

Numbers and numerosities

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Abstract

We develop new aspects of the theory of numerosity; more exactly, we emphasize its relation with the ordinal numbers, cardinal numbers, hyperreal numbers and surreal numbers. In particular, we combine the notion of numerosity with the idea of continuum and we get a definition of Euclidean line which includes all the sets of infinite numbers mentioned above.

keywords: Euclid's principle, Hume's principle, numerosities, ordinal numbers, cardinal numbers, hyperreal numbers, surreal numbers, Euclidean line.

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1 Introduction

The theory of numerosity, introduced in [1, 7] is a development of the classical cardinality for measuring the size of infinite sets. Unlike Cantor's theory, where all countably infinite sets have the same cardinality, numerosity theory allows a finer distinction between different infinite sets and it preserves properties more aligned with our intuitive understanding of "number of elements." The notion of numerosity has been developed in several direction; see e.g. [5, 8, 3, 11, 17, 18, 9, 29, 30, 31, 32] and the references therein.

In this paper we summarize some old results and present new aspects. In the first part we introduce the theory of numerosity in a new way. First we analyze the idea of number and we define the structure of *counting system*; in this context, we present the notion of *numerosity* as a natural extension of the notion of natural number. In particular, we emphasize its relation with the ordinal numbers, cardinal numbers, hyperreal numbers and surreal numbers. The relation between numerosities, hyperreal numbers and surreal numbers is a new fulfillment and suggests further developments. Then, we build a model of the numerosities based on labelled sets as in [1, 7, 9] which allows to develop new features.

1.1 Preliminaries and notation

In principle it would be desirable to define the operation of "counting" for the class of all sets; however, in order to develop the theory, it is convenient to work in a "universe" which is itself a set provided that it is very large. Hence we will work in a universe in which the classes are replaced by sets of strongly inaccessible cardinality.

Definition 1 A cardinal number χ is **inaccessible** if it is not a sum of fewer than χ cardinals that are less than χ and $\zeta < \chi$ implies $2^\zeta < \chi$. χ is **strongly inaccessible** if it is inaccessible and uncountable.

The first inaccessible cardinal number is $\aleph_0$. The first strongly inaccessible cardinal number will be denoted by κ . The existence of sets of strongly inaccessible cardinality is established by the Axiom of Inaccessibility which is independent from ZFC. We will assume this axiom and, in this paper, a set having strongly inaccessible cardinality will be called **pseudoclass**.

We will assume that there exists a set of atoms¹ **Ato** having cardinality κ . Moreover we will denote by **Card** the pseudoclass of the first κ cardinal numbers and by **Ord** the pseudoclass of the first κ ordinal numbers. The cardinality of a set X will be denoted by $|X|$.

Given any set E , let $V_\beta(E)$, $\beta \in \mathbf{Ord}$, be the superstructure on E , namely the family of sets which is inductively defined as follows:

$$\begin{aligned} V_0(E) &= E; \\ V_{\beta+1}(E) &= V_\beta(E) \cup \wp(V_\beta(E)); \\ V_\beta(E) &= \bigcup_{\gamma < \beta} V_\gamma(E) \text{ if } \beta \text{ is a limit ordinal.} \end{aligned}$$

If $A \in V_\beta(\mathbf{Ato}) \setminus \bigcup_{\gamma < \beta} V_\gamma(\mathbf{Ato})$, we say that A has rank β and we will write $\text{rank}(A) = \beta$. Now we set:

$$\mathbf{U}_{\max} := \bigcup_{\gamma < \beta} \{E \in V_\gamma(\mathbf{Ato}) \mid |E| < \kappa\}$$

It is well known that $\mathbf{U}_{\max}$ is a model of *ZFC* and we can work in it without the problems related to the theory of classes.

Definition 2 A set $\mathbf{U} \subseteq \mathbf{U}_{\max} \setminus \mathbf{Ato}$ is called *universe (of sets)* if

1. $A, B \in \mathbf{U} \Rightarrow A \cup B \in \mathbf{U}$
2. $A, B \in \mathbf{U} \Rightarrow A \times B \in \mathbf{U}$

¹In set theory, an atom a is any entity that is not a set, namely a is an atom if and only if

$$\forall x, x \notin a$$

3. $A \in \mathbf{U}$ and $B \subseteq A \Rightarrow B \in \mathbf{U}$

Moreover we shall use the following notation:

- **Inf** will denote the family of infinite subsets in $\mathbf{U}_{\max}$.
- **Fin** will denote the family of the finite subsets in $\mathbf{U}_{\max}$.
- if $E \in \mathbf{U}$, $\wp(E)$ will denote the family of subsets of E .
- $\wp_{\omega}(E) := \wp(E) \cap \mathbf{Fin}$ will denote the family of finite subsets of E
- $\mathfrak{F}(A, B)$ will denote the family of functions $f : A \rightarrow B$, namely $\mathfrak{F}(A, B) = B^A$.

2 Counting systems

2.1 The numbers

One of the aims in counting the elements of sets is the comparison of their sizes. We denote by $\preceq$ is a total preorder² relation over $\mathbf{U}$ and, as usual, we set

$$X \cong Y :\Leftrightarrow (X \preceq Y \text{ and } Y \preceq X).$$

Definition 3 A *comparison system* is a couple $(\mathbf{U}, \preceq)$ where $\mathbf{U}$ is a universe and $\preceq$ is a total preorder relation, called **comparison relation**, which satisfies the following properties:

1. *Null Set Principle*: $A \cong \emptyset$, if and only if $A = \emptyset$.
2. *Union Principle*: If $A \cap B = A' \cap B' = \emptyset$, and $A \cong A'$, $B \cong B'$, then

$$A \cup B \cong A' \cup B'$$

3. *Product Principle*: If $A \cong A'$ and $B \cong B'$; then

$$A \times B \cong B \times A \cong A' \times B'.$$

4. *Unit Principle*:

$$\{a\} \times B \cong B$$

5. *Comparison Principle*: $A \preceq B$, if and only if there exists a set $A' \subseteq B$ such that

$$A \cong A'.$$

If we have a comparison system, then it is possible to build the notion of number:

²We recall that a total preorder is a *transitive* and *reflexive* relation such that $x \preceq y$ or $y \preceq x$.

Definition 4 A set of numbers $\mathcal{N}$ is a set of atoms such that there exists a bijection

$$\Phi : \mathbf{U} / \cong \rightarrow \mathcal{N}$$

Then given a set A , the number of its elements is given by

$$\mathbf{n}(A) = \Phi([A]_{\cong})$$

Notice that in every set of numbers there are two distinguished elements:

$$0 := \mathbf{n}(\emptyset)$$

and

$$1 := \mathbf{n}(\{\emptyset\}).$$

Since we have required $\preceq$ to be a total preorder relation, the following fact follows:

Proposition 5 The numbers is a linearly ordered set with respect to the following order relation: given $\alpha = \mathbf{n}(A)$ and $\beta = \mathbf{n}(B)$

$$\alpha \leq \beta :\Leftrightarrow A \preceq B.$$

2.1.1 Operations with numbers

Given a set of numbers we can define also the two basic operations: the sum and the product.

Definition 6 Given two numbers $\alpha = \mathbf{n}(A)$ and $\beta = \mathbf{n}(B)$ with $A \cap B = \emptyset$, we set

$$\alpha + \beta := \mathbf{n}(A \cup B)$$

By the Union Principle, it follows that the operation "+" is well posed, namely it is independent of the choice of A and B . It is immediate to check that the sum satisfies the commutative property and the associative property since

$$A \cup B = B \cup A$$

and

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Moreover we have that $0 = \mathbf{n}(\emptyset)$ is the identity element for the sum. It is obvious that two numbers can always be added and we have that $\alpha + \beta = \mathbf{n}(A) + \mathbf{n}(B)$: in fact if $A \cap B \neq \emptyset$, we can replace A with the set $A \times \{c\}$ with $c \notin A \cup B$; then, $(A \times \{c\}) \cap B = \emptyset$, and hence,

$$\alpha + \beta = \mathbf{n}((A \times \{c\}) \cup B) = \mathbf{n}(A \times \{c\}) + \mathbf{n}(B) = \mathbf{n}(A) + \mathbf{n}(B)$$

Definition 7 Given two numbers $\alpha = \mathbf{n}(A)$ and $\beta = \mathbf{n}(B)$ with A, B as in Def. 3-(3), we set

$$\alpha \cdot \beta := \mathbf{n}(A \times B)$$

By the Union Principle, it follows that also the operation " $\cdot$ " is well posed. In Def. 3, we have imposed the commutative property. The associative property is satisfied if we identify the couple $((a, b), c)$ with $(a, (b, c))$. This property will be fulfilled by all our examples.

In conclusion, the numbers as defined by Def. 4, satisfy the basic algebraic properties (BAC):

- commutative property with respect to $+$ and $\cdot$
- associative property with respect to $+$ and $\cdot$
- existence of the identity elements " 0 " and " 1 " with respect to $+$ and $\cdot$
- distributive property

Definition 8 *The triple $(\mathbf{U}, \mathcal{N}, \mathbf{n})$ constructed over a comparison relation, is called **counting system**.*

2.2 The main counting systems

2.2.1 The finite counting systems

Now let us see some examples of counting systems:

Example 1 - we take

- $\mathbf{U} = \mathbf{U}_{\max}$:
- $\mathcal{N} = \{0, 1, 2, M\}$ where the number M is read "**many**".

Then, there exists a unique comparison relation which induces the following arithmetic:

$+$	$[0]$	$[1]$	$[2]$	$[M]$
$[0]$	0	1	2	M
$[1]$	1	2	M	M
$[2]$	2	M	M	M
$[M]$	M	M	M	M

;

$\cdot$	$[0]$	$[1]$	$[2]$	$[M]$
$[0]$	0	0	0	0
$[1]$	0	1	2	M
$[2]$	0	2	M	M
$[M]$	0	M	M	M

Actually, this example, does not provide the "smallest" counting system since we can take $\mathcal{N} = \{0, 1\}$. In this case, $(\mathbf{U}, \{0, 1\}, \mathbf{n})$ reduces to a Boolean algebra where, in this case, the addition " $+$ " corresponds to "*or*" ($\vee$) and the product " $\cdot$ " corresponds to "*and*" ($\wedge$):

$+$ $\equiv \vee$	$[0]$	$[1]$
$[0]$	0	1
$[1]$	1	1

;

$\cdot$ $\equiv \wedge$	$[0]$	$[1]$
$[0]$	0	0
$[1]$	0	1

. (1)

2.2.2 The Euclid's Principle and the Natural numbers

If we want to exclude these interesting, but mathematically trivial examples, we need to add some other principle: for example the *V common notion* of Euclid's elements:

The whole is greater than the part.

In our formalism

- **Euclid's principle** - *Given two sets F and G such that F is a proper part of G , then $F \prec G$.*

The most important counting system which satisfies Euclid's Principle is the counting system of natural numbers $(\mathbf{Fin}, \mathbb{N}, |\cdot|)$ where

- $\mathbf{Fin}$ is the family of finite sets.
- $\mathbb{N}$ is the set of natural numbers.
- $|A| = \mathbf{n}(A)$ is the number of elements of a finite set.

$(\mathbf{Fin}, \mathbb{N}, |\cdot|)$ is the smallest counting system which satisfies the Euclid's Principle namely every counting system which satisfies the Euclid's Principle contains $(\mathbf{Fin}, \mathbb{N}, |\cdot|)$ as a subsystem.

The system of natural numbers satisfies a good algebra; hence, we can easily build $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ which satisfy the very rich algebra which is used by everybody.

2.2.3 The Euler's infinite

In order to get a theory which includes infinite sets, Euler introduced the symbol " ∞ " which is similar to the number " M " in example 1. Using Euler's infinite, we obtain the counting system $(\mathbf{U}_{\max}, \mathbb{N} \cup \{\infty\}, \mathbf{n})$ with the relations

$$\begin{aligned} n + \infty &= \infty, \\ 0 \cdot \infty &= 0, \\ n \cdot \infty &= \infty \text{ for } n \neq 0 \end{aligned}$$

This system, satisfies the Euclid's principle only for finite sets, and hence it does not have a good algebra: in particular, the equation

$$x + \infty = \infty,$$

has infinitely many solutions; moreover we cannot define infinitesimal numbers such as

$$\frac{1}{\infty}$$

in a consistent way. For this reason the Euler's " ∞ " did not even get the dignity of "number".

2.2.4 The Hume's Principle and the cardinal numbers

Until the XIX century, the idea of number and "counting system" was rooted not only on the Euclid's principle, but also on the Hume's Principle:

The number of elements in F is equal to the number of elements in G if there is a one-to-one correspondence between F and G .

In our formalism

- **Hume's principle** - Given two sets F and G , then $F \cong G$ if there is a bijection

$$\phi : F \rightarrow G.$$

Euclid's principle and Hume's Principle are satisfied by $(\mathbf{Fin}, \mathbb{N}, |\cdot|)$ but they lead to a contradiction if our universe contains an infinite set.

Cantor had the great idea to drop Euclid's principle and to use only the relation " $\cong$ " suggested by Hume and introduced the *cardinal numbers* counting system which we will denote by

$$(\mathbf{U}_{\max}, \mathbf{Card}, |\cdot|).$$

This counting system is much richer than $(\mathbf{U}, \mathbb{N} \cup \{\infty\}, \mathbf{n})$ since for every set A ,

$$|\wp(A)| > |A|$$

Actually Cantor proved that the set of infinite cardinal number form a sequence

$$\aleph_0 < \aleph_1 < \dots < \aleph_\beta < \dots$$

where β is an ordinal number.

Cardinal numbers do not satisfy Euclid's principle, but retain all the basic algebraic properties (BAP). Unfortunately however, even if the BAP are satisfied, the lack of Euclid's Principle gives rise to an algebra very different from the algebra of $(\mathbf{Fin}, \mathbb{N}, |\cdot|)$; for example, the equation

$$a + x = b \quad (a \leq b)$$

does not have a unique solution whenever a is infinite; therefore it is not always possible to define the difference of two cardinal numbers. Furthermore, their arithmetic is poor since, given two cardinal numbers α and β , if only one of them is infinite, we have that

$$\alpha + \beta = \alpha \cdot \beta = \max(\alpha, \beta)$$

Then, also in this case, we cannot define infinitesimal number such as

$$\frac{1}{\aleph_\beta}.$$

2.2.5 The ordinal numbers

Cantor introduced also the notion of ordinal number. We can define a triple $(\mathbf{W}, \mathbf{Ord}, \sigma\tau\mathfrak{d})$ as follows:

- $\mathbf{W}$ is the class of well ordered sets.
- $\mathbf{Ord}$ is the family of ordinal numbers.
- $\forall A \in \mathbf{W}, \sigma\tau\mathfrak{d}(A) \in \mathbf{Ord}$ is the order type of A .

Similarly, we can define a order relation on sets in $\mathbf{W}$ as follows:

- $A \preceq B$ if and only there exists an injection $\Phi : A \rightarrow B$ which preserves the order, namely $\forall a_1, a_2 \in A$,

$$a_1 < a_2 \Rightarrow \Phi(a_1) < \Phi(a_2).$$

If we equip the ordinal numbers with the operations introduced by Cantor, $(\mathbf{W}, \mathbf{Ord}, \sigma\tau\mathfrak{d})$ is not a counting system since these operations are different from the ones given by definitions 6 and 7. However, if we use the *natural operations* introduced by Hessenberg they form a counting system. We will come back on this point in sections 3.1 and 3.1.1.

2.2.6 The numerosities

We have seen that it is not possible to have a counting theory which contains infinite sets in and which at the same time preserves both Euclid's principle and Hume's principle. However, we can give up Humes's principle and keep Euclid's principle.

Definition 9 *A counting system $(\mathbf{U}, \mathbf{Num}, \text{num})$ which preserves the Euclid's principle is called numerosity theory.*

Euclid's Principle is not only inherent to our idea of number, but also implies an important algebraic property: consider the equation

$$a + x = b, \quad a \leq b; \tag{2}$$

by virtue of the Comparison Principle, this equation always admits a solution; Euclid's principle implies that this solution is unique. This fact allows us to define the class of "signed" numbers $\mathcal{Z}$ which is the analogue of the set of integers $\mathbb{Z}$. $\mathcal{Z}$ can be (informally) defined in the following way

$$\mathcal{Z} = \mathbf{Num} \cup \{-x \mid x \in \mathbf{Num}\}$$

The uniqueness of the solution of (2) and the basic algebraic properties allow to prove that $\mathcal{Z}$ is an ordered integrity domain and therefore it is possible to define the related field of quotients $\mathcal{Q}$, i.e. the set of numbers of the form

$$x = \pm \frac{\text{num}(A)}{\text{num}(B)}; \quad B \neq \emptyset$$

$\mathcal{Q}$ turns out to be a non-Archimedean field which derives, like the field of rational numbers, from the idea of "number of elements" of a set. A further step leads to a field which contains the real number actually to a field isomorphic to a field of hyperreal numbers and to the field of surreal numbers. This point will be seen and discussed in section 4.

