivalent to the evaluation functor

$$\mathrm{ev}_x \colon \mathrm{Fun}(\mathrm{Exit}(X, P), \mathcal{V}) \to \mathcal{V}.$$

at $x \colon [0] \to \mathrm{Exit}(X, P)$.

Proof. Taking stalks at x is by definition the pullback along $x : * \to X$, *i.e.*

$$F_x \simeq x^*F \in \mathrm{Sh}(*; \mathcal{V}) \simeq \mathcal{V}.$$

By the functoriality of the exodromy equivalence, this is equivalent to evaluation at $x \colon [0] \to \mathrm{Exit}(X, P)$. [6] □

Proof of Theorem 6.7. By Proposition 3.11, a sheaf F is proper in $\mathrm{Sh}_\Lambda(X)$ if and only if it is proper in $\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(P, \mathrm{Sp})$. By Corollary 5.4, this is equivalent to F taking values in Sp^ω, which in turn is equivalent to F having perfect stalks by Lemma 6.8. □

[5] Here the dualizability condition is assumed to ensure that we have

$$\mathrm{Cons}_P(X; \mathcal{V}) \simeq \mathrm{Cons}_P(X; \mathrm{Spc}) \otimes \mathcal{V} \simeq \mathrm{Fun}(\mathrm{Exit}(X, P), \mathcal{V}).$$

See [9, §4] for detaield discussions on the exodromy equivalence with coefficients.

[6] Here we distinguish the topological space $*$ consisting of a single point from its homotopy type $[0]$.

The exact same argument using Proposition 4.6 can be used prove the analogous statement for $\mathcal{V}$-properness.

Theorem 6.9. Let $\mathcal{V}$ be a compactly generated stable ∞-category. An object $F \in \mathrm{Sh}_\Lambda(X) \otimes \mathcal{V}$ is $\mathcal{V}$-proper if and only if viewed as an object in $\mathrm{Sh}(X;\mathcal{V})$, it has stalks valued in $\mathcal{V}^\omega$.

Proof. By Proposition 4.6, $F \in \mathrm{Sh}_\Lambda(X) \otimes \mathcal{V}$ is $\mathcal{V}$-proper if and only if

$$\iota^{\mathcal{V}}_{\Lambda,P,*}(F) \in \mathrm{Cons}_P(X) \otimes \mathcal{V} \simeq \mathrm{Fun}(P,\mathrm{Sp}) \otimes \mathcal{V} \simeq \mathrm{Fun}(P,\mathcal{V})$$

is $\mathcal{V}$-proper. By Theorem 5.3, this is equivalent to $F(p) \in \mathcal{V}^\omega$ for every $p \in P$, which in turn is equivalent to F has stalks valued in $\mathcal{V}^\omega$ when viewed as an object in $\mathrm{Sh}(X;\mathcal{V})$ by Lemma 6.8. □

The above result, together with the following results of Kuo-Li [15], leads to our main theorem.

Recollection 6.10 ([15, Theorem 1.1, Theorem 1.2, Corollary 1.7])**.** Let X and Y be real analytic manifolds, $\Lambda \subseteq T^*X$ and $\Sigma \subseteq T^*Y$ closed conic subanalytic istropics. Then the Kunneth formula holds:

$$\mathrm{Sh}_\Lambda(X) \otimes \mathrm{Sh}_\Sigma(Y) \simeq \mathrm{Sh}_{\Sigma\times\Sigma}(X \times Y).$$

The dual of $\mathrm{Sh}_\Lambda(X)$ is $\mathrm{Sh}_{-\Lambda}(X)$. And the equivalence

$$\mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y) \simeq \mathrm{Sh}_\Lambda(X)^\vee \otimes \mathrm{Sh}_\Sigma(Y) \simeq \mathrm{Fun}^L(\mathrm{Sh}_\Lambda(X), \mathrm{Sh}_\Sigma(Y))$$

is given by the assignment

$$K \mapsto (-) * K.$$

Theorem 6.11. Let $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ be a sheaf kernel. The convolution functor

$$- * K \colon \mathrm{Sh}_\Lambda(X) \to \mathrm{Sh}_\Sigma(Y)$$

preserves compact objects if and only if for every $x \in X$, the restriction $K|_{\{x\}\times Y} \in \mathrm{Sh}_\Sigma(Y)$ is a compact object.

Proof. By definition, the convolution preserves compact objects precisely if $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y) \simeq \mathrm{Sh}_{-\Lambda}(X) \otimes \mathrm{Sh}_\Sigma(Y)$ is a $\mathrm{Sh}_\Sigma(Y)$-proper object. By Theorem 6.9, this is equivalent to K having compact stalks when viewed as a sheaf on X valued in $\mathrm{Sh}_\Sigma(Y)$. In light of the naturality of the Kunneth formula, the stalk of said sheaf at $x \in X$ is equivalent to $K|_{\{x\}\times Y} \in \mathrm{Sh}_\Sigma(Y)$, whence the result. □

Corollary 6.12. Let $K \in \mathrm{Sh}_{-\Lambda\times\Sigma}(X \times Y)$ be a sheaf kernel. If K has perfect stalks and $\mathrm{supp}(K|_{\{x\}\times Y})$ is compact for every $x \in X$, then convolution with K preserves compact objects.

Proof. Combine Corollary 6.6 and Theorem 6.11. □

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