The numerosity counting system will be denoted by $(\Lambda_b, \mathbf{Num}, \mathbf{num})$, where

$$\Lambda := \{E \in V_\omega(\mathbf{Ato}) \mid |E| < \kappa\}; \quad \Lambda_b = \Lambda \setminus \mathbf{Ato} \quad (3)$$

namely, the sets in Λ_b have accessible cardinality and finite rank. The latter limitation is necessary as the following proposition shows:

Proposition 10 *If $(\mathbf{U}, \mathbf{Num}, \mathbf{num})$ is a numerosity counting system, and $V_\beta(E) \subset \mathbf{U} \cup \mathbf{Ato}$, then $\beta = \omega$.*

Proof: We argue indirectly and we assume that $\mathbf{U}$ contains a set of infinite rank such as

$$A = \{a, (a, a), (a, a, a), \dots\}, \quad a \in E$$

then, taking $B = \{a\}$ we have that

$$A \times B = \{(a, a), (a, a, a), (a, a, a, a), \dots\} \subset A$$

This fact contradicts the definition of counting system since, by the Euclid's principle we have that

$$\mathbf{num}(A \times B) < \mathbf{num}(A)$$

while by Def. 3-(3,4) we have that

$$\mathbf{num}(A \times B) = \mathbf{num}(A) \times \mathbf{num}(B) = \mathbf{num}(A) \times 1 = \mathbf{num}(A)$$

Hence, we must have $\beta \leq \omega$. Moreover, since

$$\mathbf{rank}(A \times B) > \max\{\mathbf{rank}(A), \mathbf{rank}(B)\},$$

we must have $\beta = \omega$.

□

Remark 11 *If we weaken the request (3) of Def. 3, it is possible to develop a numerosity theory also for sets of infinite rank (see e.g. [8]). However, in this paper, we prefer to avoid the technicalities connected to this choice.*

The existence of a numerosity counting system, namely the consistency of the Euclid's principle with the notion of counting system, will be proved in section 5. In the next two sections will dig into the very rich consequences that the Euclid's principle implies.

3 Numerosities and transfinite numbers

The first peculiarity of $\mathbf{Num}$ is that this set contains in a natural way other sets of numbers such as $\mathbf{Ord}$ and $\mathbf{Card}$.

3.1 Numerosities and ordinal numbers

In this section, we will identify a subset of the numerosities with the initial segment of cardinality κ of the class of ordinal numbers. Let us see how. We remember that by the definition of number,

$$\mathbf{Num} \subset \mathbf{Ato}; \quad (4)$$

by this assumption it makes sense to talk of the numerosity of a set of numerosities.

Definition 12 *The set of the ordinal numerosities (which we will denote by $\mathbf{Ord}$) is defined as follows: $\beta \in \mathbf{Ord}$ if and only if*

$$\beta = \mathbf{num}(\mathbf{O}(\beta))$$

where $\forall \beta \in \mathbf{Num}, \mathbf{O}(\beta) := \{x \in \mathbf{Ord} \mid x < \beta\}$.

It is easy to see that $\mathbf{Ord} \neq \emptyset$ since $0 = \mathbf{num}(\emptyset) = \mathbf{num}(\mathbf{O}(\emptyset)) \in \mathbf{Ord}$; moreover

- if $\beta \in \mathbf{Ord}$, then $\beta + 1 = \mathbf{num}(\mathbf{O}(\beta) \cup \{\beta\}) \in \mathbf{Ord}$
- if $\beta \in \mathbf{Ord}$, then $\beta := \mathbf{num}\left(\bigcup_{\gamma < \beta} \mathbf{O}(\gamma)\right) \in \mathbf{Ord}$

This construction of the ordinal numerosities is similar to the construction of Von Neumann ordinals. While for Von Neumann an ordinal number β_{VN} is the set of all the ordinal numbers contained in β_{VN} , an ordinal numerosities β is the numerosity of the set of ordinal numerosities smaller than β .

Obviously, not all numerosities are ordinal: for example, $\mathbf{num}(\mathbb{N}^+)$ is not ordinal. In fact, if

$$\alpha := \mathbf{num}(\mathbb{N}^+) \quad (5)$$

were an ordinal then:

$$\begin{aligned} \alpha &= \mathbf{num}(\{x \in \mathbf{Ord} \mid x < \mathbf{num}(\mathbb{N}^+)\}) = \mathbf{num}(\mathbb{N}) \\ &= \mathbf{num}(\mathbb{N}^+ \cup \{0\}) = \alpha + 1. \end{aligned}$$

From now on, we will identify the ordinal numbers with the ordinal numerosities.

3.1.1 Operations with ordinal numerosities

In this section we will compare the operations between numerosities with the Cantorian operations between ordinals. Since we use the ordinary symbols $+$ and $\cdot$ for the operations on numerosities, the Cantorian multiplication and addition on $\mathbf{Ord}$ will be denoted by $\oplus$ and $\odot$. Moreover, we will denote by $\beta^{(\gamma)}$ the ordinal exponentiation. We recall that $\beta^{(\gamma)}$ is defined by induction as follows:

- (i) $\beta^{(0)} = 1$
- (ii) $\beta^{(\gamma+1)} = \beta^{(\gamma)} \cdot \beta$
- (iii) $\beta^{(\gamma)} = \sup_{\mathbf{Ord}} \left\{ \beta^{(x)} \mid x \in \mathbf{O}(\gamma) \right\}$ when γ is a limit ordinal³.

From (i) and (ii), it follows that $\forall n \in \mathbb{N}$,

$$\omega^{(n)} = \omega^n. \quad (6)$$

However, if $\gamma \notin \mathbb{N}$, the exponential numerosity β^γ will be defined in a different way (see section 4.1.3) and this fact legitimize the choice of the symbol $\beta^{(\gamma)}$.

We recall that each ordinal γ can be written in the Cantor normal form, namely

$$\beta = \left(\omega^{(j_n)} \odot b_n \right) \oplus \left(\omega^{(j_{n-1})} \odot b_{n-1} \right) \oplus \dots \oplus \left(\omega^{(j_0)} \odot b_0 \right) := \bigoplus_{k=0}^n \left(\omega^{(j_k)} \odot b_k \right)$$

where $b_k \in \mathbb{N}$ and $k_1 > k_2 \Rightarrow j_{k_1} > j_{k_2}$.

Using the Cantor normal form, the natural (or Hessenberg) operations " $+$ " and " $\cdot$ " are defined as follows: given

$$\beta = \bigoplus_{k=0}^n \left(\omega^{(j_k)} \odot b_k \right), \quad \gamma = \bigoplus_{k=0}^n \left(\omega^{(j_k)} \odot c_k \right) \quad (7)$$

(where some coefficient can be null in order to have the same set of j_k 's), we have

$$\beta +_{\mathbf{H}} \gamma := \bigoplus_{k=0}^n \left[\omega^{(j_k)} \odot (b_k + c_k) \right]; \quad \beta \cdot_{\mathbf{H}} \gamma := \bigoplus_{k,l=0}^n \left[\omega^{(j_k + j_l)} \odot (b_k c_l) \right]$$

It is interesting and somewhat surprising that the natural operations coincide with the numerosity operations, namely

$$\beta +_{\mathbf{H}} \gamma = \beta + \gamma \quad \text{and} \quad \beta \cdot_{\mathbf{H}} \gamma = \beta \cdot \gamma \quad (8)$$

Let us prove this fact.

Lemma 13 *If $\beta = \bigoplus_{k=0}^n \left(\omega^{(j_k)} \odot b_k \right)$, then $\beta = \sum_{k=0}^n b_k \omega^{(j_k)}$.*

³Here, we have used the obvious notation

$$\sup_{\mathbf{Ord}} A := \min \{ \gamma \in \mathbf{Ord} \mid \forall x \in A, \gamma \geq x \}$$

Proof: We set $S_m = \left\{ \xi \mid \xi < \bigoplus_{k=0}^m (\omega^{\langle j_k \rangle} \odot b_k) \right\}$; then, we have that

$$\mathbf{O}(\beta) = (S_n \setminus S_{n-1}) \cup (S_{n-1} \setminus S_{n-2}) \cup \dots \cup (S_1 \setminus S_0) \cup S_0$$

and

$$\begin{aligned} S_m - S_{m-1} &= \left\{ \xi \mid \bigoplus_{k=0}^{m-1} (\omega^{\langle j_k \rangle} \odot b_k) \leq \xi < \bigoplus_{k=m-1}^m (\omega^{\langle j_k \rangle} \odot b_k) \right\} \\ &= \left\{ \xi \mid \xi < \omega^{\langle j_m \rangle} \odot b_m \right\} \end{aligned}$$

Therefore

$$\begin{aligned} \text{num}(S_m \setminus S_{m-1}) &= b_m \cdot \text{num}(\left\{ \xi \mid \xi < \omega^{\langle j_m \rangle} \right\}) \\ &= b_m \cdot \text{num}(\mathbf{Ord}(\omega^{\langle j_m \rangle})) = \omega^{\langle j_m \rangle} \end{aligned}$$

and hence

$$\begin{aligned} \beta &= \text{num}(\mathbf{O}(\beta)) = \text{num}(S_n \setminus S_{n-1}) + \dots + \text{num}(S_1 \setminus S_0) + \text{num} S_0 \\ &= b_n \omega^{\langle j_n \rangle} + \dots + b_1 \omega^{\langle j_1 \rangle} + b_0 \omega^{\langle j_0 \rangle} \end{aligned}$$

□

Theorem 14 *The identities (8) are satisfied.*

Proof: By Th. 13,

$$\begin{aligned} \beta +_{\mathbf{H}} \gamma &= \bigoplus_{k=0}^n \left[\omega^{\langle j_k \rangle} \odot (b_k + c_k) \right] = \sum_{k=0}^n (b_k + c_k) \omega^{\langle j_k \rangle} \\ &= \sum_{k=0}^n b_k \omega^{\langle j_k \rangle} + \sum_{k=0}^n c_k \omega^{\langle j_k \rangle} = \beta + \gamma \end{aligned}$$

Moreover,

$$\begin{aligned} \beta \cdot_{\mathbf{H}} \gamma &= \bigoplus_{k,l=0}^n \left[\omega^{\langle j_k +_{\mathbf{H}} j_l \rangle} \odot (b_k c_l) \right] = \bigoplus_{k=0}^n \omega^{\langle j_k \rangle} \odot b_k + \bigoplus_{l=0}^n \omega^{\langle j_l \rangle} \odot c_l \\ &= \sum_{k=0}^n b_k \omega^{\langle j_k \rangle} + \sum_{l=0}^n c_l \omega^{\langle j_l \rangle} = \beta \gamma. \end{aligned}$$

□

Remark 15 *Within the theory of ordinal numbers the description of $\beta +_{\mathbf{H}} \gamma$ and $\beta \cdot_{\mathbf{H}} \gamma$ in terms of well ordered set is rather involved; particularly the description of the set whose order type is $\beta \cdot_{\mathbf{H}} \gamma$. On the contrary, using ordinal numerosities $\beta \cdot_{\mathbf{H}} \gamma$ is easily described as the numerosity of the set $\mathbf{O}(\beta) \times \mathbf{O}(\gamma)$.*

3.2 Numerosities and cardinal numbers

As usual, a cardinal number χ can be identified with the ordinal number

$$\chi_0 = \min \{ \gamma \in \mathbf{Ord} \mid |\mathbf{O}(\gamma)| \geq \chi \}$$

Thanks to this identification, from now on we will assume that

$$\mathbb{N} \subset \mathbf{Card} \subset \mathbf{Ord} \subset \mathbf{Num}. \quad (9)$$

If a numerosity coincide with a cardinal number will be called *cardinal numerosity*. For example, we have that

$$\omega = \aleph_0 = \mathbf{num}(\mathbb{N}).$$

and in general, $\forall j < \kappa$, $\omega_j = \aleph_j$. Also, $\forall j \in \mathbf{Ord}$, the numerosities $\beth_j$ ⁴ are well defined. If we assume the Continuum Hypotesis, then, $\beth_j = \omega_j$, but this assumption is not relevant for the numerosity theory. In section 6.3.2, we will analyze the numerosity

$$\beth_1 = \mathbf{num}(\wp(\mathbb{N})). \quad (10)$$

Caveat! - From now on, the symbols $\beth_j$'s will denote numerosities and the operations will denote the operations in the framework on numerosities.

3.3 Three different ways of counting

In this section, we will analyze the operation of counting from a more intuitive perspective. In section 3.4, we will formalize this operation introducing the notion of Λ -limit which, among the other things, will allow us to move from discrete to continuous, namely from counting to measuring.

In everyday life, there are several possible ways of counting the number of elements of finite sets which, of course, yield the same result. However, when these ways of counting are formalized and extended to infinite sets, they may give different counting systems. Basically, there are three different approaches.

- The first way of counting consists in associating to each element of a set an element of another one. If in this way one gets a 1-1 correspondence and claims that the two sets have the same number of elements. This intuition corresponds to the equipotency relation and to the Cantorian theory of cardinal numbers.
- In the second way of counting, one arranges the elements of a given set in a row and compares such a row with the sequence of natural numbers. This intuition leads to the notion of order type and to the theory of ordinal numbers.

⁴We recall that the Beth numbers are defined by transfinite recursion as follows

- $\beth_0 := \aleph_0$
- $\beth_{\beta+1} := 2^{\beth_\beta}$
- $\beth_\beta := \sup \{ \beth_\gamma \mid \gamma < \beta \}$ if β is a limit number.

- However, there exists a third way of counting which consists in arranging the elements of a given sets into smaller groups to be counted separately. As we will see in the next section this intuition is strongly related to the notion of numerosity.

Please, note that the three ways of counting discussed above imply more and more complex logical operations.

- The first way corresponds to the concept of number of a two years old kid, who associate numbers to sets of fingers of his hands; *e.g.*, the number 3 corresponds to the set

$$\{\text{index finger, middle finger, ring finger}\}.$$

- The second way of counting corresponds to the concept of number of a four years old child: she/he has already memorized the sequence of the first natural numbers and she/he is able to count objects by arranging them in a row.
- The third way of counting is much more sophisticated and requires several operations, such as collecting similar objects together, and comparing different groups. This is the way of counting of a grown child.

Clearly, the third way of counting is only possible if the objects of a given set have a "some feature" that allows us to bunch "similar objects". So we are lead to a structure formalized by the notion of *label* that will be considered in the next section.

3.4 The label-lattice

If we want to formalize the third way of counting to any set, we need a criterion to collect groups of elements. Informally, we may collect elements sharing the same "label". Now, let us formalize the notion of label.

Definition 16 A *labelling* is a family of sets $\mathfrak{L}$ which satisfies the following relations:

1. $\mathfrak{L} \subset \mathbf{Fin}$;
2. $\lambda, \mu \in \mathfrak{L} \Rightarrow \lambda \cap \mu \in \mathfrak{L}$ and $\exists \sigma \in \mathfrak{L}, \lambda \cup \mu \subseteq \sigma$.
3. $\forall a \in \Lambda, \exists \lambda \in \mathfrak{L}, a \in \lambda$.

Given a labelling $\mathfrak{L}$, the label of an element $a \in \Lambda$ is defined as follows:

$$\ell(a) = \bigcap \{\mu \in \mathfrak{L} \mid a \in \mu\}. \quad (11)$$

The set $\mathfrak{L} = \wp_\omega(\Lambda)$, is the the maximum labelling. Actually, there exists infinitely many labellings; now, we will consider a generic labelling $\mathfrak{L}$ since it is sufficient to our purposes. In section 6 we will consider peculiar labellings which provide the numerosity theory with "special properties".

By Def. 16-(2), $(\mathfrak{L}, \subseteq)$ can be equipped with a lattice structure by setting

$$\lambda \wedge \mu := \lambda \cap \mu; \lambda \vee \mu := \bigcap \{\sigma \in \mathfrak{L} \mid \lambda \cup \mu \subseteq \sigma\};$$

it will be called **label-lattice**. In particular, it is a directed set; then function $\varphi : \mathfrak{L} \rightarrow R$ is a *net* (with values in R); the set of such nets will be denoted by $\mathfrak{F}(\mathfrak{L}, R)$. If R is a commutative ring, then also $\mathfrak{F}(\mathfrak{L}, R)$ is a commutative ring with the operations defined by

$$(\varphi + \psi)(\lambda) = \varphi(\lambda) + \psi(\lambda); (\varphi \cdot \psi)(\lambda) = \varphi(\lambda) \cdot \psi(\lambda).$$

Given $A \in \Lambda$, the **counting net** $\varphi_A \in \mathfrak{F}(\mathfrak{L}, \mathbb{Z})$ is defined as follows:

$$\varphi_A(\lambda) = |\{x \in A \mid x \in \lambda\}| = |A \cap \lambda|.$$

Now we extend the subtraction in **Num** when $\text{num}(A) < \text{num}(B)$ by setting,

$$\text{num}(A) - \text{num}(B) = -\text{num}(B \setminus A'), \text{ with } A' \subset A, \text{ num}(A') = \text{num}(A).$$

We will denote by $\mathcal{Z}$ the relative ring and we will call it ring of the **signed numerosities**.

If $\mathfrak{F}_{\mathcal{Z}}(\mathfrak{L}, \mathbb{Z})$ is the ring generated by the counting nets in $\mathfrak{F}(\mathfrak{L}, \mathbb{Z})$, we denote by

$$J_{\mathcal{Z}} : \mathfrak{F}_{\mathcal{Z}}(\mathfrak{L}, \mathbb{Z}) \rightarrow \mathcal{Z} \tag{12}$$

the ring homomorphism such that $J_{\mathcal{Z}}(\varphi_A) = \text{num}(A)$. A number $J_{\mathcal{Z}}(\varphi) \in \mathcal{Z}$ can be seen as a sort of limit of the net $\lambda \mapsto \varphi(\lambda)$; hence, it is natural to employ the following notation⁵

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = J_{\mathcal{Z}}(\varphi).$$

In order to distinguish the above limit (which we will call Λ -*limit*) from the Cauchy limit of a net, we have used the notation " $\lambda \uparrow \Lambda$ " rather than " $\lambda \rightarrow \Lambda$ ". Hence the numerosity of a set can be expressed as follows:

$$\text{num}(A) = \lim_{\lambda \uparrow \Lambda} |A \cap \lambda|. \tag{13}$$

This notation is very significant since $\text{num}(A)$ appears as the limit of the cardinality of finite subsets of A that grow with the growth of λ . Furthermore, the idea of Λ -limit expresses well the intuitive idea exposed in section 3.3: in order to count the elements of a large set, it is convenient to count the elements of small groups and unify the result. Finally, since $J_{\mathcal{Z}}$ is a ring homomorphism, the Λ -limit satisfies the some of the properties of the Cauchy limit:

⁵This kind of limit generalizes the α -limit defined in [5] and, as we will see, agrees with the notion of Λ -limit used in other papers such as [9].

- if there exists λ_0 such that $\forall \lambda \geq \lambda_0, \varphi(\lambda) = \psi(\lambda)$, then

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = \lim_{\lambda \uparrow \Lambda} \psi(\lambda)$$

- if $C_z(\lambda)$ is the net identically equal to $z \in \mathbb{Z}$, then

$$\lim_{\lambda \uparrow \Lambda} C_z(\lambda) = z$$

- for every $\varphi, \psi \in \mathfrak{F}_c(\mathfrak{L}, \mathbb{Z})$,

$$\begin{aligned} \lim_{\lambda \uparrow \Lambda} [\varphi(\lambda) \pm \psi(\lambda)] &= \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \pm \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \\ \lim_{\lambda \uparrow \Lambda} [\varphi(\lambda) \cdot \psi(\lambda)] &= \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \end{aligned}$$

- for every $\varphi, \psi \in \mathfrak{F}_c(\mathfrak{L}, \mathbb{Z})$,

$$\varphi(\lambda) \geq \psi(\lambda) \Rightarrow \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \geq \lim_{\lambda \uparrow \Lambda} \psi(\lambda).$$

At this point, the notion of Λ -limit might appear technically irrelevant, but it will play an important role when it will be extended to the ring of $\mathbb{R}$ -valued nets (see Sec. 4.1.1).

3.5 The Hume principle revisited

The notion of Λ -limit suggests the following definition:

Definition 17 *If $A \in \Lambda$ and $B \subseteq \Lambda$, a bijective map*

$$\Phi : A \rightarrow B$$

*is called **comparison map** if*

$$\lim_{\lambda \uparrow \Lambda} |A \cap \lambda| = \lim_{\lambda \uparrow \Lambda} |\Phi(A) \cap \lambda|.$$

From this definition, immediately follow that for every comparison map $\Phi : A \rightarrow \Lambda$

$$\text{num}(\Phi(A)) = \text{num}(A).$$

Then the following facts hold:

Proposition 18 *Let $\Phi : A \rightarrow B$ be a map which eventually preserves the labels, namely $\forall a \in A$*

$$\ell(\Phi(a)) = \ell(a) \tag{14}$$

then Φ is a comparison map and $\text{num}(\Phi(A)) = \text{num}(A)$

Proof: Trivial.

□

Proposition 19 Let $\Phi : A \rightarrow B$ be a bijective map such that $\forall a \in A$,

$$\ell(\Phi(a)) \cap B = \ell(a) \quad (15)$$

then Φ is a comparison map and $\mathbf{num}(\Phi(A)) = \mathbf{num}(A)$.

Proof: We have that

$$\begin{aligned} |B \cap \lambda| &= |\{x \in A \mid \Phi(x) \in \lambda\}| = |\{x \in A \mid \ell(\Phi(x)) = \lambda\}| \\ &= |\{x \in A \mid \ell(x) = \lambda\}| = |A \cap \lambda| \end{aligned}$$

□

Now let us see the relation of **Num** with respect to the Hume Principle. The Hume Principle (HP) can be applied to the numerosities and to the ordinal numbers provided that we restrict the class of permitted maps, namely, we have the following situation:

- HP for Cardinals: $|A| = |B|$ if and only if there is a bijection $\Phi : A \rightarrow B$.
- HP for Ordinals: if A and B are two well ordered sets, then $\mathbf{ord}(A) = \mathbf{ord}(B)$ if and only if there is a bijection $\Phi : A \rightarrow B$ which respects the order, namely $\forall a \in A, \forall b \in B, a < b \Rightarrow \Phi(a) < \Phi(b)$.
- HP for Numerosities: if $A, B \in \Lambda$, then $\mathbf{num}(A) = \mathbf{num}(B)$ if there exists comparison map $\Phi : A \rightarrow B$.

4 Numerosities and the continuum

In the previous section we have compared the numerosity with the other numbers used to "count" infinite sets. In this section we will examine the relation of the numerosities with the numbers used to "measure" continuous magnitudes. In particular we will relate the numerosity to a peculiar field of **hyperreal numbers** (see e.g. [27]) and to the field of **surreal numbers** (see e.g. [14]).

4.1 Euclidean numbers

The Euclidean line is fundamental not only for geometry, but also for analysis and applied mathematics because, once the origin and the unit element have been chosen, all magnitudes can be represented by its points that we will call **Euclidean numbers**. The Euclidean line is generally identified with the real line, but this identification seems too restrictive to us since infinite and infinitesimal magnitudes cannot be described by real numbers. Actually the existence of infinitesimal numbers has been one of the main problem in all history of mathematics. We recall some recent studies on this argument: [25, 26, 13, 20]. In this paper, we propose a vision of the Euclidean line strictly related to the numerosities.

Definition 20 *The Euclidean line $\mathbb{E}$ is a real closed⁶ field which contains the numerosities and such that $\forall \xi \in \mathbb{E}, \exists \zeta \in \mathbf{Num}$ such that*

$$|\xi| \leq \zeta. \quad (16)$$

In this section we will construct the Euclidean via the Λ -limit and we will examine some of its properties. In particular, we will see that it is unique up to isomorphism.

We recall that recently other paths approaching non Archimedean fields have been investigated (see e.g. [12], [23] and their references).

4.1.1 Numerosities and hyperreal numbers

Probably the most relevant property of every numerosity system is that it is isomorphic to a peculiar subset of the hypernatural numbers $\mathbb{N}^*$ as defined in Nonstandard Analysis (NSA), provided that the hyperreal field $\mathbb{R}^*$ is chosen in a suitable way. In this section we will construct $\mathbb{R}^*$ exploiting the ring of signed numerosities and the notion of Λ -limit.

First of all we need the following lemma:

Lemma 21 *If $\varphi \in \mathfrak{F}_{\mathbb{Z}}(\mathfrak{L}, \mathbb{Z})$, then there exists a set $K \in \Lambda$ such that $\forall \lambda \in \mathfrak{L}$*

$$\varphi(\lambda) = \varphi(K \cap \lambda)$$

Proof: By definition, every $\varphi \in \mathfrak{F}_{\mathbb{Z}}(\mathfrak{L}, \mathbb{Z})$ can be written as follows:

$$\varphi(\lambda) = \sum_{m=0}^n z_m \varphi_{A_m}(\lambda), \quad z_m \in \mathbb{Z}, \quad A_m \in \Lambda$$

Since φ_{A_m} is a counting function, then $\varphi_{A_m}(\lambda) = |A_m \cap \lambda| = \varphi_{A_m}(A_m \cap \lambda)$ and hence

$$\varphi(\lambda) = \varphi(K \cap \lambda)$$

where $K = A_1 \cup \dots \cup A_n$.

□

We set

$$\mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R}) = \{\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \mid \exists K \in \Lambda, \varphi(K \cap \lambda) = \varphi(\lambda)\};$$

it is immediate to see that $\mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R})$ is a subring of $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$; also $\mathfrak{F}_{\mathbb{Z}}(\mathfrak{L}, \mathbb{Z})$ is a subring of $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$; now, we denote by $\mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})$ the subring of $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$ generated by $\mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R})$ and $\mathfrak{F}_{\mathbb{Z}}(\mathfrak{L}, \mathbb{Z})$.

Theorem 22 *Given a numerosity theory $(\Lambda, \mathbf{Num}, \mathbf{num})$ and a labelling $\mathfrak{L}$, there is an ordered field $\mathbb{E} \subset \mathbf{Ato}$ and a surjective ring homomorphism*

$$J : \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R}) \rightarrow \mathbb{E} \quad (17)$$

such that

⁶A field is called *real closed* if every polynomial of odd degree has at least one root.

- $\forall A \in \Lambda$,

$$J(\varphi_A) = \mathbf{num}(A).$$

- $\mathbf{Num} \subset \mathbb{E}$ and the operations $+$ and $\cdot$ coincide;
- $\mathbb{R} \subset \mathbb{E}$ and the operations $+$ and $\cdot$ coincide.

Proof: Let $J_{\mathcal{Z}}$ be the homomorphism defined by (12). It is easy to see that set $\ker(J_{\mathcal{Z}})$ is a prime ideal in $\mathfrak{F}_{\mathcal{Z}}(\mathfrak{L}, \mathbb{Z})$; hence the set

$$\mathcal{I}_{\mathbb{E}} = \{\varphi\psi \mid \varphi \in \ker(J_{\mathcal{Z}}), \psi \in \mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R})\}$$

is an ideal in $\mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})$. Actually, $\mathcal{I}_{\mathbb{E}}$ is a maximal ideal in $\mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R})$ since $\mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R})/\mathcal{I}_{\mathbb{E}}$ is a field. Let us check this fact. Take $[\varphi] \in \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})/\mathcal{I}_{\mathbb{E}}$, $[\varphi] \neq 0$; we need to prove that $[\varphi]$ has an inverse; let $\chi \in \mathfrak{F}_{\mathcal{Z}}(\mathfrak{L}, \mathbb{Z})$ be the characteristic function of $\varphi^{-1}(0)$, then

$$\forall \lambda \in \mathfrak{L}, \varphi(\lambda) \cdot \chi(\lambda) = 0 \text{ and } \varphi(\lambda) + \chi(\lambda) \neq 0$$

Since $\ker(J_{\mathcal{Z}})$ is a prime ideal and $\varphi(\lambda) \cdot \chi(\lambda) = 0$, then $\chi \in \ker(J_{\mathcal{Z}}) \subset \mathcal{I}_{\mathbb{E}}$; thus

$$[\varphi] \cdot \left[\frac{1}{\varphi + \chi} \right] = [\varphi + \chi] \cdot \left[\frac{1}{\varphi + \chi} \right] = 1.$$

So, $\mathfrak{F}_{\mathbb{B}}(\mathfrak{L}, \mathbb{R})/\mathcal{I}_{\mathbb{E}}$ is a field and the projection

$$\Pi : \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R}) \rightarrow \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})/\mathcal{I}_{\mathbb{E}}$$

is a ring homomorphism. Now, we define a field $\mathbb{E} \subset \mathbf{Ato}$ isomorphic to $\mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})/\mathcal{I}_{\mathbb{E}}$. In order to do this, it is sufficient to take an injective map

$$\Psi : \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})/\mathcal{I}_{\mathbb{E}} \rightarrow \mathbf{Ato} \quad (18)$$

such that $\forall A \in \Lambda$, $\Psi(\varphi_A) = \mathbf{num}(A)$ and $\forall r \in \mathbb{R}$, $\Psi([C_r]) = r$ (here $C_r \in \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})$ is a net identically equal to r and β). $\mathbb{E} = \text{Im}(\Psi)$ is naturally equipped with a field structure by setting $a+b = \Psi(\Psi^{-1}(a) + \Psi^{-1}(b))$ and $ab = \Psi(\Psi^{-1}(a) \cdot \Psi^{-1}(b))$.

In conclusion, we have constructed a ring homomorphism

$$J := \Psi \circ \Pi : \mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R}) \rightarrow \mathbb{E}$$

which satisfies the requests of the theorem.

□

Thanks to the homomorphism (17), the notion of Λ -limit can be extended to every net in $\mathfrak{F}_{\mathbb{E}}(\mathfrak{L}, \mathbb{R})$ by setting

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = J_{\mathbb{E}}(\varphi).$$

Clearly, it satisfies the properties listed at the end of Sec. 3.4 and the following one which is not shared by the Cauchy limit:

- if there exists λ_0 such that $\forall \lambda \geq \lambda_0, \varphi(\lambda) > 0$, then

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) > 0.$$

By Th. 22 and by well known results (see e.g. [6]), $\mathbb{E}$ is a hyperreal field. Then, we can exploit the usual notation and techniques of Nonstandard Analysis:

- if $\xi, \zeta \in \mathbb{E}$ we set $\xi \sim \zeta$ if $\xi - \zeta$ is infinitesimal;
- if $\xi \in \mathbb{E}$ is a bounded number then, the standard part of ξ , $st(\xi)$, is the only real number $r \sim \xi$. If ξ is unbounded, then we write $st(\xi) = \pm\infty$.
- every real function $f \in \mathfrak{F}(\mathbb{R}, \mathbb{R})$ can be extended to $\mathbb{E}$ by setting, for every $\xi := \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$

$$f^{\otimes}(\xi) := \lim_{\lambda \uparrow \Lambda} f(\varphi(\lambda)). \quad (19)$$

As usual, when the meaning is clear from the contest, we omit the " $\otimes$ " and we will simply write $f(\xi)$.

- if $E_\lambda \in V_n(\mathbb{R})$ is a net of sets, then their Λ -limit is defined by induction over n as follows: if $n = 1$, and $E_\lambda \subset \mathbb{R}$

$$\lim_{\lambda \uparrow \Lambda} E_\lambda := \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \forall \lambda, \varphi(\lambda) \in E_\lambda \right\} \in V_1(\mathbb{E})$$

and if $E_\lambda \in V_n(\mathbb{R})$

$$\lim_{\lambda \uparrow \Lambda} E_\lambda := \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \forall \lambda, \varphi(\lambda) \in E_\lambda \right\} \in V_{n+1}(\mathbb{E}).$$

- the hyperreal triple $(\otimes, \mathbb{R}, \mathbb{R}^{\otimes})$ induces a nonstandard universe $(\otimes, V_\omega(\mathbb{R}), V_\omega(\mathbb{E}))$ in the sense of Keisler (see [27]); the map

$$\otimes : V_\omega(\mathbb{R}) \rightarrow V_\omega(\mathbb{E}) \quad (20)$$

is defined as follows:

$$A^{\otimes} := \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \forall \lambda, \varphi(\lambda) \in A \right\}$$

Notice that this definition is equivalent to the following:

$$A^{\otimes} = \lim_{\lambda \uparrow \Lambda} C_A(\lambda)$$

where $C_A(\lambda)$ is the net identically equal to A .

- If $K \in \Lambda$, and $\{r_k\}_{k \in K}$ is a real net (i.e. $r_k \in \mathbb{R}$), then the **hyperfinite sum** is defined as follows:

$$\sum_{k \in K^{\otimes}} r_k = \lim_{\lambda \uparrow \Lambda} \left(\sum_{k \in K \cap \lambda} r_k \right).$$

4.1.2 The structure of Euclidean line

If $K \in \mathbf{Inf}$, we set

$$\mathbb{R}[K] := \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda \cap K) \mid \varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \right\} = \left\{ \sum_{k \in K^{\otimes}} r_k \mid r_k \in \mathfrak{F}(K^{\otimes}, \mathbb{R}) \right\}$$

by virtue of our construction, $\mathbb{R}[K]$ is a hyperreal field.

Remark 23 *Actually every hyperreal field obtained by an ultrapower $\mathbb{R}^I/\mathcal{U}$ is isomorphic to some $\mathbb{R}[K]$ provided that its cardinality is less than κ . The other hyperreal fields can be obtained as the inductive limit of a suitable set of $\mathbb{R}[K]$'s.*

Theorem 24 *The hyperreal field defined in Th. 22 is isomorphic the Euclidean line as defined by Def. 20.*

Proof. Since $\mathbb{E} = \mathbb{R}^{\otimes}$ is a hyperreal field, $\forall \xi \in \mathbb{E}$, there exists $\nu \in \mathbb{N}^{\otimes}$ such that $|\xi| \leq \nu$. By lemma 21, $\nu \in \mathbb{R}[K]$ for some $K \in \Lambda$. Hence

$$|\xi| \leq \nu \leq \text{num}(K).$$

Moreover, since $\mathbb{E}$ is a hyperreal field, it is real closed and κ -saturated. Hence $\mathbb{E}$ is a real closed κ -saturated field of cardinality κ ; hence by well known results, it is unique up to isomorphisms.

□

The next theorem characterizes the Euclidean numbers as hyperfinite sums of real numbers.

Theorem 25 *For every $\xi \in \mathbb{E}$, there exists a family of real numbers $\{r_k\}_{k \in K}$, $K \in \Lambda$ such that*

$$\xi = \sum_{k \in K^{\otimes}} r_k.$$

Proof. Let $\{\lambda_j\}_{j \in \text{Ord}}$ be a well ordering of $\mathfrak{L}$. Given $\xi = \lim_{\lambda \uparrow \Lambda} \varphi_{\xi}(K \cap \lambda) \in \mathbb{E}$, we set

$$\begin{aligned} r_0 &:= \varphi_{\xi}(K \cap \lambda_0) \\ r_j &= \varphi_{\xi}(K \cap \lambda_j) - \sum_{k \in \lambda_j \cap K, k < j} r_k \end{aligned} \tag{21}$$

Hence, $\forall \lambda \in \mathfrak{L}$,

$$\varphi_{\xi}(K \cap \lambda_j) = \sum_{k \in \lambda_j \cap K} r_k$$

The conclusion follows taking the Λ -limit of both sides.

□

By our construction, we have that

$$\mathbb{E} \subseteq \mathbf{Ato}$$

This fact implies that

$$V_\omega(\mathbb{E}) \subseteq \Lambda.$$

The map

$$\otimes : V_\omega(\mathbb{R}) \rightarrow V_\omega(\mathbb{E})$$

is a nonstandard Universe in the sense of Keisler ([27], Def. 15.8). If we assume $\mathbb{E} = \mathbf{Ato}$, and hence

$$V_\omega(\mathbb{E}) = \Lambda, \quad (22)$$

we are in the usual framework of NSA. Nevertheless, we can use nonstandard methods even if we do not assume (22).

Theorem 26 *Given a hyperreal field $\mathbb{R}[K]$, there is $\gamma \in \mathbf{Ord}$ such that*

$$\mathbb{R}[K] \subseteq \mathbb{R}(\gamma) := \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(|\lambda \cap \mathbf{O}(\gamma)|) \mid \varphi \in \mathfrak{F}_{\mathbb{E}}(\mathbb{N}, \mathbb{R}) \right\}.$$

Proof: By the comparison principle there exists $\gamma \in \mathbf{Ord}$ and $K' \subset \mathbf{O}(\gamma)$ such that $\mathbf{num}(K') = \mathbf{num}(K)$. Then, denoting by χ_A the characteristic function of A , it holds

$$\sum_{k \in K \cap \lambda} \chi_{\{k\}}(k) = |K \cap \lambda| = |K' \cap \lambda| = \sum_{k \in K' \cap \lambda} \chi_{\{k\}}(k)$$

and hence

$$\sum_{p \in K \cap \lambda} r_k = \sum_{k \in K \cap \lambda} r_k \chi_{\{k\}}(k) = \sum_{k \in K' \cap \lambda} r_k \chi_{\{k\}}(k) = \sum_{p \in K' \cap \lambda} r_k$$

So, for every $\xi \in \mathbb{R}[K]$, we have that

$$\xi = \lim_{\lambda \uparrow \Lambda} \sum_{p \in \lambda \cap K} r_k = \lim_{\lambda \uparrow \Lambda} \sum_{p \in K' \cap \lambda} r_k \in \mathbb{R}(\gamma).$$

□

Thanks to the above theorem, the field of Euclidean numbers can be filtered as follows:

$$\mathbb{E} = \bigcup_{\gamma < \kappa} \mathbb{R}(\gamma). \quad (23)$$

This representation of the Euclidean numbers is quite useful; in fact every Euclidean number can be seen as the Λ -limit of a net defined on $\wp_\omega(\mathbf{O}(\gamma))$; moreover, every numerosity is equal to the numerosity of a subset of $\mathbf{O}(\gamma)$.

4.1.3 Exponentiation of numerosities

Since we have identified the numerosities with hypernatural numbers, if f is a real function and β is a numerosity, then $f(\beta)$ is a hyperreal number and it could be the numerosity of some set. In particular, if β and γ are numerosities, then β^γ is a numerosity defined in a different way than the ordinal numerosity $\beta^{(\gamma)}$. Let us investigate this point.

Proposition 27 1. $\forall E \in \Lambda$,

$$\text{num}(\wp_\omega(E)) = 2^{\text{num}(E)};$$

2. if X is a finite set, then $\forall E \in \Lambda$,

$$\text{num}(\mathfrak{F}(X, E)) = \text{num}(X)^{\text{num}(E)} = |X|^{\text{num}(E)};$$

3. if $X, E \in \Lambda \setminus \{\emptyset\}$, we set

$$\mathfrak{F}_{fin}(X, E) := \{f : D \rightarrow E \mid D \in \wp_\omega(X)\};$$

then, if $a \in E$,

$$\text{num}(\mathfrak{F}_{fin}(X, E \setminus \{a\})) = \text{num}(E)^{\text{num}(X)}.$$

Proof: (1) - Let $\lambda \in \mathfrak{L}$; $a \in \wp_\omega(E) \cap \lambda$ if and only if $a \subseteq E \cap \lambda$; then

$$|\wp_\omega(E) \cap \lambda| = 2^{|E \cap \lambda|}$$

Hence

$$\begin{aligned} \text{num}(\wp(E)) &= \lim_{\lambda \uparrow \Lambda} |\wp(E) \cap \lambda| = \lim_{\lambda \uparrow \Lambda} 2^{|E \cap \lambda|} \\ &= 2^{\lim_{\lambda \uparrow \Lambda} |E \cap \lambda|} = 2^{\text{num}(E)} \end{aligned}$$

(2) - Take λ so large that $X \subseteq \lambda$. $f \in \mathfrak{F}(X, E) \cap \lambda$ if and only if $\text{Im } f \subseteq E \cap \lambda$; then

$$|\mathfrak{F}(X, E) \cap \lambda| = |X|^{|E \cap \lambda|}$$

Hence

$$\begin{aligned} \text{num}(\mathfrak{F}(X, E)) &= \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}(X, E) \cap \lambda| = \lim_{\lambda \uparrow \Lambda} |X|^{|E \cap \lambda|} \\ &= |X|^{\lim_{\lambda \uparrow \Lambda} |E \cap \lambda|} = |X|^{\text{num}(E)} \end{aligned}$$

(3) We set

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \cap \lambda; \\ a & \text{otherwise} \end{cases}$$

the map $f \in \mathfrak{F}_{fin}(X \cap \lambda, (E \cap \lambda) \setminus \{a\}) \rightarrow \tilde{f} \in \mathfrak{F}(X \cap \lambda, E \cap \lambda)$ is a bijection. Moreover, if $\tilde{f} \in \mathfrak{F}(X \cap \lambda, E \cap \lambda)$

$$|\mathfrak{F}_{fin}(X \cap \lambda, E \setminus \{a\}) \cap \lambda| = |\mathfrak{F}(X \cap \lambda, E \cap \lambda)| = |X \cap \lambda|^{|E \cap \lambda|}$$

Then

$$\begin{aligned} \text{num}(\mathfrak{F}_{fin}(X, E \setminus \{a\})) &= \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}_{fin}(X, E \setminus \{a\}) \cap \lambda| \\ &= \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}_{fin}(X \cap \lambda, (E \cap \lambda) \setminus \{a\})| \\ &= \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}(X \cap \lambda, E \cap \lambda)| \\ &= \lim_{\lambda \uparrow \Lambda} |E \cap \lambda|^{|X \cap \lambda|} = \text{num}(E)^{\text{num}(X)}. \end{aligned}$$

□

The cardinality of the continuum is $|\wp(\mathbb{N})|$ but we have seen that $2^{\mathbf{num}(\mathbb{N})}$ is the numerosity of the denumerable set $\wp_\omega(\mathbb{N})$. Actually the numerosity of $\wp(\mathbb{N})$ is given by $\beth_1$ as defined in section 3.2. Thus, in the theory of numerosities, there are three different kind of exponentiation: if $\beta = \mathbf{num}(E)$ and $\gamma = \mathbf{num}(C)$, we have:

- **hyperreal exponentiation:**

$$\beta^\gamma = \mathbf{num}(\mathfrak{F}_{fin}(C, B \setminus \{b\})) = \lim_{\lambda \uparrow \Lambda} |B \cap \lambda|^{C \cap \lambda} \quad (b \in B);$$

- **ordinal exponentiation:**

$$\beta^{\langle \gamma \rangle} = \sup_{\mathbf{Ord}} \left\{ \mathbf{num}(\mathbf{O}(\beta^{\langle x \rangle})) \mid x \in \mathbf{O}(\gamma) \right\};$$

- **cardinal exponentiation:** if β and γ are cardinal numerosities,

$$\beta^{\uparrow \gamma} := \mathbf{num}(\mathfrak{F}(C, B)) = \lim_{\lambda \uparrow \Lambda} |\mathfrak{F}(C, B) \cap \lambda|.$$

If β and γ are ordinal numerosities, $\beta^{\langle \gamma \rangle} = \mathbf{num}(\mathbf{Ord}(\beta^{\langle x \rangle}))$; if β and γ are cardinal numerosities, $\beta^{\uparrow \gamma} = \mathbf{num}(B^C)$. If $\gamma = n \in \mathbb{N}$, then

$$\beta^n = \beta^{\langle n \rangle} = \beta^{\uparrow n} = \underbrace{\beta \cdot \dots \cdot \beta}_{n \text{ times}}$$

but it is not surprising that these operations give different results when they are generalized, since they correspond to different ways of counting infinite sets. We have the following result:

Theorem 28 *If β and γ are ordinal numerosities,*

$$\beta^{\langle \gamma \rangle} \leq \beta^\gamma. \quad (24)$$

Proof: For $\gamma = 0$, the result is obvious. Now let us argue by induction: we have that

$$\beta^{\langle \gamma+1 \rangle} = \beta^{\langle \gamma \rangle} \cdot \beta \leq \beta^\gamma \cdot \beta = \beta^{\gamma+1}$$

and if γ is a limit ordinal, we have that, $\forall x < \gamma$,

$$\beta^\gamma = \mathbf{num}(\mathfrak{F}_{fin}(\mathbf{O}(\gamma), \mathbf{O}(\beta) \setminus \{0\})) > \mathbf{num}(\mathfrak{F}_{fin}(\mathbf{O}(x), \mathbf{O}(\beta) \setminus \{0\})) = \beta^x$$

Hence

$$\beta^{\langle \gamma \rangle} = \sup_{\mathbf{Ord}} \left\{ \beta^{\langle x \rangle} \mid x \in \mathbf{O}(\gamma) \right\} \leq \sup_{\mathbf{Ord}} \{ \beta^x \mid x \in \mathbf{O}(\gamma) \} \leq \beta^\gamma.$$

□

Example:

- $2^\omega = \text{num}(\wp_\omega(\mathbb{N}))$;
- $2^{(\omega)} = \sup_{\text{Ord}} \{2^x \mid x \in \mathbb{N}\} = \omega$;
- $2^{\uparrow\omega} = \text{num}(\wp(\mathbb{N})) = \beth_1$.

4.2 Numerosities and surreal numbers

In this section we will see how the Euclidean numbers can be identified with **No**, namely the field of surreal numbers, (see [14, 15, 24]) or, to be more precise the pseudoclass of surreal numbers "born" before the day κ . Hence **No**, using the terminology introduced in section 1.1, is a pseudoclass. It is well known that every hyperreal field can be embedded in **No** and that the Keisler hyperreal field of cardinality κ (see [27]) is isomorphic to the pseudoclass **No**; however, if we consider the hyperreal field of Euclidean numbers $\mathbb{E}$ (see Def. 20), there is a peculiar isomorphism which relates ordinal numbers, numerosities and surreal numbers.

In this section, we assume the reader to be familiar with the surreal numbers; however, we will recall some basic feature of **No** and we will fix the notation.

A surreal number can be represented by a sequence of +’s and –’s. For example $+ - - + +$ represents a number and we will write

$$x = (+ - - + +)$$

In a formal way, a surreal number can be identified with a map

$$x : \mathbf{O}(\gamma) \rightarrow \{+, -\};$$

the ordinal number γ is called **birthday** of x and it will be denoted by $\mathbf{b}(x)$.

For example, the ordinal numbers (which are supposed to be a sub-pseudoclass of **No**) can be represented as follows:

- $0 = ()$
- $1 = (+)$
- $2 = (++)$ etc.
- $\omega = (+ + \dots)$
- etc.

The surreal numbers can be lexicographically ordered with the convention that "undefined values" are greater than $-$ and less than $+$; for example

$$(-) < (-+) < () < (+-) < (+-+) < (+) < (++)$$

We now recall some facts relative to the surreal numbers which we will use later:

- The sequences of all $+$'s is can be identified with the ordinal numbers; hence we may assume $\mathbf{Ord} \subset \mathbf{No}$;
- $\mathbf{No}$ is a field and the operations $+$ and $\cdot$ restricted to $\mathbf{Ord}$ agree with the Hessenberg's operations.

Now, let us recall the "basic" theorem of the theory of surreal numbers. In order to do this, it is convenient the following notation: let $A, B \subset \mathbf{No}$, $(|A|, |B| < \kappa)$, be two sets such that $\forall a \in A, \forall b \in B, a < b$; then we set

$$I(A, B) = \{x \in \mathbf{No} \mid \forall a \in A, \forall b \in B, a < x < b\}$$

namely, $I(A, B)$ is an interval between A and B .

Theorem 29 (Conway simplicity theorem) *Let $A, B \subset \mathbf{No}$, $(|A|, |B| < \kappa)$, be two sets such that $\forall a \in A, \forall b \in B, a < b$; then there exists a unique $c \in I(A, B)$ such that*

$$\forall x \in I(A, B) \setminus \{c\}, \mathbf{b}(c) < \mathbf{b}(x).$$

The number c , as defined by the above theorem is denoted by:

$$c = A|B \quad (25)$$

Given a number $x \in \mathbf{No}$, the **Conway canonical form** of x is given by

$$x = L(x)|R(x)$$

where

$$\begin{aligned} L(x) &= \{t \in \mathbf{No} \mid t < x, \mathbf{b}(t) < \mathbf{b}(x)\} \text{ is the set of "left" numbers born before } x, \\ R(x) &= \{t \in \mathbf{No} \mid t > x, \mathbf{b}(t) < \mathbf{b}(x)\} \text{ is the set of "right" numbers born before } x. \end{aligned}$$

Now we set

$$\mathbf{No}(\gamma) = \{x \in \mathbf{No} \mid x < \gamma\}$$

and we recall an important result of Van den Dries and Ehrlich [19]:

Theorem 30 *If γ is an ε -number, i.e. $\gamma = \omega^{(\gamma)}$, then $\mathbf{No}(\gamma)$ is a subfield of $\mathbf{No}$.*

In order to proceed, we need to analyze some features of the ordinal numbers.

Theorem 31 *If γ is an ε -number, i.e. $\gamma = \omega^{(\gamma)}$, then $\mathbf{No}(\gamma)$ is a subfield of $\mathbf{No}$.*

Definition 32 *A ordinal number $\theta \neq 0$ is called **indecomposable**⁷ if*

$$\alpha, \beta, \gamma \in \mathbf{Ord}(\theta) \Rightarrow \alpha + \beta\gamma < \theta$$

⁷Sometimes in the literature, the word "indecomposable" is used as synonymous of "irreducible"; a number γ is irreducible if

$$\alpha, \beta < \gamma \Rightarrow \alpha + \beta < \gamma.$$

Here, we use it with a different meaning.

An ordinal $\gamma \neq 0$ is called **component** of β if there exist $x, y \in \mathbf{Ord}$ such that

$$\gamma + x = \beta \quad \text{or} \quad x + \gamma y = \beta.$$

In order to fix the ideas, we recall that the indecomposable ordinals have the following representation:

$$\theta_\gamma = \omega^{\langle \omega \rangle^{\langle \gamma \rangle}}, \quad \gamma \in \mathbf{Ord}. \quad (26)$$

The set $\mathbf{Ord}(\theta_j)$ is closed for the operations $+$ and $\cdot$ and every ordinal number $\beta \in \mathbf{Ord}(\theta_{j+1})$ can be written as follows:

$$\beta = \sum_{k=0}^n b_k \theta_j^k, \quad b_k \in \mathbf{Ord}(\theta_j) \quad (27)$$

Remark 33 *If $\beta > 0$, there is a unique polynomial of degree > 1 which allows to represent the ordinal number β in the form (27). In the context of numerosities this representation of an ordinal number is more convenient than the Cantor normal form, since the operations involved in it coincide with the operations defined for the numerosities (and hence for the Euclidean numbers).*

Now, we denote by $\mathbf{S}(\theta_\gamma)$ the field generated by $\mathbf{O}(\theta_\gamma) \cup \{\theta_\gamma\}$, namely smallest subfield of $\mathbf{No}$ containing $\mathbf{O}(\theta_\gamma) \cup \{\theta_\gamma\}$. Every $\sigma \in \mathbf{S}(\theta_\gamma)$ can be written as follows:

$$\sigma = \frac{\sum_{k=0}^n b_k \theta_\gamma^k}{\sum_{k=0}^n c_k \theta_\gamma^k}, \quad b_k, c_k \in \mathbf{S}(\theta_\gamma) \quad (28)$$

Moreover, for every number $\beth_j$ we define the field

$$\mathbf{S}(\beth_j^-) = \bigcup_{\theta_\gamma < \beth_j} \mathbf{S}(\theta_\gamma)$$

Since the ordinal numbers are surreal numbers, we may assume that

$$\mathbf{S}(\beth_j^-) \subset \mathbf{No}(\beth_{j+1})$$

In fact, if $\theta_\gamma = \mathbf{b}(\theta_\gamma) < \beth_{j+1}$, then $\mathbf{O}(\theta_\gamma) \cup \{\theta_\gamma\} \subset \mathbf{No}(\beth_{j+1})$; since $\beth_{j+1}$ is an ε -number, $\mathbf{No}(\beth_{j+1})$ is a field, and hence $\mathbf{S}(\theta_\gamma) \subset \mathbf{No}(\beth_{j+1})$.

The ordinal numbers can be identified also with the Euclidean numbers and by (19) and (23), we have that $f(\theta_\gamma) \in \mathbb{R}(\theta_\gamma)$ for every real function f ; hence $\mathbf{S}(\theta_\gamma) \subset \mathbb{R}(\theta_\gamma)$ and, if $\theta_\gamma < \beth_{j+1}$, $\mathbf{S}(\theta_\gamma) \subset \mathbb{R}(\beth_j)$. In conclusion,

$$\mathbf{S}(\beth_{j+1}^-) \subset \mathbb{R}(\beth_j).$$

Lemma 34 *For every $\xi \in \mathbb{R}(\beth_j)^+$, there exists $\sigma \in \mathbf{S}(\beth_{j+1}^-)^+$ such that $\sigma < \xi$.*

Proof: If $\xi \in \mathbb{R}(\sqsubset_j)$, then $1/\xi \in \mathbb{R}(\sqsubset_j)$ and by (16), $1/\xi < \theta_\gamma$ for some $\theta_\gamma < \sqsubset_j$. Then

$$\xi < \frac{1}{\theta_\gamma} \in \mathbf{S}(\sqsubset_{j+1}^-)^+$$

□

Corollary 35 $\mathbf{S}(\sqsubset_{j+1}^-)$ is dense in $\mathbb{R}(\sqsubset_j)$ with respect to the order topology, namely if $\xi \in \mathbb{R}(\sqsubset_j)$, then $\forall \varepsilon \in \mathbb{R}(\sqsubset_j)^+$, $\exists \sigma, \tau \in \mathbf{S}(\sqsubset_{j+1}^-)$, such that $\sigma \leq \xi \leq \tau$ and $\tau - \sigma \leq \varepsilon$.

Proof: It follows from lemma 34 and standard arguments.

□

Definition 36 A section of $\mathbf{S}(\sqsubset_j^-)$ is a pair of non empty sets (A, B) such that

- $\forall a \in A, \forall b \in B, a < b$.
- $A \cup B = \mathbf{S}(\sqsubset_j^-)$ or $\exists \sigma \in \mathbf{S}(\sqsubset_j^-), A \cup \{\sigma\} \cup B = \mathbf{S}(\sqsubset_j^-)$.

The set of all the sections of $\mathbf{S}(\sqsubset_j^-)$ will be denoted by $\mathbf{Sec}(\sqsubset_j^-)$. Every $\xi \in \mathbb{R}(\sqsubset_j)$ determines a section in $\mathbf{S}(\sqsubset_{j+1}^-)$ and we will use the following notation:

$$\mathcal{L}(\xi) = \{t \in \mathbf{S}(\sqsubset_{j+1}^-) \mid t < \xi\}; \quad \mathcal{R}(\xi) = \{t \in \mathbf{S}(\sqsubset_{j+1}^-) \mid t > \xi\}. \quad (29)$$

Also the converse is true:

Lemma 37 If (A, B) is a section of $\mathbf{S}(\sqsubset_{j+1}^-)$, there exist a unique number $\xi \in \mathbb{R}(\sqsubset_j)$ such that $(\mathcal{L}(\xi), \mathcal{R}(\xi)) = (A, B)$.

Proof: Given (A, B) , since $\mathbb{E}$ is κ -stured, there exists $\xi_0 \in \mathbb{E}$ such that

$$\forall a \in A, \forall b \in B, a < \xi_0 < b.$$

However, we cannot conclude that $\xi_0 \in \mathbb{R}(\sqsubset_j)$.

By definition of Euclidean number, we have that $\forall \sigma \in A, \forall \tau \in B$,

$$\xi_0 = \lim_{\lambda \uparrow \Lambda} \varphi_{\xi_0}(\lambda); \quad \sigma = \lim_{\lambda \uparrow \Lambda} \varphi_\sigma(\lambda); \quad \tau = \lim_{\lambda \uparrow \Lambda} \varphi_\tau(\lambda)$$

Choosing $\varphi_\sigma(\lambda), \varphi_{\xi_0}(\lambda), \varphi_\tau(\lambda)$ properly, we may assume that $\forall \lambda \in \mathfrak{L}$,

$$\varphi_\sigma(\lambda) < \varphi_{\xi_0}(\lambda) < \varphi_\tau(\lambda)$$

then, in particular, we have that

$$\varphi_\sigma(\lambda \cap \mathbf{O}(\sqsubset_j)) < \varphi_{\xi_0}(\lambda \cap \mathbf{O}(\sqsubset_j)) < \varphi_\tau(\lambda \cap \mathbf{O}(\sqsubset_j))$$

Since $\sigma, \tau \in \mathbb{R}(\sqsubset_j)$, taking the Λ -limit, we get

$$\sigma < \lim_{\lambda \uparrow \Lambda} \varphi_{\xi_0}(\lambda \cap \mathbf{O}(\sqsubset_j)) < \tau$$

The conclusion follows taking

$$\xi = \lim_{\lambda \uparrow \Lambda} \varphi_{\xi_0}(\lambda \cap \mathbf{O}(\sqsupset_j));$$

in fact, the uniqueness is guaranteed by Corollary 35.

□

By the Conway Simplicity Theorem, the map

$$\{\cdot\}|\{\cdot\} : \mathbf{Sec}(\sqsupset_{j+1}) \rightarrow \mathbf{No}(\sqsupset_{j+1}).$$

is well defined. So, we can define a map

$$i : \mathbb{R}(\sqsupset_j) \rightarrow \mathbf{No}(\sqsupset_{j+1}), \quad i(\xi) := \mathcal{L}(\xi)|\mathcal{R}(\xi) \quad (30)$$

Lemma 38 *The map $i : \mathbb{R}(\sqsupset_j) \rightarrow \mathbf{No}(\sqsupset_{j+1})$ is surjective.*

Proof: Given $x \in \mathbf{No}(\sqsupset_{j+1})$, we set

$$\mathcal{L}(x) := \{t \in \mathbf{S}(\sqsupset_{j+1}^-) \mid t < \xi\}; \quad \mathcal{R}(x) = \{t \in \mathbf{S}(\sqsupset_{j+1}^-) \mid t > \xi\}.$$

Notice that this definition is similar to (29), but in this case $x \in \mathbf{No}$ and the relations $<$ and $>$ are in $\mathbf{No}$. In any case, $(\mathcal{L}(x), \mathcal{R}(x))$ is a section in $\mathbf{S}(\sqsupset_{j+1}^-)$ and hence, by Lemma 37, there is a unique ξ between $\mathcal{L}(x)$ and $\mathcal{R}(x)$ and we have that

$$i(\xi) = \mathcal{L}(\xi)|\mathcal{R}(\xi) = \mathcal{L}(x)|\mathcal{R}(x) = x$$

□

In conclusion, a section in $\mathbf{S}(\sqsupset_{j+1}^-)$ individuates a number $x \in \mathbf{No}$ and a number in $\xi \in \mathbb{E}$ and these points can be identified. It remains to show that the operations $+$ and $\cdot$ in $\mathbf{No}$ coincide with the operations in $\mathbb{E}$. We recall that the operations in $\mathbf{No}$ satisfy the following equalities:

$$\begin{aligned} L(x+y) &= \{x^L + y, x + y^L \mid x^L \in L(x), y^L \in L(y)\} \\ R(x+y) &= \{x^R + y, x + y^R \mid x^R \in L(x), y^R \in L(y)\} \end{aligned}$$

and

$$\begin{aligned} L(xy) &= \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \mid x^L \in L(x), \dots, y^R \in L(y)\} \\ R(xy) &= \{x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L \mid x^L \in L(x), \dots, y^R \in L(y)\} \end{aligned}$$

Lemma 39 *If $\xi, \zeta \in \mathbb{R}(\sqsupset_j)$, then*

$$i(\xi + \zeta) = i(\xi) + i(\zeta)$$

$$i(\xi\zeta) = i(\xi) \cdot i(\zeta)$$

Proof: Let $x = i(\xi)$ and $y = i(\zeta)$. Since

$$\mathcal{L}(\xi)|\mathcal{R}(\xi) = x = L(x)|R(x)$$

we have that

$$I(\mathcal{L}(\xi), \mathcal{R}(\xi)) = I(L(x), R(x))$$

and similarly

$$(\mathcal{L}(\zeta), \mathcal{R}(\zeta)) = (L(y), R(y))$$

Then, it is easy to check that

$$I(\mathcal{L}(\xi + \zeta), \mathcal{R}(\xi + \zeta)) = I(L(x + y), R(x + y)).$$

In conclusion

$$i(\xi) + i(\zeta) = x + y = L(x + y)|R(x + y) = \mathcal{L}(\xi + \zeta)|\mathcal{R}(\xi + \zeta) = i(\xi + \zeta)$$

Using a similar arguments we can prove that $i(\xi) \cdot i(\zeta) = i(\xi\zeta)$ provided that $\xi, \zeta > 0$; if ξ or ζ is not positive the conclusion follows from standard algebraic manipulations.

□

In conclusion, we have proved the following theorem:

Theorem 40 *The map (30) is a field isomorphism.*

This result can be resumed as follows:

Corollary 41 *Let **Ord** be the pseudoclass of ordinal numbers equipped with the natural operations $+$ and $\cdot$ and let $\mathbb{F}$ be the smallest real closed field containing **Ord**; then $\mathbb{F}$ is isomorphic to both $\mathbb{E}$ and **No**.*

From now on the sets $\mathbb{E}$ and **No** will be identified and every Euclidean number will be considered also a surreal number.

Remark 42 *Th. 40 is new and suggests many directions for developing the theory of both hyperreal and surreal numbers. For example, given a real function, we can study the relationship between its natural extension in the set of hyperreal numbers and its extension in the set of surreal numbers (when and where it exists). Or, the relationship between hyperfinite and surreal sums.*

5 A construction of the numerosities

A numerosity theory is based on a counting system $(\Lambda, \mathbf{Num}, \mathbf{num})$ which satisfies the Euclid's Principle. Even if the Euclid's principle is a natural request, it is necessary to prove that it is consistent with the definition of counting system. We will prove such a consistency presenting a model based on a peculiar type of labellings $\mathfrak{L}$ called *label-tree*.

Figure 1: Schematic representation of a pivotal tree. Horizontal arrows connect points such that $a \equiv b$.

5.1 The label-trees

Our construction of a numerosity theory is based on a special labelling. In turn, this labeling is based on a peculiar partial order relation.

Definition 43 *If $X \subseteq \Lambda$, the triple $(X, \sqsubseteq, \oplus)$ is called “**pivotal tree**” if:*

1. $\emptyset \in X$ and $\forall x \in X, \emptyset \sqsubseteq x$.
2. $\oplus : X \setminus \{\emptyset\} \rightarrow X \setminus \{\emptyset\}$ is an injective map; the image $a^\oplus$ of an element will be called *successor* of “ a ”; also, we will use the notation $x^{m\oplus} := [x^{(m-1)\oplus}]^\oplus$;
3. $(X, \sqsubseteq)$ is a directed set⁸ such that
 - (a) $\forall b \in \mathbf{Fin}, (a \in b \text{ or } a \subseteq b) \Rightarrow a \sqsubseteq b$;
 - (b) $a \sqsubseteq b \Rightarrow \exists m \geq 1, a^{m\oplus} \equiv b$;
 - (c) $\forall a \in X$, the set $\{x \in X \mid x \sqsubseteq a\}$ is finite.

Let us analyze the structure of a pivotal tree. A pivotal tree can be regarded as an oriented graph in which the vertices are the element of X and the oriented arcs have the form $(a, a^\oplus)$. Notice that every chain (by virtue of (3b) and (3c)) has a minimum point but not a maximum. In general $\sqsubseteq$ is a preorder

⁸ $(X, \sqsubseteq)$ is called directed set if $\sqsubseteq$ is a preorder relation and

$$\forall x, y \in X, \exists z \in X, (x \sqsubseteq z \text{ and } y \sqsubseteq z);$$

As usual, we will employ also the following notation:

$$b \sqsupseteq a :\Leftrightarrow a \sqsubseteq b; \quad a \equiv b :\Leftrightarrow (a \sqsubseteq b \text{ and } b \sqsubseteq a).$$

relation and not a partial order relation, then $(X, \sqsubseteq)$ is not a lattice; however, if we combine $\sqsubseteq$ and $\oplus$, by Def. 43-(3b), we get a partial order relation

$$b \sqsubseteq_{\oplus} a :\Leftrightarrow \exists m \geq 1, b^{m\oplus} = a.$$

Then, we can define the *join* $\vee$ and the *meet* with respect to $\sqsubseteq_{\oplus}$, namely,

$$\begin{aligned} x \vee y &= \min\{z \mid x \sqsubseteq_{\oplus} z \text{ and } y \sqsubseteq_{\oplus} z\}. \\ x \wedge y &= \max\{z \mid z \sqsubseteq_{\oplus} x \text{ and } z \sqsubseteq_{\oplus} y\} \end{aligned}$$

Notice that, if $x \wedge y \neq \emptyset$, then $\exists m \geq 1, x^{m\oplus} = y$ or $y^{m\oplus} = x$.

Examples: A trivial example of pivotal tree is given by $(\mathbf{Ord}, \sqsubseteq, \oplus)$; if we set $\gamma^{\oplus} = \gamma + 1$ and

$$b \sqsubseteq a :\Leftrightarrow \exists m \in \mathbb{N}, b^{m\oplus} = b + m = a.$$

Let see an other example. We set

$$a \sqsubseteq b :\Leftrightarrow \exists b_1, \dots, b_n, \in \mathbf{Fin}, (a \in b_1 \in b_2 \dots \in b_n = b \text{ or } a \subseteq b), \quad (31)$$

moreover, we take a well ordering $\{a_j\}_{j \in \mathbf{Ord}}$ of Λ consistent with $\sqsubseteq$, namely

$$a_j \sqsubseteq a_k \Rightarrow j < k$$

We define $\oplus$ and $\sqsubseteq$ as follows,

$$a_j \sqsubset a_k :\Leftrightarrow (j < k \text{ and } a_j \sqsubseteq a_k); \quad a_j \equiv a_k :\Leftrightarrow a_j = a_k \quad (32)$$

$$a_j^{\oplus} := a_k \text{ where } k = \min\{p \in \mathbf{Ord} \mid a_p \sqsupseteq a_k\}$$

then it is easy to check that $(\Lambda, \sqsubseteq, \oplus)$ is a pivotal tree.

Theorem 44 *Given a pivotal tree $(\Lambda, \sqsubseteq, \oplus)$, there is a label lattice $(\mathfrak{L}, \subseteq)$, called **label-tree**, and a label map*

$$\ell : \Lambda \rightarrow \mathfrak{L}, \quad \ell(a) := \bigcup \{x \in \Lambda \mid x \sqsubseteq a\} = \{x \in \Lambda \mid \exists b \sqsubseteq a, x \sqsubseteq b\}$$

such that

1. if $\lambda, \mu \in \mathfrak{L}$, then $\lambda \cap \mu = \lambda \wedge \mu \in \mathfrak{L}$ and $\lambda \vee \mu \in \mathfrak{L}$;
2. $a \sqsubseteq b \Rightarrow \ell(a) \subseteq \ell(b)$;
3. $\ell(\ell(a)) = \ell(a)$;
4. $\ell(a \wedge b) = \ell(a) \wedge \ell(b)$,
5. $\ell(a \vee b) = \ell(a) \vee \ell(b)$,
6. $\ell(\{a, b\}) = \ell(\{a\}) \vee \ell(\{b\})$;

$$7. \ell((a, b)) = \ell(\{\{a\}\}) \vee \ell(\{\{b\}\}).$$

Proof: Given $a \in \Lambda$, we set

$$\lambda_a := \bigcup \{x \in \Lambda \mid x \sqsubseteq a\}$$

and $\mathfrak{L} = \{\lambda_a \mid a \in \Lambda\}$.

First of all we have to prove that $\mathfrak{L}$ is a labelling, namely that $(\mathfrak{L}, \subseteq)$ the requests of Def. 16 are satisfied. (1) and (3) are immediate. Let us see (2). We have that

$$\begin{aligned} \lambda_a \cap \lambda_b &= \left(\bigcup \{x \in \Lambda \mid x \sqsubseteq a\} \right) \cap \left(\bigcup \{x \in \Lambda \mid x \sqsubseteq b\} \right) \\ &= \bigcup \{x \in \Lambda \mid x \sqsubseteq a \cap b\} = \lambda_{a \cap b} \in \mathfrak{L} \end{aligned} \quad (33)$$

Similarly,

$$\begin{aligned} \lambda_a \cup \lambda_b &= \left(\bigcup \{x \in \Lambda \mid x \sqsubseteq a\} \right) \cup \left(\bigcup \{x \in \Lambda \mid x \sqsubseteq b\} \right) \\ &= \bigcup \{x \in \Lambda \mid x \sqsubseteq a \text{ or } x \sqsubseteq b\} \\ &\subseteq \bigcup \{x \in \Lambda \mid x \sqsubseteq_{\oplus} a \vee b\} = \lambda_{a \vee b}. \end{aligned} \quad (34)$$

Now, let us prove that

$$\ell(a) := \bigcap \{\mu \in \mathfrak{L} \mid a \in \mu\} = \lambda_a$$

It holds

$$\ell(a) = \bigcap \{\lambda_x \mid x \in \Lambda, a \in \lambda_x\}$$

Since $a \in \lambda_a$, we have that $\ell(a) \subseteq \lambda_a$. If $a \in \mu$, and $b \sqsubseteq a$, then $b \sqsubseteq \mu$; therefore $b \in \lambda_a \Rightarrow b \in \mu$. Then

$$\lambda_a \subseteq \bigcap \{\mu \in \mathfrak{L} \mid a \in \mu\} = \ell(a).$$

(1) follows from (33) and (34)

(2) - Trivial.

(3) - We have that

$$\ell(a) = \{x \in \Lambda \mid x \sqsubseteq a\} = \{x \in \Lambda \mid x \sqsubseteq \lambda_a\} = \ell(\lambda_a) = \ell(\ell(a)).$$

(4) - We have that

$$\begin{aligned} \ell(a) \wedge \ell(b) &= \lambda_a \cap \lambda_b = \bigcup \{x \in \Lambda \mid x \sqsubseteq a \text{ and } x \sqsubseteq b\} \\ &= \bigcup \{x \in \Lambda \mid x \sqsubseteq a \wedge b\} = \ell(a \wedge b) \end{aligned}$$

(5) - We have that

$$\begin{aligned} \ell(a) \vee \ell(b) &= \lambda_a \cup \lambda_b = \{x \in \Lambda \mid x \sqsubseteq a \text{ or } x \sqsubseteq b\} \\ &= \bigcup \{x \in \Lambda \mid x \sqsubseteq a \vee b\} = \ell(a \vee b) \end{aligned}$$

(6)- By Def.43-(1), we have that

$$\{a\} \sqsubseteq \{a, b\}, \{b\} \sqsubseteq \{a, b\}$$

and hence

$$\{a\} \vee \{b\} \sqsubseteq \{a, b\} = \{a\} \cup \{b\} \sqsubseteq \{a\} \vee \{b\};$$

then $\{a\} \vee \{b\} = \{a, b\}$ and

$$\ell(\{a, b\}) = \ell(\{a\} \vee \{b\}) = \ell(\{a\}) \vee \ell(\{b\})$$

(7) - By (6), identifying (a, b) with a Kuratowsky pair,

$$\begin{aligned} \ell((a, b)) &= \ell(\{\{a\}, \{a, b\}\}) = \ell(\{\{a\}\}) \vee \ell(\{\{a, b\}\}) \\ &= \ell(\{\{a\}\}) \vee \ell(\{\{a\}\}) \vee \ell(\{\{b\}\}) = \ell(\{\{a\}\}) \vee \ell(\{\{b\}\}). \end{aligned}$$

□

By this theorem, if we restrict $(\Lambda, \vee, \wedge)$ to $\mathfrak{L}$, we get a sublattice $(\mathfrak{L}, \vee, \wedge)$. The label map

$$\ell : \Lambda \rightarrow \mathfrak{L}$$

is an homomorphism from the pivotal tree $(\Lambda, \vee, \wedge)$ to the label-tree $(\mathfrak{L}, \vee, \wedge)$.

From the point of view of the graph theory, a label $\ell(a)$ is the union of all the chains which start at $\emptyset$ and end to a .

Examples: In order to familiarize the reader with the notion of label tree, we will give some trivial example using the partial order relation (32) with the further assumption that $\forall n \in \mathbb{N}, a_n = n$

- $\ell(3) = \{3, \emptyset\};$
- $\ell(\mathbb{N}) = \{\mathbb{N}, \emptyset\};$
- $\ell(\{2\}) = \{\{2\}, 2, \emptyset\};$
- $\ell(\{1, 2\}) = \{\{1, 2\}, \{1\}, \{2\}, 2, 1, \emptyset\}.$

5.2 Basic properties of label-trees

In this section we will prove some technical lemmas which will be used in the next sections.

Lemma 45 *If $(\mathfrak{L}, \sqsubseteq)$ is a label-tree and $\lambda, \mu \in \mathfrak{L}$, then*

$$\lambda \wedge \mu = \lambda \text{ or } \lambda \wedge \mu = \mu \text{ or } \lambda \wedge \mu = \emptyset.$$

Proof: Set $\lambda = \ell(a)$ and $\mu = \ell(b)$. If $a \sqsubseteq b$, then $\ell(a) \subseteq \ell(b)$; hence

$$\lambda \wedge \mu = \lambda \cap \mu = \ell(a) \cap \ell(b) = \ell(b) = \mu;$$

if $b \sqsubseteq a$, we argue in the same way. If a and b are not comparable, we claim that $\ell(a) \cap \ell(b) = \{\emptyset\}$; in fact, if it would exist $c \in \ell(a) \cap \ell(b)$, we would have that $a = c^{m \oplus}$ and $b = c^{n \oplus}$; namely a and b would be comparable.

□

Lemma 46 *If $(\mathfrak{L}, \subseteq)$ is a label-tree, there is a well ordering $\{\lambda_j\}_{j \in \Delta}$ of $\mathfrak{L}$ such that*

$$\lambda_k \subset \lambda_j \Rightarrow k < j.$$

Proof: We set $F_0 = \mathbf{Ato} \cup \mathbf{Inf}$ and, for $n \geq 0$, we will denote by F_{n+1} the family of elements in Λ which have a "minimal predecessor" in F_n , namely

$$a \in F_{n+1} : \Leftrightarrow [(\exists x \in F_n, x^\oplus = a) \text{ and } (y^\oplus = a \Rightarrow y \in F_m, m > n)].$$

Next, we take a well-ordering $\{a_j\}_{j \in \Delta}$ of $\Lambda \setminus \{\emptyset\}$ consistent with the sets F_n namely if

$$[a_k \in F_n \text{ and } a_j \in F_m, m > n] \Rightarrow k < j.$$

Then, $\{\lambda_j\}_{j \in \Delta}$ satisfies our request. In fact if $\lambda_k = \ell(a_k) \subset \lambda_j = \ell(a_j)$, we have that $a_k \in \ell(a_j)$ and hence $a_j \equiv a_k^{m \oplus}$. Then $k < j$.
□

Lemma 47 *Let $\{\lambda_j\}_{j \in \Delta}$ be of a well ordering $\mathfrak{L}$ as in lemma 46. Then, there exists finite sequence of sets $\{S_{j_0}, S_{j_1}, \dots, S_{j_n}\}$ such that*

$$p \neq q \Rightarrow S_{j_p} \cap S_{j_q} = \emptyset \tag{35}$$

and

$$\lambda_j = S_{j_0} \cup \dots \cup S_{j_n}$$

Proof: We set $j_0 = j$ and

$$j_1 := \max\{k \mid \lambda_k \subset \lambda_{j_0}\}, \quad S_{j_0} = \lambda_{j_0} \setminus \lambda_{j_1};$$

then, if $\lambda_{j_1} \neq \emptyset$,

$$\lambda_j = S_{j_0} \cup \lambda_{j_1}$$

If we iterate this operation with λ_{j_1} , we get

$$\lambda_j = S_{j_0} \cup S_{j_1} \cup \lambda_{j_2}$$

This process ends when $\{k \mid \lambda_k \subset \lambda_{j_n}\} = \emptyset$.
□

5.3 A numerosity counting system

In this section we will construct a *numerosity counting system* exploiting a *fine ultrafilter* over the label-tree.

Definition 48 *Given a label-tree, $(\mathfrak{L}, \subseteq)$ a **fine ultrafilter** $\mathcal{U}$ over $\mathfrak{L}$ is a family of subsets of $\mathfrak{L}$ which satisfies the following properties:*

1. $\mathfrak{L} \in \mathcal{U}$
2. if $Q \in \mathcal{U}$ and $P \supset Q$, then $P \in \mathcal{U}$,

3. if $P, Q \in \mathcal{U}$, then $P \cap Q \in \mathcal{U}$,
4. if $Q \in \mathcal{U}$, then $\wp_\omega(\Lambda) \setminus Q \notin \mathcal{U}$,
5. $\forall \lambda \in \mathfrak{L}, C[\lambda] \in \mathcal{U}$; here $C[\lambda]$ denotes the cone with the vertex in λ , namely

$$C[\lambda] := \{\mu \in \mathfrak{L} \mid \lambda \subseteq \mu\}. \quad (36)$$

It is well known that the existence of such ultrafilter is a consequence of Zorn's lemma. As usual, a set $Q \in \mathcal{U}$ is called **qualified**.

Definition 49 We set

$$A \preceq_n B$$

if there exists a qualified set $Q \in \mathcal{U}$ such that $\forall \lambda \in Q$

$$|A \cap \lambda| \leq |B \cap \lambda|. \quad (37)$$

The following result holds.

Theorem 50 The couple $(\Lambda, \preceq_n)$ is a comparison system

Proof: First of all, let us see that $\preceq_n$ is a preorder relation; if $A \preceq_n B$ and $B \preceq_n C$, there are two qualified sets $Q_1, Q_2 \in \mathcal{U}$ such that

$$\forall \lambda \in Q_1, |A \cap \lambda| \leq |B \cap \lambda| \quad \text{and} \quad \forall \lambda \in Q_2, |B \cap \lambda| \leq |C \cap \lambda|$$

then,

$$\forall \lambda \in Q_1 \cap Q_2, |A \cap \lambda| \leq |C \cap \lambda|;$$

since $Q_1 \cap Q_2 \in \mathcal{U}$, $A \preceq_n C$.

Now let us prove the points (1) - (5) of Def. 3.

3-(1) - Trivial.

3-(2) - If $A \cap B = A' \cap B' = \emptyset$, and $A \cong A'$, $B \cong B'$, then there are two qualified sets $Q_1, Q_2 \in \mathcal{U}$ such that

$$\forall \lambda \in Q_1, |A \cap \lambda| = |A' \cap \lambda|, \quad \forall \lambda \in Q_2, |B \cap \lambda| = |B' \cap \lambda|,$$

then, $\forall \lambda \in Q_1 \cap Q_2$,

$$\begin{aligned} |(A \cup B) \cap \lambda| &= |(A \cap \lambda) \cup (B \cap \lambda)| = |A \cap \lambda| + |B \cap \lambda| \\ &= |A' \cap \lambda| + |B' \cap \lambda| = |(A' \cup B') \cap \lambda|. \end{aligned}$$

3-(3) - Let

$$C[\ell((a, b))] := \{\lambda \in \mathfrak{L} \mid \lambda \supseteq \ell((a, b))\} \quad (38)$$

denote the cone over $\ell((a, b))$. By Prop.44 (7)-(7), and the request of Def. 3-(3), $\forall \lambda \in C[\ell((a, b))]$, we have that,

$$\begin{aligned} (A \times B) \cap \lambda &= \{x \in A \times B \mid x \in \lambda\} \\ &= \{(a, b) \mid a \in A, b \in B \text{ and } (a, b) \in \lambda\} \\ &= \{(a, b) \mid a \in A, b \in B \text{ and } \ell((a, b)) \in \lambda\} \\ &= \{(a, b) \mid a \in A, b \in B \text{ and } \ell(\{\{a\}\}) \vee \ell(\{\{b\}\}) \in \lambda\} \\ &= \{a \in A \mid \ell(\{\{a\}\}) \in \lambda\} \times \{b \in B \mid \ell(\{\{b\}\}) \in \lambda\} \end{aligned}$$

Since $\ell(a) \subseteq \ell(\{\{a\}\})$, we have that $(a \in A, \ell(\{\{a\}\}) = \lambda \Leftrightarrow \ell(a) \subseteq \lambda)$, then,

$$\begin{aligned} (A \times B) \cap \lambda &= \{a \in A \mid \ell(a) \in \lambda\} \times \{b \in B \mid \ell(b) \in \lambda\} \\ &= (A \cap \lambda) \times (B \cap \lambda). \end{aligned}$$

In conclusion, $\forall \lambda \in \mathfrak{L}$,

$$|(A \times B) \cap \lambda| = |(A \cap \lambda) \times (B \cap \lambda)| = |A \cap \lambda| \cdot |B \cap \lambda|$$

3-(4) - Let $C[\ell(a)]$ denotes the cone defined as in 38. By (3), for every label $\lambda \in C[\ell(a)]$

$$|(\{a\} \times B) \cap \lambda| = |\{a\} \cap \lambda| \cdot |B \cap \lambda| = 1 \cdot |B \cap \lambda| = |B \cap \lambda|.$$

The conclusion follows by the fact that $C[\ell(a)]$ is qualified.

3-(5) - If $A \preceq_n B$, $\exists Q \in \mathcal{U}$ such that $\forall \lambda \in Q$

$$|A \cap \lambda| \leq |B \cap \lambda|. \quad (39)$$

Let $\{\lambda_j\}_{j \in \Delta}$ be of a well ordering Q as in lemma 46. By lemma 47, we have that $\lambda_j = S_{j_0} \cup \dots \cup S_{j_n}$ with $j_0 = j$. We set $X_{j_m} := A \cap S_{j_m}$ ($0 < m < n$) and hence

$$A \cap \lambda_j = X_{j_0} \cup X_{j_1} \cup \dots \cup X_{j_n}$$

with $j_0 > \dots > j_n$ and by (35)

$$j_h \neq j_m \Leftrightarrow X_{j_h} \cap X_{j_m} = \emptyset.$$

Put $j_0 = \min(\Delta)$. Since $|A \cap \lambda_{j_0}| \leq |B \cap \lambda_{j_0}|$, $|X_{j_0}| < |B \cap \lambda_{j_0}|$; then we can take $Y_{j_0} \subset B \cap \lambda_{j_0}$ such that $|Y_{j_0}| = |X_{j_0}|$. Now, we claim that for every $k \leq j$, there are sets $Y_k \subset B$ such that

$$Y_k \subseteq (B \cap \lambda_k) \setminus [Y_{k_0} \cup \dots \cup Y_{k_0}]; \quad |Y_k| = |X_k|; \quad (40)$$

and

$$j_n \neq j_m \Leftrightarrow Y_{j_n} \cap Y_{j_m} = \emptyset.$$

We argue by induction over $j \in \Delta$. If $j_0 = \min(\Delta)$, (40) holds by the definition of Y_{j_0} . Now, we assume that (40) holds for every $k < j$, then

$$\begin{aligned} |X_j| &= |(A \cap \lambda_j) \setminus [X_{j_n} \cup \dots \cup X_{j_0}]| = |A \cap \lambda_j| - |X_{j_0}| - |X_{j_1}| - \dots - |X_{j_n}| \\ &\leq |B \cap \lambda_j| - |Y_{j_0}| - |Y_{j_1}| - \dots - |Y_{j_n}| = |(B \cap \lambda_j) \setminus [Y_{j_0} \cup \dots \cup Y_{j_n}]| \end{aligned}$$

Hence, it is possible to take a set $Y_j \subseteq (B \cap \lambda_j) \setminus [Y_{j_1} \cup \dots \cup Y_{j_n}]$ such that $|Y_j| = |X_j|$. Hence (40) holds for every $j \in \Delta$. Finally, we set

$$A' := \bigcup_{j \in \Delta} Y_j$$

and we have that

$$|A'_j \cap \lambda_j| = |Y_j \cup Y_{j_1} \cup \dots \cup Y_{j_n}| = |X_j \cup X_{j_1} \cup \dots \cup X_{j_n}| = |A_j \cap \lambda_j|$$

□

Corollary 51 *If we set*

$$\text{num}(A) := \Phi([A]_{\cong_n}).$$

and

$$\mathbf{Num} := \{x \in \mathbb{E} \mid \exists A \in *, x = \text{num}(A)\}$$

then $(\Lambda, \mathbf{Num}, \text{num})$ is a numerosity theory.

Proof: It is immediate to see that $(\Lambda, \mathbf{Num}, \text{num})$ satisfies the Euclid's Principle.

□

6 Special properties of numerosities

The properties of numerosities described in the previous sections are shared by every numerosity counting system. However if we want to answer to some specific questions, the information given by Def. 3 is not sufficient. For example, we cannot compare the numerosity of $\wp(\mathbb{N})$ and the numerosity of $\mathbb{R}$. In general given two sets A and B , not always it is possible to compare the Λ -limits of $|A \cap \lambda|$ and $|B \cap \lambda|$ since we do not have enough information. Actually different answers are consistent with the definitions/axioms of the theory. Therefore we can add new axioms and to check that they are consistent. However, from a technical point of view, it is easier to take an appropriate label-tree $\mathfrak{L}$ so that we can compare $|A \cap \lambda|$ and $|B \cap \lambda|$ for every sufficiently large label $\lambda \in \mathfrak{L}$. In other words, rather than adding new axioms, we can choose a suitable label-tree. Of course, this choice is absolutely arbitrary since it correspond to the addition of independent axioms. Anyway, we can enrich the theory with new properties which we will call "special".

6.1 The general strategy

In order to define appropriate label-trees, we will exploit the following theorem:

Theorem 52 *Given a set $\mathfrak{S} \subset \wp_\omega(\Lambda)$, if $\{\sigma \in \mathfrak{S} \mid a \in \sigma\} \neq \emptyset$, we put*

$$\ell_{\mathfrak{S}}(a) = \bigcap \{\sigma \in \mathfrak{S} \mid a \in \sigma\};$$

then, there exist a label-tree $(\mathfrak{L}_{\mathfrak{S}}, \subseteq)$ and a fine ultrafilter over $\mathfrak{L}_{\mathfrak{S}}$, such that

$$\ell_{\mathfrak{S}}(a) \in \mathfrak{L}. \tag{41}$$

and the set

$$Q_{\mathfrak{S}} := \{\ell(a) \mid a \in \bigcup \mathfrak{S}\}$$

is qualified relatively⁹ to $\bigcup \mathfrak{S}$. We will refer to $(\mathfrak{L}_{\mathfrak{S}}, \subseteq)$ as to a label tree *induced* by $\mathfrak{S}$.

Proof: First of all we define a preorder relation over Λ as follows:

$$a \sqsubseteq_{\mathfrak{S}} b :\Leftrightarrow (\ell_{\mathfrak{S}}(a) \subseteq \ell_{\mathfrak{S}}(b) \text{ or } a \in b) \quad (42)$$

Next we define a map $\oplus : \Lambda \rightarrow \Lambda$ so that $(\Lambda, \sqsubseteq_{\mathfrak{S}}, \oplus)$ be a pivotal tree as follows: we set

$$\begin{aligned} F_0 &= \mathbf{Ato} \cup \mathbf{Inf} \cup \{\emptyset\} \\ F_{n+1} &= \wp_n(F_n) \cup F_n \end{aligned}$$

where $\wp_n(X) := \{x \in \wp(X) \mid |x| \leq n\}$. Then $\Lambda = \bigcup_{n \in \mathbb{N}} F_n$. Let $\{a_j\}_{j \in \mathbf{Ord}}$ be a well ordering of Λ consistent with the F_n 's, namely such that, $a_0 = \emptyset$ and

$$(a_j \in F_n \text{ and } a_h \in F_m \setminus F_n) \Rightarrow j < k.$$

Now we set, for $j \geq 1$,

$$a_j^{\oplus} = a_k; \quad k = \min\{q \mid k > j, a_j \sqsubseteq_{\mathfrak{S}} a_k\}$$

Notice that this definition is well posed, in fact $a_j \sqsubseteq_{\mathfrak{S}} \{a_j\}$ and since $\{a_j\} = a_p$ for some $p > j$, we have that $\{q \mid k > j, a_j \sqsubseteq_{\mathfrak{S}} a_k\} \neq \emptyset$. It is easy to see that $(\Lambda, \sqsubseteq_{\mathfrak{S}}, \oplus)$ is a pivotal tree and hence by Th. 44 there exists a label-tree $(\mathfrak{L}_{\mathfrak{S}}, \subseteq)$. (41) is satisfied by construction. Now, it is sufficient to take a fine ultrafilter over $\mathfrak{L}_{\mathfrak{S}}$ which contains $Q_{\mathfrak{S}}$.

□

The introduction of $\mathfrak{S}$ and Th. 52 allows to compare the numerosities of suitable set exploiting the following proposition:

Proposition 53 *If $A, B \subset \bigcup \mathfrak{S}$, and $\forall \lambda = Q_{\mathfrak{S}}, |A \cap \lambda| = |B \cap \lambda|$ then $\text{num}(A) = \text{num}(B)$.*

Proof: Since $Q_{\mathfrak{S}}$ is qualified relatively to $\bigcup \mathfrak{S}$, there exists a qualified set $\hat{Q} \in \mathcal{U}$; thus $\forall \hat{\lambda} \in \hat{Q}$, we have that $|A \cap \hat{\lambda}| = |A \cap \lambda|$ and $|B \cap \hat{\lambda}| = |B \cap \lambda|$. Hence $\forall \hat{\lambda} \in \hat{Q}$, $|A \cap \lambda| = |B \cap \lambda|$; the conclusion follows by taking the Λ -limit.

□

Clearly, if we expand $\mathfrak{S}$, we get more information. Of course, it is necessary to do it in a suitable way.

⁹Given an ultrafilter $\mathcal{U}$ over $\mathfrak{L}$ and set X , we say that a set Q is qualified relatively to X if there exists a set $\hat{Q} \in \mathcal{U}$ such that

$$Q = \{\hat{\lambda} \cap X \mid \hat{\lambda} \in \hat{Q}\}$$

Definition 54 We say that a set $\mathfrak{S}_2$ is *compatible* with $\mathfrak{S}_1$, if $\forall a, b \in \mathfrak{S}_1 \cap \mathfrak{S}_2$,

$$a \cap b \in \mathfrak{S}_1$$

If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are compatible, we can expand $\mathfrak{S}_1$ by putting $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2$. The compatibility guarantees that for every $a \in \bigcup \mathfrak{S}_1$

$$\ell_{\mathfrak{S}}(a) = \ell_{\mathfrak{S}_1}(a).$$

6.2 The numerosities of some subsets of $\mathbb{R}$

6.2.1 Numerosity of the natural numbers

Our goal is to define a label-tree that provides "nice" properties to the subset of $\mathbb{N}$ following the procedure described in section 6.1. We set

$$\mathfrak{S}(\mathbb{N}) := \{\{0, \dots, n\} \mid n \in \Pi\} \text{ where } \Pi := \{m!^{m!} \mid m \in \mathbb{N}\}$$

and we apply Th. 52 to induce the label-tree $(\mathfrak{L}_{\mathbb{N}}, \subseteq)$. With this choice we have that $\forall n \in \mathbb{N}$,

$$\ell_{\mathbb{N}}(n) = \ell_{\mathfrak{S}}(\{n\}) = \{0, 1, 2, \dots, m!^{m!}\}$$

where $m = \min\{m \in \mathbb{N} \mid m!^{m!} \geq n\}$.

In order to simplify some algebraic manipulations, it is useful to introduce the number

$$\alpha := \text{num}(\mathbb{N}^+) = \lim_{\lambda \uparrow \Lambda} |\mathbb{N}^+ \cap \lambda| = \omega - 1 \quad (43)$$

The reason for such a choice of $\mathfrak{S}(\mathbb{N})$ is to ensure the following properties of α :

Theorem 55 Let $p \in \mathbb{N}^+$. Then

1. if for $i = 0, \dots, n-1$

$$A_i = \{n \in \mathbb{N}^+ \mid n \equiv i \pmod{p}\}.$$

then $\text{num}(A_i) = \frac{\alpha}{p}$;

2. if

$$E_p = \{x^p \in \mathbb{N}^+ \mid x \in \mathbb{N}^+\}$$

then

$$\text{num}(E_p) = \alpha^{\frac{1}{p}}.$$

Proof: (1) - For every $\lambda = \{0, 1, \dots, m!^{m!}\} \in Q_{\mathfrak{S}(\mathbb{N})}$, $m \geq p$, we have that

$$|A_i \cap \lambda| = \frac{m!^{m!}}{p} = \frac{|\mathbb{N}^+ \cap \lambda|}{p}$$

Hence,

$$\text{num}(A_i) = \lim_{\lambda \uparrow \Lambda} |A_i \cap \lambda| = \lim_{\lambda \uparrow \Lambda} \frac{|\mathbb{N}^+ \cap \lambda|}{p} = \frac{\alpha}{p}.$$

(2) - We have that

$$\begin{aligned} |E_p \cap \lambda| &= |\{x \in \mathbb{N}^+ \mid x^p \leq m!^{m!}\}| = |\{x \in \mathbb{N}^+ \mid x \leq m!^{\frac{m!}{p}}\}| \\ &= m!^{\frac{m!}{p}} = |\mathbb{N}^+ \cap \lambda|^{\frac{1}{p}} \end{aligned}$$

Then

$$\text{num}(E_p) = \lim_{\lambda \uparrow \Lambda} |\mathbb{N}^+ \cap \lambda|^{\frac{1}{p}} = \alpha^{\frac{1}{p}}$$

□

Remark 56 As we already observed, the choice of $\mathfrak{S}(\mathbb{N})$ is not intrinsic, and it has been done so to make it possible to have the properties listed in Proposition 55. Some additional motivations for this choice of $\mathfrak{S}(\mathbb{N})$ can be found in [5]. Different algebraic properties of the numerosity can be rather easily obtained by changing the choice of Π . Or if you like, you can assume that there is an other copy of the natural numbers $\mathbf{N} \subset \mathbf{Ato}$, $\mathbb{N} \cap \mathbf{N} = \emptyset$, and a labelling $\mathfrak{S}(\mathbf{N}) := \{\{0, \dots, n\} \mid n \in \Pi_1\}$.

6.2.2 Numerosity of the rational numbers

In order to get reasonable properties for the sets of rational numbers, we set

$$\mathfrak{s}(n) := \left\{ \frac{s}{n} \mid s \in \mathbb{Z}, -n^2 \leq s < n^2 \right\}$$

and

$$\mathfrak{S}(\mathbb{Q}) := \{\mathfrak{s}(n) \mid n \in \Pi\}.$$

Clearly, $\mathfrak{S}(\mathbb{Q})$ is compatible with $\mathfrak{S}(\mathbb{N})$ and hence we can take the set $\mathfrak{S}(\mathbb{Q}) \cup \mathfrak{S}(\mathbb{N}) = \mathfrak{S}(\mathbb{Q})$ and, via Th. 52 the induced lattice-tree $(\mathfrak{L}_{\mathbb{Q}}, \subseteq)$.

For every $q \in \mathbb{Q}$, it holds

$$\ell_{\mathbb{N}}(q) = \ell_{\mathbb{N}}(\{q\}) = \mathfrak{s}(n)$$

where $n = \min\{n \in \mathbb{N} \mid q \in \mathfrak{s}(n)\}$. This choice of $\mathfrak{S}(\mathbb{Q})$, allows to get the following result:

Theorem 57 *The following properties hold:*

1. $\text{num}(\mathbb{Q} \cap (0, 1]) = \alpha$;
2. for all $p, q \in \mathbb{Q}$ with $p < q$, $\text{num}(\mathbb{Q} \cap (p, q]) = (p - q) \alpha$;
3. for all $p, q \in \mathbb{R}$ with $p < q$,

$$\frac{\text{num}(\mathbb{Q} \cap (p, q])}{\alpha} \sim (p - q);$$

4. $\text{num}(\mathbb{Q}^+) = \alpha^2$;
5. $\text{num}(\mathbb{Q}) = 2\alpha^2 + 1$;
6. if $E \subset \mathbb{Q}$ is a bounded set, then, $\forall q \in \mathbb{Q}$

$$\text{num}(q + E) = \text{num}(E).$$

Proof: (1) Take $\lambda \in C[\ell_{\mathbb{Q}}(n)]$; we have that

$$|(\mathbb{Q} \cap (0, 1]) \cap \lambda| = n = |\mathbb{N}^+ \cap \lambda|;$$

the conclusion follows by taking the Λ -limit.

(2) - If $p = \frac{s_1}{n_1}$, $q = \frac{s_2}{n_2}$, take $\lambda \in C[\mathfrak{s}(m)]$ with m larger than $|p|, |q|, n_1 n_2$. Then

$$(\mathbb{Q} \cap \lambda) = (p - q)m = (p - q)|\mathbb{N}^+ \cap \lambda|;$$

the conclusion follows by taking the Λ -limit.

(3) - Take $\varepsilon \in \mathbb{R}^+$ and four numbers $p_{\varepsilon}^{\pm}, q_{\varepsilon}^{\pm} \in \mathbb{R}$ such that

$$p_{\varepsilon}^{-} < p < p_{\varepsilon}^{+} < q_{\varepsilon}^{-} < q < q_{\varepsilon}^{+}; \quad p_{\varepsilon}^{+} - p_{\varepsilon}^{-} \leq \varepsilon, \quad q_{\varepsilon}^{+} - q_{\varepsilon}^{-} \leq \varepsilon;$$

then,

$$\text{num}((p_{\varepsilon}^{+}, q_{\varepsilon}^{-}]) \leq \text{num}((p, q]) \leq \text{num}((p_{\varepsilon}^{-}, q_{\varepsilon}^{+}])$$

and by (2)

$$[(p - q) - 2\varepsilon] \alpha \leq \text{num}((p, q]) \leq [(p - q) + 2\varepsilon] \alpha;$$

hence,

$$\left| \frac{\text{num}((p, q])}{\alpha} - (p - q) \right| \leq 2\varepsilon$$

(4) - The map $\Phi : (0, 1] \times \mathbb{N}^+ \rightarrow \mathbb{Q}^+$ defined by

$$\Phi(x, n) = (n - 1) + x$$

is a comparison map, since, for a suitable $m \in \mathbb{N}$,

$$\ell(\Phi(x, n)) = \ell((n - 1) + x) = \mathfrak{s}(m) = \ell((x, n))$$

and hence, by Cor. 18,

$$\begin{aligned} \text{num}(\mathbb{Q}^+) &= \text{num}((0, 1] \times \mathbb{N}^+) = \text{num}([0, 1)) \cdot \text{num}(\mathbb{N}^+) \\ &= \text{num}([0, 1)) \cdot \text{num}(\mathbb{N}^+) = \alpha^2. \end{aligned}$$

(5) - Since $|\mathbb{Q}^+ \cap \lambda| = |\mathbb{Q}^- \cap \lambda|$, we have that $\text{num}(\mathbb{Q}^+) = \text{num}(\mathbb{Q}^-)$. Then

$$\text{num}(\mathbb{Q}^+) = \text{num}(\mathbb{Q}^+) + \text{num}(\mathbb{Q}^-) + \text{num}(\{0\}) = 2\alpha^2 + 1.$$

(6) - If $E \subset \mathfrak{s}(n_1)$ and $q \leq n_2$, then there exists n sufficiently large that $E \subset \mathfrak{s}(n)$ and $q + E \subset \mathfrak{s}(n)$; then $\forall \lambda \in C[\mathfrak{s}(n)]$, we have that

$$|E \cap \lambda| = |(q + E) \cap \lambda|$$

□

6.2.3 Numerosity of the real numbers

For every $\Xi \in \wp_\omega((0, 1])$ and every $n \in \Pi$, we set

$$\mathfrak{s}(\Xi, n) = \bigcup_{q \in \mathfrak{s}(n)} \left(q + \frac{1}{n} \Xi \right), \quad n \in \Pi$$

$$\mathfrak{S}(\mathbb{R}) := \{ \mathfrak{s}(\Xi, n) \mid \Xi \in \wp_\omega((0, 1]), \quad n \in \Pi \};$$

hence an element $\mathfrak{s} \in \mathfrak{S}(\mathbb{R})$ is contained in $[-n, n]$ and is the union of $2n^2$ copies of $\frac{1}{n}\Xi = \{ \frac{r}{n} \mid r \in \Xi \}$, each of them contained in an interval $[q, q + \frac{1}{n}]$. Clearly, $\mathfrak{S}(\mathbb{R})$ and $\mathfrak{S}(\mathbb{Q})$ are compatible and once again, we apply Th. 52 to the lattice-three $(\mathfrak{S}(\mathbb{R}) \cup \mathfrak{S}(\mathbb{Q}), \subseteq) = (\mathfrak{S}(\mathbb{R}), \subseteq)$ and we get the induced label-tree $(\mathfrak{L}_\mathbb{R}, \subseteq)$.

Now, we need to give a name to $\text{num}((0, 1])$; inspired by (43) and Prop. 57-(i), we put

$$\beta := \text{num}((0, 1])$$

Theorem 58 *The following properties hold*

1. for all $n \in \mathbb{N}$, $\text{num}((n, n+1]) = \beta$;
2. for all $p, q \in \mathbb{Q}$ with $p < q$, $\text{num}([p, q]) = (p - q)\beta$;
3. for all $p, q \in \mathbb{R}$ with $p < q$, $\text{num}([p, q]) \sim (p - q)\beta$;
4. $\text{num}(\mathbb{R}^+) = \alpha\beta$;
5. $\text{num}(\mathbb{R}) = 2\alpha\beta + 1$;
6. if $E \subset \mathbb{E}$ is a bounded set, then, $\forall q \in \mathbb{Q}$

$$\text{num}(q + E) = \text{num}(E).$$

Proof: It is similar to the proof of Prop. 57 with minor changes.

□

It is interesting to compare the numerosity of measurable subsets of $\mathbb{R}$ with their Lebesgue measure.

Definition 59 *For every $A \in \Lambda$, and every $\gamma \in \mathcal{N}$, we set*

$$m_\gamma(A) := st \left(\frac{\text{num}(A)}{\gamma} \right)$$

We will call $m_\gamma(A)$ γ -measure of A .

Lemma 60 *The γ -measure satisfies the following properties:*

1. it is finitely additive: for all sets A, B

$$m_\gamma(A \cup B) = m_\gamma(A) + m_\gamma(B) - m_\gamma(A \cap B);$$

2. it is superadditive, namely given a denumerable partition $\{A_n\}_{n \in \mathbb{N}}$ of a set $A \subset \mathbb{R}$, then

$$m_\gamma(A) \geq \sum_{n=0}^{\infty} m_\gamma(A_n).$$

Proof - (i) This is a trivial consequence of the additivity of the numerosity.

(ii) We have that for all $N \in \mathbb{N}$,

$$\text{num}(A) \geq \text{num}\left(\bigcup_{n=0}^N A_n\right) = \sum_{n=0}^N \text{num}(A_n),$$

hence

$$st\left(\frac{\text{num}(A)}{\gamma}\right) \geq st\left(\sum_{n=0}^N \frac{\text{num}(A_n)}{\gamma}\right) = \sum_{n=0}^N st\left(\frac{\text{num}(A_n)}{\gamma}\right);$$

therefore,

$$m_\gamma(A) \geq \sum_{n=0}^N m_\gamma(A_n).$$

The conclusion follows taking the Cauchy limit in the above inequality for $N \rightarrow \infty$.

□

Definition 61 A set $A \subset \mathbb{R}^N$ such that $m_\gamma(A) < \infty$, is called m_γ -measurable if

$$m_\gamma(A) = \sup \{m_\gamma(F) \mid F \subset A, F \text{ is closed}\}$$

Lemma 62 The family $\mathfrak{M}_\gamma^{<\infty}$ of m_γ -measurable sets form a σ -ring.

Proof: First let us prove that $\mathfrak{M}_\gamma^{<\infty}$ is a ring. Let $A, B \in \mathfrak{M}_\gamma^{<\infty}$; we have show that $A \cap B \in \mathfrak{M}_\gamma^{<\infty}$. Take $\varepsilon \in \mathbb{R}^+$ and two sets $F \subset A$ and $G \subset B$ such that $m_\gamma(A) - m_\gamma(F) \leq \varepsilon/2$, $m_\gamma(B) - m_\gamma(G) \leq \varepsilon/2$. Then

$$\begin{aligned} m_\gamma(A \cap B) - m_\gamma(F \cap G) &= m_\gamma((A \cap B) \setminus (F \cap G)) \\ &\leq m_\gamma([A \setminus F] \cup [B \setminus G]) \\ &\leq m_\gamma(A \setminus F) + m_\gamma(B \setminus G) \leq \varepsilon \end{aligned}$$

By the arbitrariness of ε the conclusion follows. Now let us prove the σ -additivity. Let $\{A_n\}_{n \in \mathbb{N}}$ be a denumerable partition $\{A_n\}_{n \in \mathbb{N}}$ of a set A and let $\varepsilon \in \mathbb{R}^+$; since $\sum_{n=0}^{\infty} m_\gamma(A_n) \leq m_\gamma(A)$, this series is convergent and hence there exists m such that $\sum_{n=m+1}^{\infty} m_\gamma(A_n) \leq \varepsilon/2$; now take a family of closed

sets F_n such that $m_\gamma(A_n) - m_\gamma(F_n) \leq 2^{-(n+2)}\varepsilon$. Then, putting $F = \bigcup_{n=0}^m F_n$ we have that F is closed and

$$\begin{aligned}
m_\gamma(A) - m_\gamma(F) &= m_\gamma\left(\bigcup_{n=0}^{\infty} A_n\right) - m_\gamma\left(\bigcup_{n=0}^m F_n\right) \\
&\leq \sum_{n=0}^{\infty} m_\gamma(A_n) - \sum_{n=0}^m m_\gamma(F_n) \\
&= \sum_{n=0}^m [m_\gamma(A_n) - m_\gamma(F_n)] + \sum_{n=m+1}^{\infty} m_\gamma(A_n) \\
&\leq \sum_{n=0}^m 2^{-(n+2)}\varepsilon - \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

□

From this lemma, the following theorem immediately follows:

Theorem 63 *If $\mathfrak{M}_\gamma$ is the σ -algebra generated by $\mathfrak{M}_\gamma^{<\infty}$; then $m_\gamma : \mathfrak{M}_\gamma \rightarrow [0, \infty]$ is a measure.*

By the above theorem and well known results of measure theory we get the following facts:

- the Borellian sets are m_γ -measurable;
- if A is m_γ -measurable, then

$$m_\gamma(A) = \inf \{m_\gamma(H) \mid A \subset H, H \text{ is open}\} = \sup \{m_\gamma(F) \mid F \subset A, F \text{ is closed}\}$$

From here, it follows that

Corollary 64 *If we take $\gamma = \beta$, then m_β is equal to the Lebesgue measure m_L .*

Corollary 65 *If $A \subset \mathbb{R}$ is a Lebesgue measurable set, then*

$$\text{num}(A) = (1 + \varepsilon)\beta m_L(A), \quad \varepsilon \sim 0.$$

Corollary 66 *If $A \subset \mathbb{R}^N$ is a Lebesgue measurable set, then*

$$m_N(A) = st \left(\frac{\text{num}(A)}{\beta^N} \right).$$

where m_N denotes the N -dimensional Lebesgue measure.

Proof: By Th. 58, we have that for all $p, q \in \mathbb{Q}$ with $p < q$, $\text{num}([p, q]) = (q - p)\beta$; hence

$$\text{num}([p_1, q_1] \times \dots \times [p_N, q_N]) = (q_1 - p_1) \cdot \dots \cdot (q_N - p_N) \beta^N$$

Then, if we take the γ -measure with $\gamma = \beta^N$, on the σ -algebra of the β^N -measurable sets, we have that

$$m_N(A) = m_{\beta^N}(A) = st\left(\frac{\text{num}(A)}{\beta^N}\right).$$

□

6.3 Complementary examples

In this section, we will give some examples to show how some special properties of the numerosities can be implemented in different circumstances.

6.3.1 Exponentiation of ordinal numerosities

In this section, we will examine the relation of the ordinal exponentiation $\beta^{\langle \gamma \rangle}$ and the numerosity exponentiation β^γ . It is not possible to require that $\forall \gamma \in \mathbf{Ord}$, $\beta^{\langle \gamma \rangle} = \beta^\gamma$ since the map $\langle \gamma \rangle \mapsto \beta^{\langle \gamma \rangle}$ has fixed points. For example, for every $n \in \mathbb{N}$, we have that $n^{\langle \omega \rangle} = \omega \neq n^\omega$. However, it is natural to investigate

when an ordinal number written in the Cantor normal form $\sum_{k=0}^n b_k \omega^{\langle j_k \rangle}$ is equal

to the numerosity $\sum_{k=0}^n b_k \omega^{j_k}$. The fix points of the ordinal exponentiation $\omega^{\langle \omega \rangle}$ are called ε -numbers and they are denoted by the symbol ε_j , $j \in \mathbf{Ord}$. In particular the smallest of them is given by:

$$\varepsilon_0 := \sup_{\mathbf{Ord}} \left\{ \omega^{\langle \omega^{\langle \omega^{\langle \dots \rangle} \rangle} \rangle} \right\}.$$

Then, $\omega^{\varepsilon_0} > \varepsilon_0 = \omega^{\langle \varepsilon_0 \rangle}$ and, by theorem 24, it follows that, if $\beta \geq \varepsilon_0$,

$$\beta = \sum_{k=0}^n b_k \omega^{\langle j_k \rangle} < \sum_{k=0}^n b_k \omega^{j_k}.$$

However, it is natural to require that

$$\sum_{k=0}^n b_k \omega^{\langle j_k \rangle} = \sum_{k=0}^n b_k \omega^{j_k} \quad \text{when } \beta < \varepsilon_0$$

This request can be satisfied if we choose a suitable set $\mathfrak{S}_{\mathbf{Ord}}$ and the induced label tree. To this aim, it is convenient to set

$$\mathfrak{s}(\beta) := \left\{ \sum_{k=0}^n c_k \omega^{\langle j_k \rangle} \mid c_k \subseteq_{\mathbb{N}} b_k \right\}, \quad \beta = \sum_{k=0}^n b_k \omega^{j_k}$$

and

$$\mathfrak{S}(\mathbf{Ord}) := \{\mathfrak{s}(\beta) \mid \beta \in \mathbf{Ord}\}.$$

The set $\mathfrak{S}(\mathbf{Ord})$ is compatibel with $\mathfrak{S}(\mathbb{R})$ since $\mathfrak{S}(\mathbf{Ord}) \cap \mathfrak{S}(\mathbb{R}) = \mathfrak{S}(\mathbb{N}) \subset \mathfrak{S}(\mathbb{R})$. Then we put $\mathfrak{S}(\mathbf{O}) := \mathfrak{S}(\mathbf{Ord}) \cup \mathfrak{S}(\mathbb{R})$ and we denote with $(\mathfrak{L}_{\mathbf{O}}, \subseteq)$ the label-tree induced by $\mathfrak{S}(\mathbf{O})$.

Now let us check some properties of the label $\ell_{\mathbf{O}}(a)$.

Lemma 67 *The label $\ell_{\mathbf{O}}(\gamma)$ of a ordinal number satisfies the following properties:*

- (i) $\ell_{\mathbf{O}}(\omega^{\langle j \rangle}) = \{\omega^{\langle j \rangle}\}$
- (ii) $\ell_{\mathbf{O}}(c\omega^{\langle j \rangle}) = \ell_{\mathbb{N}}(c) \vee \ell_{\mathbf{O}}(\omega^{\langle j \rangle})$;
- (iii) if $\gamma = \sum_{k=0}^n b_k \omega^{\langle j_k \rangle}$, then $\ell_{\mathbf{O}}(\gamma) = \bigvee_{k=0}^n [\ell_{\mathbb{N}}(c_k) \vee \ell_{\mathbf{O}}(\omega^{\langle j \rangle})]$

Proof: - (i) follows from the fact that the $\omega^{\langle j \rangle}$'s have no predecessor. Also we have that

$$\begin{aligned} \ell_{\mathbf{O}}(c\omega^{\langle j \rangle}) &= \{x \in \mathbf{Ord} \mid x \subseteq_{\mathbf{O}} c\omega^{\langle j \rangle}\} \\ &= \{b\omega^{\langle j \rangle} \mid b \subseteq_{\mathbb{N}} c\} = \ell_{\mathbb{N}}(c) \vee \ell_{\mathbf{O}}(\omega^{\langle j \rangle}). \end{aligned}$$

$$(iii) - \text{ If } \gamma \in \mathbf{O}(\Theta_1) = \sum_{k=0}^n b_k \omega^{\langle j_k \rangle},$$

$$\begin{aligned} \ell_{\mathbf{O}}(\gamma) &= \mathfrak{s}(\gamma) := \left\{ \sum_{k=0}^n b_k \omega^{\langle j_k \rangle} \mid b_k \omega^{\langle j_k \rangle} \subseteq_{\mathbb{N}} c_k \omega^{\langle j_k \rangle} \right\} \\ &= \bigvee_{k=0}^n [\ell_{\mathbb{N}}(c_k) \vee \ell_{\mathbf{O}}(\omega^{\langle j \rangle})]. \end{aligned}$$

□

The next theorem characterizes the numerosity exponentiation between ordinals.

Theorem 68 *If $\gamma \in \mathbf{Ord}$, then*

$$\omega^\gamma = \mathbf{num}(\mathfrak{F}_{fin}(\mathbf{O}(\gamma), \mathbb{N}^+))$$

Proof: Let us consider the map $\Psi : \mathfrak{F}_{fin}(\mathbf{O}(\gamma), \mathbb{N}^+) \rightarrow \mathbf{Ord}$ defined as follows

$$\Psi(f) = \sum_{\xi \in D_f} f(\xi) \omega^{\langle \xi \rangle}.$$

where $D_f \in \wp_\omega(\mathbf{O}(\gamma))$ is the domain of f . By Lemma 67-(iii), we get

$$\ell_{\mathbf{O}}(\Psi(f)) = \bigvee_{\xi \in D_f} [\ell_{\mathbf{O}}(\xi) \vee \ell_{\mathbf{O}}(f(\xi))].$$

Since the function f is identified with its graph which is a finite set, by Prop.44-(q4-q1), we have that

$$\begin{aligned} \ell_{\mathbf{O}}(f) &= \ell_{\mathbf{O}}(\{(\xi, f(\xi)) \mid \xi \in D_f\}) \\ &= \bigvee_{\xi \in D_f} \ell_{\mathbf{O}}((\xi, f(\xi))) = \bigvee_{\xi \in D_f} [\ell_{\mathbf{O}}(\{\{\xi\}\}) \vee \ell_{\mathbf{O}}(\{\{f(\xi)\}\})] \end{aligned}$$

Then

$$\begin{aligned} \ell_{\mathbf{O}}(f) \cap \mathbf{Ord} &= \left(\bigvee_{\xi \in D_f} [\ell_{\mathbf{O}}(\{\{\xi\}\}) \vee \ell_{\mathbf{O}}(\{\{f(\xi)\}\})] \right) \cap \mathbf{Ord} \\ &= \bigvee_{\xi \in D_f} ([\ell_{\mathbf{O}}(\{\{\xi\}\}) \cap \mathbf{Ord}] \vee [\ell_{\mathbf{O}}(\{\{f(\xi)\}\}) \cap \mathbf{Ord}]) \\ &= \bigvee_{\xi \in D_f} [\ell_{\mathbf{O}}(\xi) \vee \ell_{\mathbf{O}}(f(\xi))] = \ell_{\mathbf{O}}(\Psi(f)) \end{aligned}$$

Hence, by Prop.19, Ψ is a comparison bijection and by Prop.27

$$\mathbf{num}(\mathfrak{F}_{fin}(\mathbf{O}(\gamma), \mathbb{N}^+)) = \mathbf{num}(\mathbb{N})^{\mathbf{num}(\mathbf{O}(\gamma))} = \omega^\gamma$$

□

Theorem 69 For every $\gamma \in \mathbf{Ord}$, if $\gamma < \varepsilon_0$,

$$\omega^\gamma = \omega^{\langle \gamma \rangle} \tag{44}$$

Proof: We argue by induction over γ . If $\gamma = 0$, (44) holds trivially, and it holds also if γ is a successor. Now let us assume that (44) holds $\forall \beta < \gamma$ and let us prove it for γ :

$$\omega^\gamma = \mathbf{num}(\mathfrak{F}_{fin}(\mathbf{O}(\gamma), \mathbb{N}^+)) = \mathbf{num}\left(\bigcup_{\beta < \gamma} \mathfrak{F}_{fin}(\mathbf{O}(\beta), \mathbb{N}^+)\right) \leq \min\{\tau \in \mathbf{Ord} \mid \tau > \omega^\beta\}$$

Since $\tau < \varepsilon_0$,

$$\omega^\gamma \leq \min\{\tau \in \mathbf{Ord} \mid \tau > \omega^\beta\} \leq \sup_{\mathbf{Ord}} \{\omega^\beta \mid \beta < \gamma\}$$

By the inductive assumption, if $\beta < \gamma$, then $\omega^\beta = \omega^{\langle \beta \rangle}$ and hence

$$\omega^\gamma \leq \sup_{\mathbf{Ord}} \{\omega^{\langle \beta \rangle} \mid \beta < \gamma\} = \omega^{\langle \gamma \rangle}.$$

The conclusion follows from Th.28.

□

In conclusion, the set $\mathbf{O}(\varepsilon_0)$ is closed for exponentiation and the natural ordinal operations and the numerosity operation coincide.

6.3.2 $\beth_1$ versus β

Probably the first set having the cardinality of continuum which comes to your mind is either $[0, 1]$ or $\wp(\mathbb{N})$; we have seen that $\text{num}([0, 1]) = \beta + 1$ and $\text{num}(\wp(\mathbb{N})) = \beth_1$. It is natural to establish a relation between them. This can be done by choosing a suitable comparison map. Probably the most natural way to map $\wp(\mathbb{N})$ over $[0, 1]$ is the binary expansion of a real number given by

$$\psi(B) := \sum_{n \in B} 2^{-(n+1)}.$$

Hence, if $B \subset \wp(\mathbb{N})$ contains the number n , then, the $(n+1)$ -th digit of the dual expansion of $\psi(B)$ is "1". ψ cannot be a comparison map since it is not injective. However, its restriction to infinite set

$$\psi_{\text{r}} : \mathbf{Inf}(\mathbb{N}) \rightarrow (0, 1], \quad \mathbf{Inf}(\mathbb{N}) := \wp(\mathbb{N}) \cap \mathbf{Inf}$$

is bijective. If we want ψ_{r} to be a comparison map, it is sufficient to introduce the set

$$\mathfrak{S}(\psi_{\text{r}}) = \{\{x, \psi_{\text{r}}(x)\} \mid x \in \mathbf{Inf}(\mathbb{N})\};$$

The sets $\mathfrak{S}(\psi_{\text{r}})$ and $\mathfrak{S}(\mathbf{O})$ are trivially compatible since $\mathfrak{S}(\psi_{\text{r}}) \cap \mathfrak{S}(\mathbf{O}) = \emptyset$. We will denote by $(\mathfrak{L}_{\mathbf{O}, \psi_{\text{r}}}, \subseteq)$ the induced label tree. Using the labelling $\mathfrak{L}_{\mathbf{O}, \psi_{\text{r}}}$, we get the following results:

Theorem 70 *The numerosity of the unit interval is given by*

$$\text{num}([0, 1]) = \beth_1 - 2^\omega + 1$$

Proof: By our construction ψ_{r} is a comparison map, then applying the rules of numerosity:

$$\begin{aligned} \text{num}([0, 1]) &= \text{num}((0, 1]) + 1 = \text{num}(\psi_{\text{r}}^{-1}((0, 1])) + 1 \\ &= \text{num}(\mathbf{Inf}(\mathbb{N})) + 1 = \text{num}(\wp(\mathbb{N}) \setminus \wp_\omega(\mathbb{N})) + 1 \\ &= \text{num}(\wp(\mathbb{N})) - \text{num}(\wp_\omega(\mathbb{N})) + 1 \\ &= \beth_1 - 2^\omega + 1. \end{aligned}$$

□

Corollary 71 *We have that*

$$\beta = \beth_1 - 2^\omega$$

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